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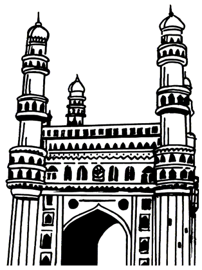

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UNIT - I

1. Let H be the set of all vectors of the from (a – 3b, b – a, a, b) where a and b are arbitrary scalars let H = {(a – 3b, b – a, a, b)}; a, b in R. Show that H is a subspace of R^4.

Sol : **(Nov./Dec.-2018)**

Refer Unit-I, Q.No. 10

---

2. If $W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} \middle/ a, b \in R \right\}$ **then find a matrix A such that w = Col A.**

Sol : **(Nov./Dec.-19)**

Refer Unit-I, Q.No. 27

---

3. Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ **and** $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

(a) Determine if u is null A? Could u be in Col A?

(b) Determine if v is in Col A. Could v be in Null A?

(OR)

Determine if $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$ **is in Col A where** $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$.

Sol : **(July-21)**

Refer Unit-I, Q.No. 29

---

4. Find a spanning set for the null space of the matrix,

$$A = \begin{bmatrix} -3 & 6 & -1 & 1-7 \\ 1 & -2 & 2 & 3-1 \\ 2 & -4 & 5 & 8-4 \end{bmatrix}$$

Sol : **(July-21)**

Refer Unit-I, Q.No. 31

5. **An Indexed set $\{v_1, v_2, \dots v_p\}$ of two or more vectors with $v_1 \neq 0$ is linearly dependent if and only if $\exists$ some v_j (with $j > 1$) is a linear combination of its preceding vectors $v_1, v_2, \dots v_{j-1}$.**

Sol : **(Nov./Dec.-2018)**

Refer Unit-I, Q.No. 37

6. **State and prove the spanning set theorem.**

Statement:

Let $S = \{v_1, v_2 \dots v_p\}$ be a set in v and $H = \text{span}\{v_1, v_2, \dots v_p\}$.

(i) If one of the vectors in S i.e., v_k is a linear combination of the remaining vectors in S then the set formed from S by reaming v_k still spans H.

(ii) If $H \neq \{0\}$ then some subset of S is a basis for H.

Sol : **(June/July-19, Nov./Dec.-18)**

Refer Unit-I, Q.No. 38

7. **Verify whether the vectors $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$ are linearly Indepenent.**

Sol : **(June/July-19)**

Refer Unit-I, Q.No. 40

8. **Let $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ then determine if $\{v_1, v_2, v_3\}$ is a basis for R^3.**

Sol : **(July-21, Nov./Dec-18)**

Refer Unit-I, Q.No. 44

9. **Suppose $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $H = \left\{ \begin{bmatrix} S \\ S \\ 0 \end{bmatrix} \Big/ S \in R \right\}$ then is (v_1, v_2) a basis for H?**

Sol : **(June/July-19)**

Refer Unit-I, Q.No. 45

10. **If a vector space V has a basis $B = \{b_1, b_2 \dots b_n\}$, then any set in V containing more than n vectors must be linearly dependent.**

Sol : **(July-21)**

Refer Unit-I, Q.No. 57

UNIT - II

1. State and prove the Rank theorem?

Sol : **(July-2021, June-July-2019, Nov.-Dec.-2018)**

Refer Unit-II, Q.No. 1

2. If A = $\begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$ then find rank A and dim null A ?

Sol : **(July-2021)**

Refer Unit-II, Q.No. 5

3. Given a Matrix A = $\begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}$ then find Rank of A and dim Null A

Sol : **(Nov.-2018, Dec.-2018)**

Refer Unit-II, Q.No. 6

4. Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ and consider the bases of R^2 given by $\beta = \{b_1, b_2\}$ and $c = \{c_1, c_2\}$. Find the change of coordinates matrix from β to e.

Sol : **(Nov.-2018, Dec.-2018)**

Refer Unit-II, Q.No. 9

5. Show that the eigen values of a Triangular Matrix are the entries of its Main diagonal.

Sol : **(June-2019/July-2019, Nov/Dec.-2018)**

Refer Unit-II, Q.No. 19

6. Find the characteristic polynomial and the real eigen values of the matrix A = $\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$

Sol : **(June / July-2019)**

Refer Unit-II, Q.No. 23

7. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

Sol : **(Nov./Dec.-2018)**

Refer Unit-II, Q.No. 24

8. Find the eigen values of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ **and compare this result with eigenvalue of** A^T

Sol : **(Nov./ Dec.-2019, Nov./ Dec.-2018)**

Refer Unit-II, Q.No. 28

9. Find the characteristic equation of the matrix $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Sol : **(June /July-2019, Nov./Dec.-2019)**

Refer Unit-II, Q.No. 31

UNIT - III

1. Show that an n×n matrix with n distinct eigen values is diagonalizable.

Sol : **(June/July-19, Nov./Dec.-18)**

Refer Unit-III, Q.No. 3

2. Diagonalize $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ **if possible**

Sol : **(July-21)**

Refer Unit-III, Q.No. 5

3. Determine whether the following matrix is diagonalizable or not,

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Sol : **(June/July-19)**

Refer Unit-III, Q.No. 9

4. Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ **if possible.**

Sol : **(Nov./Dec.-19)**

Refer Unit-III, Q.No. 12

5. If $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ **find a formula for** A^K **given that** $A = PDP^{-1}$ **where** $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

Sol : **(Nov./Dec.-19)**

Refer Unit-III, Q.No. 13

6. Find the complex eigen values of the matrix $A = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$.

Sol : **(July-2021)**

Refer Unit-III, Q.No. 26

UNIT - IV

1. Show that $\{u_1, u_2, u_3\}$ **is an orthogonal set where**

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix};\ u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix};\ u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

Sol : **(Dec.-18)**

Refer Unit-IV, Q.No. 28

2. Verify the set of vectors are orthogonal.

(a) $\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$

Sol : **(July-21, July-19)**

Refer Unit-IV, Q.No. 29

3. Find the projection of y = $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$ onto u = $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ Also write y as the sum of two orthogonal vectors are in Span {u} and one orthogonal to u.

Sol : **(Dec.-19)**

Refer Unit-IV, Q.No. 42

4. Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

Sol : **(July-21)**

Refer Unit-IV, Q.No. 43

UNIT I	**Vector Spaces :** Vector Spaces and Subspaces -Null Spaces, Column Spaces, and Linear Transformations - Linearly Independent Sets; Bases - Coordinate Systems - The Dimension of a Vector Space

1.1 Vector Space

Q1. Define vector space.

Sol :

A vector space is a non empty set V of objects called vectors on which are defined two operations called addition and multiplication by scalars subject to ten axioms.

(I) $(V_1 +)$ is abelian vector addition.

1. $u + v \in v$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is zero vector 0 in v such that $u + 0 = u$.
5. For each u in v there is a vector –u in v such that $u + (–u) = 0$.

(II) Scalar Multiplication

6. The scalar multiple of u by c that is $cu \in v$.
7. $c(u + v) = cu + cv$
8. $(c + d) u = cu + du$
9. $c(du) = (cd) u$
10. $1.u = u$

The space R^n; where $n \geq 1$ is a vector space.

Q2. For $n \geq 0$ the set p_n of polynomials of degree atmost n consists of all polynomials of the form.

$P(t) = a_0 + a_1t + a_2t^2 + ... + a_nt^n$

Where coefficient a_0, a_1 .... a_n and variable t are real numbers Here degree is n.

Sol :

Given p_n set of polynomials

Let $p(t), q(t) \in p_n$

$p(t) = a_0 + a_1t + a_2t^2 + ... a_nt^n$

$q(t) = b_0 + b_1t + b_2t^2 + ... b_nt^n$

Vector addition

1. $p(t) + q(t) = a_0 + a_1t + a_2t^2 + ... a_nt^n + b_0 + b_1t + b_2t^2 + ... b_nt^n$

   $= (a_0 + b_0) + (a_1 + b_1) t + (a^2 + b^2) t^2 + ... + (a_n + b_n) t^n \in p_n$ [closure]

2. $p(t) + [q(t) + r(t)] = [p(t) + q(t)] + r(t)$ [Associative]

   let $r(t) = c_0 + c_1t + c_2t^2 + ... c_nt^n$.

3. Let $0(t) = 0 + 0t + 0t^2 + ... + 0t^n$ be the zero polynomial.

   $p(t) + 0(t) = p(t)$ [Additive Identity]

4. Since (–1) p(t) acts as negative of p(t)

   $\therefore \forall\, p(t) \in p_n \exists\ (-1)\ p(t) \in p_n \ni$

   $p(t) + (-1)\, p(t) = 0(t)$ [Additive Inverse]

5. $p(t) + q(t) = q(t) + p(t)$ [Cumulative law]

Scalar Multiplications

6. Scalar multiple c_p is a polynomial defined by

   $c.p(t) = c[a_0 + a_1t + ... + a_n t^n]$

   $= ca_0 + ca_1t + ... + ca_n t^n \in p_n$.

7. $(c + d)\, p(t) = cp(t) + dp(t)$

8. $c(p(t) + q(t)) = cp(t) + cq(t)$

9. $c[d(p(t))] = cd[p(t)]$

10. $1.p(t) = p(t)$ [Mul. Identity]

    $\therefore$ Thus p_n is a vector space.

Q3. Define vector subspace and give examples.

Sol : **(Nov./Dec.-2019)**

Let H be a non empty subset of a vector space V that it is said to be a vector subspace of V if it satisfy the following conditions.

1. The zero vector of v is in H

   $\Rightarrow O \in H$

2. H is closed under vector addition $\forall U, V \in H$

   $\Rightarrow U + V \in H$

3. H is closed under scalar multiplication for each $U \in H\ \exists$ scalar $C \ni CU \in H$.

Example

1. The set containing of only the zero vector in a vector space V is a subspace of V called the zero subspace of written as {0}.

2. The vector space R^2 is not a subspace of R^3 because R^2 is not even a subset of R^3. Since vectors in R^3 all have three entries where vectors in R^2 have only two entries.

Q4. Define Linear Combination

Sol :

Sum of scalar multiples of vectors and span $\{v_1, v_2, ...v_p\}$ densest the set of all vectors that can be written as linear combination of $v_1, v_2,.....v_p$.

L.C $= a_1v_1 + a_2v_2 + ... a_pv_p \ \forall \ a_1, a_2,....a_p$ are scalars.

Q5. Given v_1 and v_2 in a vector space V, let H = span $\{v_1, v_2\}$ show that H in a subspace of V.

Sol : **(June/July-19)**

Given H = span $\{v_1, v_2\}$

The zero vector is in H since $o = ov_1 + ov_2$

To show let is closed under vector addition

Let $u = a_1 u_1 + a_2 u_2$ $\quad w = b_1v_1 + b_2v_2$

$u + w = (a_1 v_1 + a_2 v_2) + (b_1 v_1 + b_2 v_2)$

$= (a_1 + b_1) v_1 + (a_2 + b_2) v_2$

$\therefore u + w \in H.$

To show H is closed under scalar multiplication let c be any scalar and $u = a_1 v_1 + a_2 v_2$

$cu = c(a_1 v_1 + a_2 v_2)$

$= (ca_1) v_1 + (ca_2)v_2$

$\therefore cu \in H.$

$\therefore$ H is closed under vector addition and scalar multiplication.

$\therefore$ Thus H is a subspace of v.

Q6. If $v_1, v_2, ... v_p$ are in a vector space v then span $\{v_1, v_2, ... v_p\}$ is a subspace of v.

Sol :

(i) The zero vector is in H

$\therefore O = ov_1 + ov_2 + ... ov_p$

(ii) Let u, v any two arbitrary vectors in H

$u = a_1 v_1 + a_2 v_2 + ... a_p v_p$, $\quad v = b_1 v_1 + b_2 v_2 + ... b_p v_p$

Where $a_1 \ a_2 .... a_p, b_1, b_2, ... b_p$ are scalars.

Now

$u + v = (a_1 v_1 + a_2 v_2 + ... + a_p v_p) + (b_1 v_1 + b_2v_2 + ... + b_pv_p)$

$= (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + ... + (a_p + b_p)v_p$

$\therefore \ u + v \in H$

$\therefore$ H is closed under vector addition.

for any scalar c

$cu = c(a_1 v_1 + a_2 v_2 + ... + a_p v_p)$

$= (ca_1)v_1 + (ca_2)v_2 + ... + (ca_p)v_p$

$\therefore$ $cu \in H$

$\therefore$ H is closed under scalar multiplication

$\therefore$ H is a subspace of V.

Q7. Let w_1 is a subspace of V(F) and w_2 is a subspace of V(F) then $w_1 \cap w_2$ is also subspace of V(F).

(OR)

Inter section of two subspaces is again a subspace.

Sol :a **(July-21)**

Let given w_1 and w_2 are two subspaces of a vectors space V(F).

$\Rightarrow$ $o \in w_1$ and $o \in w_2$ [zero vector]

$\Rightarrow$ $o \in w_1 \cap w_2$

$\therefore$ $w_1 \cap w_2 \neq \{0\}$ [non empty]

$\therefore$ (i) Let $u_1 v \in w_1$ and w_2

$\Rightarrow$ $u + v \in w_1$ (vector addition)

$u + v \in w_2$ (vector addition)

$\therefore$ $u + v \in w_1 \cap w_2$

(ii) Let $u \in w_1$ and w_2

$\exists$ any scalar $c \ni$

$cu \in w_1$, $cu \in w_2$

$cu \in w_1 \cap w_2$ (Scalar)

$\therefore$ $w_1 \cap w_2$ is a subspace of V(F).

Q8. If H and K are subspaces of a vector space then H + K is also subspace of vector space V(F).

Sol :

Given V(F) is a vector space

H and k are subspaces of V

Define H + K = {w; U + v = w; for some u in H and v in k}

Given H is a subspace of V(F)

$0 \in H$ ... (1)

k is a subspace of v(F)

$0 \in k$ ... (2)

from (1) and (2) $0 \in H + k$

$\therefore H + k \neq \{ \}$ (non empty)

Let $w_1, w_2 \in H + K$

$w_1 = u_1 + v_1$ where $u_1 \in H, v_1 \in K$

$w_2 = u_2 + v_2$ where $u_2 \in H, v_2 \in K$

$w_1 + w_2 = (u_1 + v_1) + (u_2 + v_2)$

$= (u_1 + u_2) + (v_1 + v_2)$

$\in H + K$ [since $u_1 + u_2 \in H, v_1 + v_2 \in K$]

$\therefore H + K$ is closed under vector addition.

Let $cu_1 \in H$, $cu_1 \in K$

$cu_1 + cv_1 \in H + K$

$c(u_1 + v_1) \in H + K$

$cw_1 \in H + K$

$H + K$ is closed under scalar multiplication.

$\therefore$ $H + K$ is a subspace of V(F).

Q9. The union of two subspaces is again a subspace if and only if one is contained in another.

(OR)

The union of two subspaces is again subspaces $\Leftrightarrow H_1 \subseteq H_2$ (or) $H_2 \subseteq H_1$.

Sol :

Let H_1 and H_2 be two subspaces of V(F).

Case (1)

If $H_1 \cup H_2$ is a subspace of a vector space V(F).

Then we have to show that $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

If possible assume that $H_1 \not\subseteq H_2$ or H_2 H_1.

Since $H_1 \not\subseteq H_2$ so $\exists$ $a \in H_1$ and $a \notin H_2$

Since $H_2 \not\subseteq H_1$ so $\exists$ $b \in H_1$ and $b \notin H_2$

But $a \in H_1 \cup H_2$ and $b \in H_1 \cup H_2$

$\Rightarrow$ $a + b \in H_1 \cup H_2$

$\Rightarrow$ $a + b \in H_1$ and $a + b \in H_2$

Since $a + b \in H_1$ and $a \in H_1$, as

As H_1 in a subspace of V(F)

$(-1)\, a + a + b = b \in H_1$ [Closure] ... (1)

Similarly $a+b \in H_2$, $b \in H_2$

As H_2 is a subspace of V(F)

$\Rightarrow$ $a + b + (-1)\, b \in H_2$ [closure]

$\Rightarrow$ $a + b - b \in H_2$

$\Rightarrow$ $a \in H_2$ ... (2)

Which is a contradiction to our assumption that $a \notin H_2$ and $b \notin H_1$

So our assumption is using

$\therefore$ either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

Case (ii)

If $H_1 \subseteq H_2$ (or) $H_2 \subseteq H_1$ then we have to show $H_1 \cup H_2$ is a subspace of V(F).

Since $H_1 \subseteq H_2 \Rightarrow H_1 \cup H_2 = H_2$

We know that H_2 is a subspace of V(F)

So $H_1 \subseteq H_2$ is also subspace of V(F).

Case (iii)

If $H_2 \subseteq H_1 \Rightarrow H_1 \cup H_2 = H_1$

We know that H_1 is a subspace of V(F)

So $H_1 U H_2$ is also subspace of V(F)

$\therefore$ $H_1 U H_2$ is subspace of vector space V(F).

Q10. Let H be the set of all vectors of the from (a – 3b, b – a, a, b) where a and b are arbitrary scalars let H = {(a – 3b, b – a, a, b)}; a, b in R. Show that H is a subspace of R^4.

Sol : **(Nov./Dec.-2018)**

Given H = {(a – 3b, b – a, a, b)}

Write vectors in H as column vectors, then an arbitrary vector in H has the form.

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = av_1 + av_2$$

$\therefore$ H = Span $\{v_1, v_2\}$ where v_1, v_2 are the vectors. Thus H is a subspace of R^4.

Q11. Show that, H = $\left\{\begin{bmatrix} s \\ t \\ 0 \end{bmatrix}; s, t \in IR\right\}$ is a subspace of R^3.

Sol :

Given H = $\left\{\begin{bmatrix} s \\ t \\ 0 \end{bmatrix}; s, t \in IR\right\}$

H can be written as a linear combination of vectors.

$$H = \left\{ S\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Big/ S, t \in R \right\}$$

$$H = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$$= \text{span}\{u, v\} \text{ where } u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, V = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ H is a subspace of R^3

Q12. Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ o & d \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

Sol :

Given, H is a set of all matrices of the form $\begin{bmatrix} a & b \\ o & d \end{bmatrix}$,

Let $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$; $U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \in H$; $V = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \in H$

Then

(i) $u + v = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$$

$u + v \in H$

(ii) Let c = 1

$$\text{Then } cu = 1\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} \in H$$

$\therefore$ H is a subspace of $M_{2\times2}$

Q13. Let $V_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$ $V_2 = \begin{bmatrix}2\\1\\3\end{bmatrix}$ $V_3 = \begin{bmatrix}4\\2\\6\end{bmatrix}$ and $W = \begin{bmatrix}8\\4\\7\end{bmatrix}$ Is w in the subspace spanned by $\{v_1, v_2, v_3\}$? Why ?

Sol :

Given vectors are $V_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}, V_2 = \begin{bmatrix}2\\1\\3\end{bmatrix}, V_3 = \begin{bmatrix}4\\2\\6\end{bmatrix}, W = \begin{bmatrix}8\\4\\7\end{bmatrix}$

Let $a_1, a_2, a_3 \in R$

W = Linear combination of vectors

$= a_1 v_1 + a_2 v_2 + a_3 v_3$

$$\begin{bmatrix}8\\4\\7\end{bmatrix} = a_1\begin{bmatrix}1\\0\\-1\end{bmatrix} + a_2\begin{bmatrix}2\\1\\3\end{bmatrix} + a_3\begin{bmatrix}4\\2\\6\end{bmatrix}$$

The augmented matrix $\begin{bmatrix}1 & 2 & 4 & 8\\0 & 1 & 2 & 4\\-1 & 3 & 6 & 7\end{bmatrix}$

Apply Row Operation:

$$R_3 \to R_3 + R_1 \sim \begin{bmatrix}1 & 2 & 4 & 8\\0 & 1 & 2 & 4\\0 & 5 & 10 & 15\end{bmatrix}$$

$$R_3 \to \frac{R_3}{5} \sim \begin{bmatrix}1 & 2 & 4 & 8\\0 & 1 & 2 & 4\\0 & 1 & 2 & 3\end{bmatrix}$$

$$R_3 \to R_3 - R_2 \sim \begin{bmatrix}1 & 2 & 4 & 8\\0 & 1 & 2 & 4\\0 & 0 & 0 & -1\end{bmatrix}$$

$$R_1 \to R_1 - 2R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{-1} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$R_2 \to R_2 - 4R_3 \sim \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Two columns c_2 and c_3 are indentical and has no solution.

$\therefore$ W is not a subspace spanned by $\{v_1, v_2, v_3\}$.

Q14. Given v_1 and v_2 in a vector space v and let H = span $\{v_1, v_2\}$ then H is a subspace of V.

Sol :

i) The zero vector is in H.

ii) Let u, v be any two arbitrary vectors in H.

$u = s_1 v_1 + s_2 v_2$ and $v = t_1 v_1 + t_2 v_2$

Where s_1, s_2, t_1, t_2 are scalars

consider $u + v = (s_1 v_1 + s_2 v_2) + (t_1 v_1 + t_2 v_2) = (s_1 + t_1)v_1 + (s_2 + t_2)v_2$

$\Rightarrow u + v \in H$

= H is classed under vector addition.

iii) For any scalar c,

$cu = c(s_1 v_1 + s_2 u_2) = (cs_1) v_1 + (cs_2)v_2 \quad \Rightarrow \quad cu \in H$

$\therefore$ H is classed under scalar multiplication

$\therefore$ H is a subspace of v.

Q15. Let H be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Find a vector v in R^3 such that H = span{v}. Why does this show that H is a subspace of R^3.

Sol :

The given vector space is R^3

The given set is $H = \left\{ \begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix} \middle/ S \in R \right\}$

Let $v = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \in R^3$

Consider $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix} \in H$

Then $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix} = S\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ where $S \in R$

Thus every vector in H can be written as the linear combination of the vector v in R^3.

$\therefore$ H = span {v}

H is a subspace of V = R^3.

Q16. Let w be the set of all vectors of the form $\begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix}$ where b and c are arbitrary real numbers. Find the vectors u and v such that w = span {u, v}, why does this show that w is a subspace of R^3.

Sol :

The given vector space is R^3.

The given set is $w = \left\{ \begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix} \middle/ b, c \in R \right\}$

Here w is a non-empty subset of R^3 consisting of zero vector consider the vector $\begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix}$ of w.

$$\begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \text{ for } b, c \in R.$$

$= b.u + c.v$ where $u = (5, 1, 0)$, $v = (2, 0, 1)$

Thus every element in w can be written as the linear combination of u, v.

$\therefore$ w = span {u.v}

$\therefore$ w is a subspace of R^3.

Q17. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$; $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$; $v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

(a) Is w in $\{v_1, v_2, v_3\}$? How many vectors are in $\{v_1, v_2, v_3\}$?

(b) How many vectors are in span $\{v_1, v_2, v_3\}$ why ?

(c) Is w is subspace spanned by $\{v_1, v_2, v_3\}$ why ?

Sol :

The given vectors are $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$; $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$; $v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

(a) There are only three vectors in $\{v_1, v_2, v_3\}$. w is not in $\{v_1, v_2, v_3\}$.

(b) As there are many linear combinations that are possible with the vectors v_1, v_2, v_3.

$\therefore$ There will be an infinite number of elements in span $\{v_1, v_2, v_3\}$

(c) Consider the given vectors, w = (3, 1, 2), v_1 = (1, 0, –1), v_2 = (2, 1, 3), v_3 = (4, 2, 6)

$(3, 1, 2) = 1(1, 0, -1) + 1(2, 1, 3) + 0(4, 2, 6)$

$\therefore\ w \in \text{span}\{v_1, v_2, v_3\}$.

Q18. Let w be the set of all vectors of the form $\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}$ where, a, b, c are real numbers.

Find a set S of vectors that spans w or give an example to show that w is not a vector space.

Sol :

The given set is $w = \left\{ \begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix} \Big/ a, b, c \in R \right\}$ consider the general vector of w say $\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}$. This vector can be expressed in the form of a linear combination of vectors is,

$$\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix} = a\begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Thus the set S = {(4, 0, 1, –2), (3, 0, 1, 0), (0, 0, 1, 1)} spans the set w.

Q19. Show that w is in the subspace of R^4 spanned by v_1, v_2, v_3, where w = $\begin{bmatrix}-9\\7\\8\\4\end{bmatrix}$; $v_1 = \begin{bmatrix}7\\-4\\-2\\9\end{bmatrix}$,

$v_2 = \begin{bmatrix}-4\\5\\-1\\-7\end{bmatrix}$, $v_3 = \begin{bmatrix}-9\\4\\-4\\-7\end{bmatrix}$.

Sol :

Let c_1, c_2, c_3 be any three scalars.

Suppose that w can be written as the linear combinations of v_1, v_2, v_3.

Then $w = c_1 v_1 + c_2 v_2 + c_3 v_3$

$(-9, 7, 8, 4) = c_1(7, -4, -2, 9) + c_2(-4, 5, -1, -7) + c_3(-9, 4, 4, -7)$ ... (1)

$7c_1 - 4c_2 - 9c_3 = -9$ ... (2)

$-4c_1 - 5c_2 + 4c_3 = 7$ ... (3)

$-2c_1 - c_2 + 4c_3 = 8$ ... (4)

$9c_1 - 7c_2 - 7c_3 = 4$ ... (5)

Consider the augment matrix

$$= \begin{bmatrix} 7 & -4 & -9 & : & -9 \\ -4 & 5 & 4 & : & 7 \\ -2 & -1 & 4 & : & 8 \\ 9 & -7 & -7 & : & 4 \end{bmatrix}$$

$R_2 \to 7R_2 + 4R_1$; $R_3 \to 7R_3 + 2R_1$; $R_4 \to 7R_4 - 9R_1$

$$= \begin{bmatrix} 7 & -4 & -9 & -9 \\ 0 & 19 & -8 & 13 \\ 0 & -15 & 10 & 10 \\ 0 & -13 & 32 & 137 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{5} \Rightarrow \begin{bmatrix} 7 & -4 & -9 & -9 \\ 0 & 19 & -8 & 13 \\ 0 & -3 & 2 & 2 \\ 0 & -13 & 32 & 137 \end{bmatrix}$$

$R_3 \to 19R_3 + 3R_2$; $R_4 \to 19R_4 + 13R_2$

$$\Rightarrow \begin{bmatrix} 7 & -4 & -9 & -9 \\ 0 & 19 & -8 & 13 \\ 0 & 0 & 14 & 77 \\ 0 & 0 & 504 & 2772 \end{bmatrix}$$

$R_2 \to \frac{R_3}{14}$; $R_4 \to \frac{R_4}{504}$

$$\Rightarrow \begin{bmatrix} 7 & -4 & -9 & -9 \\ 0 & 19 & -8 & 13 \\ 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 1 & \frac{11}{2} \end{bmatrix}$$

$R_1 \to R_1 + 9R_3; R_2 \to R_2 + 8R_3 ; R_4 \to R_4 - R_3$

$$\Rightarrow \begin{bmatrix} 7 & -4 & 0 & \frac{81}{2} \\ 0 & 19 & 0 & 57 \\ 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{R_2}{19}$

$$\Rightarrow \begin{bmatrix} 7 & -4 & 0 & \frac{81}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + 4R_2$

$$\Rightarrow \begin{bmatrix} 7 & 0 & 0 & \frac{105}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{R_1}{7}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{15}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solutions are,

$$c_1 = \frac{15}{2}, c_2 = 3; \ c_3 = \frac{11}{2}$$

w is subspace of R^4 spanned by v_1, v_2, v_3

if $w = c_1 v_1 + c_2 v_2 + c_3 v_3$

i.e., $w = \frac{15}{2} v_1 + 3v_2 + \frac{11}{2} v_3$

$\therefore$ w is in a subspace of R^4.

Q20. Determine weather the following vectors form a sub space or not.

(i) $\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$ **; a, b are scalars**

(ii) $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$ **; a, b, c are scalars**

Sol :

(i) $W = \begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$

Here w cannot be expressed as linear combination of vectors and w does not contain zero vector.

$\therefore$ W is not a vector space.

(ii) $W = \begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$

$$W = a\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ W can i.e.,, written as a linear consideration of vectors.

$\therefore$ W forms a subspace.

Q21. Define Null Space

Sol : **(Nov./Dec.-19)**

The null space of an m × n matrix A, written as Null A is the set of all solutions of the homogeneous equation A× = 0.

Null A = {X : X is in R^n and AX = 0}

Q22. The null space of m×n matrix A is a subspace of R^n.

Sol :

Certainly Null A is a subset of R^n (since A has n columns).

Since O $\in$ Null A

$\Rightarrow$ Null A $\neq$ { }

Let u, v $\in$ Null A

AU = 0 and AV = 0

(i) To show that U + V is in Null A

A(U+V) = AU + AV

= 0 + 0

= 0

$\therefore$ U + V $\in$ Null A

(ii) Let c be any scalar

A(cu) = C(Au) = c(0) = 0

$\therefore$ cu $\in$ Null A

Null A is a subspace of R^n.

Q23. Define Column Space

Sol :

The column space of an m × n matrix A, written as Col. A is the set of all linear combination of the columns of A. If A = $\{a_1, a_2, ... a_n\}$ then Col A = span $\{a_1, a_2, ... a_n\}$.

Q24. The column space of an m×n matrix A is a subspace of R^m.

Sol :

Col A can be written as Ax for some × Since the notation Ax stands for linear combination of the columns of A.

That is Col A = {b : b = Ax for some x in R^n}

Q25. Let A = $\begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and w = $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Determine whether 'w' is in col A. Is w in null A ?

Sol :

The given matrix A is of order 2× 2.

The given vector w is of order 2×1.

Consider

$$AW = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -12+12 \\ -6+6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

$\therefore$ W is a solution of A× = 0

$\therefore$ W is in Null A.

Consider

$$[A : W] = \begin{bmatrix} -6 & 12 & : & 2 \\ -3 & 6 & : & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -6 & 12 & : & 2 \\ 0 & 0 & : & 0 \end{bmatrix}$$

This system is consistent

$\therefore$ w is in Col A.

Q26. Let H b the set of all vectors in R4 whose co-ordinates a, b, c, d satisfy the equations a – 2b + 5c = d and c – a = b. Show that H is a subspace of R^4.

Sol :

The given vector space is R^4

The given equations are $a - 2b + 5c = d$, $c - a = b$

Rewriting these equations we get, $a - 2b + 5c - d = 0$

$$-a - b + c = 0$$

These equations can be written as $AX = 0$

$$\text{Where } A = \begin{bmatrix} 1 & -1 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}_{2\times 4} ; x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{4\times 1}$$

Let H be the set of all solutions × of AX = 0.

Then H is a subspace of R^4.

Q27. If $W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} \Big/ a, b \in R \right\}$ **then find a matrix A such that w = Col A.**

Sol : **(Nov./Dec.-19)**

Consider any general vector of w say $\begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix}$ for a, $b \in R$. This can be written as the linear combination of the vectors $\begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ as below.

$$\begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} = a\begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ where } a, b \in R.$$

$$\text{Thus } w = \text{span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

With these vectors as columns form a matrix say A.

$$A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$$

Then w = col A.

Q28. Find a non-zero vector in Null A and a non-zero vector in Col A. If,

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}.$$

Sol :

The given matrix is $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

i) Finding a non-zero vector in Null A : Consider the matrix,

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

Let us reduce it to echelon form by using Row operation.

$R_2 \to R_2 + R_1$; $R_3 \to 2R_3 - 3R_1$

$$= \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 2 & -10 & 9 \end{bmatrix}$$

$$R_2 \to R_3 + 2R_2 \Rightarrow \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

$$R_2 \to -R_2 \Rightarrow \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

$$R_1 \to R_1 - 4R_2 \Rightarrow \begin{bmatrix} 2 & 0 & 18 & 17 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{17} \Rightarrow \begin{bmatrix} 2 & 0 & 18 & 17 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 17R_3$; $R_2 \to R_2 + 4R_3$

$$= \begin{bmatrix} 2 & 0 & 18 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{2} \Rightarrow \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The general Solution is,

$x_1 + 9x_3 = 0 \Rightarrow x_1 = -9x3$

$x_2 - 5x_3 = 0 \Rightarrow x_2 = 5x_3$

$x_4 = 0$

and x_3 is a free variable.

Let $x_3 = 1$ (non - zero value)

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ 1 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

The non-zero vector in Null A is

x = (–9, 5, 1, 0)

(ii) To find a non-zero vector in Col A

Any column in the matrix A will be a non-zero vector in Col A.

i.e., $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$ (or) $\begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}$ (or) $\begin{bmatrix} -2 \\ 7 \\ -8 \end{bmatrix}$ (or) $\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$.

Q29. Let $\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, **u =** $\begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ **and**

v = $\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

(a) Determine if u is null A? Could u be in Col A?

(b) Determine if v is in Col A. Could v be in Null A?

(OR)

Determine if $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$ **is in Col A where**

$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$.

Sol : **(July-21)**

The given matrix is $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}_{3\times 4}$

with $m = 3, n = 4$.

(a) Consider $Au = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$

$Au = \begin{bmatrix} 6-8+2+0 \\ -6-10-7+0 \\ 9-14+8+0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ u does not satisfy the equation $AX = 0$

$\therefore$ u is not in Null A

As Col A is a Subspace of $R^m = R^3$.

As us is of order 4×1 u can be in Col A.

(b) Here $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}_{3\times 1}$

As Null A is a subspace of $R^n = R^4$ and as the order v is 3×1 v can not be in Null A.

Consider $[A : V] = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & -1 & 5 & +4 & 2 \\ 0 & 1 & -5 & \frac{9}{2} & \frac{3}{2} \end{bmatrix}$

$\sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & 2 \\ 0 & 1 & -5 & \frac{9}{2} & \frac{3}{2} \end{bmatrix}$

$\sim \begin{bmatrix} 2 & 4 & -2 & 1 & : & 3 \\ 0 & 1 & -5 & -4 & : & 2 \\ 0 & 0 & 0 & \frac{17}{2} & : & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & 4 & -2 & 1 & : & 3 \\ 0 & 1 & -5 & -4 & : & 2 \\ 0 & 0 & 0 & 17 & : & 1 \end{bmatrix}$

From this enchelon form the system is consistent hence the vector is in Null A.

Q30. Let A be the matrix of order 2×3 that is

$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ **and** $U = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$.

Determine if U belongs to the null space of A.

Sol :

To test if U satisfies $AU = 0$

$AU = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5-9+4 \\ -25+27-2 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus U is in Null A.

