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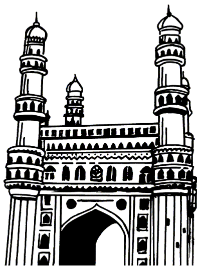

B.C.A.

II Year III Sem

Latest Edition

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Important Questions

UNIT - I

1. **If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 - \tan^{-1}\frac{x}{y}$; $xy \neq 0$. prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.**

Ans :

Refer Unit-I, Page No. 3, Q.No. 6.

---

2. **If $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$: $x^2 + y^2 + z \neq 0$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.**

Ans :

Refer Unit-I, Page No. 4, Q.No. 7.

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3. **If $x^x y^y z^z = C$ Show that $x = y = z$ $\frac{\partial^2 z}{\partial x \partial y} = -(x \log_e x)^{-1.}$**

Ans ;

Refer Unit-I, Page No. 6, Q.No. 9.

---

4. **State and prove Euler's theorem on homogeneous functions.**

Ans :

Refer Unit-I, Page No. 9, Q.No. 13.

---

5. **If $u = \tan^{-1}\left(\frac{x^3 + Y^3}{x - y}\right)$, $x \neq y$, then show that,**

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u.$

Ans :

Refer Unit-I, Page No. 11, Q.No. 14.

---

6. **Verify Euler's theorem for**

(i) $z = ax^2 + 2hxy + by^2$

(ii) $z = \sin\text{-}1\left(\frac{x}{y}\right) + \tan\text{-}1\left(\frac{y}{x}\right)$.

Ans :

Refer Unit-I, Page No. 14, Q.No. 16.

UNIT - II

1. State and prove Theorem on Total Differentials.State and prove Theorem on Total Differentials.

Ans :

Refer Unit-II, Page No. 23, Q.No. 1.

2. If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$, Then Show that $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{1}{\left(1+x^2+y^2\right)^{3/2}}$.

Ans :

Refer Unit-II, Page No. 31, Q.No. 10.

3. Write Working Rule to find the maximum or minimum value of f(x,y).

Ans :

Refer Unit-II, Page No. 34, Q.No. 14.

4. Discuss the maximum or minimum value of u, when $u = x^3 + y^3 - 3axy$.

Ans :

Refer Unit-II, Page No. 35, Q.No. 16.

5. Explain Lagrange's method of undermined multipliers.

Ans :

Refer Unit-II, Page No. 42, Q.No. 22.

6. Discuss the maxima and minima of the function u = sin x sin y sin z, where x, y, z are the angles of a triangle

Ans :

Refer Unit-II, Page No. 46, Q.No. 25.

UNIT - III

1. Explain the concept of Linear Equations in Linear Algebra.

Ans :

Refer Unit-III, Page No. 57, Q.No. 1.

2. What is system of linear equation and explain briefly ?

Ans :

Refer Unit-III, Page No. 58, Q.No. 3.

3. What is homogeneous system of linear equations and explain it cases ?

Ans :

Refer Unit-III, Page No. 59, Q.No. 4.

4. Find the system of Linear equation,

$$2x + 4y - 3z = 4$$

$$3y + 4x + 5z = 2$$

$$4z + 4x + 3y = 1$$

Ans :

Refer Unit-III, Page No. 65, Q.No. 12.

5. Find the system of line or equation

$$2x + y + z = 2$$

$$4x + y + = 6$$

$$9x + 2y + z = 2$$

Ans :

Refer Unit-III, Page No. 68, Q.No. 13.

6. What is linear independent and explain with example?

Ans :

Refer Unit-III, Page No. 78, Q.No. 25.

7. What is linear dependent and explain with example

Ans :

Refer Unit-III, Page No. 80, Q.No. 26.

8. What are the Characteristics of Linearly Dependent ?

Ans :

Refer Unit-III, Page No. 81, Q.No. 27.

9. Determining Linear Dependence Using the Determinant Method

$$2x + y - z = 5 \quad \ldots (1)$$

$$4x - 3y + 2z = 1 \quad \ldots (2)$$

$$x + 2y - z = 3 \quad \ldots (3)$$

Ans :

Refer Unit-III, Page No. 81, Q.No. 28.

UNIT - IV

1. The union of two subspaces is again subspaces $\Leftrightarrow H_1 \subseteq H_2$ (or) $H_2 \subseteq H_1$.

Ans :

Refer Unit-IV, Page No. 92, Q.No. 7.

2. Let H be the set of all vectors of the from (a – 3b, b – a, a, b) where a and b are arbitrary scalars let H = {(a – 3b, b – a, a, b)}; a, b in R. Show that H is a subspace of R^4.

Ans :

Refer Unit-IV, Page No. 93, Q.No. 8.

3. Show that w is in the subspace of R^4 spanned by v1, v2, v3 where,

$$w = \begin{bmatrix} -9 \\ 7 \\ 4 \\ 8 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 7 \\ -4 \\ -2 \\ 9 \end{bmatrix},$$

$$v_2 = \begin{bmatrix} -4 \\ 5 \\ -1 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -9 \\ 4 \\ 4 \\ -7 \end{bmatrix}$$

Ans :

Refer Unit-IV, Page No. 95, Q.No. 11.

4. Find the characteristic polynomial and the real eigen values of the matrix

$$A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$$

Ans :

Refer Unit-IV, Page No. 101, Q.No. 21.

5. Find the characteristic polynomial and the eigen values of the matrices

(i) $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$

Ans :

Refer Unit-IV, Page No. 105, Q.No. 23.

6. **Find the characteristic equation of the matrix A =** $\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Ans :

Refer Unit-IV, Page No. 109, Q.No. 30.

7. **Is l = 3 an eigenvalue of** $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ **if so find the one corresponding eigen vector.**

Ans :

Refer Unit-IV, Page No. 110, Q.No. 31.

UNIT - V

1. **State and prove the diagonalization theorem.**

Ans :

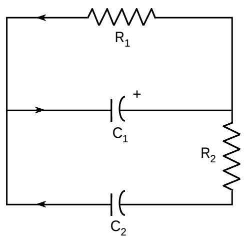
Refer Unit-V, Page No. 119, Q.No. 2.

2. **Diagonalize the matrix A =** $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ **if possible.**

Ans :

Refer Unit-V, Page No. 123, Q.No. 9.

3. **Find formulas for the voltages v_1 and v_2 (as functions of time t) for the circuit shown below, assuming that $R_1 = \frac{1}{5}$ Ohm, $R_2 = \frac{1}{3}$ ohm, C_1 = 4 farads, C_2 = 3 farads and the Initial charge on each capacitor is 4 volts.**



Ans :

Refer Unit-V, Page No. 125, Q.No. 10.

4. Construct the general solution of X' = AX involving complex eigen functions and then obtain the general real solution. Describe the shape of typical trajectories.

(i) $A = \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$

Ans :

Refer Unit-V, Page No. 130, Q.No. 12.

5. Make a change of variable that decouples the equation X' = AX write the equation X(t) = Py(t) and show the calculate that leads tot he uncoupled system Y' = DY, specifying P and D where $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$**.**

Ans :

Refer Unit-V, Page No. 133, Q.No. 13.

6. A particle moving in a planar force field has a position vector X that satisfies X^1 = AX. The 2 × 2 matrix A has eigen value 4 and 2 with corresponding eigen vectors $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ **and** $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$**. Find the position of the particle at time t, assuming that** $X(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$**.**

Ans :

Refer Unit-V, Page No. 136, Q.No. 14.

UNIT I

Partial Differentiation: Introduction - Functions of two variables - Neighborhood of a point (a, b) - Continuity of a Function of two variables, Continuity at a point - Limit of a Function of two variables - Partial Derivatives - Homogeneous Functions.

1.1 Partial Differentiation

1.1.1 Introduction

Q1. Define Partial Differentiation.

Ans :

The process of determining the partial derivatives of a function of more than one independent variables is knows as partial differentaiaion. It is denoted by symbols like $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial t}$ etc.

1.1.2 Functions of Two Variables

Q2. Explain Functions of Two Variables.

Ans :

A function which contains more than one variable is called function of several variables.

Function of Two Variables

A function which contains two variables is called function of two variables.

It is of the form,

$$z = f(x, y)$$

Example

$$z = x^2 + y^2$$

Function of Three Variables

A function which has three variables is called function of three variables.

It is of the form,

$$v = f(x, y, z)$$

Example

$$v = x^2 + y^2 + z^2$$

1.1.3 Neighborhood of a Point (a, b)

Q3. Write a short notes on Neighborhood of a Point (a, b)

Ans :

Let δ repredents a positive number i.e., $\delta > 0$.

The point (x, y) for which

$$a - \delta \le x \le a + \delta,$$

$$b - \delta \le y \le b + \delta$$

Determine a square which is bounded by the lines,

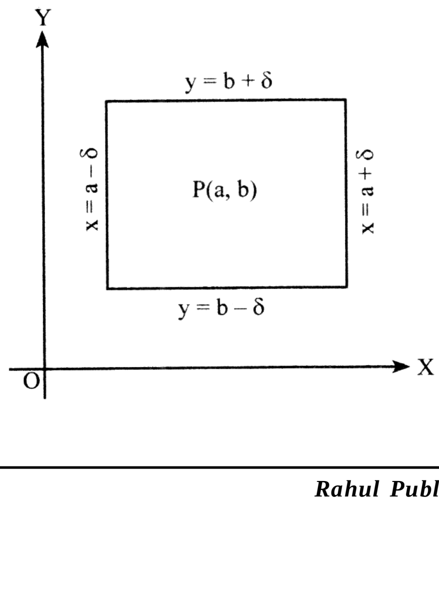
$$x = a - \delta$$

$$x = a + \delta$$

$$x = b - \delta$$

$$x = b + \delta$$

Its centre is located at the point (a, b) as shown in figure.



Neighbourhood of a Point (a, b)

From the figure, the square is termed as neighbourhood of the point (a, b)

Hence, the set

$\{(x, y) : a - \delta \leq x \leq a + \delta, b - \delta \leq y \leq b + \delta\}$

is called as the neighbourhood of the point (a, b)

1.1.4 Continuity of a Function of Two Variables, Continuity at a Point

Q4. Explain Continuity of a Function of Two Variables.

Ans :

A function f(x, y) is said to be continuous at (a, b) i.e., x = a, y = b, if a positive number δ exists such that,

$$|f(x,y) - f(a,b)| < \varepsilon \forall (x,y)$$

Where,

ε – Any preassigned positive number (small)

f(x, y) lies between f(a, b) – ε and f(a, b) + ε

If points lie on y = b then

f(x) = f(x, b)

$\therefore \quad |f(x,b) - f(a,b)| < \varepsilon$

$\Rightarrow \quad |f(x) - f(a,b)| < \varepsilon$

Hence continuous function of two variables is also a continuous function of each variable separately.

1.1.5 Limit of a Function of Two Variables

Q5. Explain Limit of a Function of two variables

Ans :

The limit of a function f(x, y) can be defind as '*l*' as x tends to a and y tends to b, such that for a small preassigned positived number ε there exists a corresponding positive number δ.

Where,

$|f(x,y) - l| < \varepsilon \forall (x,y)$ such that

$a - \delta \leq x \leq a + \delta ; b - \delta \leq y \leq b + \delta$

i.e., $\{(x, y): x \in [a - \delta, a + \delta], y \in [b - \delta, b + \varepsilon]\} \sim (a, b)$

f is continuous at (a, b), if

lim f(x, y) = f(a, b) as $(x, y) \to (a, b)$

Then for continuity at (a, b),

lim f(a, + h, b + k) = f(a, b) as $(h, k) \to (0, 0)$

6. If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 - \tan^{-1}\frac{x}{y}$; $xy \neq 0$. prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

Sol : **(Imp.)**

Given that $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

Partially Diff w.r.to y. Then we have

$$\frac{\partial u}{\partial y} = x^{\cancel{2}} \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{\cancel{x}} - 2y \tan^{-1}\frac{x}{y} + y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2} \frac{x}{y^2}$$

$$= \frac{x.x^2}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y} + \frac{y^4}{y^2+x^2}\frac{x}{y^2}$$

$$= \frac{x^3}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y} + \frac{xy^2}{y^2+x^2}$$

$$= \frac{x\left(x^2+y^2\right)}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y}$$

$$\frac{\partial u}{\partial y} = x - 2y \tan^{-1}\frac{x}{y}$$

Partially differentiate with respect to x.

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[x - 2y \tan^{-1}\frac{x}{y}\right]$$

$$= 1 - 2y \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}$$

$$= 1 - \frac{2y^2}{y^2+x^2}$$

$$= \frac{x^2+y^2-2y^2}{x^2+y^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$$

7. If $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$: $x^2 + y^2 + z \neq 0$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Sol : **(Imp.)**

$$u = \left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2} 2x$$

$$= -x\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left(-x\left(x^2 + y^2 + z^2\right)^{-3/2}\right)$$

$$= -\left(x^2 + y^2 + z^2\right)^{-1/3} + \frac{3}{\not{2}}x\left(x^2 + y^2 + z^2\right)^{-5/2}(\not{2}x)$$

$$= -\left(x^2 + y^2 + z^2\right)^{-1/3} + 3x^2\left(x^2 + y^2 + z^2\right)^{-5/2} \quad \text{....(1)}$$

$$\frac{\partial u}{\partial y} = \frac{-1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2} 2y$$

$$= -\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left[-y\left(x^2 + y^2 + z^2\right)^{-3/2}\right]$$

$$= -\left(x^2 + y^2 + z^2\right)^{-3/2} + \frac{3y}{2}\left(x^2 + y^2 + z^2\right)^{\frac{-3}{2}-1} 2y$$

$$= -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3y^2\left(x^2 + y^2 + z^2\right)^{-5/2} \quad \text{......(2)}$$

$$\frac{\partial u}{\partial z} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2} - 2z$$

$$= -z\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left[-z\left(x^2+y^2+z^2\right)^{-3/2}\right]$$

$$= \left(x^2+y^2+z^2\right)^{-3/2} + \frac{3}{2}\, z\left(x^2+z^2+y^2\right)^{-5/2}\, 2z$$

$$= -\left(x^2-y^2+z^2\right)^{-3/2} + 3z^2\left(x^2+y^2+z^2\right)^{-5/2} \quad .....(3)$$

Adding (1) & (2) & (3)

$$\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial y^2}+\frac{\partial^2 y}{\partial z^2} = -\left(x^2+y^2+z^2\right)^{-3/2} + 3x^2\left(x^2+y^2+z^2\right)^{-5/2} - \left(x^2+y^2+z^2\right)^{-3/2} + 3y^2\left(x^2+y^2+z^2\right)^{-5/2}$$

$$-\left(x^2+y^2+z^2\right)^{-3/2} + 3z^2\left(x^2+y^2+z^2\right)^{-5/2}$$

$$= -3\left(x^2+y^2+z^2\right)^{-3/2} + 3(x^2+y^2+z^2)\left(x^2+y^2+z^2\right)^{-5/2}$$

$$= -3\left(x^2+y^2+z^2\right)^{-3/2} + 3\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\therefore \frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial y^2}+\frac{\partial^2 y}{\partial z^2} = 0$$

8. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$

Sol :

We have $u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\,(3x^2 - 3yz)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}\,3y^2 - 3xz$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}\,3z^2 - 3zy$$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{x^3+y^3+z^3-3xyz}$$

$$= 3(x^2+y^2+z^2-xy-yz-zx)$$

$$= \frac{3}{x+y+z}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$\therefore \quad \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 = -\frac{9}{(x+y+z)^2}$$

---

9. If $x^x y^y z^z = C$ Show that $x = y = z$ $\frac{\partial^2 z}{\partial x \partial y} = -(x \log_e x)^{-1}$.

Sol : **(Imp.)**

Given that $x^x\ y^y\ z^z = C$

Taking Log on both sides

We have, $\log (x^x y^y z^z) – \log C$

$\log x^x + \log y^y + \log z^z = \log C$

$x \log x + y \log y + z \log z = \log C$

Differentiate partially with respect to x weget

$$x.\frac{1}{x} + 1.\log x + (z.\frac{1}{z} + 1.\log z)\frac{\partial z}{\partial x} = 0$$

$$1 + \log x + (1 + \log z)\frac{\partial z}{\partial x} = 0$$

$$(1 + \log z)\frac{\partial z}{\partial x} = -[1 + \log x]$$

$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}$$

Differentiate partially with respect to 'y' we get

$$y\cdot\frac{1}{y} + 1.\log y + z.\frac{1}{z} + 1.\log z\ \frac{\partial z}{\partial y} = 0$$

$$1 + \log y + (1 + \log z)\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log z}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{1+\log x}{(1+\log z)^2} \cdot \frac{1}{z}\frac{\partial z}{\partial y}$$

$$= -\frac{(1+\log x)}{(1+\log z)^2} \cdot \frac{1}{z} \cdot \frac{(1+\log y)}{(1+\log z)}$$

$$= -\frac{(1+\log x)^2}{(1+\log x)^3} \cdot \frac{1}{x} \qquad \text{Since } x = y = z$$

$$= -\frac{1}{x(1+\log x)}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x \log ex} = -(x \log ex)^{-1}$$

10. If u = 3(lx + my + nz)^2 – (x^2 + y^2 + z^2) & l^2 + m^2 + n^2 = 1

Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Sol :

$$u = 3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$$

$$\frac{\partial u}{\partial x} = 6\,(lx + my + nz)\,l - 2x$$

$$\frac{\partial^2 u}{\partial x^2} = 6\,l\,.\,l - 2$$

$$= 6\,l^2 - 2$$

$$\frac{\partial u}{\partial y} = 6(lx + my + nz)\,m - 2y$$

$$= 6m\,(lx + my + nz) - 2y$$

$$\frac{\partial u^2}{\partial y^2} = 6m.m - 2$$

$$= 6m^2 - 2$$

$$\frac{\partial u}{\partial z} = 6(lx + my + nz)\, n - 2z$$

$$= 6n\,(lx + my + nz) - 2z$$

$$\frac{\partial^2 u}{\partial z^2} = 6n^2 - 2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6l^2 - 2 + 6m^2 - 2 + 6n^2 - 2$$

$$= 6l^2 + 6m^2 + 6n^2 - 6$$

$$= 6\,[l^2 + m^2 + n^2] - 6$$

$$= 6(1) - 6$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

1.2 HOMOGENEOUS FUNCTIONS

Q11. Define Homogenous function with example.

Ans :

If the sum of indices of different variables contained in each term of an algebraic expression be n, it is called a homogenous function of degree n.

Let u = f(x, y) be a function of x and y. If this sum of the power of x and y in each term of f(x, y) be equal, then f(x, y) is called homogenous function.

Eg. $x^2 + y^3 + 3x^2 y$ is homogenous functio of order 3.

→ $x^4 + y^4 + 4x^2 y^2$ is homogenous function of 4th orders.

Consider the function

$f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + ... + a_{n-1} xy^{n-1} + a_n y^n$ ....(1)

We see that the expression f(x,y) is polynomial in (x,y) such that the degree of each of the terms is the same f is called a homogenous function of degree n.

- An expression in (x,y) is homogenous of degree n, if it is expressible as

  $x^n f(y/x)$

- The polynomial function (1) which can be rewritten as

$$x^n \left[a_0 + a_1 \frac{y}{x} + a_2 \left(\frac{y}{x}\right)^2 + ... + a_n \left(\frac{y}{x}\right)^n \right]$$

is a homogenous expression of order n

12. Find the degree of given Homogenous function for f(x,y) = x^n sin (y/x).

Sol :

Given that

$f(x,y) = x^n \sin(y/x)$

$f(x,y) = \left(\sqrt{y}+\sqrt{x}\right)/(y+x)$

The degree of the expression $x^n \sin(y/x)$ is n

The degree of the expression $\Rightarrow \dfrac{\sqrt{y}+\sqrt{x}}{y+x} - \dfrac{\sqrt{x}\left[1+\dfrac{\sqrt{y}}{\sqrt{x}}\right]}{x\left[1+\dfrac{y}{x}\right]}$

$$= x^{1/2-1}\frac{\left[1+\dfrac{\sqrt{y}}{\sqrt{x}}\right]}{1+y/x}$$

$$= x^{-1/2}\frac{1+\sqrt{y}/\sqrt{x}}{1+y/z}$$

So, that it is of degree $\dfrac{-1}{2}$

---

13. State and prove Euler's theorem for a homogeneous functions.

(OR)

State and prove Euler's theorem on homogeneous functions.

(OR)

If Z = f(x, y) be a homogeneous function of x, y of degree n then show that,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nz \ \forall \ x, y \in \text{ the domain of the function.}$$

Sol : **(Imp.)**

Statement

If z = f(x, y) is a homogeneous function in x and y of degree n then,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \ \forall \ x, y \in \text{ domain of the function.}$$

Proof

Let, z = f(x, y) be a homogeneous function in x and y of degree n. Then it can be expressed as,

$$z = x^n f\left[\frac{y}{x}\right] \quad .....(1)$$

Partially differentiating equation (1) with respect to x,

$$\frac{\partial z}{\partial x} = x^n \frac{\partial}{\partial x}\left[f\left(\frac{y}{x}\right)\right] + f\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left[x^n\right]$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = x^n f'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right)nx^{n-1}$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = x^n f'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right)nx^{n-1}$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) - yx^{n-2}f'\left(\frac{y}{x}\right) \qquad ....(2)$$

Multiplying equation (2) by x on both sides,

$$x\frac{\partial z}{\partial x} = n.x.x^{n-1} f\left(\frac{y}{x}\right) - yx.x^{n-2}f'\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad x\frac{\partial z}{\partial x} = nx^n f\left(\frac{y}{x}\right) - yx^{n-1}f'\left(\frac{y}{x}\right) \qquad ....(3)$$

Partially differentiating equation (1) with respect to y,

$$\frac{\partial z}{\partial y} = x^n \frac{\partial}{\partial y}\left[f\left(\frac{y}{x}\right)\right]$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = x^n f'\left(\frac{y}{x}\right).\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = x^{n-1} f'\left(\frac{y}{x}\right) \qquad ....(4)$$

Multiplying by y on both sides of equation (4),

$$y\frac{\partial z}{\partial y} = yx^{n-1} f'\left(\frac{y}{x}\right)$$

Adding equations (3) and (5),

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^{n-1} f'\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right) + yx^{n-1} f'\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n f'\left(\frac{y}{x}\right)$$

$\Rightarrow \quad x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz \qquad [\because$ From equation (1)]

$\therefore \quad x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$

14. If u = $\tan^{-1}\left(\frac{x^3+Y^3}{x-y}\right)$, x ≠ y, then show that,

(i) $x\frac{\partial u}{\partial x}$ **+** $y\frac{\partial u}{\partial y}$ **= sin 2u**

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$ **= (1 – 4 sin² u) sin2u.**

Sol : **(Imp.)**

Given that,

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right) \text{ or } \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right) \quad x \neq y$$

$$\Rightarrow \quad \tan u = \left(\frac{x^3+y^3}{x-y}\right)$$

Let, tan u = f where, $f = \left(\frac{x^3+y^3}{x-y}\right)$

The degree of this homogeneous function is obtained as,

$$f(kx, ky) = \frac{k^3x^3+k^3y^3}{kx-ky}=\frac{k^3(x^3+y^3)}{k(x-y)}=k^{3-1}\left(\frac{x^3+y^3}{x-y}\right)=k^2\left(\frac{x^3+y^3}{x-y}\right) = k^2(f)$$

Degree is 2.

(i) According to Euler's theorem,

$$x\,.\,\frac{\partial f}{\partial x} + y\,.\,\frac{\partial f}{\partial y} = 2f$$

Since,

$$f = \tan u$$

$$\Rightarrow \quad \frac{\partial f}{\partial u} = \sec^2 u$$

Condier

$$\frac{\partial f}{\partial x},$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial y}$$

$$\frac{\partial f}{\partial y} = \sec 2u.\frac{\partial u}{\partial y}$$

Substituting equations (2) and (3) in equation (1),

$$x\left(\sec 2u.\frac{\partial u}{\partial x}\right) + y\left(\sec 2u.\frac{\partial u}{\partial y}\right) = 2(\tan u)$$

$$\sec 2u.\left(x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2 \text{ tna } u$$

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 2 \tan u . \frac{1}{\sec^2 u}$$

$$= \frac{2\sin u}{\cos u} . \cos 2u = 2 \sin u. \cos u = \sin 2u.$$

$$\therefore \quad x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \sin 2u$$

(ii) Since, $g(u) = n\frac{f(u)}{f'(u)} = \frac{2\tan u}{\sec^2 u}$

$g(u) = \sin 2u$

And $g'(u) - 1 = 2 \cos 2u - 1$

From Euler's theorem,

$$x2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$= g(u)(g'(u) - 1)$$

$$= \sin 2u(2 \cos 2u - 1) = 2 \sin 2u.\cos 2u - \sin 2u$$

$$= \sin 2(2u) - \sin 2u = \sin 4u - \sin 2u$$

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy.\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u = \sin 4u - \sin 2u$$

$$= \sin 2u\,(2(\cos^2 \mu - \sin^2 \mu) - 1)$$

$$= \sin 2u\,(2\cos^2 \mu - \sin^2 \mu - 1)$$

$$= \sin 2u\,(2 - 2\sin^2 \mu - 2\sin^2 \mu - 1)$$

$$= \sin 2u(1 - 4\sin^2 \mu)$$

$$\therefore \quad \frac{x^2\partial^2 \mu}{\partial x^2} + 2xy\frac{\partial^2 \mu}{\partial x \partial y} + y^2\frac{\partial^2 \mu}{\partial y^2} = \sin 2\mu\,(1 - 4 \sin^2 \mu).$$

15. If u = log $\left\{\frac{x^4+y^4}{x+y}\right\}$, show by Euler's theorem that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ = 3.

Sol :

Given,

$$u = \log\left(\frac{x^4+y^4}{x+y}\right)$$

$$\Rightarrow \quad e^u = \frac{x^4+y^4}{x+y}$$

Let, $z = \frac{x^4+y^4}{x+y}$

Degree of $z = \frac{(kx)^4+(ky)^4}{kx+ky} = \frac{k^4}{k}\left(\frac{x^4=y^4}{x+y}\right) = k^3\left(\frac{x^4+y^4}{x+y}\right) = k^3 e^u$

z is a homogeneous function of degree 3

∴ From Euler's theorem,

$$x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=3z \quad ....(1)$$

Here,

$z = e^n$

$$\Rightarrow \quad \frac{\partial z}{\partial u} = e^u$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} = eu.\frac{\partial u}{\partial x}$$

$$\therefore \quad \frac{\partial z}{\partial x} = e^u.\frac{\partial z}{\partial x} \quad ....(2)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial z}{\partial y} = eu.\frac{\partial z}{\partial y}$$

$$\therefore \quad \frac{\partial z}{\partial y} = e^u.\frac{\partial u}{\partial y} \quad ....(3)$$

Substituting equations (2) and (3) in equation(1),

$$x\left(e^u.\frac{\partial u}{\partial x}\right) + y\left(e^u\frac{\partial u}{\partial y}\right) = 3\,(e^u)$$

$$\Rightarrow \quad e^u\left(x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}\right) = 3e^u$$

$$\Rightarrow \quad x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 3$$

$$\therefore \quad x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 3$$

16. Verify Euler's theorem for

(i) $z = ax^2 + 2hxy + by^2$

(ii) $z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$.

Sol : **(Imp.)**

(i) $z = ax^2 + 2hxy + by^2$

Given function is,

$$z = ax^2 + 2hxy + by^2$$

Let,

$$z = f(x, y)$$
$$= ax^2 + 2hxy + by^2$$

Degree of homogeneous function is obtained as,

$$f(kx, ky) = a(kx)^2 + 2h(kx)(ky) + b(ky)$$
$$= ak^2x^2 + 2hk^2xy + bk^2y^2$$
$$= k^2(ax^2 + 2hxy + by^2)$$
$$= k^2(f)$$

$\therefore$ Degree is 2.

According to Euler's theorem,

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 2z$$

Consider,

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y}$$

$$= x.\frac{\partial u}{\partial x}(ax^2 + 2hxy + by^2) + y.\frac{\partial}{\partial y}(ax^2 + 2hxy + by^2)$$

$$= x(2ax + 2hy + 0) + y(0 + 2hx + 2by)$$

$$= x(2ax + 2hy) + y(2hx + 2by)$$

$$= 2(ax^2 + hxy) + 2(hxy + by^2)$$

$$= 2[ax^2 + hxy + hxy + by^2]$$

$$= 2[ax^2 + 2hxy + by^2]$$

$$= 2z$$

$\therefore \quad x.\dfrac{\partial z}{\partial x} + y.\dfrac{\partial z}{\partial y} = 2z$

$\therefore$ From equations (1) and (2), Euler's theorem is verified.

(ii) $z = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y}{x}\right)$

Given function is,

$$z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

Degree of homogeneous function is obtained as,

$$f(kx, ky) = \sin^{-1}\left(\frac{kx}{ky}\right) + \tan^{-1}\left(\frac{ky}{kx}\right)$$

$$= \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$= k^0(f)$$

$\therefore$ Degree is 0.

According to Euler's theorem,

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 0.z \qquad .....(1)$$

Consider,

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = x.\frac{\partial}{\partial x}\left[\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)\right]+y.\frac{\partial}{\partial y}\left[\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$= x\left[\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left[\frac{y-0}{y^2}\right]+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left[\frac{0-y}{x^2}\right]\right] + y\left[\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left[\frac{0-x}{y^2}\right]+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left[\frac{x-0}{x^2}\right]\right]$$

$$= \frac{x}{y\sqrt{1-\left(\frac{x}{y}\right)^2}} - + \frac{y}{x\left(1+\left(\frac{y}{x}\right)^2\right)} - \frac{x}{y\sqrt{1-\left(\frac{x}{y}\right)^2}} + \frac{y}{x\left(1+\left(\frac{y}{x}\right)^2\right)}$$

$= 0$

$$\therefore \quad x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = 0 \qquad ....(2)$$

$\therefore$ From equations (1) and (2), Euler's theorem is verified.

Short Question and Answers

1. Define Partial Differentiation.

Ans :

The process of determining the partial derivatives of a function of more than one independent variables is knows as partial differentaiaion. It is denoted by symbols like $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}$ etc.

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2. Functions of Two Variables.

Ans :

A function which contains more than one variable is called function of several variables.

Function of Two Variables

A function which contains two variables is called function of two variables.

It is of the form,

$z = f(x, y)$

Example

$z = x^2 + y^2$

---

3. Limit of a Function of two variables

Ans :

The limit of a function f(x, y) can be defind as '*l*' as x tends to a and y tends to b, such that for a small preassigned positived number ε there exists a corresponding positive number δ.

Where,

$|f(x,y) - l| < \varepsilon \forall (x,y)$ such that

$a - \delta \leq x \leq a + \delta$; $b - \delta \leq y \leq b + \delta$

i.e., $\{(x, y): x \in [a - \delta, a + \delta], y \in [b - \delta, b + \varepsilon]\} \sim (a, b)$

f is continuous at (a, b), if

$\lim f(x, y) = f(a, b)$ as $(x, y) \to (a, b)$

Then for continuity at (a, b),

$\lim f(a, + h, b + k) = f(a, b)$ as $(h, k) \to (0, 0)$.

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4. Geometrical Interpretation of Partial Derivatives

Ans :

Assume that z = f(x, y) is a function of two variables which represents a surface in three-dimensional space. Compute the partial derivative z_x and z_y. Evaluate these partial derivatives at the point (a, b) then

z_x is the slope (measured along the x-axis) of line L_1, which is tangent to the surface at the point (a, b, f(a, b)), and z_y is the slope (measured along the y-axis) of the L_2, which is a tangent to the tangent to the surface at the point (a, b, f(a, b)) line L_1 lies in the plane y = b, line L_2 lies in the plane x = a.

5. Homogenous function with example.

Sol :

If the sum of indices of different variables contained in each term of an algebraic expression be n, it is called a homogenous function of degree n.

Let u = f(x, y) be a function of x and y. If this sum of the power of x and y in each term of f(x, y) be equal, then f(x, y) is called homogenous function.

Eg. $x^2 + y^3 + 3x^2 y$ is homogenous functio of order 3.

$\rightarrow x^4 + y^4 + 4x^2 y^2$ is homogenous function of 4th orders.

6. Euler's theorem

Sol :

If z = f(x, y) is a homogeneous function in x and y of degree n then,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \;\forall\; x, y \in \text{domain of the function.}$$

7. Define Partial derivatives.

Sol :

Consider z = f(x,y) then

$$\lim \frac{f(a+h+b) - f(a,b)}{h}; \; h \to 0$$

If it exists, is said to be the partial derivatives of f w.r. to x at (a,b) and is denoted by

$$\left(\frac{\partial z}{\partial x}\right)_{(a,b)} \text{ or } f_x(a,b)$$

$$\Rightarrow \lim \frac{f(a,b+k) - f(a,b)}{k} \text{ as } k \to 0$$

If it exists is called the partial derivative of f(x,y) w.r. to y at (a,b) & is denoted or f_y (a,b) by $\left(\partial z/\partial y\right)_{(a,b)}$

Partial derivative of Higher order

From the above first order partial derivatives, we form the partial derivative of higher order.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{yx}$$

$$\frac{\partial z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{xy}$$

8. If $z = \tan^{-1}\left(\frac{y}{x}\right)$ verify that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

Sol :

$z = \tan^{-1}(y/n)$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(y/x\right)^2}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$= -\frac{x^2}{x^2+y^2}\frac{y}{x^2}$$

$$= -\frac{y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{\left(x^2+y^2\right)0 + y(2x)}{\left(x^2+y^2\right)^2} = \frac{2xy}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\left(y/x\right)^2}\ \frac{1}{x}$$

$$= \frac{x^2}{x^2+y^2}\ \frac{1}{x} \Rightarrow \frac{x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \Rightarrow \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$\Rightarrow \frac{\left(x^2+y^2\right)0 - x(2y)}{\left(x^2+y^2\right)^2}$$

$$= \frac{-2xy}{\left(x^2+y^2\right)}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

9. Find the degree of given Homogenous function for f(x,y) = x^n sin (y/x).

Sol :

Given that

$f(x,y) = x^n \sin(y/x)$

$f(x,y) = \left(\sqrt{y}+\sqrt{x}\right)/(y+x)$

The degree of the expression $x^n \sin(y/x)$ is n

The degree of the expression

$$\Rightarrow \frac{\sqrt{y}+\sqrt{x}}{y+x} - \frac{\sqrt{x}\left[1+\frac{\sqrt{y}}{\sqrt{x}}\right]}{x\left[1+\frac{y}{x}\right]}$$

$$= x^{1/2-1}\frac{\left[1+\frac{\sqrt{y}}{\sqrt{x}}\right]}{1+y/x}$$

$$= x^{-1/2}\frac{1+\sqrt{y}/\sqrt{x}}{1+y/z}$$

So, that it is of degree $\frac{-1}{2}$

10. Write a short notes on Neighborhood of a Point (a, b)

Ans :

Let δ repredents a positive number i.e., $\delta > 0$.

The point (x, y) for which

$a - \delta \le x \le a + \delta$,

$b - \delta \le y \le b + \delta$

Determine a square which is bounded by the lines,

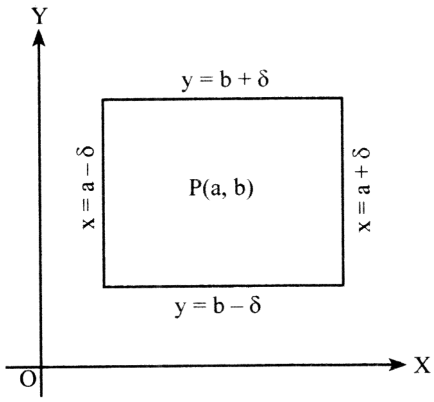
$x = a - \delta$

$x = a + \delta$

$x = b - \delta$

$x = b + \delta$

Its centre is located at the point (a, b) as shown in figure.



Neighbourhood of a Point (a, b)

From the figure, the square is termed as neighbourhood of the point (a, b)

Hence, the set

$\{(x, y) : a - \delta \leq x \leq a + \delta, b - \delta \leq y \leq b + \delta\}$

is called as the neighbourhood of the point (a, b).

Choose the Correct Answer

1. If z = xyf (x/y) then $x\frac{\partial z}{\partial x} + y\frac{\partial u}{\partial y} =$ [c]

   (a) 0 (b) $\frac{1}{z}$

   (c) 2z (d) z

2. If $\sin^{-1}\left(\frac{x-y}{\sqrt{x}+\sqrt{y}}\right)$ then the degree of homogenous function is [d]

   (a) 0 (b) $\frac{-1}{2}$

   (c) 2 (d) $\frac{1}{2}$

3. If z = f(y/x) then $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right)$ is [d]

   (a) 1 (b) 2

   (c) -2 (d) 0

4. If $f = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is [d]

   (a) f (b) 2f

   (c) sinf (d) tanx

5. If x = r cos ϕ, y = r sin then $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2$ is [a]

   (a) 1 (b) r

   (c) –r (d) –1

6. If f(x,y) is homogenous function of x and y of degree n. then [a]

   (a) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$ (b) $y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = nf$

   (c) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n$ (d) None

7. If $z = \log(x^2 + y^2)$ then $x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ is [a]

(a) 2 (b) 1

(c) 3 (d) 4

8. If $u = y^x$ then $\frac{\partial u}{\partial x}$ is [b]

(a) xy^x (b) $y^x \log y$

(c) $y^x \log x - 1$ (d) yx^{y-1}

9. If u is a homogenous function of x and y of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is [c]

(a) n (b) n – 1

(c) nu (d) n (n – 1) u

10. If u be a homogenous function of degree n then $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} =$ [b]

(a) $n\frac{\partial u}{\partial x}$ (b) $(n-1)\frac{\partial u}{\partial x}$

(c) $(n+1)\frac{\partial u}{\partial x}$ (d) None

Fill in the Blanks

1. $\frac{dy}{dx}$ = ________
2. $u = e^{x-y}$ then $\frac{\partial^2 u}{\partial x^2}$ = ________
3. First order partial derivative of $\tan^{-1}(x + y)$ = ________
4. If $z = \tan^{-1}(y/x)$ then $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$ = ________
5. $x^2 \frac{\partial z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$ = ________
6. If $u = \sin^{-1}x$ then $\frac{\partial u}{\partial x}$ = ________
7. The domain of the function $f(x,y) = \log(x+y)$ is ________
8. Second under derivation of e^{x+y} = ________
9. If $z = \frac{x^3 + y^3}{x + y}$ then the degree of the function is ________
10. $z = \cos xy$ then $\frac{\partial z}{\partial x}$ = ________

ANSWERS

1. $\frac{-f_x}{f_y}$
2. e^{x-y}
3. $\frac{1}{1+(x+y)^2}$
4. 0
5. $n(n-1)z$
6. $\frac{-1}{\sqrt{1+x^2}}$
7. $\{(x,y) : 0 < x\ x + y\}$
8. e^{x+y}
9. 2
10. $-y \sin x$

UNIT II	Theorem on Total Differentials - Composite Functions - Differentiation of Composite Functions - Implicit Functions - Maxima and Minima of functions of two variables – Lagrange's Method of undetermined multipliers.

2.1 Theorem on Total Differentials

Q1. State and prove Theorem on Total Differentials.

Ans : **(Imp.)**

Statement

Let z = f(x, y) be the function, then total differential dz is given as, $dz = \frac{\partial z}{\partial x}.dx + \frac{\partial z}{\partial y}.dy$

Proof

Given function is,

$z = f(x, y)$ is function (1)

Let (x, y) and $(x + \Delta x, y + \Delta y)$ be any two points

And Δx, Δy be the changes in the independent variables x and y.

Also Δz be the consequent change in z.

