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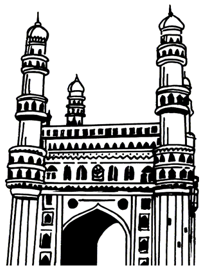

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UNIT I	Sphere: Denition-The Sphere Through Four Given Points-Equations of a Circle- Intersection of a Sphere and a Line-Equation of a Tangent Plane-Angle of Intersection of Two Spheres-Radical Plane.

1.1 Sphere

1.1.1 Definition

A sphere is the locus of point which remains at a constant distance from a fixed point. The constant distance is called the radius and the fixed point the center of the sphere.

1.1.2 The Equation of a Sphere

Let (a, b, c) be the centre and r to be the radius of a given spherer.

Equating the radius r to the distance of any point (x, y, z) on the sphere from its centre (a, b, c).

Then we have

$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$

$x^2 + a^2 - 2ax + y^2 + b^2 - 2by + z^2 + c^2 - 2cz = r^2$

$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + (a^2 + b^2 + c^2 - r^2) = 0$

which is the equation of sphere.

- Here x, y, z are second degree
- Coefficient of x^2, y^2, z^2 are all equal
- The product terms xy, yz, zx are absent.
- The locus of the equation $ax^2 + ay^2 + az^2 + 2ax + 2vy + 2wz + d = 0$.

  $a \neq 0$ is thus a sphere if $u^2 + v^2 + w^2 - ad \geq 0$.

1.1.3 General Equation of Sphere

From locus of equation $ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0$ can be written in the form of

$$x^2 + y^2 + z^2 + \frac{2u}{a}x + \frac{2v}{a}y + \frac{2w}{a}z + \frac{d}{a} = 0, \; a \neq 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z\,2' = 0$$

which is the general equation of sphere.

- The family of sphere is $x^2 + y^2 + 2ux + 2vy + 2wz + d = 0$

  where u, v, w d are parameters, such a that $u^2 + v^2 + w^2 - d \geq 0$

- The point sphere is when the radius of the sphere is '0' if $u^2 + v^2 + w^2 - d = 0$

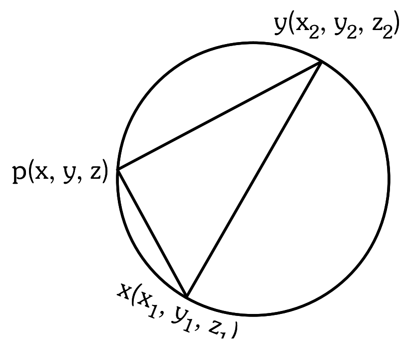
- Equation to a sphere on line joining (x_1, y_1, z_1), (x_2, y_2, z_2) as diameters

  Let P(x, y, z) be a point on the sphere. Then is a right angled triangle at p.

  Now direction cosines x_1, are proportional to $x - x_1, y - y_1, z - z_1$ and direction cosines of YP are proportional to $x - x_2, z - z_2$

  But XP and YP are at right angle to each other.

$\therefore \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$ is the equation of the sphere.



1. Find the radius and centre of the sphere $x^2+y^2+z^2-6x+8y-10z+1=0$.

Sol :

We know that the general equation.

$$x^2 + y^2 + z^2 + 24x + 2vy + 2wz + d = 0$$

Comparing with the general equation of sphere

We have $u = -3$, $v = 4$, $w = -5$ and $d = 1$

Hence, centre is $(-u, -v, -w) = (3, -4, 5)$

$$\text{Radius} = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{(3)^2 + (-4)^2 + (5)^2 - 1}$$

$$= \sqrt{9 + 16 + 25 - 1}$$

$$= \sqrt{49} = 7$$

$\therefore$ Centre = $(3, -4, 5)$

and radius = 7.

2. A plane passes through a fixed point (a, b, c) show that the locus of the foot of the perpendicular to it from the origin in the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$.

Sol :

Any plane through (a, b, c) in $l(x - a) + m(y - b) + n(z - c) = 0$ ... (1)

and the line perpendicular to it from the origin in $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ ... (2)

The foot of the perpendicular in the point of intersection (1) & (2)

To find the locus of the foot of perpendicular one should eliminate l, m, n between (1) & (2)

$$x(x - a) + y(y - b) + z(z - c) = 0$$

$$x^2 - xa + y^2 - yb + z^2 - zc = 0$$

$$x^2 + y^2 + z^2 - ax - by - cz = 0.$$

3. Obtain the equation of the sphere described on the join of the points A (2, –3, 4) B(–5, 6, –7).

Sol :

Let P(x, y, z) be a point on the sphere.

A = (2, –3, 4) and B = (–5, 6, –7)

The equation of the sphere described on the join of the points.

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

$$= (x - 2)(x - (-5)) + (y - (-3))(y - 6) + (z - 4)(z - (-7)) = 0$$

$$= (x - 2)(x + 5) + (y + 3)(y - 6) + (z - 4)(z + 7) = 0$$

$$= x^2 + 5x - 2x - 10 + y^2 - 6y + 3y - 18 + z^2 + 7z - 4z - 28 = 0$$

$$= x^2 + y^2 + z^2 + 3x - 3y + 3z - 56 = 0$$

$$= x^2 + y^2 + z^2 + 3(x - y + z) - 56 = 0$$

which is required equation of sphere.

1.2 The Sphere through Four Given Points

The general equation of a sphere contains four parameter and, as such a sphere can be uniquely determined so as to satisfy four conditions.

We can find a sphere through four non-coplanar points.

$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$ be the equation of the sphere through the four given points.

We have then the linear equation

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$

4. Find the equation to the sphere through the points (0, 0, 0), (0, 1 –1), (–1, 2, 0), (1, 2, 3).

Sol :

Let the equation of the spherer be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... (1)

(0, 0, 0) is passes through the sphere ... (1)

Then $d = 0$ ... (2)

$(0, 1, -1) \Rightarrow \quad 0 + 1 + (-1)^2 + 2u(0) + 2v(1) + 2w(-1) + d = 0$

$2 + 2v + 2w + d = 0 \quad \dots (3)$

$(-1, 2, 0) \Rightarrow \quad (-1)^2 + (2)^2 + 0 + 2u(-1) + 2v(2) + 2w(0) + d = 0$

$1 + 4 - 24 + 4v + d = 0$

$5 - 24 + 4v + d = 0 \quad \dots (4)$

$(1, 2, 3) \Rightarrow \quad 1 + 4 + 9 + 2u(1) + 2v(2) + 2w(3) + d = 0$

$14 + 2u + 4v + 6w + d = 0 \quad \dots (5)$

sub (2) in (3) (4) (5) we get,

$2 + 2v - 2w = 0 \quad \dots (i)$

$5 - 24 + 4v = 0 \quad \dots (ii)$

$14 - 24 + 4v + 6w = 0 \quad \dots (iii)$

Solving (i) and (ii)

$0.u + 2v - 2w + 2 = 0$

$-\quad 24 + 4v + 0.w + 5 = 0$

$-\quad 24 + 6v - 2w + 7 = 0 \quad \dots (iv)$

Solving (iv) and (iii)

$-\quad 2u + 6v - 2w + 7 = 0$

$2u + 4v + 6w + 14 = 0$

$-\quad 10v + 4w + 21 = 0 \quad \dots (v)$

Now solve 5x(i) and (v)

$10v - 10w + 10 = 0$

$10v + 4w + 21 = 0$

$-14w - 11 = 0$

$-14w = 11$

$$w = -\frac{11}{14}$$

Sub $w = \frac{-11}{14}$ in equation (i)

$$2V - 2W + 2 = 0 \quad \Rightarrow \quad 2V - 2\left(-\frac{11}{14}\right) + 2 = 0$$

$$2V + \frac{22}{14} + 2 = 0$$

$$2V = -\frac{25}{7}$$

$$V = -\frac{25}{14}$$

Sub $v = -\frac{25}{14}$ in equation (ii)

Then we get $5 \; - 2u + 4\left(\frac{-25}{14}\right) = 0$

$$-2u = \frac{15}{7}$$

$$u = -\frac{15}{7}$$

$\therefore\; u = -\frac{15}{7},\ v = \frac{-25}{14} \quad w = \frac{-11}{14}$ and $d = 0$

Hence the equation of sphere becomes

$$x^2 + y^2 + z^2 - 2\left(\frac{15}{14}\right)x - 2\left(\frac{25}{14}\right)y - 2\left(\frac{11}{14}\right)z = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - \frac{15}{7}\,x - \frac{25}{7}\,y - \frac{11}{7}\,z = 0$$

$$\Rightarrow \quad 7\,(x^2 + y^2 + z^2) - 15x - 25y - 11z = 0$$

5. Find the equation of the sphere thought he four points (4, –1, 2), (0, –2, 3), (1, –5, –1), (2, 0, 1).

Sol :

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{... (1)}$$

When four given points part through the given equation of sphere. Then we get as,

$(4, -1, 2) \Rightarrow (4)^2 + (-1)^2 + (2)^2 + 2u(4) + 2v(-1) + 2w(2) + d = 0$

$16 + 1 + 4 + 8u - 2v + 4w + d = 0$

$8u - 2v + 4w + d + 21 = 0$ ... (2)

$(0, -2, 3) \Rightarrow 0 + (-2)^2 + (3)^2 + 2u(0) + 2v(-2) + 2w(3) + d = 0$

$4 + 9 - 4v + 6w + d = 0$

$-4v + 6w + d + 13 = 0$ ... (3)

$(1, -5, -1) \Rightarrow 1^2 + (-5)^2 + (-1)^2 + 2u(1) + 2v(-5) + 2w(-1) + d = 0$

$1 + 25 + 1 + 2u - 10V - 2w + d = 0$

$2u - 10v - 2w + d + 27 = 0$ ... (4)

$(2, 0, 1) \Rightarrow (2)^2 + 0 + (1)^2 + 2u(2) + 2v(0) + 2w(1) + d = 0$

$4 + 1 + 4u + 2w + d = 0$

$4u + 2w + d + 5 = 0$ ... (5)

By Solving (2) – (3)

$8u - 2v + 4w + d + 21 - (-4v + 6w + d + 13) = 0$

$8u - 2v + 4w + d + 21 + 4v - 6w - d - 13 = 0$

$8u + 2v - 2w + 8 = 0$

$\Rightarrow 4u + v - w + 4 = 0$ ... (6)

Solving (3) – (4)

$-4v + 6w + d + 13 - (2u - 10v - 2w + d + 27) = 0$

$\Rightarrow -4v + 6w + d + 13 - 2u + 10v + 2w - d - 27 = 0$

$-2u + 6v + 8w - 14 = 0$

$u - 3v - 4w + 7 = 0$ ... (7)