Q31. Find a spanning set for the null space of the matrix,

$A = \begin{bmatrix} -3 & 6 & -1 & 1-7 \\ 1 & -2 & 2 & 3-1 \\ 2 & -4 & 5 & 8-4 \end{bmatrix}$

Sol : **(July-21)**

Working Rule

Step 1

Find the general solution of AX = 0.

Step 2

Reduce the augmented matrix [A0] to Echelon form using Row operations.

Step 3

Write basic variables in forms of the free variables.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1-7 & 0 \\ 1 & -2 & 2 & 3-1 & 0 \\ 2 & -4 & 5 & 8-4 & 0 \end{bmatrix}$$

$$R_1 : R_1 + 4R_2 \sim \begin{bmatrix} \boxed{1} & -2 & 7 & 13 & -11 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix}$$

$$\begin{matrix} R_3 : R_3 - 2R_1 \\ R_2 : R_2 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 7 & 13 & -11 & 0 \\ 0 & 0 & -5 & -10 & 10 & 0 \\ 0 & 0 & -9 & -18 & 18 & 0 \end{bmatrix}$$

$$\begin{matrix} R_2 : \frac{R_2}{-5} \\ R_3 : \frac{R_3}{-9} \end{matrix} \sim \begin{bmatrix} 1 & -2 & 7 & 13 & -11 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{bmatrix}$$

$$R_3 : R_3 - R_2 \sim \begin{bmatrix} 1 & -2 & 7 & 13 & -11 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 - 2x_2 + 7x_3 + 13x_4 - 11x_5 = 0$

$x_3 + 2x_4 - 2x_5 = 0$

$x_3 = -2x_4 + 2x_5$

$x_1 = 2x_2 - 7(-2x_4 + 2x_5) - 13x_4 + 11x_5$

$x_1 = 2x_2 + x_4 - 3x_5$

Decompose the vector giving general solution into a linear combination of vectors.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_1 U + x_4 V + x_5 W$$

Every linear combination of U, V and W is an element of Null A This {U, v, w} is a spanning set for Null A.

Q32. Find an explict description of Null A by listing vectors that span the Null space.

$$\mathbf{A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}}.$$

Sol :

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

Null A is given by AX = 0

The augmented matrix for Null A is

$$[A0] = \begin{bmatrix} 1 & 3 & 5 & 0 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - 3R_2 \sim \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix}$$

The general solution is

$$\left.\begin{matrix} x_1 - 7x_3 + 6x_4 = 0 \\ x_2 + 4x_3 - 2x_4 = 0 \end{matrix}\right\} \begin{matrix} \Rightarrow x_1 = 7x_3 - 6x_4 \\ \Rightarrow x_2 = -4x_3 + 2x_4 \end{matrix}$$

x_3, x_4 are free variable

$$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7x_3 - 6x_4 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

x = span (u, v)

The spanning set for Null A $= \left\{ \begin{bmatrix} +7 \\ -4 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Note :

If 'A' is a matrix of order m × n, then

- Nul A is a subspace of R^n
- Col A is a subspace of R^m

1.2 Linear Transformations

Q33. Define linear transformation Kernel, Range.

Sol :

T from a vector space v into a vector space W is a rule that assigns to each vector x is V a unique vector T(x) in W, such that,

(i) T(u + v) = T(u) + T(v) for all u, v in V

(ii) T(cu) = cT(u) for all u in V and all scalars c.

Kernel (or Nullspace) Kernel T is a set of all u in V such that T(u) = 0.

Range

Range of T is the set of all vectors in W of the form T(x) for some x in V.

Q34. Define T: $p_2 \to R^2$ by T(p) = $\begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$. For instance if p(t) = 3 + 5t + 7t² Then T(p) = $\begin{bmatrix} 3 \\ 15 \end{bmatrix}$ show that T is a linear Transformation.

Sol :

Given T: $p_2 \to R^2$

and T(p) = $\begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$

let p, q ∈ = p_2

$$T(p+q) = \begin{bmatrix} (p+q)(0) \\ (p+q)(1) \end{bmatrix}$$

$$= \begin{bmatrix} p(0)+q(0) \\ p(1)+q(1) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(0) \\ q(1) \end{bmatrix}$$

= T(P) + T(q)

∴ T(p+q) = T(p) + T(q)

Let c be any scalar.

$$T(cp) = \begin{bmatrix} cp(0) \\ cp(1) \end{bmatrix} = c\begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = CT(p)$$

∴ T(cp) = CT(p)

∴ T is a linear Transformation

Q35. Let $M_{2\times2}$ **be the vector space of all 2×2 matrices and define T:$M_{2\times2} \to M_{2\times2}$ by T(A) = A + A^T where A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.**

(i) Show that T is a linear transformation.

(ii) Let B be any element of $M_{2\times2}$ such that,

B^T= B Find an A in M_{2x2} such that T(A) = B.

(iii) Show that the range of T is the set of B in $M_{2\times2}$ with the properly that B^T = B.

(iv) Describe the kernel of T.

Sol : **(Nov./Dec.-2019)**

Given $M_{2\times2}$ is a vector space

T:$M_{2\times2} \to M_{2\times2}$ defined by T(A) = A + A^T ;

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(i) Let $A, B \in M_{2\times 2}$

$$T(A+B) = (A+B) + (A+B)^T$$
$$= A + B + A^T + B^T$$
$$= (A + A^T) + (B + B^T)$$
$$= T(A) + T(B)$$

Let $\exists$ c a scalar such that

$$T(CA) = (CA) + (CA)^T$$
$$= CA + CA^T$$
$$= C(A + A^T)$$
$$T(CA) = CT(A)$$

$\therefore$ T is a linear transformation.

(ii) Given $B = M_{2\times 2}$ such that $B^T = B$

Let $A = \frac{1}{2}B$

$$T(A) = A + A^T$$
$$= \frac{1}{2}B + \left(\frac{1}{2}B\right)^T$$
$$= \frac{1}{2}B + \frac{1}{2}B \quad [B^T = B]$$
$$= B$$

(iii) Let B be in the range of T

Then $B = T(A)$
$$= A + A^T$$
$$B^T = (A + A^T)^T$$
$$= A^T + (A^T)^T$$
$$= A^T + A = B$$
$$\therefore B^T = B$$

(iv) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be in kernel of T

$$T(A) = A + A^T = 0$$

$$A+A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Comparing on both sides.

$$2a = 0 \;;\; b + c = 0$$
$$2d = 0$$

kernel of T is $\left\{\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}, \forall b \in R\right\}$.

1.3 Linearly Independent Sets

Q36. Define linearly independent and linearly dependent.

Sol :

An indexed set of vectors $\{v_1, \dots v_p\}$ in v is said to be linearly independent if the vector equation.

$$c_1 v_1 + c_1 v_1 + \dots c_p v_p = 0$$

has only the trivial solution $c_1 = 0, c_2 = 0 \dots c_p = 0$.

An indexed set of vectors $\{v_1 \dots v_p\}$ in v is said to be linearly dependent if the vector equation

$c_1 v_1 + c_1 v_1 + \dots c_p v_p = 0$ has a non trivial solution that is not all $c_i = 0$.

37. An Indexed set $\{v_1, v_2, \dots v_p\}$ of two or more vectors with $v_1 \neq 0$ is linearly dependent if and only if $\exists$ some v_j (with $j>1$) is a linear combination of its preceding vectors $v_1, v_2, \dots v_{j-1}$.

Sol : **(Nov./Dec.-2018)**

Let v be any vector space $\{v_1, v_2, \dots v_p\}$ be any indexed set in v with $v_1 \neq 0$.

Necessary Condition

Let $\{v_1, v_2, \dots v_p\}$ be a linearly dependent set in v. Consider the linear combination of these vectors equated to a zero vector.

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0 \quad \dots (1)$$

where $c_1, c_2, \dots c_p$ are scalars and $v_i \neq 0$.

Here atleast one of the scalars say $e_j \neq 0$ for $j>1$ and suppose that $c_j = 0$ for $n>j$.

Then the above liner combination can be written as

$$c_1 v_1 + c_2 v_2 + \dots + c_j v_j = 0$$

$$c_j v_j = (-c_1) v_1 + (-c_2) v_2 + ... + (-c_{j-1}) v_{j-1}$$

$$v_j = \left(\frac{-c_1}{c_j}\right) v_1 + \left(\frac{-c_2}{c_j}\right) v_2 + ... + \left(\frac{-c_{j-1}}{c_j}\right) v_{j-1}$$

Thus the vector v_j can be written as the linear combination of its proceeding vectors.

Sufficient Condition

In the indexed set $\{v_1, v_2, ... v_p\}$ let the vector v_j $(j > 1)$ can be written as the linear combination of its proceeding vectors.

$\Rightarrow \exists$ scalars $c_1, c_2 .... c_{j-1}$ such that $v_j = c_1 v_1 + c_2 v_2 + ... + c_{j-1} v_{j-1}$

$$c_1 v_1 + c_2 v_2 + ... + c_{j-1} v_{j-1} + (-1) v_j = 0$$

Thus in this linear combination there exists a non-zero scalar coefficient –1 of v_j so the vectors $v_1, v_2,$ ... v_j are linearly independent.

$\therefore$ The set is $\{v_1, v_2, ... v_j\}$ is L.D.

The index set $\{v_1, v_2, ... v_p\}$ being the super set of this L.D is also L.D.

Q38. State and prove the spanning set theorem.

Statement:

Let $S = \{v_1, v_2 .... v_p\}$ be a set in v and $H = \text{span} \{v_1, v_2, ... v_p\}$.

(i) If one of the vectors in S i.e., v_k is a linear combination of the remaining vectors in S then the set formed from S by reaming v_k still spans H.

(ii) If $H \neq \{0\}$ then some subset of S is a basis for H.

Sol : **(June/July-19, Nov./Dec.-18)**

Let, $S = \{v_1, v_2 ....v_p\}$ be set in v

$H = \text{span} \{v_1, v_2, ... v_p\}$

If v_p is the linear combination of $v_1 ...v_{p-1}$ then

$$v_p = a_1v_1 + a_2v_2 + ... a_{p-1} v_{p-1} \quad ... (1)$$

where,

$a_1, a_2 ...... a_{p-1}$ scalars.

Consider an arbitrary element X in H such that

$$X = c_1v_1 + c_2 v_2 + ... c_{p-1} v_{p-1} + c_pv_p \quad ... (2)$$

where,

$c_1, c_2 ...... c_p$ scalars

from (1) and (2)

$$X = c_1 v_1 + c_2 v_2 + ... c_{p-1} v_{p-1} + c_p (a_1 v_1 + a_2 v_2 + ... a_{p-1} v_{p-1})$$

$$= (c_1 + ac_p) v_1 + (c_2 + ac_p) v_2 + ... (c_{p-1} + a_{p-1} c_p) v_{p-1}$$

Thus $v_1, v_2, ... v_{p-1}$ still spans H.

(ii) Consider the original spanning set S as linearly independent then it consists of basis.

Two or more vectors in the spanning set can repeat the process until it is linearly independent. Thus the basis of S gets reduced to one-zero vector this is due to existence of span vectors in H.

i.e., $H \neq \{0\}$

Q39. Define Basis

Sol :

Let v be a vector space any linearly independent subset of v that spans v is called as a "Basis of v".

(or)

If $\exists$ an indexed set $B = \{b_1, b_2, ... b_n\}$ which is a subset of v such that

(i) B is linearly independent

(ii) $v = \text{span}\{b_1, b_2, ... b_n\}$.

Q40. Verify whether the vectors $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$ are linearly Indepenent.

Sol : **(June/July-19)**

Given vectors are,

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$$

Let $v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$

Consider the matrix,

$A = [v_1\ v_2\ v_3]$

$$= \begin{bmatrix} 2 & 2 & -8 \\ -1 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1 \quad ; \quad R_3 \to 2R_3 - R_1$

$$= \begin{bmatrix} 2 & 2 & -8 \\ 0 & -4 & 2 \\ 0 & 2 & 16 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$

$$= \begin{bmatrix} 2 & 2 & -8 \\ 0 & -4 & 2 \\ 0 & 0 & 34 \end{bmatrix}$$

$R_1 \to \frac{R_1}{2}, R_2 \to \frac{R_2}{-4} ; R_3 \to \frac{R_3}{34}$

$$= \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$= \begin{bmatrix} 1 & 0 & -\frac{7}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Since the matrix, A contains pivot element in each column.

$\therefore$ The set is linearly independent.

Q41. Determine whether the set $S = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis of R^3 or not ? If not determine whether S is L - I or not ? Whether S spans R^3 or not ?

Sol :

Given vector space is R^3

The given set is $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Here $S \subseteq R^3$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = 1(1 - 0) - 1\,(0 - 0) + 1\,(0 - 0)$$

$$= 1 \neq 0$$

$\therefore$ S is linearly independent.

Here A is an inversible matrix of 3×3 then the columns of the matrix A forms a basis of R^3.

Q42. If P_n is a vector space of all polynomials of degree $\leq$ n in t then the S = {1, t, t^2, ... t^n} is a standard basis of P_n.

Sol :

Given that P_n is a vector space of all polynomials in t of degree $\leq$ n.

The given set is $S = \{1, t, t^2, ... t^n\}$

Here $S \subseteq P_n$

Clearly any polynomial in P_n is a linear combination of elements of S and hence P_n = span S.

Let $c_0, c_1, c_2, ... , c_n$ be any n+1 number of scalars and $1, t, t^2, ... t^n \in s$ consider the linear combination of these vectors and equate to zero vector.

$$c_0.1 + c_1.t + c_2.t^2 + ... + c_n.t^n = o(t)$$

i.e., $c_0.1 + c_1.t + c_2.t^2 + ... + c_n.t^n = 0.1 + 0.t + 0.t^2 + ... + o.t^n$

This is only possible if

$$c_0 = 0, c_1 = 0, c_2 = 0, ... c_n = 0$$

$\therefore$ S is linear independent

$\therefore$ S is basis of p_n consisting of unit vectors

$\therefore$ S is a standard basis of P_n.

Q43. Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$ Determine if $\{v_1, v_2\}$ is a basis of R^3 is $\{v_1, v_2\}$ a basis of R^3?

Sol :

The given vectors are $v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$

The given set is $S = \{v_1, v_2\}$

Here $S \subseteq R^3$

As v_1 is not a scalar multiple of v_2 and v_2 is not a scalar multiple of v_1 so v_1, v_2 are linearly independent.

$\therefore$ S is linearly independent

Any basis of R^3 should contain exactly 3 elements.

S cannot be basis of R^3 but can be extended to form a basis.

Here as S is not a subset of R^2.

$\therefore$ S cannot be a basis of R^2.

Q44. Let $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ then determine if $\{v_1, v_2, v_3\}$ is a basis for R^3.

Sol : **(July-21, Nov./Dec-18)**

The given vector space is R^3.

The given vectors are $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$

The given set is $S = \{v_1, v_2, v_3\}$

Construct a matrix A with these vectors as columns then,

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$$

Consider $|A| = 3(5 - 7) + 4(0 + 6) - 2(0 + 6) = -6 + 24 - 12 \neq 0$

$\therefore$ A is invertible.

Thus the set S is linearly independent subset of R^3 consisting of exactly 3 vectors.

$\therefore$ S is a basis of R^3.

Q45. Suppose $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $H = \left\{ \begin{bmatrix} S \\ S \\ 0 \end{bmatrix} \Big/ S \in R \right\}$ then is (v_1, v_2) a basis for H?

Sol : **(June/July-19)**

Given vectors are,

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$H = \left\{ \begin{bmatrix} S \\ S \\ 0 \end{bmatrix} \Big/ S \in R \right\}$$

$$H = \begin{bmatrix} S \\ S \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad H = S\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + S\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$H = Sv_1 + Sv_2$$

$\therefore$ H is a linear combination of v_1 and v_2.

The set of vectors $v = \{v_1, v_2\}$ forms a basis for H if the following conditions are satisfied.

(a) They are linearly independent set.

(b) $H = \text{span}\{v_1, v_2\}$

Consider, $[v_1 \; v_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$c_1v_1 + c_2v_2 = 0$$

$$c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix} = 0$$

i.e., $c_1 = c_2 = 0$.

The vectors are linearly independent and form a span for H i.e., H = span $\{v_1, v_2\}$.

$\therefore$ The vectors v_1, v_2 form the basis for plane of form $\begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}$ but H = $\begin{bmatrix} S \\ S \\ 0 \end{bmatrix}$, $S \in R$ represents a line.

Hence, $\{v_1, v_2\}$ does not form the basis for H.

Q46. Let H = span$\{u_1, u_2, u_3\}$ and k = span $\{v_1, v_2, v_3\}$ where $u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 2 \\ 2 \\ 7 \\ -3 \end{bmatrix}$,

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 8 \\ -4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -2 \\ 9 \\ -5 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 4 \\ 6 \\ -2 \end{bmatrix}$ Find bases for H, K and H + K.

Sol :

Given H = span $\{u_1, u_2, u_3\}$

K = span $\{v_1, v_2, v_3\}$

Consider matrix,

$$[u_1\ u_2\ u_3] = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 3 & -1 & 7 \\ -1 & 1 & -3 \end{bmatrix}$$

Converting the above matrix in reduced echelon form,

$$R_2 \to R_2 - 2R_1\ ;\ R_3 \to R_3 - 3R_1\ ;\ R_4 \to R_4 + R_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{2} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$R_3 \to R_3 + R_2\ ;\ R_4 \to R_4 - R_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix has 2 pivot columns.

$\therefore$ $\{u_1, u_2\}$ forms a basis for H.

i.e., $\left\{\begin{bmatrix}1\\2\\3\\-1\end{bmatrix}\begin{bmatrix}0\\2\\-1\\1\end{bmatrix}\right\}$ is a basis for H.

consider $[v_1\ v_2\ v_3] = \begin{bmatrix}1 & 2 & -1\\0 & -2 & 4\\8 & 9 & 6\\-4 & -5 & -2\end{bmatrix}$

Converting the above matrix into reduced echelon form,

$$R_3 \to R_3 - 8R_1 \quad ; \ R_4 \to R_4 + 4R_1 = \begin{bmatrix}1 & 2 & -1\\0 & -2 & 4\\0 & -7 & 14\\0 & 3 & -6\end{bmatrix}$$

$$R_4 \to \frac{R_2}{-2};\ R_3 \to \frac{R_3}{7};\ R_4 \to \frac{R_4}{3} = \begin{bmatrix}1 & 2 & -1\\0 & 1 & -2\\0 & -1 & 2\\0 & 1 & -2\end{bmatrix}$$

$$R_3 \to R_3 + R_2 \quad ; \ R_4 \to R_4 - R_2 = \begin{bmatrix}1 & 2 & -1\\0 & 1 & -2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$$R_1 \to R_1 - 2R_2 = \begin{bmatrix}1 & 0 & 3\\0 & 1 & -2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

The first and second column of matrix are pivot column,

$\therefore$ $\{v_1, v_2\}$ is a basis for k.

i.e., $\left\{\begin{bmatrix}1\\0\\8\\-4\end{bmatrix}\begin{bmatrix}2\\-2\\9\\-5\end{bmatrix}\right\}$ is a basis for k.

Consider,

$$[u_1, u_2, u_3, v_1, v_2, v_3] = \begin{bmatrix}1 & 0 & 2 & 1 & 2 & -1\\2 & 2 & 2 & 0 & -2 & 4\\3 & -1 & 7 & 8 & 9 & 6\\-1 & 1 & -3 & -4 & -5 & -2\end{bmatrix}$$

Converting the above matrix into reduced echelon from

$$R_2 \to R_2 - 2R_1 ; R_3 \to R_3 - 3R_1 ; R_4 \to R_4 + R_1 = \begin{bmatrix} 1 & 0 & 2 & 1 & 2 & -1 \\ 0 & 2 & -2 & -2 & -6 & 6 \\ 0 & -1 & 1 & 5 & 3 & 9 \\ 0 & 1 & -1 & -3 & -3 & -3 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{2} \sim \begin{bmatrix} 1 & 0 & 2 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & -3 & 3 \\ 0 & -1 & 1 & 5 & 3 & 9 \\ 0 & 1 & -1 & -3 & -3 & -3 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 ; R_4 \to R_4 - R_2 \sim \begin{bmatrix} 1 & 0 & 2 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & -3 & 3 \\ 0 & 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 & -6 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{4} ; R_4 \to \frac{R_4}{-2} \sim \begin{bmatrix} 1 & 0 & 2 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & -3 & 3 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

$$R_1 \to R_1 - R_3 ; R_2 \to R_2 + R_3 ; R_4 \to R_4 - R_3 \sim \begin{bmatrix} 1 & 0 & 2 & 1 & 2 & -1 \\ 0 & 1 & -1 & 0 & -3 & 6 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here first, second and fourth columns are pivot columns.

$\therefore$ $\{u_1, u_2, v_1\}$ is a basis for H + K.

i.e., $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 8 \\ -4 \end{bmatrix} \right\}$ is a basis for H + K.

1.4 Co-ordinate System

Q47. The Unique Representation theorem:-

Let B = $\{b_1, b_2, ..., b_n\}$ be a basis for a vector space V. Then for each x in V, there exists a unique set of scalars $c_1, c_2, ..., c_n$ such that

$x = c_1 b_1 + ... + c_n b_n$ ... (1)

Sol :

Given $B = \{b_1, b_2, ... b_n\}$ is a basis for a vector space V. Since B spans V, there exist scalars such that (1) holds. Suppose x also has the representation.

$$x = d_1 b_1 + d_2 b_2 + ... + d_n b_n \qquad ...(2)$$

for scalars $d_1, d_2, ..., d_n$.

Then, subtracting (2) from (1), we have

$$\vec{O} = x - x = (c_1 - d_1)b_1 + (c_2 - d_2)b_2 + ... + (c_n - d_n) b_n \qquad ...(3)$$

Since B is linearly independent, the weights in (3) must all be zero. That is, $c_j = d_j$, for $1 \le j \le n$.

Q48. Define B-coordinates of x, co-ordinate mapping, change of co-ordinates matrix.

Ans :

Co-ordinates of 'x' relative to the basis B (or) B-Co-ordinates of x

Suppose $B = \{b_1, b_2 .... b_n\}$ is a basis for V and x is in V. The co-ordinates of x relative to the basis B (or the B-co-ordinate of x) are the weights $c_1, c_2 ..... c_n$ such that

$$x = c_1 b_1 + c_2 b_2 + ... + c_n b_n$$

The Co-ordinate Mapping

If $c_1, c_2, ..., c_n$ are the B-co-ordinates of x, then the vector in R^n

$[X]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is the co-ordinate vector x relative to B (or the B - co-ordinate vector of x). The mapping $x \to [X]_B$ is the co-ordinate mapping (determined by B).

Change-of-Co-ordinates Matrix

The vector equation

$x = c_1b_1 + c_2b_2 + ... + c_n b_n$ is equivalent to

$x = P_B[x]_B$

Where $P_B = [b_1, b_2, ... b_n]$ is the change-of-co-ordinates matrix from B to the standard basis in IR^n.

Note: P_B^{-1} exists

Q49. Let $B = \{b_1, b_2, ..., b_n\}$ be a basis for a vector space V. Then the co-ordinate mapping $X \to [X]_B$ is a one-to-one linear transformation from V onto R^n.

Sol :

Take two typical vectors in V, say

$\vec{u} = c_1 b_1 + c_2 b_2 + ... + c_n b_n$

$W = d_1 b_1 + d_2 b_2 + ... + d_n b_n$

Then, using vector operations,

$u + w = (c_1 + d_1)b_1 + (c_2 + d_2) b_2 + ... + (c_n + d_n)b_n$

The B-coordinates of u + w is

$$[u + w]_B = \begin{bmatrix} c_1 + d_1 \\ c_2 + d_2 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = [u]_B + [w]_B \qquad \text{... (1)}$$

So, the co-ordinate mapping preserves addition.

If 'r' is any scalar, then

$$r\vec{u} = r(c_1, b_1 + c_2 b_2 + ... + c_n b_n)$$
$$= (rc_1)b_1 + (rc_2)b_2 + ... + (rc_n)bn$$

$$\Rightarrow [ru]_B = \begin{bmatrix} rc_1 \\ rc_2 \\ \vdots \\ rc_n \end{bmatrix} = r\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r[u]_B \qquad \text{...(2)}$$

Thus the co-ordinate mapping also preserves scalar multiplication.

Hence, from (1) and (2) co-ordinate mapping is a linear transformation.

Since co-ordiante mapping is invertible.

$\therefore$ The co-ordinate mapping is a one-to-one mapping and maps V onto IR^n.

PROBLEMS

Q50. Let $v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$. Let B = [$v_1$, v_2] and H = span {v_1, v_2}. B is a basis for H. If x is in H, then find the co-ordinate vector of x relative to B.

Sol :

If $x \in H$ then $x \in$ span [v_1, v_2] then $\exists$ two scalars c_1 and c_2 such that,

$$x = c_1 v_1 + c_2 v_2$$

$$\Rightarrow \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix} = c_1\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{... (1)}$$

By solving (1), we get $c_1 = 2$, $c_2 = 3$

Then $[X]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is the co-ordinate vector of x relative to B.

Q51. If B = {$1 - t^2$, $t - t^2$, $2-t + t^2$} is a basis for P_2 then find the co-ordinate vector of $P(t) = 1 + 3t - 6t^2$ relative to B.

Sol :

Given B = {$1 - t^2, t - t^2, 2 - t + t^2$} is a basis for P_2 where P_2 is the vector space consisting of all polynomials of degree $\leq$ 2.

The given polynomial is $P(t) = 1 + 3t - 6t^2$

Since degree of $p(t) = 2 \Rightarrow P(t) \in P_2$.

As $P(t) \in P_2$ and B is a basis of P_2, So P(t) can be written as the linear combination of elements of B.

Then $\exists$ c_1, c_2, c_3 scalars such that $P(t) = c_1(1 - t^2) + c_2(t - t^2) + c_3(2 - t + t^2)$

$\Rightarrow 1 + 3t - 6t^2 = c_1(1 - t^2) + c_2(t - t^2) + c_3(2 - t + t^2)$

$\Rightarrow 1 + 3t - 6t^2 = (c_2 + 2c_3) + (c_2 - c_3)t - (c_1 + c_2 + c_3)$

By equating the coefficient of similar powers of t, we get,

$c_2 + 2c_3 = 1$

$c_2 - c_3 = 3$

$-(c_1 + c_2 + c_3) = -6 \Rightarrow c_1 + c_2 + c_3 = 6$

Solving above equations, we get

$c_1 = 3, c_2 = 5/2, c_3 = -1/2$

The co-ordinate vector of P(t) relative to B is

$$[P(t)]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5/2 \\ -1/2 \end{bmatrix}.$$

Q52. If the set B = {$1 + t^2$, $t + t^2$, $1 + 2t + t^2$} is a basis for P_2 then find the co-ordinate vector of $p(t) = 1 + 4t + 7t^2$ relative to B.

Sol :

Given B = {$1 + t^2, t + t^2, 1 + 2t + t^2$} is the basis of P_2 where P_2 is the vector space of consisting of all polynomials of degree $\leq$ 2.

The given polynomial is $p(t) = 1 + 4 + 7t^2$

$\therefore$ $p(t) \in P_2$

The degree of p(t) is 2.

Since, B is a basis of P_2 and $p(t) \in P_2$, then p(t) can be written as the linear combination of elements of B.

$\Rightarrow$ $\exists$ scalars c_1, c_2, c_3 such that

$p(t) = c_1(1 + t^2) + c_2(t + t^2) + c_3(1 + 2t + t^2)$

$\Rightarrow$ $1 + 4t + 7t^2 = c_1 + c_1t^2 + c_2t + c_2t^2 + c_3 + 2c_3t + c_3t^2$

$\Rightarrow$ $1 + 4t + 7t^2 = (c_1 + c_3) + (c_2 + 2c_3)t + (c_1 + c_2 + c_3)t^2$

Equating the coefficients of similar powers of t, we have

$c_1 + c_3 = 1$

$c_2 + 2c_3 = 4$

$c_1 + c_2 + c_3 = 7$

Solving above equations, we get

$c_1 = 2, c_2 = 6, c_3 = -1.$

$\therefore$ The co-ordinate vector of p(t) relative to the basis B is,

$$[p(t)]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

Q53. The set B = {1 + t, 1 + t², t + t²} is a basis for P_2. Find the co-ordinate vector of p(t) = 6 + 3t – t² relative to B.

Sol : **(June/July-2019)**

Given B = $\{1 + t, 1 + t^2, t + t^2\}$ is the basis of P_2 where P_2 is the vector space consisting of all polynomials of degree ≤ 2.

The given polynomial is $p(t) = 6 + 3t - t^2$, the degree of p(t) = 2.

$\therefore$ $p(t) \in P_2$

Since B is a basis of P_2 and $p(t) \in P_2$, then p(t) can be written as the linear combination of elements of B.

$\Rightarrow$ $\exists$ scalars c_1, c_2, c_3 such that

$$p(t) = c_1(1+t) + c_2(1+t^2) + c_3(t+t^2)$$

$\Rightarrow$ $6 + 3t - t^2 = c_1 + c_1t + c_2 + c_2t^2 + c_3t + c_3t^2$

$6 + 3t - t^2 = (c_1 + c_2) + (c_1 + c_3)t + (c_2 + c_3)t^2$

Equating the coefficients of similar powers of t, we have $c_1 + c_2 = 6$, $c_1 + c_3 = 3$, $c_2 + c_3 = -1$.

Solving above equations, we get $c_1 = 5$, $c_2 = 1$, $c_3 = -2$.

$\therefore$ Then $[p(t)]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$ is the co-ordiante vector of p(t) relative to B.

Q54. Let $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\beta = [b_1, b_2]$ then find the co-ordinate vector $[x]_\beta$ of x relative to β.

Sol : **(Nov./Dec.-19)**

Given basis $\beta = [b_1, b_2]$

$$b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \; b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and vector } x = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

The β-co-ordinates c_1, c_2 of x is $c_1 b_1 + c_2 b_2 = x$

$$\Rightarrow \quad c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$$

$$\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$$

The augmented matrix is $\begin{bmatrix}2 & -1 & 4\\1 & 1 & 5\end{bmatrix}$

$$R_2 \to 2R_2 - R_1 \sim \begin{bmatrix}2 & -1 & 4\\0 & 3 & 6\end{bmatrix}$$

$$R_1 \to \frac{R_1}{3} \sim \begin{bmatrix}2 & -1 & 4\\0 & 1 & 2\end{bmatrix}$$

$$R_1 \to R_1 + R_2 \sim \begin{bmatrix}2 & 0 & 6\\0 & 1 & 2\end{bmatrix}$$

$$R_1 \to \frac{R_1}{2} \sim \begin{bmatrix}1 & 0 & 3\\0 & 1 & 2\end{bmatrix}$$

$\therefore$ The solutions are $c_1 = 3$, $c_2 = 2$.

$\therefore$ The co-ordinate vector x relative to β is $[x]_\beta = \begin{bmatrix}3\\2\end{bmatrix}$.

1.5 The Dimension of a Vector Space

Q55. Finite - Dimensional and Infinite dimensional.

Sol :

If v is spanned by a finite set, then V is said to be finite - dimensional, and the dimension of V, written as dim V, is the number of vectors in a basis for V.

If v is not spanned by a finite set then v is said to be infinite dimensional.

Q56. Find the dimension of a vector space V = R^3.

Sol :

Consider the vector space V = IR^3

Since $B = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is the standard basis for V = IR^3.

$\therefore$ Dim IR^3 = Dim V = number of elements (or vectors) in a basis B for V = IR^3.

$\Rightarrow$ Dim IR^3 = 3.

Similarly Dim IR^2 = 2, Since B for IR^2 = $\left\{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right\}$.

Note :

1. The standard basis for IR^n contains 'n' vectors so dim IR^n = n.
2. The standard basis for vector space P_n contains (n+1) vectors.

Q57. If a vector space V has a basis B = {b_1, b_2 .... b_n}, then any set in V containing more than n vectors must be linearly dependent.

Sol : **(July-21)**

Given V is a vector space and B = {b_1, b_2 .... b_n} is a basis of V. $\therefore$ dim V = n.

Suppose

{u_1, u_2, ... , u_p} is a set in V with more than 'n' vectors.

$\Rightarrow$ $[u_1]_B$, $[u_2]_B$ ..... $[u_1]_B$ are the co-ordinate vectors of V relative to B, since there are more vectors (p) than entries (n) in each vector, then they form a linearly dependent set in IR^n.

So, there exists scalares c_1, c_2, ... c_p not all zeros, such that,

$$c_1[u_1]_B + c_2[u_2]_B + ... + cp[up]_B = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} \quad ... (1)$$

Since the co-ordinate mapping is a linear transformation.

$$(1) \Rightarrow [c_1 u_1 + c_2 u_2 + ... + c_p u_p]_B = \begin{bmatrix}0\\0\\0\\0\\\vdots\\0\end{bmatrix} \quad ... (2)$$

The zero vector on the R.H.S of (2) displays the n-weights needed to build the vector.

$c_1 u_1 + c_2 u_2 + ... + c_p u_p$ (which is linear combination) from the basis vectors in B.

That is $c_1 u_1 + c_2 u_2 + ... + c_p u_p = 0.b_1 + 0.b_2 + ... + 0.b_n = 0$

Since the c_1's are not all zero.

$\Rightarrow$ {u_1, u_2, ...u_p} are linearly independent.

Q58. If a vector space has a basis of n-vectors, then every basis of V must consist of exactly n-vectors.

Sol : **(Nov./Dec.-18)**

Given V is a vector space

Let B_1 be a basis of V consisting of n vectors.

Let B_2 be another basis of V consisting of m vectors.

We have to show m = n

Case (i)

Since B_1 is a basis of V consisting of n-elements and B_2 is a linealry independent set.

So, by above theorem

B_2 cannot have more than n-vectors i.e., $m \leq n$ ... (1)

Case (ii)

Since B_2 is a basis of V consisting of m-elements (or vectors) and B_1 is a linealry independent set.

So, by above theorem

B_1 cannot have more than m-vectors i.e., $n \leq m$ ... (2)

Therefore from (1) and (2) m = n.

Thus B_2 also consists of exactly n-vectors.

$\Rightarrow$ Every basis of V must consist of exactly n-vectors.

Note

If a non-zero vector space V is spanned by a finite set S, i.e., S = span V.

Then a subset of S is a basis for V.

1.5.1 Subspaces of a Finite - Dimensional Space

Q59. Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and dim H $\leq$ dim V.