Then,

$z + \Delta z = f(x + \Delta x, y + \Delta y)$

$\Rightarrow \Delta z = f(x + \Delta x, y + \Delta y) - z$

$\Rightarrow \Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ [$\because$ From equation (1)]

Adding and subtracting $f(x + \Delta x, y)$

$\Delta z = [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)] + [f(x + \Delta x, y) - f(x, y)]$ ... (2)

Let $f(y) = f(x + \Delta x, y)$

From Lagrange's mean value theorem,

$f(a + h) - f(a) = hf'(a + \theta h)$

$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = \Delta y f_y(x + \Delta x, y + \theta_1 \Delta y)$

Let, $f_y(x + \Delta x, y + \theta_1 \Delta y) - f_y(x, y) = \varepsilon_2$ ... (3)

Whre

ε_2 depends on Δx and Δy

As $\Delta x, \Delta y \to 0$, $f_y(x, y) \to 0$

And, let f(x) = f(x, y) where y is constant

From Lagrange's mean value theorem,

$$f(x + \Delta x, y) - f(x, y) = \Delta x f_x (x + \theta_2 \Delta x, y)$$

Let, $f_x(x + \theta_2 \Delta x, y) - f_x (x, y) = \varepsilon_1$ ... (4)

Where

ε_1 depends on Δx

As $\Delta x \to 0$, $f_x(x, y) \to \theta$

Substituting equation (3) and (4) in equation (2),

$$\Delta z = \Delta y\, f(x + \Delta x, y + \theta_1 \Delta y) + \Delta x f_x (x + \theta_2 \Delta x, y) = \Delta y [\varepsilon_2 + f_y (x, y)] + \Delta x [\varepsilon_1 + f_y (x, y)]$$

$$= \Delta y\, \varepsilon_2 + \Delta y f_y (x, y) + \Delta x \varepsilon_1 + \Delta x f_x(x, y) = \Delta x f_x (x, y) + \Delta y f_y (x, y) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Since z = f(x, y)

$$\Rightarrow \quad \frac{\partial z}{\partial x} = f_x(x, y), \quad \frac{\partial z}{\partial y} = f_y(x, y)$$

$$\therefore \quad \Delta z = \Delta x \, , \frac{\partial z}{\partial x} + \Delta y \, . \frac{\partial z}{\partial y} + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Here

$\Delta z = \Delta x \, . \frac{\partial z}{\partial x} + \Delta y \, . \frac{\partial z}{\partial y}$ is called the differential of z, denoted by dz

i.e., $dz = \frac{\partial z}{\partial x} \, . \Delta x + \frac{\partial z}{\partial x} \, . \Delta y$ ... (5)

Consider,

z = x

Differentiating with respect to 'x'

dz = dx

$\Rightarrow$ $1 \, . \Delta x = dx$ [$\because$ The differential dx and dy are actual changes Δx and Δy]

$\Rightarrow$ $dx = \Delta x$

Similarly,

z = y

Differentiating with respect to 'y'.

dz = dy

$\Rightarrow$ $1 \, . \Delta y = dy$

$\Rightarrow$ $dy = \Delta y$

From equation (5),

$$dz = \frac{\partial z}{\partial x} \, . dx + \frac{\partial z}{\partial x} \, . dy$$

2.1.1 Composite Functions

Q2. Explain about Composite Functions

Ans :

Statement

Let $z = f(x, y)$ possess continuous partial derivatives and let $x = \phi(t)$, $y = \Psi(t)$ possess continuous derivatives.

Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Proof :

Given,

$z = f(x, y)$ has partial derivatives

And $x = \phi(t)$, $y = \Psi(t)$ have continuous derivatives.

Let, t and $t + \Delta t$ be any two values

and $\Delta x, \Delta y, \Delta z$ be the changes in x, y, z respectively, consequent to the change Δt in t.

i.e., $x + \Delta x = \phi(t + \Delta t)$

$y = \Delta y = \Psi(t + \Delta t)$

Then

$z = \Delta z = f(x + \Delta x, y + \Delta y)$

$\Rightarrow \Delta z = f(x + \Delta x, y + \Delta y) - z$

$\Rightarrow \Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ $[\because z = f(x, y)]$

$\Rightarrow \Delta z = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]$

Applying lagrange's mean value theorem,

$\Delta z = \Delta x\, f_x(x + \theta_1 \Delta x, y + \Delta y) + \Delta y f_y (x, y + \theta_2 \Delta y), 0 < \theta_1, \theta_2 < 1$

$$\Rightarrow \frac{\Delta z}{\Delta t} = \frac{\Delta x}{\Delta t} f_x(x + \theta_1 \Delta x, y + \Delta y) + \frac{\Delta y}{\Delta t} f_y (x, y + \theta_2 \Delta y) \quad \text{... (1)}$$

Let $\Delta t \to 0$

$\Rightarrow \Delta x \to 0, \Delta y \to 0$

Since partial derivatives are continuous

$$\lim_{(\Delta x, \Delta y) \to (0, 0)} f_y(x + \theta_1 \Delta x, y + \Delta y) = f_z(x, y)$$

$$= \frac{\partial z}{\partial x} \quad \text{... (2)}$$

And $\lim_{\Delta y \to 0} f_y(x, y + \theta_2 \Delta y) = f_y(x, y)$

$$= \frac{\partial z}{\partial y} \quad \text{... (2)}$$

Substituting equations (2) and (3) in equation (1),

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

2.1.2 Differentiation of Composite Functions

Q3. State and prove theorem of differentiation Composite Function.

Ans :

Statement

Let = f(x, y) possess continuous partial derivatives and let $x = \phi(t)$, $y = \psi$ (t) Possess continuous derivatives.

Then

$$\frac{dz}{dt} = \frac{\partial^2}{\partial x}, \frac{dx}{dt} + \frac{\partial^2}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Proof :

Given,

z = f(x, y) has partial derivatives

And x = x = $\phi(t)$, $y = \psi$ (t) have continuous derivatives.

Let, t and t + Δt be any two values

and Δx, Δy, Δz be the changes in x, y, z respectively, consequent to the change Δt in t.

i.e., $x + \Delta x = \phi(t + \Delta t)$

$y + \Delta y = \psi(t + \Delta t)$

Then,

$z + \Delta z = f(x + \Delta x, y + \Delta y)$

$\Rightarrow \quad \Delta z = f(x + \Delta x, y + \Delta y) - z$

$\Rightarrow \quad \Delta z = f(x + \Delta x, y + \Delta y) - f(x,y) \qquad [\because z = f(x, y)]$

$\Rightarrow \quad \Delta z = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]$

Applying lagrange's mean value theorem,

$$\Delta z = \Delta x\, f_x(x + \theta_1 \Delta x, y + \Delta y) + \Delta y\, f_y(x, y + \theta_2 \Delta y),\ 0 < \theta_1, \theta_2 < 1$$

$$\Rightarrow \quad \frac{\Delta z}{\Delta t} = \frac{\Delta x}{\Delta t} f_x(x + \theta_1 \Delta x, y + \Delta y) + \frac{\Delta y}{\Delta t} f_y(x, y + \theta_2 \Delta y) \qquad ....(1)$$

Let $\Delta t \to 0$

$\Rightarrow \Delta x \to 0, \Delta y \to 0$

Since partial derivatives are continuous

$$\lim_{(\Delta x,\ \Delta y) \to (0,\ 0)} f_x(x + \theta_1 \Delta x, y + \Delta y) = f_x(x, y)$$

$$= \frac{\partial z}{\partial x} \quad \text{.... (2)}$$

And $\lim_{\Delta y \to 0} f_y(x, y + \theta_2 \Delta y) = f_y(x, y)$

$$= \frac{\partial z}{\partial y} \quad \text{.... (3)}$$

Substituting equations (2) and (3) in equation (1),

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

2.1.3 Implicit Functions

Q4. Define Implicit Function

Ans :

Let f be a function of two variables since $f(x, y) = 0$ ... (1)

We can obtain y as function of x, the equation (1) defines y as an implicit function of x.

Assuming that the conditions under which the equation (1) defines y as a derivable function of x are satisfied.

We shall Now obtain the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terns of the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y^2}$ of 'f'

with respect to x & y

Then

$$\frac{dy}{dx} = \frac{-\partial f / \partial x}{\partial f / \partial y} = \frac{-fx}{fy} \text{ if } fy \neq 0$$

$$\frac{\partial^2 y}{dx^2} = \frac{f_{x^2}\left(f_y\right)^2 - 2f_{yx}f_xf_y + f_{y^2}\left(f_x\right)^2}{\left(f_y\right)^3}$$

Q5. Prove that if $y^3 - 3ax^2 + x^3 = 0$ then $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$

Sol :

$$y^3 - 3ax^2 + x^3 = 0$$

$$y^3 = 3ax^2 - x^3$$

Differentiate with respect to 'x'

$$3y^2\frac{dy}{dx} = y^3 = 3ax^2 - x^3$$

Different with respect to 'x'

$$3y^2\frac{dy}{dx} = 6ax - 3x^2$$

$$\frac{dy}{dx} = \frac{6ax - 3x^2}{3y^2}$$

$$\frac{dy}{dx} = \frac{2ax - x^2}{y^2}$$

Again, different w.r. to x

$$\frac{d^2y}{dx^2} = \frac{y^2(2a-2x) - 2y\left(2ax - x^2\right)\frac{dy}{dx}}{\left(y^2\right)^2}$$

$$= \frac{(2a-2x)y^2 - 2y\left(2ax-x^2\right)\left[\frac{2ax-x^2}{y^2}\right]}{y^4}$$

$$= \frac{2(a-x)y^3 - 2\left(2ax-x^2\right)^2}{y^4 \cdot y}$$

$$= \frac{2(a-x)y^3 - 2\left(2ax-x^2\right)^2}{y^5}$$

$$= \frac{2(a-x)\left(3ax^2-x^3\right) - 2\left(2ax-x^2\right)^2}{y^5}$$

$$\frac{d^2y}{dx^2} = \frac{-2a^2x^2}{y^5}$$

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$$

Q6. If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$ find $\frac{\partial u}{\partial r}$

Sol :

Given

$$u = x^2 - y^2, \quad x = 2r - 3s + 4, \quad y = -r + 8s - 5$$

Different with respect to 'r'

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

Consider $x = 2r - 3s - 4$; $y = -r + 8s - 5$

$$\frac{\partial x}{\partial r} = 2 ; \frac{\partial y}{\partial r} = -1$$

Consider $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial u}{\partial r} = 2x(2) + (-2y)(-1)$$

$$\frac{\partial u}{\partial r} = 4x + 2y$$

Q7. If z = (cosy)/x and x = u² – v, y = eᵛ. Find $\frac{\partial z}{\partial v}$.

Sol :

Given

$$z = \frac{\cos y}{x}, \ x = u^2 - v, \ y = e^v$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Consider

$$x = u^2 - v, \quad y = e^v$$

$$\frac{\partial x}{\partial v} = -1, \ \frac{\partial y}{\partial v} = e^v$$

$$z = \frac{\cos y}{x}$$

$$\frac{\partial z}{\partial x} = \cos y\left(\frac{-1}{x^2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{-\sin y}{x}$$

$$\therefore \frac{\partial z}{\partial v} = \cos y\left(\frac{-1}{x^2}\right)(-1) + \left(\frac{-\sin y}{x}\right)(e^v)$$

$$= \frac{\cos y}{x^2} - \frac{e^v \sin y}{x}$$

Since $y = e^v$

$$\frac{\partial z}{\partial v} = \frac{\cos y - xy \ \sin y}{x^2}$$

Q8. Find $\frac{dy}{dx}$ for x sin(x – y) – (x + y) = 0

Sol :

Given,

$$x \sin(x - y) - (x + y) = 0$$

with respect to $\frac{dy}{dx} = \frac{-fx}{fy}$

$$f(x, y) = x \sin(x - y)(x + y) = 0$$

$$x \sin(x - y) = x + y$$

$$\sin(x - y) = \frac{x + y}{x}$$

Partially different with respect to 'x' & 'y'

$$fy = \frac{\partial f}{\partial x}$$

$$= x \cos(x - y)(1) + 1.\sin(x - y) - 1$$

$$= x \cos(x - y) + \sin(x - y) - 1$$

$$fx = \frac{\partial f}{\partial y}$$

$$= x \cos(x - y)(-1) + 0.\sin(x - y) - (1)$$

$$= -x \cos(x - y) - 1$$

$$= -[x \cos(x - y) + 1]$$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$= \frac{-[x\cos(x - y) + \sin(s + y) - 1]}{-[x\cos(x - y) + 1]}$$

$$= \frac{x\cos(x - y) + \frac{x + y}{x} - 1}{x\cos(x - y) + 1}$$

$$= \frac{x^2\cos(x - y) + x + y - x}{x^2\cos(x - y) + x}$$

$$= \frac{x^2\cos(x - y) + y}{x^2\cos(x - y) + x}$$

Q9. If u = $\frac{(x+y)}{1-xy}$; x = tan (2r – s²), y = cot (r² s), find $\frac{\partial u}{\partial s}$

Sol :

Given,

$$u = \frac{x+y}{1-xy}$$

$$x = \tan(2r - s^2),\ y = \cot(r^2 s)$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

Consider

$$x = \tan(2r - s^2)$$

Partial different with respect to 's'

$$\frac{\partial x}{\partial s} = \sec^2(2r - s^2)(-2s)$$

$$= -2s\sec^2(2r - s^2)$$

Consider

$$y = \cot(r^2 s)$$

Partial differentiate with respect to 's'

$$\frac{\partial y}{\partial s} = -\text{cosec}^2(r^2 s)(r^2)$$

$$= -r^2\,\text{cosec}^2(r^2 s)$$

Consider

$$u = \frac{x+y}{1-xy} \Rightarrow (x + y).(1 - xy)^{-1}$$

$$= (x + y)\left[\frac{-1}{(1-xy)^2}\right](-y) + (1 - xy)(1)$$

$$\frac{\partial u}{\partial y} = (x + y)\left[\frac{-1}{(1-xy)^2}(-x)\right] + (1 - xy)^{-1}(1)$$

Consider

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \left[x+y\left[\frac{-1}{(1-xy)^2}\right](-y)+\frac{1}{1-xy}\right](-2s \sec^2 (2r - s^2) + x + y \left[\frac{-1}{(1-xy)^2}(-x)+\frac{1}{1-xy}\right] \left(-r^2 \cos ec^2\right)\left(r^2 s\right)]$$

$$= \left[\frac{1}{1-xy}+\frac{y(x+y)}{(1-xy)^2}\right][(-2 \sec^2 (2r - s^2)] + \left[\frac{x(x+y)}{(1-xy)^2}+\frac{1}{1-xy}\right] [(-r^2 \text{cosec}^2 (r^2 s)]$$

$$= \left[\frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}\right] [- 2s \sec^2 92r - s^2) + \left[\frac{x^2+\cancel{xy}+1-\cancel{xy}}{(1-xy)^2}\right] [(- r^2 \text{cosec}^2 (r^2 s)]$$

$$= \frac{(1+y^2)}{(1-xy)^2} - (2s (\sec^2 (2r - s^2)) + \frac{(1+x^2)}{(1-xy)^2} (- r^2 \ \text{cosec}^2 (r^2 s))$$

$$\frac{\partial u}{\partial s} = \frac{1}{(1-xy)^2}[(- 2s) \sec^2 (2r - s^2) (1 + y^2) - r^2 \ \text{cosec}^2 (r^2 s) (1 + x^2)].$$

Q10. If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$**, Then Show that** $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$.

Sol : **(Imp.)**

$$u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$$

Partially differentiate with respect to 'y'

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)^2}\left[y\frac{-1}{\cancel{2}(1+x^2+y^2)^{3/2}}\cancel{2}y+\frac{1}{\sqrt{1+x^2+y^2}}(1)\right]$$

$$= \frac{(1+x^2+y^2)x}{1+x^2+y^2+x^2y^2}\left[\frac{-y^2}{(1+x^2+y^2)^{3/2}}+\frac{1}{\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{x}{(1+y^2)+x^2(1+y^2)}\left[\frac{-y^2}{(1+x^2+y^2)^{1/2}}+(1+x^2+y^2)^{-1/2+1}\right]$$

$$= \frac{x}{(1+x^2)(1+y^2)}\left[\frac{-y^2}{(1+x^2+y^2)^{1/2}} + (1+x^2+y^2)^{1/2}\right]$$

$$= \frac{x}{(1+x^2)(1+y^2)}\left[\frac{-y^2+1+x^2+y^2}{(1+x^2+y^2)^{1/2}}\right]$$

$$= \frac{x}{(1+x^2)(1+y^2)}\left[\frac{1+x^2}{(1+x^2+y^2)^{1/2}}\right]$$

$$\frac{\partial u}{\partial y} = \frac{x}{\sqrt{1+x^2+y^2}\,(1+y^2)}$$

Consider

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{1+x^2+y^2}\,(1+y^2)}\right)$$

$$= \frac{1}{1+y^2}\left[\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{1+x^2+y^2}}\right]\right]$$

$$= \frac{1}{1+y^2}\left[x\cdot\frac{-1}{2(1+x^2+y^2)^{3/2}}(2x) + \frac{1}{\sqrt{1+x^2+y^2}}(1)\right]$$

$$= \frac{1}{1+y^2}\left[\frac{-x^2}{(1+x^2+y^2)^{3/2}} + \frac{1}{\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{1}{1+y^2}\left[\frac{-x^2+1+x^2+y^2}{(1+x^2+y^2)^{3/2}}\right]$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$$

Q11. If $z = xy\, f\left(\frac{y}{x}\right)$ and z is constant. Then show that $\frac{f'(y/x)}{f(y/x)} = \frac{x(y+xy')}{y(y-xy')}$.

Sol :

$$z = xy\, f\left(\frac{y}{x}\right)$$

$$z = x \cdot x \frac{y}{x} f\left(\frac{y}{x}\right)$$

$$z = x^2 \frac{y}{x} f\left(\frac{y}{x}\right)$$

$$= x^2\, z\left(\frac{y}{x}\right) \quad \text{where } t\left(\frac{y}{x}\right) = \frac{y}{x} f\left(\frac{y}{x}\right)$$

z is a homogenous function of degrees '2'

By Euler's theorem we have

$$x\frac{\partial z}{\partial r} + y\frac{\partial z}{\partial y} = 2z$$

Consider

$$z = xy\, f\left(\frac{y}{x}\right)$$

Differentiate with respect to 'x'

$$\Rightarrow \quad 0 = xy\, f'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2} + \frac{1}{x}\cdot\frac{dy}{dx}\right] + f\left(\frac{y}{x}\right)\cdot x\frac{dy}{dx} + y \cdot f\left(\frac{y}{x}\right)\cdot 1$$

$$= xy\, f'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2} + \frac{1}{x}\frac{dy}{dx}\right] + xf\left(\frac{y}{x}\right)\frac{dy}{dx} + y \cdot f\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow -f'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2} + y\frac{dy}{dx}\right] = f\left(\frac{y}{x}\right)\left[x\frac{dy}{dx} + y\right]$$

$$\Rightarrow \frac{-f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x\frac{dy}{dx} + y}{\frac{y}{x}\left[-y + x\frac{dy}{dx}\right]}$$

$$\Rightarrow \frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x\left(x\frac{dy}{dx} + y\right)}{y\left(y - x\frac{dy}{dx}\right)}$$

$$\therefore \quad \frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x\left(x\frac{dy}{dx} + y\right)}{y\left(y - x\frac{dy}{dx}\right)}$$

2.2 Maxima and Minima of Functions of Two Variables

Q12. Define Maxima and Minima of functions of two variables.

Sol :

Let f(x, y) be a function of two independent variables x, y such that it is continuous and finite for all values of x and y in the neighbourhood of their values a & b.

The values of f(a, b) is called maximum or minimum value of f(x, y) according as f(a + h, b + k).

* Condition for the existence of maxima or minima.

We know by Taylor's expansion in two variables, that

$$f(x + h, y + k) = f(x, y) + h\left(\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + ...$$

(or)

$$f(x + h,, y + k) - f(x, y) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \text{(terms of second and higher order)}$$

Q13. Write Lagrange's condition for maximum and minimum values of a function of two variables.

Sol :

If r,s,t denote the values of $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$

When x = a, y = b then supposing that the necessary condition for the maximum & minimum are satisfied.

i.e $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ when x = a, y = b

We can write $f(a + h, b + k) - f(a,b) = \frac{1}{2!}[rh^2 + 2shk + tk^2] + R$

Where R consists of terms of higher order of h and k.

- Lagrange's condition for minimum is $rt - s^2 > 0$, and $r > 0$
- Lagrange's condition for maximum is $rt - s^2 > 0$ and $r < 0$

But it $rt - s^2 < 0$. then there is neither a maximum nor a minimum.

Q14. Write Working Rule to find the maximum or minimum value of f(x,y).

Sol : **(Imp.)**

Step 1 :

Let the given function be f(x,y)

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ and equation them to zero.

Solve the equation $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ for x and y.

Let the solution be (a,b), (c, d)

Step 2 :

Calculate $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at (a,b) and (c,d). Calculate $rt - s^2$ in each care.

i.e., at (a,b) & (c,d)

Step 3 :

If $rt - s^2 > 0$ and $r < 0$ at (a,b) then f has a maximum value at x = (a,b)

If or at (c,d) if $rt - s^2 > 0$ and $r < 0$ then f has maximum value at (c,d)

Step 4 :

If $rt - s^2 > 0$, and $r > 0$ at (a,b) then f has a minimum value at (a,b) or if $rt - s^2 > 0$ and $r > 0$ at (c,d) & has a minimum at (c,d).

Step 5 :

If $rt - s^2 < 0$. at (a,b) then f has neither , maximum , nor minimum . Then (a,b) is saddle point.

If $rt - s^2 < 0$ at (c,d), then (c,d) is saddle point.

Step 6 :

If $rt - s^2 = 0$, we can't decide whether f, has maximum or minimum for the investigation is needed.

Q15. Define Stationary points and Extreme points.

Sol :

Points at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ are called stationary points for the function f(x,y)

If it is a maximum or a minimum is known as an extreme point and the value of the function at an extreme point is known as an extreme value.

Q16. Discuss the maximum or minimum value of u, when $u = x^3 + y^3 - 3axy$.

Sol : **(Imp.)**

$u = x^3 + y^3 - 3axy$

$\frac{\partial u}{\partial x} = 3x^2 - 3ay$; $\frac{\partial u}{\partial y} = 3y^2 - 3ax$.

for a max or min of u, we must have

$\frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0$

$3x^2 - 3ay = 0 \Rightarrow x^2 - ay = 0$ ... (1)

$3y^2 - 3ax = 0 \Rightarrow y^2 - ax = 0$ ... (2)

Solve (1) & (2)

from $x^2 = ay \Rightarrow y = \frac{x^2}{a}$

Sub y in (2)

$$\Rightarrow \left(\frac{x^2}{a}\right) - ax = 0$$

$$\Rightarrow \frac{x^4}{a^2} - ax = 0$$

$x^4 - a^3x = 0$

$x(x^3 - a^3) = 0$

$x = 0, x = a$

Similarly $y = 0, y = a$.

Similarly $y = 0, y = a$.

Thus (0,0) & (a,a) are the stationary points of u.

Now, $r = \frac{\partial^2 u}{\partial x^2} = 6x \quad t = \frac{\partial^2 u}{\partial y^2} = 6y$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -3a$$

for $x = 0, y = 0 \Rightarrow r = 0, t = 0, s = -3a$.

$rt - s^2 = (0)(0) - (-3a)^2$

$= 9a^2 < 0$

$\Rightarrow$ u is neither maximum nor minimum at $x = 0$ & $y = 0$

Also $x = a, y = b$

$r = 6a, s = -3a, t = 6a$

Now, $rt - s^2 = (6a)(6a) - (-3a^2)$

$= 36a^2 - 9a^2$

$= 27a^2 > 0$

Also $r = 6a$ which is positive if $a > 0$

(i) u is maximum at $x = a, y = a$, if $r < 0$

(ii) u is minimum at $x = 0, y = 0$ if $r > 0$.

Q17. Show that minimum value of $u = xy + (a^3 / x) + (a^3 / y)$ is $3a^2$.

Sol :

$u = xy + (a^3/x) + (a^3/y)$

We have

$$\frac{\partial u}{\partial x} = y - \frac{a^3}{x^2}; \quad \frac{\partial u}{\partial y} = x - \frac{a^3}{y^2}$$

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{2a^3}{x^3};$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 1;$$

$$t = \frac{\partial^2 u}{\partial y^2} = \frac{2a^3}{y^3}$$

Now, for maximum or minimum we must have $\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0$.

So from $\frac{\partial u}{\partial x} = 0 \Rightarrow y - \frac{a^3}{x^2} = 0$

$x^2 y = a^3$ ... (1)

From $\frac{\partial u}{\partial y} = 0 \Rightarrow x - \frac{a^3}{y^2} = 0$

$y^2 x = a^3$ ... (2)

From (1) & (2)

We get

$x^2 y = y^2 x$

$x^2 y - y^2 x = 0 \Rightarrow xy(x - y) = 0$

$x = 0, y = 0, \& x = y$

From (1) & (2) we see that $x = 0$ & $y = 0$. do not hold as it gives $a = 0$

Hence we must have $x = y$ & from (1) we get

$x^2 y = a^3 \Rightarrow x^2 . x = a^3$

$x^3 = a^3 \Rightarrow x = a$

At $x = y = a$ we have

$$r = 2\frac{a^3}{x^3} = 2\frac{a^3}{a^3} = 2$$

$$s = 1; \; t = \frac{2a^3}{y^3} = \frac{2a^3}{a^3} = 2$$

$rt - s^2 = (2)(2) - 1^2 = 3 > 0$

Also, r = 2 > 0

Hence there is minimum at x = y = a.

hence the minimum value of,

$$u = a.a + \frac{a^3}{a} + \frac{a^3}{a}$$

$$= a^2 + \frac{a^3}{a} + \frac{a^3}{a}$$

$$= a^2 + a^2 + a^2$$

$$= 3a^2$$

Q18. Discuss the maximum or minimum value of u given by u = $x^3 y^2 (1 - x - y)$.

Sol :

$$u = x^3 y^2 (1 - x - y)$$

$$\frac{\partial u}{\partial x} = 3x^2 y^2 (1 - x - y) + x^3 y^2 (-1)$$

$$= 3x^2 y^2 - 3x^3y^2 - 3x^2 y^3 - x^3y^2$$

$$\frac{\partial u}{\partial x} = 3x^2 y^2 - 4x^3y^2 - 3x^2y^3$$

$$\frac{\partial u}{\partial y} = 2x^3y (1 - x - y) + x^3y^2 (-1)$$

$$= 2x^3 y - 2x^4 y - 2x^3y^2 - x^3y^2$$

$$\frac{\partial u}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$$

$$r = \frac{\partial^2 u}{\partial x^2} = 6xy^2 - 12x^2 y^2 - 6xy^3$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$$

Now, for maximum or minimum we must have

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0$$

From $\frac{\partial u}{\partial x} = 0 \Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$$x^2y^2(3 - 4x - 3y) = 0$$

Hence we get x = 0, y = 0

$$4x + 3y = 3 \quad \text{... (2)}$$

Also from $\frac{\partial u}{\partial y} = 0,$ we get

$$2xy^3 - 2x^4 y - 3x^3 y^2 = 0$$

$$x^3y (2 - 2x - 3xy) = 0$$

$$x = 0, \; y = 0 \text{ and } 2x + 3y = 2 \text{ ... (3)}$$

By solving (1) & (2)

	x		y	1
3		-3	4	3
3		-2	2	3

$$\frac{x}{-6+9} = \frac{y}{-6+8} = \frac{1}{12-6}$$

$$\frac{x}{3} = \frac{y}{2} = \frac{1}{6}$$

$$x = \frac{3}{6};\; y = \frac{2}{6}$$

$$x = \frac{1}{2};\; y = \frac{1}{3}$$

Hence the solution are

$$x = 0,\; y = 0,\; x = \frac{1}{2},\; y = \frac{1}{3},$$

when

$$r = 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3$$

$$= 3\frac{1}{9} - 3\left(\frac{1}{9}\right) - 3\left(\frac{1}{27}\right)$$

$$= \frac{1}{3} - \frac{1}{3} - \frac{1}{9} = -\frac{1}{9}$$

$$S = 6\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right) - 8\left(\frac{1}{3}\right)^3\left(\frac{1}{2}\right) - 9\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2$$

$$= 2\cdot\frac{1}{\not{4}_2} - 4\cdot\frac{1}{\not{8}_2} - \frac{1}{4}$$

$$= \frac{1}{2} - \frac{1}{2} - \frac{1}{4} = -\frac{1}{4}$$

$$t = 2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{3}\right)^4 - 6\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} - \frac{2}{81} - \frac{1}{4} = \frac{-2}{81}$$

From there we have $rt - s^2 > 0$ and $r < 0$

So, there is maximum at $x = \frac{1}{2}, y = \frac{1}{3}$

Q19. Find a point within a triangle such that the sum of the square of its distance from the three vertices is a minimum.

Sol :

Let (x_r, y_r), $r = 1,2,3$ be the vertices of the triangle and (x,y) be any point side the triangle.

Let $u = \sum_{r=1}^{3}\left[(x-x_r)^2 + (y-y_r)^2\right]$

For maximum or minimum of u, we have

$$\frac{\partial u}{\partial x} = \Sigma 2\,(x - x_r) = 0$$

$$(x - x_1) + (x - x_2) + (x - x_3) = 0$$

$$x - x_1 + x - x_2 + x - x_3 = 0$$

$$3x = x_1 + x_2 + x_3$$

$$x = \frac{x_1 + x_2 + x_3}{3}$$

Similarly $\frac{\partial u}{\partial y} = 0 \Rightarrow \Sigma 2(y - y_r) = 0$

$$(y - y_1) + (y - y_2) + (y - y_3) = 0$$

$$y - y_1 + y - y_2 + y - y_3 = 0$$

$$3y = y_1 + y_2 + y_3$$

$$y = \frac{y_1 + y_2 + y_3}{3}$$

$$r = \frac{\partial^2 u}{\partial x^2} = 6$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 0, \ t = \frac{\partial^2 u}{\partial y^2} = 6$$

So that $rt - s^2 \Rightarrow (6)(6) - 0 = 36 > 0$

Hence u is minimum when

$$x = \frac{x_1 + x_2 + x_3}{3}, \ y = \frac{y_1 + y_2 + y_3}{3}$$

Thus, the required point is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

Q20. Find the maximum value of $(ax + by + cz)\, e^{-\alpha^2 x^2 - \beta^2 y^2 - \gamma^2 z^2}$.

Sol :

$u = (ax + by + cz)\, e^{-\alpha^2 x^2 - \beta^2 y^2 - \gamma^2 z^2}$

$\log u = \log (ax + by + cz) - (\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2)$

Differentiating partially w.r.to 'x'

We get

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{a}{ax + by + cz}(-2\alpha^2 x) = 0$$

Similarly $\frac{1}{u}\frac{\partial u}{\partial y} = \frac{b}{ax + by + cz} - 2\beta^2 y = 0$

$$\frac{1}{u}\frac{\partial u}{\partial z} = \frac{c}{ax + by + cz} - 2\gamma^2 \beta = 0$$

$$x(ax + by + cz) = \frac{a}{2\alpha^2} \quad \text{... (1)}$$

$$y(ax + by + cz) = \frac{b}{2\beta^2} \quad \text{... (2)}$$

$$z(ax + by + cz) = \frac{c}{2\gamma^2} \quad \text{... (3)}$$

Multiplying (1), (2), (3) by a,b,c and adding. we get

$$(ax + by + cz)^2 = \frac{1}{2}\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right)$$

$$(ax + by + cz) = \sqrt{\frac{1}{2}\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right)} = A$$

$$x = \frac{a}{2\alpha^2 A},\ y = \frac{b}{2\beta^2 A},\ z = \frac{c}{2\gamma^2 A}$$

Again $\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{a^2}{(ax+by+cz)^2} - 2\alpha^2$

$$\frac{\partial^2 u}{\partial x^2} = -u\left[\frac{a^2}{(ax+by+cz)^2} + 2\alpha^2\right]$$

Since $\frac{\partial u}{\partial \alpha} = 0$

Hence for these values of x, y, z will be maximum, maximum value of u is given by

$$= \sqrt{\left[\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right]e^{\frac{-1}{4a^2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)}}$$

$$= \sqrt{\left[\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right]e^{1/2}}$$

$$\therefore \quad U = \sqrt{\frac{1}{2e}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)}$$

Q21. Discuss the maxima or minima of $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Sol :

Given $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y$$

$$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y$$

for max and min of 'u' we must have $\frac{\partial u}{\partial x} = 0,\ \frac{\partial u}{\partial y} = 0$

$4x^3 - 4x + 4y = 0$ ... (1)

$4y^3 + 4x - 4y = 0$ ... (2)

Solve (1) & (2)

$x^3 + y^3 = 0$

$(x + y)(x^2 - xy + y^2) = 0$

$x + y = 0 \Rightarrow x = -y$

Substituting y = – x in (1) we get

$$4x^3 - 8x = 0 \Rightarrow 4x(x^2 - 2x) = 0$$

$$x = 0, \pm\sqrt{2}$$

The values of y are 0, $-\sqrt{2}$, $\sqrt{2}$

Therefore the stationary points are (0,0), $(\sqrt{2},\sqrt{2})$, $(-\sqrt{2},-\sqrt{2})$

Now, $r = \dfrac{\partial^2 u}{\partial x^2} = 12x^2 - 4$

$s = \dfrac{\partial^2 u}{\partial x \partial y} = 4$

$t = \dfrac{\partial^2 u}{\partial y^2} = 12y^2 - 4$

At (0,0) = $rt - s^2$

= (12 (0) – 4) (12 (0) – 4)–16

= (–4) (–4) –16

= 16 – 16 = 0

i.e, no conclusion can be drawn about max or min.

At $(\sqrt{2},-\sqrt{2})$

$rt - s^2$ = (12(2)–4) (12(2) – 4) – $(4)^2$

= 400 – 16

= 384 > 0

$rt - s^2 > 0$, $r < 0$.

∴ f has minimum value at $(\sqrt{2},\sqrt{2})$, $(-\sqrt{2},-\sqrt{2})$

Now, $r = \dfrac{\partial^2 u}{\partial x^2} = 12x^2 - 4$

$s = \dfrac{\partial^2 u}{\partial x \partial y} = 4$

$t = \dfrac{\partial^2 u}{\partial y^2} = 12y^2 - 4$

At (0,0)

$rt - s^2$ = (12(0) – 4) (12 (0) – 4) – 16

= 0

i.e., no. conclusion can be shown about max or min.

At $(\sqrt{2},-\sqrt{2})$

$rt - s^2$ (2) – 4) (12(2) – 4) – $(4)^2$

400 – 16

384 > 0

$rt - s^2 > 0, r > 0$

$\therefore$ f has minimum value at $(\sqrt{2},-\sqrt{2})$.

2.3 Lagrange's Method of undetermined Multipliers

Q22. Explain Lagrange's method of underermind multipliers.

Ans : **(Imp.)**

Let u be a function of n variables given as,

$$u = f(x_1, x_2, x_3, ... x_n) \quad ... (1)$$

Let the variables are connected by a relation such that there are n – r independent variables.

i.e., $\phi_1(x_1, x_2, x_3, ... x_n) = 0 \quad ... (2)$

$\phi_2(x_1, x_2, x_3, ... x_n) = 0 \quad ... (3)$

. . .

$\phi_r(x_1, x_2, x_3, ... x_n) = 0 \quad ... (4)$

For maximum or minimum, du must be zero

$$\text{i.e., } du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + ... + \frac{\partial u}{\partial x_n} dx_n = 0 \quad ... (5)$$

Differentiating equations (2),(3),(4) and multiplying the resultants with $\lambda_1, \lambda_2, ... \lambda_r$ respectively,

$$\lambda_1 d\phi_1 = \lambda_1 \frac{\partial \phi_1}{\partial x_1} dx_1 + \lambda_1 \frac{\partial \phi_1}{\partial x_2} dx_2 + ... + \lambda_1 \frac{\partial \phi_1}{\partial x_n} dx_n = 0 \quad ... (6)$$

$$\lambda_2 d\phi_2 = \lambda_2 \frac{\partial \phi_2}{\partial x_1} dx_1 + \lambda_2 \frac{\partial \phi_2}{\partial x_2} dx_2 + ... + \lambda_2 \frac{\partial \phi_2}{\partial x_n} dx_n = 0 \quad ... (7)$$

$$\lambda_r d\phi_r = \lambda_r \frac{\partial \phi_r}{\partial x_1} dx_1 + \lambda_r \frac{\partial \phi_r}{\partial x_2} dx_2 + ... + \lambda_r \frac{\partial \phi_r}{\partial x_n} dx_n = 0 \quad ... (8)$$

Where, $\lambda_1, \lambda_1, ... \lambda_r$ are multipliers.

Adding equations (5), (6), (7), (8),

$$A_1 dx_1 + A_2 dx_2 + ... + A_n dx_n = 0 \quad ... (9)$$

Where,

$$A_1 = \frac{\partial u}{\partial x_1} + \lambda_1 \frac{\partial \phi_1}{\partial x_1} + \lambda_2 \frac{\partial \phi_2}{\partial x_2} + ... + \lambda_r \frac{\partial \phi_r}{\partial x_n}; \ i = 1, 2, ...n$$

The values of $\lambda_1, \lambda_2, ... \lambda_r$ are chose in such a way that $A_1 = A_2 = A_3 = A_r = 0$ ... (10)

Substituting equation (10) in equation (9),

$$A_{r+1}\, dx_{r+1} + A_{r+2}\, dx_{r+2} + ... + A_n\, dx_n = 0$$

Let the independent variables be x\, x_{r+2}, ... x_n such that dx_{r+1}, dx_{r+2} .... dx_n are independent.

$$\Rightarrow \quad A_{r+1} + A_{r+2} + ... + A_n = 0$$

$$\therefore \quad A_1 = A_2 = A_3 = A_r = A_{r+1} = A_{r+2} = ... + A_n = 0 \quad ... (11)$$

From equations (2),(3) and (4),

$$\phi_1 = \phi_2 = ... = \phi_r = 0 \quad ... (12)$$

From equations (11) and (12), The possible values of u are obtained by determining the multipliers $(\lambda_1, \lambda_2, ... \lambda_r)$ from (n + r) equations.

Q23. Find the minimum value of x + y + z, subject to the condition $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$ = 1.