Solving (4) – (5)

$(2u - 10v - 2w + d + 27) - (4u + 2w + d + 5) = 0$

$2u - 10v - 2w + d + 27 - 4u - 2w - d - 5 = 0$

$-2u - 10v - 4w + 22 = 0$

$\Rightarrow \quad u + 5v + 2w - 11 = 0 \quad \ldots (8)$

Now, to get the values of u, v, w

$3 \times$ equation (6) + equation (7)

$3[4u + v - w + 4] + [u - 3v + 4w + 7] = 0$

$12u + 3v - 3w + 12 + u - 31 - 4w + 7 = 0$

$134 - 7w + 19 = 0 \quad \ldots (9)$

By solving of $5 \times$ equation (7) + $3 \times$ equation (8)

$5[u - 3v - 4w + 7] + 3[u + 5v + 2w - 11] = 0$

$5u - 15v - 20w + 35 + 3u + 15v + 6w - 33 = 0$

$8u - 14w + 2 = 0$

$4u - 7u + 1 = 0 \quad \ldots (10)$

equation (9) – equation (10)

we get $13u - 7w + 19 - [4u - 7w + 1] = 0$

$13u - 7w + 19 - 4u + 7w - 1 = 0$

$94 + 18 = 0$

$$4 = \frac{-18}{9} = -2$$

$\therefore u = -2$

Sub $u = -2$ in equation (10)

$4u - 7w + 1 = 0$

$4(-2) - 7w + 1 = 0$

$-8 + 1 - 7w = 0$

$-7 - 7w = 0 \Rightarrow w = -1$

Sub $u = -2$, $w = -1$ in equation (8)

i.e., $u + 5v + 2w - 11 = 0$

$(-2) + 5v + 2(-1) - 11 = 0$

$$-2 + 5v - 2 - 11 = 0$$

$$5v - 15 = 0$$

$$v = \frac{15}{5} = 3$$

$$\therefore v = 3$$

Sub u = – 2, w = –1 in (5) then we get,

$$4u + 2w + d + 5 = 0$$

$$4(-2) + 2(-1) + d + 5 = 0$$

$$-8 - 2 + 5 + d = 0$$

$$-10 + 5 + d = 0$$

$$d = 5$$

∴ The equation spare in

$x^2 + y^2 + z^2 + 2(-2)\,x + 2\,(3)\,y + 2(-1)\,z + 5 = 0.$

$x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0.$

---

6. A variable plane through a fixed point (a, b, c) cute the coordinate axes in the point A, B, C show that the locus of the centres of the sphere OABC is a/x + b/y + c/z = 2.

Sol :

Let the sphere OABC be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$ ... (1)

So that u, v, w are different for different spheres.

The points A, B, C where it cuts the three axes are

(–2u, 0, 0) (0, –2v, 0), (0, 0, –2w)

The equation of the plane ABC is

$$\frac{x}{-24} + \frac{y}{-2v} + \frac{z}{-2w} = 1$$

Since this plane through (a, b, c), we have

$$\frac{a}{-2u} + \frac{b}{-2v} + \frac{c}{-2w} = 1 \quad \text{... (2)}$$

If x, y, z be the centre of the sphere then

$$x = -u,\ y = -v,\ z = -w \qquad \text{... (3)}$$

from (2) and (3), we obtain

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$ as the required locus.

7. Find the equation of the sphere pussing through the origin and the points (1, 0, 0), (0, 2, 0), (0, 0, 3).

Sol :

The equation of sphere in

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{... (A)}$$

The equation passing through the origin is

(0, 0, 0) $\Rightarrow$ $d = 0$ ... (1)

(1, 0, 0) $\Rightarrow$ $1 + 2u(1) + d = 0$ ... (2)

$\Rightarrow$ $1 + 2u + d = 0$

(0, 2, 0) $\Rightarrow$ $(2)^2 + 2v(2) + d = 0$ ... (3)

$4 + 4v + d = 0$

(0, 0, 3) $\Rightarrow$ $3^2 + 2w(3) + d = 0$

$\Rightarrow$ $9 + 6w + d = 0$ ... (4)

Sub d = 0 in (1) then we get $1 + 2u + 0 = 0$

$$2u = -1$$

$$u = -\frac{1}{2}$$

Sub d = 0 in (2) then we get $4 + 4v + 0 = 0$

$$4v = -4$$

$$v = -1$$

Sub d = 0 in (3) then, we get $9 + 6w + d = 0$

$$9 + 6w + 0$$

$$6w = -9$$

$$w = -\frac{9}{6} = -\frac{3}{2}$$

$\therefore$ $u = -\frac{1}{2}$, v = –1, w = –3 and d = 0 in (A)

$$x^2 + y^2 + z^2 + 2\left(\frac{-1}{2}\right) x + 2(-1)y + 2\left(\frac{-3}{2}\right)z + 0 = 0$$

$$x^2 + y^2 + z^2 - x - 2y - 3z = 0$$

$\therefore$ It is a required equation of sphere.

8. A sphere of constant radius k passes through. The origin and cuts the axes in A, B and C. Find the locus of the centroid of the triangle ABC.

Sol :

Let the coordinates of A, B and C be (a, 0, 0) (0, b, 0) and (0, 0, C) respectively.

The sphere also passes through the origin (0, 0, 0).

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

As it passes through (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c) we have d = 0.

$$a^2 + 2ua + d = 0 \Rightarrow u = -\frac{1}{2}a$$

$$b^2 + 2vb + d = 0 \Rightarrow v = -\frac{1}{2}b$$

$$c^2 + 2wc + d = 0 \Rightarrow w = -\frac{1}{2}c$$

$\therefore$ The required equation of sphere is $x^2 + y^2 + z^2 - ax - by - cz = 0$.

It's radius = $\sqrt{\left(\frac{1}{2}a\right)^2 + \left(\frac{1}{2}b\right)^2 + \left(\frac{1}{2}c\right)^2} = k$

$$a^2 + b^2 + c^2 = 4k^2 \quad \text{... (1)}$$

If (x_1, y_1, z_1) be the coordinates of the centroid of ΔABC, then $x_1 = \frac{1}{3}a, y_1 = \frac{1}{3}b,$ $z_1 = \frac{1}{3}c$.

$a = 3x_1, b = 3y_1, c = 3z_1$

Sub all the above values in (1) then we get,

$(3x)^2 + (3y)^2 + (3z)^2 = 4k^2$

$9(x^2 + y^2 + z^2) = 4k^2$

$\therefore$ it is required locus.

9. Find the equation of the sphere through the four points (0, 0, 0), (–a, b, c), (a, –b, c), (a, b, –c) and determine its radius.

Sol :

The equation of sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{... (A)}$$

(0, 0, 0) is passes through the equation of sphere.

$\Rightarrow d = 0$ ... (1)

$(-a, b, c) \Rightarrow (-a)^2 + b^2 + c^2 + 2u(-a) + 2v(b) + 2w(c) + d = 0$

$a^2 + b^2 + c^2 - 2ua + 2vb + 2wc + d = 0$ ... (2)

$(a, -b, c) \Rightarrow a^2 + (-b)^2 + c^2 + 2u(a) + 2v(-b) + 2w(c) + d = 0$

$a^2 + b^2 + c^2 + 2ua - 2vb + 2wc + d = 0$ ... (3)

$(a, b, -c) \Rightarrow a^2 + b^2 + (-c)^2 + 2u(a) + 2v(b) + 2w(-c) + d = 0$

$a^2 + b^2 + c^2 + 2ua + 2vb - 2wc + d = 0$ ... (4)

$d = 0 \Rightarrow$ by (2)

$\Rightarrow a^2 + b^2 + c^2 - 2ua + 2vb + 2wc = 0$ ... (5)

$d = 0 \Rightarrow$ by (3)

$$a^2 + b^2 + c^2 + 2ua - 2vb + 2wc = 0 \quad \text{... (6)}$$

$d = 0 \Rightarrow$ by (4)

$$a^2 + b^2 + c^2 + 2ua + 2vb - 2wc = 0 \quad \text{... (7)}$$

from (5) and (6)

$$a^2 + b^2 + c^2 - 2ua + 2vb + 2wc = 0$$

$$a^2 + b^2 + c^2 + 2ua - 2vb + 2wc = 0$$

$$2a^2 + 2b^2 + 2c^2 + 4wc = 0 \Rightarrow 2(a^2 + b^2 + c^2) + 4wc = 0$$

$$w = \frac{-2(a^2 + b^2 + c^2)}{4c}$$

from (6) and (7)

$$a^2 + b^2 + c^2 - 2ua - 2vb + 2wc = 0$$

$$a^2 + b^2 + c^2 + 2ua + 2vb - 2wc = 0$$

$$2(a^2 + 2b^2 + 2c^2) + 4ua = 0$$

$$u = \frac{-2(a^2 + b^2 + c^2)}{4a}$$

from (5) and (7)

$$a^2 + b^2 + c^2 - 2ua + 2vb + 2wc = 0$$

$$a^2 + b^2 + c^2 + 2ua + 2vb - 2wc = 0$$

$$2(a^2 + b^2 + c^2) + 4vb = 0$$

$$v = \frac{-2(a^2 + b^2 + c^2)}{4b}$$

$$\therefore\ u = -\frac{(a^2 + b^2 + c^2)}{2a},\ v = -\frac{(a^2 + b^2 + c^2)}{2b},\ w = -\frac{(a^2 + b^2 + c^2)}{2c}$$

above value sub in (A).

$$= x^2 + y^2 + z^2 - \not{2}\left[\frac{(a^2 + b^2 + c^2)}{\not{2}\,a}\right]x - \not{2}\left[\frac{(a^2 + b^2 + c^2)}{\not{2}\,b}\right]y$$

$$-\not{2}\left[\frac{(a^2 + b^2 + c^2)}{\not{2}\,c}\right]z = 0$$

$$= \frac{x^2+y^2+z^2}{a^2+b^2+c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0$$

is a required equation of sphere.

$$\text{Radius} = \sqrt{u^2+v^2+w^2-d}$$

$$= \sqrt{\left(-\frac{(a^2+b^2+c^2)}{2a}\right)^2 + \left(-\frac{(a^2+b^2+c^2)}{2b}\right)^2 + \left(-\frac{(a^2+b^2+c^2)}{2c}\right)^2 - 0}$$

$$= \sqrt{\frac{(a^2+b^2+c^2)^2}{4a^2} + \frac{(a^2+b^2+c^2)^2}{4b^2} + \frac{(a^2+b^2+c^2)^2}{4c^2}}$$

$$= \sqrt{\left(\frac{(a^2+b^2+c^2)}{4}\right)^2 \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}$$

$$= \frac{(a^2+b^2+c^2)}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

$$\text{Radius} = \frac{1}{2}(a^2+b^2+c^2)\sqrt{a^{-2}+b^{-2}+c^{-2}}.$$

1.3 Equation of a Circle

A circle is the intersection of its plane with some sphere through it a circle can be represented by two equations, representing a sphere and the other a plane. Thus, the two equations.