Sol :

Given V is a finite dimensional vector space and H is a subspace of 'V'.

Case (i)

If H = {0}, then dim H = 0 $\Rightarrow$ dim H = 0 $\leq$ dim V.

Case (ii)

If H $\neq$ {0}, Let dim V = n

Then any subset of 'V' consisting of more than n–vectors is always linearly dependent.

As every subset of H is also a subset of V.

$\Rightarrow$ Every subset of H is, linearly independent.

$\Rightarrow$ Any linearly independent set of vectors in H can contain at most n-vectors.

Let S = $\{u_1, u_2 \ldots\ldots u_k\}$ be largest linearly independent subset of H. ($k \leq n$)

Let $\beta \in H$

Then the set $\{u_1, u_2, \ldots\ldots, u_k, \beta\}$ is a linearly dependent subset of H.

$\Rightarrow$ The vector β can be written as the linear combination of its preceding vectors $u_1, u_2, \ldots, u_k$.

$\Rightarrow$ $\beta \in L(s)$

Thus every element of H can be written as the linear combination of elements of S.

$\therefore$ $L(s) = H \Rightarrow$ H is a finite dimensional vector space also S is a basis of H.

$\dim(H) = k \leq n = \dim v \Rightarrow \dim H \leq \dim V$ Hence proved.

PROBLEMS

Q60. Let H = span$\{v_1, v_2\}$ where $v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ and $v_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ Find the dimension of H.

Sol :

Given $H = \text{span}\{v_1, v_2\}$ where $v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ and $V_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ since v_1 and v_2 are not multiples and hence they are linearly independent. $\therefore$ dim H = 2.

Q61. Find the dimension of the subspace.

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } R \right\}$$

Sol : **(Nov./Dec.-19)**

$$\text{Given } H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } R \right\}.$$

Let 'W' be the subspace of H, which is general vector,

$$\therefore \quad W = \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix}$$

This vector can be written as the linear combination of four vectors v_1, v_2, v_3, v_4 where,

$$v_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

Clearly $v_1 \neq 0$, v_2 is not multiple of v_1, but v_3 is a multiple of v_2 i.e., $v_3 = 2v_2$.

$\Rightarrow$ The vectors v_1, v_2, v_3, v_4 are linearly dependent.

By the spanning set theorem, we can discard v_3, we get, the set $S = \{v_1, v_2, v_4\}$ still spans H.

$\Rightarrow$ $L(s) = H$ and V_4 is not a linear combination of v_1 and v_2.

So, $\{v_1, v_2, v_4\}$ is linearly independent.

$\Rightarrow$ S is a basis of H.

$\therefore$ H is a finite dimensional vector space and dim H = 3.

Q62. If $H = \left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s, t \in R \right\}$ **then find a basis of H and state the dimension of H.**

Sol :

The given vector space in $H = \left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s, t \in R \right\}$ consider any general vector of H, say w.

$$W = \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \text{ for } s, t \in R$$

Thus every element in H can be written as the linear combination of elements if the set,

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\} = \{v_1, v_2\}$$

Also the vectors v_1 and v_2 are linearly independent.

$\therefore$ S is a basis of H, dim (H) = 2.

Q63. Define The Dimensions of Nul A and Col A.

Ans :

The dimension of Nul A is the number of free variables in the equation A × = 0 and the dimension of Col A is the number of pivot columns in A.

Q64. Find the dimensions of the null space and column space of,

$$\mathbf{A} = \begin{Bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{Bmatrix}$$

Sol:

Echelon form of A is,

$$\begin{Bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{Bmatrix}$$

There are three free variables x_2, x_4 and x_5. Hence the dimension of Nul A is 3(number of free variables) the dimension Col A = 2, because a has two pivot column.

(They are 1st and 3rd columns).

Q65. Determine the dimensions of Null A and Col A for the matrices.

(i) $\mathbf{A} = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(ii) $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ -6 & 10 \end{bmatrix}$

Sol:

i) Given matrix is,

$$A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in echelon form

Since, there are 3 pivot columns

$\therefore$ dimension of Col A = 3

Also there are two columns without pivots.

$\Rightarrow$ Equation Ax = 0 contains 2 free variables

$\therefore$ Dimension of Null A = 2.

(ii) Given matrix is,

$$A = \begin{bmatrix} 3 & 4 \\ -6 & 10 \end{bmatrix}$$

Row reducing the above matrix

$$R_2 \to \frac{R_2}{2} \sim \begin{bmatrix} 3 & 4 \\ -3 & 5 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 \sim \begin{bmatrix} 3 & 4 \\ 0 & 9 \end{bmatrix}$$

$$R_2 \sim \frac{R_2}{9} \sim \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}$$

$$R_1 \sim R_1 - 4R_2 \sim \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_1 \sim \frac{R_3}{3} \cap A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix has two pivot columns.

$\therefore$ Dimensions of Col A = 2.

Since the matrix has no columns without pivots.

$\Rightarrow$ The equation Ax = 0 has only the trivial solution.

$\therefore$ Dimension of Null A = 0.

Q66. Let H be a non-zero subspace of v and let T(H) be the set of images of vectors in H. Then T(H) is a subspace of w. Prove that dim T(H) $\leq$ dim H.

Ans :

Given H is a Non-zero subspace of a finite dimensional vector space v.

And T: $V \rightarrow W$ is a linear transformation

Let $\{u_1, u_2, ... u_p\}$ be the basis for H.

$\Rightarrow$ dim H = P ... (1)

Let $y \in T(H)$

Then there exist $x \in H$ such that,

$T(x) = y$ ... (2)

x can be written as linear combinations of the basis vectors $x = c_1 u_1 + c_2 u_2 + ... + c_p u_p$.

Since T is linear transformation

$\Rightarrow \quad T(x) = T(c_1 u_1 + c_2 u_2 + ... + c_p u_p)$

$\Rightarrow \quad y = c_1T(u_1) + c_2T(u_2) + ... + c_pT(u_p)$ ($\because$ from (2))

$\therefore$ y is a liner combination of $T(u_1), T(u_2), ... T(u_p)$ and $\{T(u_1), T(u_2), ... T(u_p)\}$ spans T(H).

From spanning set theorem,

$\{T(u_1), T(u_2), ... T(u_p)\}$ is a basis for T(H), containing not more than P vectors.

$\Rightarrow$ dim $T(H) \leq P$

$\leq$ dim H (from equation (1))

$\therefore$ dim $T(H) \leq$ dim H.

Q67. (a) Let H be an n-dimensional subspace of an n-dimensional vector space v. Show that H = V.

(b) Explain why the space p of all polynomials is an polynomials is an infinite dimensionals space.

Ans :

(a) Given

V is an n-dimensional vector space and H is an n-dimensional subspace of V.

Case (i)

If dim V = dim H = 0 then v = {0}

and H = {0}

i.e., V and H has no basis.

$\therefore$ H = V.

Case (ii)

If dim v = dim H > 0

Then H contains a basis $S = \{v_1, v_2, ...v_n\}$

Then by basis theorem,

S is also a basis for V

$\Rightarrow$ H and V both span S.

$\therefore$ H = V

(b) Let dim P is finite

i.e., dim P = k (< ∞) ... (1)

p_n is a subspace of p $\forall n$

dim p_{k-1} = k

$\Rightarrow$ dim p_{k-1} = dim p ($\therefore$ from (1))

$\Rightarrow$ $p_{k-1} = p$

This is not true.

$\therefore$ The dimension of P cannot be finite.

Hence, P is an infinite dimensional space.

Q68. Show that the s = {(1, 0, 0, –1) (0, 1, 0, –1), (0, 0, 1, –1) and (0, 0, 0, 1)} in R^4 is linearly independent.

Sol : **(Nov./Dec.-18)**

Given set vectors in R^4 are,

$$\text{Let } v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Consider the matrix } A = [v_1\ v_2\ v_3\ v_4] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$R_4 \to R_4 + R_1 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

$$R_4 \to R_4 + R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$R_4 \to R_4 + R_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the matrix A contains pivot in each column

$\therefore$ The given set of vectors are linearly independent.

Choose the Correct Answers

1. The union of two subspaces is subspace of vector space iff [a]
   (a) One is contained in another (b) One is not contained in another
   (c) Both (a) and (b) (d) None

2. If A = $[a_1, a_2, ... a_n]$ then Col A = [c]
   (a) $[a_1, a_2, ... a_n]$ (b) $\{x / x \text{ is in } R^n\}$
   (c) Span $\{a_1, a_2, ... a_n\}$ (d) None

3. If T is a linear transformation from a vector space V into a vector space w then [c]
   (a) $T(u + v) = T(u) + T(v) + u, v \in V$ (b) $T(u) = CT(u) \ \forall \ u \in V$
   (c) Both (a) and (b) (d) None

4. Let B = $\{b_1, b_2, ... b_n\}$ is a basis for a vector space V. For each $x \in V$, then exist a unique set of scales $c_1, c_2, ... c_n$ such that $x = c_1b_1 + c_2b_2 + c_nb_n$. This is called [d]
   (a) Spanning set theorem (b) Rank theorem
   (c) Basis theorem (d) Unique representation theorem.

5. Col A = R^m if the equation [a]
   (a) Ax = b has a solution $\forall \ b \in R^m$ (b) Ax = 0 has only the trivial solution
   (c) Nul A = {0} (d) All the above

6. H is a subspace of V and B = $\{b_1, b_2, ... b_n\} \in V$ is a basis for H if [c]
   (a) B is linearly independent (b) H = span $\{b_1, b_2, ..., b_n\}$
   (c) Both (a) and (b) (d) None

7. The Set S = {(1, –2, 1), (2, 1, –1), (7, –4, 1)} forms [c]
   (a) Linearly dependent (b) Linearly span
   (c) Linearly independent (d) None

8. If a vector space V is not spanned by the finite set S then V is said ________. [b]
   (a) Finite dimensional (b) Infinite dim
   (c) Finite and Infinite dim (d) None

9. The set S = {(1, 0, 0, –1), (0, 1, 0, –1), (0, 0, 1, –1), (0, 0, 0, 1)} in R^4 is ________. [b]
   (a) Linearly dependent (b) Linearly independent
   (c) Both (a) and (b) (d) None

10. Let B = $\{b_1, b_2, ... b_n\}$ be a basis for a vector space V. Then the co-ordinate mapping $x \to [x]_B$ is a ________ theory transformation form v onto R^n. [b]
    (a) Onto (b) One-one
    (c) Both (a) and (b) (d) None

Fill in the Blanks

1. The intersection of the two subspaces of V(F) is again a ________.
2. The set V = {0} is a vector space and it is said to be a ________.
3. Let V be any vector space if it is any non-empty subset of V and H is also a vector space then H is called as a ________ of V.
4. If $v_1, v_2, \ldots, v_p$ are in a vector space v then

   span $\{v_1, v_2, \ldots, v_p\}$ is a ________ of v.
5. The null space of an m × n matrix A is a subspace of ________.
6. The ________ of an m × n matrix A B a subspace of R^m.
7. If T is a matrix transformation that T(x) = Ax for any matrix A then Range of T is ________ and kernal of T is ________.
8. Let V be a vector space. Any linearly independent subset of v that spams V is called as a ________.
9. If P_n is a vector space of all polynomials of degree ≤ n in 't' then the set S = $\{1, t, t^2, \ldots t^n\}$ is a ________ of P_n.
10. The no. of elements in the basis of a vector space is called as ________.

ANSWERS

1. Subspace
2. Zero space
3. Subspace
4. subspace
5. R^n
6. Colum space
7. Col A, Null of A
8. Basis of V
9. Standard basis
10. Dimension of the vector space v

UNIT II	Rank-Change of Basis - Eigenvalues and Eigenvectors - The Characteristic Equation

2.1 Rank - Change of Basis

Q1. State and prove the Rank theorem?

Sol : **(July-2021, June-July-2019, Nov.-Dec.-2018)**

Statement

The dimension of the column space and the Row Space of an $m \times n$ matrix A are equal. This common dimension is said to be the rank of Matrix A.

Rank of A is also equal to the number of pivot positions in A and satisfies the equation.

Rank A + dim Null A = n ; where n is the dimension of a vector space.

Proof :

Let $[A]_{m\times n}$ be a matrix of order $m \times n$

Rank of A = the number of pivot elements in A

Rank of A = the number of pivot positions in an echelon form B of A.

For each pivot position there is non zero row in B

These number of non zero rows will form a basis of the row space of A.

Rank of A = The dimension of the row space of A

= The no. of pivot columns .... (1)

Dimension of Null A = The number of free variables in the equation Ax = 0

= The no. of columns of A that are not pivot elements of A

= number of non-pivot columns ....(2)

The total no. of columns in A

= The no. of pivot elements + The no. of non pivot columns ...... (3)

from equation (1), (2) and (3) we get

Rank of A + dim null A = n

Q2. State Invertible Matrix Theorem :

Sol :

Statement

If A is an invertiable matrix of order n × n then

Here A is a square matrix

If A is Invertiable then

(i) The columns of A forms a basis of R^n

(ii) Col A = R^n

(iii) dim col A = n

(iv) Rank = n

(v) Null A = {0}

(vi) dim Null A = 0

Q3. If A is a 7 × 9 Matrix with a two dimensional Null space, what is the rank of A ?

Sol :

The given Matrix A is of order 7 × 9

n = no. of columns = 2 [given two dimensional Null Space]

dim Null A = 2

From Rank theorem

rank of A + dim Null A = n

rank of A + 2 = 9

rank A = 9 – 2

= 7

Q4. If a 7 × 5 matrix A has rank 2, then find dim null A, dim row A and rank A^T.

Sol :

The given matrix is of order 7 × 5

n = no. of columns of A = 5

rank of A = 2

dim row A = rank of A = 2

rank of A^T = rank of A = 2

From Rank Theorem :

Rank of A + dim null A = n

2 + dim null A = 5

dim null A = 3

Q5. If A = $\begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$ then find rank A and dim null A ?

Sol : **(July-2021)**

Given Matrix is

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$$

Order of Matrix A = 4 × 5

Converting matrix A to Echelon form :

$R_2 : R_2 + R_1$

$R_3 : R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 14 & -35 & 42 \end{bmatrix}$$

$$R_3 : R_3 + 7R_2 \sim \begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are two pivot columns.

Rank A = 2

From Rank nullity theorem

Rank A + dim Null A = n

2 + dim null A = 4

dim nul A = 4 – 2

∴ dim Nul A = 2

Q6. Given a Matrix A = $\begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}$ then find Rank of A and dim Null A

Sol : **(Nov.-2018, Dec.-2018)**

$$\text{Given Matrix A} = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}$$

Converting Matrix to Echelon form :

Step (1)

$$R_1 : R_1 - R_2 : \begin{bmatrix} 1 & 1 & 5 & -9 & 10 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}$$

Step (2) :

$R_2 : R_2 - R_1$

$R_3 : R_3 + 7R_1$

$R_4 : R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & 1 & 5 & -9 & 10 \\ 0 & -3 & -9 & 12 & -12 \\ 0 & 15 & 45 & -60 & +60 \\ 0 & -9 & -27 & +36 & -36 \end{bmatrix}$$

Step (3) :

$R_3 : R_3 + 5R_2$

$R_4 : R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 5 & -9 & 10 \\ 0 & -3 & -9 & 12 & -12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are two pivot columns in Matrix A

Rank A = 2

From Rank theorem : Rank A + dim Null A = n

2 + dim Null A = 5

dim Null A = 3

Q7. Let the matrix A is Row Equivalent to B. Without calculation List rank A and dim Nul A. Then find bases for col A, Row A and Null A.

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Sol :

Given Matrices are

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And Matrix A is Row equivalent to Matrix B.

Here Matrix B is in echelon form.

Also it has two pivot columns

dim col A = 2

Since rank A = dim col A

= 2

Also there are two non-pivot columns in Matrix B

$\Rightarrow$ The Equation Ax = 0 has two free variaties

$\therefore$ dim Nul A = 2

The basis for col A is pivot columns of A

i.e., $\left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}$

The basis for row A is the non-zero rows of matrix B.

i.e., {(1, 0, –1, 5), (0,–2,5,–6)}

Basis for Nul A :

Row reducing the matrix B

$$B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 1 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{-2} \sim B =$$

$$\begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & \frac{-5}{2} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is

$$x_1 - x_3 + 5x_4 = 0$$

$$\Rightarrow \quad x_1 = x_3 - 5x_4$$

$$x_2 - \frac{5}{2}x_3 + 3x_4 = 0$$

$$\Rightarrow x_2 = \frac{5}{2}x_3 - 3x_4$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 5x_4 \\ \frac{5}{2}x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ \frac{5}{2} \\ 1 \\ 0 \end{bmatrix}$$

$$+ x_4 \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

The basis for Null A is , $\left\{ \begin{bmatrix} 1 \\ \frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Q8. Find the base for the row space of the matrix

$$\begin{bmatrix} \mathbf{-2} & \mathbf{-5} & \mathbf{8} & \mathbf{0} & \mathbf{-17} \\ \mathbf{1} & \mathbf{3} & \mathbf{-5} & \mathbf{1} & \mathbf{5} \\ \mathbf{3} & \mathbf{11} & \mathbf{-19} & \mathbf{7} & \mathbf{1} \\ \mathbf{1} & \mathbf{7} & \mathbf{-13} & \mathbf{5} & \mathbf{-3} \end{bmatrix}$$

(June / July-2019)

Sol :

Given matrix is

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Converting the above matrix into echelon form, $R_2 \to 2R_2 + R_1$; $R_3 \to 2R_3 + 3R_1$; $R_4 \to 2R_4 + R_1$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 7 & -14 & 14 & -49 \\ 0 & 9 & -18 & 10 & -23 \end{bmatrix}$$

$R_3 \to R_3 + 7R_2 \,;\, R_4 \to R_4 - 9R_2$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & 40 \end{bmatrix}$$

$R_4 \to \dfrac{R_4}{2}$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & 7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$A = \begin{bmatrix} -2 & -6 & 10 & -2 & -10 \\ 0 & 1 & -2 & 2 & 7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to \dfrac{R_1}{-2}$

$$A = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The basis for row space of A is the non- zero rows of reduced matrix A.

i.e., {(1, 3, –5, 1, 5), (0, 1, –2, 2, –7), (0,0,0, –4, 20)}

Q9. Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ and consider the bases of R^2 given by $\beta = \{b_1, b_2\}$ and $c = \{c_1, c_2\}$. Find the change of coordinates matrix from β to e.

Sol : **(Nov.-2018, Dec.-2018)**

Given $\beta = \{b_1, b_2\}$, $c = \{c_1, c_2\}$ are the bases for R^2

Where $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$,

$c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$

The change of coordinate matrix from β to e is given by,

$$\underset{C \leftarrow \beta}{P} = [c_1 \quad c_2 \quad b_1 \quad b_2]$$

$$= \begin{bmatrix} -7 & -5 & 1 & -2 \\ 9 & 7 & -3 & 4 \end{bmatrix}$$

converting the matrix into reduced echelon form

$R_2 \to 7R_2 + 9R_1$

$$\sim \begin{bmatrix} -7 & -5 & 1 & -2 \\ 0 & 4 & -12 & 10 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{2} \sim \begin{bmatrix} -7 & -5 & 1 & -2 \\ 0 & 2 & -6 & 5 \end{bmatrix}$$

$$R_1 \to 2R_1 + 5R_2 \sim \begin{bmatrix} -14 & 0 & -28 & 21 \\ 0 & 2 & -6 & 5 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{-14},\ R_2 \to \frac{R_2}{2}$$

$$\sim\begin{bmatrix} 1 & 0 & 2 & \frac{-3}{2} \\ 0 & 1 & -3 & \frac{5}{2} \end{bmatrix}$$

$$\therefore \underset{C\leftarrow\beta}{P} = \begin{bmatrix} 2 & \frac{-3}{2} \\ -3 & \frac{5}{2} \end{bmatrix}$$

Q10. If 3 × 8 matrix A has Rank 3. Find dim Null A, dim Row A and rank A^T

Sol :

Given,

Rank of 3 × 8 matrix A = 3

From Rank theoren,

Rank A + dim Null A = n

$\Rightarrow$ 3 + dim Null A = 8

dim Null A = 8 – 3

$\therefore$ dim Null A = 5

dim Row A = Rank A

= 3

$\therefore$ dim Row A = 3

Rank A^T = dim Row A

$\therefore$ Rank A^T = 3

Q11. Colud a 6 × 9 matrix have a two - dimensional Null space?

Sol : **(June /July-2019)**

Given matrix is 6 × 9

Let the matrix be A

Here m = 6, n = 9

From Rank theorem,

Rank A + dim Null A = n .... (1)

The rank A is the number of pivot positions in matrix A.

since the number of pivot positions cannot exceed the numbers of rows or columns.

Rank A = 9

Substituting the corresponding values in eq (1)

9 + dim Null A = 9

dim Null A = 0

For the matrix A to have two dimensional Null space, it should have rank = 7

$\therefore$ The matrix A cannot have a two - dimensional Null space.

Q12. Let u = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find v in R^3 such that

$$\begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix} = uv^T$$

Sol :

Given u = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Such that $\begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix} = uv^T$

Let v^T = (x y z)

Then $\begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (x \ y \ z)$

$$\begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix} = \begin{bmatrix} x & y & z \\ 2x & 2y & 2z \end{bmatrix}$$

Comparing, x = 1; y = – 3, z = 4

$\Rightarrow v^T$ = (x y z) = (1 – 3 4)

$$\therefore v = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

Q13. Consider two bases B = {b_1, b_2} **and C = {c_1, c_2} be bases for a vector space v such that $b_1 = 4c_1 + c_2$ and $b_2 = -6c_1 + c_2$ suppose $x = 3b_1 + b_2$ i.e., suppose $[x]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[x]_C$**

Sol :

Given, $b_1 = 4c_1 + c_2$ and $b_2 = -6c_1 + c_2$ ....(1)

and $x = 3b_1 + b_2$ ....(2)

apply the co-ordinate mapping determined by C to x in (2)

Since the coordinate mapping is a linear tranformation

$$[x]_C = [3b_1 + b_2]_C$$

$$= 3[b_1]_C + [b_2]_C$$

we can write this vector equation as a matrix equation, using the vectors in the linear combination as the columns of a matrix.

$$[x]_C = [[b_1]_C \ [b_2]_C] \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{...... (3)}$$

$$[b_1]_C = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \ \& \ [b_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

$$(3) \Rightarrow [x]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Q14. Let $b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and consider the bases for R^2 given by B = {b_1 ,b_2} and c = {c_1, c_2}. Find the change of coordinates matrix from B to C.

Sol : **(June /July - 2019)**

The matrix $\underset{C \leftarrow B}{P}$ involves the coordinates vectors of b_1 and b_2 .

Let $[b_1]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[b_2]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Then, by defination

$[c_1\ c_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$ and $[c_1\ c_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b_2$

To solve both systems simultaneously , augment the co-efficient matrix with b_1 & b_2 and row reduce,

$$[c_1\ \ c_2\ ;\ b_1\ \ b_2] = \begin{bmatrix} 1 & -3 & : & -9 & -5 \\ -4 & -5 & : & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & : & 6 & 4 \\ 0 & 1 & : & -5 & -3 \end{bmatrix}$$

Thus $[b_1]_C = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$ and $[b_2]_C = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

The desired change of coordinates matrix is,

$$\underset{C\leftarrow B}{P} = [[b_1]_C[b_2]_C]$$

$$= \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

Q15. In P_2, find the change of coordinates matrix from the basis B ={1 – 2t + t^2, 3–5t+4t^2, 2t+3t^2} to the standard basis C = {1, t, t^2}. Then find the B – coordinate vector for –1+2t.

Sol :

Given basis is,

$B = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$

i.e., $B = \{b_1, b_2, b_3\}$

and standard basis,

$C = \{1, t_1\ \ t^2\}$

i.e., $C = \{c_1, c_2, c_3\}$

The coordinate vectors of b_1, b_2 and b_3 are,

$$[b_1]_C = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, [b_2]_C = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, [b_3]_C = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

The change of coordinates matrix from B to C is,

$$\underset{C \leftarrow B}{P} = [[b_1]_C \quad [b_2]_C \quad [b_3]_C]$$

$$= \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

let x = –1 + 2t

Since $\underset{C \leftarrow B}{P} [x]_B = [x]_C$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

The augmented matrix is, $\begin{bmatrix} 1 & 3 & 0 & ; & -1 \\ -2 & -5 & 2 & ; & 2 \\ 1 & 4 & 3 & ; & 0 \end{bmatrix}$

Converting the matrix into reduced echelon form,

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 3 & 0 & ; & -1 \\ 0 & 1 & 2 & ; & 0 \\ 0 & 1 & 3 & ; & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 3 & 0 & ; & -1 \\ 0 & 1 & 2 & ; & 0 \\ 0 & 0 & 1 & ; & 1 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$

$$\sim \begin{bmatrix} 1 & 0 & -6 & ; & -1 \\ 0 & 1 & 2 & ; & 0 \\ 0 & 0 & 1 & ; & 1 \end{bmatrix}$$

$R_1 \to R_1 + 6R_3$; $R_2 \to R_2 - 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & ; & 5 \\ 0 & 1 & 0 & ; & -2 \\ 0 & 0 & 1 & ; & 1 \end{bmatrix}$$

$$\therefore [x]_B = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

Q16. Let B = {b_1, b_2} and C = {c_1, c_2} be bases for R^2. Find the change of coordinates matrix from B to C and the change of coordinates matrix from C to B.

$$b_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}; b_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, c_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}; c_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Sol :

Given bases for R^2 are B = {b_1, b_2},

C = {c_1, c_2}

Where $b_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, b_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}; c_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$,

$$c_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

The change of coordinates matrix from B to C is given by,

$$\underset{C \leftarrow B}{P} = [c_1 \quad c_2 \quad b_1 \quad b_2]$$

$$= \begin{bmatrix} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{bmatrix}$$

convert the matrix into reduced echelon form

$R_2 \rightarrow R_2 + 5R_1$

$$= \begin{bmatrix} 1 & -2 & 7 & -3 \\ 0 & -8 & 40 & -16 \end{bmatrix}$$

$R_2 \rightarrow \dfrac{R_2}{-8}$

$$= \begin{bmatrix} 1 & -2 & 7 & -3 \\ 0 & 1 & -5 & 2 \end{bmatrix}$$

$R_1 \rightarrow R_1 + 2R_2$

$$= \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{bmatrix}$$

$$\therefore \underset{C \leftarrow B}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$$

The change of coordinate matrix from C to B is ,

$$\underset{B \leftarrow C}{P} = \left[\underset{C \leftarrow B}{P}\right]^{-1}$$

$$= \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}^{-1}$$

$$= \frac{1}{-1}\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

$$\therefore \underset{B \leftarrow C}{P} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

Q17. Let B = {b_1, b_2} and C = {c_1,c_2}be bases for a vector space v and suppose b_1 = $6c_1$ – $2c_2$ and b_2 = $9c_1$ – $4c_2$. Then find change of coordinate matrix B to C.

Sol : **(July - 2021)**

Given bases of a vector space v are,

B = {b_1, b_2} , C = {c_1, c_2}

Where $b_1 = 6c_1 - 2c_2$; $b_2 = 9c_1 - 4c_2$

i.e., $[b_1]_C = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $[b_2]_C = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$

The change of coordinate matrix from B to C is given by,

$$\underset{C\to B}{P}\left[[b_1]_C\,[b_2]_C\right] = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$$

$$\therefore \underset{C\to B}{P} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$$

2.2 Eigen Values and Eigen Vectors

Q18. Define Eigen values and Eigen vectors.

Sol : **(Nov/ Dec.-2018)**

Definition

Let A be any n × n matrix, λ be any scalar. If x is any n × 1 matrix such that $Ax = \lambda x$ then the scalar x is called as an eigen value of the matrix A and the non zero vector x is called as an eigen value of A corresponding to λ.

Note :

(1) Eigen values are also called as Latent roots, characteristic values.

(2) Eigen vectors are also called Latent vectors, characteristic vectors

Q19. Show that the eigen values of a Triangular Matrix are the entries of its Main diagonal.

Sol : **(June-2019/July-2019, Nov/Dec.-2018)**

Let us consider 3 × 3 triangular matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ be the triangular Matrix of order 3 × 3.

Let λ be any scalar and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to the eigen value λ

Consider $(A - \lambda I) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$$

Consider the equation $(A - \lambda I)\, x = 0$ .... (1)

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ 0x_1 + (a_{22}-\lambda)x_2 + a_{23}x_3 &= 0 \\ 0x_1 + 0x_2 + (a_{33}-\lambda)x_3 &= 0 \end{aligned}\right\} \quad ....(2)$$

λ is an eigen value of the matrix A

$\Leftrightarrow$ The system (1) has non trivial solution

$\Leftrightarrow$ The homogenous system in (2) has a non trivial soltuion i.e., $x_1 \neq 0$, $x_2 \neq 0$, $x_3 \neq 0$.

$\Leftrightarrow \lambda = a_{11}, a_{22}, a_{33}$

Thus the eigen values of the matrix A are main diagonal elements of the matrix A.

Q20. Show that. If v_1, v_2 ......v_r are eigen vectors that correspond to distinct eigen values λ_1, λ_2 ....λ_r of an $n \times n$ matrix A, then the set [v_1,v_2 ......v_r] is linearly Independent.

Sol : **(Nov. /Dec.-2019)**

Let A be an $n \times n$ Matrix

Let $\lambda_1, \lambda_2 .... \lambda_r$ be the eigen values of the matrix A. Let v_1, v_2 ....v_r be the corresponding eigen vectors of the matrix A.

To show that [v_1, v_2 ....v_r] is linearly independent.

If possible suppose that the set [v_1, v_2 .....v_r] is linearly dependent.

Then $\exists$ a vector which can be written as the linear combination of its preceeding vectors.

Let P be the least index such that the vector V_{P+1} can be written as the linear combinations of its preceeding linearly independent vectors V_1, V_2 ....V_{P-1} ,V_P.

$\Rightarrow \exists$ Scalars $C_1, C_2, ...C_{P-1}, C_P$ such that

$V_{P+1} = C_1V_1 + C_2V_2 + ... + C_{P-1}V_{P-1} + C_P V_P$ .... (1)

Multiplying both sides with A, we get

$C_1AV_1 + C_2AV_2 + ... + C_PAV_P = AV_{P+1}$ .... (2)

By the defination of eigen value and eigen vector

$\forall$ K, $AV_K = \lambda_K V_K$ .... (3)

$\Rightarrow$ from equations (2) & (3) we get

$C_1 \lambda_1 V_1 + C_2 \lambda_2 V_2 + ... + C_P \lambda_P V_P = \lambda_{P+1} V_{P+1}$ .... (4)

Multiplying Equation (1) with λ_{P+1} we get

$C_1 \lambda_{P+1} V_1 + C_2 \lambda_{P+1} V_2 + ... + C_P \lambda_{P+1} V_P = \lambda_{P+1} V_{P+1}$ .... (5)

Now sub Equation (1) and Equation (5)

$C_1 (\lambda_1 - \lambda_{P+1}) V_1 + C_2 (\lambda_2 - \lambda_{P+1}) V_2 + ...C_P (\lambda_P - \lambda_{P+1}) V_P = 0$ .... (6)

Eqn (6) is a linear combination of the vectors V_1, V_2 ..... V_P which are linearly independent and so

$\lambda_1 - \lambda_{P+1} = 0$; $\lambda_2 - \lambda_{P+1} = 0$; $\lambda_P - \lambda_{P+1} = 0$

$\therefore \lambda_1 = \lambda_{P+1}$, $\lambda_2 = \lambda_{P+1}$ .... $\lambda_P = \lambda_{P+1}$

This the eigen vectors are equal ; But the eigen values are given to be distinct.

This is a contraction

our supposition is wrong

$\therefore$ $[v_1, v_2, ... v_r]$ is a linearly independent set.

Q21. If λ is the eigen value of the matrix A then show that λ^k is the eigen value of the matrix A^k.

Sol :

Let A be the given Matrix

Let λ be the eigen value of the matrix A.

Let x be the corresponding eigen vector of the Matrix A

Then $Ax = \lambda x$ ....(1)

Consider $A^2 x = A(Ax)$

$= A(\lambda x) = \lambda(\lambda x)$

$= \lambda(Ax) = \lambda(\lambda x)$

$A^2 x = \lambda^2 x$ ....(2)

$\therefore$ λ^2 is the eigen value of the matrix A^2

Consider $A^3 x = A(A^2 x)$

$= A(\lambda^2 x)$

$= \lambda^2 (Ax)$

$= \lambda^2 (\lambda x)$

$= \lambda^3 x$

$A^3 x = \lambda^3 x$ ....(3)

$\therefore$ λ^3 is the eigen value of the matrix A^3 suppose that λ^{k-1} is the eigen value of A^{k-1}

$\Rightarrow A^{k-1} x = \lambda^{k-1} x$ .... (4)

Consider $A^k x = A(A^{k-1} x)$

$= A(\lambda^{k-1}x)$

$= \lambda^{k-1} (Ax)$ from (4)

$= \lambda^{k-1}(\lambda x)$ from (1)

$A^k x = \lambda^k x$

$\therefore$ λ^k is the eigen value of the matrix A^k

Q22. If λ is the eigen value of a matrix A then show that λ is also the eigen value of the matrix A^T.

Sol :

Proof

Given that λ is the eigen value of the matrix A let x be the corresponding eigen vector of the matrix A.

$\Rightarrow Ax = \lambda x$ .... (1)

$\Rightarrow Ax - \lambda x = 0$

$\Rightarrow (A - \lambda I) x = 0$ ....(2)

Consider $(A - \lambda I)^T = A^T - (\lambda I)^T$ $[I^T = I, \lambda$ is a scaler]

$(A - \lambda I)^T = A^T - \lambda I$ ....(3)

From (2) & (3) is the eigen value of the matrix

$\Leftrightarrow$ λ is the eigen value of the Matrix A^T.