Ans :

Given function is,

$$f(x, y, z) = x + y + z \quad ... (1)$$

It is basically a constrained extremum problem, where a function "f" is subjected to the constraint

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z} = 1$$

$$\text{Let, } \phi = \frac{a}{x}+\frac{b}{y}+\frac{c}{z} - = 0 \quad ... (2)$$

According to Lagrange's function,

$$F(x, y, z) = f(x + y + z) + \phi\lambda$$

Where,

λ – Lagrangian multiplier

$$F(x, y, z) = (x + y + z) + \lambda\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right) \quad ... (3)$$

Differentiating equation (3) with respect to x, y, z and equating to zero

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(x + y + z) + \lambda\frac{\partial}{\partial x}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right) = 1 + \lambda\left[\frac{-a}{x^2}\right]$$

$$\therefore \quad \frac{\partial F}{\partial x} = \frac{x^2 - a\lambda}{x^2}$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(x + y + z) + \frac{\partial}{\partial y}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right) = 1 + \lambda\left[\frac{-b}{y^2}\right]$$

$$\therefore \quad \frac{\partial F}{\partial y} = \frac{y^2 - b\lambda}{y^2}$$

$$\frac{\partial F}{\partial z} = \frac{\partial}{\partial z}(x + y + z) + \lambda \frac{\partial}{\partial z}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right) = 1 + \lambda\left[\frac{-c}{z^2}\right]$$

$$\therefore \quad \frac{\partial F}{\partial z} = \frac{z^2 - c\lambda}{z^2}$$

The condition for minimum values is,

$$\frac{\partial F}{\partial x} = 0 \; ; \; \frac{\partial F}{\partial y} = 0 \; ; \frac{\partial F}{\partial z} = 0$$

$$\frac{x^2 - a\lambda}{x^2} = 0 \; ; \; \frac{y^2 - b\lambda}{y^2} = 0 \; ; \; \frac{z^2 - c\lambda}{z^2} = 0$$

$$\Rightarrow \quad x^2 = a\lambda \; ; \quad \Rightarrow y^2 = b\lambda \; ; \quad \Rightarrow z^2 = c\lambda \; ;$$

$$\Rightarrow \quad x = \pm\sqrt{a\lambda} \Rightarrow \quad y = \pm\sqrt{b\lambda} \Rightarrow c = \pm\sqrt{c\lambda}$$

Substituting the corresponding values in equation (2),

$$\frac{a}{\sqrt{a\lambda}} + \frac{b}{\sqrt{b\lambda}} + \frac{c}{\sqrt{c\lambda}} = 1$$

$$\Rightarrow \frac{\sqrt{a}}{\sqrt{\lambda}} + \frac{\sqrt{b}}{\sqrt{\lambda}} + \frac{\sqrt{c}}{\sqrt{\lambda}} = 1$$

$$\Rightarrow \sqrt{\lambda} = (\sqrt{a} + \sqrt{b} + \sqrt{c})$$

$$\Rightarrow \quad \lambda = (\sqrt{a} + \sqrt{b} + \sqrt{c})^2$$

$$\Rightarrow \quad x = \pm\sqrt{\lambda a} = \pm\left(\sqrt{(\sqrt{a})^2(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\right) = \pm\left((\sqrt{a})(\sqrt{a}+\sqrt{b}+\sqrt{c})\right)$$

$$\Rightarrow \quad y = \left(\tfrac{\pi}{3}\right) = \pm\left(\sqrt{(\sqrt{b})^2(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\right) = \pm\left((\sqrt{b})(\sqrt{a}+\sqrt{b}+\sqrt{c})\right)$$

$$\Rightarrow \quad z = \pm\sqrt{\lambda c} = \pm\left(\sqrt{(\sqrt{c})^2(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\right) = \pm\sqrt{c}\left((\sqrt{b})(\sqrt{a}+\sqrt{b}+\sqrt{c})\right)$$

The stationary point is,

$$\therefore \quad (x, y, z) = \left(\sqrt{a}(\sqrt{a}+\sqrt{b}+\sqrt{c}), \sqrt{b}(\sqrt{a}+\sqrt{b}+\sqrt{c}), \sqrt{c}(\sqrt{a}+\sqrt{b}+\sqrt{c})\right)$$

The minimum value is obtained by substituting stationary point in equation (1), Minimum value = $\sqrt{a}(\sqrt{a}+\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{a}+\sqrt{b}+\sqrt{c})+\sqrt{c}(\sqrt{a}+\sqrt{b}+\sqrt{c})$

$= (\sqrt{a}+\sqrt{b}+\sqrt{c})\,(\sqrt{a}+\sqrt{b}+\sqrt{c}) = (\sqrt{a}+\sqrt{b}+\sqrt{c})^2$

$\therefore$ Minimum value $= (\sqrt{a}+\sqrt{b}+\sqrt{c})^2$

Q24. Find the minimum value of $x^2+y^2+z^2$ when $x+y+z=3a$.

Ans : **(Imp.)**

Let the given function be,

$f(x, y, z) = x^2+y^2+z^2$ and $\phi = x+y+z-3a$

The auxiliary function is given as, ... (1)

$F(x, y, z) = f(x, y, z) + \lambda\phi$

Substituting the corresponding values in equation (1),

$F(x, y, z) = (x^2+y^2+z^2) + \lambda(x+y+z-3a)$ ... (2)

Partially differentiating equation (2) with respect to x,y and z respectively,

$$\frac{\partial F}{dx} = \frac{\partial}{dx}(x^2+y^2+z^2) + \lambda\frac{\partial}{dx}(x+y+z-3a) = (2x+0+0)+\lambda(1+0+0-0)$$

$$\frac{\partial F}{dx} = 2x+\lambda \quad \text{... (3)}$$

$$\frac{\partial F}{dy} = \frac{\partial}{dy}(x^2+y^2+z^2) + \lambda\frac{\partial}{dy}(x+y+z-3a) = (0+2y+0)+\lambda(0+1+0-0)$$

$$\frac{\partial F}{dy} = 2y+\lambda \quad \text{... (4)}$$

$$\frac{\partial F}{dz} = \frac{\partial}{dz}(x^2+y^2+z^2) + \lambda\frac{\partial}{dz}(x+y+z-3a) = (0+0+2z)+\lambda(0+0+1-0)$$

$$\frac{\partial F}{dz} = 2z+\lambda \quad \text{... (5)}$$

Equating equations (3),(4) and (5) to zero,

$2x+\lambda=0 \qquad 2y+\lambda=0 \qquad 2z+\lambda=0$

$2x=-\lambda \qquad 2y=-\lambda \qquad 2z=-\lambda$

$x=-\dfrac{\lambda}{2} \qquad y=-\dfrac{\lambda}{2} \qquad z=-\dfrac{\lambda}{2}$

$\therefore \quad x = y = z = -\dfrac{\lambda}{2}$

Substituting the corresponding values of x,y and z in $x+y+z=3a$, ... (6)

$$\Rightarrow -\frac{\lambda}{2}-\frac{\lambda}{2}-\frac{\lambda}{2}=3a \quad \Rightarrow -\frac{3\lambda}{2} = 3a \qquad \Rightarrow -\frac{\lambda}{2} = a$$

$\Rightarrow \quad \lambda = -2a$

Substituting the value of λ in equation (6),

$$x = y = z = \frac{-(-2a)}{2}$$

$\therefore \quad x = y = z = a$

The minimum value of the given function is given by,

$f(x, y, z) = x^2 = y^2 + z^2$

$= a^2 + a^2 + a^2$

$f(x, y, z) = 3a^2 \qquad (\therefore\ x = y = z = 0)$

$\therefore$ The minimum value is , $3a^2$.

Q25. Discuss the maxima and minima of the function u = sin x sin y sin z, where x, y, z are the angles of a triangle

Ans :

Given that,

$u = \sin x \sin y \sin z$ ..... (1)

In a triangle,

$x + y + z = \pi$ ..... (2)

Differentiating equation (1) on both sides,

$$du = \sin y \sin z, \frac{d}{dx}(\sin x) + \sin x \sin z \frac{d}{dy}(\sin y) + \sin x \sin y. \frac{d}{dy}(\sin z)$$

$\Rightarrow \quad du = \sin y \sin z, (\cos x\, dx) + \sin x \sin z (\cos y\, dy) + \sin x \sin y (\cos x\, dz)$

$\Rightarrow \quad du = \text{con } x \sin y \sin z\, dx + \sin x \cos y \sin z\, dy + \sin x \sin y \cos z\, dz$

For maximum or minimum of u, du = 0,

$\text{con } x \sin y \sin z\, dx + \sin x \cos y \sin z\, dy + \sin x \sin y \cos z\, dz = 0$ ..... (3)

Differentiating equation (2),

$dx + dy + dz = 0$

Multiplying above equation by λ,

$\lambda\, dx + \lambda\, dy + \lambda\, dz = 0$ ..... (4)

Adding equation (3) and (4),

$\text{con } x \sin y \sin z\, dx + \sin x \cos y \sin z\, dy + \sin x \sin y \cos z\, dz + \lambda\, dx + \lambda\, dy + \lambda\, dz = 0$

$\Rightarrow \quad (\text{con } x \sin y \sin z + \lambda)\, dx + (\sin x \cos y \sin z + \lambda)\, dy + (\sin x \sin y \cos z + \lambda)\, dz = 0$

Equating the coefficients of dx, dy and dz to zero,

$\text{con } x \sin y \sin z + \lambda = 0 \Rightarrow -\lambda = \text{con } x \sin y \sin z$

$\sin x \cos y \sin z + \lambda = 0 \Rightarrow -\lambda = \sin x \cos y \sin z$

$\sin x \sin y \cos z + \lambda = 0 \Rightarrow -\lambda = \sin x \sin y \cos z$

$\Rightarrow \text{con } x \sin y \sin z = \sin x \cos y \sin z = \sin x \sin y \cos z$

Dividing by sin x sin y sin z on both sides,

$$\frac{\cos x \sin y \sin z}{\sin x \sin y \sin z} = \frac{\sin x \text{ con } y \sin z}{\sin x \sin y \sin z} = \frac{\sin x \sin y \cos z}{\sin x \sin y \sin z}$$

$\Rightarrow \cot x = \cot y = \cot z$

$\Rightarrow x = y = z = \dfrac{\pi}{3}$ [$\because x + y + z = \pi$]

Consider x,y as independent variables and z as a function of x and y.

Differentiating equation (2) partially with respect to 'x',

$$1 + 0 + \frac{dz}{dx} = 0$$

$\Rightarrow \dfrac{dz}{dx} = -1$ ..... (5)

Differentiating equation (2) partially with respect to 'y',

$$0 + 1 + \frac{dz}{dx} = 0$$

$\Rightarrow \dfrac{dz}{dy} = -1$ .....(6)

Differentiating equation (1) partially with respect to 'y',

$$\frac{\partial u}{\partial x} = \sin y \sin z, \frac{\partial}{\partial x}(\sin x) + \sin x \sin z, \frac{\partial}{\partial x}(\sin y) + \sin x \sin y. \frac{\partial}{\partial x}(\sin z)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \sin y \sin z \cos x + 0 + \sin x \sin y \cos z \frac{\partial z}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \sin y \sin z \cos x + \sin x \sin y \cos z \frac{\partial z}{\partial x}$$

$\Rightarrow \dfrac{\partial u}{\partial x} = \sin y \sin z \cos x + \sin x \sin y \cos z\,(-1)$ [$\because$ From equation (5)]

$\Rightarrow \dfrac{\partial u}{\partial x} = \sin y \sin z \cos x - \sin x \sin y \cos z$ .....(7)

Differentiating equation (7) partially with respect to 'x',

$$r = \frac{\partial^2 u}{\partial x^2} = \sin y\left[\sin z.\frac{\partial}{\partial x}(\cos x) + \cos x.\frac{\partial}{\partial x}(\sin z)\right] - \sin y\left[\sin x.\frac{\partial}{\partial x}(\cos z) + \cos z.\frac{\partial}{\partial x}(\sin x)\right]$$

$$\Rightarrow \quad r = \sin y\left[\sin z.(-\sin x) + \cos x \cos z\frac{\partial z}{\partial x}\right] - \sin y\left[\sin x.(-\sin z).\frac{\partial z}{\partial x} + \cos z \cos x\right]$$

$$\Rightarrow \quad r = \sin y\,[-\sin x \sin z + \cos x \cos z\,(-1)] - \sin y[-\sin x \sin z\,(-1) + \cos z \cos x]$$

[∵ From equation (6)]

$$\Rightarrow \quad r = \sin y\,[-\sin x \sin z - \cos x \cos z\,] - \sin y\,[\sin x \sin z + \cos x \cos z\,]$$

$$\Rightarrow \quad r = -\sin x \sin y \sin z \cos x \cos z \sin y - \sin y \sin x \sin z - \sin y \cos z \cos x$$

$$\Rightarrow \quad r = -2\sin x \sin y \sin z - 2\cos x \cos z \sin y$$

$$r\Big|\frac{\pi}{3} = -2\sin\frac{\pi}{3}\ \sin\frac{\pi}{3}\ \sin\frac{\pi}{3} - 2\cos\frac{\pi}{3}\ \cos\frac{\pi}{3}\ \sin\frac{\pi}{3}$$

$$= -2\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2} - 2\ \frac{1}{2}.\ \frac{1}{2}.\frac{\sqrt{3}}{2}$$

$$= \frac{-3\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = \frac{-4\sqrt{3}}{4}$$

$$r\Big|\frac{\pi}{3} = -\sqrt{3}$$

Differentiating equation (7) with respect to 'y'

$$s = \frac{\partial^2 u}{\partial y \partial x} = \cos x\left[\sin z.\frac{\partial}{\partial y}(\sin y) + \sin y.\frac{\partial}{\partial y}(\sin z)\right] - \sin x\left[\cos z.\frac{\partial}{\partial y}(\sin y) + \sin y.\frac{\partial}{\partial y}(\cos z)\right]$$

$$\Rightarrow \quad s = \sin z \cos x \cos y + \sin y \text{ coa } x \cos z.\frac{\partial z}{\partial y} - \sin x \cos z \cos y - \sin x \sin y\,(-\sin z).\frac{\partial z}{\partial y}$$

$$\Rightarrow \quad s = \sin z \cos x \cos y + \sin y \text{ coa } x \cos z.(-1) - \sin x \cos z \cos y + \sin x \sin y\ \sin z(-1)$$

[∵ From equation (6)]

$$\Rightarrow \quad s = \sin z \cos x \cos y - \sin y \text{ coa } x \cos z. - \sin x \cos y \cos z - \sin x \sin y\ \sin z$$

$$s\Big|\frac{\pi}{3} = \sin\frac{\pi}{3}\ \cos\frac{\pi}{3}\ \cos\frac{\pi}{3} - \sin\frac{\pi}{3}\ \text{coa}\frac{\pi}{3}\ \cos\frac{\pi}{3} - \sin\frac{\pi}{3}\ \cos\frac{\pi}{3}\ \cos\frac{\pi}{3} - \sin\frac{\pi}{3}\ \sin\frac{\pi}{3}\ \sin\frac{\pi}{3}$$

$$\Rightarrow s\Big|\frac{\pi}{3} = \frac{\sqrt{3}}{2}.\frac{1}{2}.\frac{1}{2} - \frac{\sqrt{3}}{2}.\frac{1}{2}.\frac{1}{2} - \frac{\sqrt{3}}{2}.\frac{1}{2}.\frac{1}{2} - \frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2}$$

$$\Rightarrow s_{\left|\frac{\pi}{3}\right.} = \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{3\sqrt{3}}{8} = \frac{-4\sqrt{3}}{8}$$

$$\Rightarrow s_{\left|\frac{\pi}{3}\right.} = \frac{-\sqrt{3}}{2}$$

By symmetry, $t_{\left|\frac{\pi}{3}\right.} = \frac{\partial^2 u}{\partial y^2} = -\sqrt{3}$

$$rt - s^2 = \sqrt{3}(-\sqrt{3}) - \left(\frac{-\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \quad rt - s^2 = 3 - \frac{3}{4}$$

$$\Rightarrow \quad rt - s^2 = \frac{9}{4} > 0$$

Since, $rt - s^2$ is + ve and r is –ve, the function u is maximum

$$\text{Maximum value} = \sin\left(\frac{\pi}{3}\right) \sin\left(\frac{\pi}{3}\right) \sin\left(\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$$

$\therefore$ Maximum value of u is $\frac{3\sqrt{3}}{8}$

Short Question and Answers

1. Differentiation Composite Function.

Ans :

Let = f(x, y) possess continuous partial derivatives and let x = ϕ(t), y = ψ (t) Possess continuous derivatives.

Then

$$\frac{dz}{dt} = \frac{\partial^2}{\partial x}, \frac{dx}{dt} + \frac{\partial^2}{\partial y} \cdot \frac{dy}{dt}$$

2. Define Implicit Function.

Ans :

Let f be a function of two variables since f(x, y) = 0 ... (1)

We can obtain y as function of x, the equation (1) defines y as an implicit function of x.

Assuming that the conditions under which the equation (1) defines y as a derivable function of x are satisfied.

We shall Now obtain the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terns of the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y^2}$ of 'f'

with respect to x & y

Then

$$\frac{dy}{dx} = \frac{-\partial f / \partial x}{\partial f / \partial y} = \frac{-fx}{fy} \text{ if } fy \neq 0$$

$$\frac{\partial^2 y}{dx^2} = \frac{f_{x^2}\left(f_y\right)^2 - 2f_{yx}f_x f_y + f_{y^2}\left(f_x\right)^2}{\left(f_y\right)^3}$$

3. Define Maxima and Minima of functions of two variables.

Sol :

Let f(x, y) be a function of two independent variables x, y such that it is continuous and finite for all values of x and y in the neighbourhood of their values a & b.

The values of f(a, b) is called maximum or minimum value of f(x, y) according as f(a + h, b + k).

Condition for the existence of maxima or minima.

We know by Taylor's expansion in two variables, that

$$f(x + h, y + k) = f(x, y) + h\left(\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots$$

(or)

$$f(x + h,, y + k) - f(x, y) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \text{(terms of second and higher order)}$$

4. Write Lagrange's condition for maximum and minimum values of a function of two variables.

Sol :

If r,s,t denote the values of $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$

When x = a, y = b then supposing that the necessary condition for the maximum & minimum are satisfied.

i.e $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ when x = a, y = b

We can write $f(a + h, b + k) - f(a,b) = \frac{1}{2!}[rh^2 + 2shk + tk^2] + R$

Where R consists of terms of higher order of h and k.

- Lagrange's condition for minimum is $rt - s^2 > 0$, and $r > 0$
- Lagrange's condition for maximum is $rt - s^2 > 0$ and $r < 0$

But it $rt - s^2 < 0$. then there is neither a maximum nor a minimum.

5. Define Stationary points and Extreme points.

Sol :

Points at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ are called stationary points for the function f(x,y)

If it is a maximum or a minimum is known as an extreme point and the value of the function at an extreme point is known as an extreme value.

6. If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$ find $\frac{\partial u}{\partial r}$.

Sol :

Given

$$u = x^2 - y^2, \quad x = 2r - 3s + 4, \quad y = -r + 8s - 5$$

Different with respect to 'r'

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

Consider $x = 2r - 3s - 4$; $y = -r + 8s - 5$

$$\frac{\partial x}{\partial r} = 2 \; ; \; \frac{\partial y}{\partial r} = -1$$

Consider $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x, \; \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial u}{\partial r} = 2x(2) + (-2y)(-1)$$

$$\frac{\partial u}{\partial r} = 4x + 2y$$

7. Define Equality of $f_{xy}(a, b)$, $f_{yx}(a, b)$.

Sol :

If has been seen that the two repeated second order partial derivatives are generally equal. They are not, however, always equal as is shown below.

We have $f_{xy}(a, b) = \lim\limits_{h\to 0} \dfrac{f_y(a+h,b) - f_y(a,b)}{h}$

also, $f_y(a + h, b) = \lim\limits_{k\to 0} \dfrac{f(a+h,b+k) - f(a+h,b)}{k}$

$f_y(a, b) = \lim\limits_{k\to 0} \dfrac{f(a,b+k) - f(a,b)}{k}$

$\therefore \quad f_{xy}(a, b) = \lim\limits_{h\to 0} \lim\limits_{k\to 0} \dfrac{\phi(h,k)}{hk}$

$\therefore \quad f_{xy}(a, b) = \lim\limits_{h\to 0} \lim\limits_{k\to 0} \dfrac{f(a+k,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)}{hk}$

$= \lim\limits_{h\to 0} \lim\limits_{k\to 0} \dfrac{\phi(h,,k)}{hk}$

It many similarly show that

$f_{yx}(a, b) = \lim\limits_{k\to 0} \lim\limits_{h\to 0} \dfrac{\phi(h,k)}{hk}$.

8. State and prove Taylor's theorem for a function of two variables.

Ans :

Statement

If f(x, y) is a function which possisses continuous partial derivatives of the third order in a neighbourhood of a point (a, b) and if (a + h, b + k) is a point of this neighbourhood, then there exists a positive number θ which is less than I, such that f(a + h, b + k) =

$$f(a, b) + [hf_x(a, b) + kf_y(a,b)] + \frac{1}{2!}[h^2 fx^2 (a, b) + 2hk\, f_{xy} (a, b) + k^2 fy^2 (a, b)]$$

$$+ \frac{1}{3!}[h^2k\, f_{y2x}(u, v) + 3hk^2 f_{y2x}(u,v) + k^3\, f_{y2x}(u, v)]$$

Where,

$u = a + h, v = b + \theta k$

9. Write Working Rule to find the maximum or minimum value of f(x,y).

Sol :

Step 1 :

Let the given function be f(x,y)

find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ and equation them to zero.

Solve the equation $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ for x and y.

Let the solution be (a,b), (c, d)

Step 2 :

Calculate $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at (a,b) and (c,d). Calculate $rt - s^2$ in each care.

i.e., at (a,b) & (c,d)

Step 3 :

If $rt - s^2 > 0$ and $r < 0$ at (a,b) then f has a maximum value at x = (a,b)

If or at (c,d) if $rt - s^2 > 0$ and $r < 0$ then f has maximum value at (c,d)

Step 4 :

If $rt - s^2 > 0$, and $r > 0$ at (a,b) then f has a minimum value at (a,b) or if $rt - s^2 > 0$ and $r > 0$ at (c,d) & has a minimum at (c,d).

Step 5 :

If $rt - s^2 < 0$. at (a,b) then f has neither , maximum , nor minimum . Then (a,b) is saddle point.

If $rt - s^2 < 0$ at (c,d), then (c,d) is saddle point.

Step 6 :

If $rt - s^2 = 0$, we can't decide whether f, has maximum or minimum for the investigation is needed.

Choose the Correct Answer

1. The function haas neither maximum nor minimum value if [b]

   (a) $rt - s^2 > 0$ (b) $rt - s^2 < 0$

   (c) $r > 0$ (d) $r < 0$

2. If f(x,y) has continuous second order partial derivatives f_{xy} and f_{yx} then [a]

   (a) $f_{xy} = f_{yx}$ (b) $f_{xy} \neq f_{yx}$

   (c) $f_{xy} < f_{yx}$ (d) $f_{xy} > f_{yx}$

3. The function is maximum value if [a]

   (a) $rt - s^2 > 0, \ r < 0$ (b) $rt - s^2 > 0, r > 0$

   (c) $rt - s^2 < 0, r < 0$ (d) $rt - s^2 < 0, r > 0$

4. The conclution for maximum or minimum values are [b]

   (a) $h\frac{\partial f}{\partial x} = 0, k\frac{\partial f}{\partial y} = 0$ (b) $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

   (c) $\frac{\partial^2 f}{\partial x^2} = 0, \frac{\partial^2 f}{\partial y^2} = 0$ (d) None

5. Maximum value of (log x) / x [a]

   (a) $x = \pi/3$ (b) $x = \pi$

   (c) $x = \pi/2$ (d) $x = 0$

6. The maximum value of sin x + cos x is [d]

   (a) 2 (b) 1

   (c) $\sqrt{2}$ (d) $1 + \sqrt{2}$

7. Sinx (1 + cos x) is maximum at [a]

   (a) $x = \pi/3$ (b) $x = \pi$

   (c) $x = \pi/2$ (d) $x = 0$

8. If $y^3 - 3ax^2 + x^3 = 0$ then $\frac{d^2y}{dx^2}$ = [c]

   (a) $\frac{-2(x-a)}{y^5}$ (b) $\frac{2ax - x^2}{y^2}$

   (c) $\frac{-2a^2x^2}{y^5}$ (d) None

9. Euler's theorem on homogenous function if 'F' is homogenous x, y, z of degree n, then [d]

(a) $x\frac{\partial F}{\partial x} - y\frac{\partial F}{\partial y} = nz$

(b) $x\frac{\partial F}{\partial y} + x\frac{\partial F}{\partial x} = nz$

(c) $x\frac{\partial F}{\partial x} - y\frac{\partial F}{\partial y} - z\frac{\partial F}{\partial z} = nF$

(d) $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = nF$

10. If f(x,y) = c then $\frac{dy}{dx}$ [d]

(a) $\frac{\partial f / \partial x}{\partial f / \partial y}$

(b) $\frac{\partial f}{\partial x}$

(c) $\frac{\partial f}{\partial y}$

(d) $\frac{-fx}{fy}$

Fill in the Blanks

1. If $u = y^x$, then $\frac{\partial u}{\partial x}$ = ________
2. The value of f(a,b) is called ________ value of f(x,y)
3. If Δx is increment in x then Δy is ________ in y
4. Legrange's condition for maximum are ________ and ________
5. Legrange's condition for maximum are ________ and ________
6. If $u = (\tan x)^y + y \cot x$ then $\frac{dy}{dx}$ = ________
7. If $u = x^2 - y^2$ then $\frac{\partial u}{\partial x}$ = ________
8. If $z = (\cos y) / x$ and $x = u^2 - v$, $y = e^v$ then $\frac{\partial z}{\partial v}$ = ________
9. $\frac{dz}{dt}$ by composite function ________
10. If $x = e^u + e^{-v}$ then $\frac{\partial u}{\partial x}$ = ________

ANSWERS

1. $y^x \log x$
2. Maxima or minima
3. Consequent increment
4. $rt - s^2 > 0$, $r < 0$
5. $rt - s^2 > 0$ and $r < 0$
6. a
7. 2x
8. $(\cos y - xy \sin y) / x^2$
9. $\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$
10. e^u

UNIT III

Linear Equations in Linear Algebra – Systems of Linear Equations – Consistent and Inconsistent Systems; Solution sets of Linear Systems – trivial and Non trivial Solutions; Linear Independence – Linear Independence of Matrix Columns and Characterization of Linearly Dependent sets.

3.1 Linear Equations in Linear Algebra

Q1. Explain the concept of Linear Equations in Linear Algebra.

Ans : **(Imp.)**

Introduction of Linear Equation

Linear equations are a fundamental concept in the field of linear algebra, which is a branch of mathematics that deals with vector spaces, linear transformations, and systems of linear equations. At its core, linear algebra seeks to understand and analyze the relationships between vectors and how they can be manipulated and transformed in a coherent and systematic manner.

A linear equation is an equation that describes a linear relationship between variables. In its simplest form, a linear equation involves terms that are either constants or the product of a constant and a single variable raised to the power of 1. The fundamental property of linear equations is that when graphed, they form straight lines, hence the name "linear."

Linear equations play a pivotal role in various areas of science, engineering, economics, computer graphics, and more. They are used to model and solve problems that involve proportional relationships, such as the growth of populations, the flow of currents in electrical circuits, and the optimization of resources in economic systems.

When dealing with multiple linear equations involving multiple variables, we encounter systems of linear equations. These systems represent interconnected relationships between variables, where finding a solution means determining the values of the variables that satisfy all the given equations simultaneously.

Linear algebra provides powerful tools and techniques for understanding, analyzing, and solving systems of linear equations. Matrices, which are rectangular arrays of numbers, are used to compactly represent systems of linear equations. Linear transformations, which are operations that map vectors from one space to another while preserving the linear structure, play a significant role in understanding the behavior of systems of equations.

Solving systems of linear equations involves methods such as substitution, elimination, and Gaussian elimination, where the goal is to find the values of the variables that make all equations in the system true. In cases where exact solutions may not exist, techniques like least squares approximation can be employed to find the best possible solution.

Overall, linear equations form the cornerstone of linear algebra, providing a framework for understanding relationships between variables in a wide range of applications. From fundamental concepts to advanced techniques, the study of linear equations in linear algebra is essential for building a solid foundation in mathematical reasoning and problem-solving.

Q2. What is general form of linear equation?

Ans : **(Imp.)**

General form of a linear equation in one variable: let's call it "x," is:

$$\boxed{ax + b = 0}$$

Here, "a" and "b" are constants, and "x" is the variable. The equation is linear because each term involves a constant or the product of a constant and the variable "x" raised to the power of 1.

In this form, the equation represents a straight line on a graph, with "a" determining the slope of the line and

If we're dealing with a linear equation in two variables: "x" and "y," the general form would be:

$$ax + by + c = 0$$

Here, "a," "b," and "c" are constants, and "x" and "y" are the variables. Similarly, the equation is linear because each term involves constants or the products of constants and the variables "x" and "y" raised to the power of 1. In this case, the equation represents a straight line in a two-dimensional coordinate plane.

These general forms of linear equations are foundational in mathematics and have widespread applications in various fields. They serve as the building blocks for more complex systems of linear equations and are essential for understanding linear relationships and their graphical representations

3.2 Systems of Linear Equation

Q3. What is system of linear equation and explain briefly ?

Ans : **(Imp.)**

A system of linear equations is a collection of two or more linear equations involving the same set of variables. These equations describe relationships between variables in a linear manner. The general form of a system of linear equations with "n" equations and "m" variables can be represented as follows:

Standard Form

$$a_{11} x_1 + a_{12} x_2 + ... + a_{1m} x_m = b_1$$
$$a_{21} x_1 + a_{22} x_2 + ... + a_{2m} x_m = b_2$$
$$: \quad : \quad : \quad : \quad :$$
$$: \quad : \quad : \quad : \quad :$$
$$a_{n1} x_1 + a_{n2} x_2 + ... + a_{na} x_m = b_m$$

Here, each "a_{ij}" represents the coefficients of the variables, "x_i" represents the variables, and "b_i" represents constants on the right-hand side of each equation ..

1. Matrix Equation Form

This system can also be represented as a matrix equation using matrix multiplication:

AX = B

Where,

- "A" is an "n × m" matrix containing the coefficients of the variables.
- "X" is an "m × 1" column matrix containing the variables.
- "B" is an "n × 1" column matrix containing the constants on the right-hand side of each equation.

2. Augmented Matrix Form

Another way to represent the system is in augmented matrix form:

[A | B]

Here, "A" is the coefficient matrix, and "B" is the column matrix containing the constants. This form is often used when performing operations like Gaussian elimination to solve the system.

Solving a system of linear equations involves finding values for the variables "x_1" "x_2" ... "x_m" that simultaneously satisfy all of the equations in the system. Depending on the number of equations and variables and the nature of the coefficients, the system may have unique solutions, infinitely many solutions, or no solutions.

Theorem

The system AX=B has

(i) A unique solution if and only if rank (A) = Rank(A|B) = number of variables

(ii) Infinitely many solutions $\rho(A) = \rho(A|B) <$ number of variables and

(iii) No solution (inconsistent) if $\rho(A) \neq \rho(A|B)$ i.e $\rho(A) < \rho(A|B)$

Various methods can be employed to solve systems of linear equations, including:

- **Substitution Method**

  Solve one equation for one variable and substitute the expression into the other equations.

- **Elimination Method**

  Add or subtract equations to eliminate one variable and solve for the other.

- **Matrix Methods**

  Use techniques such as Gaussian elimination or matrix inversion.

- **Graphical Method**

  Graph the equations on a coordinate plane to find the intersection point(s), if they exist.

Solving systems of linear equations is a fundamental concept in mathematics and has a wide range of applications in fields such as engineering, physics, economics, and computer science.

Q4. What is homogeneous system of linear equations and explain it cases ?

Ans : **(Imp.)**

Homogeneous System of Linear Equations

A homogeneous system of linear equations is a system in which all equations are set equal to zero. In other words, each equation follows the form

$$a_1 x_1 + a_2 x_2 + ... + a_n x_n = 0$$

where "a_1" "a_2" ..., "a_n" are coefficients and "x_1" "x_2" ..., "x_n" are the variables. Here's how the two cases break down .

1. Trivial Case

In the trivial case, the system has only one solution, and that solution is the trivial solution, where all variables are equal to zero. This means the origin (0, 0, ..., 0) is the solution to the system. This solution is always present for any homogeneous system since setting all variables to zero will satisfy all equations of the system.

Mathematically, the trivial solution is represented as:

$$x_1 = 0,$$
$$x_2 = 0,$$
$$\vdots$$
$$x_n = 0$$

2. Nontrivial Case

In the nontrivial case, the system has solutions other than the trivial solution. This means that there exist non-zero values for the variables that satisfy the system of equations. In this case, the system has infinitely many possible solutions, forming a solution space. The solution space can be represented as a line, plane, or hyperplane in higher dimensions.

Mathematically, the nontrivial solutions are represented as non-zero values for at least one of the variables:

$$x_1 \neq 0 \text{ or } x_2 \neq 0 \text{ or ... or } x_n \neq 0$$

The distinction between these two cases is essential when studying systems of linear equations. Homogeneous systems often arise in various mathematical and practical contexts, such as in physics, engineering, and computer graphics, and understanding these cases helps in determining the nature of solutions and the behavior of the system.

3.2.1 Consistent and Inconsistent Systems; Solution sets of Linear Systems

Q5. What is non-homogeneous system of linear equations and explain it cases?

Ans : **(Imp.)**

Non-homogeneous System of Linear Equations

A non-homogeneous system of linear equations is a system in which at least one equation is not set equal to zero. In other words, the system involves both the coefficients of the variables and constants on the right-hand side of the equations. The general form of a non-homogeneous system of linear equations with "n" equations and "m" variables can be represented as follows:

1. **Standard Form**

$$a_{11} x_1 + a_{12} x_2 + ... + a_{1m} x_m = b_1$$
$$a_{21} x_1 + a_{22} x_2 + ... + a_{2m} x_m = b_2$$
$$\vdots$$
$$a_{n1} x_1 + a_{n2} x_2 + ... + a_{nm} x_m = b_n$$

Here, each "a_{ij}" represents the coefficients of the variables, "xb"" represents the variables, and "bb"" represents constants on the right-hand side of each equation.

2. **Matrix Equation Form**

This system can be represented as a matrix equation using matrix multiplication:

$$AX = B$$

Where,

- "A" is an "n × m" matrix containing the coefficients of the variables.
- "X" is an "m × 1" column matrix containing the variables.
- "B" is an "n × 1" column matrix containing the constants on the right-hand side of each equation.

3. **Augmented Matrix Form**

The non-homogeneous system can also be represented in augmented matrix form:

$$[A \mid B]$$

Here, "A" is the coefficient matrix, and "B" is the column matrix containing the constants. This form is often used when performing operations like Gaussian elimination to solve the system.

Solving a non-homogeneous system of linear equations involves finding values for the variables $x_1, x_2 ..... x_n$ that simultaneously satisfy all of the equations in the system. The presence of non-zero constants on the right-hand side makes the problem more complex than solving a homogeneous system.

when dealing with non-homogeneous systems of linear equations, there are three possible scenarios or conditions that determine the nature of the solutions. These conditions are related to the coefficients of the equations and the constants on the right-hand side. Let's go through each of these conditions:

1. **Consistent and Unique Solution**

In this case, the non-homogeneous system has a unique solution. This means that there is a specific set of values for the variables that satisfies all of the equations in the system. Geometrically, this corresponds to the intersection of "n" hyperplanes in an "m"-dimensional space at a single point.

in mathematically , If $|A| \neq 0$,system is consistent and unique solution given by $X = A^{-1}B$.

2. **Consistent and Infinite Solutions**

In this scenario, the non-homogeneous system has infinitely many solutions. The equations in the system are not contradictory, and there is more than one way to choose the values of the variables that satisfy the system. Geometrically, this corresponds to the intersection of "n" hyperplanes in an "m"-dimensional space along a line, plane, or hyperplane.

in mathematically , If $|A| = 0$ and $(\text{adj } A)B = 0$,system is consistent and infinite solutions .

3. **Inconsistent**

An inconsistent non-homogeneous system has no solution. This happens when the equations are contradictory and cannot be satisfied simultaneously. Geometrically, this means that the "n" hyperplanes in an "m"-dimensional space do not intersect at all.

in mathematically , If $|A| = 0$ and $(\text{adj } A)B \neq 0$ system is inconsistent.

The nature of the solution depends on the relationships between the coefficients and constants, and whether the equations can be linearly combined to produce consistent results. These conditions can be determined using methods such as Gaussian elimination, which simplifies the system into row-echelon form or reduced row-echelon form, revealing the possibilities for solutions.

Q6. What are the methods are there to finding non-homogeneous systems?

Ans :

Methods for solving non-homogeneous systems include:

➢ **Substitution Method**

Solve one equation for one variable and substitute the expression into the other equations.

➢ **Elimination Method**

Add or subtract equations to eliminate one variable and solve for the other.

➢ **Matrix Methods**

Use techniques such as Gaussian elimination with augmented matrices or matrix inversion.

The solution to a non-homogeneous system can be a unique solution, infinitely many solutions, or no solution, depending on the coefficients, constants, and the relationships between equations. Non-homogeneous systems of linear equations have wide applications in various fields, just like homogeneous systems, and their solution methods are crucial for understanding real-world scenarios involving linear relationships.

PROBLEMS

7. Using matrix inversion method ,solve the following system of equations :

5x + 2y = 4

7x + 3y = 5

Sol :

Given equation is

5x + 2y = 4

7x + 3y = 5

Which can be written in the matrix form as,

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

AX = B

Where

$$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}, X = c \quad B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} = 15 - 14 = 1$$

So, the given system of equation is consistent and independent i.e., it has unique solution is given by

AX = B or $X = A^{-1} \cdot B$

The co-factors of the elements of [A] are

$C_{11} = +3 = 3$

$C_{12} = -(7) = -7$

$C_{21} = -(2) = -2$

$C_{22} = +5 = 5$

$A^{-1} = 1/|A|$ adj.A and

$$\text{adj } A = (CF)t = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times 4 + (-2 \times 5) \\ -7 \times 4 + 5 \times 5 \end{bmatrix} = \begin{bmatrix} 12 - 10 \\ -28 + 25 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Hence, the required solution x=2,y=-3

8. Solve the following linear system

2x + 3y = 8

x – 2y = – 4

Sol :

First, solve one equation for one variable using the elimination method. Let's solve the second equation for x :

x = 2y – 4

Now, substitute this expression for x into the first equation:

2(2y – 4) + 3y = 8

4y – 8 + 3y = 8

7y – 8 = 8

7y = 16

$$y = \frac{16}{7}$$

Now that we have found the value of y, substitute it back into the expression for x:

$$x = 2\left(\frac{16}{7}\right) - 4$$

$$x = \frac{32}{7} - 4$$

$$x = \frac{(32-28)}{7}$$

$$x = \frac{4}{7}$$

So, the unique solution is $x = \frac{4}{7}$ and $y = \frac{16}{7}$.

9. Solve the following linear system:

3x – 2y = 7

4x + 3y = 22

Sol :

We can use the elimination method here. Multiply the first equation by 3 and the second equation by 2 to make the coefficients of y's in both equations equal.

9x – 6y = 21

8x + 6y = 44

Now, add the two equations together to eliminate y:

9x – 6y + 8x + 6y = 21 + 44

17x = 65

x = 65/17

x = 13/3

Now, substitute the value of x into one of the original equations. Let's use the first equation:

3(13/3) – 2y = 7

13 – 2y = 7

– 2y = 7 – 13

– 2y = – 6

y = – 6/– 2

y = 3

So, the unique solution is x = 13/3 and y = 3.

10. Solve the following linear system:

5x + 2y = 11

3x – y = 7

Sol :

Let's use the substitution method for this system. Solve the second equation for y

y = 3x – 7

Now, substitute this expression for y into the first equation

5x + 2(3x – 7) = 11

5x + 6x – 14 = 11

11x – 14 = 11

11x = 11 + 14

11x = 25

x = 25/11

Now that we have found the value of x, substitute it back into the expression for y:

y = 3(25/11) – 7

y = 75/11 – 7

y = 75/11 – 77/11

y = – 2/11

Method 2

11. Solve the following system of linear equations.

x + 2y + z = 7

x + 3z = 11

2x – 3y = 1

Sol :

Given AX = B

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

|A| = 1(0 + 9) – 1(0 + 3) + 2(6)

$|A| = 9 - 3 + 12$

$|A| = 18$

$|A| \neq 0$ so, the given system of equation has a unique solution given by $X = A^{-1}B$

Now ,calculate the adjoint of matrix .