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$

$lx + my + nz = p$. Taken together represent a circle.

10. Show that the centre of all sections of the sphere $x^2 + y^2 + z^2 = r^2$ by planes through a point (x^1, y^1, z^1) lie on the sphere, $x(x - x^1) + y(y - y^1) + z(z - z^1) = 0$.

Sol :

The plane which cuts the sphere in a circle with centre (f, g, h) is $f(x - f) + g(y - g) + h(z - h) = 0$

It will pass through (x', y', z') if $f(x' - x) + g(y' - g) + h(z' - h) = 0$

It and according the locus of (f, g, h) is the sphere

$x(x^1 - x) + y(y^1 - y) + z(z^1 - z) = 0.$

11. If r be the radius of the circle $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ $lx + my + nz = 0$. Prove that

$(r^2 + d)(l^2 + m^2 + n^2) = (mw - nv)^2 + (nu - lw)^2 + (lv - mu)^2$

Sol :

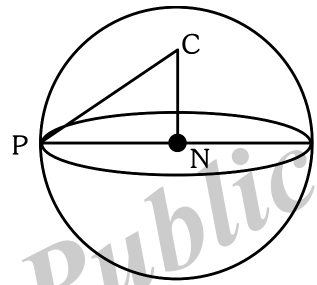
The equation of the given sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

having centre at $(-u, -v, -w)$

and radius $r = \sqrt{u^2 + v^2 + w^2 - d}$

Now, distance CN of centre of sphere from the plane is length of perpendicular from centre of sphere on the plane.



$$lx + my + nz = 0$$

$$CN = \frac{lu + mv + nw}{\sqrt{l^2 + m^2 + n^2}}$$

$$CP = \sqrt{u^2 + v^2 + w^2 - d}, \; NP = r$$

$$r^2 = CP^2 - CN^2 = u^2 + v^2 + w^2 - d - \frac{(lu + mv + nw)^2}{l^2 + m^2 + n^2}$$

$$\Rightarrow (r^2+d)(l^2+m^2+n^2) = (u^2+v^2+w^2)(l^2+m^2+n^2) - (lu+mv+nw)^2$$

$$\Rightarrow (r^2+d)(l^2+m^2+n^2) = (mw - nv)^2 (nu - lw)^2 - (lv - mv)^2$$

1.3.1 Sphere Through the Given Circle

The equation $S + KU = 0$ represents a sphere through the circle with equation $S = 0, U = 0$.

Where $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$

$U \equiv lx + my + nz - P$

Thus, the set of sphere through the circle

$$S = 0, U = 0$$

$$S + KU = 0$$

K is the parameter, is also the equation S + KS' = 0 represents a sphere through the circle with equation S = 0, S' = 0.

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$$

$$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d'$$

for all values of k.

The set of spehre through the circle S = 0, S' = 0.

The general equation of a sphere through the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0, z = 0$$

is $x^2 + y^2 + z^2 + 2gx + 2fy + 2kz + c = 0$

Where k is the parameter.

12. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and the point (1, 2, 3).

Sol :

The sphere through the circle in

$$x^2 + y^2 + z^2 - 9 + k(2x + 3y + 4z - 5) = 0$$

Passes through the given circle for all value of k.

It will pass through (1, 2, 3) if

$$(1)^2 + (2)^2 + (3)^2 - 9 + k(2(1) + 3(2) + 4(3) - 5) = 0$$

$$1 + 4 + 9 - 9 + k(2 + 6 + 12 - 5) = 0$$

$$5 + k(15) = 0$$

$$k = \frac{-5}{15} = \frac{-1}{3}$$

∴ The required equation of the sphere is

$$x^2 + y^2 + z^2 - 9 + \left(\frac{-1}{3}\right)(2x + 3y + 4z - 5) = 0$$

$3(x^2 + y^2 + z^2 - 9) - 2x - 3y - 4z + 5 = 0$

$3(x^2 + y^2 + z^2) - 27 - 2x - 3y - 4z + 5 = 0$

$3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0.$

13. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 + 2x + 3y + 6 = 0$, $x - 2y + 4z - 9 = 0$ and the centre of the sphere.

$x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0.$

Sol :

The given equation of sphere in

$x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$

$\therefore$ The centre of the sphere in $(-u, -v, -w) = (1, -2, 3)$

The sphere through the circle is

$(x^2 + y^2 + z^2 + 2x + 3y + 6) + k(x - 2y + 4z - 9) = 0$

It will passes through (1, –2, 3).

$= ((1)^2 + (-2)^2 + (3)^2 + 2(1) + 3(-2) + 6) + k(1 - 2(-2) + 4(3) - 9) = 0$

$(1 + 4 + 9 + 2 - 6 + 6) + k(1 + 4 + 12 - 9) = 0$

$16 + k(8) = 0$

$8k = -16$

$k = \dfrac{-16}{8} = -2$

$k = -2$

$(x^2 + y^2 + z^2 + 2x + 3y + 6) - 2(x - 2y + 4z - 9) = 0$

$x^2 + y^2 + z^2 + 2x + 3y + 6 - 2x + 4y - 8z + 18 = 0$

$x^2 + y^2 + z^2 + 7y - 8z - 24 = 0$

1.4 Intersection of a Sphere and a Line

Let $x^2 + y^2 + z^2 - 2ux + 2vy + 2wz + d = 0$ and $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ be the equations of a sphere and a line respectively. ... (B)

The point $(lr + \alpha, mr + \beta, nr + \gamma)$ which lies on the given line(B) for all value of r, will also lie on the given sphere (A).

For those of the values of r which satisfies the equation.

$$r^2(l^2 + m^2 + n^2) + 2r[l(\alpha + u) + m(\beta + v) + n(\lambda + w)]$$
$$+ (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0$$

and this latter being a quadratic equation in r, gives two values say r_1, r_2 of r. We suppose that the equation has real roots so that r_1, r_2 are real then ($lr_1 + \alpha$, $mr_1 + \beta$, $nr_1 + r$), are the two points of intersection.

14. Show that the equation of the sphere passing through the three points (3, 0, 2), (–1, 1, 1) (2, –5, 4) and having its centre on the plane 2x + 3y + 4z = 6 is $x^2 + y^2 + z^2 + 4y - 6z = 1$.

Sol :

The equation of the sphere in

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{... (1)}$$

The three points (3, 0, 2), (–1, 1), (2, –5, 4) are passing through the equation of sphere.

$\therefore$ by equation (1)

(3, 0, 2) $\Rightarrow$ $(3)^2 + (0)^2 + (2)^2 + 2u(3) + 2v(0) + 2w(2) + d = 0$

$9 + 4 + 6u + 4w + d = 0$

$6u + 4w + 13 + d = 0 \quad \text{... (2)}$

(–1, 1, 1) is passes through equation (1), then we get

$= \quad (-1)^2 + (1)^2 + (1)^2 + 2u(-1) + 2v(1) + 2w(1) + d = 0$

$1 + 1 + 1 - 2u + 2v + 2w + d = 0$

$-2u + 2v + 2w + 3 + d = 0 \quad \text{... (3)}$

(2, –5, 4) is passes through the equation (1) then we get

$= \quad (2)^2 + (-5)^2 + (4)^2 + 2u(2) + 2v(-5) + 2w(4) + d = 0$

$4u - 10V + 8w + 4 + 25 + 16 + d = 0$

$4u - 10v + 8w + 45 + d = 0 \quad \text{... (4)}$

We know that the centre of equation is (–u, –v, –w) is passes through 2x + 3y + 4z = 6.

$\Rightarrow \quad 2(-u) + 3(-v) + 4(-z) = 6$

$-2u - 3v - 4w - 6 = 0 \quad \text{... (5)}$

By solving equation (2) – (3) we get

$6u + 4w + 13 + d - (-2u + 2v + 2w + 3 + d) = 0$

$6u + 4w + 13 + d + 24 - 2v - 2w - 3 - d = 0$

$8u - 2v + 2w + 10 = 0$

$\Rightarrow \quad 4u - v + w + 5 = 0 \qquad \ldots (6)$

By solving (3) – (4) we get

$-2u + 2v + 2w + 3 + d - (4u - 10v + 8w + 45 + d) = 0$

$-2u + 2v + 2w + 3 + d - 4u + 10v - 8w - 45 - d = 0$

$-6u + 12v - 6w - 42 = 0$

$\Rightarrow \quad -u + 2v - w - 7 = 0 \qquad \ldots (7)$

Now, by solving

$2 \times (5) + (6)$ then we get

$\Rightarrow \quad -4u - 6v - 8w - 12 = 0$

$\Rightarrow \quad -4u - 6v - 8w - 12 + 4u - v + w + 5 = 0$

$-7v - 7w - 7 = 0$

$7v + 7w + 7 = 0 \qquad \ldots (8)$

By solving $(6) + 4 \times (7)$ then we get

$4u - v + w + 5 + 4[-u + 2v - w - u] = 0$

$4u - v + w + 5 - 4u + 8v - 4w - 28 = 0$

$7v - 3w - 23 = 0 \qquad \ldots (9)$

To get the values of v and w. We used to solve (8) – (9).

$7v + 7w + 7 - [7v - 3w - 23] = 0$

$7v + 7w + 7 - 7v + 3w + 23 = 0$

$10w + 30 = 0$

$10w = -30$

$$w = \frac{-30}{10} = -3$$

$\therefore \boxed{w = -3}$

Sub w = –3 in equation (9) then we get the value of v

$\therefore \quad 7v - 3(-3) - 23 = 0$

$7v + 9 - 23 = 0$

$7v = 14$

$v = 14/7 = 2$

$\therefore \quad \boxed{v = 2}$

Sub v = 2 and w = –3 in equation (7)

$u - 2v + w + 7 = 0$

$u - 2(2) + (-3) + 7 = 0$

$u - 4 - 3 + 7 = 0$

$\boxed{u = 0}$

$\therefore$ u = 0 and w = –3 sub in (2)

i.e., $6u + 4w + 13 + d = 0$

$6(0) + 4(-3) + 13 + d = 0$

$-12 + 13 + d = 0$

$d = -1$

$\therefore$ By equation of sphere will get it as

$x^2 + y^2 + z^2 + 2(0)x + 2(2)y + 2(-3)z + (-1) = 0$

$x^2 + y^2 + z^2 + 4y - 6z - 1 = 0$

$\therefore$ Hence proved.