Q23. Find the characteristic polynomial and the real eigen values of the matrix A = $\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$

Sol : **(June / July-2019)**

Given matrix is,

$$A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$$

The characteristic polynomial is given by,

$$\det\left(A - \lambda I\right) = \left| \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$= \begin{vmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{vmatrix} = (-4 - \lambda)(1 - \lambda) + 6$$

$$= -4 + 4\lambda - \lambda + \lambda^2 + 6$$

$$= \lambda^2 + 3\lambda + 2$$

$\therefore$ The characteristic polynomial is $\lambda^2 + 3\lambda + 2$

The characteristic equation is ,

det $(A - \lambda I) = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 1)(\lambda + 2) = 0$

$\therefore \lambda = -1, -2$

$\therefore$ The real eigen values are –1 and –2

Q24. Find the eigen values and eigen vectors of A = $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

Sol : **(Nov./Dec.-2018)**

Given matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

Eigen values

The characteristic equation is given by det $(A - \lambda I) = 0$

$$\Rightarrow \left| \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4 - \lambda)[(1 - \lambda)(8 - \lambda) + 6] + 1[2(8 - \lambda) - 2(6)] + 6[-2 - 2(1 - \lambda)] = 0$

$\Rightarrow (4 - \lambda)[8 - \lambda - 8\lambda + \lambda^2 + 6] + [16 - 2\lambda - 12] + [-2-2 + 2\lambda] = 0$

$\Rightarrow (4 - \lambda)[\lambda^2 - 9\lambda + 14] + [-2\lambda + 4] + [-24 + 12\lambda] = 0$

$\Rightarrow 4\lambda^2 - 36\lambda + 56 - \lambda^3 + 9\lambda^2 - 14\lambda - 2\lambda + 4 - 24 + 12\lambda = 0$

$\Rightarrow -\lambda^3 + 13\lambda^2 - 40\lambda + 36 = 0$

$\Rightarrow \lambda^3 - 13\lambda^2 + 40\lambda - 36 = 0$

By trial and error method $\lambda = 2$ satisfies the equation $f(2) = 2^3 - 13(2)^2 + 40(2) - 36 = 0$

Then 2 | 1   -13   40   -36

2	1	-13	40	-36
	0	2	-22	36
	1	-11	18	0

$\Rightarrow \lambda^2 - 11\lambda + 18 = 0$

$\Rightarrow \lambda^2 - 9\lambda - 2\lambda + 18 = 0$

$\Rightarrow \lambda(\lambda - 9) - 2(\lambda - 9) = 0$

$\Rightarrow (\lambda - 9)(\lambda - 2) = 0$

$\Rightarrow \lambda = 2, 9$

$\therefore$ The eigen values are 2, 2 and 9.

To find Eigen vectors :

If $\lambda = 2$

$[A - \lambda I]\, x = 0$

$$\Rightarrow \left[\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the Argumented matrix

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix}$$

Apply Row operations

$$\begin{matrix} R_2 : R_2 - R_1 \\ R_3 : R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The Euqtions are

$$2x_1 - x_2 + 6x_3 = 0$$

$$2x_1 = +x_2 - 6x_3$$

$$x_1 = +\frac{1}{2}x_2 - 3x_3 \; ; \; x_2, x_3 \text{ are free variables}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The eigen vectors corresponding to eigen value $\lambda = 2$ are

$$V_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

If $\lambda = 9$

Consider $A - 9\,I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{bmatrix}$

The Argmented matrix [A – 9 I 0] is

$$\begin{bmatrix} -5 & -1 & 6 & 0 \\ 2 & -8 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

$$R_3 : R_3 - R_2 \sim \begin{bmatrix} -5 & -1 & 6 & 0 \\ 2 & -8 & 6 & 0 \\ 0 & 7 & -7 & 0 \end{bmatrix}$$

$R_3 \to \frac{R_3}{7}$, $R_2 \to \frac{R_2}{2}$ $\sim \begin{bmatrix} -5 & -1 & 6 & 0 \\ 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

$R_1 \to R_1 + R_3 \sim \begin{bmatrix} -5 & 0 & 5 & 0 \\ 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

$R_1 \to \frac{R_1}{-5} \sim \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

$R_2 \to R_2 - R_1 \sim \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

$R_2 \to \frac{R_2}{-4} \sim \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

$R_3 \to R_3 - R_2 \sim \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The Equations are $-x_1 + x_3 = 0$

$\Rightarrow x_1 = x_3$

$x_2 - x_3 = 0$

$x_2 = x_3$

Here x_3 is a free variable

$\therefore$ The general solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector corresponding to the eigen value $\lambda = 9$ is $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2.3 The Characteristic Equation

Q25. Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Also find algebraic multiplicity of the eigen values.

Sol : **(Nov./Dec.-2018)**

Given Matrix is $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The characteristic equation of A is given as $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5 - \lambda)\ [(3 - \lambda)(5 - \lambda)(1 - \lambda)] = 0$

$\Rightarrow (5 - \lambda)^2\ (3 - \lambda)(1 - \lambda) = 0$

$\Rightarrow (25 + \lambda^2 - 10\lambda)\ (3 - 3\lambda - \lambda + \lambda^2) = 0$

$\Rightarrow (\lambda^2 - 10\lambda + 25)\ (\lambda^2 - 4\lambda + 3) = 0$

$\Rightarrow \lambda^4 - 4\lambda^3 + 3\lambda^2 - 10\lambda^3 + 40\lambda^2 - 30\lambda + 25\lambda^2 - 100\lambda + 75 = 0$

$\Rightarrow \lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$

Since the given matrix is upper triangular,

The given values are $\lambda = 1$, $\lambda = 3$ with multiplicity 1 and $\lambda = 5$ with multiplicity 2.

Q26. Is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$? If so, find the corresponding eigenvalue

Ans :

Given matrix is,

$$A = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$$

and $x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Consider, $Ax = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -3 \\ 6 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

$$= (-2)\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\Rightarrow Ax = -2x$

$\Rightarrow$ Ax is a multiple of x

$\therefore \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector of A and the corresponding eigenvalue is $\lambda = -2$

Q27. (a) Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.

(b) Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

Ans :

(a) Given,

λ is an eigenvalue of invertible matrix A.

If λ is eigenvalue of A, then there exists a non - zero vector x such that $Ax = \lambda x$ [$\because$ A is invertible]

$\Rightarrow A^{-1} Ax = A^{-1}(\lambda x)$ [$\because A^{-1} A = I, Ix = x$]

$\Rightarrow x = \lambda (A^{-1}x)$

$\Rightarrow \lambda^{-1} x = A^{-1}x$

$\therefore \lambda^{-1}$ is an eigenvalue of A^{-1}

(b) Let, A^2 be zero motion.

If $Ax = \lambda x, x \neq 0$

Then, $A^2x = A(Ax)$

$= A(\lambda x)$

$= \lambda(Ax)$

$= \lambda(\lambda x)$

$= \lambda^2 x$

$\Rightarrow A^2x = \lambda^2 x$

Since, $x \neq 0$

$\Rightarrow \lambda \neq 0$

$\therefore$ The matrix A has only eigen value '0'

Q28. Find the eigen values of A = $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ and compare this result with eigenvalue of A^T

OR

Find the eigenvalues of the matrix

$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Sol : **(Nov./ Dec.-2019, Nov./ Dec.-2018)**

Given matrix is,

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

The characteristic equation is given by,

$\det (A - \lambda I) = 0$

$$\Rightarrow \left| \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2 - \lambda)(-6 - \lambda) - (3)(3) = 0$

$\Rightarrow -12 - 2\lambda + 6\lambda + \lambda^2 - 9 = 0$

$\Rightarrow \lambda^2 + 4\lambda - 21 = 0$

$\lambda^2 + 7\lambda - 3\lambda - 21 = 0$

$\Rightarrow \lambda(\lambda + 7) - 3(\lambda + 7) = 0$

$\Rightarrow (\lambda - 3)(\lambda + 7) = 0$

$\Rightarrow \lambda - 3 = 0; \lambda + 7 = 0$

$\Rightarrow \lambda = 3; \lambda = -7$

$\therefore$ The eigen values of A are 3, –7

The tranpose of A is given by

$$A^T = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

$\Rightarrow A^T = A$

$\therefore$ The eigen value of A^T are same as the given values of A.

Q29. Find eigenvalues for matrix A = $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

Sol :

Given matrix is A = $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

The characteristic equation is given by,

det $(A - \lambda I) = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix} = 0$$

$\Rightarrow (1 - \lambda)(2 - \lambda) - 30 = 0$

$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 30 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 28 = 0$

$\Rightarrow \lambda(\lambda - 7) + 4(\lambda - 7) = 0$

$\Rightarrow (\lambda - 7)(\lambda + 4) = 0$

$\Rightarrow \lambda = 7, -4$

$\therefore$ The eigen values are 7, – 4

Q30. Find the eigenvector for A = $\begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}$ corresponding to eigenvalue $\lambda = -5$

Sol :

Given matrix is, A = $\begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}$

Eigenvalue $\lambda = -5$

Let, x = $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the required eigenvector

Then, $(A - \lambda 1)x = 0$

$$\Rightarrow \left[\begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left| \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right| = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e., $x_1 + 2x_2 = 0$

$\Rightarrow \quad x_1 = -2x_2$

Let, $x_2 = k$

$\Rightarrow x_1 = -2k$

$$\Rightarrow x = \begin{bmatrix} -2k \\ k \end{bmatrix} = k\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector corresponding to $\lambda = -5$ is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Q31. Find the characteristic equation of the matrix $\mathbf{A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$

Sol : **(June /July-2019, Nov./Dec.-2019)**

$$\text{Given matrix is } A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic equation of A is given as,

$|A - \lambda 1| = 0$

$$\Rightarrow \left| \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 5 & -\lambda - 2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) = 0$

$\Rightarrow (5-\lambda)^2 (3-\lambda)(1-\lambda) = 0$

$\Rightarrow (25 + \lambda^2 - 10\lambda)(3 - 4\lambda + \lambda^2) = 0$

$\Rightarrow 25(3 - 4\lambda + \lambda^2) + \lambda^2(3 - 4\lambda + \lambda^2)$

$\quad - 10\lambda(3 - 4\lambda + \lambda^2) = 0$

$\Rightarrow \lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$

$\therefore$ The characteristic equation is,

$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$

Q32. Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ if so find the one corresponding eigen vector.

Sol : **(Nov./ Dec.-2019)**

Given motion is,

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and $\lambda = 3$

Consider,

$$A - 3I = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

The augmented matrix $[(A - 3I)0]$ is,

$$= \begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$R_2 \to 2R_2 + 3R_1$

$$= \begin{bmatrix} -2 & 2 & 2 & 0 \\ 0 & -4 & 8 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \rightarrow 4R_3 + R_2$

$$= \begin{bmatrix} -2 & 2 & 2 & 0 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow \frac{R_1}{R_2},\ R_2 \rightarrow \frac{R_2}{4}$

$$= \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow R_1 + R_2$

$$= \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $(A - 3I)x = 0$ has a not trivial soltuion.

$\therefore$ 3 is an eigen value

The General solutions are

$x_1 - 3x_3 = 0$

$\Rightarrow x_1 = 3x_3$

$x_2 - 2x_3 = 0$

$\Rightarrow x_2 = 2x_3$

And x_3 is a free variable

Let $x_3 = 1 \neq 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_3 \\ 2x_3 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1) \\ 2(1) \\ 1 \end{bmatrix} \quad [\because x_3 = 1]$$

$x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is eigenvector corresponding to the eigenvalue 3.

Q33. Find the eigenvalue of A = $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Sol : **(June /July-2019)**

Given matrix is $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

The characteristic equation is given by

$\det(A - \lambda I) = 0$

$$\Rightarrow \left| \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4 - \lambda)[(1 - \lambda)(1-\lambda) - 0] - 0 + 1$

$[-2(0) - (-2)(1-\lambda) = 0$

$\Rightarrow (4-\lambda)[1 - \lambda - \lambda + \lambda^2] + 1[0 + 2 - 2\lambda) = 0$

$\Rightarrow (4 - \lambda)[1 - 2\lambda + \lambda^2] + [2 - 2\lambda] = 0$

$\Rightarrow 4 - 8\lambda + 4\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 + 2 - 2\lambda = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1$	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$\Rightarrow (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 - 2\lambda - 3\lambda + 6) = 0$

$\Rightarrow (\lambda - 1)(\lambda(\lambda - 2) - 3(\lambda - 2)) = 0$

$\Rightarrow (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\Rightarrow (\lambda - 1) = 0, (\lambda - 2) = 0, (\lambda - 3) = 0$

$\Rightarrow \lambda = 1, 2, 3$

$\therefore$ The eigen values are 1, 2 and 3.

Q34. If 2 is the eigen value of the matrix A = $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ then find a basis for the eigen.

Sol :

The given matrix is $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}, \lambda = 2$

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector of the matrix A corresponding to the root $\lambda = 2$

Then $(A - \lambda I)\, x = 0$

Here $A - \lambda I = A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ .....(1)

$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$

$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (After row operations) .... (2)

Here the rank of the matrix $A - 2I$ is $r = 1$ and the no. of variables $n = 3$

So, we need to choose $n - r = 3 - 1 = 2$

No. of variables say $x_2 = k_1, x_3 = k_2$

Then from (1) & (2), we get

$2x_1 - x_2 + 6x_3 = 0$

$2x_1 - k_1 + 6k_2 = 0$

$2x_2 = k_1 - 6k_2$

$x_1 = \frac{1}{2}k_1 - 3k_2$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}k_1 - 3k_2 \\ k_1 \\ k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The eigen vector corresponding to the root $\lambda = 2$ for the matrix A.

The eigen space is a two - dimensional sub space of R^3.

The set $\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis

Q35. Is $\lambda = 2$ an eigen value of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$ or not ?

Sol :

The given matrix is $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}, \lambda = 2$

Consider the equation $Ax = \lambda x$ for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

i.e., $Ax - \lambda x = 0 \Rightarrow (A - \lambda I)x = 0$

Here $A - \lambda I = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

Consider $(A - \lambda I)\, x = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

Choosing $x_2 = k\ (k \in R)$

$x_1 = -2k$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ -2k \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Thus $(A - \lambda I)x = 0$ has non trivial solutions

$\therefore \lambda = 2$ is an eigen value of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$$

Q36. Prove $\lambda = 4$ is an eigen value of $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$ And find the corresponding Eigen vector and characteristic equation of A.

Sol : **(July-2021)**

Given matrix is $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$

Let $\lambda = 4 \Rightarrow A - \lambda I = A - 4I$

$$= \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & +1 \end{bmatrix}$$

The Augmented matrix [(A – 4I)0] is $\begin{bmatrix} -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 4 & 4 & 0 \end{bmatrix}$$

$$R_3 \to R_3 + 4R_2 \sim \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to -R_2 \sim \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solutions are

$-x_1 - x_3 = 0 \Rightarrow x_1 = -x_3$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

Here x_3 is a free variable

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

For eigen value 4 corresponding eigen vector $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

The characteristic of A is given as det $(A - \lambda I) = 0$

$$\Rightarrow \left| \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 0 & -1 \\ 2 & 3-\lambda & 1 \\ -3 & +4 & 5-\lambda \end{bmatrix} = 0$$

$\Rightarrow (3-\lambda)[(3-\lambda)(5-\lambda) - 4] - 0\,[2(5-\lambda)+3] - 1[8 + 3(3-\lambda)] = 0$

$\Rightarrow (3-\lambda)[15 - 8\lambda + \lambda^2 - 4] - 8 - 9 + 3\lambda = 0$

$\Rightarrow (3-\lambda)(\lambda^2 - 8\lambda + 11] + 3\lambda - 17 = 0$

$\Rightarrow 3\lambda^2 - 24\lambda + 33 - \lambda^3 + 8\lambda^2 - 11\lambda + 3\lambda - 17 = 0$

$\Rightarrow -\lambda^3 - 11\lambda^2 + 32\lambda - 16 = 0$

$\therefore$ Characteristic equation of A is $\lambda^3 + 11\lambda^2 - 32\lambda + 16 = 0$

Q37. The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. **Find the eigenvalues and their multiplicties**

Sol :

Given , charactristic polynomial of 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$.

$\Rightarrow$ det $(A - \lambda I) = \lambda^6 - 4\lambda^5 - 12\lambda^4$.

Since characteristic equation is given by

det $(A - \lambda I) = 0$

$\lambda^6 - 4\lambda^5 - 12\lambda^4 = 0$

$\Rightarrow \lambda^4 (\lambda^2 - 4\lambda - 12) = 0$

$\Rightarrow \lambda^4 = 0$; $\lambda^2 - 4\lambda - 12 = 0$

; $\lambda^2 - 6\lambda + 2\lambda - 12 = 0$

; $\lambda(\lambda - 6) + 2(\lambda - 6) = 0$

; $(\lambda - 6)(\lambda + 2) = 0$

$\lambda = 0, 0, 0, 0$; $\lambda = 6, -2$

$\therefore$ Eigen values are $\lambda = 0, 0, 0, 0, -2, 6$

$\lambda = 0$ with multiplicity 4

$\lambda = 6$ with multiplicity 1

$\lambda = -2$ with multiplicity 1

Q38. If A and B are two similar matrices then show that they have the same charactistic polynomial & hence has the same eigen value.

Sol :

Given that A and B are two similar matrices of order $n \times n$.

Let λ be any scalar

Let I be an Identify Matrix

Then $\exists$ an invertible matrix P such that $B = P^{-1}AP$

To show that A and B have same characteristic polynomial and has same eigen values.

Consider $B - \lambda I = P^{-1}AP - \lambda I$

$= P^{-1}AP - \lambda P^{-1}P$

$\Rightarrow P^{-1}[AP - \lambda P]$

$\Rightarrow P^{-1}[A - \lambda I]P$

$|B - \lambda I| = |P^{-1}[A - \lambda I]P|$

$= |P^{-1}||A - \lambda I||P|$

$= |A - \lambda I||P^{-1}P|$

$\Rightarrow |A - \lambda I||I|$

$= |A - \lambda I|$

$\therefore$ The characteristic polynomials of B and A are Equal.

$\therefore$ The Characteristic equations of B and A are equal

$\therefore$ B and A have same eigen values

Q39. Find the characteristic polynomial of matrix $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$ using either a co-factor expansion or the special formula for 3 × 3 determinants.

Sol :

Given matrix is $A = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$

consider,

det $(A - \lambda I) =$ det

$$\left[\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right]$$

$$= \det \begin{bmatrix} 4-\lambda & 0 & 0 \\ 5 & 3-\lambda & 2 \\ -2 & 0 & 2-\lambda \end{bmatrix}$$

Let the co-factor expansion be along the first row

$$\det(A - \lambda I) = (4 - \lambda) \det \begin{bmatrix} 3-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}$$

$$= (4 - \lambda) [(3 - \lambda)(2 - \lambda) - 0]$$

$$= (4 - \lambda) (\lambda^2 - 5\lambda + 6)$$

$\det (A - \lambda I) = - \lambda^3 + 9\lambda^2 - 26\lambda + 24$

$\therefore$ The characteristic polynomial of matrix A is, $- \lambda^3 + 9\lambda^2 - 26\lambda + 24$.

Q40. Find the characteristic equation and eigen values of A = $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Sol :

Given matrix is $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

The characteristic equation is given by, $\det (A - \lambda I) = 0$

$$\left| \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$$

(2 –1) (–6 –l) – 9 = 0

$\Rightarrow \lambda^2 + 4\lambda - 21 = 0$

$(\lambda + 7) (\lambda - 3) = 0$

$\lambda = 3, \lambda = -7$

$\therefore$ The eigen values are 3 and –7

Choose the Correct Answers

1. If A is any matrix then the number of pivot columns in A is called as ________ of the matrix A. [a]

   (a) Rank (b) Dimensions

   (c) (a) and (b) (d) None

2. The dimensions of the column space and the row space of an $m \times n$ matrix A are, [a]

   (a) Equal (b) Unequal

   (c) Can't be aid (d) None

3. Let A be an $n \times n$ square matrix. If A is invertible then [c]

   (a) Col A = R^n (b) Rank = n

   (c) Both (a) and (b) (d) None

4. If A is a 7×9 matrix with a two dimensional null space then the rank of A is ________. [c]

   (a) 9 (b) 6

   (c) 7 (d) 5

5. The rank of A is the dimension of ________ of A. [b]

   (a) Row space of A (b) Column space of A

   (c) Basis (d) None

6. If λ is the eigen value of then matrix A then ________ is the eigen value of the matrix A^k. [a]

   (a) λ^k (b) λ^{k+1}

   (c) Both (d) None

7. If $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$. The eigen values of a are [b]

   (a) 1, 2, 0 (b) 1, 2, 3

   (c) 1, 0, 0 (d) 0, 6, 3

8. If A is an invertible matrix of order 8×8. Then din Col A = [d]

   (a) 6 (b) 7

   (c) 0 (d) 8

9. The characteristic equation of the matrix is $\begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$. [c]

(a) $\lambda^2 + 14\lambda + 12 = 0$ (b) $\lambda^3 - 14\lambda + 12 = 0$

(c) $\lambda^3 - 14\lambda - 12 = 0$ (d) $-\lambda^3 - 14\lambda + 12 = 0$

10. The eigen values of the matrix $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$. [b]

(a) $\lambda = 5, \lambda = 9$ (b) $\lambda = -5, \lambda = 9$

(c) $\lambda = -5, \lambda = -9$ (d) $\lambda = +5, \lambda = -9$

Fill in the Blanks

1. If A is any m × n matrix then rank of A + dim (Null A) = ________.
2. If $v_1, v_2, \ldots v_r$ are eigen vectors that correspond to district eigen values $\lambda_1, \lambda_2, \ldots, \lambda_r$, of an n × n matrix A then the set $\{v_1, v_2, \ldots v_r\}$ is ________.
3. If two matrices A and B are ________ then their row spaces are the same.
4. The rank of A is the ________ of a.
5. Let A be any n × n matrix, then det A^T = ________.
6. If A and B are two n × n matrices then |AB| = ________.
7. If n × n matrices A and B are similar than they have same ________.
8. The eigen values of triangular matrix are the entries on its ________.
9. If A is not invertible then ________.
10. The rank of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is ________.

Answers

1. n
2. linearly independent
3. row equivalent
4. dimension of the column space
5. det A
6. |A|, |B|
7. Eigen values
8. Main diagonal
9. |A| = 0
10. 3

UNIT III

Diagonalization - Eigen vectors and Linear Transformations - Complex Eigenvalues - Applications to Differential Equations

3.1 Diagonalization - Eigen Vectors and Linear Transformations

Q1. Define Diagonalization.

Sol :

A square matrix A is said to be diagonalizable if $\exists$ a non-singular matrix (Invertible) P such that $A = PDP^{-1}$ where D is a Diagonal matrix.

We say that P diagonalizes A.

$$A = PDP^{-1} \Leftrightarrow AP = PD.$$

Q2. State and prove the diagonalization theorem.

Statement: An n × n matrix A is diagonalizable if and only if A has n linearly independent eigen vectors.

Proof :

Let A be any $n \times n$ square matrix.

Let P be any $n \times n$ matrix with columns $v_1, v_2, ... v_n$

$$p = [v_1, v_2, ... v_n].$$

Let D by any diagonal matrix of order $n \times n$ with diagonal elements, $\lambda_1, \lambda_2, ... \lambda_n$.

$$\text{Then, } D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Consider

$$AP = A[v_1, v_2, ... v_n]$$

$$= [Av_1, Av_2, ... Av_n] \quad ...(1)$$

$$PD = P\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= [\lambda_1 v_1 \;\; \lambda_2 v_2 \; .... \; \lambda_n v_n] \quad ...(2)$$

Part I

Suppose that the matrix A is diagonalizable.

$\Rightarrow$ A can be written as $A = PDP^{-1}$.

$$\Rightarrow AP = PD$$

$\Rightarrow$ $[Av_1 \; Av_2 \; ... \; Av_n] = [\lambda_1 v_1 \;\; \lambda_2 v_2 \; .... \; \lambda_n v_n]$ ...(3)

$\Rightarrow$ Equating the corresponding columns on both sides we get,

$$Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2, Av_n = \lambda_n v_n \quad ...(4)$$

Each of the expression in (4) is of the form $AX = \lambda X$ which indicates that λ is the eigen value of A and X is the corresponding eigen vector of A.

Thus $\lambda_1, \lambda_2, ... \lambda_n$ are the eigen values of A and $v_1, v_2, ... v_n$ are the corresponding eigen vectors of A.

Since P is invertible the columns of P are linearly independent and these columns are non-zero.

$\Rightarrow$ The vectors $v_1, v_2, ... v_n$ are linearly independent. Thus the matrix A has n linearly independent eigen vectors.

Part II

Let us suppose that the matrix A has 'n' linearly independent eigen vectors.

Let $v_1, v_2, \ldots v_n$ be the n linearly independent eigen vectors of A corresponding to the eigen value $\lambda_1, \lambda_2, \ldots \lambda_n$ let $P = [v_1, v_2, \ldots v_n]$. Since the columns of P and linearly independent.

$|p| \neq 0$

$\Rightarrow$ p^{-1} exists

Consider,

$$AP = A[v_1, v_2, \ldots v_n] = [Av_1\ Av_2 \ldots Av_n]$$

$$= [\lambda_1 v_1\ \lambda_2 v_2 \ldots \lambda_n v_n]$$

$$= [v_1, v_2, \ldots v_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

AP = PD where D is the diagonal matrix.

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$\Rightarrow$ $P^{-1}AP = P^{-1}PD$

$\Rightarrow$ $P^{-1}AP = D$

$\therefore$ A is Diagonalizable

Q3. Show that an $n \times n$ matrix with n distinct eigen values is diagonalizable.

Sol : **(June/July-19, Nov./Dec.-18)**

Let A be any square matrix of order $n \times n$.

Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the n distinct eigen values of the matrix A.

Let $v_1, v_2, \ldots v_n$ be the corresponding eigen vectors of the matrix A.

Then $\{v_1, v_2, \ldots v_n\}$ is a linearly independent set of A .

$\Rightarrow$ A is diagonalizable.

Q4. Determine whether the following matrix is diagonalizable or not $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$

Sol :

The given matrix is $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$

This is a 3×3 matrix. It is a triangular matrix the eigen values of A are 5, 0, –2.

Thus there are '3' distinct eigen values of the matrix A and hence the matrix A is diagonalizable.

Q5. Diagonalize $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ if possible

Sol : **(July-21)**

Given matrix is, $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

The characteristic equation of A is

$$\left| \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(3-\lambda)\,[(3-\lambda)-1]-1[3-\lambda-1]+1[1-(3-\lambda)] = 0$

$\Rightarrow$ $(3-\lambda)\,[9-3\lambda-3\lambda+\lambda^2-1]-1[3-\lambda-1]+1[1-(3-\lambda)] = 0$

$\Rightarrow$ $3\lambda^2-18\lambda+24-3\lambda^3+6\lambda^2+8\lambda+2\lambda-4=0$

$\Rightarrow$ $-\lambda^3-9\lambda^2-24\lambda^2+20 = 0$

$\Rightarrow$ $\lambda^3-9\lambda^2-24\lambda-20 = 0$

$\Rightarrow$ $(\lambda-2)\,(\lambda^2-7\lambda+10) = 0$

$\Rightarrow$ $(\lambda-2)\,(\lambda-5)\,(\lambda-10) = 0$

$\Rightarrow$ $(\lambda-2)^2\,(\lambda-5) = 0$

$\Rightarrow$ $\lambda-2,\ \lambda = 5$

$\therefore$ The eigen values are $\lambda = 2$ with multiplicity 2 and $\lambda = 2$ with multiplicity 1.

If $\lambda = 2$

consider $A-2I = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The augmented matrix [(A – 2I) 0] is,

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The quation is,

$x_1 + x_2 + x_3 = 0$

$\Rightarrow$ $x_1 = -x_2 - x_3$

and x_2, x_3 are free variables

The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The basis vector for eigen space is,

$$\{v_1, v_2\} = \left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$$

If $\lambda = 5$

Consider

$$A - 5I = \begin{bmatrix}3&1&1\\1&3&1\\1&1&3\end{bmatrix} - 5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}$$

The augmented matrix [(A – 5I) 0] is,

$$= \begin{bmatrix}-2&1&1&0\\1&-2&1&0\\1&1&-2&0\end{bmatrix}$$

$R_2 \to 2R_2 + R_1$, $R_3 \to 2R_3 + R_1$

$$= \begin{bmatrix}-2&1&1&0\\0&-3&3&0\\0&3&-3&0\end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$= \begin{bmatrix}-2&1&1&0\\0&-3&3&0\\0&0&0&0\end{bmatrix}$$

Theh equation are,

$-2x_1 + x_2 + x_3 = 0$

$\Rightarrow$ $-2x_1 = -x_2 - x_3$ ...(1)

$-3x_2 + 3x_3 = 0$

$\Rightarrow$ $-3x_2 = -3x^3$

$\Rightarrow$ $x_2 = x_3$ ...(2)

Sub (2) in (1)

$-2x_1 = -x_3 - x_3 \Rightarrow x_1 = x_3$

$\therefore$ x_3 is a free variable

$\therefore$ The general solution is

$$x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_3\\x_3\\x_3\end{bmatrix} = x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$\therefore$ The basic vector for eigen space is,

$$v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

let $P = [v_1\ v_2\ v_3] = \begin{bmatrix}-1&-1&1\\1&0&1\\0&0&1\end{bmatrix}$

and $D = \begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix} = \begin{bmatrix}2&0&0\\0&2&0\\0&0&5\end{bmatrix}$

The matrix D has the eigen values correspondingv to eigen vectors v_1, v_2 and v_3 respectively.

Consider,

$$AP = \begin{bmatrix}3&1&1\\1&3&1\\1&1&3\end{bmatrix}\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}$$

$$= \begin{bmatrix}-2&-2&5\\2&0&5\\0&2&5\end{bmatrix}$$

Consider,

$$PD = \begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&2&0\\0&0&5\end{bmatrix}$$

$$= \begin{bmatrix}-2&-2&5\\2&0&5\\0&2&5\end{bmatrix}$$

$\Rightarrow$ AP = PD

$\therefore$ The matrix A is diagonalizable

Q6. If p = $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, D = $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ then compute A^2, A^4 if A = PDP^{-1}.

Sol : **(July-21)**

Here D is a diagonal matrix

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} 2^k & 0 \\ 0 & 1^k \end{bmatrix} = \begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix} \ \forall k \geq 1$$

$$P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \Rightarrow |P| = 15 - 14 = 1 \neq 0$$

$\therefore$ P is Invertiable and $P^{-1} = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$

i) Consider $A = PDP^{-1}$

$$A^2 = PD^2P^{-1}$$

$$A^2 = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 46 & -105 \\ 18 & -41 \end{bmatrix}$$

ii) Consider $A^4 = PD^4P^{-1}$

$$A^4 = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$$

Q7. If the eigen vales of a matrix A are 2 and 1. The corresponding eigen vectors of A are $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then find A^8.

Sol :

Let A be any square matrix of order 2×2. The eigen values of A are 2 and 1 say $\lambda_1 = 2$ and $\lambda_2 = 1$.

Let $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are the corresponding eigen vectors of the matrix A.

Here the vectors v_1 and v_2 are linearly independent.

$\therefore$ The matrix A is diagonalizable.

$\Rightarrow$ $\exists$ a non-singular matrix p and a diagonal matrix D such that $A = PDP^{-1}$ where,

$$P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Consider $A = PDP^{-1}$

$$A^8 = PD^8P^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2^8 & 0 \\ 0 & 1^8 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}$$

Q8. Verify whether A is invertible if A is diagonalizable.

Sol :

Suppose that A is diagonalizable.

$\Rightarrow$ $\exists$ an Invertible matrix p and diagonal matrix D such that $A = PDP^{-1}$.

$A^{-1} = (PDP^{-1})^{-1}$

$A^{-1} = (P^{-1})^{-1} D^{-1} P^{-1}$

$A^{-1} = P D^{-1} P^{-1}$

$A^{-1} = PEP^{-1}$ where $E = D^{-1}$ is also a diagonal matrix.

$\therefore$ A is invertible.

Q9. Determine whether the following matrix is diagonalizable or not,

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Sol : **(June/July-19)**

The given matrix is $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Let λ be any scalar and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be the unit vector.

Step I

The characterize equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(2 - \lambda)\,[(-6 - \lambda)\,(1 - \lambda) + 9]$

$- 4[-4\,(1-\lambda) + 6]$

$+ 3[-12 - 2(-6 - \lambda)] = 0$

$\Rightarrow$ $-\lambda^3 - 3\lambda^2 + 4 = 0$

$\Rightarrow$ $(\lambda - 1)\,(\lambda + 2)^2 = 0$

$\Rightarrow$ $\lambda = 1, \lambda = -2$ are the eigen values of the matrix A.

Step II

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector of the matrix A then consider $(A - \lambda I)x = 0$.

$$\begin{bmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad ...(1)$$

when $\lambda = 1$; $\begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_2 \to R_2 + 4R_1, R_3 \to R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 4 & 3 \\ 0 & 9 & 9 \\ 0 & -9 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + R_2$

$$= \begin{bmatrix} 1 & 4 & 3 \\ 0 & 9 & 9 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \rightarrow \frac{R_2}{9}$

$$= \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \rightarrow R_1 - 4R_2$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The equations are $x_1 - x_3 = 0$

$x_1 = x_3$

$x_2 + x_3 = 0$

$x_2 = -x_3$

Choosing $x_3 = k$, we get $x_2 = -k$, $x_1 = k$

Here k is any scalar.

∴ The eigen vector of the matrix a corresponding tot he eigen value $\lambda = 1$ is,

$$\begin{bmatrix} k \\ -k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \text{(where } k \in R\text{)}$$

When $\lambda = -2$

$$\begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 4 & 3 \\ 0 & 0 & 0 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 4 & 3 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 4 & 3 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

⇒ $4x_1 + 4x_2 + 3x_3 = 0$

$x_1 + x_2 = 0$

⇒ $x_1 = -x_2 = k$ (say)

$4k - 4k + 3k_3 = 0 \Rightarrow k_3 = 0$

∴ The eigen vector corresponding the root $\lambda =$

$$-2 \text{ is} \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Step-III

Thus the eigen vectors of the matrix A are

$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ which are linearly independent.

Here the square matrix A is of order 3 but A has only two independent eigen values.

∴ The matrix A is not diagonalizable.

Q10. Diagonalize the matrix $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ if possible.

Sol :

The given matrix is $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$.

Let λ be any scalar and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the unit matrix. The characteristic equation is.

$|A - \lambda I| = 0$

$$\begin{bmatrix} 5-\lambda & 1 \\ 0 & 5-\lambda \end{bmatrix} = 0$$

$(5 - \lambda)^2 = 0$

$\lambda = 5, 5$

Thus the matrix A of order 2 × 2 has only one distinct eigen value.

∴ The matrix A is not diagonalizable.

Q11. Diagonalize the matrix A = $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ if possible.