Let C_{ij} be the co-factors of elements a_{ij} in $A = [a_{ij}]$. Then

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix} = 9$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = -3$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 1 \\ -3 & 0 \end{vmatrix} = 3$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = 7$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -2$$

$$\text{adj } A = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}$$

$$A = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

Now, X = $A^{-1}B$

$$X = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63 & -33 & +6 \\ 42 & -22 & -2 \\ -21 & +77 & -2 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

X= 2, y=1 and z= 3 is the required solution

12. Find the system of Linear equation,

2x + 4y – 3z = 4

3y + 4x + 5z = 2

4z + 4x + 3y = 1

Sol : **(Imp.)**

Given Ax = B

2x + 4y – 3z = 4

4x + 3y + 5z = 2

4x + 3y + 4z = 1

$$\begin{bmatrix} 2 & 4 & -3 \\ 4 & 3 & 5 \\ 4 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

AX = B

X = $A^{-1}B$

$$A^{-1} = \frac{\text{Adj}\,A}{|A|}$$

$$|A| = 2\begin{vmatrix} 3 & 5 \\ 3 & 4 \end{vmatrix} - 4\begin{vmatrix} 4 & 5 \\ 4 & 4 \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ 4 & 3 \end{vmatrix}$$

= 2(12 – 15) – 4 (16 – 20) – 3(12 – 12)

= 2(– 3) – 4(–4) – 3(0)

= – 6 + 16

= 10

$\boxed{|A| = 10}$

$$A = \begin{bmatrix} 2 & 4 & -3 \\ 4 & 3 & 5 \\ 4 & 3 & 4 \end{bmatrix}$$

Now, calculate the adjoint of matrix Let C_{ij} be the co-factors of elements a_{ij} in $A = [a_{ij}]$

Then,

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 3 \\ 5 & 4 \end{vmatrix} = 12 - 15 = -3$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} = -9 - 16 = +25$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 4 & 3 \\ -3 & 5 \end{vmatrix} = 20 + 9 = 29$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 5 & 4 \\ 4 & 4 \end{vmatrix} = 20 + 6 = 4$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 4 & -3 \\ 4 & 2 \end{vmatrix} = 8 + 12 = 20$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} -3 & 5 \\ 2 & 4 \end{vmatrix} = 12 + 10 = 22$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 4 \\ 3 & 3 \end{vmatrix} = 12 - 12 = 0$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 4 & 2 \\ 3 & 4 \end{vmatrix} = 16 - 6 = 10$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 4 \\ 4 & 3 \end{vmatrix} = 6 - 16 = -10$$

$$\text{Adj } A = \begin{bmatrix} -3 & -25 & 29 \\ 4 & 20 & -22 \\ 0 & 10 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}\,A}{|A|}$$

$$= \frac{1}{10}\begin{bmatrix} -3 & -25 & 29 \\ 4 & 20 & -22 \\ 0 & 10 & -10 \end{bmatrix} = \begin{bmatrix} \frac{-3}{10} & \frac{-25}{10} & \frac{29}{10} \\ \frac{4}{10} & \frac{20}{10} & \frac{-22}{-10} \\ \frac{0}{10} & \frac{10}{10} & \frac{-10}{10} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{-3}{10} & \frac{-5}{2} & \frac{29}{10} \\ \frac{2}{5} & 2 & \frac{-11}{-10} \\ 0 & 1 & -1 \end{bmatrix}$$

$$X = A^{-1} B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-3}{10} & \frac{-5}{2} & \frac{29}{10} \\ \frac{2}{5} & 2 & \frac{-11}{-10} \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3}{10}\times 4 - \frac{5}{2}\times 2 + \frac{29}{10}\times 1 \\ \frac{2}{5}\times 4 + 2\times 2 - \frac{11}{5}\times 1 \\ 0\times 4 + 1\times 2 - 1\times 1 \end{bmatrix} = \begin{bmatrix} \frac{-12}{10} - \frac{10}{2} + \frac{29}{10} \\ \frac{8}{5} + 4 - \frac{11}{5} \\ 0 + 2 - 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{12 - 50 + 29}{10} \\ \frac{8 + 20 - 11}{5} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-33}{10} \\ \frac{17}{5} \\ 1 \end{bmatrix}$$

$$\therefore \quad X = \frac{-33}{10}, Y = \frac{17}{5}, Z = 1$$

13. Find the system of linear or equation

2x + y + z = 2

4x + y + = 6

9x + 2y + z = 2

Sol : **(Imp.)**

Given Ax = B

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ 9 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

AX = B

$X = A^{-1} B$

$$A^{-1} = \frac{\text{Adj}\, A}{|A|}$$

$$|A| = 2\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & 0 \\ 9 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 9 & 2 \end{vmatrix}$$

$= 2(1 - 0) - 1(4 - 0) + (8 - 9)$

$= 2 - 4 + 1$

$= -3$

$\boxed{|A| = -3}$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ 9 & 2 & 1 \end{bmatrix}$$

Now, calculate the adjoint of matrix

Let C_{ij} be the co-factors of elements a_{ij} in $A = [a_{ij}]$

Then,

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix}2 & 1\\1 & 1\end{vmatrix} = 2-1 = 1$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix}1 & 1\\1 & 0\end{vmatrix} = 0-1 = -1$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix}0 & 1\\4 & 9\end{vmatrix} = 0-4 = -4$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}1 & 1\\9 & 2\end{vmatrix} = 2-9 = -7$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix}1 & 0\\2 & 4\end{vmatrix} = 4-0 = 4$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix}4 & 9\\1 & 2\end{vmatrix} = 8-9 = -1$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix}9 & 2\\2 & 1\end{vmatrix} = 9-4 = 5$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix}2 & 4\\1 & 1\end{vmatrix} = 2-4 = -2$$

$$\text{Adj } A = \begin{bmatrix}1 & 1 & -1\\-4 & -7 & 4\\-1 & 5 & -2\end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}\,A}{|A|}$$

$$= \frac{-1}{3}\begin{bmatrix}1 & 1 & -1\\-4 & -7 & 4\\-1 & 5 & -2\end{bmatrix} = \begin{bmatrix}-\frac{1}{3} & -\frac{1}{3} & \frac{1}{3}\\ \frac{4}{3} & \frac{7}{3} & -\frac{4}{3}\\ \frac{1}{3} & -\frac{5}{3} & \frac{2}{3}\end{bmatrix}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{-4}{3} & \frac{-7}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{5}{3} & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} - \frac{6}{3} + \frac{2}{3} \\ \frac{8}{3} + \frac{42}{3} + \frac{8}{3} \\ \frac{2}{3} + \frac{30}{3} + \frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2-6+2}{3} \\ \frac{8+42-8}{3} \\ \frac{2-30+4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-6}{3} \\ \frac{42}{3} \\ \frac{24}{3} \end{bmatrix} = \begin{bmatrix} -2 \\ 14 \\ -8 \end{bmatrix}$$

$\therefore \quad x = -2, \ y = 14, \ z = -8$

14. Find the system of linear equation.

x + 2y + 3z = 4

4x + 5y + 6z = 8

7x + 8y + 9z = 12

Sol :

Given AX = B

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$

AX = B

$X = A^{-1} B$

$$A^{-1} = \frac{\text{Adj} A}{|A|}$$

$$|A| = 1\begin{vmatrix}5 & 6\\ 8 & 9\end{vmatrix} - 2\begin{vmatrix}1 & 6\\ 7 & 9\end{vmatrix} + 3\begin{vmatrix}4 & 5\\ 7 & 8\end{vmatrix}$$

$$= 1\,(45 - 48) - 2\,(36 - 42) + 3\,(32 - 35)$$

$$= 1(-3) - 2(-6) + 3\,(-3)$$

$$= -3 + 12 - 9$$

$$= 0$$

$\boxed{|A| = 0}$, So find the (Adj A)B

(i) If (Adj A) B = 0, consistent and infinite

(ii) If (Adj A) B $\neq$ 0, In consistent

$$A = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$$

Now, calculate the adjoint of matrix

Let C_{ij} be the co-factors of elements a_{ij} in A = $[a_{ij}]$

Then,

$$C_{11} = (-1)^{1+1}\begin{vmatrix}5 & 8\\ 6 & 9\end{vmatrix} = 45 - 48 = -3$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix}8 & 2\\ 9 & 3\end{vmatrix} = 24 - 18 = -6$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix}2 & 5\\ 3 & 6\end{vmatrix} = 12 - 15 = -3$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix}6 & 9\\ 4 & 7\end{vmatrix} = 42 - 36 = 6$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}9 & 3\\ 7 & 1\end{vmatrix} = 9 - 21 = -12$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix}3 & 6\\ 1 & 4\end{vmatrix} = 12 - 6 = 6$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix}4 & 7\\5 & 8\end{vmatrix} = 32 - 35 = -3$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix}7 & 1\\8 & 2\end{vmatrix} = 14 - 8 = 6$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix}1 & 4\\2 & 5\end{vmatrix} = 5 - 8 = -3$$

$$\text{Adj } A = \begin{bmatrix}-3 & 6 & -3\\6 & -12 & 6\\-3 & 6 & -3\end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= \begin{bmatrix}-3 & 6 & -3\\6 & -12 & 6\\-3 & 6 & -3\end{bmatrix}$$

$$X = A^{-1}B$$

$$X = \begin{bmatrix}-3 & 6 & -3\\6 & -12 & 6\\-3 & 6 & -3\end{bmatrix}\begin{bmatrix}4\\8\\12\end{bmatrix}\begin{bmatrix}-3\times4+6\times8-3\times12\\6\times4-12\times8+6\times12\\-3\times4+6\times8-3\times12\end{bmatrix}$$

$$= \begin{bmatrix}-12+48-36\\24-96+72\\-12+48-36\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

∴ Given the system is consistent and infinitely solutions.

15. Find the system of linear equation.

x + 4y + 7z = 14

3x + 8y – 2z – 13

7x + 8y + 26z – 5

Sol :

Given AX = B

$$\begin{bmatrix}1 & -4 & 7\\3 & 8 & -2\\7 & -8 & 26\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}14\\13\\5\end{bmatrix}$$

$$AX = B$$

$$X = A^{-1} B$$

$$A^{-1} = \frac{\text{Adj}\,A}{|A|}$$

$$|A| = \begin{vmatrix} 8 & -2 \\ -8 & 26 \end{vmatrix} + 4\begin{vmatrix} 3 & -2 \\ 7 & 26 \end{vmatrix} + 7\begin{vmatrix} 3 & 8 \\ 7 & -8 \end{vmatrix}$$

$$= 1(208 - 16) + 4\,(78 + 14) + 7(-24 - 56)$$

$$= 1(192) + 4\,(92) + 7(-80)$$

$$= 0$$

Now, calculate the adjoint of matrix

Let C_{ij} be the co-factors of elements a_{ij} in $A = [a_{ij}]$,

Then,

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 8 & -8 \\ -2 & 26 \end{vmatrix} = 208 - 16 = 192$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} -8 & -4 \\ 26 & 7 \end{vmatrix} = -56 - (-104) = 48$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -4 & 8 \\ 7 & -2 \end{vmatrix} = 8 - 56 = -48$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 26 \\ 3 & 7 \end{vmatrix} = -14 - 78 = -92$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 26 & 7 \\ 7 & 1 \end{vmatrix} = 26 - 49 = -23$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 7 & -2 \\ 1 & 3 \end{vmatrix} = 21 - (-2) = 23$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 7 \\ 8 & -8 \end{vmatrix} = -24 - 56 = -80$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 7 & 1 \\ -8 & -4 \end{vmatrix} = -28-(-8) = -20$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ -4 & 8 \end{vmatrix} = 8+12 = 20$$

$|A| = 0$. So find the (Adj A) B

(i) If (Adj A) B = 0, Consistent and infinite

(ii) If (Adj A) B $\neq$ 0, In consistent

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 192 & 48 & -48 \\ -92 & -23 & 23 \\ -80 & -20 & 20 \end{bmatrix}\begin{bmatrix} 14 \\ 13 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 192\times14+48\times13-48\times5 \\ -92\times14-23\times13+23\times5 \\ -80\times14-20\times13+20\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 2688+624-240 \\ 1288-299+115 \\ -11200-260+100 \end{bmatrix} = \begin{bmatrix} 3072 \\ 1104 \\ -11360 \end{bmatrix}$$

$\therefore$ X = 3072, Y = 1104, Z = – 11360

Given System is inconsistent

16. The following system of equations

$$x_1 + x_2 + x_3 = 3$$

$$x_1 + x_3 = 0$$

$$x_1 - x_2 + x_3 = 1$$

Sol :

Given AX = B

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Consider the augmented matrix [A|B]

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{array}\right]$$

$R_1 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -3 \\ 0 & -2 & 0 & -2 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

$\rho(A) = 2$, $\rho(A|B) = 3$

here $\rho(A) \neq \rho(A|B)$

Solution does not exist.

17. Solve the following linear system

$x+2y = 5$

$4x+8y = 12$

$3x+6y+3z = 15$

Sol :

Given AX = B

$$= \begin{bmatrix} 1 & 2 & 0 \\ 4 & 8 & 0 \\ 3 & 6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 15 \end{bmatrix}$$

$$\text{Consider } [A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 4 & 8 & 0 & 12 \\ 3 & 6 & 3 & 15 \end{array}\right]$$

$R_2 \to R_2 \to 4R_1$

$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

$\rho(A) = 2$, $\rho(A|B) = 3$

Here $\rho(A) \neq \rho(A|B)$

No solution

18. Solve the following linear system using.

$x + y + z = 6$

$2x + 3y + 2z = 14$

$x + 2y + 3z = 15$

Sol :

Given AX=B

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 15 \end{bmatrix}$$

$$\text{Consider } [A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 2 & 14 \\ 1 & 2 & 3 & 15 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 9 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 7 \end{array}\right]$$

Divide the third row by 2

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3.5 \end{array}\right]$$

Perform back-substitution to find the solutions:

$z = 3.5$

$x + z = 4$

$x + 3.5 = 4$

$x = 0.5$

$y = 2$

So, the unique solution is $x = 0.5$, $y = 2$, and $z = 3.5$.

3.3 Trivial and Nontrivial Solutions in Systems of Linear Equations

Q19. What is trivial and nontrivial solution?

Ans :

i) Trivial Solution

A trivial solution is a solution to a system of equations where all variables are set to zero. In other words, it's the solution where no variable takes on a nonzero value. Trivial solutions are always valid for any system of equations.

ii) Nontrivial Solution

A nontrivial solution is a solution where at least one variable takes on a nonzero value. Nontrivial solutions provide meaningful solutions to the system of equations.

Problems

20. Find the following equations trivial.

$3x + 2y = 0$ ... (1)

$2x - 3y = 0$ ... (2)

Sol :

Trivial

The trivial solution occurs when both variables are set to zero. Let's solve for x and y .

$3x + 2y = 0$ ... (1)

$2x - 3y = 0$ ... (2)

Dividing equation (2') by 2:

$x - (3/2)y = 0$

$x = (3/2)y$

Substituting x in equation (1)

$3((3/2)y) + 2y = 0$

$(9/2)y + 2y = 0$

$(9/2 + 4/2)y = 0$

$(13/2)y = 0$

$\boxed{y = 0}$

Substituting $y = 0$ in $x = (3/2)y$

$\boxed{x = 0}$

Therefore, the trivial solution is $x = 0$ and $y = 0$.

21. Find the following equations trivial

$2x + y + z = 0$

$3x - 2y + z = 0$

$x + 3y - z = 0$

Sol :

We want to find the values of x, y, and z that satisfy all three equations simultaneously. Let's consider the trivial solution where all variables are set to zero: $x = 0$, $y = 0$, and $z = 0$.

For Equation 1

$2(0) + 0 + 0 = 0$

$0 + 0 + 0 = 0$ (Satisfied)

For Equation 2

$3(0) - 2(0) + 0 = 0$

$0 - 0 + 0 = 0$ (Satisfied)

For Equation 3

0 + 3(0) – 0 = 0

0 + 0 – 0 = 0 (Satisfied)

Since all three equations are satisfied by x = 0, y = 0, and z = 0, this is called the trivial solution.

22. Given equation to find non trivial solution.

3x + 2y = 6 ... (1)

2x – 3y = -5 ... (2)

Sol :

To find a nontrivial solution, we need to determine values for x and y that satisfy both equations and are not both zero.

One way to approach this is by solving one equation for one variable and then substituting that value into the other equation.

Solve Equation (1) for x

3x + 2y = 6

3x = 6 – 2y

x = (6 – 2y) / 3

Put in equation (2)

2x – 3y = -5

2((6 – 2y)/3) – 3y = -5

(12 – 4y)/3 – 3y = -5

12 – 4y – 9y = -15

12 – 13y = – 15

– 13y = – 27

y = 27 / 13

y = 27 / 13substituting in x = (6 – 2y) / 3 for x

x = (6 – 2(27/13)) / 3

x = (78 – 54)/39

x = 24/39

x = 8/13

So, a nontrivial solution for the system of equations is.

x = 8/13

y = 27/13

These values satisfy both Equation 1 and Equation 2 and are not both zero. This is a nontrivial solution because both x and y are nonzero.

The nontrivial solution for the given system of equations is x = 8/13 and y = 27/13

23. Given equation to find non trivial solution.

2x – y = 3 ... (1)

4x + y = 7 ... (2)

Sol :

Solve Equation (1) for x

2x – y = 3

2x = 3 + y

x = (3 + y)/2

Substitute x into Equation (2)

4x + y = 7

4((3 + y)/2) + y = 7

(12 + 4y)/2 + y = 7

12 + 4y + 2y = 14

6y = 2

y = 1/3

y = 1/3 substituting in x = (3 + y)/2

x = (3 + (1/3))/2

x = (10/3)/2

x = 5/3

So, a nontrivial solution for the system of equations is:

$x = 5/3$

$y = 1/3$

The nontrivial solution for the given system of equations is $x = 5/3$ and $y = 1/3$.

24. find the following equations non trivial

$2x - y + z = 5$ ... (1)

$4x + 2y - z = 8$ ... (2)

$x - 3y + 2z = 1$ ... (3)

Sol : **(Imp.)**

We want to find a nontrivial solution for this system, which means values for x, y, and z that satisfy all three equations and are not all equal to zero.

Solve Equation (1)for x

$2x - y + z = 5$

$2x = 5 + y - z$

$x = (5 + y - z)/2$ ... (4)

Substitute x into Equation (2)

$4x + 2y - z = 8$

$4((5 + y - z) / 2) + 2y - z = 8$

$(10 + 4y - 2z) + 2y - z = 8$

$10 + 4y - 2z + 2y - z = 8$

$6y - 3z = -2$

$6y = 3z - 2$

$y = (3z - 2)/6$

$y = (z - 2/3)/2$

$y = (z - 1/3)$ ... (5)

Substitute x into Equation (3)

$x - 3y + 2z = 1$

$((5 + y - z)/2) - 3y + 2z = 1$

$(5 + y - z) - 6y + 4z = 2$

$5 + y - z - 6y + 4z = 2$

$-5y + 3z = -3$

$-5y = -3 - 3z$

$y = (3 + 3z)/5$

$y = (3/5) + (3/5)z$

From equation (4) and (5)

$x = (5 + y - z)/2$

$x = (5 + (z - 1/3) - z)/2$

$x = (5 + z - 1/3 - z)/2$

$x = (4 - 1/3)/2$

$x = (11/3)/2$

$x = 11/6$

So, a nontrivial solution for the system of equations is $x = 11/6$, $y = z - 1/3$

These values satisfy all three equations and are not all equal to zero. This is a nontrivial solution because at least one of the variables (x, y, or z) is nonzero.

3.4 Linear Independence and Dependence of Matrix Columns

Q25. What is linear independent and explain with an example?

Ans : **(Imp.)**

Linearly Independent Columns

In linear algebra, the linear independence and dependence of matrix columns are crucial concepts. They determine whether the columns of a matrix form a linearly independent set or a linearly dependent set. Here's a detailed explanation of these concepts.

Matrix columns are linearly independent if none of the columns can be expressed as a linear combination of the others. In other words, no column can be obtained by multiplying another column by a scalar and adding it to a different column. Mathematically, if "A" is an "m x n" matrix with columns "v_1," "v_2" ..., "v_n" then the columns are linearly independent if the equation:

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$$

only has the trivial solution:

$$c_1 = c_2 = ... = c_n = 0$$

In this case, the columns span the full "n"-dimensional space of vectors.

Example

Suppose we have the following matrix "A" with three columns

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

We want to determine whether the columns of matrix "A" are linearly independent.

To check for linear independence, we need to see if the equation:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

where "v_1," "v_2" and "v_3'' are the columns of matrix "A," has a nontrivial solution. In other words, we want to find constants "c_1" "c_2" and "c_3" not all zero, such that the linear combination results in the zero vector.

Let's set up the equation and solve for the constants:

$$c_1 * \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} c_2 * \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} c_3 * \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system of equations gives us,

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = 0$$

Since the only solution is the trivial solution (all constants equal to zero), the columns of matrix "A" are linearly independent. This means that none of the columns can be expressed as a linear combination of the others

In this example, the matrix "A" has linearly independent columns because no nontrivial linear combination of the columns results in the zero vector.

Determinant Test for Linear Independence

For a set of vectors or a matrix, you can use the determinant to test for linear independence. Consider a set of vectors $\{v_1 , v_2, ..., v_n\}$. If you arrange these vectors as columns in a matrix "A," then the vectors are linearly independent if and only if the determinant of the matrix "A" is nonzero. Mathematically:

If det(A) $\neq$ 0, then the vectors $\{v_1, v_2, ..., v_n\}$ are linearly independent.

Example of Linear Independence

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = 1(1 – 0) – 2(0 – 0) + 3(0 – 0) = 1$$

$$|A| = 1$$

$$|A| \neq 0$$

The determinant of an upper triangular matrix (a matrix where all entries below the diagonal are zero) is simply the product of its diagonal entries. Therefore, the determinant of matrix "A" is:

$$\det(A) = 1 * 1 * 1 = 1$$

Since the determinant of matrix "A" is not equal to zero (det (A) $\neq$ 0), this confirms that the columns of matrix "A" are indeed linearly independent.

The determinant test is a powerful method to determine linear independence, especially for square matrices. When the determinant is nonzero, the columns are linearly independent. If the determinant is zero, it indicates that the columns are linearly dependent. In your example, the fact that the determinant of matrix "A" is not equal to zero directly supports the conclusion that the columns are linearly independent .

Q26. What is linear dependent and explain with an example

Ans : **(Imp.)**

Linearly Dependent Columns

Matrix columns are linearly dependent if at least one column can be obtained as a linear combination of the others. In other words, there exist scalars, not all of which are zero, such that the linear combination of columns results in the zero vector. Mathematically, if there exist constants "c_1," "c_2" ..., "c_n" not all zero, such that:

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$$

Then the columns are linearly dependent

Example of linear dependent

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

We want to determine whether the columns of matrix "B" are linearly dependent.

To check for linear dependence, we need to see if there exist constants "c_1," "c_2" and "c_3" not all zero, such that the linear combination of columns results in the zero vector. We'll set up the equation

$$c_1 * \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} c_2 * \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} c_3 * \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

solving this system of equations, we find:

1. The first row gives us: $c_1 + 2c_2 + c_3 = 0$
2. The second row gives us: $2c_1 + 4c_2 + 2c_3 = 0$
3. The third row gives us: $3c_1 + 6c_2 + 3c_3 = 0$

Now, let's simplify these equations:

1. $c_1 + 2c_2 + c_3 = 0$
2. $2c_1 + 4c_2 + 2c_3 = 0$

Notice that the third equation is simply twice the first equation. In other words, it's a linear combination of the first and second equations. Therefore, we have a nontrivial solution to the system of equations:

$$c_1 = -2c_2 - c_3$$

$$2c_1 + 4c_2 + 2c_3 = 0$$

Since there exists a nontrivial solution (not all constants are zero) that makes the linear combination equal the zero vector, the columns of matrix "B" are linearly dependent.

In this example, the matrix "B" has linearly dependent columns because a nontrivial linear combination of the columns results in the zero vector.

Determinant Test for Linear Dependent

The determinant method is a useful approach to determine whether the columns of a matrix are linearly dependent. Specifically, if the determinant of the matrix is zero, it indicates that the columns are linearly dependent.

Example

We want to determine whether the columns of matrix "A" are linearly dependent using the determinant method**.**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$|A| = 1(12 - 12) - 2(6 - 6) + 3(4 - 4)$$

$$|A| = 0$$

Since the determinant of matrix "A" is zero (det(A) = 0), the vectors v_1, v_2, and v_3 are linearly dependent.

Keep in mind that the determinant method is a convenient way to determine linear independence or dependence when you're dealing with small sets of vectors or columns. For larger sets, computational tools may be more practical.

3.4.1 Characterization of Linearly Dependent sets

Q27. What are the Characteristics of Linearly Dependent ?

Ans : **(Imp.)**

1. **Determinant of a Square Matrix**

   For a square matrix "A," its columns are linearly dependent if and only if the determinant of "A" is zero. This determinant test is useful in determining linear dependence.

2. **Rank Deficiency**

   Columns are linearly dependent if the rank of the matrix is less than the number of columns. A rank-deficient matrix cannot fully span the space.

3. **Nontrivial Linear Combination**

   Linear dependence means that there is a nontrivial linear combination of columns that results in the zero vector.

4. **Inconsistent System**

   If the homogeneous system of equations formed by the columns has more unknowns than equations, the columns are linearly dependent.

5. **Repeated Columns**

   If a matrix has repeated columns, it is trivially linearly dependent since one column can be expressed as a scalar multiple of the other repeated column.

PROBLEMS

28. To determine whether the given system of equations is linearly dependent using the determinant method :

3x + 2y = 6 ... (1)

2x – 3y = – 5 ... (2)

Sol : **(Imp.)**

We can represent the coefficient matrix as follows:

Now, let's calculate the determinant of this matrix:

Determinant = (3 * – 3) – (2 * 2)

Determinant = – 9 – 4

Determinant = – 13

Since the determinant is nonzero (– 13), the system of equations is linearly independent. This means that the equations are not proportional to each other and do not lie on the same line in the coordinate plane.

In summary, using the determinant method, we found that the given system of equations is linearly independent, which implies that the equations are not linearly dependent and do not represent the same line.

29. Determining Linear Dependence Using the Determinant Method

2x + y – z = 5 ... (1)

4x – 3y + 2z = 1 ... (2)

x + 2y – z = 3 ... (3)

Sol :

$$|A| = \begin{bmatrix} 2 & 1 & -1 \\ 4 & -3 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

$= 2(3 - 4) - 1(-4 - 2) + (8 - 3)$
$= 2(1) - 1(-6) + (5)$
$= 2 + 6 + 5$
$= 13$

Since the determinant is nonzero (-13), the system of equations is linearly independent .

30. Determine if a system of linear equations is linearly dependent or independent using the determinant method?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Sol :

$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$
$= 1(-3) - 2(-6) + 3(-3)$
$= -3 + 12 - 9$
$= 12 - 12$
$= 0$
$|A| = 0$

$\therefore$ A islinear dependent

31. Determine if a system of linear equations is linearly dependent or independent using the determinant method?

$$A = \begin{bmatrix} 4 & 6 & 9 \\ 0 & 2 & 4 \\ 3 & 2 & 4 \end{bmatrix}$$

Sol :

$$|A| = \begin{bmatrix} 4 & 6 & 9 \\ 0 & 2 & 4 \\ 3 & 2 & 4 \end{bmatrix}$$

$= 4(0) - 6(-12) + 9(-6)$
$= 0 + 72 - 36$
$|A| = 36$
$|A| \neq 0$

$\therefore$ A is in linear dependent.

Short Question and Answers

1. Linear Equation

Ans :

Linear equations are a fundamental concept in the field of linear algebra, which is a branch of mathematics that deals with vector spaces, linear transformations, and systems of linear equations. At its core, linear algebra seeks to understand and analyze the relationships between vectors and how they can be manipulated and transformed in a coherent and systematic manner.

A linear equation is an equation that describes a linear relationship between variables. In its simplest form, a linear equation involves terms that are either constants or the product of a constant and a single variable raised to the power of 1. The fundamental property of linear equations is that when graphed, they form straight lines, hence the name "linear."

2. What is general form of linear equation?

Ans :

General form of a linear equation in one variable: let's call it "x," is:

$$ax + b = 0$$

Here, "a" and "b" are constants, and "x" is the variable. The equation is linear because each term involves a constant or the product of a constant and the variable "x" raised to the power of 1.

In this form, the equation represents a straight line on a graph, with "a" determining the slope of the line and

If we're dealing with a linear equation in two variables: "x" and "y," the general form would be:

$$ax + by + c = 0$$

Here, "a," "b," and "c" are constants, and "x" and "y" are the variables. Similarly, the equation is linear because each term involves constants or the products of constants and the variables "x" and "y" raised to the power of 1. In this case, the equation represents a straight line in a two-dimensional coordinate plane.

3. Matrix Equation Form

Ans :

This system can also be represented as a matrix equation using matrix multiplication:

AX = B

Where,

- "A" is an "n × m" matrix containing the coefficients of the variables.
- "X" is an "m × 1" column matrix containing the variables.
- "B" is an "n × 1" column matrix containing the constants on the right-hand side of each equation.

4. Augmented Matrix Form

Ans :

Another way to represent the system is in augmented matrix form:

[A | B]

Here, "A" is the coefficient matrix, and "B" is the column matrix containing the constants. This form is often used when performing operations like Gaussian elimination to solve the system.

Solving a system of linear equations involves finding values for the variables "x_1" "x_2" ... "x_m" that simultaneously satisfy all of the equations in the system. Depending on the number of equations and variables and the nature of the coefficients, the system may have unique solutions, infinitely many solutions, or no solutions.

5. Homogeneous System of Linear Equations

Ans :

A homogeneous system of linear equations is a system in which all equations are set equal to zero. In other words, each equation follows the form

$a_1 x_1 + a_2 x_2 + ... + a_n x_n = 0$

where "a_1" "a_2" ..., "a_n" are coefficients and "x_1" "x_2" ..., "x_n" are the variables. Here's how the two cases break down.

6. Non-homogeneous System of Linear Equations

Ans :

A non-homogeneous system of linear equations is a system in which at least one equation is not set equal to zero. In other words, the system involves both the coefficients of the variables and constants on the right-hand side of the equations.

7. What are the methods are there to finding non-homogeneous systems?

Ans :

Methods for solving non-homogeneous systems include:

- **Substitution Method**

  Solve one equation for one variable and substitute the expression into the other equations.

- **Elimination Method**

  Add or subtract equations to eliminate one variable and solve for the other.

- **Matrix Methods**

  Use techniques such as Gaussian elimination with augmented matrices or matrix inversion.

8. Trivial Solution

Ans :

A trivial solution is a solution to a system of equations where all variables are set to zero. In other words, it's the solution where no variable takes on a nonzero value. Trivial solutions are always valid for any system of equations.

9. Nontrivial Solution

Ans :

A nontrivial solution is a solution where at least one variable takes on a nonzero value. Nontrivial solutions provide meaningful solutions to the system of equations.

10. Linearly Independent Columns

Ans :

In linear algebra, the linear independence and dependence of matrix columns are crucial concepts. They determine whether the columns of a matrix form a linearly independent set or a linearly dependent set. Here's a detailed explanation of these concepts.

Matrix columns are linearly independent if none of the columns can be expressed as a linear combination of the others. In other words, no column can be obtained by multiplying another column by a scalar and adding it to a different column. Mathematically, if "A" is an "m x n" matrix with columns "v_1," "v_2" ..., "v_n" then the columns are linearly independent if the equation:

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$$

only has the trivial solution:

$$c_1 = c_2 = ... = c_n = 0$$

In this case, the columns span the full "n"-dimensional space of vectors.

Choose the Correct Answer

1. A system of linear equations with exactly one solution is called: [a]
   (a) Consistent (b) Inconsistent
   (c) Homogeneous (d) Dependent

2. The solution set of a homogeneous system of linear equations always includes: [d]
   (a) No solutions (b) One solution
   (c) More than one solution (d) The zero vector

3. A matrix with more columns than rows is called a: [c]
   (a) Square matrix (b) Row matrix
   (c) Column matrix (d) Rectangular matrix

4. If the determinant of a square matrix is zero, then the matrix is: [c]
   (a) Invertible (b) Orthogonal
   (c) Singular (d) Symmetric

5. The rank of a matrix is defined as: [d]
   (a) The number of rows in the matrix
   (b) The number of columns in the matrix
   (c) The sum of the elements in the matrix
   (d) The maximum number of linearly independent rows or columns

6. The reduced row-echelon form (RREF) of a matrix is: [c]
   (a) A matrix with all elements set to zero
   (b) A matrix with ones along the main diagonal
   (c) A unique matrix that's obtained through row operations
   (d) A matrix with only positive integer values

7. The solution set of a consistent system of linear equations can be described as: [c]
   (a) A single point in space (b) A line
   (c) A plane (d) A hyperplane

8. The span of a set of vectors is: [c]
   (a) The number of vectors in the set
   (b) The sum of the vectors in the set
   (c) The set of all possible linear combinations of the vectors
   (d) The determinant of the vectors

9. Linearly independent vectors in a vector space are always: [d]
   (a) Parallel to each other
   (b) Orthogonal to each other
   (c) Collinear with the origin
   (d) Linearly dependent

10. The column space of a matrix is also known as its: [d]
    (a) Null space
    (b) Row space
    (c) Rank space
    (d) Range

11. A matrix A is said to be orthogonal if: [c]
    (a) Its determinant is zero
    (b) It is a square matrix
    (c) A * A ^ T = I (identity matrix)
    (d) It has all elements equal to one

12. The determinant of an upper triangular matrix is equal to: [b]
    (a) The sum of its diagonal elements
    (b) The product of its diagonal elements
    (c) Zero
    (d) One

13. The solution set of an inconsistent system of equations is: [c]
    (a) Non-empty
    (b) Infinite
    (c) Empty
    (d) Unique

14. If a matrix has linearly dependent columns, its determinant is: [a]
    (a) Always zero
    (b) Always nonzero
    (c) Equal to the identity matrix
    (d) Equal to the zero matrix

15. The dot product of two vectors is zero if: [a]
    (a) The vectors are orthogonal
    (b) The vectors have the same magnitude
    (c) The vectors are parallel
    (d) The vectors are linearly dependent

Fill in the Blanks

1. A system of linear equations that has at least one solution is called a ________ system.
2. The ________ solution of a system of linear equations is the one where all variables are set to zero.
3. The solution to an inconsistent system is ________.
4. A set of vectors is said to be linearly ________ if none of the vectors in the set can be written as a linear combination of the others.
5. The columns of an identity matrix are ________.
6. If the determinant of a square matrix is zero, then its columns are ________.
7. In a matrix, a linearly dependent set of columns will lead to a ________ determinant.
8. A system of linear equations with more equations than unknowns is typically ________.
9. A system of linear equations with the same number of equations and unknowns can be categorized as ________.
10. The solution space of a homogeneous system of linear equations always contains the ________ vector.

Answers

1. consistent
2. trivial
3. empty
4. independent
5. linearly independent
6. linearly dependent
7. zero
8. overdetermined
9. square
10. zero

UNIT IV	Vector spaces and Subspaces, Linearly independent sets; bases. Eigenvalues and Eigenvectors - The Characteristic Equation.

4.1 Vector Spaces

Q1. Define vector space.

Sol :

A vector space is a non empty set V of objects called vectors on which are defined two operations called addition and multiplication by scalars subject to ten axioms.

(I) $(V_1 +)$ is abelian vector addition.

1. $u + v \in v$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is zero vector 0 in v such that $u + 0 = u$.
5. For each u in v there is a vector –u in v such that $u + (-u) = 0$.

(II) Scalar Multiplication

6. The scalar multiple of u by c that is $cu \in v$.
7. $c(u + v) = cu + cv$
8. $(c + d)\, u = cu + du$
9. $c(du) = (cd)\, u$
10. $1.u = u$

The space R^n; where $n \geq 1$ is a vector space.

Q2. For $n \geq 0$ the set p_n of polynomials of degree atmost n consists of all polynomials of the form.

$$P(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n$$

Where coefficient $a_0, a_1 .... a_n$ and variable t are real numbers Here degree is n.

Sol :

Given p_n set of polynomials

Let $p(t), q(t) \in p_n$

$p(t) = a_0 + a_1 t + a_2 t^2 + ... a_n t^n$

$q(t) = b_0 + b_1 t + b_2 t^2 + ... b_n t^n$

Vector addition

1. $p(t) + q(t) = a_0 + a_1t + a_2t^2 + ... a_nt^n + b_0 + b_1t + b_2t^2 + ... b_nt^n$
   $= (a_0 + b_0) + (a_1 + b_1)\, t + (a^2 + b^2)\, t^2 + ... + (a_n + b_n)\, t^n \in p_n$ [closure]
2. $p(t) + [q(t) + r(t)] = [p(t) + q(t)] + r(t)$ [Associative]
   let $r(t) = c_0 + c_1t + c_2t^2 + ... c_nt^n$.
3. Let $0(t) = 0 + 0t + 0t^2 + ... + 0t^n$ be the zero polynomial.
   $p(t) + 0(t) = p(t)$ [Additive Identity]
4. Since (–1) p(t) acts as negative of p(t)
   $\therefore \quad \forall\, p(t) \in p_n \;\exists\; (-1)\, p(t) \in p_n \ni$
   $p(t) + (-1)\, p(t) = 0(t)$ [Additive Inverse]
5. $p(t) + q(t) = q(t) + p(t)$ [Cumulative law]

Scalar Multiplications

6. Scalar multiple c_p is a polynomial defined by
   $c.p(t) = c[a_0 + a_1t + ... + a_nt^n]$
   $= ca_0 + ca_1t + ... + ca_nt^n \in p_n$.
7. $(c + d)\, p(t) = cp(t) + dp(t)$
8. $c(p(t) + q(t)) = cp(t) + cq(t)$
9. $c[d(p(t))] = cd[p(t)]$
10. $1.p(t) = p(t)$ [Mul. Identity]
   $\therefore$ Thus p_n is a vector space.

4.2 SUBSPACES

Q3. Define vector subspace and give examples.

Sol :

Let H be a non empty subset of a vector space V that it is said to be a vector subspace of V if it satisfy the following conditions.

1. The zero vector of v is in H
   $\Rightarrow O \in H$
2. H is closed under vector addition $\forall U, V \in H$
   $\Rightarrow U + V \in H$
3. H is closed under scalar multiplication for each $U \in H \;\exists$ scalar $C \ni CU \in H$.

Example

1. The set containing of only the zero vector in a vector space V is a subspace of V called the zero subspace of written as {0}.
2. The vector space R^2 is not a subspace of R^3 because R^2 is not even a subset of R^3. Since vectors in R^3 all have three entries where vectors in R^2 have only two entries.

Q4. Given v_1 and v_2 in a vector space V, let H = span $\{v_1, v_2\}$ show that H in a subspace of V.

Sol :

Given H = span $\{v_1, v_2\}$

The zero vector is in H since $o = ov_1 + ov_2$

To show let is closed under vector addition

Let $u = a_1 u_1 + a_2 u_2$ $\quad w = b_1 v_1 + b_2 v_2$

$u + w = (a_1 v_1 + a_2 v_2) + (b_1 v_1 + b_2 v_2)$

$= (a_1 + b_1) v_1 + (a_2 + b_2) v_2$

$\therefore u + w \in H.$

To show H is closed under scalar multiplication let c be any scalar and $u = a_1 v_1 + a_2 v_2$

$cu = c(a_1 v_1 + a_2 v_2)$

$= (ca_1) v_1 + (ca_2) v_2$

$\therefore cu \in H.$

$\therefore$ H is closed under vector addition and scalar multiplication.

$\therefore$ Thus H is a subspace of v.

Q5. If $v_1, v_2, \ldots, v_p$ are in a vector space v then span $\{v_1, v_2, \ldots, v_p\}$ is a subspace of v.