15. Obtain the equation of the sphere passing through the three points (1, 0, 0), (0, 1, 0), (0, 0) and has its radius as small as possible.

Sol :

The equation of sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{... (1)}$$

(1, 0, 0) is passes through the equation (1)

Then $(1)^2 + 0 + 0 + 2u(1) + 0 + 0 + d = 0$

$1 + 2u + d = 0$

$$2u = -d - 1 \Rightarrow u = -\frac{(d+1)}{2}$$

(0, 1, 0) is passes through the equation (1)

Then $1 + 2v(1) + d = 0$

$$1 + 2v + d = 0$$

$$2v = -d - 1$$

$$v = -\frac{1}{2}(d + 1)$$

(0, 0, 1) is passes through the equation (1)

Then $1 + 2w\,(1) + d = 0$

$$2w + d + 1 = 0$$

$$2w = -d - 1$$

$$w = -\frac{1}{2}\,(d + 1)$$

$\therefore$ Radius of the sphere $r = \sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{\left(-\frac{1}{2}(d+1)\right)^2 + \left(-\frac{1}{2}(d+1)\right)^2 + \left(-\frac{1}{2}(d+1)\right)^2 - d}$$

$$= \sqrt{\frac{(d+1)^2}{4} + \frac{(d+1)^2}{4} + \frac{(d+1)^2}{4} - d}$$

$$= \sqrt{\frac{3(d+1)^2}{4} - d}$$

$$= \sqrt{\frac{3}{4}(d^2 + 1 + 2d) - d}$$

$$= \frac{1}{2}\sqrt{3d^2 + 3 + 6d - 4d}$$

$$= \frac{1}{2}\sqrt{3d^2 + 2d + 3}$$

$$= \frac{1}{2}\sqrt{3\left(d^2 + \frac{2}{3}d + 1\right)}$$

$$= \frac{1}{2}\sqrt{3\,d^2 + 2\,d\,\frac{1}{3} + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2 + 1}$$

$$= \frac{1}{2}\sqrt{3\left(d + \frac{1}{3}\right)^2 + \frac{1}{9} + 1}$$

$$= \sqrt{3\left(d + \frac{1}{3}\right)^2 + \frac{10}{9}}$$

Lake for r $d + \frac{1}{3} = 0$

$$d = -\frac{1}{3}$$

$$u = -\frac{1}{2}(d + 1)$$

$$= -\frac{1}{2}\left(-\frac{1}{3} + 1\right)$$

$$= -\frac{1}{2}\left(\frac{2}{3}\right) \Rightarrow -\frac{1}{3}$$

$$\therefore\ u = -\frac{1}{3}$$

$$v = -\frac{1}{2}(d + 1) \Rightarrow -\frac{1}{2}\left(-\frac{1}{3} + 1\right)$$

$$= -\frac{1}{2}\left(\frac{2}{3}\right) \Rightarrow -\frac{1}{3}$$

Similarly $w = -\frac{1}{3}$

$\therefore$ Substituting $u = -\frac{1}{3}, v = -\frac{1}{3}, w = -\frac{1}{3}$ and $d = -\frac{1}{3}$ in equation (1)

Then we get

$$x^2 + y^2 + z^2 + 2\left(\frac{-1}{3}\right)x + 2\left(\frac{-1}{3}\right)y + 2\left(\frac{-1}{3}\right)z + \left(\frac{-1}{3}\right) = 0$$

$$3(x^2 + y^2 + z^2) - 2x - 2y - 2z - 1 = 0$$

$$3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$$

16. Find the centre and the radius of the circle $x + 2y + 2z = 15$, $x^2 + y^2 + z^2 - 2y - 4z = 11$.

Sol :

Given that the equation of sphere is $x^2 + y^2 + z^2 - 2y - 4z = 11$ ... (1)

and the equation of plane is $x + 2y + 2z = 15$ ... (2)

centre of equation of the sphere in (–u, –v, –w) i.e., (0, 1, 2).

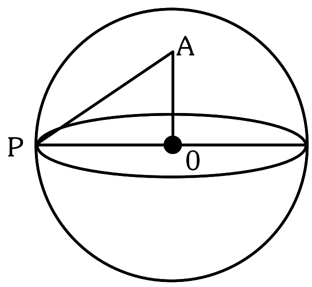
$\therefore$ The radius of equation (1) in $\sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{0 + (1)^2 + 2^2 + 11} = \sqrt{1 + 4 + 11} = \sqrt{16} = 4$$

$\therefore$ r = 4

Now the distance OA of centre of the sphere from the plane is length of perpendicular from centre of the sphere on the plane.

$x + 2y + 2z = 15$



OA is perpendicular equation to (2)

$$\therefore \text{ OA} = \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{2}$$

$$\frac{x}{1} = t \qquad \frac{y-1}{2} = t \qquad \frac{z-2}{2} = t$$

$$x = t \qquad y = 2t + 1 \qquad z = 2t + 2$$

$\therefore$ A = (t, 2t + 1, 2t +2)

Sub in (2) we get,

$$\Rightarrow t + 2(2t + 1) + 2(2t + 2) = 15$$

$$\Rightarrow t + 4t + 2 + 4t + 4 - 15 = 0$$

$$t + 8t - 9 = 0$$

$$9t - 9 = 0$$

$$t = \frac{9}{9} = 1$$

$\therefore$ t = 1

$\therefore$ A = (t, 2t + 1, 2t + 2) = (1, 2(1) + 1, 2(1) + 2)

Centre is = (1, 3, 4)

$$\text{OA} = \sqrt{(1-0)+(3-1)^2+(4-2)^2} = \sqrt{1+2^2+2^2}$$

$$= \sqrt{1+4+4} = \sqrt{9}$$

OA = 3

$\therefore$ $Ap^2 = PO^2 + OA^2$

$(4)^2 = r^2 + (3)^2$

$16 = r^2 + 9$

$7 = r^2$

$\sqrt{7} = r$

$\therefore$ $r = \sqrt{7}$

17. Show that the following sets of points are con cyclic.

(i) (5, 0, 2), (2, –6, 0), (7, –3, 8), (4, –9, 6).

(ii) (–8, 5, 2), (–5, 2, 2), (–7, 6, 6) (–4, 3, 6).

Sol :

(i) (5, 0, 2), (2, –6, 0), (7, –3, 8), (4, –9, 6)

The general equation of sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{... (1)}$$

Origin in passes through the sphere

Then $(0, 0, 0) \Rightarrow d = 0$ ... (2)

(5, 0, 2) is passes through the sphere

$$(5)^2 + (0)^2 + (2)^2 + 2u(5) + 2v(0) + 2w(2) + 0 = 0$$

$$25 + 4 + 10u + 4w = 0$$

$$\Rightarrow 10u + 4w + 29 = 0 \quad \text{... (3)}$$

(2, –6, 0) is passes through the sphere

$$(2)^2 + (-6)^2 + 0 + 2u(2) + 2v(-6) + 2w(0) + 0 = 0$$

$$4 + 36 + 4u - 12v = 0$$

$$4u - 12v + 40 = 0$$

$$u - 3v + 10 = 0 \quad \text{... (4)}$$

(7, –3, 8) is passes through the sphere

$$(7)^2 + (-3)^2 + (8)^2 + 2u(7) + 2v(-3) + 2w(8) + 0 = 0$$

$$49 + 9 + 64 + 14u - 6v + 16w = 0$$

$$14u - 6v + 16w + 122 = 0$$

$$\Rightarrow 7u - 3v + 8w + 61 = 0 \quad \text{... (5)}$$

(4, –9, 6) is passes through the sphere

$$(4)^2 + (-9)^2 + (6)^2 + 2u(4) + 2v(-9) + 2w(6) + 0 = 0$$

$$16 + 81 + 36 + 8u - 18v + 12w = 0$$

$$\Rightarrow 8u - 18v + 12w + 133 = 0 \quad \text{... (6)}$$

By solving (3) + (4)

$$10u + 4w + 29 + (u - 3v + 10) = 0$$

$$10u + 4w + 29 + u - 3v + 10 = 0$$

$$11u + 4w - 3v + 39 = 0 \quad \text{... (7)}$$

Solving 2x (7) – (5)

$$2(11u - 3v + 4w + 39) - (7u - 3v + 8w + 61) = 0$$

$$22u - 6v + 8w + 78 - 7u + 3v - 8w - 61 = 0$$

$$15u - 3v + 17 = 0 \qquad \ldots (8)$$

Solving (8) – (4)

$$15u - 3v + 17 - (u - 3v + 10) = 0$$

$$15u - 3v + 17 - u + 3v - 10 = 0$$

$$14u + 7 = 0$$

$$u = -\frac{7}{14} = -\frac{1}{2}$$

$$u = -\frac{1}{2}$$

$u = -\frac{1}{2}$ substitute in (4) then we get

$$u - 3v + 10 = 0$$

$$-\frac{1}{2} - 3v + 10 = 0$$

$$-3v + \frac{19}{2} = 0$$

$$3v = \frac{19}{2}$$

$$v = \frac{19}{6}$$

Substitute $u = -\frac{1}{2}$ in (3) then we get

$$10u + 4w + 29 = 0$$

$$10\left(\frac{-1}{2}\right) + 4w + 29 = 0$$

$-5 + 29 + 4w = 0$

$4w = -24$

$w = -6$

$\therefore\ u = -\frac{1}{2}, v = \frac{19}{6}, w = -6$ in equation (1)

i.e., $x^2 + y^2 + z^2 + 2\left(\frac{-1}{2}\right) x + 2\left(\frac{19}{6}\right) y + 2(-6)z = 0$

$$x^2 + y^2 + z^2 - x + \frac{19}{3} y - 12 z = 0 \qquad \text{... (9)}$$

Substitute (4, –9, 6) in (9)

$(4)^2 + (-9)^2 + 6^2 - 4 + \frac{19}{3}(-9) - 12(6) = 0$

$16 + 81 + 36 - 4 - 57 - 72 = 0$

$133 - 133 = 0$

$0 = 0$

$\therefore$ The given set of points are concyclic.