Sol :

The given matrix is A = $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

Let λ be any scalar and I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be a unit matrix. The characteristic equation is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$(3 - \lambda)(5 - \lambda) + 1 = 0 \quad \Rightarrow \lambda^2 + 2\lambda + 16 = 0$

There are no two distinct real roots of the matrix A. The matrix Ais not diagonalizable.

Q12. Diagonalize the matrix A = $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ if possible.

Sol : **(Nov./Dec.-19)**

Given matrix is, A = $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

The characteristic equation of A is,

det $(A - \lambda I) = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(1-\lambda)\,[(-5-\lambda)(1-\lambda)-(-3)3]-3[(-3)(1-\lambda)-(3)(-3)]+3[(-3)(3)-3(-5-\lambda)]=0$

$\Rightarrow$ $(1-\lambda)\,[-5+5\lambda-\lambda+\lambda^2+9)-3[-3+3\lambda+9]+3[-9+15+3\lambda]=0$

$\Rightarrow$ $(1-\lambda)\,[\lambda^2+4\lambda+4]-3(3\lambda+6)+3(3\lambda+6)=0$

$\Rightarrow$ $(1-\lambda)(\lambda^2+4\lambda+4)=0$

$\Rightarrow$ $(\lambda-1)(\lambda^2+4\lambda+4)=0$

$\Rightarrow$ $(\lambda-1)(\lambda+2)(\lambda+2)=0$

$\Rightarrow$ $\lambda=-2,-2,1$

If $\lambda=-2$

Consider,

$$A-(-2)I = A+2I$$

$$= \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

The augmented matrix $[(A+2I)\ 0]$ is,

$$\begin{bmatrix} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 = \begin{bmatrix} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 = \begin{bmatrix} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation is,

$$3x_1 + 3x_2 + 3x_3 = 0$$

$$3x_1 = -3x_2 - 3x_3$$

$$x_1 = -x_2 - x_3$$

and x_2, x_3 are free variables

The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = -2$ is,

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

If $\lambda = 1$,

Consider,

$$A - \lambda I = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

The augmented matrix [(A–I) 0] is

$$\begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 = \begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -3 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 = \begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 = \begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{3}, R_2 \to \frac{R_2}{-3} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equations are,

$x_2 + x_3 = 0$

$\Rightarrow$ $x_2 = -x_3$

$x_1 - x_3 = 0$

$\Rightarrow$ $x_1 = x_3$

and x_3 is a free variable.

$\therefore$ The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = 1$ is,

$$v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

let $P = [v_1 \ v_2 \ v_3]$

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix D has the eigen values corresponding to eigen vectors, v_1, v_2 and v_3 respectively.

Consider,

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ -2 & 0 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

Consider,

$$PD = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ -2 & 0 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

$\Rightarrow$ AP = PD

$\therefore$ The matrix A is diagonalizable.

3.1.1 Linear Transformations

Q13. If A = $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ find a formula for A^K given that $A = PDP^{-1}$ where P = $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, D = $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

Sol : **(Nov./Dec.-19)**

Given

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-2+1} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} +2 & +1 \\ -1 & -1 \end{bmatrix}$$

Since given $A = PDP^{-1}$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$\Rightarrow$ $A^k = PD^kP^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5^k & 3^k \\ -5^k & -2\times 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 5^k - 3^k & 5^k - 3^k \\ -5^k \times 2 + 2\times 3^k & -5^k + 2\times 3^k \end{bmatrix}$$

$$A^k = \begin{bmatrix} 2\times 5^k - 3^k & 5^k - 3^k \\ -2\times 5^k + 2\times 3^k & -5^k + 2\times 3^k \end{bmatrix}$$

Q14. Define Linear Transformation, Kernal of Linear Transformation, Range of a Linear Transformation.

Sol :

Linear Transformation

Let v and w be two vector spaces defined over a field F.

$T: V \rightarrow W$ be any mapping such that

$T(u+v) = T(u) + T(v)$

$T(cu) = CT(u)$ $\forall u, v \in V$ and for any scalar c then T is called as a linear transformation from v to w.

Kernal of a Linear Transformation

Let $T: V \rightarrow W$ be any linear transformation. Then the set consisting of all these elements of v whose images are equal to the zero vector of w is called as the kernal of T.

The Kernal of T is also called as Null space of T

Kernal T or $K_T = \{U/U \in V$ and $T(U) = 0; 0 \in W\}$

Range of a Linear Transformation

Let $T: V \rightarrow W$ be any linear transformation. The set of all images of elements of v under the transformation T is called as Range of T.

Q15. If $T: V \rightarrow W$ is a linear transformation then,

(i) Kernal of T is a subspace of v

(ii) Range of T is a subspace of W.

Sol :

Let V and w be two vector spaces.

$T : V \rightarrow W$ be a linear transformation.

(i) By definition of Kernal

Kernal of T $= \{\alpha \in v / T(\alpha) = 0 ; 0 \in W\}$

AS $O \in V$, $T(0) = 0$

$O \in$ kernal of T

Kernal T is non empty subset of V.

let $v_1\, v_2 \in$ Kernal T

$\Rightarrow \quad T(v_1) = 0, \; T(v_2) = 0$

As $v_1\, v_2 \in V \Rightarrow v_1 + v_2 \in V$

$T(v_1 + v_2) = T(v_1) + T(v_2)$

$= 0 + 0$

$= 0$

$v_1 + v_2 \in$ kernal T ... (1)

For any scalar k, for any $V \in$ kenal T

$\Rightarrow \quad T(v) = 0$

$\Rightarrow \quad T(kv) = kT(v)$

$= 0 \in$ kernal T

Kernal T is a subspace of w.

(ii) To Prove Range of T

Range of T $= \{T(v)/v \in V\}$

As V is a vector space

$\Rightarrow O \in V$

$T(O) \in$ Range of T ($\because$ T is a linear Transformation)

Consider any $\alpha, \beta \in$ range set of T

$\alpha \in R(T) \Rightarrow \alpha = T(v_1)$ where $v_1 \in V$

$\beta \in R(T) \Rightarrow \beta = T(v_2)$ where $v_2 \in V$

Consider $\alpha + \beta = T(v_1) + T(v_2) = T(v_1 + v_2)$

$\alpha + \beta \in$ Range of T

For any scalar k ; for $\alpha \in$ Range of T

$k\alpha = k.T(x_1) = T(kx_1)$

$k\alpha \in$ Range set of T ($\because$ T is a linear Transformation

$\therefore$ Range of T contains zero vector, closed under vector addition and scalar multiplication. Range of T is a subspace of W.

Q16. Suppose B = {b_1 b_2} is a basis for v and c = {c_1, c_2, c_3} is a basis flow W let T: V $\rightarrow$ W be a linear Transformation with the property that,

$$T(b_1) = 3c_1 - 2c_2 + 5c_3$$

$$T(b_2) = 4c_1 + 7c_2 - c_3$$

Find the matrix M for T relative to B and C.

Sol :

The c-coordinate vectors of the images of b_1 and b_2 are $[T(b_1)]_C = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$ and $[T(b_2)]_C = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$.

Hence $M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$.

Q17. The mapping T:$p_2 \rightarrow p_2$ defined by $T(a_0 + a_1t + a_2t^2) = a_1 + 2a_2t$ is a linear transformation.

(a) Find the B-matrix for T, when B is a basis {1, t, t^2}.

(b) Verify that $[T(p)]_B = [T]_B[P]_B$ for each p in p_2.

Sol :

(a) Given $T(a_0 + a_1t + a_2t^2) = a_1 + 2a_2t$

$T(1) = 0$

$T(t) = 1$

$T(t^2) = 2t$

$$[T(1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [T(t)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [T(t^2)]_B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$[T]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Given $p(t) = a_0 + a_1t + a_ct^2$

$$[T(p)]_B = [a_1 + 2a_2t]_B = \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = [T]_B\ [P]_B.$$

Q18. Let T: $p_2 \to p_3$ be such that T(p(t)) = (t + 3) p(t).

(i) Find T(3 – 2t + t^2)

(ii) Show that T is a linear transformation.

(iii) Find the matrix for T relation to the bases {1, t, t^2} and {1, t, t^2, t^3}.

Sol :

Given T: $p_2 \to p_3$ is defined by

$T(p) = (t + 3)\ p(t)$ ... (1)

(i) $p(t) = 3 - 2t + t^2$

$T(p) = (t + 3)\ (3 - 2t + t^2)$

$= 3t - 2t^2 + t^3 + 9 - 6t + 3t^2$

$= 9 - 3t + t^2 + t^3$

$T(p) = 9 - 3t + t^2 + t^3$

(ii) Let p, q be two polynomials in p_2 and c be any scalar.

Consider

(a) $T[(p(t) + q(t)] = (t + 3)[p(t) + q(t)]$

$= (t + 3)\ p(t) + (t + 3)\ q(t)$

$= T[p(t)] + T[q(t)]$

(b) $T[c.p(t)] = (t + 3)\ [c.p(t)]$

$= c(t + 3)\ p(t)$

$= cT(p(t))$

∴ T is linear transformation.

(iii) Let B = {1, t, t^2} be basis for p_2

and c = {1, t, t^2, t^3} be basis for p_3

Since $T(b_1) = T(1) = (t + 3)\ (1)$

$= t + 3 = 3 + t$

$$[T(b_1)]_c = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Q19. Suppose A = PDP^{-1} where D is a diagonal n × n matrix. If B is a basis for R^n formed from the columns of p then D is the B-matrix for the Transformation X → AX.

Sol :

Denote the columns of P by $b_1\ b_2\ ...\ b_n$

So that B = {$b_1, b_2, ... b_n$} and

$p = [b_1\ b_2\ ...b_n]$

P is the change of coordinate matrix P_B.

Where $p[X]_B = x$ and $[x]_B = p^{-1} x$

If T(X) = AX, for X in R^n then

$[T]_B = [(Tb_1)]_B\ ....\ [T(b_n)]_B$

$= [(Ab_1]_B\ ....\ [Ab_n]_B]$

$= [P^{-1}\ Ab_1\ ....\ P^{-1}\ Abn]$

$= P^{-1}\ A[b_1\ ....\ b_n]$

$= P^{-1}\ AP$

Since $A = PDP^{-1}$ we hve $[T]_B = p^{-1}\ AP = D$.

Q20. Let A = $\begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}$ $b_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$; $b_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; Find B-Matrix.

Sol :

If $p = [b_1\ b_2]$ then the B-Matrix is p^{-1} AP

Compute $AP = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 12-18 & 8-9 \\ 12-16 & 8-8 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -1 \\ -4 & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -6 & -1 \\ -4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$

Hence Eigen value of A is on the diagonal.

$$T(b_2) = T(t) = (t+3)(t)$$

$$= t^2 + 3t$$

$$= 3t + t^2$$

$$= 0 + 3t + 1t^2 + 0t^3$$

$$[T(b_2)]_c = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$T(b_3) = T(t^2) = (t+3)\,t^2 = t^3 + 3t^2$$

$$= 0 + 0t + 3t^2 + t^3$$

$$[T(b_3)]_c = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

The matrix for T relative to B and C is

$$[[T(b_1)]_c \; [T(b_2)]_c \; [T(b_3)]_c] = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Q21. T: $R^3 \to R^3$ is defined by $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y \\ x-z \end{bmatrix}$ check whether T is a linear transformation or not.

Sol :

Given $T: R^3 \to R^3$ defined by

$$T = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y \\ x-z \end{bmatrix} \quad \text{... (1)}$$

let $\alpha = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $\beta = \begin{bmatrix} x_2 \\ y_2 \\ z_z \end{bmatrix}$

Consider,

$$T(\alpha+\beta) = T\left[\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right]$$

$$= T\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+x_2+y_1+y_2 \\ y_1+y_2 \\ x_1+x_2-z_1-z_2 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1+y_1)+(y_2+y_2) \\ y_1+y_2 \\ (x_1-z_1)+(x_2-z_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1+y_1 \\ y_1 \\ x_1-z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ y_2 \\ x_2-z_2 \end{bmatrix}$$

$$= T(\alpha) + T(\beta)$$

$\therefore \quad T(\alpha + \beta) = T(\alpha) + T(\beta)$ ... (2)

For any scalar C,

$$\text{Consider } T[C\alpha] = T\left[C\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right] = T\begin{bmatrix} Cx_1 \\ Cy_1 \\ Cz_1 \end{bmatrix}$$

$$= \begin{bmatrix} cx_1 + cy_1 \\ cy_1 \\ cx_1 - cz_1 \end{bmatrix} = C\begin{bmatrix} x_1 + y_1 \\ y_1 \\ x_2 - z_1 \end{bmatrix} = CT(\alpha)$$

$\therefore \quad T[C\alpha] = CT(\alpha)$ ... (3)

From (2) and (3)

$T: R^3 \rightarrow R^3$ is linear transformation.

3.2 Complex Eigen Value - Applications to Differential Equations

3.2.1 Complex Eigen Values

Q22. If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find the Eigen values of A and find a basis for each eigen space.

Sol :

Given matrix is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

The characteristic equation is det $[A - \lambda I] = 0$

$$\det\left[\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] = 0$$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \sqrt{-1} = \pm i$$

$$\lambda = i, -i$$

If $\lambda = i$

Consider $[A - \lambda I]\, X = 0$ is

$$\left[\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = i$

$$\left[\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -ix_1 - x_2 &= 0 \\ x_1 - ix_2 &= 0 \end{aligned}\right\} \quad \boxed{x_1 = ix_2}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ix_2 \\ x \end{bmatrix} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Thus i and –i are eigen values with

If $\lambda = -i$

Consider $[A - \lambda I]\, X = 0$ is

$$\left[\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$ix_1 - x_2 = 0$

$x_1 + ix_2 = 0$

$\boxed{x_1 = ix_2}$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Thus and – i are eigen values with $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ are corresponding eigen vectors.

Q23. Let $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$ Find the eigen values of A and find a basis for each eigen space.

Sol :

The characteristics equation of A is

det $[A - \lambda I] = 0$

$$\left|\begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} 0.5 - \lambda & -0.6 \\ 0.75 & 1 - \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (0.5 - \lambda)(1.1 - \lambda) - (-0.6)(0.75) = 0$

$\Rightarrow \quad \lambda^2 - 1.6\lambda + 1 = 0$

$$\lambda = \frac{1}{2}\left[1.6 \pm \sqrt{(-1.6)^2 - 4}\right]$$

$\lambda = 0.8 \pm 0.6i$

Construct

$[A - (0.8 - 0.6i)I]X$

$$= \left[\begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix} - \begin{bmatrix} 0.8 - 0.6i & 0 \\ 0 & 0.8 - 0.6i \end{bmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.3 + 0.6i & -0.6 \\ 0.75 & 0.3 + 0.6i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$(-0.3 + 0.6i)\, X_1 - 0.6\, X_2 = 0$

$0.75\, X_1 + (0.3 + 0.6i)\, X_2 = 0$

$0.75\, X_1 = -(0.3 + 0.6i)\, X_2$

$X_1 = (0.4 - 0.8\,i)\, X_2$

Choose $X_2 = 5$ to eliminate the decimals and obtain $X_1 = -2 - 4i$ corresponding to $\lambda = 0.8 - 0.6i$ is $V_1 = \begin{bmatrix} -2 - 4i \\ 5 \end{bmatrix}$.

For $\lambda = 0.8 + 0.6\,i$ corresponding eigen vector $V_2 = \begin{bmatrix} -2 + 4i \\ 5 \end{bmatrix}$.

Q24. Solve the initial value problem $X^1(t)$ = AX(t) for $t \geq 0$ with X(0) = (3, 2). Classify the nature of the origin as an attractor, repeller, or saddle point of the dynamical system described by X' = AX. Find the direction of greatest attraction and / or regulation.

(i) $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

Sol :

Given matrix is $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

The characteristic equation is given by det $(A - \lambda I) = 0$.

$$\left| \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$\Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2 - \lambda)(-2 - \lambda) + 3 = 0$

$\Rightarrow \lambda^2 - 2\lambda + 2\lambda - 4 + 3 = 0$

$\Rightarrow \lambda^2 - 1 = 0$

$\Rightarrow (\lambda - 1)(\lambda + 1) = 0$

$\Rightarrow \lambda = 1, -1$

If $\lambda = 1$

Consider

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}$$

Argumented Matrix $[(A - \lambda I)0]$ is,

$$\begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation is $X_1 + 3X_2 = 0$

$\Rightarrow X_1 = -3x_2$

X_2 is a free variable

$\therefore$ The general solution is,

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -3X_2 \\ X_2 \end{bmatrix} = X_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

IP $\lambda = -1$

$$A + \lambda I = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$$

The Augmented Matrix $[(A + \lambda I)0]$ is

$$= \begin{bmatrix} 3 & 3 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

$$R_2 : 3R_2 + R_1 \sim \begin{bmatrix} 3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 : \frac{R_1}{3} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The equations is $X_1 + X_2 = 0$

$X_1 = -X_2$

$\therefore$ X_2 is a free variable

$\therefore$ The general solution is,

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -X_2 \\ X_2 \end{bmatrix} = X_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Initial condition $X(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Let the constant C_1, C_2 satisfy X(0) Such that $c_1v_1 + c_2v_2 = X(0)$.

$$\Rightarrow \quad C_1\begin{bmatrix}-3\\1\end{bmatrix} + C_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

$$[v_1\ v_2\ x(0)] = \begin{bmatrix}-3 & 1 & 3\\ 1 & 1 & 2\end{bmatrix}$$

$$R_2 \to 3R_2 + R_1 \sim \begin{bmatrix}-3 & -1 & 3\\ 0 & 2 & 9\end{bmatrix}$$

$$R_1 \to \frac{R_1}{-3};\ R_2 \to \frac{R_2}{2} \sim \begin{bmatrix}1 & \frac{1}{3} & -1\\ 0 & 1 & \frac{9}{2}\end{bmatrix}$$

$$R_1 \to R_1 - \frac{1}{3}R_2 \sim \begin{bmatrix}1 & 0 & \frac{-5}{2}\\ 0 & 1 & \frac{9}{2}\end{bmatrix}$$

$$C_1 = -\frac{5}{2},\ C_2 = \frac{9}{2}$$

The general solution of $X^1 = AX$ is

$$X(t) = c_1v_1e^{\lambda_1 t} + c_2v_2e^{\lambda_2 t}$$

$$\Rightarrow \quad X(t) = -\frac{5}{2}\begin{bmatrix}-3\\1\end{bmatrix}e^{t} + \frac{9}{2}\begin{bmatrix}-1\\1\end{bmatrix}e^{-t}$$

Since the matrix A has both positive and negative eigen values.

$\therefore$ The origin is a saddle point of the dynamical system described by $X' = AX$.

The direction of greatest attraction is the line through V_2 and origin corresponding to negative eigen value. The direction of greatest repulsion is the line through V_1 and the origin corresponding to the positive eigen value.

Q25. Find eigen vector for $A = \begin{bmatrix}4 & 3\\ -3 & 4\end{bmatrix}$ corresponding to eigen value $\lambda = 4 + 3i$

Sol :

Given matrix is $A = \begin{bmatrix}4 & 3\\ -3 & 4\end{bmatrix}$

Eigen value $\lambda = 4 + 3i$

Consider

$$A - \lambda I = A - (4 + 3i)I$$

$$= \begin{bmatrix}4 & 3\\ -3 & 4\end{bmatrix} - \begin{bmatrix}(4+3i) & 0\\ 0 & 4+3i\end{bmatrix} = \begin{bmatrix}-3i & 3\\ -3 & -3i\end{bmatrix}$$

The equation $[A - (4 + 3i)I]\, x = 0$ gives

$$\begin{bmatrix}-3i & 3\\ -3 & -3i\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \quad -3x_1 + (-3i)\,x_2 = 0$$

$$-3x_1 = 3ix_2$$

$$x_1 = -ix_2$$

And x_2 is free variable.

$\therefore$ The general solution is

$$x = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}ix_2\\x_2\end{bmatrix} = x_2\begin{bmatrix}-i\\1\end{bmatrix}$$

Here $v = \begin{bmatrix}-i\\1\end{bmatrix}$

$\therefore$ The eigen vector $v = \begin{bmatrix}-i\\1\end{bmatrix}$.

Q26. Find the complex eigen values of the matrix $A = \begin{bmatrix}3 & -3\\ 3 & 3\end{bmatrix}$.

Sol : **(July-2021)**

Given matrix is, $A = \begin{bmatrix}3 & -3\\ 3 & 3\end{bmatrix}$

The characteristics equation is given by,

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \left|\begin{bmatrix}3 & -3\\ 3 & 3\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right| = 0$$

$$\Rightarrow \quad \left|\begin{bmatrix}3 & -3\\ 3 & 3\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right| = 0$$

$$= \begin{vmatrix}3-\lambda & -3\\ 3 & 3-\lambda\end{vmatrix} = 0$$

$\Rightarrow \quad (3-\lambda)^2 + 9 = 0$

$\Rightarrow \quad \lambda^2 - 6\lambda + 18 = 0$

$$\lambda = \frac{(-6) \pm \sqrt{(-6)^2 - 4(1)(18)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36-72}}{2} = \frac{6 \pm 6i}{2}$$

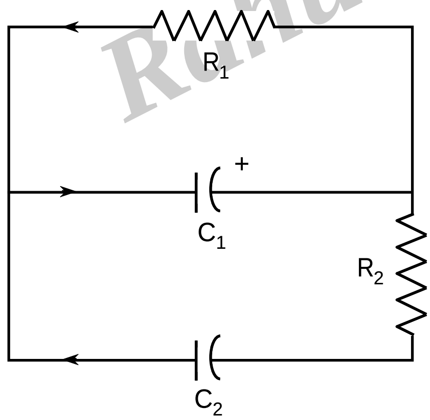
$$= \frac{2(3 \pm 3i)}{2}$$

$$\lambda = 3 \pm 3i$$

$\therefore$ The complex eigen values are $3 \pm 3i$.

3.3 Applications to Differential Equations

Q27. Find formulas for the voltages v_1 and v_2 (as functions of time t) for the circuit shown below, assuming that $R_1 = \frac{1}{5}$ Ohm, $R_2 = \frac{1}{3}$ ohm, $C_1 = 4$ farads, $C_2 = 3$ farads and the Initial charge on each capacitor is 4 volts.



Sol :

Given $R_1 = \frac{1}{5}$ ohm, $R_2 = \frac{1}{3}$ ohm, $C_1 = 4$ farads, $C_2 = 3$ farads and $x(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

$$\text{Since} \quad A = \begin{bmatrix} \dfrac{-\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)}{C_1} & \dfrac{1}{(R_2 C_1)} \\ \dfrac{1}{(R_2 C_2)} & \dfrac{-1}{(R_2 C_2)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-\left(\dfrac{1}{\frac{1}{5}} + \dfrac{1}{\frac{1}{3}}\right)}{4} & \dfrac{1}{\left(\frac{1}{3}(4)\right)} \\ \dfrac{-1}{\left(\frac{1}{3}(3)\right)} & \dfrac{-1}{\left(\frac{1}{3}(3)\right)} \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{(5+3)}{4} & \dfrac{3}{4} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & \dfrac{3}{4} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0.75 \\ 1 & -1 \end{bmatrix}$$

The characteristic equation is given by , det $(A - \lambda I) = 0$.

$$\Rightarrow \quad \left| \begin{bmatrix} -2 & 0.75 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \quad \left| \begin{bmatrix} -2-\lambda & 0.75 \\ 1 & -1-\lambda \end{bmatrix} \right| = 0$$

$\Rightarrow \quad (-2-\lambda)(-1-\lambda) - 0.75 = 0$

$\Rightarrow \quad 2 + 2\lambda + \lambda + \lambda^2 - 0.75 = 0$

$\Rightarrow \quad \lambda^2 + 3\lambda + 1.25 = 0$

It is in the quadratic from $ax^2 + bx + c = 0$

$$\lambda = \frac{-3 \pm \sqrt{3^2 - 4(1.25)}}{2(1)}$$

$$= \frac{-3 \pm 2}{2}$$

$$\lambda = -0.5, -2.5$$

$\therefore$ Eigen values are $\lambda_1 = -0.5$; $\lambda_2 = -2.5$

If $\lambda_1 = -0.5$

Consider $A + (0.5)\,I = \begin{bmatrix} -2 & 0.75 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} -1.5 & 0.75 \\ 1 & -0.5 \end{bmatrix}$

The equations $(A + (0.5)\,I) \times = 0$ gives

$x_1 - 0.5\,x_2 = 0$

$\Rightarrow \quad x_1 = 0.5\,x_2$

And x_2 is free variable,

$\therefore$ The general solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore \quad v = x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = -0.5$ is $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

If $\lambda_2 = -2.5$

Consider, $A + (2.5)\,I = \begin{bmatrix} -2 & 0.75 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.75 \\ 1 & 1.5 \end{bmatrix}$

The equation $(A + (2.5)I) \times = 0$ gives

$x_1 + (1.5)\,x_2 = 0$

$\Rightarrow \quad x_1 = -1.5\,x_2$

$\Rightarrow \quad \frac{-3}{2}\,x_2$

And x_2 is free variable.

$\therefore$ The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$\therefore \; v_2 = x_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = -2.5$ is $v_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

The general solution is,

$x(t) = c_1 v_1 e^{\lambda} t + c_2 v_2 e^{\bar{\lambda}} 2^t$

$x(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.51} + c_2 \begin{bmatrix} -3 \\ e \end{bmatrix} e^{-2.5t}$

where c_1, c_2 are complex numbers

The constants c_1, c_2 satisfy the initial condition $x(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ is,

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Consider augmented matrix,

$$[v_1 \ v_2 \ x(0)] = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 2 & 4 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1 = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 8 & -4 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{8} \Rightarrow \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

$$R_1 \to R_1 + 3R_2 = \begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & \frac{-1}{2} \end{bmatrix}$$

$$\therefore c_1 = \frac{5}{2} ; c_2 = -\frac{1}{2}$$

$$\therefore \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = x(t) = \frac{5}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.5t} - \frac{1}{2} \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{-2-5t}$$

Q28. Construct the general solution of X' = AX involving complex eigen functions and then obtain the general real solution. Describe the shape of typical trajectories.

(i) $A = \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$

Sol :

Given matrix is, $A = \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix}$

The characteristic equation is given by, $\det(A - \lambda I) = 0$

i.e., $\left| \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$

$$= \begin{vmatrix} -3-\lambda & -9 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(-3 - \lambda)(-3 - \lambda) + 18 = 0$

$\Rightarrow$ $\lambda^2 + 9 = 0$

$\Rightarrow$ $\lambda = \pm 3i$

$\therefore$ Eigen value is $\lambda = 3i, -3i$

Consider

$A - (3i)I = \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 3i & 0 \\ 0 & 3i \end{bmatrix} = \begin{bmatrix} -3-3i & -9 \\ 2 & 3-3i \end{bmatrix}$

The equation $(A - (3i)I)x = 0$ gives,

$2x_1 + (3 - 3i)x_2 = 0$

$\Rightarrow$ $x_1 = -\frac{(3-3i)}{2} x_2$

And x_2 is free variable

$\therefore$ The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{(3-3i)x_2}{2} \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-3(3-3i)}{2} \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} -(3-3i) \\ 2 \end{bmatrix}$$

$\therefore$ The eigen vector $v = \begin{bmatrix} -3+3i \\ 2 \end{bmatrix}$

The complex functions are $ve^{\lambda t}$ and $\bar{v}e^{\bar{\lambda}t}$

$\therefore$ The general complex solution of $x' = Ax$ is,

$x(t) = c_1 ve^{\lambda t} + c_2 \bar{v} e^{\bar{\lambda} t}$

$\Rightarrow$ $x(t) = c_1 \begin{bmatrix} -3+3i \\ 2 \end{bmatrix} e^{3it} + c_2 \begin{bmatrix} -3-3i \\ 2 \end{bmatrix} e^{(-3i)t}$

where c_1, c_2 are complex numbers

let

$$ve^{(3i)t} = \begin{bmatrix} -3+3i \\ 2 \end{bmatrix} (\cos 3t + i \sin 3t)$$

$$= \begin{bmatrix} -3\cos 3t - 3i\sin 3t + 3i\cos 3t - 3\sin 3t \\ 2\cos 3t + 2i\sin 3t \end{bmatrix}$$

$$ve^{(3i)t} = \begin{bmatrix} -3\cos 3t - 3\sin 3t \\ 2\cos 3t \end{bmatrix} + i\begin{bmatrix} -3\sin 3t - 3\cos 3t \\ 2\sin 3t \end{bmatrix}$$

$\therefore$ The general real solution is,

$$x(t) = c_1\begin{bmatrix} -3\cos 3t - 3\sin 3t \\ 2\cos 3t \end{bmatrix} + c_2\begin{bmatrix} -3\sin 3t - 3\cos 3t \\ 2\sin 3t \end{bmatrix}$$

where c_1, c_2 are real numbers.

Since real parts of the eigen values are zero. The trajectories are ellipses about the origin.

(ii) Given matrix is, $A = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$

The characteristic equation is given by, det $(A - \lambda I) = 0$

$$\text{i.e., } \left|\begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\Rightarrow \left|\begin{bmatrix} 4-\lambda & -3 \\ 6 & -2-\lambda \end{bmatrix}\right| \Rightarrow (4-\lambda)(-2-\lambda) + 18 = 0$$

$$\Rightarrow -8 - 4\lambda + 2\lambda + \lambda^2 + 18 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 10 = 0$$

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$$

$\therefore$ Eigen value is, $\lambda = 1 + 3i$

Consider,

$$A - (1+3i)I = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} - \begin{bmatrix} (1+3i) & 0 \\ 0 & 1+3i \end{bmatrix} = \begin{bmatrix} 3-3i & -3 \\ 6 & -3-3i \end{bmatrix}$$

The equation $(A - (1+3i)I)x = 0$ gives,

$$6x_1 + (-3-3i)x_2 = 0$$

$$\Rightarrow 6x_1 - (3+3i)x_2 = 0$$

$$\Rightarrow x_1 = \frac{1+i}{2}x_2$$

Ans x_2 is free variable.

$\therefore$ The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{(1+i)}{2} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

$$v = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

$\therefore$ Eigen vector $v = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

The complex functions are $ve^{\lambda t}$ and $\bar{v}e^{\bar{\lambda} t}$

The general complex solution of $x' = Ax$ is,

$$x(t) = c_1 v e^{\lambda t} + c_2 \bar{v} e^{\bar{\lambda} t}$$

$$x(t) = c_1 \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{(1+3i)t} + \begin{bmatrix} 1-i \\ 2 \end{bmatrix} e^{(1-3i)t}$$

where c_1, c_2 are complex numbers,

let $ve^{(1+3i)t} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{t+3it} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^t . e^{3it}$

$$= \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^t (\cos 3t + i \sin 3t)$$

$$= \begin{bmatrix} \cos 3t + i \sin 3t + i(\sin 3t + \sin 3t) \\ 2\cos 3t + i\,(2\sin 3t) \end{bmatrix} e^t$$

$$= \begin{bmatrix} \cos 3t + \sin 3t + i(\sin 3t + \cos 3t) \\ 2\cos 3t + i\,(2\sin 3t) \end{bmatrix} e^t$$

$$= \begin{bmatrix} \cos 3t + \sin 3t \\ 2\cos 3t \end{bmatrix} e^t + i \begin{bmatrix} \sin 3t + \cos 3t \\ 2\sin 3t \end{bmatrix} e^t$$

$\therefore$ The general real solution is,

$$x(t) = c_1 \begin{bmatrix} \cos 3t - \sin 3t \\ 2\cos 3t \end{bmatrix} e^t + c_2 \begin{bmatrix} \sin 3t + \cos 3t \\ 2\sin 3t \end{bmatrix} e^t$$

Where c_1, c_2 are real numbers.

Since real parts of the eigen values are positive.

The trajectories spiral out away from the origin.

Q29. Make a change of variable that decouples the equation X' = AX write the equation X(t) = Py(t) and show the calculate that leads tot he uncoupled system Y' = DY, specifying P and D where A = $\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$.

Sol :

Given matrix is, $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

The characteristic equation is given by, det $(A - \lambda I) = 0$.

i.e., $\left| \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0 \qquad \Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} = 0$

$\Rightarrow (1 - \lambda)(-4 - \lambda) + 6 = 0 \qquad \Rightarrow \lambda^2 + 3\lambda + 2 = 0 \qquad \Rightarrow \lambda = -1, -2$

$\therefore$ The eigen values are –1, –2.

If $\lambda = -2$

Consider, $A + 2I = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}$

The augmented matrix $[(A + 2I)\ 0]$ is $\begin{bmatrix} 3 & -2 & 0 \\ 3 & -2 & 0 \end{bmatrix}$

$R_2 \to R_2 - R_1 = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$R_1 \to \frac{R_1}{3} = \begin{bmatrix} 1 & \frac{-2}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ The equation is,

$x_1 - \frac{2}{3} x_2 = 0$

$x_1 = \frac{2}{3} x_2$

and x_2 is a free variable

$\therefore$ The general solution is,

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\therefore v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

If $\lambda = -1$,

Consider,

$$A + I = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$$

The augmented matrix [(A + I) 0] is $\begin{bmatrix} 2 & -2 & 0 \\ 3 & -3 & 0 \end{bmatrix}$

$$R_2 \to \frac{R_1}{2}, R_2 \to \frac{R_2}{-3} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The equation is,

$x_1 - x_2 = 0$

$\therefore \quad x_1 = x_2$

and x_2 is a free variable

$\therefore$ The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Initial condition $x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Let the constants c_1, c_2 satisfy x(0) such that $c_1v_1 + c_2v_2 = x(0)$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Consider the augmented matrix,

$$[v_1\ v_2\ x(0)] = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$R_2 \to 2R_2 - 3R_1 = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -5 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{2}, R_2 \to \frac{R_2}{-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 5 \end{bmatrix}$$

$$R_1 \to 2R_1 - \frac{1}{2}R_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \end{bmatrix}$$

$\therefore c_1 = -1; c_2 = 5$

The general solution of $x^1 = Ax$ is

$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

$\Rightarrow$ $x(t) = -1\begin{bmatrix}2\\3\end{bmatrix} e^{-2t} + 5\begin{bmatrix}1\\1\end{bmatrix} e^{-t}$.

Since both eigen values of matrix A are negative.

$\therefore$ The origin is an attractor of the dynamical system described by $x^1 = Ax$.

The direction of greatest attraction is the line through v_1 and the origin.