Sol :

(i) The zero vector is in H

$\therefore O = ov_1 + ov_2 + \ldots ov_p$

(ii) Let u, v any two arbitrary vectors in H

$u = a_1 v_1 + a_2 v_2 + \ldots a_p v_p$, $\quad v = b_1 v_1 + b_2 v_2 + \ldots b_p v_p$

Where $a_1 a_2 \ldots a_p, b_1, b_2, \ldots b_p$ are scalars.

Now

$u + v = (a_1 v_1 + a_2 v_2 + \ldots + a_p v_p) + (b_1 v_1 + b_2 v_2 + \ldots + b_p v_p)$

$= (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \ldots + (a_p + b_p)v_p$

$\therefore u + v \in H$

$\therefore$ H is closed under vector addition.

for any scalar c

$cu = c(a_1 v_1 + a_2 v_2 + \ldots + a_p v_p)$

$= (ca_1)v_1 + (ca_2)v_2 + \ldots + (ca_p)v_p$

$\therefore cu \in H$

$\therefore$ H is closed under scalar multiplication

$\therefore$ H is a subspace of V.

Q6. If H and K are subspaces of a vector space then H + K is also subspace of vector space V(F).

Sol :

Given V(F) is a vector space

H and k are subspaces of V

Define H + K = {w; U + v = w; for some u in H and v in k}

Given H is a subspace of V(F)

$0 \in H$ ... (1)

k is a subspace of v(F)

$0 \in k$ ... (2)

from (1) and (2) $0 \in H + k$

$\therefore H + k \neq \{\ \}$ (non empty)

Let $w_1, w_2 \in H + K$

$w_1 = u_1 + v_1$ where $u_1 \in H, v_1 \in K$

$w_2 = u_2 + v_2$ where $u_2 \in H, v_2 \in K$

$w_1 + w_2 = (u_1 + v_1) + (u_2 + v_2)$

$= (u_1 + u_2) + (v_1 + v_2)$

$\in H + K$ [since $u_1 + u_2 \in H, v_1 + v_2 \in K$]

$\therefore$ H + K is closed under vector addition.

Let $cu_1 \in H$, $cu_1 \in K$

$cu_1 + cv_1 \in H + K$

$c(u_1 + v_1) \in H + K$

$cw_1 \in H + K$

$H + K$ is closed under scalar multiplication.

$\therefore$ H + K is a subspace of V(F).

Q7. The union of two subspaces is again a subspace if and only if one is contained in another.

(OR)

The union of two subspaces is again subspaces $\Leftrightarrow H_1 \subseteq H_2$ (or) $H_2 \subseteq H_1$.

Sol : **(Imp.)**

Let H_1 and H_2 be two subspaces of V(F).

Case (1)

If $H_1 \cup H_2$ is a subspace of a vector space V(F).

Then we have to show that $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

If possible assume that $H_1 \not\subseteq H_2$ or H_2 H_1.

Since $H_1 \not\subseteq H_2$ so $\exists$ $a \in H_1$ and $a \notin H_2$

Since $H_2 \not\subseteq H_1$ so $\exists$ $b \in H_1$ and $b \notin H_2$

But $a \in H_1 \cup H_2$ and $b \in H_1 \cup H_2$

$\Rightarrow$ $a + b \in H_1 \cup H_2$

$\Rightarrow$ $a + b \in H_1$ and $a + b \in H_2$

Since $a + b \in H_1$ and $a \in H_1$, as

As H_1 in a subspace of V(F)

$(-1)\ a + a + b = b \in H_1$ [Closure] ... (1)

Similarly $a + b \in H_2$, $b \in H_2$

As H_2 is a subspace of V(F)

$\Rightarrow$ $a + b + (-1)\ b \in H_2$ [closure]

$\Rightarrow$ $a + b - b \in H_2$

$\Rightarrow$ $a \in H_2$ ... (2)

Which is a contradiction to our assumption that $a \notin H_2$ and $b \notin H_1$

So our assumption is using

$\therefore$ either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

Case (ii)

If $H_1 \subseteq H_2$ (or) $H_2 \subseteq H_1$ then we have to show $H_1 \cup H_2$ is a subspace of V(F).

Since $H_1 \subseteq H_2 \Rightarrow H_1 \cup H_2 = H_2$

We know that H_2 is a subspace of V(F)

So $H_1 \subseteq H_2$ is also subspace of V(F).

Case (iii)

If $H_2 \subseteq H_1 \Rightarrow H_1 \cup H_2 = H_1$

We know that H_1 is a subspace of V(F)

So $H_1 U H_2$ is also subspace of V(F)

$\therefore$ $H_1 U H_2$ is subspace of vector space V(F).

Q8. Let H be the set of all vectors of the from (a – 3b, b – a, a, b) where a and b are arbitrary scalars let H = {(a – 3b, b – a, a, b)}; a, b in R. Show that H is a subspace of R^4.

Sol : **(Imp.)**

Given H = {(a – 3b, b – a, a, b)}

Write vectors in H as column vectors, then an arbitrary vector in H has the form.

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = av_1 + av_2$$

$\therefore$ H = Span $\{v_1, v_2\}$ where v_1, v_2 are the vectors. Thus H is a subspace of R^4.

Q9. Let $V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $V_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ $V_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $W = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$ Is w in the subspace spanned by $\{v_1, v_2, v_3\}$? Why ?

Sol :

Given vectors are

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, V_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, W = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$$

Let $a_1, a_2, a_3 \in R$

W = Linear combination of vectors

$= a_1 v_1 + a_2 v_2 + a_3 v_3$

$$\begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix} = a_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + a_2\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + a_3\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$$

The augmented matrix $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ -1 & 3 & 6 & 7 \end{bmatrix}$

Apply Row Operation:

$$R_3 \to R_3 + R_1 \sim \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 5 & 10 & 15 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{5} \sim \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \sim \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{-1} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$R_2 \to R_2 - 4R_3 \sim \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Two columns c_2 and c_3 are indentical and has no solution.

$\therefore$ W is not a subspace spanned by $\{v_1, v_2, v_3\}$.

Q10. Given v_1 and v_2 in a vector space v and let H = span $\{v_1, v_2\}$ then H is a subspace of V.

Sol :

(i) The zero vector is in H.

(ii) Let u, v be any two arbitrary vectors in H.

$u = s_1 v_1 + s_2 v_2$ and $v = t_1 v_1 + t_2 v_2$

Where s_1, s_2, t_1, t_2 are scalars

consider u + v

$= (s_1 v_1 + s_2 v_2) + (t_1 v_1 + t_2 v_2)$

$= (s_1 + t_1)v_1 + (s_2 + t_2)v_2$

$\Rightarrow$ $u + v \in H$

= H is classed under vector addition.

(iii) For any scalar c,

$cu = c(s_1 v_1 + s_2 u_2)$

$= (cs_1) v_1 + (cs_2)v_2$

$\Rightarrow$ $cu \in H$

$\therefore$ H is classed under scalar multiplication

$\therefore$ H is a subspace of v.

Q11. Show that w is in the subspace of R^4 spanned by v1, v2, v3 where,

$$w = \begin{bmatrix} -9 \\ 7 \\ 4 \\ 8 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 7 \\ -4 \\ -2 \\ 9 \end{bmatrix},$$

$$v_2 = \begin{bmatrix} -4 \\ 5 \\ -1 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -9 \\ 4 \\ 4 \\ -7 \end{bmatrix}$$

Ans : **(Imp.)**

Given,

$$w = \begin{bmatrix} -9 \\ 7 \\ 4 \\ 8 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 7 \\ -4 \\ -2 \\ 9 \end{bmatrix},$$

$$v_2 = \begin{bmatrix} -4 \\ 5 \\ -1 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -9 \\ 4 \\ 4 \\ -7 \end{bmatrix}$$

Consider the augmented matrix,

$$[v1\ v2\ v3\ w\ ] = \begin{bmatrix} 7 & -4 & -9 & -9 \\ -4 & 5 & 4 & 7 \\ -2 & -1 & 4 & 4 \\ 9 & -7 & -7 & 8 \end{bmatrix}$$

$R_2 \to 7R_2 + 4R_1$

$R_3 \to 7R_3 + 2R_1$

$R_4 \to 7R_4 - 9R_1$

$$= \begin{bmatrix} 7 & -4 & -9 & -9 \\ 0 & 19 & -8 & 13 \\ 0 & -15 & 10 & 10 \\ 0 & -13 & 32 & 137 \end{bmatrix}$$

$R_3 \to \dfrac{R_3}{5}$

$$= \begin{bmatrix} 7 & -4 & -9 & -9 \\ 0 & 19 & -8 & 13 \\ 0 & -3 & 2 & 2 \\ 0 & -13 & 32 & 137 \end{bmatrix}$$

$R_3 \to 19R_3 + 3R_2$

$R_4 \to 19R_4 + 13R_2$

$$= \begin{bmatrix} 7 & -4 & -9 & -9 \\ 0 & 19 & -8 & 13 \\ 0 & 0 & 14 & 77 \\ 0 & 0 & 504 & 2772 \end{bmatrix}$$

$R_4 \to \dfrac{R_3}{14}, R_4 \to \dfrac{R_4}{504}$

$$= \begin{bmatrix} 7 & -4 & -9 & -9 \\ 0 & 19 & -8 & 13 \\ 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 1 & \frac{11}{2} \end{bmatrix}$$

$R_1 \to R_1 + 9R_3$

$R_2 \to R_2 + 8R_3$

$R_4 \to R_4 - R_3$

$$= \begin{bmatrix} 7 & -4 & 0 & \frac{81}{2} \\ 0 & 19 & 0 & 57 \\ 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \dfrac{R_2}{19}$

$$= \begin{bmatrix} 7 & -4 & 0 & \frac{81}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + 4R_2$

$$= \begin{bmatrix} 7 & 0 & 0 & \frac{105}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{R_1}{7}$

$$= \begin{bmatrix} 1 & 0 & 0 & \frac{15}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solutions are,

$$x_1 = \frac{15}{2}$$

$$x_2 = 3$$

$$x_3 = \frac{11}{2}$$

w is subspace of R^4 spanned by v_1, v_2, v_3 if x_1, $v_1 + x_2 v_2 + x_3 v_3 = w$

if $x_1 v_1 + x_2 v_2 + x_3 v_3 = w$

i.e., $\frac{15}{2} v_1 + 3v_2 + \frac{11}{2} v_3 = w$

$\therefore$ w is in a subspace of R^4.

Q12. Determine if y is in the subspace of R^4 spanned by the columns of A, where

$$\mathbf{A} = \begin{bmatrix} 5 & -5 & -9 \\ 8 & 8 & -6 \\ -5 & -9 & 3 \\ 3 & -2 & -7 \end{bmatrix} \qquad y = \begin{bmatrix} 6 \\ 7 \\ 1 \\ -4 \end{bmatrix}$$

Ans : **(Imp.)**

Given,

$$A = \begin{bmatrix} 5 & -5 & -9 \\ 8 & 8 & -6 \\ -5 & -9 & 3 \\ 3 & -2 & -7 \end{bmatrix}$$

$$y = \begin{bmatrix} 6 \\ 7 \\ 1 \\ -4 \end{bmatrix}$$

y is in subspace of R4 spanned by columms of A if $x_1 v_1 + x_2 v_2 + x_3 v_3 = y$ has a solution.

Consider the augmented matrix [A y]

$$[A\ y] = \begin{bmatrix} 5 & -5 & -9 & 6 \\ 8 & 8 & -6 & 7 \\ -5 & -9 & 3 & 1 \\ 3 & -2 & -7 & -4 \end{bmatrix}$$

$R_2 \to 5R_2 + 8R_2$

$R_3 \to R_3 + R_1$

$R_4 \to 5R_4 + 3R_1$

$$= \begin{bmatrix} 5 & -5 & -9 & 6 \\ 0 & 80 & 42 & -13 \\ 0 & -14 & -6 & 7 \\ 0 & 5 & -8 & -38 \end{bmatrix}$$

$R_3 \to \frac{R_3}{2}$

$$= \begin{bmatrix} 5 & -5 & -9 & 6 \\ 0 & 80 & 42 & -13 \\ 0 & 0 & 54 & 189 \\ 0 & 0 & -850 & -2975 \end{bmatrix}$$

$R_3 \to \frac{R_3}{54}$

$R_4 \to \frac{R_4}{850}$

$$= \begin{bmatrix} 5 & -5 & -9 & 6 \\ 0 & 80 & 42 & -13 \\ 0 & 0 & 1 & \frac{7}{2} \\ 0 & 0 & -1 & \frac{-7}{2} \end{bmatrix}$$

$R_1 \to R_1 + 9R_3$

$R_2 \to R_2 - 42R_3$

$R_4 \to R_4 + R_3$

$$= \begin{bmatrix} 5 & -5 & 0 & \frac{75}{2} \\ 0 & 80 & 0 & -160 \\ 0 & 0 & 1 & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{R_1}{5}, R_2 \to \frac{R_2}{80}$

$$= \begin{bmatrix} 1 & -1 & 0 & \frac{15}{2} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$= \begin{bmatrix} 1 & 1 & 0 & \frac{11}{2} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The general solutions are,

$$x_1 = \frac{11}{2}$$

$$x_2 = -2$$

$$x_3 = \frac{7}{2}$$

$\therefore$ y is in subspace of R^4 spanned by columns of A

4.3 Linearly Independent Sets

Q13. Define linearly independent and linearly dependent.

Sol :

An indexed set of vectors $\{v_1, \dots v_p\}$ in v is said to be linearly independent if the vector equation.

$$c_1 v_1 + c_1 v_1 + \dots c_p v_p = 0$$

has only the trivial solution $c_1 = 0, c_2 = 0 \dots c_p = 0$.

An indexed set of vectors $\{v_1 \dots v_p\}$ in v is said to be linearly dependent if the vector equation

$c_1 v_1 + c_1 v_1 + \dots c_p v_p = 0$ has a non trivial solution that is not all $c_i = 0$.

Q14. An Indexed set $\{v_1, v_2, \dots v_p\}$ of two or more vectors with $v_1 \neq 0$ is linearly dependent if and only if $\exists$ some v_j (with $j > 1$) is a linear combination of its preceding vectors $v_1, v_2, \dots v_{j-1}$.

Sol :

Let v be any vector space $\{v_1, v_2, \dots v_p\}$ be any indexed set in v with $v_1 \neq 0$.

Necessary Condition

Let $\{v_1, v_2, \dots v_p\}$ be a linearly dependent set in v. Consider the linear combination of these vectors equated to a zero vector.

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0 \quad \dots (1)$$

where $c_1, c_2, \dots c_p$ are scalars and $v_i \neq 0$.

Here atleast one of the scalars say $e_j \neq 0$ for $j > 1$ and suppose that $c_j = 0$ for $n > j$.

Then the above liner combination can be written as

$$c_1 v_1 + c_2 v_2 + ... + c_j v_j = 0$$

$$c_j v_j = (-c_1) v_1 + (-c_2) v_2 + ... + (-c_{j-1}) v_{j-1}$$

$$v_j = \left(\frac{-c_1}{c_j}\right) v_1 + \left(\frac{-c_2}{c_j}\right) v_2 + ... + \left(\frac{-c_{j-1}}{c_j}\right) v_{j-1}$$

Thus the vector v_j can be written as the linear combination of its proceeding vectors.

Sufficient Condition

In the indexed set $\{v_1, v_2, ... v_p\}$ let the vector v_j $(j > 1)$ can be written as the linear combination of its proceeding vectors.

$\Rightarrow \exists$ scalars $c_1, c_2 .... c_{j-1}$ such that $v_j = c_1 v_1 + c_2 v_2 + ... + c_{j-1} v_{j-1}$

$$c_1 v_1 + c_2 v_2 + ... + c_{j-1} v_{j-1} + (-1) v_j = 0$$

Thus in this linear combination there exists a non-zero scalar coefficient –1 of v_j so the vectors $v_1, v_2, ... v_j$ are linearly independent.

$\therefore$ The set is $\{v_1, v_2, ... v_j\}$ is L.D.

The index set $\{v_1, v_2, ... v_p\}$ being the super set of this L.D is also L.D.

Q15. State and prove the spanning set theorem.

Statement:

Let $S = \{v_1, v_2 .... v_p\}$ be a set in v and $H = \text{span}\{v_1, v_2, ... v_p\}$.

(i) If one of the vectors in S i.e., v_k is a linear combination of the remaining vectors in S then the set formed from S by reaming v_k still spans H.

(ii) If $H \neq \{0\}$ then some subset of S is a basis for H.

Sol :

Let, $S = \{v_1, v_2 ....v_p\}$ be set in v

$H = \text{span}\{v_1, v_2, ... v_p\}$

If v_p is the linear combination of $v_1 ...v_{p-1}$ then

$$v_p = a_1 v_1 + a_2 v_2 + ... a_{p-1} v_{p-1} \quad ... (1)$$

where,

$a_1, a_2 ...... a_{p-1}$ scalars.

Consider an arbitrary element X in H such that

$$X = c_1 v_1 + c_2 v_2 + ... c_{p-1} v_{p-1} + c_p v_p \quad ... (2)$$

where,

$c_1, c_2 ...... c_p$ scalars

from (1) and (2)

$$X = c_1 v_1 + c_2 v_2 + ... c_{p-1} v_{p-1} + c_p (a_1 v_1 + a_2 v_2 + ... a_{p-1} v_{p-1})$$

$$= (c_1 + ac_p) v_1 + (c_2 + ac_p) v_2 + ... (c_{p-1} + a_{p-1} c_p) v_{p-1}$$

Thus $v_1, v_2, ... v_{p-1}$ still spans H.

(ii) Consider the original spanning set S as linearly independent then it consists of basis.

Two or more vectors in the spanning set can repeat the process until it is linearly independent. Thus the basis of S gets reduced to one-zero vector this is due to existence of span vectors in H.

i.e., $H \neq \{0\}$

4.4 Bases

Q16. Define Basis

Sol :

Let v be a vector space any linearly independent subset of v that spans v is called as a "Basis of v".

(or)

If $\exists$ an indexed set $B = \{b_1, b_2, ... b_n\}$ which is a subset of v such that

(i) B is linearly independent

(ii) $v = \text{span}\{b_1, b_2, ... b_n\}$.

Q17. Verify whether the vectors $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$ are linearly Indepenent.

Sol :

Given vectors are,

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$$

Let $v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$

Consider the matrix,

$A = [v_1 \; v_2 \; v_3]$

$$= \begin{bmatrix} 2 & 2 & -8 \\ -1 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1 \quad ; \quad R_3 \to 2R_3 - R_1$

$$= \begin{bmatrix} 2 & 2 & -8 \\ 0 & -4 & 2 \\ 0 & 2 & 16 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$

$$= \begin{bmatrix} 2 & 2 & -8 \\ 0 & -4 & 2 \\ 0 & 0 & 34 \end{bmatrix}$$

$R_1 \to \frac{R_1}{2}, R_2 \to \frac{R_2}{-4} ; R_3 \to \frac{R_3}{34}$

$$= \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$= \begin{bmatrix} 1 & 0 & -\frac{7}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Since the matrix, A contains pivot element in each column.

$\therefore$ The set is linearly independent.

Q18. Determine whether the set $S = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis of R^3 or not ? If not determine whether S is L - I or not ? Whether S spans R^3 or not ?

Sol :

Given vector space is R^3

The given set is $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Here $S \subseteq R^3$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = 1(1-0) - 1(0-0) + 1(0-0)$$
$$= 1 \neq 0$$

$\therefore$ S is linearly independent.

Here A is an inversible matrix of 3 × 3 then the columns of the matrix A forms a basis of R^3.

4.5 Eigenvalues and Eigenvectors

Q19. Define Eigen values and Eigen vectors.

Sol :

Definition

Let A be any n × n matrix, λ be any scalar. If x is any n × 1 matrix such that $Ax = \lambda x$ then the scalar x is called as an eigen value of the matrix A and the non zero vector x is called as an eigen value of A corresponding to λ.

Note :

1. Eigen values are also called as Latent roots, characteristic values.
2. Eigen vectors are also called Latent vectors, characteristic vectors

Q20. Show that the eigen values of a Triangular Matrix are the entries of its Main diagonal.

Sol :

Let us consider 3 × 3 triangular matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ be the triangular Matrix of order 3 × 3.

Let λ be any scalar and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to the eigen value λ

Consider $(A - \lambda I) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$$

Consider the equation $(A - \lambda I)\, x = 0$ .... (1)

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + a_{13}x_3 = 0 \\ 0x_1 + (a_{22}-\lambda)x_2 + a_{23}x_3 = 0 \\ 0x_1 + 0x_2 + (a_{33}-\lambda)x_3 = 0 \end{array}\right\} \quad ....(2)$$

λ is an eigen value of the matrix A

$\Leftrightarrow$ The system (1) has non trivial solution

$\Leftrightarrow$ The homogenous system in (2) has a non trivial soltuion i.e., $x_1 \neq 0$, $x_2 \neq 0$, $x_3 \neq 0$.

$\Leftrightarrow$ $\lambda = a_{11}, a_{22}, a_{33}$

Thus the eigen values of the matrix A are main diagonal elements of the matrix A.

Q21. Find the characteristic polynomial and the real eigen values of the matrix $A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$.

Sol :

Given matrix is,

$$A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$$

The characteristic polynomial is given by,

$$\det(A - \lambda I) = \left|\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right|$$

$$= \begin{vmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{vmatrix} = (-4-\lambda)(1-\lambda) + 6$$

$$= -4 + 4\lambda - \lambda + \lambda^2 + 6$$

$$= \lambda^2 + 3\lambda + 2$$

$\therefore$ The characteristic polynomial is $\lambda^2 + 3\lambda + 2$

The characteristic equation is ,

$\det(A - \lambda I) = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 1)(\lambda + 2) = 0$

$\therefore \quad \lambda = -1, -2$

$\therefore$ The real eigen values are –1 and –2

Q22. Find the eigen values and eigen vectors of A = $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

Sol : **(Imp.)**

Given matrix A = $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

Eigen values

The characteristic equation is given by det $(A - \lambda I) = 0$

$$\Rightarrow \left| \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4-\lambda)[(1-\lambda)(8-\lambda) + 6] + 1[2(8-\lambda) - 2(6)] + 6[-2 - 2(1-\lambda)] = 0$

$\Rightarrow (4-\lambda)[8 - \lambda - 8\lambda + \lambda^2 + 6] + [16 - 2\lambda - 12] + [-2-2 + 2\lambda] = 0$

$\Rightarrow (4-\lambda)[\lambda^2 - 9\lambda + 14] + [-2\lambda + 4] + [-24 + 12\lambda] = 0$

$\Rightarrow 4\lambda^2 - 36\lambda + 56 - \lambda^3 + 9\lambda^2 - 14\lambda - 2\lambda + 4 - 24 + 12\lambda = 0$

$\Rightarrow -\lambda^3 + 13\lambda^2 - 40\lambda + 36 = 0$

$\Rightarrow \lambda^3 - 13\lambda^2 + 40\lambda - 36 = 0$

By trial and error method $\lambda = 2$ satisfies the equation $f(2) = 2^3 - 13(2)^2 + 40(2) - 36 = 0$

Then

2	1	- 13	40	- 36
	0	2	- 22	36
	1	- 11	18	0

$\Rightarrow \quad \lambda^2 - 11\lambda + 18 = 0$

$\Rightarrow \quad \lambda^2 - 9\lambda - 2\lambda + 18 = 0$

$\Rightarrow \quad \lambda(\lambda - 9) - 2(\lambda - 9) = 0$

$\Rightarrow \quad (\lambda - 9)(\lambda - 2) = 0$

$\Rightarrow \quad \lambda = 2, 9$

$\therefore$ The eigen values are 2, 2 and 9.

To find Eigen vectors :

If $\lambda = 2$

$[A - \lambda I]\, x = 0$

$$\Rightarrow \left[\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the Argumented matrix

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix}$$

Apply Row operations

$$\begin{matrix} R_2 : R_2 - R_1 \\ R_3 : R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The Euqtions are

$2x_1 - x_2 + 6x_3 = 0$

$2x_1 = + x_2 - 6x_3$

$x_1 = + \frac{1}{2} x_2 - 3x_3$; x_2, x_3 are free variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The eigen vectors corresponding to eigen value $\lambda = 2$ are

$$V_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

If $\lambda = 9$

Consider $A - 9\,I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{bmatrix}$

The Argmented matrix [A – 9 I 0] is

$$\begin{bmatrix} -5 & -1 & 6 & 0 \\ 2 & -8 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

$$R_3 : R_3 - R_2 \sim \begin{bmatrix} -5 & -1 & 6 & 0 \\ 2 & -8 & 6 & 0 \\ 0 & 7 & -7 & 0 \end{bmatrix}$$

$$\begin{matrix} R_3 \to \frac{R_3}{7} \\ R_2 \to \frac{R_2}{2} \end{matrix} \sim \begin{bmatrix} -5 & -1 & 6 & 0 \\ 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$R_1 \to R_1 + R_3 \sim \begin{bmatrix} -5 & 0 & 5 & 0 \\ 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{-5} \sim \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$R_2 \to R_2 - R_1 \sim \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{-4} \sim \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \sim \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The Equations are $-x_1 + x_3 = 0$

$\Rightarrow x_1 = x_3$

$x_2 - x_3 = 0$

$x_2 = x_3$

Here x_3 is a free variable

$\therefore$ The general solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector corresponding to the eigen value $\lambda = 9$ is $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

4.6 The Characteristic Equation

Q23. Find the characteristic polynomial and the eigen values of the matrices

(i) $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$

Ans : **(Imp.)**

(i) Given matrix,

$$A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$$

The characteristic polynomaial is given by, det$(A - \lambda I)$,

$$= \left[\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]$$

$$= \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$$

$= (2-\lambda)^2 - 49$

$= 4 + \lambda^2 - 4\lambda - 49$

$= \lambda^2 - 4\lambda - 45$

$\therefore$ The characteristic polynomial is, $\lambda^2 - 4\lambda - 45$.

The characteristic equation is, det $(A - \lambda I) = 0$

$\Rightarrow \lambda^2 - 4\lambda - 45 = 0$

$\Rightarrow \lambda^2 - 9\lambda + 5\lambda - 45 = 0$

$\Rightarrow \lambda(\lambda - 9) + 5(\lambda - 9) = 0$

$\Rightarrow (\lambda - 9) + (\lambda + 5) = 0$

$\Rightarrow \lambda - 9 = 0,\ \lambda + 5 = 0$

$\lambda = 9, -5$

$\therefore$ The eigen values are 9 and – 5.

(ii) Given matrix,

$$A = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$$

The characteristic polynomial is given by, det$(A - \lambda I)$,

$$= \left[\begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]$$

$$= \begin{vmatrix} 3-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix}$$

$= (3-\lambda)(-1-\lambda) + 2$

$= 3 - 3\lambda + \lambda + \lambda^2 + 2$

$= \lambda^2 - 2\lambda - 1$

$\therefore$ The characteristic polynomial is, $\lambda^2 - 2\lambda - 1$

The characteristic equation is det (A – λI) = 0

$\Rightarrow \quad \lambda^2 - 2\lambda - 1 = 0$

It is in quadratic form ax2 + bx + c = 0

Here,

a = 1, b = –2, c = –1

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4+4}}{2}$$

$$= \frac{2 \pm \sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

$$\lambda = 1 \pm \sqrt{2}$$

∴ The eigen values are $1 \pm \sqrt{2}$

(iii) Given matrix,

$$A = \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$$

The characteristic polynomial is given by, det(A – λI)

$$= \left[\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]$$

$$= \begin{bmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{bmatrix}$$

$= (5 - \lambda)(4 - \lambda) + 12$

$= 20 - 5\lambda - 4\lambda + \lambda^2 + 12$

$= \lambda^2 - 9\lambda + 32$

∴ $\lambda^2 - 9\lambda + 32$ is the characteristic polynomial.

The characteristic equation is det(A – λI) = 0

$\Rightarrow \quad \lambda^2 - 9\lambda + 32 = 0$

it is in quadratic form $ax^2 + bx + c = 0$

Here,

a = 1, b = –9, c = 32

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{9 \pm \sqrt{81 - 4(32)}}{2}$$

$$= \frac{9 \pm \sqrt{-47}}{2}$$

$$\lambda = \frac{9 \pm i\sqrt{47}}{2}$$

∴ A has no real eigen valucs.

(iv) Given matrix,

$$A = \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$$

The characteristic polynomial is given by, det (A – λI),

$$= \left[\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]$$

$$= \left[\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]$$

$= (7 - \lambda)(3 - \lambda) + 4$

$= 21 - 7\lambda - 3\lambda + \lambda^2 + 4$

$= \lambda^2 - 10\lambda + 25$

∴ $\lambda^2 - 10\lambda + 25$ is the characteristic polynomial.

The characteristic equation is det(A – λI) = 0

$\Rightarrow \quad \lambda^2 - 10\lambda + 25 = 0$

$\Rightarrow \quad \lambda^2 - 5\lambda - 5\lambda + 25 = 0$

$\Rightarrow \quad \lambda(\lambda - 5) - 5(\lambda - 5) = 0$

$\Rightarrow \quad (\lambda - 5)^2 = 0$

$\Rightarrow \quad \lambda = 5, 5$

∴ A has only one eigen value 5.

Q24. Find the characteristic equation of A =

$$\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

. Also find algebraic multiplicity of the eigen values.

Sol :

Given Matrix is A = $\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The characteristic equation of A is given as $|A - \lambda I| = 0$

$$\left[\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right] = 0$$

$$\begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5 - \lambda)\ [(3 - \lambda)(5 - \lambda)(1 - \lambda)] = 0$

$\Rightarrow (5 - \lambda)^2\ (3 - \lambda)(1 - \lambda) = 0$

$\Rightarrow (25 + \lambda^2 - 10\lambda)\ (3 - 3\lambda - \lambda + \lambda^2) = 0$

$\Rightarrow (\lambda^2 - 10\lambda + 25)\ (\lambda^2 - 4\lambda + 3) = 0$

$\Rightarrow \lambda^4 - 4\lambda^3 + 3\lambda^2 - 10\lambda^3 + 40\lambda^2 - 30\lambda + 25\lambda^2 - 100\lambda + 75 = 0$

$\Rightarrow \lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$

Since the given matrix is upper triangular,

The given values are $\lambda = 1$, $\lambda = 3$ with multiplicity 1 and $\lambda = 5$ with multiplicity 2.

Q25. Is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$? If so, find the corresponding eigenvalue

Ans :

Given matrix is,

$$A = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$$

and $x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Consider, $Ax = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & -3 \\ 6 & -12 \end{bmatrix}$

$= \begin{bmatrix} -2 \\ -6 \end{bmatrix}$

$= (-2)\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\Rightarrow Ax = -2x$

$\Rightarrow Ax$ is a multiple of x

$\therefore \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector of A and the corresponding eigenvalue is $\lambda = -2$

Q26. (a) Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.

(b) Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

Ans :

(a) Given,

λ is an eigenvalue of invertible matrix A.

If λ is eigenvalue of A, then there exists a non - zero vector x such that $Ax = \lambda x$ [$\because$ A is invertible]

$\Rightarrow \quad A^{-1}Ax = A^{-1}(\lambda x)$

$[\because A^{-1}A = I, Ix = x]$

$\Rightarrow \quad x = \lambda(A^{-1}x)$

$\Rightarrow \quad \lambda^{-1}x = A^{-1}x$

$\therefore \quad \lambda^{-1}$ is an eigenvalue of A^{-1}

(b) Let, A^2 be zero motion.

If $Ax = \lambda x, x \neq 0$

Then, $A^2x = A(Ax)$

$= A(\lambda x)$

$= \lambda(Ax)$

$= \lambda(\lambda x)$

$= \lambda^2 x$

$\Rightarrow \quad A^2x = \lambda^2 x$

Since, $x \neq 0$

$\Rightarrow \quad \lambda \neq 0$

$\therefore$ The matrix A has only eigen value '0'

Q27. Find the eigen values of A = $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ and compare this result with eigenvalue of A^T

OR

Find the eigenvalues of the matrix

A = $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Sol :

Given matrix is,

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

The characteristic equation is given by,

$\det(A - \lambda I) = 0$

$$\Rightarrow \left| \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (2-\lambda)(-6-\lambda) - (3)(3) = 0$

$\Rightarrow \quad -12 - 2\lambda + 6\lambda + \lambda^2 - 9 = 0$

$\Rightarrow \quad \lambda^2 + 4\lambda - 21 = 0$

$\lambda^2 + 7\lambda - 3\lambda - 21 = 0$

$\Rightarrow \quad \lambda(\lambda+7) - 3(\lambda + 7) = 0$

$\Rightarrow \quad (\lambda - 3)(\lambda + 7) = 0$

$\Rightarrow \quad \lambda - 3 = 0; \lambda + 7 = 0$

$\Rightarrow \quad \lambda = 3; \lambda = -7$

$\therefore$ The eigen values of A are 3, –7

The tranpose of A is given by

$$A^T = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

$\Rightarrow \quad A^T = A$

$\therefore$ The eigen value of A^T are same as the given values of A.

Q28. Find eigen values for matrix A = $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

Sol :

Given matrix is $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

The characteristic equation is given by,

$\det(A - \lambda I) = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix} = 0$$

$\Rightarrow$ $(1-\lambda)(2-\lambda)-30 = 0$

$\Rightarrow$ $2-\lambda-2\lambda+\lambda^2-30 = 0$

$\Rightarrow$ $\lambda^2-3\lambda-28 = 0$

$\Rightarrow$ $\lambda(\lambda-7)+4(\lambda-7) = 0$

$\Rightarrow$ $(\lambda-7)(\lambda+4) = 0$

$\Rightarrow$ $\lambda = 7, -4$

$\therefore$ The eigen values are 7, – 4

Q29. Find the eigenvector for A = $\begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}$ corresponding to eigenvalue λ = –5

Sol :

Given matrix is, A = $\begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}$

Eigenvalue $\lambda = -5$

Let, x = $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the required eigenvector

Then, $(A-\lambda 1)\,x = 0$

$$\Rightarrow \left[\begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left|\begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right| = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1+2x_2 \\ 3x_1+6x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e., $x_1 + 2x_2 = 0$

$\Rightarrow$ $x_1 = -2x_2$

Let, $x_2 = k$

$\Rightarrow$ $x_1 = -2k$

$$\Rightarrow x = \begin{bmatrix} -2k \\ k \end{bmatrix} = k\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector corresponding to λ = –5 is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Q30. Find the characteristic equation of the matrix A = $\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Sol : **(Imp.)**

Given matrix is A = $\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The characteristic equation of A is given as,

$|A-\lambda 1| = 0$

$$\Rightarrow \left|\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix} 5 & -\lambda-2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) = 0$

$\Rightarrow$ $(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$

$\Rightarrow$ $(25+\lambda^2-10\lambda)(3-4\lambda+\lambda^2) = 0$

$\Rightarrow$ $25(3-4\lambda+\lambda^2)+\lambda^2(3-4\lambda+\lambda^2)$
$-10\lambda(3-4\lambda+\lambda^2) = 0$

$\Rightarrow$ $\lambda^4-14\lambda^3+68\lambda^2-130\lambda+75 = 0$

$\therefore$ The characteristic equation is,

$\lambda^4-14\lambda^3+68\lambda^2-130\lambda+75 = 0$

Q31. Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ if so find the one corresponding eigen vector.

Sol : **(Imp.)**

Given motion is,

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and $\lambda = 3$

Consider,

$$A - 3I = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

The augmented matrix [(A – 3 I)0] is,

$$= \begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$R_2 \rightarrow 2R_2 + 3R_1$

$$= \begin{bmatrix} -2 & 2 & 2 & 0 \\ 0 & -4 & 8 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \rightarrow 4R_3 + R_2$

$$= \begin{bmatrix} -2 & 2 & 2 & 0 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow \frac{R_1}{R_2}, \; R_2 \rightarrow \frac{R_2}{4}$

$$= \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow R_1 + R_2$

$$= \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation (A – 3 I) x = 0 has a not trivial soltuion.

$\therefore$ 3 is an eigen value

The General solutions are

$x_1 - 3x_3 = 0$

$\Rightarrow x_1 = 3x_3$

$x_2 - 2x_3 = 0$

$\Rightarrow x_2 = 2x_3$

And x_3 is a free variable

Let $x_3 = 1 \neq 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_3 \\ 2x_3 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1) \\ 2(1) \\ 1 \end{bmatrix} \quad [\because x_3 = 1]$$

$x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is eigenvector corresponding to the eigenvalue 3.

Q32. Find the eigenvalue of A = $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

Sol :

Given matrix is A = $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

The characteristic equation is given by

det $(A - \lambda I) = 0$

$$\Rightarrow \left| \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4-\lambda)[(1-\lambda)(1-\lambda) - 0] - 0 + 1$

$[-2(0) - (-2)(1-\lambda)] = 0$

$\Rightarrow (4-\lambda)[1-\lambda-\lambda+\lambda^2] + 1[0 + 2 - 2\lambda) = 0$

$\Rightarrow (4-\lambda)[1 - 2\lambda + \lambda^2] + [2 - 2\lambda] = 0$

$\Rightarrow 4 - 8\lambda + 4\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 + 2 - 2\lambda = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

λ = 1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$\Rightarrow (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 - 2\lambda - 3\lambda + 6) = 0$

$\Rightarrow (\lambda - 1)(\lambda(\lambda - 2) - 3(\lambda - 2)) = 0$

$\Rightarrow (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\Rightarrow (\lambda - 1) = 0, (\lambda - 2) = 0, (\lambda - 3) = 0$

$\Rightarrow \lambda = 1, 2, 3$

$\therefore$ The eigen values are 1, 2 and 3.

Short Question and Answers

1. Define vector space.

Sol :

A vector space is a non empty set V of objects called vectors on which are defined two operations called addition and multiplication by scalars subject to ten axioms.

(I) $(V_1 +)$ is abelian vector addition.

1. $u + v \in v$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is zero vector 0 in v such that $u + 0 = u$.
5. For each u in v there is a vector –u in v such that $u + (-u) = 0$.

(II) Scalar Multiplication

6. The scalar multiple of u by c that is $cu \in v$.
7. $c(u + v) = cu + cv$
8. $(c + d)\, u = cu + du$
9. $c(du) = (cd)\, u$
10. $1.u = u$

The space R^n; where $n \geq 1$ is a vector space.

2. Define vector subspace and give examples.

Sol :

Let H be a non empty subset of a vector space V that it is said to be a vector subspace of V if it satisfy the following conditions.

1. The zero vector of v is in H

$\Rightarrow O \in H$

2. H is closed under vector addition $\forall U, V \in H$

$\Rightarrow U + V \in H$

3. H is closed under scalar multiplication for each $U \in H$ $\exists$ scalar $C \ni CU \in H$.

Example

1. The set containing of only the zero vector in a vector space V is a subspace of V called the zero subspace of written as {0}.
2. The vector space R^2 is not a subspace of R^3 because R^2 is not even a subset of R^3. Since vectors in R^3 all have three entries where vectors in R^2 have only two entries.

3. Define Null Space

Sol :

The null space of an m × n matrix A, written as Null A is the set of all solutions of the homogeneous equation A× = 0.

Null A = {X : X is in R^n and AX = 0}

4. Define column space of a m × n matrix A.

Ans :

The column space of a m × n matrix A is the set of linear combinations of the columns of A.

i.e., A = {a_1, a_2, ... a_n} then ColA = Span {a_1, a_2, ... a_n} It is denoted by ColA.

5. Define Basis

Sol :

Let v be a vector space any linearly independent subset of v that spans v is called as a "Basis of v".

(or)

If ∃ an indexed set B = {b_1, b_2, ... b_n} which is a subset of v such that

(i) B is linearly independent

(ii) v = span {b_1, b_2, ... b_n}.