(ii) (–8, 5, 2), (–5, 2, 2), (–7, 6, 6), (–4, 3, 6)

Sol :

The equation of sphere in $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... (1)

If the origin is passes through the sphere

$(0, 0, 0) \Rightarrow d = 0$

(–8, 5, 2) is passes through the sphere then

$(-8)^2 + (5)^2 + (2)^2 + 2u(-8) + 2v(5) + 2w(2) + 0 = 0$

$64 + 25 + 4 - 16u + 10v + 4w = 0$

$-16u + 10v + 4w + 93 = 0$ ... (2)

(–5, 2, 2) is passes through the sphere then

$(-5)^2 + (2)^2 + (2)^2 + 2u(-5) + 2v(2) + 2w(2) + 0 = 0$

$25 + 4 + 4 - 10u + 4v + 4w = 0$

$-10u + 4v + 4w + 33 = 0$ ... (3)

(–7, 6, 6) is passes through the sphere then

$$(-7)^2 + (6)^2 + (6)^2 + 2u(-7) + 2v(6) + 2w(6) + 0 = 0$$

$$49 + 36 + 36 - 14u + 12v + 12w = 0$$

$$-14u + 12v + 12w + 121 = 0 \quad \ldots (4)$$

Solving 3 × (3) – (4)

$$3(-10u + 4v + 4w + 33) - (-14u + 12v + 12w + 121) = 0$$

$$-30u + 12v + 12w + 99 + 14u - 12v - 12w - 121 = 0$$

$$-16u - 22 = 0$$

$$u = -\frac{22}{16}$$

$$u = -\frac{11}{8}$$

By solving (2) – (3)

$$-16u + 10v + 4w + 93 - (-10u + 4v + 4w + 33) = 0$$

$$-16u + 10v + 4w + 93 + 10u - 4v - 4w + 33 = 0$$

$$-6u + 6v + 60 = 0$$

$$-u + v + 10 = 0 \quad \ldots (5)$$

Sub $u = -\frac{11}{8}$ in (5)

$$-\left(-\frac{11}{8}\right) + v + 10 = 0$$

$$\frac{11}{8} + 10 + v = 0$$

$$v = -\frac{91}{8}$$

$u = \frac{-11}{8}$ and, $v = -\frac{91}{8}$ in (3)

$$\therefore -10\left(-\frac{11}{8}\right) + 4\left(-\frac{91}{8}\right) + 4w + 33 = 0$$

$$\frac{55}{4} - \frac{91}{2} + 33 + 4w = 0$$

$$\frac{55 - 182 + 132 + 16w}{4} = 0$$

$$187 - 182 + 16w = 0$$

$$5 + 16\,w = 0$$

$$w = -\frac{5}{16}$$

$\therefore\ u = -\frac{11}{8},\ v = -\frac{91}{8},\ w = \frac{-5}{16}$ sub in equation of sphere

$$x^2 + y^2 + z^2 + 2\left(\frac{-11}{8}\right)x + 2\left(\frac{-91}{8}\right)y + 2\left(\frac{-5}{16}\right)z = 0$$

$$16(x^2 + y^2 + z^2) - 44x - 36\,4y - 10\,z = 0$$

$\therefore$ The fourth points i.e., (–4, 3, 6) sub in above equation.

$$16((4)^2 + (3)^2 + (6)^2) - 44(-4) - 364(3) - 10(6) = 0$$

$$16(16 + 9 + 36) + 176 - 1092 - 60 = 0$$

$$976 + 176 - 1152 = 0$$

$$1152 - 152 = 0$$

$$0 = 0.$$

$\therefore$ The set of points are concyclic

1.5 Equation of a Tangent Plane

To find the equation of the tangent plane at any point (α, β, γ) of the sphere, $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

$\therefore$ The point (α, β, γ) lies on the sphere

$$\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d = 0 \quad \text{... (1)}$$

The point of intersection of any line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \qquad \text{... (2)}$$

Through (α, β, γ) with the sphere are $(l\alpha + \alpha, m\alpha + \beta, nr + \gamma)$ where the values of r are the boots of the quadratic equation.

$r^2(l^2 + m^2 + n^2) + 2r[l(\alpha + u) + m(\beta + v) + n(\gamma + w)] + (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0$

By condition (1) one root of this quadratic equation is zero so that one of the points of intersection coincides with (α, β, γ).

The second pint of intersection may also coincide with (α, β, γ) the second value of r must also vanish and this requires.

$$l(\alpha + u) + m(\beta + v) + n(\gamma + w) = 0$$

Thus, the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ meets the sphere in two coincident points at (α, β, γ) and so is a tangent line.

18. Find the equation of the sphere which touches the sphere $x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$ at the point (1, 1, –1) and passes through the origin.

Sol :

The tangent of the required sphere at (1, 1, –1) is

$-x + 5y - 6 = 0$

The equation of the required sphere in

$$x^2 + y^2 + z^2 - x + 3y + 2z - 3 + k(x + 5y - 6) = 0$$

This will pass through the origin

Then $-3 + k(-6) = 0$

$-6k = 3$

$k = -1/2$

$$x^2 + y^2 + z^2 - x + 3y + 2z - 3 - \frac{1}{2}(x + 5y - 6) = 0$$

$$2(x^2 + y^2 + z^2) - 2x + 6y + 4z - 6 - x - 5y + 6 = 0$$

$$2(x^2 + y^2 + z^2) - 3x + y + 4z = 0.$$

19. Show that the plane lx + my + nz = p will touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ $(ul + vm + wn + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + m^2 - d)$.

Sol :

Equating the radius $\sqrt{u^2+v^2+w^2-d}$ of the sphere tot he length of the perpendicular from the centre (–u, –v, –w) to the plane lx + my + nz = p.

$$= \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

$$= \left|\frac{l(-u)+m(-v)+n(-w)-p}{\sqrt{l^2+m^2+n^2}}\right| = \sqrt{u^2+v^2+w^2-d}$$

$$= \frac{lu+mv+nw+p}{\sqrt{l^2+m^2+n^2}} = \sqrt{u^2+v^2+w^2-d}$$

Squaring on both sides

$$\frac{(lu+mv+nw+p)^2}{l^2+m^2+n^2} = u^2+v^2+w^2-d$$

$(lu + mv + nw + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$

$(ul + vm + wn + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$.

20. Show that the spheres $x^2 + y^2 + z^2 = 64$ and $x^2 + y^2 + z^2 - 12x + 4y - 6z + 48 = 0$ touch internally and find their point of contact.

Sol :

Two spheres will touch internally if the difference of their radii is equal to the distance between their centres.

The distance between two centres (0, 0, 0) and (6, –2, 3)

The radius of 1st spehre $= \sqrt{0+0+0+64} = 8$

The radius of 2nd sphere $= \sqrt{(6)^2+(-2)^2+(3)^2-48}$

$= \sqrt{36+4+9-48}$

$= \sqrt{48-48} = 1.$

$\therefore$ The difference of radi in 8 – 1 = 7.

Let (α, β, γ) be their point of contact.

Then tangent planes to two spheres at this point are $\alpha x + \beta y + \gamma z - 64$ ... (1)

$\alpha x + \beta y + \gamma z - 6(x + \alpha) + 2(y + \beta) - 3(z + \gamma) + 48 = 0$

$\alpha x + \beta y + \gamma z - 6x - 6\alpha + 2y + 2\beta - 3z - 3\gamma + 48 = 0$

$(\alpha - 6)x + (\beta + 2)y + (\gamma - 3)z - 6\alpha + 2\beta - 3\gamma + 48 = 0$ ... (2)

comparing (1) and (2)

$$\frac{\alpha-6}{\alpha} = \frac{\beta+2}{\beta} = \frac{\gamma-3}{\gamma} = \frac{-6\alpha+2\beta-3\gamma+48}{-64} = k$$

$$\frac{\alpha-6}{\alpha} = k \qquad \frac{\beta+2}{\beta} = k \qquad \frac{\gamma-3}{\gamma} = k$$

$\alpha - 6 = \alpha k \qquad \beta + \beta k = \beta k \qquad \gamma - 3 = \gamma k$

$\alpha - \alpha k = 6 \qquad \beta - \beta k = -2 \qquad \gamma(1 - k) = 3$

$$\alpha = \frac{6}{1-k} \qquad \beta = -\frac{2}{1-k} \qquad \gamma = \frac{3}{1-k}$$

$-6\alpha + 2\beta - 3\gamma + 48 = -64k$

Sub α, β, γ in above equation

$$-6\left(\frac{6}{1-k}\right) + 2\left(\frac{-2}{1-k}\right) - 3\left(\frac{3}{1-k}\right) + 48 = -64k$$

$-36 - 4 - 9 + 48(1 - k) + 64k(1 - k) = 0$

$-49 + 48 - 48k + 64k - 64k^2 = 0$

$-64k^2 + 16k - 1 = 0$

$64k^2 - 8k - 8k + 1 = 0$

$(8k - 1)(8k - 1) = 0$

$$\boxed{k = \frac{1}{8}}$$

Then we get $k = \frac{1}{8}$

$$\alpha = \frac{6}{1-k}$$

$$= \frac{6}{1-\frac{1}{8}} = \frac{6}{\frac{7}{8}} = \frac{48}{7}$$

$$\beta = \frac{-2}{1-k} \Rightarrow \frac{-2}{1-\frac{1}{8}} \Rightarrow -\frac{16}{7}$$

$$\gamma = \frac{3}{1-k} \Rightarrow \frac{3}{1-\frac{1}{8}} \Rightarrow \frac{24}{7}$$

$\therefore$ The point of content is $\left(\frac{48}{7}, \frac{-16}{7}, \frac{24}{7}\right)$.

21. If the tangent plane tot he sphere $x^2 + y^2 + z^2 = r^2$ makes intercepts a, b, c on the coordinate axes, show that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{r^2}$.

Sol :

The equation to the tangent plane at (α, β, γ) to the given sphere in

$$x\alpha + y\beta + z\gamma = r^2 \quad \text{... (1)}$$

a in the intercept made by the plane (1) on x-axis.

$$a\alpha = r^2$$

$$\alpha = \frac{r^2}{a}$$

b in the intercept made by the plane (1) on y axis then

$$b\beta = r^2$$

$$\beta = \frac{r^2}{b}$$

c is the intercept made by the plane (1) on z-axis so,

$$c\gamma = r^2$$

$$\gamma = \frac{r^2}{c}$$

Also, as (α, β, γ) is a point on the sphere,

$$\alpha^2 + \beta^2 + \gamma^2 = r^2$$

$$\left(\frac{r^2}{a}\right)^2 + \left(\frac{r^2}{b}\right)^2 + \left(\frac{r}{c}\right)^2 = r^2$$

$$\frac{r^4}{a^2} + \frac{r^4}{b^2} + \frac{r^4}{c^2} = r^2$$

$$r^4\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = r2$$

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{r^2}{r^4}$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{r^2}$$

22. Find the value of a for which the plane $x + y + z = a\sqrt{3}$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.