To decouple the equation $x^1 = Ax$

Let $P = [v_1 \ v_2]$

$$P = \begin{bmatrix}2 & 1\\3 & 1\end{bmatrix}$$

and $D = \begin{bmatrix}-2 & 0\\0 & -1\end{bmatrix}$

Given $x^1 = AX$ ... (1)

Since $A = PDP^{-1}$

$\Rightarrow D = PAP^{-1}$.

Substituting, $x(t) = py(t)$ in equation (1)

i.e., $\frac{d}{dt}(py) = A(py)$

$= PDP^{-1}(Py) = PD(P^{-1}P)y = PDy$

$\therefore$ P has constant entries.

$$\frac{d}{dt}(py) = PDy$$

$\Rightarrow P\left(\frac{d}{dt}(y)\right) = PDy$

$\Rightarrow P^{-1}P\left(\frac{d}{dt}(y)\right) = P^{-1}PDy$

$\Rightarrow y' = Dy$

i.e., $\begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix} = \begin{bmatrix}-2 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix}$

Q30. Suppose that a particle is moving in a planar force held and its position vector x satisfies $X^1 = AX$ and $X(0) = X_0$ where $A = \begin{bmatrix}4 & -5\\-2 & 1\end{bmatrix}$, $x_0 = \begin{bmatrix}2.9\\2.6\end{bmatrix}$ solve this initial value problem for $t \geq 0$ and sketch the trajectory of the particle.

Sol :

The given matrix is $A = \begin{bmatrix}4 & -5\\-2 & 1\end{bmatrix}$

Let λ be any scalar, I is a unit matrix.

The characteristic equation $|A - \lambda I| = 0$

$$\begin{bmatrix} 4-\lambda & -5 \\ -2 & 1-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)(1-\lambda) - 10 = 0$$

$$\lambda^2 - 5\lambda - 6 = 0$$

$$\lambda_1 = 6;\ \lambda_2 = -1$$

Let $V = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the corresponding eigen vectors of A.

Then $(A - \lambda I)V = 0$

$$\begin{bmatrix} 4-\lambda & -5 \\ -2 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

when $\lambda = 6 = \lambda_1$

$$\begin{bmatrix} -2 & -5 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 5x_2 = 0$$

$$2x_1 + 5x_2 = 0$$

Choosing $x_2 = k$; $2x_1 = -5k$

$$x_1 = -\frac{5}{2}k \text{ (k is a scalar)}$$

$$\therefore\ X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2}k \\ k \end{bmatrix} = k\begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix}$$

Q31. A particle moving in a planar force field has a position vector X that satisfies $X^1 = AX$. The 2×2 matrix A has eigen value 4 and 2 with corresponding eigen vectors $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find the position of the particle at time t, assuming that $X(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$.

Sol :

Given, A is a 2×2 matrix

Eigen values are 4 and 2

Eigen vectors $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The initial condition $x(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

The eigen functions for the differential equation $X^1 = Ax$ are $v_1e^{\lambda_1 t}$ and $v_2e^{\lambda_2 t}$

i.e., v_1e^{4t}, v_2e^{2t}

The general solution of x' = Ax has the form

$$x(t) = c_1v_1e^{t} + c_2v_2e^{2t}$$

$$x(t) = c_1\begin{bmatrix}-3\\1\end{bmatrix}e^{4t} + c_2\begin{bmatrix}-1\\1\end{bmatrix}e^{2t} \quad \ldots (1)$$

Let the constants c_1, c_2 satisfy the initial condition $x(0) = \begin{bmatrix}-6\\1\end{bmatrix}$

i.e., $c_1\begin{bmatrix}-3\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-6\\1\end{bmatrix}$

$\Rightarrow -3c_1 - c_2 = -6$

$c_2 + c_2 = 1$

The augmented matrix is $[v_1, v_2, x(0)]$

$$= \begin{bmatrix}-3 & -1 & -6\\1 & 1 & 1\end{bmatrix}$$

$$R_2 \to 3R_2 + R_1 \Rightarrow \begin{bmatrix}-3 & -1 & -6\\0 & 2 & -3\end{bmatrix}$$

$$R_1 \to \frac{R_1}{-3} \text{ and } R_2 \to \frac{R_2}{2} = \begin{bmatrix}1 & \frac{1}{3} & 2\\0 & 1 & \frac{-3}{2}\end{bmatrix}$$

$$R_1 \to R_1 - \frac{1}{3}R_2 = \begin{bmatrix}1 & 0 & \frac{5}{2}\\0 & 1 & \frac{-3}{2}\end{bmatrix}$$

$c_1 = \frac{5}{2}$; $c_2 = \frac{-3}{2}$

Substituting the corresponding values in equation (1)

$$\therefore x(t) = \frac{5}{2}\begin{bmatrix}-3\\1\end{bmatrix}e^{4t} - \frac{3}{2}\begin{bmatrix}-1\\1\end{bmatrix}e^{2t}.$$

Choose the Correct Answers

1. The matrix A to be diagonalizable is [a]

   (a) $A = PDP^{-1}$ (b) $AP = PD$

   (c) $A = PD^2P^2$ (d) None

2. If $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$ then $A^8 =$ [c]

   (a) $\begin{bmatrix} 2^8 & 0 \\ 1 & 1^8 \end{bmatrix}$ (b) $\begin{bmatrix} 256 & 0 \\ 0 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}$ (d) $\begin{bmatrix} 2^8 & 4^8 \\ -3^8 & -1^8 \end{bmatrix}$

3. The matrix $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ is [b]

   (a) diagonalizable (b) not diagonalizable

   (c) linear independent (d) none

4. The eigen vector for $A = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$ corresponding to eigen value $\lambda = 4 + 3$. [a]

   (a) $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -i \\ -i \end{bmatrix}$

   (c) $\begin{bmatrix} i \\ i \end{bmatrix}$ (d) None

5. The complex eigen values of then matrix $A = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$ is [c]

   (a) 3 + 3i (b) 3 – 3i

   (c) 3 ± 3i (d) None

6. If A is both diagonalizable and invertible then A^{-1} is [a]

   (a) diagonalizable (b) invertible

   (c) both (d) none

7. If the eigen value $\lambda = a + bi$ then $\bar{\lambda} =$ [c]

(a) bi (b) b – ai

(c) 0 – bi (d) –a

8. The eigen values for the matrix A = $\begin{bmatrix} 0-1 & 0-1 \\ -0-1 & 0-1 \end{bmatrix}$. [b]

(a) a + bi (b) a $\pm$ bi

(c) a – bi (d) None

9. If $T(b_1) = 3c_1 - 2c_2 + 5c_3$ and $T(b_2) = 4c_1 + 7c_2 - c_3$ then the matrix M for T relative to B and C is [b]

(a) $\begin{bmatrix} 3 & -2 \\ 7 & 5 \\ 5 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 5 \\ 4 & 7 \\ -2 & 3 \end{bmatrix}$

10. If $T(a_0 + a_1t + a_2t^2) = a_1 + 2a_2t$ then T(1) = [a]

(a) 0 (b) 1

(c) 2t (d) –1

Fill in the Blanks

1. An n × n matrix with A district eigen values is ________.
2. If the matrix A is diagonalizable then A is ________.
3. Ab n × n matrix is diagonalizable if and only if A has ________ eigen vectors.
4. A is diagonalizable if and only if there are enough eigen vectors to form a ________ if R^T.
5. A square matrix a is diagonalizable if A is similar to ________ matrix.
6. A square matrix A of order n × n is diagonalizable if there are A district ________ of A.
7. ν is a finite dimensional vector space of dimension 'n'. and T : v → v is a linear transformation. B is an ordered basis for ν. Then $[T(x)]_B$ = ________ $\forall\ x \in$ v.
8. If the origin is an attractor then the solution of the system is ________.
9. The parametric equations of the solution of given system represents a curve known as ________.
10. A ________ arise when the matrix A has both positive and negative eigen values.

ANSWERS

1. diagonalizable
2. invertible
3. n linearly independent
4. basis
5. diagonal
6. eigen values
7. $[T]_B\ [x]_B$
8. stable
9. trajectory
10. saddle point

UNIT IV

Orthogonality and Least Squares : Inner Product, Length, and Orthogonality - Orthogonal Sets - Orthogonal Projections - The Gram - Schmidt Process.

4.1 Orthogonality and Least Squares

4.1.1 Inner product length and Orthogonality

Q1. Define inner product.

Sol : **(Dec.-18)**

Let R^n be the vector space of all ordered n-tuples of real numbers. Let u and v any two vectors in R^n then the product u^Tv is called as the inner product of u and v.

Inner product of u and v is denoted by u.v also called as "dot product".

The Inner product space of u and v is given by

$$u.v = [u_1\ u_2\ ....u_n]_{1\times n} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{n\times 1}$$

$$= [u_1v_1 + u_2v_2 + ...u_nv_n]$$

Q2. Write the properties of Inner product.

Sol : **(Dec.-18)**

Properties of Inner Product

If u, v and w are vectors in R^n and 'C' be any scalar then,

i) u.v = v.u

ii) (u + v).w = u.w + v.w

iii) (cu).v = c(u.v) = u.(cv)

iv) $u.u \geq 0$ and u.u = 0 if and only if u = 0

Q3. Define Norm (or) length of a vector.

Sol :

Let R^n be the vector space of all ordered n-tuples of real numbers.

let $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ be any vector in R^n

Then the positive square root of the inner product u.u is called as the length of the vector u and is denoted by $||u||$

$$||u|| = \sqrt{u.u} = \sqrt{u_1^2 + u_2^2 + ... + u_n^2}$$

$$||u||^2 = u.u$$

The norm of a vector is also defined as the length of the point from the origin.

Q4. Find the norm of the vector (1, –2, 2, 0).

Sol : **(July-19)**

Given vector is,

$$u = (1, -2, 2, 0) = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

The Norm (length) of vector is given by,

$||u|| = \sqrt{1^2 + (-2)^2 + 2^2 + 0}$

$= 3$

$\therefore\ ||u|| = 3$

Q5. Define Normalizing a vector.

Sol :

Let v be any non-zero vector in a vector space

If $||v|| = 1$ then v is a unit vector.

If v is not a unit vector then by dividing v with its norm

$\therefore$ unit vector $u = \frac{v}{||v||}$

This processing of converting a non-zero vector into a unit vector is called as normalization.

Q6. If V = [1, –2, 2, 0] then find a unit vector u in the same direction as v.

Sol :

The given vector space is R^4

The given non-zero vector is

$v = [1, -2, 2, 0]$

Consider,

$$||v||^2 = v.v = [1, -2, 2, 0]\begin{bmatrix}1\\-2\\2\\0\end{bmatrix}$$

$= 1^2 + (-2)^2 + 2^2 + 0^2$

$||v||^2 = 9$

$||v|| = 3 \neq 1$

$\therefore$ v is not a unit vector

Dividing v with $||v||$ we get a unit vector say,

$$u = \frac{v}{||v||} = \frac{[1, -2, 2, 0]}{3}$$

$$u = \begin{bmatrix}1/3\\-2/3\\2/3\\0\end{bmatrix}$$

Consideration,

$$||u||^2 = u.u = \left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = \frac{9}{9} = 1$$

$\therefore$ u is a unit vector

Q7. Let w be the subspace of R^2 spanned by $x = \left(\frac{2}{3}, 1\right)$. Find a unit vector z that is a basis is for w.

Sol :

The Given vector space is R^2

$w = \text{span } x = \text{span}\left[\frac{2}{3}, 1\right]$, is a subspace of R^2

w can forms all scalar multiples of x

Here $x = \begin{bmatrix}2/3\\1\end{bmatrix}$

consider $y = \begin{bmatrix}2\\3\end{bmatrix}$

consider $||y||^2 = 2^2 + 3^2 = 13$

$||y|| = \sqrt{13}$

Normalizing 'y' we get another vector

i.e., say $z = \frac{y}{||y||} = \frac{1}{\sqrt{13}}\begin{bmatrix}2\\3\end{bmatrix}$

$$z = \begin{bmatrix}2/\sqrt{13}\\3/\sqrt{13}\end{bmatrix}$$

Here z is a unit vector in R^2

Which is linearly independent and is a scalar multiple of x.

w = span [x] ; w = span[z]

As z is a basis for w.

Q8. Define unit Vector and Normalization.

Sol :

(i) Unit Vector : A vector whose length or norm is 1 is called as "unit vector".

If $||u|| = 1$ then u is called as a unit vector.

(ii) Normalization : If u is a non-zero vector then by dividing the vector 'u' with its norm (or) length then we can get the unit vector travelling in the same direction. This process of getting a unit vector is called as normalization.

$\frac{u}{||u||}$ is the unit vector

Q9. If $u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$; $v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ are any two vectors in R^2 then find

(a) u.u **(b) v.u** **(c) $\frac{v.u}{u.u}$** **(d) $\frac{u}{u.u}$** **(e) $||u||$**

(f) $||v||$

Sol :

The given vector space is R^2

let $u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ be the given vectors

(a) Consider $u.u. = u^T u = [-1, 2]\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$u.u = 1 + 4 = 5$

(b) Consider $v.u = v^T u = [4\ 6]\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 12 = 8$

(c) Consider $\frac{v.u}{u.u} = \frac{8}{5}$

(d) Consider $\frac{u}{u.u} = \frac{1}{5}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$

(e) Consider $||u| = \sqrt{u_1^2 + u_2^2} = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

(f) Consider $||v| = \sqrt{v_1^2 + v_2^2} = \sqrt{4^2 + 6^2} = \sqrt{52}$

Q10. Find a unit vector in the direction of $v = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

Sol : **(July-21)**

Given vector is, $v = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

The unit vector u in the direction v is given as,

$$u = \frac{1}{\|v\|}.v$$

$$||v|| = \sqrt{v.v} = \sqrt{(-6)^2 + (4)^2 + (-3)^2} = \sqrt{61}$$

$$u = \frac{1}{\sqrt{61}}\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$$

$$\text{unit vector } u = \begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$$

Q11. Define distance between two points.

Sol :

Consider the vector space R^n

let u and v be any two vectors in R^n

Then the distance between u and v is defined as the norm (or) length of the vector u – v and is denoted by distance (u, v)

$\therefore$ Distance (u, v) = $||u - v||$

when $u = (u_1, u_2, u_3)$; $v = (v_1, v_2, v_3)$ are any two vectors in R^3

Then distance (u, v) = $||u - v|| = \sqrt{(u - v)(u - v)}$

$$= \sqrt{(u_1 - v_1, u_2 - v_2, u_3 - v_3).(u_1 - v_1, u_2 - v_2, u_3 - v_3)}$$

$$\text{distance }(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

Q12. Find a unit vector in the direction of the vector $\overline{v}$ = (1, –2, 2, 0).

Sol :

Given vector is, $\overline{v} = (1, -2, 2, 0) = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$

The unit vector $\overline{u}$ in the direction of $\overline{v}$ is,

$$\overline{u} = \frac{1}{\|v\|}\overline{v}$$

$$= \frac{1}{\sqrt{(1)^2 + (-2)^2 + 2^2 + 0}} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

$\therefore$ unit vector u = $\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$

Q13. Find the distance between the vectors u = $\begin{bmatrix} 10 \\ -3 \end{bmatrix}$, v = $\begin{bmatrix} -1 \\ -5 \end{bmatrix}$.

Sol :

The given vector space is R^2

The distance between two vectors u and v is,

distance (u, v) = $\|u - v\|$

distance (u, v) = ||(10 + 18, (–3 + 5)||

= ||(11, 2)||

= $\sqrt{11^2 + 2^2}$ = $\sqrt{121 + 4}$

= $\sqrt{125}$

Q14. Find the distance between the vector u = $\begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and v = $\begin{bmatrix} -4 \\ -3 \\ 8 \end{bmatrix}$ in R^3.

Sol :

The given vector space is R^3

The distance between the vectors u and v is given by dis (u, v) = ||u – v||

dis (u, v) = ||(0 + 4, – 5 + 3, 2 – 8)||

= ||(4, – 2, –6)||

= $\sqrt{4^2 + (-2)^2 + (-6)^2}$ = $\sqrt{16 + 4 + 36}$ = $\sqrt{56}$

Q15. Find $[dist_e(u, -v)]^2$

Sol : **(Dec.-19)**

[dis t(u, –v)] = $||u - (-v)||^2$

= $||u + v||^2$ = (u + v).(u + v)

= u(u + v) + v(u + v)

= u.u + u.v + u.v + v.u + v.v

= $||u||^2 + ||v||^2$ + 2.u.v

$[dis\ t(u_1 - v)]^2 = ||u||^2 + ||v||^2 + 2.u.v$

Q16. Define orthogonality.

Sol :

Let R^n be the vector space consisting of all ordered n-tuples. let u, v be any two vectors in R^n.

(i) The two vectors u and v are said to be orthogonal if the distance between u, v and u, –v are same i.e., distance (u, v) = distance (u, –v).

(ii) The two vectors u and v are said to be orthogonal if u.v = 0

Q17. State and prove Pythagorean theorem.

Sol :

Statement : Two vectors u and v are orthogonal if and only if $||u + v||^2 = ||u||^2 + ||v||^2$

Proof :

Part (i) : Let u and v be two orthogonal vectors

i.e., $u.v = 0$

$||u + v||^2 = (u + v).(u + v)$

$= u(u + v) + v(u + v) = u.u + u.v + v.u + v.v$

$= ||u||^2 + u.v + v.u + ||v||^2$

$||u + v||^2 = ||u||^2 + ||v||^2$ $[\because u.v = 0]$

Part (ii) : Let $||u + v||^2 = ||u||^2 + ||v||^2$

$||u||^2 + ||v||^2 + 2.u.v = ||u||^2 + ||v||^2$

$2(u.v) = 0$

$u.v = 0$

$\Rightarrow$ u and v are orthogonal.

Q18. State and prove parallelogram law.

Sol :

Statement: If u and v are any two vectors in R^n then

$$||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2$$

Proof :

Let R^n be the given vector space

let $u, v \in R^n \Rightarrow u + v, u - v \in R^n$

By the definition of norm of a vector we have

$||u||^2 = u.u$; $||v||^2 = v.v$

$||u + v||^2 = (u + v).(u + v)$

$= u(u + v) + v(u + v)$

$= u.u + u.v + v.u + v.v$

$||u + v||^2 = u^2 + 2(u.v) + v^2$ ....(1)

lly

$||u - v||^2 = u^2 - 2(u.v) + v^2$ ....(2)

Add equations (1) & (2)

$||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2$

Q19. Define orthogonal complements.

Sol :

Let R^n be the vector space of all ordered n-tuples

let w be a subspace of R^n

The vector z is orthogonal to every vector in w then z is said to be orthogonal to w.

The set consisting of all the vectors in z which are orthogonal to w is called as "orthogonal complement of w" and is denoted by w.

Q20. In the vector space R^2 verify whether the vectors $a = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$, $b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ are orthogonal (or) not?

Sol :

The given vectors in R^2 are

$$a = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$a.b = [a]^T b = [8\ -5]\begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$= -16 + 15 = -1 \neq 0$$

$\therefore$ a & b are not orthogonal

Q21. In a vector space R^4 verify whether the vectors

$$u = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, v = \begin{bmatrix} -4 \\ +1 \\ -2 \\ 6 \end{bmatrix}$$ **are orthogonal or not?**

Sol :

$$\text{Consider } u.v = u^T v = [3\ 2\ -5\ 0]\begin{bmatrix} -4 \\ +1 \\ -2 \\ 6 \end{bmatrix}$$

$u.v = -12 + 2 + 10 + 0 = 0$

u and v are orthogonal

Q22. Determine which pairs of vectors are orthogonal

(i) $u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

(ii) $u = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

Sol :

(i) Given vectors are,

$$u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

Consider,

$$u.v = u^T v$$

$$= [12\ 3\ -5]\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$= 12(2) + 3(-3) - 5(3)$$

$$= 24 - 9 - 15$$

$$= 24 - 24$$

$u.v = 0$

$\therefore$ u and v are orthogonal

(ii) Given vectors are,

$$u = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$$

Consider,

$$u.v = u^T v$$

$$= [-3\ 7\ 4\ 0]\begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$$

$= -3(1) + 7(-8) + 4(15) + 0(-7)$

$= -3 - 56 + 60 + 0$

$u.v = 1 \neq 0$

$\therefore$ u and v are not orthogonal

Q23. Let u = $\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$, v = $\begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$ compute

u.v, ||u||², ||v||², ||u + v||² & ||u + v||

Sol : **(July-21)**

Given vectors are,

$$u = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}, v = \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$$

Consider,

$$u.v = u^T v = [2 \ -5 \ -1]\begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$$

$= 2(-7) - 5(-4) - 1(6)$

$= -14 + 20 - 6$

$\Rightarrow$ $u.v = 0$

$||u||^2 = 2^2 + (-5)^2 + (-1)^2$

$= 4 + 25 + 1$

$||u||^2 = 30$

$||v||^2 = 30$

$||v||^2 = (-7)^2 + (-4)^2 + 6^2$

$||v||^2 = 0$

Consider,

$$u + v = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} + \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$$

$$u + v = \begin{bmatrix} 2-7 \\ -5-4 \\ -1+6 \end{bmatrix}$$

$$u + v = \begin{bmatrix} -5 \\ -9 \\ 5 \end{bmatrix}$$

Then, $||u + v|| = \sqrt{(-5)^2 + (-9)^2 + 5^2} = \sqrt{131}$

$||u + v||^2 = (-5)^2 + (-9)^2 + 5^2$

$= 25 + 81 + 25$

$||u + v||^2 = 131 = 30 + 101$

$||u + v||^2 = ||u||^2 + ||v||^2$

$\therefore \ ||u + v||^2 = ||u||^2 + ||v||^2$

Q24. Suppose that a vector y is orthogonal to the vectors u and v. Show that y is orthogonal to u + v.

Sol :

In a vector space v let us consider three vectors u, v & y.

Then u + v is also a vector in the same vector space.

Given that y is orthogonal to u

$\Rightarrow$ $y.u = 0$

y is orthogonal to v

$\Rightarrow$ $y.v = 0$

consider $y(u + v) = y.u + y.v = 0 + 0 = 0$

$\therefore$ y is orthogonal to u + v.

Q25. Suppose that y is orthogonal to u and v. Show that y orthogonal to every w in span[u, v].

Sol :

Given that y is orthogonal to u and v.

$\Rightarrow$ $y.u = 0$ and $y.v = 0$

To show that y is orthogonal to every w in span[u, v]

let $w \in$ span [u, v]

$\Rightarrow$ $\exists$ scalars c_1, c_2 such that

$w = c_1 u + c_2 v$

Consider,

$$y.w = y.[c_1u + c_2v]$$
$$= y.(c_1u) + y(c_2v)$$
$$= c_1(y.u) + c_2(y.v)$$
$$= c_1.0 + c_2.0$$
$$y.w = 0$$

$\therefore$ y is orthogonal to w

Since w is orbitary, y is orthogonal to every $w \in$ span[u,v]

Q26. Let $u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, w = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix},$

$x = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$ **Compute**

(i) u.u, v. u and $\frac{v.u}{u.u}$ **(ii)** $\frac{1}{u.u}u$

(iii) $\left(\frac{x.w}{x.x}\right)x$ **(iv) ||w||**

Sol :

Given vectors are,

$$u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, w = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, x = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

(i) $u.u = ||u||^2 = (-1)^2 + 2^2 = 1 + 4 = 5$

$$v.u = v^Tu = [4\ 6]\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
$$= 4(-1) + 6(2)$$
$$= -4 + 12$$
$$= 8$$

and $\frac{v.u}{u.u} = \frac{8}{5}$

(ii) $\frac{1}{u.u}u = \frac{1}{5}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$$

(iii) $x.w = x^Tw = [6\ -2\ 3]\begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$

$$= 6(3) - 2(-1) + 3(-5)$$
$$= 18 + 2 - 15 = 5$$
$$x.x = ||x||^2 = (6)^2 + (-2)^2 + 3^2$$
$$= 36 + 4 + 9$$
$$x.x = 49$$

Then,

$$\left(\frac{x.w}{x.x}\right)x = \frac{5}{49}\begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}$$

(iv) $||w|| = \sqrt{(3)^2 + (-1)^2 + (-5)^2}$

$$= \sqrt{9 + 1 + 25}$$
$$||w|| = \sqrt{35}$$

4.2 Orthogonal Sets

Q27. Define orthogonal sets.

Sol : **(Dec.-19)**

A set of vectors $\{u_1, u_2, \ldots u_p\}$ in R^n is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal i.e., if $u_i, u_j = 0, \forall i \neq j$

Q28. Show that $\{u_1, u_2, u_3\}$ is an orthogonal set where

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix};\ u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix};\ u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

Sol : **(Dec.-18)**

The given vector space is R^3

The given set is $\{u_1, u_2, u_3\}$

Consider the three possible pairs of distinct vectors,

namely $\{u_1, u_2\}$, $\{u_1, u_3\}$ & $\{u_2, u_3\}$

Consider $u_1.u_2 = u^T, u_2 = [3\ 1\ 1]\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -3 + 2 + 1 = 0$

$\therefore$ u_1 is orthogonal to u_2

consider $u_2.u_3 = u_1^T, u_3 = [3\ 1\ 1]\begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = 3\left(\frac{-1}{2}\right) + 1\,(-2) + 1\left(\frac{7}{2}\right)$

$$= \frac{-3}{2} - 2 + \frac{7}{2}$$

$$= \frac{-3-4+7}{2} = 0$$

$\therefore$ u_1 is orthogonal to u_3

consider $u_2.u_3 = u_2^T u_2 = =-1\ 2\ 1]\begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = (-1)\left(\frac{-1}{2}\right) + 2\,(-2) + 1\left(\frac{7}{2}\right)$

$$= \frac{1}{2} - 4 + \frac{7}{2}$$

$$= \frac{1-8+7}{2} = 0$$

$\therefore$ u_2 is orthogonal to u_3

$\therefore$ each pair of vector is orthogonal

$\therefore$ $\{u_1, u_2, u_3\}$ is an orthogonal set.

Q29. Verify the set of vectors are orthogonal.

(a) $\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$

Sol : **(July-21, July-19)**

(a) The given vector space is R^3

let the given set is $\{u_1, u_2, u_3\}$

consider three possible pairs of distinct vectors,

namely $\{u_1, u_2\}$, $\{u_1, u_3\}$ and $\{u_2, u_3\}$

where $u_1 = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}; u_2 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}; u_3 = \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$

consider $u_1.u_2 = u_1^T , u_2 = (-1\ 4\ -3)\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$

$= (-1)5 + 4(2) + (-3)1$

$= -5 + 8 - 3$

$= -8 + 8$

$= 0$

$\therefore$ u_1 is orthogonal to u_2

consider $u_1.u_3 = u_1^T , u_3 = (-1\ 4\ -3)\begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$

$= (-1)3 + 4(-4) + (-3)(-7)$

$= -3 - 16 + 21$

$= 2 \neq 0$

consider $u_2.u_3 = u_2^T , u_3 = (5\ \ 2\ \ 1)\begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$

$= 5(3) + 2(-4) + 1(-7)$

$= 15 - 8 - 7$

$= 0$

Since $u_1.u_2 = u_2.u_3 \neq u_1.u_3$

The set $\{u_1, u_2, u_3\}$ is not an orthogonal set.

(b) Given set of vectors are,

$$u = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}; v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}; w = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

consider $u.v = u^T v = [1 \ \ -2 \ \ 1]\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$= 1(0) + (-2)1 + (1)2$

$= 0 - 2 + 2$

$u.v = 0$

consider $u.w = u^T w = [1 \ \ -2 \ \ 1]\begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$

$= 1(-5) + (-2)(-2) + (1)$

$= -5 + 4 + 1$

$u.w = 0$

consider $v.w = v^T w = [0 \ \ 1 \ \ 2]\begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$

$= 0(-5) + 1(-2) + 2(1)$

$= 0 - 2 + 2$

$v.w = 0$

each pair of distinct vectors is orthogonal

$\therefore$ The given set is orthogonal.

Q30. Show that $\{u_1, u_2, u_3\}$ is an orthogonal set where

$$u_1 = \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}; u_2 = \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}; u_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

Sol :

The given vector space is R^3

The given set is $\{u_1, u_2, u_3\}$

consider the three possible pairs of distinct vectors,

namely $\{u_1, u_2\}$, $\{u_1, u_3\}$ and $\{u_2, u_3\}$

consider $u_1.u_2 = u_1^T u_2 = [2 \ -7 \ -1]\begin{bmatrix}-6\\-3\\9\end{bmatrix}$

$= 2(-6) + (-7)(-3) + (-1)9$

$= -12 + 21 - 9$

$u_1.u_2 = -21 + 21 = 0$

consider $u_1.u_3 = u_1^T u_3 = [2 \ -7 \ -1]\begin{bmatrix}3\\1\\-1\end{bmatrix}$

$= 2(3) + (-7)1 + (-1)(-1)$

$= 6 - 7 + 1$

$u_1.u_3 = 7 - 7 = 0$

consider $u_2.u_3 = u_1^T u_3 = [-6 \ -3 \ 9]\begin{bmatrix}3\\1\\-1\end{bmatrix}$

$= -6(3) + (-3)1 + 9(-1)$

$= -18 - 3 - 9$

$u_2.u_3 = -30 \neq 0$

Since $u_1.u_2 = u_1.u_3 \neq u_2.u_3$

The set $\{u_1, u_2, u_3\}$ is not an orthogonal set.

Q31. If $s = \{u_1, u_2.....u_n\}$ is an orthogonal set of non-zero vectors in R^n then s is linearly independent.

Sol : **(July-19, Dec.-18)**

The given vector space is R^n

$s = \{u_1, u_2.....u_n\}$ is an orthogonal set consulting of non-zero vectors in R^n $(u_i \neq 0)$

i.e., $u_i \neq 0 \quad \forall\, i =$ to n

$u_i, u_j = 0 \quad \forall\, i \neq j$

To prove that $s = \{u_1, u_2,.....u_n\}$ is linearly independent

consider $c_1, c_2,....c_n$ be collection of any 'n' of scalars

consider the linear combination of these vectors and equate it zero.

$c_1u_1 + c_2u_2 + ... + c_nu_n = 0 \qquad ......(1)$

$\Rightarrow$ $(c_1u_1 + c_2u_2 + ... + c_nu_n)u_1 = 0.u_1$

$\Rightarrow$ $(c_1u_1).u_1 + (c_2u_2)u_1 + ... + (c_nu_n)u_1 = 0$

$\Rightarrow$ $c_1.(u_1.u_1) + c_2.(u_2.u_1) + ... + c_n(u_n.u_1) = 0$

$\Rightarrow$ $c_1(u_1.u_1) = 0$ $(\because u_1$ is non zero)

$\Rightarrow$ $c_1 = 0$

Ily, we can prove that $c_2 = c_3 = c_4.....c_n = 0$

from (1), The vectors $u_1, u_2,.......u_n$ are linearly independent

$\therefore$ S is linearly independent.

Thus, every orthogonal set consisting of non-zero vectors is always orthogonal.

Q32. Define orthogonal basis.

Sol :

Let w be a subspace of a vector space R^n. If s is an orthogonal set that spans w then s is called as an orthogonal basis for w.

Q33. Let $\{u_1, u_2.....u_n\}$ be an orthogonal basis for a subspace w of R^n. For each y in w weight on the linear combination $y = c_1u_1 + c_2u_2 + ...c_nu_n$ is given by $c = \frac{y.u_i}{u_i.u_i}$ (i = 1, 2.....n)

Sol :

The given vector space is R^n

let $s = [u, u_2,.....u_n]$ is an orthogonal basis of w.

$s \leq w$

let $c_1, c_2,......c_n$ be n scalars and $y = c_1u_1 + c_2u_2 + ... + c_nu_n$

consider $y.u_1 = (c_1u_1 + c_2u_2 + ... + c_nu_n).u_1$

$= c_1.(u_1.u_1) + c_2(u_2.u_1) + ... + c_n(u_n.u_1)$

$= c_1(u_1.u_1)$

$$\Rightarrow c_1 = \frac{y.u_1}{u_1.u_1}$$

Ily $$c_i = \frac{y.u_i}{u_i.u_i} \quad \forall\, i = 1 \text{ to } n$$

Q34. Let $u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$; $u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$; $x = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$. Show that $\{u_1, u_2\}$ is an orthogonal basis for R^2. Then express x as a linear combination of the u's.

Sol :

Given vectors are,

$$u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}; u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \text{ and } x = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

Consider, $u_1.u_2 = u_1^T u_2 = [3\ 1]\begin{bmatrix} -2 \\ 6 \end{bmatrix} = -6 + 6 = 0$

$\therefore$ $\{u_1, u_2\}$ is an orthogonal set

Since u_1, u_2 are non-zero vectors.