6. Define B-coordinates of x, co-ordinate mapping, change of co-ordinates matrix.

Ans :

Suppose B = {b_1, b_2 .... b_n} is a basis for V and x is in V. The co-ordinates of x relative to the basis B (or the B-co-ordinate of x) are the weights c_1, c_2 ..... c_n such that

$$x = c_1 b_1 + c_2 b_2 + ... + c_n b_n$$

The Co-ordinate Mapping

If c_1, c_2, ..., c_n are the B-co-ordinates of x, then the vector in R^n

$[X]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is the co-ordinate vector x relative to B (or the B - co-ordinate vector of x). The mapping

$x \to [X]_B$ is the co-ordinate mapping (determined by B).

Change-of-Co-ordinates Matrix

The vector equation

$x = c_1b_1 + c_2b_2 + ... + c_n b_n$ is equivalent to

$x = P_B[x]_B$

Where $P_B = [b_1, b_2, ... b_n]$ is the change-of-co-ordinates matrix from B to the standard basis in IR^n.

Note: P_B^{-1} exists

7. If $v_1, v_2, ... v_p$ are in a vector space v then span $\{v_1, v_2, ... v_p\}$ is a subspace of v.

Sol :

(i) The zero vector is in H

$\therefore O = ov_1 + ov_2 + ... ov_p$

(ii) Let u, v any two arbitrary vectors in H

$u = a_1 v_1 + a_2 v_2 + ... a_p v_p$, $v = b_1 v_1 + b_2 v_2 + ... b_p v_p$

Where $a_1 a_2 .... a_p, b_1, b_2, ... b_p$ are scalars.

Now

$u + v = (a_1 v_1 + a_2 v_2 + ... + a_p v_p) + (b_1 v_1 + b_2 v_2 + ... + b_p v_p)$

$= (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + ... + (a_p + b_p)v_p$

$\therefore$ $u + v \in H$

$\therefore$ H is closed under vector addition.

for any scalar c

$cu = c(a_1 v_1 + a_2 v_2 + ... + a_p v_p)$

$= (ca_1)v_1 + (ca_2)v_2 + ... + (ca_p)v_p$

$\therefore cu \in H$

$\therefore$ H is closed under scalar multiplication

$\therefore$ H is a subspace of V.

8. Define Eigen values and Eigen vectors.

Sol :

Definition

Let A be any n × n matrix, λ be any scalar. If x is any n × 1 matrix such that Ax = λx then the scalar x is called as an eigen value of the matrix A and the non zero vector x is called as an eigen value of A corresponding to λ.

Note :

1. Eigen values are also called as Latent roots, characteristic values.
2. Eigen vectors are also called Latent vectors, characteristic vectors

9. Find eigen values for matrix A = $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

Sol :

Given matrix is A = $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

The characteristic equation is given by,

$\det (A - \lambda I) = 0$

$\Rightarrow \quad \left|\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$

$\Rightarrow \quad \left|\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$

$\Rightarrow \quad \begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix} = 0$

$\Rightarrow \quad (1 - \lambda)(2 - \lambda) - 30 = 0$

$\Rightarrow \quad 2 - \lambda - 2\lambda + \lambda^2 - 30 = 0$

$\Rightarrow \quad \lambda^2 - 3\lambda - 28 = 0$

$\Rightarrow \quad \lambda(\lambda - 7) + 4(\lambda - 7) = 0$

$\Rightarrow \quad (\lambda - 7)(\lambda + 4) = 0$

$\Rightarrow \quad \lambda = 7, -4$

$\therefore$ The eigen values are 7, – 4

10. If A is a 7 × 5 matrix, what is the largest possible rank of A? If A is a 5 × 7 matrix, what is the largest possible rank of A?

Sol :

The rank of A is the number of pivot positions in matrix A.

Since the number of pivot positions cannot exceed the number of rows or columns.

$\therefore$ The largest possible rank A is 5.

11. If A is a 4 × 3 matrix, what is the largest possible dimension of the row space of A? If A is a 3 × 4 matrix, what is the largest possible dimension of the row space of A?

Sol :

dim Row A = Rank A

The rank A is the number of pivot positions in matrix A.

Since the number of pivot positions cannot exceed the number of rows or columns.

$\Rightarrow$ The largest possible rank A = 3

$\therefore$ For both the matrices the largest possible dim row A = 3.

Choose the Correct Answers

1. The union of two subspaces is subspace of vector space iff [a]

   (a) One is contained in another (b) One is not contained in another

   (c) Both (a) and (b) (d) None

2. If A = $[a_1, a_2, ... a_n]$ then Col A = [c]

   (a) $[a_1, a_2, ... a_n]$ (b) $\{x / x \text{ is in } R^n\}$

   (c) Span $\{a_1, a_2, ... a_n\}$ (d) None

3. If T is a linear transformation from a vector space V into a vector space w then [c]

   (a) $T(u + v) = T(u) + T(v) + u, v \in V$ (b) $T(u) = CT(u) \ \forall \ u \in V$

   (c) Both (a) and (b) (d) None

4. Let B = $\{b_1, b_2, ... b_n\}$ is a basis for a vector space V. For each $x \in V$, then exist a unique set of scales $c_1, c_2, ... c_n$ such that $x = c_1b_1 + c_2b_2 + c_nb_n$. This is called [d]

   (a) Spanning set theorem (b) Rank theorem

   (c) Basis theorem (d) Unique representation theorem.

5. Col A = R^m if the equation [a]

   (a) Ax = b has a solution $\forall \ b \in R^m$ (b) Ax = 0 has only the trivial solution

   (c) Nul A = {0} (d) All the above

6. H is a subspace of V and B = $\{b_1, b_2, ... b_n\} \in V$ is a basis for H if [c]

   (a) B is linearly independent (b) H = span $\{b_1, b_2, ..., b_n\}$

   (c) Both (a) and (b) (d) None

7. The Set S = {(1, –2, 1), (2, 1, –1), (7, –4, 1)} forms [c]

   (a) Linearly dependent (b) Linearly span

   (c) Linearly independent (d) None

8. If a vector space V is not spanned by the finite set S then V is said ________. [b]

   (a) Finite dimensional (b) Infinite dim

   (c) Finite and Infinite dim (d) None

9. The set S = {(1, 0, 0, –1), (0, 1, 0, –1), (0, 0, 1, –1), (0, 0, 0, 1)} in R^4 is ________. [b]

(a) Linearly dependent (b) Linearly independent

(c) Both (a) and (b) (d) None

10. Let B = {b_1, b_2, ... b_n} be a basis for a vector space V. Then the co-ordinate mapping x $\rightarrow$ $[x]_B$ is a ________ theory transformation form v onto R^n. [b]

(a) Onto (b) One-one

(c) Both (a) and (b) (d) None

Fill in the Blanks

1. The intersection of the two subspaces of V(F) is again a ________.
2. The set V = {0} is a vector space and it is said to be a ________.
3. Let V be any vector space if it is any non-empty subset of V and H is also a vector space then H is called as a ________ of V.
4. If $v_1, v_2, ..., v_p$ are in a vector space v then

   span $\{v_1, v_2, ..., v_p\}$ is a ________ of v.
5. The null space of an m × n matrix A is a subspace of ________.
6. The ________ of an m × n matrix A B a subspace of R^m.
7. If T is a matrix transformation that T(x) = Ax for any matrix A then Range of T is ________ and kernal of T is ________.
8. Let V be a vector space. Any linearly independent subset of v that spams V is called as a ________.
9. If P_n is a vector space of all polynomials of degree ≤ n in 't' then the set S = $\{1, t, t^2, ... t^n\}$ is a ________ of P_n.
10. The no. of elements in the basis of a vector space is called as ________.

Answers

1. Subspace
2. Zero space
3. Subspace
4. subspace
5. R^n
6. Colum space
7. Col A, Null of A
8. Basis of V
9. Standard basis
10. Dimension of the vector space v

UNIT V	Diagonalization – Diagonalizing Matrices with distinct eigen values and non distinct eigen values; Applications to Differential Equations.

5.1 Diagonalization

5.1.1 Diagonalizing Matrices with Distinct Eigen Values and Non Distinct Eigen Values

Q1. Define Diagonalization.

Sol :

A square matrix A is said to be diagonalizable if $\exists$ a non-singular matrix (Invertible) P such that $A = PDP^{-1}$ where D is a Diagonal matrix.

We say that P diagonalizes A.

$$A = PDP^{-1} \Leftrightarrow AP = PD.$$

Q2. State and prove the diagonalization theorem.

Sol : **(Imp.)**

Statement :

An $n \times n$ matrix A is dia-gonalizable if and only if A has n linearly independent eigen vectors.

Proof :

Let A be any $n \times n$ square matrix.

Let P be any $n \times n$ matrix with columns $v_1, v_2, ... v_n$

$$p = [v_1, v_2, ... v_n].$$

Let D by any diagonal matrix of order $n \times n$ with diagonal elements, $\lambda_1, \lambda_2, ... \lambda_n$.

$$\text{Then, } D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Consider

$$AP = A[v_1, v_2, ... v_n]$$
$$= [Av_1, Av_2, ... Av_n] \quad ...(1)$$

$$PD = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= [\lambda_1 v_1 \ \lambda_2 v_2 .... \lambda_n v_n] \quad ...(2)$$

Part I

Suppose that the matrix A is diagonalizable.

$\Rightarrow$ A can be written as $A = PDP^{-1}$.

$\Rightarrow AP = PD$

$\Rightarrow [Av_1 \ Av_2 ... Av_n] = [\lambda_1 v_1 \ \lambda_2 v_2 .... \lambda_n v_n]$ ...(3)

$\Rightarrow$ Equating the corresponding columns on both sides we get,

$$Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2, Av_n = \lambda_n v_n \quad ...(4)$$

Each of the expression in (4) is of the form $AX = \lambda X$ which indicates that λ is the eigen value of A and X is the corresponding eigen vector of A.

Thus $\lambda_1, \lambda_2, ... \lambda_n$ are the eigen values of A and $v_1, v_2, ... v_n$ are the corresponding eigen vectors of A.

Since P is invertible the columns of P are linearly independent and these columns are non-zero.

$\Rightarrow$ The vectors $v_1, v_2, ... v_n$ are linearly independent. Thus the matrix A has n linearly independent eigen vectors.

Part II

Let us suppose that the matrix A has 'n' linearly independent eigen vectors.

Let $v_1, v_2, ... v_n$ be the n linearly independent eigen vectors of A corresponding to the eigen value $\lambda_1, \lambda_2, ... \lambda_n$ let $P = [v_1, v_2, ... v_n]$. Since the columns of P and linearly independent.

$|p| \neq 0$

$\Rightarrow$ p^{-1} exists

Consider,

$$AP = A[v_1, v_2, ... v_n] = [Av_1\ Av_2 ... Av_n]$$

$$= [\lambda_1 v_1\ \lambda_2 v_2 .... \lambda_n v_n]$$

$$= [v_1, v_2, ... v_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

AP = PD where D is the diagonal matrix.

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$\Rightarrow$ $P^{-1}AP = P^{-1}PD$

$\Rightarrow$ $P^{-1}AP = D$

$\therefore$ A is Diagonalizable

Q3. Show that an n × n matrix with n distinct eigen values is diagonalizable.

Sol :

Let A be any square matrix of order n×n.

Let $\lambda_1, \lambda_2, ... \lambda_n$ be the n distinct eigen values of the matrix A.

Let $v_1, v_2, ... v_n$ be the corresponding eigen vectors of the matrix A.

Then $\{v_1, v_2, ... v_n\}$ is a linearly independent set of A .

$\Rightarrow$ A is diagonalizable.

Q4. Determine whether the following matrix is diagonalizable or not A = $\begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$

Sol :

The given matrix is $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$

This is a 3×3 matrix. It is a triangular matrix the eigen values of A are 5, 0, –2.

Thus there are '3' distinct eigen values of the matrix A and hence the matrix A is diagonalizable.

Q5. If $p = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ **then compute** A^2, A^4 **if** $A = PDP^{-1}$.

Sol :

Here D is a diagonal matrix

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} 2^k & 0 \\ 0 & 1^k \end{bmatrix}$$

$$= \begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix} \quad \forall k \geq 1$$

$$P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \Rightarrow |P| = 15 - 14 = 1 \neq 0$$

$\therefore$ P is Invertiable and $P^{-1} = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$

(i) Consider $A = PDP^{-1}$

$$A^2 = PD^2P^{-1}$$

$$A^2 = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 46 & -105 \\ 18 & -41 \end{bmatrix}$$

(ii) Consider $A^4 = PD^4P^{-1}$

$$A^4 = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$$

Q6. If the eigen vales of a matrix A are 2 and 1. The corresponding eigen vectors of A are $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then find A^8.

Sol :

Let A be any square matrix of order 2 × 2. The eigen values of A are 2 and 1 say $\lambda_1 = 2$ and $\lambda_2 = 1$.

Let $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are the corresponding eigen vectors of the matrix A.

Here the vectors v_1 and v_2 are linearly independent.

∴ The matrix A is diagonalizable.

⇒ ∃ a non-singular matrix p and a diagonal matrix D such that $A = PDP^{-1}$ where,

$$P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Consider $A = PDP^{-1}$

$$A^8 = PD^8P^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2^8 & 0 \\ 0 & 1^8 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}$$

Q7. Verify whether A is invertible if A is diagonalizable.

Sol :

Suppose that A is diagonalizable.

⇒ ∃ an Invertible matrix p and diagonal matrix D such that $A = PDP^{-1}$.

$A^{-1} = (PDP^{-1})^{-1}$

$A^{-1} = (P^{-1})^{-1} D^{-1} P^{-1}$

$A^{-1} = P D^{-1} P^{-1}$

$A^{-1} = PEP^{-1}$ where $E = D^{-1}$ is also a diagonal matrix.

∴ A is invertible.

Q8. Diagonalize the matrices if possible.

(i) $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

Ans :

(i) Given matrix,

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$$

Since A is triangular,

The eigen value is 5.

If $\lambda = 5$

Consider,

$$A = 5I = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The equations are

$x_2 = 0$

And x_1 is free variable

∴ The general solution is,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since the basis for R^2 is not generated by an eigen vector.

$\therefore$ A is not digonalizable.

(ii) Given matrix,

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

Consider the characteristic equation

det (A – 7I) = 0

$$\Rightarrow \left| \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(2-\lambda)(1-\lambda) - 12 = 0$

$\Rightarrow$ $2 - 2\lambda - \lambda + \lambda^2 - \lambda 2 = 0$

$\Rightarrow$ $\lambda^2 - 3\lambda - 10 = 0$

$\Rightarrow$ $\lambda^2 - 5\lambda + 2\lambda - 10 = 0$

$\Rightarrow$ $\lambda(\lambda - 5) + 2(\lambda - 5) = 0$

$\Rightarrow$ $(\lambda - 5)(\lambda + 2) = 0$

$\Rightarrow$ $\lambda - 5 = 0, \lambda + 2 = 0$

$\Rightarrow$ $\lambda = 5, \ \lambda = -2$

If $\lambda = 5$

Consider,

$$A = 5I = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}$$

The augmented matrix [(A – 5I) 0] is,

$$\begin{bmatrix} -3 & 3 & 0 \\ 4 & -4 & 0 \end{bmatrix}$$

$$R_1 \rightarrow \frac{R_1}{-3}$$

$$R_2 \rightarrow \frac{R_2}{-4}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + R_1$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The equations are,

$x_1 - x_2 = 0$

$\Rightarrow$ $x_1 = x_2$

And x_2 is free variable

The general solution is,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ The basis vector for eigen space is

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If $\lambda = -2$

Consider,

$$A + 2I = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}$$

The augmented matrix [(A + 2I) 0] is

$$\begin{bmatrix} 4 & 3 & 0 \\ 4 & 4 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 \rightarrow R_1$$

$$= \begin{bmatrix} 4 & 3 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

The equations are,

$4x_1 + 3x_2 = 0$

$\Rightarrow$ $4x_1 = -3x_2$

$\Rightarrow$ $x_1 = \frac{-3}{4} x_2$

And x_2 is free variable.

The general solution is

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-3}{4}x_2 \\ x_2 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} \frac{-3}{4} \\ 1 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

∴ The basis vector for eigen space is

$$v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$P = [v_1 \ v_2]$$

$$= \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$$

And $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

∴ The matrix D has the eigen values corresponding to eigen vectors v_1, v_2 respectively.

Q9. Diagonalize the matrix A = $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ if possible.

Sol : **(Imp.)**

Given matrix is, $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

The characteristic equation of A is,

det (A – λI) = 0

$$\Rightarrow \left| \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0$$

⇒ $(1 - \lambda)\,[(-5 - \lambda)(1 - \lambda) - (-3)3] - 3[(-3)(1-\lambda) - (3)(-3)] + 3[(-3)(3) - 3(-5-\lambda)] = 0$

⇒ $(1-\lambda)\,[-5 + 5\lambda - \lambda + \lambda^2 + 9] - 3[-3 + 3\lambda + 9] + 3[-9 + 15 + 3\lambda] = 0$

⇒ $(1 - \lambda)\,[\lambda^2 + 4\lambda + 4] - 3(3\lambda + 6) + 3(3\lambda + 6) = 0$

⇒ $(1-\lambda)(\lambda^2 + 4\lambda + 4) = 0$

⇒ $(\lambda - 1)(\lambda^2 + 4\lambda + 4) = 0$

⇒ $(\lambda - 1)(\lambda + 2)(\lambda + 2) = 0$

⇒ $\lambda = -2, -2, 1$

If $\lambda = -2$

Consider,

$A - (-2)I = A + 2I$

$$= \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

The augmented matrix [(A + 2I) 0] is,

$$\begin{bmatrix} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 = \begin{bmatrix} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 = \begin{bmatrix} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation is,

$$3x_1 + 3x_2 + 3x_3 = 0$$
$$3x_1 = -3x_2 - 3x_3$$
$$x_1 = -x_2 - x_3$$

and x_2, x_3 are free variables

The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

∴ The eigen vector corresponding to eigen value $\lambda = -2$ is,

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

If $\lambda = 1$,

Consider,

$$A - \lambda I = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

The augmented matrix [(A–I) 0] is

$$\begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 = \begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -3 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 = \begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 = \begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{3}, R_2 \to \frac{R_2}{-3} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equations are,

$$x_2 + x_3 = 0$$
$$\Rightarrow x_2 = -x_3$$
$$x_1 - x_3 = 0$$
$$\Rightarrow x_1 = x_3$$

and x_3 is a free variable.

∴ The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

∴ The eigen vector corresponding to eigen value $\lambda = 1$ is,

$$v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Let $P = [v_1 \ v_2 \ v_3]$

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix D has the eigen values corresponding to eigen vectors, v_1, v_2 and v_3 respectively.

Consider,

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ -2 & 0 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

Consider,

$$PD = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

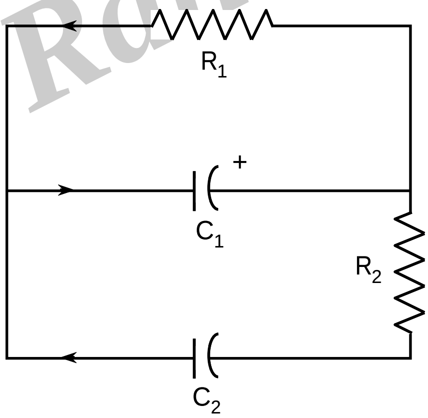
$$= \begin{bmatrix} 2 & 2 & 1 \\ -2 & 0 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

$\Rightarrow$ AP = PD

$\therefore$ The matrix A is diagonalizable.

5.2 Applications to Differential Equations

Q10. Find formulas for the voltages v_1 and v_2 (as functions of time t) for the circuit shown below, assuming that $R_1 = \frac{1}{5}$ Ohm, $R_2 = \frac{1}{3}$ ohm, C_1 = 4 farads, C_2 = 3 farads and the Initial charge on each capacitor is 4 volts.



Sol : **(Imp.)**

Given $R_1 = \frac{1}{5}$ ohm, $R_2 = \frac{1}{3}$ ohm, C_1 = 4 farads, C_2 = 3 farads and $x(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

Since $A = \begin{bmatrix} \dfrac{-\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)}{C_1} & \dfrac{1}{(R_2\, C_1)} \\ \dfrac{1}{(R_2 C_2)} & \dfrac{-1}{(R_2 C_2} \end{bmatrix}$

$$= \begin{bmatrix} \dfrac{-\left(\dfrac{1}{\frac{1}{5}} + \dfrac{1}{\frac{1}{3}}\right)}{4} & \dfrac{1}{\left(\frac{1}{3}(4)\right)} \\ \dfrac{-1}{\left(\frac{1}{3}(3)\right)} & \dfrac{-1}{\left(\frac{1}{3}(3)\right)} \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{(5+3)}{4} & \dfrac{3}{4} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & \dfrac{3}{4} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0.75 \\ 1 & -1 \end{bmatrix}$$

The characteristic equation is given by , det $(A - \lambda I) = 0$.

$$\Rightarrow \left| \begin{bmatrix} -2 & 0.75 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} -2-\lambda & 0.75 \\ 1 & -1-\lambda \end{bmatrix} \right| = 0$$

$\Rightarrow$ $(-2 - \lambda)(-1 - \lambda) - 0.75 = 0$

$\Rightarrow$ $2 + 2\lambda + \lambda + \lambda^2 - 0.75 = 0$

$\Rightarrow$ $\lambda^2 + 3\lambda + 1.25 = 0$

It is in the quadratic from $ax^2 + bx + c = 0$

$$\lambda = \frac{-3 \pm \sqrt{3^2 - 4(1.25)}}{2(1)}$$

$$= \frac{-3 \pm 2}{2}$$

$\lambda = -0.5, -2.5$

$\therefore$ Eigen values are $\lambda_1 = -0.5$; $\lambda_2 = -2.5$

If $\lambda_1 = -0.5$

Consider $A + (0.5)\,I = \begin{bmatrix} -2 & 0.75 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} -1.5 & 0.75 \\ 1 & -0.5 \end{bmatrix}$

The equations $(A + (0.5)\,I) \times = 0$ gives

$x_1 - 0.5\,x_2 = 0$

$\Rightarrow \quad x_1 = 0.5\,x_2$

And x_2 is free variable,

$\therefore$ The general solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore \quad v = x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = -0.5$ is $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

If $\lambda_2 = -2.5$

Consider, $A + (2.5)\,I = \begin{bmatrix} -2 & 0.75 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.75 \\ 1 & 1.5 \end{bmatrix}$

The equation $(A + (2.5)I) \times = 0$ gives

$x_1 + (1.5)\,x_2 = 0$

$\Rightarrow \quad x_1 = -1.5\,x_2$

$\Rightarrow \quad \frac{-3}{2}\,x_2$

And x_2 is free variable.

$\therefore$ The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$\therefore v_2 = x_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = -2.5$ is $v_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

The general solution is,

$x(t) = c_1 v_1 e^{\lambda} t + c_2\, v_2 e^{\bar{\lambda}}\, 2^{t}$

$x(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.51} + c_2 \begin{bmatrix} -3 \\ e \end{bmatrix} e^{-2.5t}$

Where c_1, c_2 are complex numbers

The constants c_1, c_2 satisfy the initial condition $x(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ is,

$$c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Consider augmented matrix,

$$[v_1\ v_2\ x(0)] = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 2 & 4 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1 = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 8 & -4 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{8} \Rightarrow \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

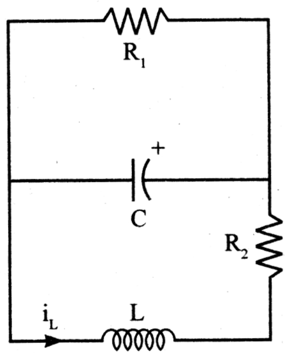
$$R_1 \to R_1 + 3R_2 = \begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & \frac{-1}{2} \end{bmatrix}$$

$$\therefore \quad c_1 = \frac{5}{2}\ ;\ c_2 = -\frac{1}{2}$$

$$\therefore \quad \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = x(t) = \frac{5}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{-0.5t} - \frac{1}{2}\begin{bmatrix} -3 \\ 2 \end{bmatrix}e^{-2-5t}$$

Q11. The circuit in figure can be describled by the equation

$$\begin{bmatrix} i'_L \\ v'_C \end{bmatrix} = \begin{bmatrix} \frac{-R_2}{L} & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{(R_1C)} \end{bmatrix}\begin{bmatrix} i_L \\ v_C \end{bmatrix}$$



where i_L is the current passing through the inductor L and v_C is the voltage drop across the capacitor C. Suppose R_1 is 5 ohms, R_2 is 0.8 ohm, C is 0.1 farad, and L is 0.4 henry.Find for-mulas for i_L and v_C, if the initial current through the inductor is 3 amperes and the intial voltage across the capacitor is 3 volts.

Ans :

Given,

R_1 = 5 ohm

R_2 = 0.8 ohm

C = 0.1 farad

L = 0.4 Henry

$$x_0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{-R_2}{L} & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{(R_1C)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-0.8}{0.4} & \frac{-1}{0.4} \\ \frac{1}{0.1} & \frac{-1}{(5(0.1))} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2.5 \\ 10 & -2 \end{bmatrix}$$

The characteristic equation is given by, det(A $-\lambda$ I) = 0

$$\Rightarrow \left| \begin{bmatrix} -2 & -2.5 \\ 10 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & -2.5 \\ 10 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-2-\lambda)(-2-\lambda)+(2.5)10 = 0$$

$$\Rightarrow 4 + 2\lambda + 2\lambda + \lambda^2 + 25 = 0$$

$$\Rightarrow \lambda_2 + 4\lambda + 29 = 0$$

It is in quadratic form $ax^2 + bx + c = 0$

Here, a = 1, b = 4, c = 0

$$\lambda = \frac{-4 \pm \sqrt{(4)^2 - 4(29)(1)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16-116}}{2}$$

$$= \frac{-4 \pm \sqrt{-100}}{2}$$

$$= \frac{-4 \pm 10i}{2}$$

$$\lambda = -2 \pm 5i$$

$\therefore$ Eigen value is, $\lambda = -2 + 5i$

Consider augmented matrix,

A – (–2 + 5i)I

$$= \begin{bmatrix} -2 & -2.5 \\ 10 & -2 \end{bmatrix} - \begin{bmatrix} -2+5i & 0 \\ 0 & -2+5i \end{bmatrix}$$

$$= \begin{bmatrix} -5i & -2.5 \\ 10 & -5i \end{bmatrix}$$

The equation (A –(– 2 + 5i)) X = 0 gives,

$10x_1 - (5i)x_2 = 0$

$\Rightarrow$ $10x_1 = (5i)x_2$

$\Rightarrow$ $x_1 = \frac{1}{2} i\, x_2$

And x_2 is free variable.

$\therefore$ The general solution is,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{i}{2} x_2 \\ x_2 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} \frac{i}{2} \\ 1 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} i \\ 2 \end{bmatrix}$$

$\therefore \quad v = \begin{bmatrix} i \\ 2 \end{bmatrix}$

$\therefore \quad$ The eigen vector $v = \begin{bmatrix} i \\ 2 \end{bmatrix}$

The complex functions are $ve^{\lambda t}$ and $\overline{v}e^{\overline{\lambda}t}$

$\therefore$ The general complex solution of $x' = Ax$ is,

$$x(t) = c_1 ve^{\lambda t} + c_2 - \overline{v}e^{\lambda t}$$

$$x(t) = c_1 \begin{bmatrix} i \\ 2 \end{bmatrix} e^{(-2+5i)t} + c_2 \begin{bmatrix} -i \\ 2 \end{bmatrix} e^{(-2+5i)t}$$

Where, c_1, c_2 are complex numbers.

Let, $ve^{(-2+5i)t} = \begin{bmatrix} i \\ 2 \end{bmatrix} e^{-2i}. e^{(5i)t}$

$$= \begin{bmatrix} i \\ 2 \end{bmatrix} e^{-2i} (\cos 5t + i \sin 5t)$$

$$= \begin{bmatrix} i\cos 5t - \sin 5t \\ 2\cos 5t + 2i \sin 5t \end{bmatrix} e^{-2t}$$

$$= \begin{bmatrix} -\sin 5t \\ 2\cos 5t \end{bmatrix} e^{-2t} + i \begin{bmatrix} \cos 5t \\ 2\sin 5t \end{bmatrix} e^{-2t}$$

$\therefore$ The general real solution is,

$$x(t) = c_1 \begin{bmatrix} -\sin 5t \\ 2\cos 5t \end{bmatrix} e^{-2t} + i \begin{bmatrix} \cos 5t \\ 2\sin 5t \end{bmatrix} e^{-2t}$$

Where, c_1, c_2 are real numbers.

The constants c1,c2 satisfy the intial conditions

$$x(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \text{is,}$$

$$c_1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Consider augmented matrix,

$$[v_1\ v_2\ x(0)] = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 3 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{2}$$

$$= \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & \frac{3}{2} \end{bmatrix}$$

$\therefore$ $c_1 = 3$

$$c_1 = \frac{3}{2} = 1.5$$

$$\therefore \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = x(t) = 1.5 \begin{bmatrix} -\sin 5t \\ 2\cos 5t \end{bmatrix} e^{-2t} + i \begin{bmatrix} \cos 5t \\ 2\sin 5t \end{bmatrix} e^{-2t}$$

$$\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} -1.5\sin 5t \\ 3\cos 5t \end{bmatrix} e^{-2t} + \begin{bmatrix} 3\cos 5t \\ 6\sin 5t \end{bmatrix} e^{-2t}$$

$$= \begin{bmatrix} -1.5\sin 5t + 3\cos 5t \\ 3\cos 5t + 6\sin 5t \end{bmatrix} e^{-2t}$$

$$\therefore \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} -1.5\sin 5t + 3\cos 5t \\ 3\cos 5t + 6\sin 5t \end{bmatrix} e^{-2t}$$

Q12. Construct the general solution of X' = AX involving complex eigen functions and then obtain the general real solution. Describe the shape of typical trajectories.

(i) $\mathbf{A} = \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix}$

(ii) $\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$

Sol : **(Imp.)**

Given matrix is, $A = \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix}$

The characteristic equation is given by, det $(A - \lambda I) = 0$

$$\text{i.e., } \left| \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$= \begin{vmatrix} -3-\lambda & -9 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(-3-\lambda)(-3-\lambda) + 18 = 0$

$\Rightarrow$ $\lambda^2 + 9 = 0$

$\Rightarrow$ $\lambda = \pm 3i$

$\therefore$ Eigen value is $\lambda = 3i, -3i$

Consider

$$A-(3i)I = \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 3i & 0 \\ 0 & 3i \end{bmatrix} = \begin{bmatrix} -3-3i & -9 \\ 2 & 3-3i \end{bmatrix}$$

The equation $(A-(3i)I)x = 0$ gives,

$$2x_1 + (3-3i)x_2 = 0$$

$$\Rightarrow \quad x_1 = -\frac{(3-3i)}{2}x_2$$

And x_2 is free variable

$\therefore$ The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{(3-3i)x_2}{2} \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} \frac{-3(3-3i)}{2} \\ 1 \end{bmatrix} = x_2\begin{bmatrix} -(3-3i) \\ 2 \end{bmatrix}$$

$$\therefore \quad \text{The eigen vector } v = \begin{bmatrix} -3+3i \\ 2 \end{bmatrix}$$

The complex functions are $ve^{\lambda t}$ and $\bar{v}e^{\bar{\lambda}t}$

$\therefore$ The general complex solution of $x' = Ax$ is,

$$x(t) = c_1 ve^{\lambda t} + c_2\bar{v}e^{\bar{\lambda}t}$$

$$\Rightarrow \quad x(t) = c_1\begin{bmatrix} -3+3i \\ 2 \end{bmatrix}e^{3it} + c_2\begin{bmatrix} -3-3i \\ 2 \end{bmatrix}e^{(-3i)t}$$

Where

c_1, c_2 are complex numbers

Let

$$ve^{(3i)t} = \begin{bmatrix} -3+3i \\ 2 \end{bmatrix}(\cos 3t + i\sin 3t)$$

$$= \begin{bmatrix} -3\cos 3t - 3i\sin 3t + 3i\cos 3t - 3\sin 3t \\ 2\cos 3t + 2i\sin 3t \end{bmatrix}$$

$$ve^{(3i)t} = \begin{bmatrix} -3\cos 3t - 3\sin 3t \\ 2\cos 3t \end{bmatrix} + i\begin{bmatrix} -3\sin 3t - 3\cos 3t \\ 2\sin 3t \end{bmatrix}$$

$\therefore$ The general real solution is,

$$x(t) = c_1\begin{bmatrix} -3\cos 3t - 3\sin 3t \\ 2\cos 3t \end{bmatrix} + c_2\begin{bmatrix} -3\sin 3t - 3\cos 3t \\ 2\sin 3t \end{bmatrix}$$

Where

c_1, c_2 are real numbers.

Since real parts of the eigen values are zero. The trajectories are ellipses about the origin.

(ii) Given matrix is, $A = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$

The characteristic equation is given by, det $(A - \lambda I) = 0$

$$\text{i.e., } \left| \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 4-\lambda & -3 \\ 6 & -2-\lambda \end{bmatrix} \right| \Rightarrow (4-\lambda)(-2-\lambda) + 18 = 0$$

$$\Rightarrow -8 - 4\lambda + 2\lambda + \lambda^2 + 18 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 10 = 0$$

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$$

$\therefore$ Eigen value is, $\lambda = 1 + 3i$

Consider,

$$A - (1+3i)I = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} - \begin{bmatrix} (1+3i) & 0 \\ 0 & 1+3i \end{bmatrix} = \begin{bmatrix} 3-3i & -3 \\ 6 & -3-3i \end{bmatrix}$$

The equation $(A - (1+3i) I) x = 0$ gives,

$$6x_1 + (-3-3i)x_2 = 0$$

$$\Rightarrow 6x_1 - (3+3i)x_2 = 0$$

$$\Rightarrow x_1 = \frac{1+i}{2} x_2$$

Ans x_2 is free variable.

$\therefore$ The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{(1+i)}{2} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

$$v = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

$\therefore$ Eigen vector $v = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

The complex functions are $ve^{\lambda t}$ and $\bar{v}e^{\bar{\lambda}t}$

The general complex solution of x' = Ax is,

$$x(t) = c_1 v e^{\lambda t} + c_2 \bar{v} e^{\bar{\lambda}t}$$

$$x(t) = c_1 \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{(1+3i)t} + \begin{bmatrix} 1-i \\ 2 \end{bmatrix} e^{(1-3i)t}$$

Where

c_1, c_2 are complex numbers,

Let $$ve^{(1+3i)t} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{t+3it} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^t.e^{3it}$$

$$= \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^t (\cos 3t + i \sin 3t)$$

$$= \begin{bmatrix} \cos 3t + i \sin 3t + i(\sin 3t + \sin 3t) \\ 2\cos 3t + i\,(2\sin 3t) \end{bmatrix} e^t$$

$$= \begin{bmatrix} \cos 3t + \sin 3t + i(\sin 3t + \cos 3t) \\ 2\cos 3t + i\,(2\sin 3t) \end{bmatrix} e^t$$

$$= \begin{bmatrix} \cos 3t + \sin 3t \\ 2\cos 3t \end{bmatrix} e^t + i \begin{bmatrix} \sin 3t + \cos 3t \\ 2\sin 3t \end{bmatrix} e^t$$

$\therefore$ The general real solution is,

$$x(t) = c_1 \begin{bmatrix} \cos 3t - \sin 3t \\ 2\cos 3t \end{bmatrix} e^t + c_2 \begin{bmatrix} \sin 3t + \cos 3t \\ 2\sin 3t \end{bmatrix} e^t$$

Where c_1, c_2 are real numbers.

Since real parts of the eigen values are positive.

The trajectories spiral out away from the origin.

Q13. Make a change of variable that decouples the equation X' = AX write the equation X(t) = Py(t) and show the calculate that leads tot he uncoupled system Y' = DY, specifying P and D where A = $\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$.

Sol : **(Imp.)**

Given matrix is, $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

The characteristic equation is given by, det $(A - \lambda I) = 0$.

i.e., $\left|\begin{bmatrix}1 & -2\\ 3 & -4\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right| = 0 \quad \Rightarrow \begin{vmatrix}1-\lambda & -2\\ 3 & -4-\lambda\end{vmatrix} = 0$

$\Rightarrow \quad (1-\lambda)(-4-\lambda) + 6 = 0 \quad \Rightarrow \lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \lambda = -1, -2$

$\therefore$ The eigen values are –1, –2.

If $\lambda = -2$

Consider, $A + 2I = \begin{bmatrix}1 & -2\\ 3 & -4\end{bmatrix} + \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}3 & -2\\ 3 & -2\end{bmatrix}$

The augmented matrix [(A + 2I) 0] is $\begin{bmatrix}3 & -2 & 0\\ 3 & -2 & 0\end{bmatrix}$

$R_2 \to R_2 - R_1 = \begin{bmatrix}3 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}$

$R_1 \to \dfrac{R_1}{3} = \begin{bmatrix}1 & \dfrac{-2}{3} & 0\\ 0 & 0 & 0\end{bmatrix}$

$\therefore$ The equation is,

$$x_1 - \frac{2}{3}x_2 = 0$$

$$x_1 = \frac{2}{3}x_2$$

and x_2 is a free variable

$\therefore$ The general solution is,

$$x = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}\frac{2}{3}x_2\\ x_2\end{bmatrix} = x_2\begin{bmatrix}2\\ 3\end{bmatrix}$$

$\therefore \quad v_1 = \begin{bmatrix}2\\ 3\end{bmatrix}$.

If $\lambda = -1$,

Consider,

$$A + I = \begin{bmatrix}1 & -2\\ 3 & -4\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}2 & -2\\ 3 & -3\end{bmatrix}$$

The augmented matrix [(A + I) 0] is $\begin{bmatrix}2 & -2 & 0\\ 3 & -3 & 0\end{bmatrix}$

$$R_2 \to \frac{R_1}{2}, R_2 \to \frac{R_2}{-3} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 \quad = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The equation is,

$$x_1 - x_2 = 0$$

$\therefore$ $x_1 = x_2$

and x_2 is a free variable

$\therefore$ The general solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Initial condition $x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Let the constants c_1, c_2 satisfy x(0) such that $c_1v_1 + c_2v_2 = x(0)$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Consider the augmented matrix,

$$[v_1 \; v_2 \; x(0)] = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$R_2 \to 2R_2 - 3R_1 = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -5 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{2}, R_2 \to \frac{R_2}{-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 5 \end{bmatrix}$$

$$R_1 \to 2R_1 - \frac{1}{2}R_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \end{bmatrix}$$

$\therefore c_1 = -1; c_2 = 5$

The general solution of $x^1 = Ax$ is

$$x(t) = c_1v_1e^{\lambda_1 t} + c_2v_2e^{\lambda_2 t}$$

$$\Rightarrow \quad x(t) = -1\begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t} + 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

Since both eigen values of matrix A are negative.

$\therefore$ The origin is an attractor of the dynamical system described by $x^1 = Ax$.

The direction of greatest attraction is the line through v_1 and the origin.

To decouple the equation $x^1 = Ax$

Let

$$P = [v_1\ v_2]$$

$$P = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

and $$D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

Given $x^1 = AX$ ... (1)

Since $A = PDP^{-1}$

$\Rightarrow D = PAP^{-1}$.

Substituting, $x(t) = py(t)$ in equation (1)

i.e., $\frac{d}{dt}(py) = A(py)$

$$= PDP^{-1}(Py) = PD(P^{-1}P)y = PDy$$

$\therefore$ P has constant entries.

$$\frac{d}{dt}(py) = PDy$$

$$\Rightarrow P\left(\frac{d}{dt}(y)\right) = PDy$$

$$\Rightarrow P^{-1}P\left(\frac{d}{dt}(y)\right) = P^{-1}PDy$$

$\Rightarrow y' = Dy$

i.e., $$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

Q14. A particle moving in a planar force field has a position vector X that satisfies $X^1 = AX$. The 2 × 2 matrix A has eigen value 4 and 2 with corresponding eigen vectors $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find the position of the particle at time t, assuming that $X(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$.