Sol :

Given sphere of equation is

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$$

Centre of the sphere (1, 1, 1)

$\therefore$ radius of sphere $\sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{1+1+1+6}$$

$$= \sqrt{9} = 3$$

$\therefore$ x + y + z = a$\sqrt{3}$ touches the sphere

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = r$$

$$\left|\frac{1(1) + 1(1) + 1(1) + (-a\sqrt{3})}{\sqrt{1+1+1}}\right| = 3$$

$$\frac{3 - a\sqrt{3}}{\sqrt{3}} = \pm 3$$

$$3 - a\sqrt{3} = \pm 3\sqrt{3}$$

$$3 \pm 3\sqrt{3} = a\sqrt{3}$$

$$a = \frac{3 \pm 3\sqrt{3}}{\sqrt{3}} \Rightarrow \frac{\sqrt{3}\,(3 \pm 3\sqrt{3})}{\sqrt{3} \times \sqrt{3}}$$

$$\sqrt{3}(\sqrt{3} \pm 3) = \frac{3\sqrt{3} \pm 9}{3}$$

$$= \frac{3(\sqrt{3} \pm 3)}{3}$$

$$= \sqrt{3} \pm 3 \cdot$$

1.5.1 Plane of Contact

To find the locus of the points of contact of the tangent planes which pass through a given point (α, β, γ) and touch the sphere.

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

The tangent plane

$$x(x^1 + u) + y(y^1 + v) + z(z^1 + w) + (ux^1 + vy^1 + wz^1 + d) = 0$$

at (x^1, y^1, z^1) will pass through the point (α, β, γ).

$$\alpha(x^1 + u) + \beta(y^1 + v) + \gamma(z^1 + w) + (ux^1 + vy^1 + wz^1 + d) = 0$$

$$\Leftrightarrow \quad x^1(\alpha + u) + y^1(\beta + v) + z^1(\gamma + w) + (u\alpha + v\beta + w\gamma + d) = 0$$

Which is the conditions that the point (x^1, y^1, z^1) should lies on the plane.

$$x(\alpha + u) + y(\beta + v) + z(\gamma + w) + (u\alpha + v\beta + w\gamma + d) = 0$$

It is called the plane of contact for the point (α, β, γ) Thus, the locus of points of content is the circle in which the plane cuts the sphere.

1.5.1.1 Pole of Plane

If π be the polar plane of a point p then p is called the pole of the plane π.

1.5.1.2 Pole of Plane

To find the pole of the plane $lx + my + nz = p$ ... (1) with respect to the sphere $x^2 + y^2 + z^2 = a^2$.

If (α, β, γ) be the required pole, t hen we bee that the equation (1).

$$\alpha x + \beta y + \gamma z = a^2 \qquad \text{... (2)}$$

Comparing (1) and (2)

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n} = \frac{a^2}{p}$$

$$\frac{\alpha}{l} = \frac{a^2}{p} \quad \frac{\beta}{m} = \frac{a^2}{p} \quad \frac{\gamma}{m} = \frac{a^2}{p}$$

$$\alpha = \frac{a^2 l}{p} \quad \beta = \frac{ma^2}{p} \quad \gamma = \frac{ma^2}{p}$$

$\therefore\ (\alpha, \beta, \gamma) = \left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right)$ is the pole of the plane $lx + my + nz = p$. With respect to the sphere $x^2 + y^2 + z^2 = a^2$.

1.5.2 Some results concerning poles and polars conjugate points and conjugate planes

If A, B are two points such that the polar plane of B w.r. to sphere passes through A, then A, B are called conjugate points w.r. to a sphere.

$\therefore$ The polar planes of A and B are called conjugate planes.

Two lines such that the polar plane of energy point on either passes through the other are called polar lines.

23. Show that the polar line of $\frac{(x+1)}{2} = \frac{y-2}{3} = z+3$ with respect to the sphere $x^2 + y^2 + z^2 = 1$ is the line $\frac{7x+3}{11} = \frac{2-7y}{5} = \frac{z}{-1}$.

Sol :

The given line in $\frac{x+1}{2} = \frac{y-2}{3} = \frac{z+3}{1} = r$

$x + 1 = 2r \qquad y - 2 = 3r \qquad z + 3 = r$

$x = 2r - 1 \qquad y = 3r + 2 \qquad z = r - 3$

$\therefore$ The point on this line $(2r-1, 3r+2, r-3)$

The polar plane of this point with respect to the sphere $x^2 + y^2 + z^2 - 1 = 0$.

$x(2r - 1) + y(3r + 2) + z(r - 3) - 1 = 0$

$2rx - x + 3ry + 2y + zr - 3z - 1 = 0$

$-x + 2y - 3z - 1 + r(2x + 3y + z) = 0$

$-x + 2y - 3z - 1 = 0, \qquad 2x + 3y + z = 0$

Let $z = 0$

$-x + 2y - 1 = 0 \qquad \ldots (1)$

$2x + 3y + z = 0 \qquad \ldots (2)$

from (1) $\qquad x = 2y - 1$

Sub in (2)

$2(2y - 1) + 3y + z = 0 \qquad$ as $z = 0$

$2(2y - 1) + 3y = 0$

$4y - 2 + 3y = 0$

$7y = 2$

$y = \frac{2}{7}$

Sub $y = \frac{2}{7}$ in $x = 2y - 1$

i.e., $x = 2\left(\frac{2}{7}\right) - 1$

$= \frac{4}{7} - 1$

$= \frac{-3}{7}$.

$\therefore$ The point on line in $\left(\frac{-3}{7}, \frac{2}{7}, 0\right)$.

From $-x + 2y - 3z = 0$ and $2x + 3y + z = 0$ of (a, b, c)

$-a + 2b - 3c = 0$ and $2a + 3b + c = 0$

by solving above two equations

$$\begin{array}{cccccccc} & a & & b & & c & \\ 2 & & -3 & & -1 & & 2 \\ 3 & & 1 & & 2 & & 2 \end{array}$$

$\Rightarrow \frac{a}{2+9} = \frac{b}{-6+1} = \frac{c}{-3-4} = 1$

$\frac{a}{11} = 1 \quad \frac{b}{-5} = 1 \quad \frac{c}{-7} = 1$

$\therefore$ $a = 11 \qquad b = -5 \qquad c = -7$

$\therefore$ $(a, b, c) = (11, -5, -7)$

$\therefore$ The required line in $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

$$\frac{x - \left(\frac{-3}{7}\right)}{11} = \frac{y - \left(\frac{2}{7}\right)}{-5} = \frac{z - 0}{-7}$$

$$\frac{x + \frac{3}{7}}{11} = \frac{y - \frac{2}{7}}{-5} = \frac{z}{-7}$$

$$\frac{7x+3}{7(11)} = \frac{7y-2}{-7(5)} = \frac{z}{-7}$$

$$\frac{7x+3}{11} = \frac{7y-2}{-5} = \frac{z}{-1}$$

$$\therefore \quad \frac{7x+3}{11} = \frac{2-7y}{5} = \frac{z}{-1} \text{ Hence Proved}$$

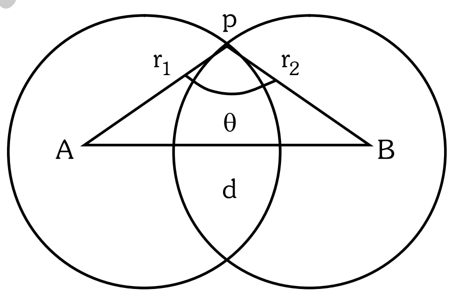
1.6 Angle of Intersection of two spheres

P is common point to two spheres any angle θ between the tangent planes at p to two spheres is called an angle of intersection of the sphere at p. The other angle between the spheres in $\pi - \theta$.

If $\theta = \frac{\pi}{2}$ the spheres are said to intersect orthogonally at p and the spheres are called orthogonal spheres.

24. If r_1 r_2 are the radii of two orthogonal spheres, then the radius of the circle of their intersection is $\frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$.

Sol :



A, B are the centre of the two orthogonal spheres. M in the centre and a in the radius of he circle common to the sphere.

$AP = r_1$ $BP = r_2$ $\underline{|APB} = 90^o \Rightarrow AB^2 = r_1^2 + r_2^2$

$\Rightarrow (AM + MB)^2 = r_1^2 + r_2^2$

$\Rightarrow AM^2 + MB^2 + 2\, AM.MB = r_1^2 + r_2^2$

$\Rightarrow \quad r_1^2 - a^2 + r_2^2 - a^2 + 2\sqrt{(r_1^2 - a^2)(r_1^2 - a^2)} = r_1^2 + r_2^2$

$\Rightarrow \quad 4(r_1^2 - a^2)(r_2^2 - a^2) = 4a^4$

$= \quad r_1^2 r_2^2 - a^2(r_1^2 + r_2^2) = 0$

$$a^2 = \frac{r_1^2 \; r_2^2}{r_1^2 + r_2^2}$$

$$a = \sqrt{\frac{r_1^2 \; r_2^2}{r_1^2 + r^2}}$$

$$a = \frac{r_1 \; r_2}{\sqrt{r_1^2 + r_2^2}}$$

25. Condition for the Orthogonality of Two Spheres

$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$S' \equiv x^2 + y^2 + z^2 + 2u^1x + 2v^1y + 2w^1z + d^1 = 0$ are two orthogonal spheres $\Leftrightarrow$ $2uu^1 + 2vv^1 + 2ww^1 = d + d^1$

Sol :

Let A, B be the centre and r_1, r_2 be the radii of the spheres S = 0, S' = 0.

Spheres S = 0, S^1 = 0 cut orthogonally

$\Leftrightarrow \quad AB = r_1^2 + r_2^2$

$\Leftrightarrow \quad (u^1 - u)^2 + (v^1 - u)^2 + (w^1 - w)^2 = u^2 + v^2 + w^2 - d + u^{1^2} + v^{1^2} + w^{1^2} - d^1$

$\Leftrightarrow \quad -2uu^1 - 2vv^1 - 2ww^1 = d + d^1$

$\Leftrightarrow \quad 2uu^1 + 2vv^1 + 2ww^1 = d + d^1$

26. Find the equation of the sphere that passes through the circle $x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0$, $3x - 4y + 5z - 15 = 0$ and cuts the sphire $x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ orthogonally.

Sol :

The equation of the sphere through the given circle is $S + \lambda U = 0$.