$\Rightarrow$ u_1, u_2 are linearly independent and form a basis for R^2

$\therefore$ $\{u_1, u_2\}$ is an orthogonal basis for R^2

let x be linear combination of vectors u_1, u_2

i.e., $x = \dfrac{x.u_1}{u_1.u_1} u_1 + \dfrac{x.u_2}{u_2.u_2} u_2$ ......(1)

Consider,

$x.u_1 = x^T u_1$

$= [-6\ 3]\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -6(3) + 3(1) = -18 + 3$

$x.u_1 = -15$

$x.u_2 = x^T u_2$

$= [-6\ \ 3]\begin{bmatrix} -2 \\ 6 \end{bmatrix} = -6(-2) + 3(6) = 12 + 18$

$x.u_2 = 30$

$u_1.u_1 = u_1^T u_1 = [3\ \ 1]\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3(3) + 1(1) = 9 + 1 = 10$

$u_1.u_1 = 10$

$u_2.u_2 = u_2^T u_2$

$= [-2\ \ 6]\begin{bmatrix} -2 \\ 6 \end{bmatrix} = -2(-2) + 6(6) = 4 + 36$

$u_2.u_2 = 40$

Sub, corresponding values in equation (1)

$$x = \frac{-15}{10}\begin{bmatrix}3\\1\end{bmatrix} + \frac{30}{40}\begin{bmatrix}-2\\6\end{bmatrix}$$

$$x = \frac{-3}{2}\begin{bmatrix}3\\1\end{bmatrix} + \frac{3}{4}\begin{bmatrix}-2\\6\end{bmatrix}$$

Q35. Assume $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for R^4 and write x as the sum of two vectors, one in span $\{u_1, u_2, u_3\}$ and the other in span $\{u_4\}$)

$$u_1 = \begin{bmatrix}0\\1\\-4\\-1\end{bmatrix};\ u_2 = \begin{bmatrix}3\\5\\1\\1\end{bmatrix};\ u_3 = \begin{bmatrix}1\\0\\1\\-4\end{bmatrix};\ u_4 = \begin{bmatrix}5\\-3\\-1\\1\end{bmatrix};\ x = \begin{bmatrix}10\\-8\\2\\0\end{bmatrix}$$

Sol :

Given vectors are,

$$u_1 = \begin{bmatrix}0\\1\\-4\\-1\end{bmatrix};\ u_2 = \begin{bmatrix}3\\5\\1\\1\end{bmatrix};\ u_3 = \begin{bmatrix}1\\0\\1\\-4\end{bmatrix};\ u_4 = \begin{bmatrix}5\\-3\\-1\\1\end{bmatrix};\ \text{and } x = \begin{bmatrix}10\\-8\\2\\0\end{bmatrix}$$

Let $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for R^4 then

$$x = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 \qquad \text{...........(1)}$$

The vector x can be represented as sum of two vectors (ie., in span $\{u_1, u_2, u_3\}$ and span $\{u_4\}$). as,

$$x = z_1 + z_2 \qquad \text{......... (2)}$$

where,

$z_1 = c_1u_1 + c_2u_2 + c_3u_3$ is in span $\{u_1, u_2, u_3\}$ and

c_4u_4 is in span $\{u_4\}$

Consider,

$$z_1 = c_1u_1 + c_2u_2 + c_3u_3$$

$$\Rightarrow\ z_1 = \frac{x.u_1}{u_1.u_1}u_1 + \frac{x.u_2}{u_2.u_2}u_2 + \frac{x.u_3}{u_3.u_3}.u_3 \qquad \text{........(3)}$$

$$x.u_1 = x^Tu_1 = [10\ \ -8\ \ 2\ \ 0]\begin{bmatrix}0\\1\\-4\\-1\end{bmatrix} = 10(0) - 8(1) + 2(-4) + 0(-1) = -16$$

$$u_1.u_1 = u_1^T u_1 = [0 \;\; 1 \;\; -4 \;\; -1]\begin{bmatrix}0\\1\\-4\\-1\end{bmatrix} = 0(0) + 1(1) + (-4)(-4) + (-1)(-1) = 18$$

$$x.u_2 = x^T u_2 = [10 \;\; -8 \;\; 2 \;\; 0]\begin{bmatrix}3\\5\\1\\1\end{bmatrix} = 10(3) - 8(5) + 2(1) + 0(1) = -8$$

$$u_2.u_2 = u_2^T u_2 = [3 \;\; 5 \;\; 1 \;\; 1]\begin{bmatrix}3\\5\\1\\1\end{bmatrix} = 3(3) + 5(5) + 1(1) + 1(1) = 36$$

$$x.u_3 = x^T u_3 = [10 \;\; -8 \;\; 2 \;\; 0]\begin{bmatrix}1\\0\\1\\-4\end{bmatrix} = 10(1) - 8(0) + 2(1) + 0(-4) = 12$$

$$u_3.u_3 = u_3^T u_3 = [1 \;\; 0 \;\; 1 \;\; -4]\begin{bmatrix}1\\0\\1\\-4\end{bmatrix} = 1(1) + 0(0) + 1(1) + (-4)(-4) = 18$$

Substituting corresponding values in equation (3)

$$z_1 = \frac{-16}{18}u_1 + \frac{-8}{36}u_2 + \frac{12}{18}u_3$$

$$\Rightarrow \quad z_1 = \frac{-8}{9}u_1 - \frac{2}{9}u_2 + \frac{2}{3}u_3 \qquad \text{........(4)}$$

Consider,

$$z_2 = c_4 u_4$$

$$z_2 = \frac{x.u_4}{u_4.u_4}u_4 \qquad \text{.........(5)}$$

$$x.u_4 = x^T u_4 = [10 \quad -8 \quad 2 \quad 0]\begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix} = 10(5) -8(-3) + 2(-1) + 0(1) = 72$$

$$u_4.u_4 = u_4^T u_4 = [5 \quad -3 \quad -1 \quad 1]\begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix} = 5(5) -3(-3) -1(-1) + 1(1) = 36$$

Sub the corresponding values in equation (5)

$$z_2 = \frac{72}{36} u_4$$

$$z_2 = 2u_4 \quad ......(6)$$

Sub, equation (4) & (6) in (2)

$$x = \frac{-8}{9} u_1 - \frac{2}{9} u_2 + \frac{2}{3} u_3 + 2u_4$$

$$x = \frac{-8}{9}\begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix} - \frac{2}{9}\begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix} + 2\begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -8/9 \\ 32/9 \\ 8/9 \end{bmatrix} + \begin{bmatrix} -2/3 \\ -10/9 \\ -2/9 \\ -2/9 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 0 \\ 2/3 \\ -8/3 \end{bmatrix} + \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 - \frac{2}{3} + \frac{2}{3} \\ \frac{-8}{9} - \frac{10}{9} + 0 \\ \frac{32}{9} - \frac{2}{9} + \frac{2}{3} \\ \frac{8}{9} - \frac{2}{9} - \frac{8}{3} \end{bmatrix} + \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$$

$\therefore$ x is represented as sum of two vectors

$$\begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$$

Q36. Define orthonormal sets.

Sol :

Let v be any vector space of R^n

let s = $[u_1, u_2, \ldots, u_p]$ be any subset of v.

If every vector of s is a unit vector and if each pair of distinct vectors of s are mutually orthogonal then s is said to be an orthonormal set.

i.e., $u_i . u_j = \begin{cases} 1 & \forall i = j \\ 0 & \forall i \neq j \end{cases}$

If w is any subspace of v and if the orthogonal set s = $\{u_1, u_2, \ldots, u_n\}$ spans w then s is an orthonormal basis for w.

Q37. If $v_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}$, $v_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $v_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$ then such that $[v_1, v_2, v_3]$ is an orthonormal basis of R^3.

Sol :

The given vector space is R^3

The given set is s = $\{v_1, v_2, v_3\}$

(i) Consider,

$$v_1.v_1 = \frac{3}{\sqrt{11}}\cdot\frac{3}{\sqrt{11}} + \frac{1}{\sqrt{11}}\cdot\frac{1}{\sqrt{11}} + \frac{1}{\sqrt{11}}\cdot\frac{1}{\sqrt{11}} = \frac{9}{11} + \frac{1}{11} + \frac{1}{11} = \frac{11}{11} = 1$$

$$v_2.v_2 = \left(\frac{-1}{\sqrt{6}}\cdot\frac{-1}{\sqrt{6}}\right) + \left(\frac{2}{\sqrt{6}}\cdot\frac{2}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}\right) = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = \frac{6}{6} = 1$$

$$v_3.v_3 = \left(\frac{-1}{\sqrt{66}}\cdot\frac{-1}{\sqrt{66}}\right) + \left[\frac{-4}{\sqrt{66}}\cdot\frac{-4}{\sqrt{66}}\right] + \left[\frac{7}{\sqrt{66}}\cdot\frac{7}{\sqrt{66}}\right] = \frac{1}{66} + \frac{16}{66} + \frac{49}{66} = \frac{66}{66} = 1$$

Thus each vector is a unit vector

(ii) Consider,

$$v_1.v_2 = \frac{-3}{\sqrt{66}} + \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}} = 0$$

$$v_1.v_3 = \frac{-3}{\sqrt{726}} - \frac{4}{\sqrt{726}} + \frac{7}{\sqrt{726}} = 0$$

$$v_2.v_3 = \frac{1}{\sqrt{396}} - \frac{8}{\sqrt{396}} + \frac{7}{\sqrt{396}} = 0$$

Thus each pair of distinct vectors of S are mutually orthogonal

$\therefore$ S = $\{u_1, u_2, u_3\}$ is an orthonormal set in R^3 .

Since each vector of S is a unit vector S is linearly Independent.

$\therefore$ S is a basis of R^3

Q38. Show that $\{v_1, v_2, v_3\}$ **is an orthonormal basis of** R^3 **where** $v_1 = \left(\frac{1}{\sqrt{18}}, \frac{4}{\sqrt{18}}, \frac{1}{\sqrt{18}}\right)$,

$$v_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right), \quad v_3 = \left(\frac{-2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$$

Sol :

Given vectors are $v_1 = \begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$; $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$; $v_3 = \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

(i) Consider

$$||v_1|| = \sqrt{\left(\frac{1}{\sqrt{18}}\right)^2 + \left(\frac{4}{\sqrt{18}}\right)^2 + \left(\frac{1}{\sqrt{18}}\right)^2} = 1$$

$$||v_2|| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2} = 1$$

$$||v_3|| = \sqrt{\left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2} = 1$$

$\therefore$ Thus each vector is a unit vector

(ii) Consider,

$$v_1.v_2 = v_1^T v_2 = \begin{bmatrix} \frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} & \frac{-1}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \left(\frac{1}{\sqrt{18}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{4}{\sqrt{18}}\right)(0) + \left(\frac{1}{\sqrt{18}}\right)\left(\frac{-1}{\sqrt{2}}\right)$$

$v_1 . v_2 = 0$

$v_1.v_3 = v_1^T v_3$

$$= \begin{bmatrix} \frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} & \frac{1}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$= \frac{1}{\sqrt{18}}\left(\frac{-2}{3}\right) + \frac{4}{\sqrt{18}}\left(\frac{1}{3}\right) + \frac{1}{\sqrt{18}}\left(\frac{-2}{3}\right)$$

$v_1.v_3 = 0$

$v_2.v_3 = v_2^T v_3$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-2}{3}\right) + 0\left(\frac{1}{3}\right) + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-2}{3}\right)$$

$$= 0$$

$\therefore$ $\{v_1, v_2, v_3\}$ is an orthogonal set

$\therefore$ $\{v_1, v_2, v_3\}$ is an orthonormal set

Since v_1, v_2, v_3 are linearly independent and forms a basis for R^3

$\therefore$ $\{v_1, v_2, v_3\}$ is an orthonormal basis for R^3

Q39. If U is an m × n matrix then U has orthonormal columns if and only if $U^T V = I$

Sol :

Let us consider the case in the vector space R^3

Let $U = \{u_1, u_2, u_3\}$ has only three columns each vector in R^m

$$U^T = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix}$$

Consider $U^TU = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix} [u_1\ u_2\ u_3]$

$$U^T U = \begin{bmatrix} u_1^Tu_1 & u_1^Tu_2 & u_1^Tu_3 \\ u_2^Tu_1 & u_2^Tu_2 & u_2^Tu_3 \\ u_3^Tu_1 & u_3^Tu_2 & u_3^Tu_3 \end{bmatrix} \quad \text{....... (1)}$$

Here the columns of U are orthogonal.

$$\left.\begin{array}{l} u_1^Tu_2 = 0 = u_2^Tu_1 \\ u_1^Tu_3 = 0 = u_3^Tu_1 \\ u_2^Tu_3 = 0 = u_3^Tu_2 \end{array}\right\} \quad \text{....... (2)}$$

The columns of U are have unit length if and only if $u_1^T u_1 = 1; u_2^T u_2 = 1; u_3^T u_3 = 1$ ......(3)

From (1), (2) & (3) $U^TU = I$

Thus, U has orthogonal polynomials $\Leftrightarrow U^T U = I$

Q40. If $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ **and** $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$ **then show that U has orthonormal columns also verify** $\|Ux\| = \|x\|$

Sol :

The given matrices are $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ & $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$

$$U^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$\text{Consider } U^TU = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & \frac{2}{3} \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ U has orthonormal columns

$$\text{Consider } Ux = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\|Ux\| = \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11}$$

$$\|x\| = \sqrt{(\sqrt{2})^2 + 3^2} = \sqrt{11}$$

$$\therefore \|Ux\| = \|x\|$$

Q41. Determine which sets of vectors are orthogonal. If a set is only orthogonal, normalize the vectors to produce on orthonormal set.

(i) $\begin{bmatrix} \frac{1}{3} \\ 1/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

Sol :

Given set of vectors are, $\left\{\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}\right\}$

Let $u = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ and $v = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$

Consider $u.v = u^T v = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \frac{1}{3}\left(\frac{-1}{2}\right) + \frac{1}{3}(0) + \frac{1}{3}\left(\frac{1}{2}\right)$

$= 0$

$\Rightarrow u.v = 0$

$\Rightarrow$ {u, v} is an orthogonal set

$\| u \|^2 = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{1}{3}$

$\| v \|^2 = \left(\frac{-1}{2}\right)^2 + (0)^2 + \left(\frac{1}{2}\right)^2$

$= \frac{1}{2}$

Since $\| u \|^2 = \| v \|^2 \neq 1$

$\therefore$ {u, v} is not an orthonormal set.

The vectors u & v are normalized to form the orthonormal set.

i.e, $\left\{ \frac{u}{\| u \|}, \frac{v}{\| v \|} \right\} = \left\{ \frac{1}{\sqrt{1/3}} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \frac{1}{\sqrt{1/2}} \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} \right\}$

$= \left\{ \sqrt{3} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \sqrt{2} \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} \right\}$

$$= \left\{ \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}, \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

(ii) Given set of vectors are

$$= \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Let $u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

Consider

$$u.v = u^T v = [\,0\ 1\ 0\,] \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$= 0(0) + 1(-1) + 0(0) = -1 \neq 0$$

{u,v} is not an orthogonal set.

4.3 Orthogonal Projection

Q42. Find the projection of y = $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$ onto u = $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ Also write y as the sum of two orthogonal vectors are in Span {u} and one orthogonal to u.

Sol : **(Dec.-19)**

The given vectors are $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

consider $y.u = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 7(4) + 6(2) = 28 + 12 = 40$

and u.u = $\begin{bmatrix} 4 \\ 2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ = 16 + 4 = 20

If $\hat{y}$ is the orthogonal projection of u then

$$\hat{y} = \frac{y.u}{u.u}u = \frac{40}{20}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

Let Z be the orthogonal complement if y orthogonal to u.

$$Z = y - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\Rightarrow$ y = $\hat{y} + z$

Thus y can be written as the sum of two orthogonal vectors $\hat{y}$ and z.

$$\left[\therefore \hat{y}.z = \begin{bmatrix} 8 \\ 4 \end{bmatrix}.[-1\ 2] = -8 + 8 = 0\right]$$

Here $\hat{y}$ is in span {u} and z is orthogonal to u.

Q43. Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

Sol : **(July-21)**

The given vectors are y = $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ and u = $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$

Consider y.u = $\begin{bmatrix} 1 \\ 7 \end{bmatrix}.\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ = – 4 + 14 = 10

u.u = $\begin{bmatrix} -4 \\ 2 \end{bmatrix}.\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ = 16 + 4 = 20

If $\bar{y}$ is the orthogonal projection of u then

$$\bar{y} = \frac{y.u}{u.u}u = \frac{10}{20}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -4/2 \\ 2/2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let L be any line passing through $u = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ & the origin then the orthogonal projection of $y = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ on to the vector $u = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ is given by

$$\bar{y} = Pro_L\, y = \frac{y.u}{u.u} u = \frac{10}{20} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Q44. Show that the set s = {u_1, u_2, u_3} where $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$ is an orthogonal basis for R^3 and express the vector $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of vectors in S.

Sol :

The given vector space is R^3

The given set is $s = \{u_1, u_2, u_3\} \subseteq R^3$

(i) Consider $u_1.u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} [3 \;\; 1 \;\; 1] = -3 + 2 + 1 = 0$

$$u_1.u_3 = [3 \;\; 1 \;\; 1] \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = \frac{-3}{2} - 2 + \frac{7}{2} = 0$$

$$u_2.u_3 = [-1 \;\; 2 \;\; 1] \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = \frac{1}{2} - 4 + \frac{7}{2} = 0$$

Thus each pair of vectors of s are mutually orthogonal

$\therefore$ S is an orthogonal set.

Since S is an orthogonal set of non-zero vectors S is a linearly independent subset of R^3 consisting of 3 vectors.

$\therefore$ S is a basis of R^3

Thus S is an orthogonal basis of R^3

(ii) The given vector is $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$

Consider $y.u_1 = [6 \ \ 1 \ -8] \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 11$

$y.u_2 = [6 \ \ 1 \ -8] \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -12$

$y.u_3 = [6 \ \ 1 \ -8] \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = -33$

$u_1.u_1 = [3 \ 1 \ \ 1] \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 9 + 1 + 1 = 11$

$u_2.u_2 = [-1 \ \ 2 \ \ 1] \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 6$

$u_3.u_3 = \begin{bmatrix} \frac{-1}{2} & -2 & \frac{7}{2} \end{bmatrix} \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = \frac{33}{2}$

The weights in the linear combination of y is u_1, u_2, u_3 are c_1, c_2, c_3 ...... (1)

Here $C_1 = \frac{y.u_1}{u_1.u_1} = \frac{11}{11} = 1$

$C_2 = \frac{y.u_2}{u_2.u_2} = \frac{-12}{6} = -2$

$C_3 = \frac{y.u_3}{u_3.u_3} = \frac{-33}{\frac{33}{2}} = -2$

$\therefore \boxed{y = 1.u_1 - 2.u_2 - 2.u_3}$

Q45. Compute orthogonal projection of (1,–1) onto the line through (–4,2) and (0,0)

Sol :

Let the vectors be,

$y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $u = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

The orthogonal projection of y onto u is given as,

$\hat{y} = \frac{y.u}{u.u} u$ .... (1)

$y.u = [1 \ \ -1] \begin{bmatrix} -4 \\ 2 \end{bmatrix} = -6$

$u.u = [-4 \ 2] \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 20$

Sub, the corresponding values in equation (1)

$\hat{y} = \frac{-6}{20} u = \frac{-3}{10} \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 12/10 \\ -6/10 \end{bmatrix}$$

$\therefore$ The orthogonal projection is $\hat{y} = \begin{bmatrix} 12/10 \\ -6/10 \end{bmatrix}$

Q46. Let $y = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the distance from y to the line through u and the origin.

Sol :

Given vectors are $y = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The distance from y to the line through u and the origin is $\| y - \hat{y} \|$

The orthogonal projection of y on to L is given as,

$$\hat{y} = \frac{y.u}{u.u} u \text{ .... (1)}$$

$$y.u = [-3 \ 9] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (-3)(1) + 9(2)$$

$$= -3 + 18$$

$$= 15$$

$$u.u = [1 \ \ 2] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1(1) + 2(2)$$

$$= 1 + 4$$

$$= 5$$

substituting the corresponding values in equation (1)

$$\hat{y} = \frac{15}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{y} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Consider $y - \hat{y} = \begin{bmatrix} \frac{-3}{9} \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} -3-3 \\ 9-6 \end{bmatrix}$$

$$y - \hat{y} = \begin{bmatrix} -6 \\ +3 \end{bmatrix}$$

Then $\|y - \hat{y}\| = \sqrt{(-6)^2 + 3^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$

$\therefore$ The distance from y to the line through u and origin is $3\sqrt{5}$

Q47. State and prove the orthogonal decomposition theorem.

Sol :

Statement:

If w is a subspace of R^n then each y in R^n can be written uniquely in the form $y = \hat{y} + z$ where $\hat{y}$ is in w and z is in w.

Additionally, if $\{u_1, u_2, ... u_p\}$ is an orthogonal basis of w, then the orthogonal projection of y onto w is,

$$\hat{y} = \text{Proj}_w y = \frac{y.u_1}{u_1.u_1} u_1 + \frac{y.u_2}{u_2.u_2} u_2 + ...+ \frac{y.u_p}{u_p.u_p} u_p \text{ and } z = y - \hat{y}$$

Proof :

Let the orthogonal basis for a subspace w be $\{u_1, u_2, ... u_p\}$

For any vector y, the orthogonal projection of y onto w is given as,

$$\hat{y} = \frac{y.u_1}{u_1.u_1} u_1 + \frac{y.u_2}{u_2.u_2} u_2 + ...+ \frac{y.u_p}{u_p.u_p} u_p \quad .... (1)$$

from eqn (1), $\hat{y}$ is a linear combination of u_i i.e., u_1, u_2 .....u_p such that y is in w.

Let $z = y - \hat{y}$

Then,

$z.u_1 = (y - \hat{y})\, u_1$

$= y.u_1 - \hat{y}\, u_1$

$= y.u_1 - \left(\frac{y.u_1}{u_1.u_1} u_1 + \frac{y.u_2}{u_2.u_2} u_2 + ... + \frac{y.u_p}{u_p.u_p} u_p \right) .u_1$

$\Rightarrow z.u_1 = y.u_1 - \frac{y.u_1}{u_1.u} u_1.u_1 - \frac{y.u_2}{u_2.u_2} u_2.u_1 .... - \frac{y.u_p}{u_p.u_p} u_p.u_1$

Since u_1 is orthogonal to $u_2, u_3, ...u_p$ then

$z.u_1 = y.u_1 - y.u_2$

$z.u_1 = 0$

i.e., z is orthogonal to u_1

Similarly, z is orthogonal to $u_2, u_3 .... u_p$

Hence, z is orthogonal to each u_i in w

i.e., z is in $w^{\perp}$

Uniqueness.

Let y can be written as sum of two vectors (in w & $w^{\perp}$) in two different ways i.e.,

$y = \hat{y} + z$ .... (2)

& $y = \hat{y}_1 + z_1$ .... (3)

from (2) & (3)

$\hat{y} + z = \hat{y}_1 + z_1$

$\Rightarrow \hat{y} - \hat{y}_1 = z_1 - z$ .... (4)

Let $v = \hat{y} - \hat{y}_1$ is in w then $v.v = 0 \Rightarrow v = 0$

If $v = 0$ then $\hat{y} - \hat{y}_1 = 0 \Rightarrow \hat{y} = \hat{y}_1$

Sub, $y = \hat{y}_1$ is eq^n (4)

$\hat{y} - \hat{y} = z_1 - z$

$z_1 - z = 0$

$z_1 = z$

Thus, $y = \hat{y} + z$

Hence, y can be written uniquely in the form $y = \hat{y} + z$

When $\hat{y}$ is in w and z is in $w^{\perp}$

Q48. Find the orthogonal projection of y onto span $\{u_1, u_2\} = w$

$$\mathbf{y = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}; u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \; \& \; u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}}$$

Sol :

Given vectors are,

$$y = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

The orthogonal projection of y onto span $\{u_1, u_2\}$ is given as,

$$\hat{y} = \frac{y.u_1}{u_1 u_1} u_1 + \frac{y.u_2}{u_2.u_2} u_2 \text{ .... (1)}$$

$$y.u_1 = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} . \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = (-1)(3) + 2(-1) + 6(2) = 7$$

$$y.u_2 = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = (-1)1 + 2(-1) + 6(-2) = -15$$

$$u_1.u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 3(3) - 1(-1) + 2(2) = 14$$

$$u_2.u_2 = \begin{bmatrix}1\\-1\\-2\end{bmatrix}\begin{bmatrix}1\\-1\\-2\end{bmatrix} = 1(1) -1(-1) - 2(-2) = 6$$

Sub, the corresponding values in eq^n.(1)

$$\hat{y} = \frac{7}{14}u_1 - \frac{15}{6}u_2$$

$$= \frac{7}{14}\begin{bmatrix}3\\-1\\2\end{bmatrix} - \frac{5}{2}\begin{bmatrix}1\\-1\\-2\end{bmatrix}$$

$$= \begin{bmatrix}3/2\\-1/2\\1\end{bmatrix} - \begin{bmatrix}.5/2\\-5/2\\-5\end{bmatrix}$$

$$= \begin{bmatrix}\frac{3}{2}-\frac{5}{2}\\ \frac{-1}{2}+\frac{5}{2}\\ 1+5\end{bmatrix} = \begin{bmatrix}-1\\2\\6\end{bmatrix} = y$$

$\therefore$ The orthogonal projection of y on to span $\{u_1, u_2\}$ = w is the closest point in w to y.

Q49. Let w be the subspace spanned by the u's & write y as the sum of a vector in w and a vector orthogonal to w.

$$\mathbf{y = \begin{bmatrix}1\\3\\5\end{bmatrix}; u_1 = \begin{bmatrix}1\\3\\-2\end{bmatrix}; u_2 = \begin{bmatrix}5\\1\\4\end{bmatrix}}$$

Sol :

Given vectors are,

$$y = \begin{bmatrix}1\\3\\5\end{bmatrix};\ u_1 = \begin{bmatrix}1\\3\\-2\end{bmatrix} \text{ and } u_2 = \begin{bmatrix}5\\1\\4\end{bmatrix}$$

consider $u_1.u_2 = \begin{bmatrix}1\\3\\-2\end{bmatrix}\begin{bmatrix}5\\1\\4\end{bmatrix} = 1(5) + 3(1) - 2(4)$

$= 0$

$\therefore \{u_1, u_2\}$ is an orthogonal set.

By orthogonal decomposition theorem, y can be written a sum of vector in w and a vector orthogonal to w (i.e., $w^{\perp}$)

i.e., $y = \hat{y} + z$ .... (1)

The orthogonal projection of y onto w is given as,

$$\hat{y} = \frac{y.u_1}{u_1.u_1}u_1 + \frac{y.u_2}{u_2.u_2}u_2 \text{ ....(2)}$$

$$y.u_1 = \begin{bmatrix}1\\3\\5\end{bmatrix}\begin{bmatrix}1\\3\\-2\end{bmatrix} = 1(1) + 3(3) + 5(-2) = 0$$

$$u_1.u_1 = \begin{bmatrix}1\\3\\-2\end{bmatrix}\begin{bmatrix}1\\3\\-2\end{bmatrix} = 1(1) + 3(3) - 2(-2) = 14$$

$$y.u_2 = \begin{bmatrix}1\\3\\5\end{bmatrix}\begin{bmatrix}5\\1\\4\end{bmatrix} = 1(5) + 3(1) + 5(4) = 28$$

$$u_2.u_2 = \begin{bmatrix}5\\1\\4\end{bmatrix}\begin{bmatrix}5\\1\\4\end{bmatrix} = 5(5) + 1(1) + 4(4) = 42$$

Substituting corresponding values in eq^n (2)

$$\hat{y} = \frac{0}{14}\begin{bmatrix}1\\3\\-2\end{bmatrix} + \frac{28}{42}\begin{bmatrix}5\\1\\4\end{bmatrix}$$

$$= 0 + \frac{2}{3}\begin{bmatrix}5\\1\\4\end{bmatrix}$$

$$\hat{y} = \begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix}$$

From eq^n (1)

$z = y - \hat{y}$

Substituting the corresponding values in above equation

$$z = \begin{bmatrix}1\\3\\5\end{bmatrix} - \begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix}$$

$$= \begin{bmatrix}1-10/3\\3-2/3\\5-8/3\end{bmatrix} \Rightarrow z = \begin{bmatrix}-7/3\\7/3\\7/3\end{bmatrix}$$

Substituting the corresponding values in eq^n (1),

$$\therefore\ y = \begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix} + \begin{bmatrix}-7/3\\7/3\\7/3\end{bmatrix}$$

Q50. State and Prove the best approximation theorem.

Sol :

Statement

Let w be a subspace of R^n, y any vector in R^n, y any vector in R^n, and $\hat{y}$ is the orthogonal projection of y onto w. Then $\hat{y}$ is the closest point in w to y, in the sense that $\|y - \hat{y}\| < \|y - v\|\ \forall$ v in w distinct from $\hat{y}$

Proof : For any vector y in w, the orthogonal projection of y onto w is $\hat{y}$

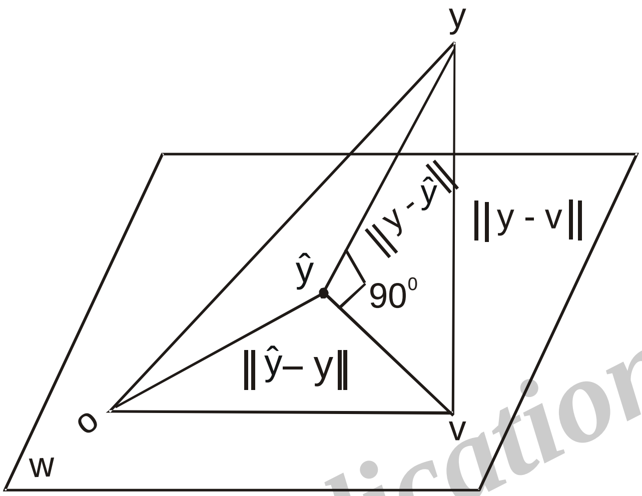
Let v be the distinct vector in subspace w.

As both the vectors $\hat{y}$ and v are in w, the vector $\hat{y}$ –v will also be in w.

By orthogonal decomposition theorem, y – $\hat{y}$ is orthogonal to w such that y – $\hat{y}$ and $\hat{y}$ – v are orthogonal.

Hence, the three vectors $\hat{y}$ – v, and y – $\hat{y}$ from a right angle triangle.

By pythagarous theorem, $\|\hat{y} - v\|^2 + \|y - \hat{y}\|^2 = \|y - v\|^2$



Since the vector v and $\hat{y}$ are distinct,

$v \neq \hat{y} \Rightarrow \hat{y} - v \neq 0 \Rightarrow \hat{y} - v > 0$

Then, $\|y - \hat{y}\|^2 < \|y - v\|^2$

$\Rightarrow \|y - \hat{y}\| < \|y - v\|$

Q51. Find the closest point to y in the subspace w panned by v_1 and v_2

$$\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}; \mathbf{v_1} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}; \mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Sol :

Given vectors are, $y = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}; v_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}; v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

Consider,

$$v_1.v_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 3(1) + 1(-1) - 1(1) + 1(-1) = 0$$

$\therefore$ $\{v_1,v_2\}$ is an orthogonal set.

From the best approximation theorem, the closest point to y in w is the orthogonal projection of y onto w.

The orthogonal projection of y onto w is given as,

$$\hat{y} = \frac{y.v_1}{v_1.v_1} v_1 + \frac{y.v_2}{v_2.v_2} v_2 \quad \text{......... (1)}$$

$$y.v_1 = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} . \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} = 3(3) + 1(1) + 5(-1) + 1(1) = 6$$

$$v_1.v_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} = 3(3) + 1(1) -1(-1) + 1(1) = 12$$

$$y.v_2 = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 3(1) + 1(-1) + 5(1) + 1(-1) = 6$$

$$v_2.v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 1(1) + (-1)(-1) + 1(1) + (-1)(-1) = 4$$

Substituting the corresponding values in (1)

$$\hat{y} = \frac{6}{12}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} + \frac{6}{4}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

$$= \begin{bmatrix}3/2\\1/2\\-1/2\\1/2\end{bmatrix} + \begin{bmatrix}3/2\\-3/2\\3/2\\-3/2\end{bmatrix} = \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}$$

$\therefore$ the closest point to y is $\begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}$

Q52. Let $y = \begin{bmatrix}5\\-9\\5\end{bmatrix}$; $u_1 = \begin{bmatrix}-3\\-5\\1\end{bmatrix}$; $u_2 = \begin{bmatrix}-3\\2\\1\end{bmatrix}$ Find the distance from y to the plane in R^3 spanned by u_1 and u_2

Sol :

Given vectors are, $y = \begin{bmatrix}5\\-9\\5\end{bmatrix}$, $u_1 = \begin{bmatrix}-3\\-5\\1\end{bmatrix}$, $u_2 = \begin{bmatrix}-3\\2\\1\end{bmatrix}$

By the best approximation theorem, the distance from y to subspace w in R^3 is $\| y - \hat{y} \|$
The orthogonal projection of y on to w is given as,

$$\hat{y} = \frac{y.u_1}{u_1.u_1}u_1 + \frac{y.u_2}{u_2.u_2}u_2 \quad ....(1)$$

$$y.u_1 = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}\begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} = 5(-3)-9(-5) + 5(1) = 35$$

$$u_1.u_1 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}\begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} = -3(-3) - 5(-5) + 1(1) = 35$$

$$y.u_2 = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = 5(-3)-9(2) + 5(1) = -28$$

$$u_2.u_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = -3(-3) + 2(2) + 1(1) = 14$$

Sub, the corresponding value in (1)

$$\hat{y} = \frac{35}{35}u_1 - \frac{28}{14}u_2$$

$$= 1\begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} - \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -3+6 \\ -5-4 \\ 1-2 \end{bmatrix}$$

$$\hat{y} = \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}$$

Consider $y - \hat{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} 5-3 \\ -9+9 \\ 5+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

Then,

$$\| y - \hat{y} \| = \sqrt{2^2 + 0^2 + 6^2}$$

$$= \sqrt{40}$$

$\therefore$ The distance from y to the plane in R^3 spanned by u_1 and u_2 is $\sqrt{40}$

Q53. Let $y = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$, $u_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$ and

w = Span $\{u_1, u_2\}$

(i) Let $U = \{u_1, u_2\}$, compute U^TU and UU^T

(ii) Compute $proj_w$ y and (UU^T) y

Sol :

The given vectors are,

$$y = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}, u_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, u_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

(i) Given matrix is,

$$U = [u_1 \ \ u_2] = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

Then,

$$U^T = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Consider,

$$U^TU = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{9}+\frac{1}{9}+\frac{4}{9} & \frac{-4}{9}+\frac{2}{9}+\frac{2}{9} \\ \frac{-4}{9}+\frac{2}{9}+\frac{2}{9} & \frac{4}{9}+\frac{4}{9}+\frac{1}{9} \end{bmatrix}$$

$$U^TU = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Consider,

$$UU^T = \begin{bmatrix} \frac{2}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{-2}{2} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{9}+\frac{4}{9} & \frac{2}{9}-\frac{4}{9} & \frac{4}{9}-\frac{2}{9} \\ \frac{2}{9}-\frac{4}{9} & \frac{1}{9}+\frac{4}{9} & \frac{2}{9}+\frac{2}{9} \\ \frac{4}{9}-\frac{2}{9} & \frac{2}{9}+\frac{2}{9} & \frac{4}{9}+\frac{1}{9} \end{bmatrix}$$

$$UU^T = \begin{bmatrix} \frac{8}{9} & \frac{-2}{9} & \frac{2}{9} \\ \frac{-2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix}$$

(ii) Since $U^TU = I$, U has an orthonormal columns. Then $UU^T y = proj_w y$

$$Proj_w y = UU^T y = \begin{bmatrix} \frac{8}{9} & \frac{-2}{9} & \frac{2}{9} \\ \frac{-2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32}{9} - \frac{16}{9} + \frac{2}{9} \\ \frac{-8}{9} + \frac{40}{9} + \frac{4}{9} \\ \frac{8}{9} + \frac{32}{9} + \frac{5}{9} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\therefore Proj_w y = UU^T y = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

4.4 The Gram -Schmidt Process

Q54. State & prove Gram-Schmidt process

Sol :

Statement:

Given a basis $\{x_1, x_2, ... x_p\}$ for a subspace w of R^n , define

$$v_1 = x_1$$

$$v_2 = \frac{x_2.v_1}{v_1.v_1}v_1$$

$$v_3 = x_3 - \frac{x_3.v_1}{v_1.v_1}v_1 - \frac{x_3.v_2}{v_2.v_2}v_2$$

$$v_p = x_p - \frac{x_p v_1}{v_1.v_1}v_1 - \frac{x_p v_2}{v_2.v_2}v_2 \text{ ....} - \frac{x_p v_{p-1}}{v_{p-1}.v_{p-1}} \cdot v_{p-1}$$

then $\{v_1, v_2, ...v_p\}$ is an orthogonal basis for w. In addition span $\{v_1, v_2, ...v_k\}$ = span $\{x_1, x_2, ... x_k\}$ for $1 \le k \le P$

Proof :

Let w_k = Span $\{x_1, x_2, ...x_k\}$ for $1 \le k \le P$

Assign $v_1 = x_1$ then Span $\{v_1\}$ = Span $\{x_1\}$

Let $\{v_1, v_2, ... v_k\}$ is an orthogonal basis for w_k then

$v_{k+1} = x_{k+1} - \text{Proj}_{w_k}\ x_{k+1}$ for $k < P$

By orthogonal decomparition theorem,

v_{k+1} is orthogonal to w_k

since $\text{Proj}_{w_k} x_{k+1}$ is in w_k, $\text{Proj}_{w_k} x_{k+1}$ will also be in w_{k+1} and also v_{k+1} is in w_{k+1} since x_{k+1} is in w_{k+1}

But $v_{k+1} \neq 0$ as x_{k+1} is not in w_k

Thus, for $(k+1)$ – dimensional space $\{v_1, v_2, ...v_{k+1}\}$ is an orthogonal set.