Sol : **(Imp.)**

Given, A is a 2 × 2 matrix

Eigen values are 4 and 2

Eigen vectors $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The initial condition $x(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

The eigen functions for the differential equation $X^1 = Ax$ are $v_1 e^{\lambda_1 t}$ and $v_2 e^{\lambda_2 t}$

i.e., $v_1 e^{4t}$, $v_2 e^{2t}$

The general solution of $x' = Ax$ has the form

$$x(t) = c_1 v_1 e^{t} + c_2 v_2 e^{2t}$$

$$x(t) = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} \quad \text{... (1)}$$

Let the constants c_1, c_2 satisfy the initial condition $x(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

i.e., $c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

$\Rightarrow -3c_1 - c_2 = -6$

$c_2 + c_2 = 1$

The augmented matrix is $[v_1, v_2, x(0)]$

$$= \begin{bmatrix} -3 & -1 & -6 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_2 \to 3R_2 + R_1 \Rightarrow \begin{bmatrix} -3 & -1 & -6 \\ 0 & 2 & -3 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{-3} \text{ and } R_2 \to \frac{R_2}{2} = \begin{bmatrix} 1 & \frac{1}{3} & 2 \\ 0 & 1 & \frac{-3}{2} \end{bmatrix}$$

$$R_1 \to R_1 - \frac{1}{3} R_2 = \begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & \frac{-3}{2} \end{bmatrix}$$

$c_1 = \frac{5}{2}$; $c_2 = \frac{-3}{2}$

Substituting the corresponding values in equation (1)

$$\therefore \quad x(t) = \frac{5}{2} \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} - \frac{3}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}.$$

Short Question and Answers

1. Define Diagonalization.

Sol :

A square matrix A is said to be diagonalizable if $\exists$ a non-singular matrix (Invertible) P such that $A = PDP^{-1}$ where D is a Diagonal matrix.

We say that P diagonalizes A.

$A = PDP^{-1} \Leftrightarrow AP = PD.$

2. Show that an n × n matrix with n distinct eigen values is diagonalizable.

Sol :

Let A be any square matrix of order $n \times n$.

Let $\lambda_1, \lambda_2, ... \lambda_n$ be the n distinct eigen values of the matrix A.

Let $v_1, v_2, ... v_n$ be the corresponding eigen vectors of the matrix A.

Then $\{v_1, v_2, ... v_n\}$ is a linearly independent set of A .

$\Rightarrow$ A is diagonalizable.

3. Determine whether the following matrix is diagonalizable or not $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$

Sol :

The given matrix is $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$

This is a 3 × 3 matrix. It is a triangular matrix the eigen values of A are 5, 0, –2.

Thus there are '3' distinct eigen values of the matrix A and hence the matrix A is diagonalizable.

4. Verify whether A is invertible if A is diagonalizable.

Sol :

Suppose that A is diagonalizable.

$\Rightarrow$ $\exists$ an Invertible matrix p and diagonal matrix D such that $A = PDP^{-1}$.

$A^{-1} = (PDP^{-1})^{-1}$

$A^{-1} = (P^{-1})^{-1} D^{-1} P^{-1}$

$A^{-1} = P D^{-1} P^{-1}$

$A^{-1} = PEP^{-1}$ where $E = D^{-1}$ is also a diagonal matrix.

$\therefore$ A is invertible.

5. Diagonalization Theorem.

Sol :

An $n \times n$ matrix A is dia-gonalizable if and only if A has n linearly independent eigen vectors.

6. Linear Transformation

Ans :

Let v and w be two vector spaces defined over a field F.

T:V $\rightarrow$ W be any mapping such that

$T(u+v) = T(u) + T(v)$

$T(cu) = CT(u)$ $\forall u, v \in V$ and for any scalar c then T is called as a linear transformation from v to w.

7. Kernal of a Linear Transformation

Ans :

Let T:V $\rightarrow$ W be any linear transformation. Then the set consisting of all these elements of v whose images are equal to the zero vector of w is called as the kernal of T.

The Kernal of T is also called as Null space of T

Kernal T or $K_T = \{U/U \in V$

and $T(U) = 0; 0 \in W\}$

8. Range of a Linear Transformation

Ans :

Let T: V $\rightarrow$ W be any linear transformation. The set of all images of elements of v under the transformation T is called as Range of T.

9. Construct a non-zero 2 × 2 matrix that is / invertible but not diagonalizable.

Sol :

Let a 2 × 2 matrix be, $A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$

The matrix A is invertible if the eigen values are not zero.

The matrix A has eigen values 2, 4.

$\therefore$ A is invertible

The matrix A to be non-diagonalizable, it must contain the eigen values which are not distinct.

Let, $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

$\therefore$ A is not diagonalizable

$\therefore$ $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ is the 2 × 2 matrix which is invertible but not diagonalizable.

Choose the Correct Answers

1. The matrix A to be diagonalizable is [a]

   (a) $A = PDP^{-1}$ (b) $AP = PD$

   (c) $A = PD^2P^2$ (d) None

2. If $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$ then $A^8 =$ [c]

   (a) $\begin{bmatrix} 2^8 & 0 \\ 1 & 1^8 \end{bmatrix}$ (b) $\begin{bmatrix} 256 & 0 \\ 0 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}$ (d) $\begin{bmatrix} 2^8 & 4^8 \\ -3^8 & -1^8 \end{bmatrix}$

3. The matrix $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ is [b]

   (a) diagonalizable (b) not diagonalizable

   (c) linear independent (d) none

4. The eigen vector for $A = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$ corresponding to eigen value $\lambda = 4 + 3$. [a]

   (a) $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -i \\ -i \end{bmatrix}$

   (c) $\begin{bmatrix} i \\ i \end{bmatrix}$ (d) None

5. The complex eigen values of then matrix $A = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$ is [c]

   (a) $3 + 3i$ (b) $3 - 3i$

   (c) $3 \pm 3i$ (d) None

6. If A is both diagonalizable and invertible then A^{-1} is [a]

   (a) diagonalizable (b) invertible

   (c) both (d) none

7. If the eigen value $\lambda = a + bi$ then $\bar{\lambda} =$ [c]

(a) bi (b) b – ai

(c) 0 – bi (d) –a

8. The eigen values for the matrix A = $\begin{bmatrix} 0-1 & 0-1 \\ -0-1 & 0-1 \end{bmatrix}$. [b]

(a) a + bi (b) a ± bi

(c) a – bi (d) None

9. If $T(b_1) = 3c_1 - 2c_2 + 5c_3$ and $T(b_2) = 4c_1 + 7c_2 - c_3$ then the matrix M for T relative to B and C is [b]

(a) $\begin{bmatrix} 3 & -2 \\ 7 & 5 \\ 5 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 5 \\ 4 & 7 \\ -2 & 3 \end{bmatrix}$

10. If $T(a_0 + a_1t + a_2t^2) = a_1 + 2a_2t$ then T(1) = [a]

(a) 0 (b) 1

(c) 2t (d) –1

Fill in the Blanks

1. An n × n matrix with A district eigen values is ________.
2. If the matrix A is diagonalizable then A is ________.
3. Ab n × n matrix is diagonalizable if and only if A has ________ eigen vectors.
4. A is diagonalizable if and only if there are enough eigen vectors to form a ________ if R^T.
5. A square matrix a is diagonalizable if A is similar to ________ matrix.
6. A square matrix A of order n × n is diagonalizable if there are A district ________ of A.
7. ν is a finite dimensional vector space of dimension 'n'. and T : v $\rightarrow$ v is a linear transformation. B is an ordered basis for ν. Then $[T(x)]_B$ = ________ $\forall$ x $\in$ v.
8. If the origin is an attractor then the solution of the system is ________.
9. The parametric equations of the solution of given system represents a curve known as ________.
10. A ________ arise when the matrix A has both positive and negative eigen values.

Answers

1. diagonalizable
2. invertible
3. n linearly independent
4. basis
5. diagonal
6. eigen values
7. $[T]_B$ $[x]_B$
8. stable
9. trajectory
10. saddle point

FACULTY OF INFORMATICS

BCA III Semester (CBCS) Examination

Model Paper - I

APPLIED MATHEMATICS

Time : 3 Hours] **[Max. Marks : 70**

Note: Answer all questions from Part-A and answer any five questions from Part-B. Choosing one question from each unit.

PART - A (10 × 2 = 20 Marks)

Answers

1. (a) Define Partial Differentiation. **(Unit-I, SQA-1)**
   (b) Homogenous function with example. **(Unit-I, SQA-5)**
   (c) Define Implicit Function. **(Unit-II, SQA-2)**
   (d) Define Maxima and Minima of functions of two variables. **(Unit-II, SQA-3)**
   (e) Non-homogeneous System of Linear Equations **(Unit-III, SQA-6)**
   (f) Augmented Matrix Form **(Unit-III, SQA-4)**
   (g) Define vector space. **(Unit-IV, SQA-1)**
   (h) Define Eigen values and Eigen vectors. **(Unit-IV, SQA-8)**
   (i) Define Diagonalization. **(Unit-V, SQA-1)**
   (j) Verify whether A is invertible if A is diagonalizable. **(Unit-V, SQA-4)**

PART - B (5 × 10 = 50 Marks)

UNIT - I

2. (a) Write a short notes on Neighborhood of a Point (a, b). **(Unit-I, Q.No. 3)**

(OR)

   (b) State and prove Euler's theorem for a homogeneous functions. **(Unit-I, Q.No. 13)**

UNIT - II

3. (a) State and prove Theorem on Total Differentials. **(Unit-II, Q.No. 1)**

(OR)

   (b) Find a point within a triangle such that the sum of the square of its distance from the three vertices is a minimum. **(Unit-II, Q.No. 19)**

UNIT - III

4. (a) What is homogeneous system of linear equations and explain it cases ? **(Unit-III, Q.No. 4)**

(OR)

   (b) Find the system of Linear equation,

$2x + 4y - 3z = 4$

$3y + 4x + 5z = 2$

$4z + 4x + 3y = 1$ **(Unit-IIII, Q.No. 12)**

UNIT - IV

5. (a) For $n \geq 0$ the set p_n of polynomials of degree atmost n consists of all polynomials of the form.

$$P(t) = a_0 + a_1t + a_2t^2 + ... + a_nt^n$$

Where coefficient $a_0, a_1 .... a_n$ and variable t are real numbers
Here degree is n. **(Unit-IV, Q.No. 2)**

(OR)

(b) An Indexed set $\{v_1, v_2, ... v_p\}$ of two or more vectors with $v_1 \neq 0$ is linearly dependent if and only if $\exists$ some v_j (with $j > 1$) is a linear combination of its preceding vectors $v_1, v_2, ... v_{j-1}$. **(Unit-IV, Q.No. 14)**

UNIT - V

6. (a) State and prove the diagonalization theorem. **(Unit-V, Q.No. 2)**

(OR)

(b) Make a change of variable that decouples the equation X' = AX write the equation X(t) = Py(t) and show the calculate that leads tot he uncoupled system Y' = DY, specifying P and D where $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$. **(Unit-V, Q.No. 13)**

FACULTY OF INFORMATICS

BCA III Semester (CBCS) Examination

Model Paper - II

APPLIED MATHEMATICS

Time : 3 Hours] **[Max. Marks : 70**

Note: Answer all questions from Part-A and answer any five questions from Part-B. Choosing one question from each unit.

PART - A (10 × 2 = 20 Marks)

Answers

1. (a) Limit of a Function of two variables **(Unit-I, SQA-3)**

   (b) Define Partial derivatives. **(Unit-I, SQA-7)**

   (c) Differentiation Composite Function. **(Unit-II, SQA-1)**

   (d) State and prove Taylor's theorem for a function of two variables. **(Unit-II, SQA-8)**

   (e) Homogeneous System of Linear Equations **(Unit-III, SQA-5)**

   (f) Linearly Independent Columns **(Unit-III, SQA-10)**

   (g) Define Basis. **(Unit-IV, SQA-5)**

   (h) Find eigen values for matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. **(Unit-IV, SQA-9)**

   (i) Linear Transformation **(Unit-V, SQA-6)**

   (j) Construct a non-zero 2 × 2 matrix that is / invertible but not diagonalizable. **(Unit-V, SQA-9)**

PART - B (5 × 10 = 50 Marks)

UNIT - I

2. (a) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u$

   $= -\frac{9}{(x+y+z)^2}$ **(Unit-I, Q.No. 8)**

(OR)

   (b) If $u = \tan^{-1}\left(\frac{x^3 + Y^3}{x - y}\right)$, $x \neq y$, then show that,

   (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

   (ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u.$ **(Unit-I, Q.No. 14)**

UNIT - II

3. (a) If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$, Then Show that $\frac{\partial^2 u}{\partial x\,\partial y}$

$= \frac{1}{\left(1+x^2+y^2\right)^{3/2}}$. **(Unit-II, Q.No. 10)**

(OR)

(b) Discuss the maxima or minima of $u = x^4 + y^4 - 2x^2 + 4\,xy - 2y^2$. **(Unit-II, Q.No. 21)**

UNIT - III

4. (a) What is non-homogeneous system of linear equations and explain it cases? **(Unit-III, Q.No. 5)**

(OR)

(b) Find the system of line or equation

$2x + y + z = 2$

$4x + y + = 6$

$9x + 2y + z = 2$ **(Unit-III, Q.No. 13)**

UNIT - IV

5. (a) Let H be the set of all vectors of the from (a – 3b, b – a, a, b) where a and b are arbitrary scalars let H = {(a – 3b, b – a, a, b)}; a, b in R. Show that H is a subspace of R^4. **(Unit-IV, Q.No. 8)**

(OR)

(b) Show that the eigen values of a Triangular Matrix are the entries of its Main diagonal. **(Unit-IV, Q.No. 20)**

UNIT - V

6. (a) Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ if possible. **(Unit-V, Q.No. 9)**

(OR)

(b) Diagonalize the matrices $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ **(Unit-V, Q.No. 8)**

FACULTY OF INFORMATICS

BCA III Semester (CBCS) Examination

Model Paper - III

APPLIED MATHEMATICS

Time : 3 Hours] [Max. Marks : 70

Note: Answer all questions from Part-A and answer any five questions from Part-B. Choosing one question from each unit.

PART - A (10 × 2 = 20 Marks)

Answers

1. (a) Find the degree of given Homogenous function for $f(x,y) = x^n \sin(y/x)$. **(Unit-I, SQA-9)**

(b) Functions of Two Variables. **(Unit-I, SQA-2)**

(c) Define Stationary points and Extreme points. **(Unit-II, SQA-5)**

(d) Define Equality of $f_{xy}(a, b)$, $f_{yx}(a, b)$. **(Unit-II, SQA-7)**

(e) What is general form of linear equation? **(Unit-III, SQA-2)**

(f) Linear Equation **(Unit-III, SQA-1)**

(g) Define Null Space. **(Unit-IV, SQA-3)**

(h) If A is a 4 × 3 matrix, what is the largest possible dimension of the row space of A? If A is a 3 × 4 matrix, what is the largest possible dimension of the row space of A? **(Unit-IV, SQA-11)**

(i) Kernal of a Linear Transformation **(Unit-V, SQA-7)**

(j) Determine whether the following matrix is diagonalizable or not

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$ **(Unit-V, SQA-3)**

PART - B (5 × 10 = 50 Marks)

UNIT - I

2. (a) If $u = \log\left\{\frac{x^4+y^4}{x+y}\right\}$, show by Euler's theorem that $X\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3$. **(Unit-I, Q.No. 15)**

(OR)

(b) If $u = x^2\tan^{-1}\frac{y}{x} - y^2 - \tan^{-1}\frac{x}{y}$; $xy \neq 0$. prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$. **(Unit-I, Q.No. 6)**

UNIT - II

3. (a) Discuss the maximum or minimum value of u, when $u = x^3 + y^3 - 3axy$. **(Unit-II, Q.No. 16)**

(OR)

(b) Explain Lagrange's method of underermind multipliers. **(Unit-II, Q.No. 22)**

UNIT - III

4. (a) What are the Characteristics of Linearly Dependent ? **(Unit-III, Q.No. 27)**

(OR)

(b) Given equation to find non trivial solution.

$2x - y = 3$ ... (1)

$4x + y = 7$ ... (2) **(Unit-IIII, Q.No. 23)**

UNIT - IV

5. (a) Show that w is in the subspace of R^4 spanned by v1, v2, v3 where,

$$w = \begin{bmatrix} -9 \\ 7 \\ 4 \\ 8 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 7 \\ -4 \\ -2 \\ 9 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -4 \\ 5 \\ -1 \\ -7 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -9 \\ 4 \\ 4 \\ -7 \end{bmatrix}$$ **(Unit-IV, Q.No. 11)**

(OR)

(b) Find the characteristic equation of $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Also find

algebraic multiplicity of the eigen values. **(Unit-IV, Q.No. 24)**

UNIT - V

6. (a) If the eigen vales of a matrix A are 2 and 1. The corresponding eigen

vectors of A are $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then find A^8. **(Unit-V, Q.No. 6)**

(OR)

(b) A particle moving in a planar force field has a position vector X that satisfies $X^1 = AX$. The 2×2 matrix A has eigen value 4 and 2 with

corresponding eigen vectors $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find the position

of the particle at time t, assuming that $X(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$. **(Unit-V, Q.No. 14)**

FACULTY OF INFORMATICS

BCA III-Semester (CBCS) (Main & Backlog) (New) Examination,

January / February - 2024

APPLIED MATHEMATICS

Time : 3 Hours] **[Max. Marks : 70**

Note : I. Note: Answer all questions from Part-A and answer any five questions from Part-B. Choosing one question from each unit.

II. Missing data, if any, may be suitably assumed.

PART - A (10 × 2 = 20 Marks)

1. (a) Find the domain of the function f(x, y) = log(x + y).

Sol :

$f(x, y) = \log(x + y)$

The domain of the function $f(x, y) = \log(x + y)$ is the set of all ordered paired (x, y) such that $x + y > 0$.

$x + y > 0$

$y > -x$

Domain: $\{(x, y) \in R^2 : x + y > 0\}$

(b) Find the first order partial derivative of $x^3 + y^3 + 3xy$.

Sol :

Let

$$f(x, y) = x^3 + y^3 + 3xy$$

Partial derivation w.r.t 'x'

$$\frac{df(x,y)}{dx} = \frac{d}{dx}(x^3 + y^3 + 3xy)$$

$$= 3x^2 + 0 + 3y$$

$$= 3x^2 + 3y$$

Partial derivation with respect to 'y'.

$$\frac{df(x,y)}{dy} = \frac{d}{dy}(x^3 + y^3 + 3xy)$$

$$= 0 + 3y^2 + 3x$$

$$= 3y^2 + 3x.$$

(c) If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$, then find $\frac{\partial u}{\partial r}$.

Sol :

To find $\frac{du}{dv}$, we will use the chair rule.

$$u = x^2 - y^2$$

$$\frac{du}{dr} = \frac{du}{dx} \times \frac{dx}{dr} + \frac{dv}{dy} \times \frac{dy}{dr}$$

First, Find $\frac{du}{dx}$, $\frac{du}{dy}$

$$\frac{du}{dx} = \frac{d}{dx}(x^2 - y^2)$$

$$= 2x$$

$$\frac{du}{dy} = \frac{d}{dy}(x^2 - y^2)$$

$$= -2y$$

$$\frac{dx}{dr} = \frac{d}{dr}(2r - 3s + 4)$$

$$= 2$$

$$\frac{du}{dr} = (-r + 8s - 5)$$

$$= -1 + 0 - 0$$

$$= -1$$

Now, use the chain rule.

$$\frac{du}{dr} = \frac{du}{dx} \times \frac{dx}{dr} + \frac{du}{dy} \times \frac{dy}{dr}$$

$$= 2x \times 2 + (-2y) \times (-1)$$

$$= 4x + 2y$$

Substitute x and y

$$\frac{du}{dr} = 4(24 - 3s + 4) + 2(-r + 8s - 5)$$

$$= 8r - 12s + 16 - 2r + 16s - 10$$

$$= 6r + 4s + 6.$$

(d) Find the stationary points of u = $x^3 y^2(1 - x - y)$.

Sol :

Let

$$u = x^3 y^2 (1 - x - y) \quad \text{... (1)}$$

Partially differentia w.r. t 'x' and d 'y'

$$\frac{du}{dx} = \frac{d}{dx}(x^3 y^2 .(1 - x - y))$$

$$= x^3 y^2 (-1) + 3x^2 y^2 (1 + x - y)$$

$$= -x^3 y^2 + 3x^2 y^2 (1 - x - y) \quad \text{... (2)}$$

$$\frac{du}{dy} = \frac{d}{dy}(x^3 y^2 .(1 - x - y))$$

$$= x^3 y^2 (-1) + 3x^2 2y (1 - x - y)$$

$$= - x^3 y^2 + 2x^2 y (1 - x - y) \quad \text{... (3)}$$

Take $\frac{du}{dx} = 0,$

$$= - x^3 y^2 + 3x^2 y^2 (1 - x - y) = 0$$

$$3x^2 y^2 (1 - x - y) = x^3 y^2 \quad \text{...(4)}$$

$$\frac{du}{dy} = 0$$

$$= - x^3 y^2 + 2x^2 y^2 (1 - x - y) = 0$$

$$2x^3 y (1 - x - y) = x^3 y^2 \quad \text{...(5)}$$

from (4) and (5), we get

$$2\cancel{x^3} y (1 - x - \cancel{y}) = 3\cancel{x^2}\, \cancel{y^2} (1 - x - \cancel{y})$$

$$2x = 3y$$

$$\boxed{y = \frac{2}{3}x}$$

From Equation (4)

$$3\, \cancel{x^2}\, \cancel{y^2}\, (1 - x - y) = \cancel{x^2}\, \cancel{y^2}$$

$$3 (1 - x - y) = x$$

$$3 - 3x - 3y = x$$

$$3 - 3y = x + 3x$$

$$3 - 3y = 4x$$

$$3 - 2x = 4x$$

$$3 = 6x$$

$$x = \frac{3}{6}$$

$$\boxed{x = 1/2}$$

$x = \frac{1}{2}$ put in equation (6)

$$y = \frac{2'}{3}\left(\frac{1}{x}\right)$$

$$\boxed{y = \frac{1}{3}}$$

Stationary Points is $\left(\frac{1}{2}, \frac{1}{3}\right)$.

(e) Let H be the set of vectors of the form $\begin{bmatrix} 3t \\ 0 \\ -7t \end{bmatrix}$, where t is real number, show that H is subspace of R^3.

To show that H is subplace of R3, we need to verify that H satisfies the three critiria for a subspace.

1. **The zero vector is H**
2. **H is closed under vector addition**
3. **H is closed under scalar multiplication.**

Sol :

$$\text{Let } H = \begin{pmatrix} 3t \\ 0 \\ -7t \end{pmatrix}$$

1. Zero vector

$$\begin{pmatrix} 3t \\ 0 \\ -7t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3(0) \\ 0 \\ -7(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The zero vector is in H.

2. H is closed under vector addition

Take two vectors u = $\begin{pmatrix} 3t_1 \\ 0 \\ -7t_2 \end{pmatrix}$ and v = $\begin{pmatrix} 2t_1 \\ 0 \\ -7t_2 \end{pmatrix}$

$$u + v = \begin{pmatrix} 3t_1 \\ 0 \\ -7t_2 \end{pmatrix} + \begin{pmatrix} 3t_2 \\ 0 \\ -7t_2 \end{pmatrix} = \begin{pmatrix} 3(t_1 + t_2) \\ 0 \\ -7(t_1 + t_2) \end{pmatrix}$$

Since $t_1 + t_2$ is real number, this vector is also of the form $\begin{pmatrix} 3t \\ 0 \\ -7t \end{pmatrix}$, where $t = t_1 + t_2$.

H is closed under vector addition.

3. H is closed under scalar multiplication

Take a vector u = $\begin{pmatrix} 3t_1 \\ 0 \\ -7t_1 \end{pmatrix}$ in H and a scalar $C \in R$

$$Cu = C\begin{pmatrix} 3t \\ 0 \\ -7t \end{pmatrix} = \begin{pmatrix} 3tc \\ 0 \\ -7tc \end{pmatrix}$$

Since it is real number, this vector is also of the rom $\begin{pmatrix} 3t' \\ 0 \\ -7t' \end{pmatrix}$ where $t' = ct$.

H is closed under scalar multiplication.

Hence H is subspace of R^3.

(f) Define column space and Null space.

Sol :

Column Space

The column space of a metric A. denoted by Col (A) or C(A), is the set of all linear combinations of the columns of A.

In other words it's the set of are possible vectors that can be expressed as a combination of the columns of A using scalar multiplication and addition.

Example : $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{Column } A = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Null Space

The null space of a matrix A denoted by N(A) or Kar (A), is the set of are vectors 'x' such that Ax = 0, where '0' is the zero vector, it's the et of all vectors that when multiplied by A_1 result in zero vector.

The Null Space represent the set of also solutions to the homogeneous equation $\boxed{A \times = 0}$.

(g) Is $\lambda = 2$ an Eigne value of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

Sol :

To determine if $\lambda = 2$ is an eigne value of the matrix A = $\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$. We need to check if the let $(A - \lambda I) = 0$.

$$(A - \lambda I) = \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-2 & 2-0 \\ 3-0 & 8-2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

$$\text{let } (A - \lambda I) = \text{let} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

$$= 6 - 6$$

$$\text{let } (A - \lambda I) = 0$$

Since the let $(A - \lambda I) = 0$, $\lambda = 2$ is an eigne value of matrix A = $\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$.

(h) Find the characteristic polynomial of A = $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$.

Sol :

Let A = $\begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}$

Z find crarecteristics polynomial of A = $\begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}$

The characteristic polynomial is given by the let $(A - \lambda I)$, where λ is the eigen value and I is use identity matrix.

$$(A - \lambda I) = \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$$

$$\text{let }(A - \lambda I) = \text{Let}\begin{pmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)(2-\lambda) - 49$$

$$= (2-\lambda)^2 - 49$$

$$= 4 + \lambda^2 - 4\lambda - 49$$

$$= \lambda^2 - 4\lambda - 45$$

$$\boxed{\text{Let }(A - \lambda I) = \lambda^2 - 4\lambda - 45}$$

(i) Define Basis of a vector space.

Sol :

Definition

Let $\overline{v}_1, \overline{v}_2, ...\overline{v}_n$ be a basis for a vector space v if

(i) $\{\overline{v}_1, \overline{v}_2, ...\overline{v}_n\}$ is linear independent.

(ii) $v = \text{span}\{\overline{v}_1, \overline{v}_2, ...\overline{v}_n\}$.

(j) Find Eigen values of A = $\begin{bmatrix} 5 & 1 \\ -8 & 1 \end{bmatrix}$.

Sol :

Find eigen value of A = $\begin{bmatrix} 5 & 1 \\ -8 & 1 \end{bmatrix}$

characteristics equation let $(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 5 & 1 \\ -8 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & 1 \\ -8 & 1-\lambda \end{bmatrix}$$

$$\text{let }(A - \lambda I) = \text{let}\begin{bmatrix} 5-\lambda & 1 \\ -8 & 1-\lambda \end{bmatrix}$$

$$= (5-\lambda)(1-\lambda) + 8$$

$$= 5 - 5\lambda - \lambda + \lambda^2 + 8$$

$$= \lambda^2 - 6\lambda + 13$$

By using characteristic equation to get eigen values.

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6 \pm 4i}{2}$$

$\lambda = 3 \pm 2i$

$\lambda_1 = 3 + 2i$, $\lambda_2 = 3 - 2i.$

Eigen values $= \lambda_1 = 3 + 2i, \lambda_2 = 3 - 2i$

PART - B (5 × 10 = 50 Marks)

UNIT - I

2. If u = 3(lx + my + nz)² **– (x² + y² + z²) and l² + m² + n² = 1, prove that** $\frac{\partial^3 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

Sol :

Let V = $3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$

Partial Differentiate with reference to x

$$\frac{du}{dx} = \frac{d}{dx}(3(ln + my + nz)^2 - (x^2 + y^2 + z^2))$$

$= 6l(lx + my + nl) - 2x$

Partial Differentiate with reference to 'y'

$$\frac{du}{dy} = \frac{d}{dy}(3(ln + my + nz)^2 - (x^2 + y^2 + z^2))$$

$= 6m(lx + my + nz) - 2y$

Partial Differentiate with reference to 'z'

$$\frac{du}{dz} = \frac{d}{dz}(3(lx + my + nz)^2 - (x^2 + y^2 + z^2))$$

$= 6n(lx + my + nz) - 2z$

Compute Second Derivation

Partial Differentiate with reference to 'x'

$$\frac{\partial^2 u}{\partial x^2} = \frac{d}{dx}(6l(lx + my + nz) - 2x)$$

$= 6l \, . \, l - 2$

$= 6l^2 - 2.$

Partial Differentiate with reference to 'y'

$$\frac{\partial^2 u}{\partial y^2} = \frac{d}{dy}\,(6m\,(lx + my + nz) - 2y)$$

$= 6m\,.\,m - 2$

$= 6m^2 - 2$

Partial Differentiate with reference to 'z'

$$\frac{\partial^2 u}{\partial z^2} = \frac{d}{dy}\,(6n\,(lx + my + nz) - 2z)$$

$= 6n.n - 2$

$= 6n^2 - 2$

Add all the second derivation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{du^2}{dz^2} = (6l^2 - 2) + (6m^2 - 2) + (6m^2 - 2)$$

$= 6(l^2 + m^2 + n^2) - 6$

$= 6(1) - 6$

$= 6 - 6$

$= 0.$

Hence proved.

(OR)

3. If $u = \sin^{-1}\left[\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right]$, **show that** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

Sol :

Let

$$u = \sin^{-1}\left[\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right] \quad \text{... (1)}$$

Assume

$$V = \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \quad \text{... (2)}$$

By solving (1) and (2) we get

$u = \sin^{-1}(v)$

Compute the partial derivatives of 'v' with respect to 'x' and 'y' we need to find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

Partial derivative of 'v' with respect to x :

$$v(x, y) = \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$

To differentiate 'v' with respect to x, we use the quotient rules

$$\frac{\partial v}{\partial x} = \frac{(\sqrt{x} + \sqrt{y}).\frac{\partial}{\partial x}(\sqrt{x} - \sqrt{y}) - (\sqrt{x} - \sqrt{y}).\frac{\partial}{\partial x}(\sqrt{x} + \sqrt{y})}{(\sqrt{x} + \sqrt{y})^2}$$

$$\Rightarrow \quad \frac{\partial v}{\partial x} = \frac{(\sqrt{x} + \sqrt{y}).\frac{1}{2\sqrt{x}} - (\sqrt{x} - \sqrt{y}).\frac{1}{2\sqrt{x}}}{(\sqrt{x} + \sqrt{y})^2}$$

$$\frac{\partial v}{dx} = \frac{\sqrt{y}}{\sqrt{x}(\sqrt{x} + \sqrt{y})^2}$$

Partial Derivative of "v" with respect ot y:

Similarly to differentiate 'v' with respect to 'y'.

$$\frac{\partial v}{\partial y} = \frac{(\sqrt{x} - \sqrt{y}).\frac{\partial}{\partial y}(\sqrt{x} - \sqrt{y}) - (\sqrt{x} - \sqrt{y}).\frac{\partial}{\partial y}(\sqrt{x} + \sqrt{y})}{(\sqrt{x} + \sqrt{y})^2}$$

$$\frac{\partial v}{\partial y} = \frac{(\sqrt{x} + \sqrt{y}).\left(-\frac{1}{2\sqrt{y}}\right) - (\sqrt{x} - \sqrt{y}).\frac{1}{2\sqrt{y}}}{(\sqrt{x} + \sqrt{y})^2}$$

$$\Rightarrow \quad \boxed{\frac{dv}{dy} = \frac{\sqrt{x}}{\sqrt{y}(\sqrt{x} + \sqrt{y})^2}}$$

By using the chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}.\frac{\partial v}{\partial x}$$

Since $u = \sin^{-1}(v)$, we have

$$\frac{\partial u}{\partial v} = \frac{1}{\sqrt{1 - V^2}}$$

so

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - v^2}}.\frac{\sqrt{y}}{\sqrt{x}(\sqrt{x} + \sqrt{y})^2}$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-v^2}}\left(-\frac{\sqrt{x}}{\sqrt{y}\left(\sqrt{x}+\sqrt{y}\right)^2}\right)$$

Combining the terms to prove identity

$$\frac{x\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

Substituting the expressions for $\frac{\partial u}{\partial z}$ and $\frac{\partial u}{\partial z}$:

$$x\frac{\partial u}{\partial x} = x.\frac{1}{\sqrt{1-v^2}} \cdot \frac{\sqrt{y}}{\sqrt{x}(\sqrt{x}+\sqrt{y})^2} \quad \text{... (3)}$$

$$y\frac{\partial u}{\partial y} = y.\frac{1}{\sqrt{1-v^2}}.\left(-\frac{\sqrt{x}}{\sqrt{y}(\sqrt{x}+\sqrt{y})^2}\right) \quad \text{... (4)}$$

Adding (3) and (4)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x\sqrt{y}}{\sqrt{x}(\sqrt{x}+\sqrt{y})^2\sqrt{1-v^2}} - \frac{y\sqrt{x}}{\sqrt{y}(\sqrt{x}+\sqrt{y})^2\sqrt{1-v^2}}$$

By Factoring common terms.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{y\sqrt{y} - y\sqrt{x}}{\sqrt{x}\sqrt{y}(\sqrt{x}+\sqrt{y})^2\sqrt{1-v^2}} = \frac{0}{\sqrt{x}\sqrt{y}\ (\sqrt{x}+\sqrt{y})^2\sqrt{1-v^2}}$$

$$\Rightarrow \quad x\sqrt{y} - y\sqrt{x} = 0$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Hence proved.

UNIT - II

4. If $z = x^2 + xy + y^2$, $x = s + t$, $y = s - t$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial t}$.

Sol :

$z = x^2 + xy + y^2$, $x = s + t$, $y = s - t$

put x, y in $z = x^2 + xy + y^2$

$z = (s + t)^2 + (s + t)(s - t) + (s - t)^2$

$z = s^2 + t^2 + 2t/s + s^2 - st + st - t^2 + s^2 + t^2 - 2st$

$z = 3s^2 + t^2$

Compute $\frac{\partial z}{\partial s} = \frac{\partial t}{\partial s}(3s^2 + t^2)$

$= 6s$

$$\boxed{\frac{\partial z}{\partial s} = 6s}$$

Compute $\frac{\partial z}{\partial t} = \frac{\partial}{\partial t}(3s^2 + t^2)$

$= 2t$

$$\boxed{\frac{\partial z}{\partial t} = 2t}$$

(OR)

5. Find the minimum value of $x^2 + y^2 + z^2$, given that $ax + by + cz = 1$.

Sol :

Objective function $f(x, y, z) = x^2 + y^2 + z^2$

Constraint $g(x, y, z) = ax + by + cz - 1 = 0$

Form the Lagrangian

$L(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(ax + by + cz - 1)$

Where λ is the Lagrange multiple partial derivatives and set them to equal to zero.

$$\frac{dL}{dx} = 2x + \lambda a = 0 \Rightarrow x = -\frac{\lambda a}{2}$$

$$\frac{dL}{dy} = 2y + \lambda b = 0 \Rightarrow y = -\frac{\lambda b}{2}$$

$$\frac{dL}{dz} = 2z + \lambda c = 0 \Rightarrow z = -\frac{\lambda c}{2}$$

$$\frac{dL}{d\lambda} = ax + by = cz - z = 0$$

Substitute the expression for x, y, z into the constraint.

$$a\left(-\frac{\lambda a}{2}\right) + b\left(-\frac{\lambda b}{2}\right) + c\left(-\frac{\lambda c}{2}\right) = 1$$

$$-\frac{\lambda}{2}(a^2 + b^2 + c^2) = 1$$

$$\lambda = \frac{-2}{a^2 + b^2 + c^2}$$

Substitute λ back to find x, y, z

$$\lambda = -\frac{2}{a^2+b^2+c^2}$$

$$x = \frac{-\lambda a}{2} = \frac{a}{a^2+b^2+c^2}$$

$$y = \frac{-\lambda b}{2} = \frac{b}{a^2+b^2+c^2}$$

$$z = \frac{-\lambda c}{2} = \frac{c}{a^2+b^2+c^2}$$

Calculate $x^2 + y^2 + z^2$

$$x^2 + y^2 + z^2 = \left(\frac{a}{a^2+b^2+c^2}\right)^2 + \left(\frac{b}{a^2+b^2+c^2}\right)^2 + \left(\frac{c}{a^2+b^2+c^2}\right)^2$$

$$= \frac{a^2}{(a^2+b^2+c^2)^2} + \frac{b^2}{(a^2+b^2+c^2)^2} + \frac{c^2}{(a^2+b^2+c^2)^2}$$

$$= \frac{a^2+b^2+c^2}{(a^2+b^2+c^2)^2}$$

$$= \frac{1}{a^2+b^2+c^2}$$

The minimum value of $x^2 + y^2 + z^2$ given the constraint,

$$ax + by + cz = 1 \text{ is } \boxed{\frac{1}{a^2+b^2+c^2}}$$

UNIT - III

6. Let $A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 6 \\ 1 & 1 & 0 & -2 \end{bmatrix}$ and $W = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$. Determine if W is in Col A, Is W in Nul A?

Sol :

We know that

$$A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 6 \\ 1 & 1 & 0 & -2 \end{bmatrix}$$

Vector w, $w = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$

Determine if w is in column (A)

Ac = w

$$\begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 6 \\ 1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

Row Reduce the augmented Matrix

$$\left[\begin{array}{cccc|c} 10 & -8 & -2 & -2 & 2 \\ 0 & 2 & 2 & -2 & 2 \\ 1 & -1 & 6 & 6 & 0 \\ 1 & 1 & 0 & -2 & 2 \end{array}\right]$$

$$R_1: \frac{1}{10} \to R_1 \quad \left[\begin{array}{cccc|c} 1 & -0.8 & -0.2 & 0.2 & 0.2 \\ 0 & 2 & 2 & -2 & 2 \\ 1 & -1 & 6 & 6 & 0 \\ 1 & 1 & 0 & -2 & 2 \end{array}\right]$$

$$R_3: R_3 - R_1 \quad \left[\begin{array}{cccc|c} 1 & -0.8 & -0.2 & 0.2 & 0.2 \\ 0 & 2 & 2 & -2 & 2 \\ 0 & -0.2 & 6.2 & 6.2 & -0.2 \\ 1 & 1 & 0 & -2 & 2 \end{array}\right]$$

$$R_4: R_4 - R_1 \quad \left[\begin{array}{cccc|c} 1 & -0.8 & -0.2 & 0.2 & 0.2 \\ 0 & 2 & 2 & -2 & 2 \\ 0 & -0.2 & 6.2 & 6.2 & -0.2 \\ 0 & 1.8 & 0.2 & -1.8 & 1.8 \end{array}\right]$$

Next, we continuous with row prediction and Eventually, the system will tell us whether there is a contradiction. If no contradiction appears, w is in col(A).

Determine is W is in Null (A)

To check iR W is in Nul (A), we need tos ec if AW = 0 we compute AW.

$$AW = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 6 \\ 1 & 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 20-16+0-4 \\ 0+4+0-4 \\ 2-2+0+12 \\ 2+2+0-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \\ 0 \end{bmatrix}$$

Since AW does not equal O, W is not in Nul (A)

(OR)

7. Determine if $\{v_1, v_2, v_3\}$ is linearly independent or linearly dependent where

$$\mathbf{v_1 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix} v_3 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}.}$$

Sol :

To determine whether the vectors v_1, v_2, v_3 are linear independent or dependent, we need to see if the equation.