$S = x^2 + y^2 + z^2 - 2x + 3y - 4z + 6$

$U = 3x - 4y + 5z - 15 = 0$

$(x^2 + y^2 + z^2 - 2x + 3y - 4z + 6) + \lambda(3x - 4y + 5z - 15) = 0$

$x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 + \lambda 3x - 4\lambda y + 5\lambda z - 15\lambda = 0$

$x^2 + y^2 + z^2 + x(-2 + 3\lambda) + y(3 - 4\lambda) + z(-4 + 5\lambda) + 6 - 15\lambda = 0 \quad \ldots (1)$

Given equation (1) cuts the sphere

$x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ orthogonally

$\therefore\ 2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

$$(u_1, v_1, w_1) = \left(\frac{-2+3\lambda}{2}, \frac{3-4\lambda}{2}, \frac{-4+5\lambda}{2}\right)$$

$(u_2, v_2, w_2) = (1, 2, -3)$

$d_1 = +6 - 15\lambda$

$d_2 = 11$

$$\therefore\quad 2\left(\frac{-2+3\lambda}{2}\right)(1) + 2\left(\frac{+3-4\lambda}{2}\right)(2) + \left(\frac{-4+5\lambda}{2}\right)(-3) = 6 - 15\lambda + 11$$

$-2 + 3\lambda + 6 - 8\lambda + 12 - 15\lambda = 17 - 15\lambda$

$16 - 20\lambda + 15\lambda - 17 = 0$

$-5\lambda - 1 = 0$

$$\lambda = -\frac{1}{5}$$

Sub $\lambda = \frac{1}{5}$ Equation (1)

$$(x^2 + y^2 + z^2 - 2x + 3y - 4z - 6) - \frac{1}{5}(3x - 4y + 5z - 15) = 0$$

$5(x^2 + y^2 + z^2) - 10x + 15y - 20z - 30 - 3x + 4y - 5z + 15 = 0$

$5(x^2 + y^2 + z^2) - 13x + 19y - 25z + 45 = 0$

1.7 Radical Plane

The locus of points each of whose powers w.r. to two non concentric spheres are equal is a plane called the radical plane of the two spheres.

A, B are two non-concentric spheres and π is their radical plane.

Theorem :

Equation to the radical plane of spheres $S = 0$, $S^1 = 0$ in $S - S^1 = 0$.

Proof :

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

$$S^1 \equiv x^2 + y^2 + z^2 + 2u^1x + 2v^1y + 2w^1z + d^1 = 0$$

$\therefore$ $(-u, -v, -w) \neq (-u^1, -v^1, -w^1)$

$B(x_1, y_1, z_1)$ is a point whose power w, r, to the spheres are equal,

$\Leftrightarrow$ $S_1 = S_1^1$

$\Leftrightarrow$ $x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$

$= x_1^2 + y_1^2 + z_1^2 + 2u^1x_1 + 2v^1y_1 + 2w^1z_1 + d = 0$

$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d - x_1^2 - y_1^2 - z_1^2 - 2u^1x_1 - 2v^1y_1$

$+ 2(w - w^1)z^1 + (d - d^1) = 0$

$\therefore$ Locus of B in $2(u - u^1)x + 2(v - v^1)y + 2(w - w^1)z + (d - d^1) = 0$

Which is a plane.

But locus of B is the radical plane of the sphere $S = 0$, $S^1 = 0$.

$\therefore$ Radical plane of the sphere $S = 0$

$S^1 = 0$ is

$2(u - u^1)x + 2(v - v^1)y + 2(w - w^1)z \neq d - d^1 = 0.$

i.e., $S - S^1 = 0$

1.7.1 Radical

If A, B, C, D are four radical lines of four spheres taken three by three intersect at a point. The point common to the three plants.

$$S_1 = S_2 = S_3 = S_4$$

is clearly common to the radical line, taken three by three, of the four spheres.

$$S_1 = 0, S_2 = 0, S_3 = 0, S_4 = 0.$$

The intersection f the two lines

$$S_1 - S_2 = 0, S_2 - S_3 = 0, S_1 - S_3 = 0, S_2 - S_4 = 0$$

This above points is called the radical centre of the four spheres.

1.7.2 Radical Line

The three radical planes of three sphere intersect in a line.

If $S_1 = 0, S_2 = 0, S_3 = 0$ be the three spheres, their radical spheres.

$$S_1 - S_2 = 0, \quad S_2 - S_3 = 0, \quad S_3 - S_1 = 0$$

Clearly $S_1 = S_2 = S_3$

$\Leftrightarrow S_1 - S_2 = 0, S_2 - S_3 = 0$

This line is called the radical line of the three sphere.

27. If $S_1 = 0$, $S_2 = 0$ be two spheres, then the equation $S_1 + \lambda S_2 = 0$. λ being the parameter, represents a system of sphere such that any two members of the systems have the same radical plane.

Sol :

Let $S_1 + \lambda_1 S_2 = 0$ and $S_1 + \lambda_2 S_2 = 0$ be any two members of the system.

Making the coefficients of second degree terms unity we write there equation in the from.

$$\frac{S_1 + \lambda_1 S_2}{1 + \lambda_1} = 0 \quad \frac{S_1 + \lambda_1 S_2}{1 + \lambda_2} = 0$$

$$\frac{S_1 + \lambda_1 S_2}{1 + \lambda_1} - \frac{S_1 + \lambda_1 S_2}{1 + \lambda_2} = 0$$

$(1 + \lambda_2) S_1 + \lambda_1 S_2 - [(1 + \lambda_1)(S_1 + \lambda_2 \ S_2)] = 0$

$S_1 + \lambda_1 S_2 + \lambda_2 S_1 + \lambda_1 \lambda_2 S_2 - [(S\lambda + \lambda_2 S_2) + (\lambda_1 S_1 + \lambda_1 \lambda_2 S_2)] = 0$

$\lambda_1 (S_2 - S_1) + \lambda_2 (S_1 - S_2) = 0$

$(S_1 - S_2)(\lambda_1 + \lambda_2) = 0$

$S_1 - S_2 = 0.$

Since this eqution is independent λ_1 and λ_2 we see that energy two members of the system have the same radical plane.

28. Show that the sphere $x^2 + y^2 + z^2 = 25$, $x^2 + y^2 + z^2 - 24x - 40y - 18z + 22\ 5 = 0$ touch externally and find the point of the contact.

Sol :

Given spheres are $x^2 + y^2 + z^2 - 25 = 0$ ... (1)

$x^2 + y^2 + z^2 - 24x - 40y - 18z + 22t = 0$ ... (2)

Now the centre of equation (1) is $(-u, -v, -w) = (0, 0, 0)$

radius is $\sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{0 + 0 + 0 + 25}$

$= \sqrt{25}$

$r = 5$

Now, the centre of the equation (2) $(u_1 - v_1, -w) = (12, 20, 9)$

$\therefore$ Radius $= \sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{(12)^2 + (20)^2 + (9)^2 - 225}$

$= \sqrt{144 + 400 + 81 - 225}$

$= \sqrt{625 - 225}$

$= \sqrt{400} = 20$

The distance between A to B in

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Here A = (0, 0, 0) and B = (12, 20, 9)

$= \sqrt{(12 - 0)^2 + (20 - 0)^2 + (9 - 0)^2}$

$= \sqrt{144 + 400 + 81}$

$= \sqrt{625}$

$= 25$

and $r_1 + r_2 = 5 + 20$

$= 25.$

from equation (1) and (2) are touches externally the tangent planes of two sphere

$(x_1, y_1, z_1) = xx_1 + yy_1 + zz_1 - 25 = 0$ ... (3)

$xx_1 + yy_1 + zz_1 - 12(x + x_1) - 20(y + y_1) - 9(z + z_1) + 225 = 0$

$xx_1 + yy_1 + zz_1 - 12x - 12x_1 - 20y - 2y_1 - 9z - 9z_1 + 225 = 0$

$x(x_1 - 12) + y(y_1 - 20) + z(z_1 - 9) - 12x_1 - 20y_1 - 9z_1 + 225 = 0$ ... (4)

By solving (3) and (4)

$$\frac{x_1 - 12}{x_1} = \frac{y_1 - 20}{y_1} = \frac{z_1 - 9}{z_1} = - \frac{12x_1 - 20y_1 - 9z_1 + 225}{-25} = k$$

$x - 12 = x_1 k$	$y_1 - 20 = ky_1$	$z_1 - 9 = kz_1$
$x_1 - kx_1 = 12$	$y_1 - ky_1 = 20$	$z_1 - kz_1 = 9$
$x_1 (1 - k) = 12$	$y_1 (1 - k) = 20$	$z_1 (1 - k) = 9$
$x_1 = \frac{12}{1-k}$	$y_1 = \frac{20}{1-k}$	$z_1 = \frac{9}{1-k}$

$-12x_1 - 20y_1 - 9z_1 + 225 = -25k$

Sub x_1, y_1, z_1 values in above equation

$$-12\left(\frac{12}{1-k}\right) - 20\left(\frac{20}{1-k}\right) - 9\left(\frac{9}{1-k}\right) + 225 = -25k$$

$-144 - 400 - 81 + 225(1 - k) + 25k(1 - k) = 0$

$-625 + 225 - 225k + 25k - 25k^2 = 0$

$-400 - 200k - 25k^2 = 0$

$-25k^2 + 200k + 400 = 0$

$25(k^2 + 8k + 16) = 0$

$k^2 + 8k + 16 = 0 \quad \Rightarrow (k + 4)^2 = 0$

$= k = -4$

Sub k = – u in x_1, y_1, z_1 values. Then we get that

$$x_1 = \frac{12}{1-k} \qquad y_1 = \frac{20}{1-k} \qquad z_1 = \frac{9}{1-k}$$

$$= \frac{12}{1-(-4)} \qquad = \frac{20}{1-(-4)} \qquad = \frac{9}{1-(-4)}$$

$$x_1 = \frac{12}{5} \qquad y_1 = \frac{20}{-5} \qquad z_1 = \frac{20}{5}$$

$\therefore$ The point of content in $\left(\frac{12}{5}, \frac{20}{5}, \frac{9}{5}\right)$.

29. Find the equation of the sphere that passes through the two points (0, 3, 0), (–2, –1, –4) and cuts orthogonally the two sphere $x^2+y^2+z^2+x-3z-2 = 0$, $2(x^2 + y^2 + z^2) + x + 3y + 4 = 0$.