By basis theorem, the set $\{v_1, v_2 .....v_{k+1}\}$ is an orthogonal basis for w_{k+1}

Hence, w_{k+1} = Span $\{v_1, v_2, ...v_{k+1}\}$ and the process is terminated when $k+1 = p$

Q55. Given set is a basis for a subspaces w = Span $\left\{\begin{bmatrix}2\\-5\\1\end{bmatrix}, \begin{bmatrix}4\\-1\\2\end{bmatrix}\right\}$

(a) Use Gram- schmidt process to produce an orthogonal basis for w.

(b) Find an orthonormal basis of the subspace spanned by vectors.

Sol :

Given basis is, w = Span $\left\{\begin{bmatrix}2\\-5\\1\end{bmatrix}, \begin{bmatrix}4\\-1\\2\end{bmatrix}\right\}$

Let $x_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

(a) By gram-Schmidt process,

$$v_1 = x_1$$

$$\Rightarrow v_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$v_2 = x_2 - \frac{x_2 v_1}{v_1.v_1} v_1$$

$$x_2.v_1 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = 4(2) - 1(-5) + 2(1) = 15$$

$$v_1.v_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = 2(2) - 5(-5) + 1(1) = 30$$

Then, $v_2 = x_2 - \frac{15}{30} v_1$

$$= \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -5/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-1 \\ -1+\frac{5}{2} \\ 2-\frac{1}{2} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$$

$$\therefore \text{ orthogonal basis for w is } \left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} \right\}$$

(b) Let u_1, u_2 be the orthonormal basis for the vectors v_1, v_2 respectively,

$$u_1 = \frac{1}{\|v_1\|} v_1$$

$$= \frac{1}{\sqrt{2^2 + (-5)^2 + 1^2}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 2/\sqrt{30} \\ -5/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix}$$

$$u_2 = \frac{1}{\|v_2\|} v_2$$

$$= \frac{1}{\sqrt{(3)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2}} \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$$

$$= \frac{2}{3\sqrt{6}}\begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\cancel{3}\sqrt{6}}(\cancel{3}) \\ \frac{\cancel{2}}{3\sqrt{6}}\left(\frac{3}{\cancel{2}}\right) \\ \frac{\cancel{2}}{3\sqrt{6}}\left(\frac{3}{\cancel{2}}\right) \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$\therefore$ Orthonormal basis for w is $\left\{ \begin{bmatrix} \frac{2}{\sqrt{30}} \\ \frac{-5}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \right\}$

Q56. Write the steps to finding QR factorication of matrix.

Sol :

Step - 1 : Assign the columns of the matrix as basis i.e., $A_{m\times n} = \{x_1, x_2, \text{----}\ x_n\}$

Step - 2 : Obtain the orthogonal basis $\{v_1, v_2, \text{----}v_n\}$ by using the Gram - Schmidt process.

Step - 3 : Normalize the orthogonal basis $\{v_1, v_2, ...v_n\}$ to obtain the orthonormal basis $\{u_1, u_2, ...u_n\}$

Step - 4 : Assign the orthonormal vectors as columns of Q i.e, $Q_{m\times n} = \{u_1, u_2, ...\ u_n\}$

Step - 5 : Obtain the upper triangular matrix R by using the transformation, $R_{m\times n} = Q^T A$]

Q57. Find the orthogonal basis for the column space of matrix $\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$. Also find the QR factorization.

Sol :

Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & 3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

Let the basis be $x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$; $x_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 5 \\ -4 \\ 3 \\ 7 \\ 1 \end{bmatrix}$

By Gram -Schmidt process,

$v_1 = x_1$

$$\Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow v_2 = x_2 - \frac{x_2 v_1}{v_1.v_1} v_1$$

$$x_2.v_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = 2(1) + 1(-1) + 4(-1) - 4(1) + 2(1) = -5$$

$$v_1.v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = 1(1) - 1(-1) - 1(-1) + 1(1) + 1(1) = 5$$

Then,

$$v_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} - \left(\frac{-5}{5}\right)\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 1-1 \\ 4-1 \\ -4+1 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad v_3 = x_3 - \frac{x_3.v_1}{v_1.v_1} v_1 - \frac{x_3 v_2}{v_2.v_2} v_2$$

$$x_3.v_1 = \begin{bmatrix} 5 \\ -4 \\ 3 \\ 7 \\ 1 \end{bmatrix} . \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = 5(1) - 4(-1) - 3(-1) + 7(1) + 1(1) = 20$$

$$x_3.v_2 = \begin{bmatrix} 5 \\ -4 \\ 3 \\ 7 \\ 1 \end{bmatrix} . \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} = 5(3) - 4(0) - 3(3) + 7(-3) + 1(3) = -12$$

$$v_2.v_2 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} . \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} = 3(3) + 0(0) + 3(3) - 3(-3) + 3(3) = 36$$

Then,

$$v_3 = x_3 - \frac{20}{5} v_1 - \frac{(-12)}{36} v_2$$

$$v_3 = \begin{bmatrix} 5 \\ -4 \\ 3 \\ 7 \\ 1 \end{bmatrix} -4 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 5 \\ -4 \\ 3 \\ 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ -4 \\ -4 \\ 4 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5-4+1 \\ -4+4+0 \\ -3+4+1 \\ 7-4-1 \\ 1-4+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

$\therefore$ The orthogonal basis for the column space is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right\}$$

Let u_1, u_2 and u_3 be the orthonormal basis for the vectors v_1, v_2 and v_3 respectively. Then, the matrix Q is $[u_1\ u_2\ \ u_3]$

$$u_1 = \frac{1}{\|v_1\|} v_1$$

$$= \frac{1}{\sqrt{(1)^2 + (-1)^2 + (-1)^2 + 1^2 + 1^2}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad u_1 = \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \\ -1/\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$u_2 = \frac{1}{\|v_2\|} v_2$$

$$= \frac{1}{\sqrt{(3)^2 + (0)^2 + 3^2 + (-3)^2 + 3^2}} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$u_3 = \frac{1}{\| v_3 \|} v_3$$

$$= \frac{1}{\sqrt{2^2 + o^2 + 2^2 + 2^2 + (-2)^2}} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

Then, the matrix $Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{\sqrt{5}} & 0 & 0 \\ \frac{-1}{\sqrt{5}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{5}} & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{5}} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$

The upper triangular matrix (R) = $Q^T A$

$$= \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{5}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}-\frac{1}{\sqrt{5}}-\frac{4}{\sqrt{5}}-\frac{4}{\sqrt{5}}+\frac{2}{\sqrt{5}} & \frac{5}{\sqrt{5}}+\frac{4}{\sqrt{5}}+\frac{3}{\sqrt{5}}+\frac{7}{\sqrt{5}}+\frac{1}{\sqrt{5}} \\ \frac{1}{2}+0-\frac{1}{2}-\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+0+\frac{4}{2}+\frac{4}{2}+\frac{2}{2} & \frac{5}{2}+0-\frac{3}{2}-\frac{7}{2}+\frac{1}{2} \\ \frac{1}{2}-0-\frac{1}{2}+\frac{1}{2}-\frac{1}{2} & 1-0+\frac{4}{2}-\frac{4}{2}-\frac{2}{2} & \frac{5}{2}-0-\frac{3}{2}+\frac{7}{2}-\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{\sqrt{5}} & \frac{-5}{\sqrt{5}} & \frac{20}{\sqrt{5}} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow \quad R = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

Q58. Find the least squares solution of Ax = b for

$$\mathbf{a} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Sol :

Given matrices are,

$$a = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

The least square solution of $Ax = b$ satisfies the normal equations

$$A^T Ax = A^T b \text{ normal } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad ....(1)$$

Consider,

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 16+0+1 & 0+0+1 \\ 0+0+1 & 0+4+1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

And

$$A^T b = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 8+0+11 \\ 0+0+11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

Sub, the corresponding values in equation (1)

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

Consider,

$$(A^T A)^{-1} = \frac{1}{17(5) - 1(1)} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$= \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

The general least square solution for $Ax = b$ is given as

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$\hat{x} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$= \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore$ The least square solution is,

$$\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Choose the Correct Answers

1. A set of vectors $\{u_1, u_2, \ldots u_p\}$ in R^n is said to be an orthogonal set if [a]

   (a) $u_i \cdot u_j = 0$ for $i \neq j$ (b) $u_i \cdot u_j \neq 0$ for $i \neq j$

   (c) $u_i \cdot u_j =$ for $i = j$ (d) None

2. R^n be any vector space and u, v, w be any three vectors in R^n then [d]

   (a) $u \cdot u = 0 \Leftrightarrow u = 0$ (b) $u \cdot v = v \cdot u$

   (c) $u \cdot (v \cdot w) = (u \cdot v) \cdot w$ (d) All the above

3. A vector whose length (or norm) is 1 is called as [a]

   (a) Unit vector (b) Normalizing a vector

   (c) Both (d) None

4. Two vectors u and v are orthogonal if and only if [c]

   (a) $\|u+v\|^2 + \|u\|^2 - \|v\|^2$ (b) $\|u-v\|^2 = \|u\|^2 - \|v\|^2$

   (c) $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ (d) $\|u-v\|^2 = \|u\|^2 + \|v\|^2$

5. If u and v are any two vectors in R^n then [b]

   (a) $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 - 2\|v\|^2$ (b) $\|u+v\|^2 + \|u-v\|^2 - 2\|u\|^2 + 2\|v\|^2$

   (c) $\left\|\frac{u+v}{u-v}\right\|^2 = \|u\|^2 - \|v\|^2$ (d) None

6. If $x \in w^{\perp}$ then x is ________ to every vector in a sdet that spans w. [b]

   (a) Orthogonal complement (b) Orthogonal

   (c) Both (d) None

7. An $m \times n$ matrix A has orthonormal columns if and only if [d]

   (a) $A - A^T = I$ (b) $A = A^T$

   (c) $A + A^T = I$ (d) $A^T \cdot A = I$

8. If A is an $m \times n$ matrix. Then $(\text{Row } A^T) =$ ________. [a]

   (a) Nul A^T (b) Nul A

   (c) Nul A^{-1} (d) None

9. A set $\{u_1, u_2, ..., u_p\}$ is an ________ set if it is an orthogonal set of unit vectors. [c]

(a) Orthogonal (b) Linearly dependent

(c) Orthonormal (d) None

10. If $\{u_1, u_2, ..., u_p\}$ is an orthonormal basis and $U = \{u_1, u_2, ..., u_p\}$ then. [c]

(a) $Proj_w y = Ipx$ (b) $Proj_w y = UU^T X$

(c) $Proj_w y = UU^T y$ (d) None

Fill in the Blanks

1. If u is any vector then length of u is ________.
2. dist (u, v) = ________.
3. The two vectors u and v are sAid to be orthogonal if ________ for u, v $\in R^n$.
4. ________ vector is orthogonal to every vector.
5. If S = $\{u_1, u_2, ... u_n\}$ is an orthogonal set of non-zero vectors in R^n then S is ________.
6. Let w be a subspace of a vector space R^n. If S is an orthogonal set that spans w then S is called as an ________ for w.
7. A square matrix U is orthogonal U^{-1} = ________.
8. A is an m × n matrix then $(\text{Col } A)^T$ = ________.
9. By orthogonal decomposition theorem, y can be written uniquely in the form ________.
10. IN QR factorization of matrix A, the upper triangular matrix R is obtained by the transformation ________.

ANSWERS

1. $\|u\|$
2. $\|u - v\|$
3. u . v = 0
4. Zero
5. Linearly independent
6. Orthogonal basis
7. U^T
8. Nul A^T
9. $y = \hat{y} + z$
10. Q^TA

FACULTY OF SCIENCE

B.Sc. III Year V-Semester(CBCS) Examination

MODEL PAPER - I

LINEAR ALGEBRA

(MATHEMATICS)

Time : 3 Hours] [Max. Marks : 80

PART - A (5 × 4 = 20 Marks)

Note : Answer any **FIVE** of the following questions.

ANSWERS

1. Given v_1 and v_2 in a vector space V, let H = span $\{v_1, v_2\}$ show that H in a subspace of V. **(Unit-I, Q.No.5)**

2. Let $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ then determine if $\{v_1, v_2, v_3\}$ is a basis for R^3. **(Unit-I, Q.No.44)**

3. Find the eigen values of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ and compare this result with eigenvalue of A^T **(Unit-II, Q.No.28)**

4. If a 7 × 5 matrix A has rank 2, then find dim null A, dim row A and rank A^T. **(Unit-II, Q.No.4)**

5. Show that an $n \times n$ matrix with n distinct eigen values is diagonalizable. **(Unit-III, Q.No.3)**

6. Find the complex eigen values of the matrix $A = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$. **(Unit-IV, Q.No.26)**

7. Define inner product. Write the properties of Inner product. **(Unit-IV, Q.No.1,2)**

8. Find $[\text{dist}_e(u, -v)]^2$ **(Unit-IV, Q.No.15)**

9. Define vector space. **(Unit-I, Q.No.1)**

10. State Invertible Matrix Theorem **(Unit-IV, Q.No.2)**

11. Determine whether the following matrix is diagonalizable or not

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$ **(Unit-III, Q.No.3)**

12. Find $[\text{dist}_e(u, -v)]^2$ **(Unit-IV, Q.No.15)**

PART - B (4 × 15 = 60 Marks)

Note : Answer **all** the questions.

13. (a) An Indexed set $\{v_1, v_2, \ldots v_p\}$ of two or more vectors with $v_1 \neq 0$ is linearly dependent if and only if $\exists$ some v_j (with $j > 1$) is a linear combination of its preceding vectors $v_1, v_2, \ldots, v_{j-1}$. **(Unit-I, Q.No.20)**

OR

(b) Show that the $s = \{(1, 0, 0, -1)\ (0, 1, 0, -1), (0, 0, 1, -1)$ and $(0, 0, 0, 1)\}$ in R^4 is linearly independent. **(Unit-I, Q.No.68)**

14. (a) State and prove the Rank theorem? **(Unit-II, Q.No.1)**

OR

(b) Prove $\lambda = 4$ is an eigen value of $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$ And find the corresponding Eigen vector and characteristic equation of A. **(Unit-II, Q.No.36)**

15. (a) Determine whether the following matrix is diagonalizable or not,

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$ **(Unit-III, Q.No.9)**

OR

(b) (i) If $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ find a formula for A^K given that $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix},\ D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$ **(Unit-III, Q.No.13)**

(ii) Find the complex eigen values of the matrix $A = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$. **(Unit-III, Q.No.26)**

16. (a) Find the projection of $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ onto $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ Also write y as the sum of two orthogonal vectors are in Span $\{u\}$ and one orthogonal to u. **(Unit-IV, Q.No.42)**

OR

(b) Verify the set of vectors are orthogonal.

(a) $\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$ **(Unit-IV, Q.No.29)**

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MODEL PAPER - II

LINEAR ALGEBRA

(MATHEMATICS)

Time : 3 Hours] [Max. Marks : 80

PART - A (5 × 4 = 20 Marks)

Note : Answer any **FIVE** of the following questions.

ANSWERS

1. Define vector subspace and give examples. **(Unit-I, Q.No.3)**

2. If $W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} \middle/ a, b \in R \right\}$ then find a matrix A such that w = Col A. **(Unit-I, Q.No.27)**

3. If A is a 7 × 9 Matrix with a two dimensional Null space, what is the rank of A? **(Unit-II, Q.No.3)**

4. Define Eigen values and Eigen vectors. **(Unit-II, Q.No.18)**

5. If $p = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ then compute A^2, A^4 if $A = PDP^{-1}$. **(Unit-III, Q.No.6)**

6. Find eigen vector for $A = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$ corresponding to eigen value $\lambda = 4 + 3i$ **(Unit-III, Q.No.25)**

7. If V = [1, –2, 2, 0] then find a unit vector u in the same direction as v. **(Unit-IV, Q.No.6)**

8. If $s = \{u_1, u_2.....u_n\}$ is an orthogonal set of non-zero vectors in R^n then s is linearly independent. **(Unit-IV, Q.No.31)**

9. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$; $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$; $v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

   (a) Is w in $\{v_1, v_2, v_3\}$? How many vectors are in $\{v_1, v_2, v_3\}$?

   (b) How many vectors are in span $\{v_1, v_2, v_3\}$ why ?

   (c) Is w is subspace spanned by $\{v_1, v_2, v_3\}$ why ? **(Unit-I, Q.No.17)**

10. If a 7 × 5 matrix A has rank 2, then find dim null A, dim row A and rank A^T. **(Unit-II, Q.No.4)**

11. Diagonalize the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ if possible. **(Unit-III, Q.No.11)**

12. Define inner product. **(Unit-IV, Q.No.1)**

PART - B (4 × 15 = 60 Marks)

Note : Answer **all** the questions.

13. (a) State and prove the spanning set theorem.

Statement:

Let $S = \{v_1, v_2 .... v_p\}$ be a set in v and $H = \text{span}\{v_1, v_2, ... v_p\}$.

(i) If one of the vectors in S i.e., v_k is a linear combination of the remaining vectors in S then the set formed from S by reaming v_k still spans H.

(ii) If $H \neq \{0\}$ then some subset of S is a basis for H. **(Unit-I, Q.No.38)**

OR

(b) If the set $B = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ is a basis for P_2 then find the co-ordinate vector of $p(t) = 1 + 4t + 7t^2$ relative to B. **(Unit-I, Q.No.52)**

14. (a) Find the characteristic equation of $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Also find algebraic multiplicity of the eigen values. **(Unit-II, Q.No.25)**

OR

(b) If $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$ then find rank A and dim null A? **(Unit-II, Q.No.5)**

15. (a) Determine whether the following matrix is diagonalizable or not,

$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. **(Unit-III, Q.No.9)**

OR

(b) If $T:V \to W$ is a linear transformation then,

(i) Kernal of T is a subspace of v

(ii) Range of T is a subspace of W. **(Unit-III, Q.No.15)**

16. (a) (i) If $s = \{u_1, u_2.....u_n\}$ is an orthogonal set of non-zero vectors in R^n then s is linearly independent. **(Unit-IV, Q.No.31)**

(ii) Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin. **(Unit-IV, Q.No.43)**

OR

(b) Given set is a basis for a subspaces $w = \text{Span}\left\{\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}\right\}$

(i) Use Gram- schmidt process to produce an orthogonal basis for w.

(ii) Find an orthonormal basis of the subspace spanned by vectors. **(Unit-IV, Q.No.55)**

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B.Sc. III Year V-Semester(CBCS) Examination

MODEL PAPER - III

LINEAR ALGEBRA

(MATHEMATICS)

Time : 3 Hours] [Max. Marks : 80

PART - A (5 × 4 = 20 Marks)

Note : Answer any **FIVE** of the following questions.

ANSWERS

1. Let w_1 is a subspace of V(F) and w_2 is a subspace of V(F) then $w_1 \cap w_2$ is also subspace of V(F). **(Unit-I, Q.No.7)**

2. Verify whether the vectors $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$ are linearly Indepenent. **(Unit-I, Q.No.40)**

3. Find eigenvalues for matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ **(Unit-II, Q.No.29)**

4. Colud a 6 × 9 matrix have a two - dimensional Null space? **(Unit-II, Q.No.11)**

5. Show that an $n \times n$ matrix with n distinct eigen values is diagonalizable. **(Unit-III, Q.No.3)**

6. Diagonalize the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ if possible. **(Unit-III, Q.No.11)**

7. Define orthogonal sets. **(Unit-IV, Q.No.27)**

8. Find the projection of $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ onto $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ Also write y as the sum of two orthogonal vectors are in Span {u} and one orthogonal to u. **(Unit-IV, Q.No.42)**

9. Define Basis **(Unit-I, Q.No.39)**

10. If λ is the eigen value of a matrix A then show that λ is also the eigen value of the matrix A^T. **(Unit-II, Q.No.22)**

11. Define Diagonalization. **(Unit-III, Q.No.1)**

12. Find the distance between the vector $u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} -4 \\ -3 \\ 8 \end{bmatrix}$ in R^3. **(Unit-IV, Q.No.14)**

PART - B (4 × 15 = 60 Marks)

Note : Answer **all** the questions.

13. (a) Let $V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, V_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, W = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$ Is w in the subspace spanned by $\{v_1, v_2, v_3\}$? Why ? **(Unit-I, Q.No.13)**

OR

(b) Find a spanning set for the null space of the matrix,

$$A = \begin{bmatrix} -3 & 6 & -1 & 1-7 \\ 1 & -2 & 2 & 3-1 \\ 2 & -4 & 5 & 8-4 \end{bmatrix}$$ **(Unit-I, Q.No.31)**

14. (a) Find the eigen values and eigen vectors of $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ **(Unit-II, Q.No.24)**

OR

(b) (i) Show that. If $v_1, v_2 ......v_r$ are eigen vectors that correspond to distinct eigen values $\lambda_1, \lambda_2 ....\lambda_r$ of an $n \times n$ matrix A, then the set $[v_1, v_2 ......v_r]$ is linearly Independent. **(Unit-II, Q.No.20)**

(i) Is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$? If so, find the corresponding eigenvalue **(Unit-II, Q.No.26)**

15. (a) Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ if possible. **(Unit-III, Q.No.12)**

OR

(b) Solve the initial value problem $X^1(t) = AX(t)$ for $t \geq 0$ with $X(0) = (3, 2)$. Classify the nature of the origin as an attractor, repeller, or saddle point of the dyna-mical system described by $X' = AX$. Find the direction of greatest attraction and /or regulation. **(Unit-III, Q.No.24)**

16. (a) Show that $\{u_1, u_2, u_3\}$ is an orthogonal set where

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}; \; u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}; \; u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$ **(Unit-IV, Q.No.28)**

OR

(b) State and prove the orthogonal decomposition theorem. **(Unit-IV, Q.No.47)**

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B.Sc. III Year V-Semester(CBCS) Examination

July - 2021

LINEAR ALGEBRA

(MATHEMATICS)

Time : 2 Hours] **[Max. Marks : 60**

PART - A - (4 × 5 = 20 Marks)

ANSWERS

Note : Answer any **Four** questions.

1. Prove that intersection of a subspace is again a subspace. **(Unit-I, Q.No.7)**

2. Determine if $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$ is in col A, where $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$. **(Unit-I, Q.No.29)**

3. Determine if $\{v_1, v_2, v_3\}$ is basis for R^3, where $v_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ **(Unit-I, Q.No.44)**

4. Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for a vector space V and suppose $b_1 = 6c_1 - 2c_2$ and $b_2 = 9c_1 - 4c_2$. Then find change of coordinate matrix B to C. **(Unit-II, Q.No.17)**

5. Find the complex Eigen values of the matrix $A = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$. **(Unit-III, Q.No.26)**

6. Compute A^4 where $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and A = PDP. **(Unit-III, Q.No.6)**

7. Compute $\|u + v\|$ where $u = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}, v = \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$. **(Unit-IV, Q.No.23)**

8. Find a unit vector in the direction of $v = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$ **(Unit-IV, Q.No.10)**

PART - B - (5 × 8 = 40 Marks)

Note : Answer any **Two** Questions.

9. Define Null space and find spanning set for the Null space of given matrix.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$ **(Unit-I, Q.No.31)**

10. If a vector space V has a basis $\beta = \{b_1, b_2, ...b_n\}$ then show that any set in V containing more than n vectors must be linear dependent. **(Unit-I, Q.No.57)**

11. State and prove that Rank theorem. Also find rank A, where

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$$ **(Unit-II, Q.No.5)**

12. Prove $\lambda = 4$ is an Eigen value of $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$ and find the corresponding Eigen vector and characteristic equation of A. **(Unit-II, Q.No.36)**

13. Diagonalize $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, if possible. **(Unit-III, Q.No.5)**

14. (i) Find the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin. **(Unit-IV, Q.No.43)**

(ii) Determine if the set {u, v, w} is orthogonal set.

Given $u = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$, $v = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$, $w = \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$ **(Unit-IV, Q.No.29)**

FACULTY OF SCIENCE

B.Sc. III Year V-Semester(CBCS) Examination

November / December - 2019

LINEAR ALGEBRA

(MATHEMATICS)

Time : 3 Hours] [Max. Marks : 60

PART - A (5 × 3 = 15 Marks)

(Short Answer Type)

Note : Answer any **FIVE** of the following questions.

ANSWERS

1. Define null space of a matrix. **(Unit-I, Q.No.26)**

2. Find a matrix A such that W = Col A. Where $W = \left\{ \begin{pmatrix} 6a - b \\ a + b \\ -7a \end{pmatrix}; a, b \in R \right\}$ **(Unit-I, Q.No.27)**

3. Find the eigen values of the matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ **(Unit-II, Q.No.28)**

4. Find the characteristics equation of the matrix $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **(Unit-II, Q.No.31)**

5. If $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, find a formula for A^k given that $A = PDP^{-1}$

   where $P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ **(Unit-III, Q.No.13)**

6. Define orthogonal set. **(Unit-IV, Q.No.27)**

7. Define vector subspace with an example. **(Unit-I, Q.No.3)**

8. Find $[\text{dist}(u, -v)]^{2.}$ **(Unit-IV, Q.No.15)**

PART - B (3 × 15 = 45 Marks)
[Essay Answer Type]

Note : Answer **all** from the following questions.

9. (a) Let M_{2x2} be the vector space of all 2 × 2 matrices and define

$T : M_{2\times2} \to M_{2\times2}$ by $T(A) = A + A^T$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Show that T is a linear transformation. **(Unit-I, Q.No.35)**

OR

(b) Find the dimension of the subspace $H = \left\{ \begin{bmatrix} a-3b+6c \\ 6a+4d \\ b-2c-d \\ 5d \end{bmatrix} : a, b, c, d \in R \right\}$ **(Unit-I, Q.No.61)**

10. (a) Show that if $v_1, v_2, ... v_r$ are eigen vectors that correspond to distinct eigen $\lambda_1, \lambda_2, ... \lambda_r$ of an n × n matrix A then the set $(v_1, v_2, ... v_r)$ is linearly independent. **(Unit-II, Q.No.20)**

OR

(b) Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. If so find the one corresponding eigen vector. **(Unit-II, Q.No.32)**

11. (a) Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ if possible. **(Unit-III, Q.No.12)**

OR

(b) Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of y onto u.

Then write y as the sum of two orthogonal vectors, one in span {u} and other one orthogonal to u. **(Unit-IV, Q.No.42)**

FACULTY OF SCIENCE

B.Sc. III Year V-Semester(CBCS) Examination

June / July - 2019

LINEAR ALGEBRA

(MATHEMATICS)

Time : 3 Hours] [Max. Marks : 60

PART - A (5 × 3 = 15 Marks)

(Short Answer Type)

Note : Answer any **FIVE** of the following questions.

ANSWERS

1. Give V_1 and V_2 in a vector space V and let H = span $\{V_1, V_2\}$. Show that H is a subspace of V. **(Unit-I, Q.No.35)**

2. Verify whether the vector $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$ are linearly independent. **(Unit-I, Q.No.40)**

3. Let $b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and consider the bases for R^2 given by $\beta = [b_1, b_2]$ and $\zeta\{c_1, c_2\}$. Find the change of coordinates matrix from β to ζ. **(Unit-I, Q.No.14)**

4. Find the characteristics polynomial and the real eigen values of the matrix.

   $A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$ **(Unit-II, Q.No.23)**

5. Prove that an n × n matrix with n distinct eigen values in diagonalizable. **(Unit-III, Q.No.3)**

6. Write the properties of inner products of vectors in R^n. Also find the norm of the vector (1, –2, 2, 0). **(Unit-IV, Q.No.2,4)**

7. Justify that can be 6 × 9 matrix have a two dimensional null space. **(Unit-IV, Q.No.11)**

8. The set b = $\{1 + t, 1 + t^2, 1 + t^2\}$ is a basis for P_2. Find the coordinate vector of $P(t) = 6 + 3t - t^2$ relative to β. **(Unit-I, Q.No.53)**

PART - B (3 × 15 = 45 Marks)
[Essay Answer Type]

Note : Answer **all** from the following questions.

9. (a) (i) State and prove the spanning set theorem. **(Unit-I, Q.No.38)**

(ii) Suppose $V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $H = \left\{ \begin{bmatrix} S \\ S \\ 0 \end{bmatrix} / R \text{ where } S \in R \right\}$

Then is $\{V_1, V_2\}$ a basis for H? **(Unit-I, Q.No.45)**

(b) (i) State and prove the Rank theorem. **(Unit-II, Q.No.1)**

(ii) Find the base for the row space of the matrix.

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$ **(Unit-II, Q.No.8)**

10. (a) (i) Find the eigen values of $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ **(Unit-II, Q.No.33)**

(ii) Prove that the eigen values of a triangular matrix are its diagonal elements. **(Unit-II, Q.No.19)**

OR

(b) Find the characteristics equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

also find algebraic multiplicity of the eigen values. **(Unit-II, Q.No.25)**

11. (a) Diagonalize the matrix, if possible. $\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ **(Unit-III, Q.No.9)**

OR

(b) (i) If $S = \{u_1, u_2, ... u_p\}$ is an orthogonal set of non zero vectors in R^n, then prove that S is linearly independent. **(Unit-IV, Q.No.31)**

(ii) Verify whether the set of vectors.

$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$ are orthogonal. **(Unit-IV, Q.No.29)**

FACULTY OF SCIENCE

B.Sc. III Year V-Semester(CBCS) Examination

November / December - 2018

LINEAR ALGEBRA

(MATHEMATICS)

Time : 3 Hours] [Max. Marks : 60

PART - A (5 × 3 = 15 Marks)

(Short Answer Type)

Note : Answer any **FIVE** of the following questions.

ANSWERS

1. Let H be the set of all vectors of the form (a – 3b, b – a, a, b), where a and b are arbitrary scalars. Show that H is a subspace of R^4. **(Unit-I, Q.No.10)**

2. Suppose $V_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, V_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ and $V_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ are three vectors in R^3.

   Then that $\{V_1, V_2, V_3\}$ is a basis for R^3. **(Unit-I, Q.No.44)**

3. Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ and consider the bases

   for R^2 given by $\beta = [b_1, b_2]$ and $\zeta\{c_1, c_2\}$. Find the change of coordinates matrix from β to ζ. **(Unit-II, Q.No.9)**

4. Find the eigne value of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ and compare this result with eigen

   value of A^T. **(Unit-II, Q.No.28)**

5. Prove that an n × n matrix with n distinct eigen values in diagnalizable. **(Unit-III, Q.No.3)**

6. Define an inner product between two vectors and write the properties of inner product. **(Unit-IV, Q.No.1,2)**

7. Given a matrix $A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}$ then rank of A and dir. Null A. **(Unit-II, Q.No.6)**

8. Let $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, x = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\beta = [b_1, b_2]$ then find the coordinate

   vector $[x]_\beta$ of x relative to β. **(Unit-I, Q.No.54)**

PART - B (3 × 15 = 45 Marks)

[Essay Answer Type]

Note : Answer **all** from the following questions.

9. (a) (i) Prove that a set $\{v_1, v_2, ..., v_p\}$ of two or more vectors with $v_1 \neq o$, is linearly dependent if and only if some V_j (with $j > 1$) is a linear combination of the preceding vectors $v_1, v_2 ..., v_{j-1}$. **(Unit-I, Q.No.37)**

(ii) Show that the set $S = \{(1, 0, 0, -1)\ (0, 1, 0, -1)\ (0, 0, 1, -1)$ and $(0, 0, 0, 1)\}$ in R_4 is linearly independent. **(Unit-I, Q.No.68)**

OR

(b) (i) State and prove the spanning set theorem. **(Unit-I, Q.No.38)**

(ii) If a vector space V has a basis of n vectors then prove that every basis of V must consist of exactly n vectors. **(Unit-I, Q.No.58)**

10. (a) (i) Find the eigen values and eigen vectors of $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. **(Unit-II, Q.No.24)**

(ii) Prove that the eigen values of a triangular matrix and its diagonal elements. **(Unit-II, Q.No.19)**

(b) Define the term

(i) Rank of matrix

(ii) Eigen values and Eigen vectors of matrix **(Unit-II, Q.No.18)**

(iii) Find the characteristic equation if $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **(Unit-II, Q.No.31)**

11. (a) Diagonalize the matrix, if possible $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ **(Unit-III, Q.No.12)**

OR

(b) (i) Show that the set $\{u_1, u_2, u_3\}$ is an orthogonal set, where

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix},\ u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \text{ and } u_3 = \begin{bmatrix} -1/2 \\ 2 \\ 7/2 \end{bmatrix}$$ **(Unit-IV, Q.No.28)**

(ii) If $S = \{u_1, u_2, ... u_p\}$ is an orthogonal set of non-zero vectors is R^n, then prove that S is linearly independent. **(Unit-IV, Q.No.31)**