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

The vectors given are

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, v_3 = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$$

Equation can be written as

$$c_1 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, c_2 = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, c_3 = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 0c_1 + 1c_2 + 4c_3 \\ c_1 + 2c_2 + c_3 \\ 5c_1 + 8c_2 + 0c_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So system of linear equation

$$1c_2 + 4c_3 = 0 \quad ...(1)$$

$$c_1 + 2c_2 + c_3 = 0 \quad ...(2)$$

$$5c_1 + 8c_2 = 0 \quad ...(3)$$

For the First Equation

$$c_2 + 4c_3 = 0$$

$$c_2 = -4c_3$$

Put equation (2) and (3)

$c_1 + 2c_2 + c_s = 0$

$c_1 + 2(-4c_3) - c_3 = 0$

$c_1 - 8c_3 - c_3 = 0$

$c_1 - 9c_3 = 0$

$c_1 = 9c_3$

$c_2 = -4c_3$ put in (3)

$5c_1 + 8c_2 = 0$

$5c_1 + 8(-4c_3) = 0$

$5(4c_3) + 8(-4c_3) = 0$

$45c_3 - 32c_3 = 0$

$13c_3 = 0$

$\boxed{c_3 = 0}$

For Determine the value of c_1 and c_2

$c_3 = 0$, we have

$c_1 = 9c_3 = 9(0) = 0$

$\boxed{c_1 = 0}$

$c_2 = -4c_3 = -4(0) = 0$

$\boxed{c_2 = 0}$

$c_1 = c_2 = c_3 = 0,$

So therefore, the vector, v_1, v_2, v_3 are linear independent.

UNIT - IV

8. Find characteristic equation of A = $\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$**.**

Sol :

Finding characteristics equation A = $\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

We know that characteristic equation

let (A – λI) = 0

$$\text{let }(A-\lambda I) = \text{let}\begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

$$= (5-\lambda)^2(3-\lambda)(1-\lambda)$$

So characteristic equation

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

This equation shows that the eigen values of the matrix A are λ = 5 (with Algebraic multiplicity 2), λ = 3, λ = 1.

(OR)

9. Find the characteristic polynomial of matrix A = $\begin{bmatrix} \mathbf{4} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{4} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}$**.**

Sol :

We know that characteristic equation, Let (A – λI) = 0.

$$A-\lambda I = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 0 & -1 \\ 0 & 4-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$\text{Let }(A-\lambda I) = \text{let}\begin{bmatrix} 4-\lambda & 0 & -1 \\ 0 & 4-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$\text{Let }(A-\lambda I) = (4-\lambda)\text{ let}\begin{bmatrix} 4-\lambda & -1 \\ 0 & 2-\lambda \end{bmatrix} - 0\,.\,\det\begin{pmatrix} 0 & -1 \\ 1 & 2-\lambda \end{pmatrix} - 1\,.\,\det\begin{pmatrix} 0 & 4-\lambda \\ 1 & 0 \end{pmatrix}$$

$$= (4-\lambda)(4-\lambda)(2-\lambda) - 0 - 1(0-(4-\lambda))$$

$$= (4-\lambda)(4-\lambda)(2-\lambda) - (-4+\lambda)$$

$= (4-\lambda)(4-\lambda)(2-\lambda) - (4-\lambda)$

$= (4-\lambda)(8-6\lambda+\lambda^2) + (4-\lambda)$

$= 4\lambda^2 - 24\lambda + 32 - \lambda^3 + 6\lambda^2 - 8\lambda + 4 - \lambda$

$= -\lambda^3 + 10\lambda^2 - 33\lambda + 36$

So characteristic polynomial of matrix A $-\lambda^3 + 10\lambda^2 - 33\lambda + 36$

Let given matrix $A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

By using characteristic equation

let $(A - \lambda I) = 0$

$$(A - \lambda I) = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix} - I \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 0 & -1 \\ 0 & 4-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{bmatrix}$$

$$= \det \begin{bmatrix} 4-\lambda & 0 & -1 \\ 0 & 4-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{bmatrix}$$

$$= (4-\lambda) \,.\, \text{let} \begin{bmatrix} 4-\lambda & -1 \\ 0 & 2-\lambda \end{bmatrix} - 0 + (-1) \begin{bmatrix} 0 & 4-\lambda \\ 1 & 0 \end{bmatrix}$$

$= (4-\lambda)(4-\lambda)(2-\lambda) + (-(4-\lambda))$

$= (4-\lambda)(4-\lambda)(2-\lambda) + (4-\lambda)$

$= (4-\lambda)[(4-\lambda)(2-\lambda) + 1]$

$= (4-\lambda)[8 - 4\lambda + \lambda^2 - 2\lambda + 1]$

$= (4-\lambda)[\lambda^2 - 6\lambda + 9]$

$= (4-\lambda)(\lambda-3)^2$

The characteristic polynomial is

$$\boxed{\text{let } (A - \lambda I) = (4-\lambda)(\lambda-3)^2}$$

UNIT - V

10. Find diagonalization of matrix A = $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$, if possible.

Sol :

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Above the matrix linear independent eigen vectors. So it is diagonalizable.

Finding the Eigen Value

Characteristic equation

let $(A - \lambda I) = 0$

$$\text{let} \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{let} \begin{pmatrix} I - \lambda & 0 \\ 6 & -1 - \lambda \end{pmatrix}$$

let $(A - \lambda I) = (1-\lambda)(-1-\lambda) = 0$

$= (1-\lambda)(-1-\lambda) = 0$

So eigen values

$\lambda_1 = 1, \lambda_2 = -1$

$$A - \lambda I = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 6 & -2 \end{pmatrix}$$

Solve $(A - I)\,v = 0$.

$$\begin{pmatrix} 0 & 0 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$6x - 2y = 0$

$6x = 2y$

$\boxed{y = 3}$

Eigen vectors for $\lambda_1 = 1$

$$v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

For $\lambda_2 = -1$ put in $(A - \lambda I)\,v = 0$

$(A + I)\ v = 0$

$$\left(\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2x = 0 \quad | \quad 6x = 0$

$x = 0$, Since there is no conditinon y, we can take $y = 1$,

So, eigen vectors for $\lambda_2 = -1$

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Form the Diagonalization

Here

$$P = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix},\ D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Verify Diagonalization

$P^{-1} A P = D$

Here,

$$P = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$$

Verify,

$P^{-1} AP = D$

$$P^{-1} AP = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$AP = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 6-3 & 0-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$P^{-1} AP = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 0+0 \\ -3+3 & 0-1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{P^{-1}AP = D}$$

So A Matrix is Diagonalization.

(OR)

11. Find the Eigen values and basis for each Eigen space in C_2 where A = $\begin{bmatrix} 0 & 5 \\ -2 & 2 \end{bmatrix}$.

Sol :

$$A = \begin{bmatrix} 0 & 5 \\ -2 & 2 \end{bmatrix}$$

Finding Eigen Values

By solving the characteristic equation

let $(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 0 & 5 \\ -2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & 5 \\ -2 & 2-\lambda \end{pmatrix}$$

$$\text{let } (A - \lambda I) = \text{let} \begin{pmatrix} -\lambda & 5 \\ -2 & 2-\lambda \end{pmatrix}$$

$$= -\lambda (2 - \lambda) - (-2) 5$$

$$= -2\lambda + \lambda^2 + 10$$

$$= \lambda^2 - 2\lambda + 10$$

$$\boxed{\lambda^2 - 2\lambda + 10 = 0}$$

By using quadratic equation

$$A = (-2)^2 - 4(1)\,(10)$$

$$= 4 - 40$$

$$= -36$$

So, eigen values are complex and given by,

$$\lambda = \frac{(-2) \pm \sqrt{-36}}{2(1)}$$

$$= \frac{2 \pm 6i}{2}$$

$$= 1 \pm 3i$$

So the eigen values are :

$$\lambda_1 = 1 + 3i, \quad \lambda_2 = 1 - 3i$$

Finding eigen vectors

For each eigen values λ, $(A - \lambda I)\,V = 0$

For $\lambda_1 = 1 + 3i$

$(A - \lambda I)\,v = 0$

$$\begin{pmatrix} -\lambda_1 & 5 \\ -2 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Substitute $\lambda_1 = 1 + 3i$

$$\begin{pmatrix} -(1+3i) & 5 \\ -2 & 2-(1+3i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} -1-3i & 5 \\ -2 & 2-1+3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$x = \frac{1-3i}{2}y$$

So, eigen vector for λ_1

$$v_1 = \begin{pmatrix} \frac{1-3i}{2} \\ 1 \end{pmatrix}$$

for $\lambda_2 = 1 - 3i$

$(A - \lambda I)\, v = 0$

$$\begin{pmatrix} 0 & 5 \\ -2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 5 \\ -2 & 2 \end{pmatrix} - (1-3i) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} -1+3i & 5 \\ -2 & 1+3i \end{pmatrix} = 0$$

System of Linear Equation form

$$\begin{pmatrix} -1+3i & 5 \\ -2 & 1+3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$(-1 + 3i)\, x + 5y = 0$ ... (1)

$-2x + (1 + 3i)y = 0$ ... (2)

First equation $(-1 + 3i)\, x + 5y = 0$

$$\begin{pmatrix} -1-3i & 5 \\ -2 & 1-3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$(-1 - 3i)\, x + 5y = 0$ ... (1)

$-2x + (1 - 3i)\, y = 0$ ... (2)

First equation

$(-1 - 3i)\, x + 5y = 0$

$(-1 - 3i)\, x = -5y$

$$x = \frac{-5y}{-1-3i}$$

$$\boxed{x = \frac{-5}{-1-3i}\, y}$$

To simplify 'x', multiply the numerator and denominator by complex conjugate of denominator (or) (multiply by $(-1+3i)$.

$$x = \frac{-5\,(-1+3i)}{(-1-3i)\,(-1+3i)}\, y$$

$$x = \frac{-5(-1+3i)}{1+9} = \frac{+5-15i}{10}$$

$$= \frac{\cancel{5}(1-ei)}{\cancel{10}} = \frac{1-3i}{2}$$

$$x = \frac{-5}{(-1+3i)}y$$

Multiply the numerator and denominator by compute conjugate of denominator (or) (multiply by (–1 –3i))

$$x = \frac{-5(-1-3i)}{(-1+3i)(-1-3i)}y$$

$$x = \frac{5+15i}{1+9}y$$

$$x = \frac{\cancel{5}(1+3i)}{\cancel{10}_2}y$$

$$\boxed{x = \frac{1+3i}{2}y}$$

So, the eigen vector for λ_2

$$v_2 = \begin{pmatrix} 1+3i \\ 1^2 \end{pmatrix}$$

FACULTY OF INFORMATICS

B.C.A III - Semester (CBCS) (Backlog) Examination,

June / July - 2024

APPLIED MATHEMATICS

Time : 3 Hours] [Max. Marks : 70

Note : I. Answer all questions from Part-A and answer any five questions from Part-B. Choosing one question from each unit.

II. Missing data, if any, may be sultably assumed.

PART - A (20 Marks)

ANSWERS

1. (a) Define continuity of a function.

Sol :

A function f(x) is said to be continuous as a point x = c it the following three conditions are satisfied.

1. f(c) is defined
2. $\underset{x \to c}{\text{Lt}} f(x)$ exists
3. $\underset{x \to c}{\text{Lt}} f(x) = f(c)$

A function f is continuous on the open interval (a, b).

(b) Find the first order partial derivatives of $ax^2 + 2hxy + by^2$.

Sol :

Given $f(x,y) = ax^2 + 2hxy + by^2$

p.d.w.r.t. 'x'

$$\frac{df}{dx} = \frac{d}{dx}(ax^2 + 2hxy + by^2)$$

$$= 2xa + 2xy$$

p.d. w.r.t 'y'

$$\frac{df}{dy} = \frac{d}{dy}(ax^2 + 2hxy + by^2)$$

$$= 2hx + 2by$$

(c) Find $\frac{dz}{dt}$ when $= xy^2 + x^2 y, x = at^2, y = 2at.$

Sol :

Given $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$

p.d.w.r.t 'x'

$$\frac{dz}{dx} = \frac{d}{dx}(xy^2 + x^2y)$$

$$= y^2 + 2xy$$

p.d.w.r.t 'y'

$$\frac{dz}{dy} = \frac{d}{dy}(x^2 y^2 + x^2y)$$

$$= 2yx + x^2$$

$$= 2xy + x^2$$

p.d.w.r.t 't'

$$\frac{dx}{dt} = \frac{d}{dt}(at^2)$$

$$= 2at$$

p.d.w.r.t 't'

$$\frac{dy}{dt} = \frac{d}{dt}(2at)$$

$$= 2a$$

Now apply the chain sale.

$$\frac{dz}{dt} = \left(\frac{dz}{dx}\right)\left(\frac{dx}{dt}\right) + \left(\frac{dz}{dy}\right)\left(\frac{dy}{dt}\right)$$

$$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$

Put $x = at^2$, $y = 2at$

$= (2at)^2 + 2(at^2)(2at)(2at) = (2(at^2)(2at) + (at^2)^2 (2a)$

$= (4a^2t^2 + 4a^2t^3)(2at) + (4a^2 t^3 + a^2t^2)(2a)$

$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^5$

$= 16a^3t^3 + 8a^3t^3 + 2a^3t^5$

$$\boxed{\frac{dz}{dt} = 16a^3t^3 + 8a^3t^3 + 2a^3t^5}$$

(d) Write Taylor's theorem for a function of two variables.

Sol :

Taylor's theorem for two variables

Let a function be defined in some domain D in IR^2 and have continuous partial derivatives up to $(n + 1)^{th}$ order in some neighborhood of a point $P(x_0, y_0)$ in D.

Then

$$F(x_0 + h, y_0 + k) = F(x_0, y_0) + \left(h\frac{d}{dx}+k\frac{d}{dy}\right)F(x_0, y_0)\ \frac{1}{2!}\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^2$$

$$F(x_0, y_0) + ... + \frac{1}{n!}\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^n F(x_0, y_0) + R_n$$

Where

$$R_n = \frac{1}{(n+1)!}\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^{n+1} F(x_0 + \theta h, y_o + \theta k),\ 0 < \theta < 1$$

(e) If $= \left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 3b + c = 0\right\}$. Is W is a subspace of R^3.

Sol :

$$\text{Given } w = \left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 3b + c = 0\right\}$$

whether w is a subspace of R^3

A subset w of a vector space v is called a subspace if. It satisfies three conditions.

1. Zero vector
2. Closed under Addition
3. Closed under scalar multiplication

1. Zero vector :

$$\text{Let } w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check, it 0 is in w

a = b = c = 0, we have

$a - 3b + c = 0$

$0 - 3(0) + 0 = 0$

Therefore, 0 in w.

2. Closed under addition

Let $u = \begin{bmatrix} a_1 \\ a_1 \\ a_1 \end{bmatrix}$ and $v = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ be any two vectors in w

$a_1 - 3b_1 + c_1 = 0$
$a_2 - 3b_2 + c_2 = 0$

$u + v = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix}$ in w

$(a_1 + a_2) - 3(b_1 + b_2) + (c_1 + c_2) = (a_1 - 3b_1 + c_1) + (a_2 - 3b_2 + c_2) = 0 + 0 = 0$

Thus u + v is in w.

3. Closed under Scalar Multiplication

Let $u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be any vector in w

$ku = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}$ is in w

$ka - 3(kb) + kc = k(a - 3b + c) = k.0 = 0$

thus ku is in w

Since w contains the zero vector is closed under addition and is closed under scalar multiplication w is a subspace of R^3.

(f) Define basis of a vector space.

Sol :

For answer refer Februar-2024, Q.No. (i).

(g) If A is a 7 × 9 matrix with a two-dimensional null space, what is the rank of A ?

Sol :

Using Rank – nullity theorem

Rank (A) + dim N(A) = n

Where

dim (A) = 2, n = 9

Rank (A) + dim N(A) = n

Rank (A) + 2 = 9

The Rank (A) = 9 – 2

$\boxed{\text{Rank }(A) = 7}$

(h) Is $\lambda = -3$, an eigen value of $\begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$. Why or why not ?

Sol :

Given

$\lambda = -3,\ A = \begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$

Characteristic equation

$\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+3 & 4 \\ 6 & 9+3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 2 & 4 \\ 6 & 12 \end{pmatrix}$$

$$= 2(12) - 6(4)$$

$$= 24 - 24$$

$$= 0$$

Since the det $= 0$, $\lambda = -3$ is indeed an eigen value of the matrix A.

why $\lambda = -3$

The det $(A - \lambda I) = 0$. Indicates that the matrix is singular. Which corresponds to the condition for λ being an eigen value, thus $\lambda = -3$ is an eigen value of the given note.

(i) Find the eigen values of $\begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$

Sol :

Given $A = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$

Characteristic equation,

$\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & -3 \\ 3 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -3 \\ 3 & 3-\lambda \end{bmatrix}$$

$$= (3 - \lambda)(3 - \lambda) - (-3)(3)$$

$$= (3 - \lambda)^2 + 9$$

$$= 9 + \lambda^2 - 6\lambda + 9$$

$$= \lambda^2 - 6\lambda + 18$$

To find the eigen value by using quadratic equation

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)}$$

$$\lambda = \frac{6 \pm \sqrt{36 - 72}}{2}$$

$$= \frac{6 \pm \sqrt{-36}}{2}$$

$$= \frac{6 \pm : \sqrt{36}}{2}$$

$$= \frac{6 \pm 6i}{2}$$

$$= \boxed{\lambda = 3 \pm 3i}$$

$$\boxed{\text{eigen values} : \lambda_1 = 3 + 3i, \lambda_2 = 3 - 3i}$$

(j) Find the characteristic equation of = $\begin{bmatrix} 2 & 3 \\ 0 & -5 \end{bmatrix}$

Sol :

Given A = $\begin{bmatrix} 2 & 3 \\ 0 & -5 \end{bmatrix}$

Characteristics equation

det (A – λI) = 0

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 0 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 0 & -5-\lambda \end{bmatrix}$$

$$\det (A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 3 \\ 0 & -5-\lambda \end{pmatrix}$$

$$= (2 - \lambda)(-5 - \lambda) - 0$$

$$= -10 - 2\lambda + 5\lambda + \lambda^2$$

$$\boxed{\text{Characteristic equation} = \lambda^2 + 3\lambda - 10}$$

PART - B (5 × 10 = 50 Marks)

Unit-I

2. If $U = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$; $x^2 + y^2 + z^2 \neq 0$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

Sol :

$$u = (x^2 + y^2 + z^2)^{-½}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[(x^2 + y^2 + z^2)^{-½}] = \frac{-1}{2}(x^2 + y^2 + z^2)^{-½} \times \frac{\partial}{\partial x}(x^2 + y^2 + z^2)$$

$$= \frac{-1}{2}(x^2 + y^2 + z^2)^{-3/2} \times \left(\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(z^2)\right)$$

$$= \frac{-1}{2}(x^2 + y^2 + z^2)^{-3/2}] \times (2x + 0 + 0) = \frac{-1}{\cancel{x}}(x^2 + y^2 + z^2)^{-3/2} \times x$$

$$= -x(x^2 + y^2 + z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}[-x(x^2 + y^2 + z^2)^{-3/2}]$$

$$= -\left[x \cdot \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-3/2} + \left(x^2+y^2+z^2\right)^{-3/2} \frac{.\partial}{\partial x}(x)\right]$$

$$= -\left[x.-\frac{3}{2}\left(x^2+y^2+z^2\right)^{-3/2-1} \times \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right) + \left(x^2+y^2+z^2\right)^{-3/2}.1\right]$$

$$= -\left[x \cdot \frac{-3}{2}\left(x^2+y^2+z^2\right)^{-5/2} \times \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(z^2)^1\right] + \left(x^2+y^2+z^2\right)^{-3/2}\right]$$

$$= -\left[x \cdot \frac{-3}{2}\left(x^2+y^2+z^2\right)^{-5/2} \times [2x+0+0] + \left(x^2+y^2+z^2\right)^{-3/2}\right]$$

$$= -\left[x \cdot \frac{-3}{\not 2}\left(x^2+y^2+z^2\right)^{-5/2} \times \not 2x + \left(x^2+y^2+z^2\right)^{-3/2}\right]$$

$= 3x^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = 3y^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 3x^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}$$

$+ 3y^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2} + 3z^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}$

$= 3(x^2 + y^2 + z^2)^{-5/2} [x^2 + y^2 - z^2] - 3(x^2 + y^2 + z^2)^{-3/2}$

$= 3(x^2 + y^2 + z^2)^{-5/2} (x^2 + y^2 + z^2)^1 - 3(x^2 + y^2 + z^2)^{-3/2}$

$= 3(x^2 + y^2 + z^2)^{-5/2+1} - 3(x^2 + y^2 + z^2)^{-3/2}$

$= 3(x^2 + y^2 + z^2)^{-3/2} - 3(x^2 + y^2 + z^2)^{-3/2} = 0$

(OR)

3. If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

Sol :

$$u = \log\left(\frac{x^2+y^2}{x+y}\right)$$

L.H.S $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x}, \left(\frac{x+y}{x^2+y^2}\right) . \frac{2x(x+y)-1(x^2+y^2)}{(x+y)^2}$$

$$\frac{2x^2+2xy-x^2-y^2}{(x+y)(x^2+y^2)} \times \frac{x^2-y^2+2xy}{(x+y)(x^2+y^2)}$$

$$\frac{\partial u}{\partial y} = \left(\frac{x+y}{x^2+y^2}\right)\frac{2y(x+y)-1(x^2+y^2)}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2xy+2y^2-x^2-y^2}{(x+y)(x^2+y^2)} = \frac{y^2-x^2+2xy}{(x+y)(x^2+y^2)}$$

$$x\frac{du}{dx}+y\frac{du}{dy} = x\left[\frac{x^2-y^2+2xy}{(x+y)(x^2+y^2)}\right] + y\left[\frac{y^2-x^2+2xy}{(x+y)(x^2+y^2)}\right]$$

$$= \frac{x^3-xy^2+2x^2y+y^3-x^2y+2xy^2}{(x+y)(x^2+y^2)}$$

$$= \frac{(x+y)(x^2+y^2-xy)-xy(x+y)+2xy(x+y)}{(x+y)(x^2+y^2)}$$

$$= \frac{(x+y)(x^2+y^2-xy-xy+2xy)}{(x+y)(x^2+y^2)} = 1 \text{ R.H.S}$$

$$\boxed{x\cdot\frac{\partial y}{\partial x}+y.\frac{\partial y}{\partial y}=1}$$

Unit-II

4. If H = f(y – z, z – x, x – y), prove that $\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0$.

Sol :

u = f (y – z, z – x, x – y)

a = y – z

b = z – x

c = x

u = f (a, b, c)

Diff. 'u' with respect to x.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial x} + \frac{\partial u}{x} + \frac{\partial u}{x}\ \frac{x}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}(1) + \frac{\partial u}{\partial b}(0) + \frac{\partial u}{\partial c}(-1)$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} - \frac{\partial u}{\partial c}}$$

Differentiate u = f (a, b, c) with respect to 'y'

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial u}{\partial a}\cdot\frac{\partial b}{\partial y} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial y}$$

$$= \frac{\partial u}{\partial a}(-1) + \frac{\partial u}{\partial b} = \frac{-\partial u}{\partial a} + \frac{\partial u}{\partial b}$$

Diff u = f (a, b, c) with respect to 'z'

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial z} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial z} + \frac{\partial u}{\partial c}\cdot\frac{x}{\partial z}$$

$$= \frac{\partial u}{\partial a}(0) + \frac{\partial u}{\partial b}(-1) + \frac{\partial u}{\partial c}(1)$$

$$\boxed{\frac{\partial u}{\partial z} = \frac{-\partial u}{\partial b} + \frac{\partial u}{\partial c}}$$

Now

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial a} - \frac{\partial u}{\partial c} - \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} - \frac{\partial u}{\partial b} + \frac{\partial u}{\partial c}\ \boxed{= 0}$$

$$\boxed{\frac{du}{dx} + \frac{du}{dsy} + \frac{du}{dz} = 0}$$

(OR)

5. Show that the minimum valued of $u = xy + \frac{a^3}{x} + \frac{a^3}{y}$ is $3a^2$.

Sol :

Given $u = xy + \frac{a^3}{x} + \frac{a^3}{y}$, p.d.w.r.t 'x'

$$\frac{du}{dx} = \frac{d}{dx}\left(xy + \frac{a^3}{x} + \frac{a^3}{y}\right)$$

$$= y + a^3 \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= y + a^3 \left(-\frac{1}{x^2}\right)$$

$$= y - \frac{a^3}{x^2}$$

p.d. w.r.t 'y'

$$\frac{du}{dy} = \frac{d}{dy}\left(xy + \frac{a^3}{x} + \frac{a^3}{y}\right)$$

$$= x - \frac{a^3}{y^2}$$

for $\frac{du}{dx} = 0, \Rightarrow y - \frac{a^3}{x^2} = 0$

$$y = \frac{a^3}{x^2}$$

$$\frac{du}{dy} = 0 \Rightarrow x - \frac{a^3}{y^2} = 0$$

$$x = \frac{a^3}{y^2}$$

Solving the system of equation

Sub eqn (1) in equation (2)

$$x = \frac{a^3}{\left(\frac{a^3}{x^2}\right)^2}$$

$$= \frac{a^3}{\frac{a^6}{x^4}}$$

$$= \frac{\cancel{a^3} x^4}{a^{\cancel{6}}}$$

$$x = \frac{x^4}{a^3}$$

$xa^3 = x^4$

$\boxed{a^3 = x^5}$

$x = a^{6/5}$

$= a^{3/2}$

using eqn (1)

$$y = \frac{a^3}{x^2}$$

$$= \frac{a^3}{\left(a^{3/2}\right)^2} = \frac{a^3}{a^{6/4}} = \frac{a^3}{a^{3/2}}$$

$y = a^{3/2}$

Critical points = $(a^{3/2}, a^{3/2})$

Put $x = y = a^{3/2}$ into the original equation

$$u = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

$$u = (a^{3/2})(a^{3/2}) + \frac{a^3}{a^{3/2}} + \frac{a^3}{a^{3/2}}$$

$= a^3 + a^{3/2} + a^{3/2}$

$= a^3 + 2a^{3/2}$

Given 'u' is minimum value

Show the minimum value is $3a^2$

$$u = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

Let Assum x = y = a

$$u = (a)\,(a) + \frac{a^3}{a} + \frac{a^3}{a}$$

$= a^2 + 2a^2$

$= 3a^2$

Unit-III

6. (a) Find a nonzero vector in Col A and a nonzero vector in Nul A where

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

Sol :

$$\text{Given } A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

Finding Non-zero vector in col A

The column space of A is the span of it column ? any non- zero linear combinations of columns of A will be a non - zero vector in col A.

$$V = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

V is Non - zero vector in col A.

Finding Non-zero vector in Nocl (A)

We need to solve the homogeneous equation

Ax = 0

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{array}\right]$$

By using echelon form

by using,

$$R_2 = R_2 + R_1 \Rightarrow \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3}{2}R_1 \Rightarrow \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 1 & -5 & \frac{9}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Rightarrow \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & \frac{17}{2} & 0 \end{array}\right]$$

$2x_1 + 4x_2 - 2x_3 + x_4 = 0$ .... (1)

$-x_2 + 5x_3 + 4x_4 = 0$ ....(2)

$\frac{17}{2}x_4 = 0$ .... (3)

Solving we system

$\frac{17}{2}x_4 = 0$

$\boxed{x_4 = 0}$

$x_4 = 0$, put in (2)

$-x_2 + 5x_3 + 4(0) = 0$

$-x_2 + 5x_3 = 0$

$x_2 = 5x_3$

Put $x_2 = 5x_3$, $x_4 = 0$ in (1)

$2x_1 + 4x_2 - 2x_3 + x_4 = 0$

$2x_1 + 4(5x_3) - 2x_3 + 0 = 0$

$2x_1 + 20x_3 - 2x_3 = 0$

$2x_1 + 18x_3 = 0$

$2x_1 = -18x_3$

$$x_1 = \frac{\overset{9}{\cancel{-18}}}{\cancel{2}} x_3$$

$$\boxed{x_1 = -9x_3}$$

Solution is

$x_1 = -9x_3$

$x_2 = 5x_3$

$x_3 = x_3$

$x_4 = 0$

Nul (A) is spanned by vector

$$V = \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

This is a non-zero vector in Nol (A)

$2x_1 + 4x_2 - 2x_3 + x_4 = 0 \Rightarrow 2(-9) + 4(5) - 2(1) + 0 = 0$

$= -18 + 20 - 2 = 0 \Rightarrow -20 + 20 = 0$

(OR)

7. Determine whether the set $S = \{v_1, v_2, v_3\}$ is a basis of R^3,

where $v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$

Sol :

Given

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$$

To determine whether the set $S = \{v_1, v_2, v_3\}$

for R^3, we need to check two things.

1. Linear Independent
2. Spanning

1. Linear Independent

$$A = \begin{bmatrix} 2 & 2 & -8 \\ -1 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

$$|A| = 2\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix} - 2\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix} - 8\begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$$

$= 2(-12-10) - 2(-4-5) - 8(-2+3)$

$= 2(-22) - 2(-9) - 8(1)$

$= -44 + 18 - 8$

$|A| = -34$

$|A| \neq 0$, So Linear Independent.

Because the vector are L.I and there are three of then in IR^3, they must also **span R^3**.

The set $s = \{v_1, v_2, v_3\}$ is basis for IR^3.

Unit-IV

8. Find eigen values and eigen vectors of $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

Sol :

Given $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

Characteristic Equation

$\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{bmatrix}$$

$$= (4-\lambda)\begin{vmatrix} 1-\lambda & 6 \\ -1 & 8-\lambda \end{vmatrix} + 1\begin{pmatrix} 2 & 6 \\ 2 & 8-\lambda \end{pmatrix} + 6\begin{pmatrix} 2 & 1-\lambda \\ 2 & -1 \end{pmatrix}$$

$= (4-\lambda)\,[(1-\lambda)(8-\lambda) + 6] + 1\,(2(8-\lambda) - 12) + 6\,(-2 - 2(1-\lambda)$

$= (4-\lambda)\,[8 - \lambda - 8\lambda + \lambda^2 + 6] + [16 - 2\lambda - 12] + 6\,(-2 - 2 + 2\lambda)$

$= (4-\lambda)\,(\lambda^2 - 9\lambda + 14) + (-2\lambda + 4) + 6(2\lambda - 4)$

$= 4\lambda^2 - 36\lambda + 56 - \lambda^3 + 9\lambda^2 - 14\lambda - 2\lambda + 4 + 12\lambda - 24$

$= -\lambda^3 + 13\lambda^2 - 40\lambda + 36$

by using factorization method

$-\lambda^3 + 13\lambda^2 - 40\lambda + 36 = 0$

$(\lambda - 2)(-\lambda^2 + 1l\lambda - 18) = 0$

$(\lambda - 2)(\lambda - 2)(\lambda - 9) = 0$

$(\lambda - 2)^2(\lambda - 9) = 0$

$\lambda_1 = 2, \lambda_2 = 2, \lambda_1 = 9$

Eigen vector

$\lambda_1 = 2$ in $A - \lambda I$

$$A - 2I = \begin{bmatrix} 4-2 & -1 & 6 \\ 2 & 1-2 & 6 \\ 2 & -1 & 8-2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

$(A - 2I)x = 0$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 - x_2 + 6x_3 = 0$

$2x_1 - x_2 + 6x_3 = 0$

$2x_1 - x_2 + 6x_3 = 0$

Take Eq (1) $2x_1 - x_2 + 6x_3 = 0$

Let $x_3 = t$

$x_2 = 2x_1 + 6t$

$x_1 = s$

$x_2 = 2s + 6t$

$\lambda_1 = 2$

$$x_1 = \begin{bmatrix} s \\ 2s+6t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}$$

This mean that any combination of these two vectors is an eigen vector corresponding to $\lambda_1 = 2$

$$x_1 = \begin{bmatrix}1\\2\\0\end{bmatrix} \; x_1^1 = \begin{bmatrix}0\\6\\1\end{bmatrix}$$

Eigen vector for $\lambda_2 = 9$

Put $\lambda_2 = 9$ in $A - \lambda_2 I$

$$A - aI = \begin{bmatrix}4-9 & -1 & 6\\ 2 & 1-9 & 6\\ 2 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}-5 & -1 & 6\\ 2 & -8 & 6\\ 2 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

System of equation

$-5x_1 - x_2 + 6x_3 = 0$ ...... (1)

$2x_1 - 8x_2 + 6x_3 = 0$ ...... (2)

$2x_1 - x_2 - x_3 = 0$ .... (3)

$2 \times (2) - (2)$

$2(-5x_1 - x_2 + 6x_3) - (2x_1 - 8x_2 + 6x_3) = 0$

$-10x_1 - 2x_2 + 12x_3 - 2x_1 + 8x_2 - 6x_3 = 0$

$-12x_1 + 6x_2 + 6x_3 = 0$

$-2x_1 + x_2 + x_3 = 0$

Now we have

1. $-5x_1 - x_2 + 6x_3 = 0$
2. $-2x_1 + x_2 + x_3 = 0$
3. $2x_1 - x_2 - x_3 = 0$

Notice that the second and third Equation are essentials

some $-2x_1 + x_2 + x_3 = 0$

So we only need to solve two equation

1. $-5x_1 - x_2 + 6x_3 = 0$
2. $-2x_1 + x_2 + x_3 = 0$

(2) $\Rightarrow x_2 = 2x_1 - x_3$ ...... (4)

Put in (1) $-5x_1 - (2x_1 - x_3) + 6x_3 = 0$

$-5x_1 - 2x_1 + x_3 + 6x_3 = 0$

$-7x_1 + 7x_3 = 0$

$x_1 = x_3$

So, $x_2 = 2x_1 - x_3 = x_1$, therefore the eigenvector corresponding 6 to $\lambda_2 = 9$ is

$$x_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Finding Eigen vectors

$\lambda_1 = 2$

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \; x_1^1 \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}$$

$\lambda_2 = 9$

$$\lambda_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(OR)

9. Find the characteristic polynomial of $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 2 & -2 & -4 & 0 \\ 6 & 6 & 0 & 1 \end{bmatrix}$

Sol :

$$\text{Given } A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 2 & -2 & -4 & 0 \\ 6 & 6 & 0 & 1 \end{bmatrix}$$

Characteristic Equation

$\det(A - \lambda I) = 0$

$$(A - \lambda I) = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 2 & -2 & -4 & 0 \\ 6 & 6 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & 0 & 0 & 0 \\ 2 & 4-\lambda & 0 & 0 \\ 2 & -2 & -4-\lambda & 0 \\ 6 & 6 & 0 & 1-\lambda \end{bmatrix}$$

$$= (5-\lambda)\begin{bmatrix} 4-\lambda & 0 & 0 \\ -2 & -4-\lambda & 0 \\ 6 & 0 & 1-\lambda \end{bmatrix}$$

$$= (5-\lambda)\,(4-\lambda)\begin{bmatrix} -4-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}$$

$= (5-\lambda)(4-\lambda)((-4-\lambda)(1-\lambda)-0)$

$= (5-\lambda)(4-\lambda)\,(-4-\lambda)\,(1-\lambda)$

$= (20-5\lambda-4\lambda+\lambda^2)\,(-4+4\lambda-\lambda+\lambda^2)$

$= -80 + \cancel{80\lambda} - \cancel{20\lambda} + \cancel{20\lambda^2} + 20\lambda - \cancel{20\lambda^2} + \cancel{5\lambda^2} - \cancel{5\lambda^3} + 16\lambda - \cancel{16\lambda^2} + \cancel{4\lambda^2} - \cancel{4\lambda^3} - \cancel{4\lambda^2} + \cancel{4\lambda^3} - \cancel{\lambda^3} + \lambda^4$

$= \lambda^4 - 6\lambda^3 - 11\lambda^2 + 96\lambda - 80$

Characteristic Equation of matrix A

$\boxed{\lambda^4 - 6\lambda^3 - 11\lambda^2 + 96\lambda - 80}$

Unit-V

10. Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, if possible.

Sol :

Given $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Characteristic Equation

det $(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - I\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix}$$

$= (1-\lambda)(\lambda^2 + 4\lambda + 4) - 3\,\cancel{(3\lambda+6)} + 3\,\cancel{(3\lambda+6)}$

$= (1-\lambda)(\lambda^2 + 4\lambda + 4)$

$= (1-\lambda)(\lambda + 2)^2 = 0$

Eigen values

$\lambda_1 = 1, \lambda_2 = -2, -2$

Finding Eigen vectors :

$(A - \lambda I)x = 0$

$$\text{Let } \lambda_1 = 1, (A - I) = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_2 + 3x_3 = 0$ .... (1)

$-3x_1 - 6x_2 - 3x_3 = 0$ .... (2)

$3x_1 + 3x_2 = 0$ .... (3)

We get (1)

$3x_2 + 3x_3 = 0$

$\cancel{3}x_2 = -\cancel{3}x_3$

$\boxed{x_2 = -x_3}$

$-3x_1 - 6x_2 - 3x_3 = 0$

$-3x_1 - 6(-x_3) - 3x_3 = 0$

$-3x_1 + 6x_3 - 3x_3 = 0$

$-3x_1 + 3x_3 = 0$

$-\not{3}x_1 = -\not{3}x_3$

$\boxed{x_1 = x_3}$

Eigen vector for $\lambda_2 = -2$

$$A + 2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

Solve the system

$$\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 + x_3 = 0$

This gives us two independent eigen vectors

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Diagonalize A

$$P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$A = PDP^{-1}$

Where P is the Matrix of eigenvectors and D is diagonal matrix of eigen valued confirm that A is diagonalizable.

(OR)

11. Construct general solution of $x' = Ax$ where $A = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}$

Sol :

Given $A = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}$

Finding eigen values

Characteristic Equation

$\det(A - \lambda I) = 0$

$(A - \lambda I) = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4-\lambda & -5 \\ -2 & 1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (4 - \lambda)(1 - \lambda) - 10$

$= 4 - 4\lambda - \lambda + \lambda^2 - 10$

$= \lambda^2 - 5\lambda - 6 = 0$

$= (\lambda - 6)(\lambda + 1) = 0$

Eigen values

$\lambda_1 = 6, \lambda_2 = 1$

Finding the Eigen vector of A

$\lambda_1 = 6, \Rightarrow (A - \lambda I)\, v = 0$

$(A - 6I)\, v = 0$

$(A - 6I) = \begin{bmatrix} 4-6 & -5 \\ -2 & 1-6 \end{bmatrix}$

$= \begin{bmatrix} -2 & -5 \\ -2 & -5 \end{bmatrix}$

System of equation

$-2x_1 - 5x_2 = 0$

$-2x_1 = 5x_2$

$\boxed{x_1 = -\frac{5}{2}x_2}$

eigen vector for $\lambda_1 = 6$

$v_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

eigen vector for $\lambda_2 = -1$

$$A + 1 = \begin{bmatrix} 4+1 & -5 \\ -2 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -5 \\ -2 & 2 \end{bmatrix}$$

System of equation

$$5x_1 - 5x_2 = 0 \Rightarrow \cancel{5}x_1 = \cancel{5}x_2$$

$$x_1 = x_2$$

eigen vector of $\lambda_2 = -1$

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Construct the General solution

The general solution to the system is a linear combination of the solutions associated with each eigen value and eigenvectors.

$x(t) = c_1 e^{\lambda 1t} x_1 + c_2 e^{\lambda 2t} x_2$

Substituting the eigen values and eigen vectors

$$x(t) = c_1 e^{6t} \begin{bmatrix} 5 \\ -2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 5c_1 e^{6t} + c_2 e^{-t} \\ -2c_1 e^{6t} + c_2 e^{-t} \end{bmatrix}$$

where c_1 & c_2 are arbitrary constant determined by initial conditions.