Sol :

Given the sphere is $x^2 + y^2 + z^2 + x - 3z - 2 = 0$ ... (1)

and another sphere is $2(x^2 + y^2 + z^2) + x + 3y + 4 = 0$ ... (2)

Let the equation of the sphere by

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... (3)

given equation passes through (0, 3, 0), (–2, –1, –4)

$\therefore$ (0, 3, 0) $= 0 + (3)^2 + +2u(0) + 2v(3) + 2w(0) + d = 0$

$9 + 6v + d = 0$ ... (4)

If (–2, –1, –4) $= (-2)^2 + (-1)^2 + (-4)^2 + 2u(-2) + 2v(-1) + 2w(-4) + d = 0$

$4 + 1 + 16 - 4u - 2v - 8w + d = 0$

$-4u - 2v - 8w + 21 + d = 0$ ... (5)

The equation of the sphere cuts the $x^2 + y^2 + z^2 + x - 3z - 2 = 0$ orthogonally.

So, here centre in $\left(-\frac{1}{2}, 0, -\frac{3}{2}\right)$ and d = –2.

$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$.

$$\therefore\ 2u\left(\frac{1}{2}\right) + 2v(0) + 2w\left(-\frac{3}{2}\right) = d - 2$$

$$u - 3w - d + 2 = 0 \qquad \text{... (6)}$$

The equation of sphere cuts the equation 2 and equation (2) we can rewrite it as

$$x^2 + y^2 + z^2 + \frac{x}{2} + \frac{3}{2}y + \frac{4}{2} = 0$$

$$x^2 + y^2 + z^2 + \frac{x}{2} + \frac{3}{2}y + 2 = 0$$

Now the equation of sphere cuts the above equation orthogonally centre $= (-u, -v, -w) = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$.

$$\therefore\ 2u\left(\frac{1}{4}\right) + 2v\left(\frac{3}{4}\right) + 2w(0) = d + 2$$

$$\frac{u}{2} + \frac{3v}{2} = d + 2$$

$$u + 3v = 2d + 4$$

$$u + 3v - 2d - 4 = 0 \qquad \text{... (7)}$$

By solving (5) + (6) then we get

$$-4u - 2v - 8w + 21 + d + u - 3w - d + 2 = 0$$

$$-3u - 2v - 11w + 23 = 0 \qquad \text{... (8)}$$

By solving 2 × (4) – (7) then we get

$$2(9 + 6v + d) + (u + 3v - 2d - 4) = 0$$

$$18 + 12v + 2d + u + 3v + 2d - 4 = 0$$

$$u + 15v + 14 = 0 \qquad \text{... (9)}$$

Now,

$$9 + 6v + d - (-4u - 2v - 8w + 21 + d) = 0$$

$$9 + 6v + d + 4u + 2v + 8w - 21 - d = 0$$

$$4u + 8v + 8w - 12 = 0 \qquad \text{... (10)}$$

Equation 11 × (10) + 8 × (8)

$$11(44 + 8v + 8w - 12) + 8(-5u - 2v - 11w + 23) = 0$$

$$44u + 88v + 88w - 132 - 24u - 16v - 88w + 184 = 0$$

$$20u + 72v + 52 = 0$$

$$5u + 18v + 13 = 0 \quad \dots (11)$$

Now, To get the values of u, v, w.

we need to solve equation

5 × equation (9) – equation (11)

$$5(u + 15v + 14) - (5u + 18v + 13) = 0$$

$$5u + 75v + 70 - 5u - 18v - 13 = 0$$

$$57v + 57 = 0$$

$$v = -\frac{57}{57} = -1$$

$\therefore$ v = –1

V = –1 sub in (11)

$$\because \quad 5u + 18v + 13 = 0$$

$$5u + 18(-1) + 13 = 0$$

$$5u - 18 + 13 = 0$$

$$5u - 5 = 0$$

$$u = 1$$

Sub u, v value in equation (10)

$$\because \quad 4u + 8v + 82 - 12 = 0$$

$$4(1) + 8(-1) + 8w - 12 = 0$$

$$4 - 8 + 8w - 12 = 0$$

$$8w - 20 + 4 = 0$$

$$8w - 16 = 0$$

$$w = 2$$

$\therefore$ Sub v = –1 in equation (4) then we will get the value of d

$\because \quad 6v + 9 + d = 0$

$6(-1) + 9 + d = 0$

$-6 + 9 + d = 0$

$d = -3$

Now, u = 1, v = –1, w = 2 and d = –3 sub in equation of sphere

i.e., $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$x^2 + y^2 + z^2 + 2x(1) + 2y(-1) + 2z(2) + (-3) = 0$

$x^2 + y^2 + z^2 + 2x - 2y + 4z - 3 = 0.$

30. Find the limiting points of the coaxial system of spheres. $x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 + \lambda(2x - 3y + 4z) = 0$.

Sol :

Given coaxial system of sphere is

$x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 + \lambda(2x - 3y + 4z) = 0$

$x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 + 2\lambda x - 3\lambda y + 4\lambda z = 0$

$x^2 + y^2 + z^2 + x(-20x + 2\lambda) - y(30 - 3\lambda) + z(-40 + 4\lambda) + 29 = 0$

$\therefore$ Centre of sphere in (–u, –v, –w)

$$= \left[-(\lambda - 10), -\left(\frac{30 - 3\lambda}{2}\right), -(2\lambda - 20)\right]$$

For limiting points radius must zero

$$(\lambda - 10)^2 + \left(\frac{30 - 3\lambda}{2}\right)^2 + (2\lambda - 20)^2 - 29 = 0$$

$$\lambda^2 + 100 - 20\lambda + \left(\frac{30 - 3\lambda}{2}\right)^2 + 4\lambda^2 + 400 - 80\lambda - 29 = 0$$

$4\lambda^2 + 400 - 80\lambda + 900 + 9\lambda^2 - 180\lambda + 16\lambda^2 + 1600 - 320\lambda - 116 = 0$

$29\lambda^2 - 580\lambda + 278u = 0$

$29(\lambda^2 - 20\lambda + 96) = 0$

$\lambda^2 - 20\lambda + 96 = 0$

$$\lambda^2 - 12\lambda - 8\lambda + 96 = 0$$

$$\lambda(\lambda - 12) - 8(\lambda - 12) = 0$$

$$(\lambda - 12)(\lambda - 8) = 0$$

$$\lambda = 12, 8$$

If $\lambda = 12$

Then centre of sphere

$$\left[-(12 - 10), -\left(\frac{30 - 3(12)}{2}\right), -(2(12) - 20)\right]$$

$$(-2, 3, -4)$$

If $\lambda = 8$

Then the centre of the sphere is

$$\left[-(8 - 10), -\left(\frac{30 - 3(8)}{2}\right), -(2(8) - 20)\right]$$

$$(2, -3, 4).$$

31. Find the eqution to the two spheres of the coaxial systems $x^2 + y^2 + z^2 - 5 + \lambda(2x + y + 3z - 3) = 0$ which touch the plane, $3x + 4y = 15$.

Sol :

The coaxial system is $x^2 + y^2 + z^2 - 5 + l(2x + y + 3z - 3) = 0$

$$x^2 + y^2 + z^2 - 5 + 2\lambda x + \lambda y + 3\lambda z - 3\lambda = 0$$

$$x^2 + y^2 + z^2 + 2\lambda x + \lambda y + 3\lambda z - 5 - 3\lambda = 0 \quad \text{... (1)}$$

Centre of equation (1) $(-u, -v, -w) = \left(-\lambda, \frac{-\lambda}{2}, \frac{-3\lambda}{2}\right)$

$$\Rightarrow \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \left|\frac{3(-\lambda) + 4\left(\frac{-\lambda}{2}\right) - 15}{\sqrt{3^2 + 4^2}}\right| = \sqrt{\lambda^2 + \frac{\lambda^2}{4} + \frac{9\lambda^2}{4} + 5 + 3\lambda}$$

$$= \left|\frac{-3\lambda - 2\lambda - 15}{\sqrt{9+16}}\right| = \sqrt{\frac{14\lambda^2}{4} + 5 + 3\lambda}$$

$$\frac{5\lambda + 15}{\sqrt{25}} = \sqrt{\frac{14\lambda^2 + 20 + 12\lambda}{2}}$$

$$\frac{5(\lambda + 3)}{5} = \sqrt{\frac{14\lambda^2 + 20 + 12\lambda}{2}}$$

$$2(\lambda + 3) = \sqrt{14\lambda^2 + 20 + 12\lambda}$$

Squaring on the both sides

$$(2(\lambda+3))^2 = (\sqrt{14\lambda^2 + 20 + 12\lambda})^2$$

$$4(\lambda+3)^2 = 14\lambda^2 + 20 + 12\lambda$$

$$4(\lambda^2 + 9 + 6\lambda) = 14\lambda^2 + 20 + 12\lambda$$

$$4\lambda^2 + 36 + 24\lambda = 14\lambda^2 + 20 + 12\lambda$$

$$14\lambda^2 - 4\lambda^2 + 12\lambda - 24\lambda + 20 - 36 = 0$$

$$10\lambda^2 - 12\lambda - 16 = 0$$

$$5\lambda^2 - 6\lambda - 8 = 0$$

$$5\lambda^2 - 10\lambda + 4\lambda - 8 = 0$$

$$5\lambda(\lambda - 2) + 4(\lambda - 2) = 0$$

$$(\lambda - 2)(5\lambda + 4) = 0$$

$$\lambda = 2, -\frac{4}{5}$$

If $\lambda = 2$ in equation (1)

$$x^2 + y^2 + z^2 - 5 + \lambda(2x + y + 3z - 3) = 0$$

$$x^2 + y^2 + z^2 - 5 + 2(2x + y + 3z - 3) = 0$$

$$x^2 + y^2 + z^2 - 5 + 4x + 2y + 6z - 6 = 0$$

$$x^2 + y^2 + z^2 + 4x + 2y + 6z - 11 = 0$$

If $\lambda = -\frac{4}{5}$ sub in equation (1)

$x^2 + y^2 + z^2 - 5 + \lambda (2x + y + 3z - 3) = 0$

$x^2 + y^2 + z^2 - 5 + \left(-\frac{4}{5}\right)(2x + y + 3z - 3) = 0$

$5(x^2 + y^2 + z^2) - 25 + (-8x + 4y + 12z + 12) = 0$

$5(x^2 + y^2 + z^2) - 8x - 4y - 12z - 13 = 0.$

32. Show that the radical plane of the sphere of a coaxial system and of the given any sphere pass through a line.

Sol :

Let $S_1 = 0$ and $S_2 = 0$ be the two sphere the equation of coaxial system of sphere the equation of coaxial system of sphere is,

$$S_1 + \lambda(S_1 - S_2) = 0$$

Where $S_1 - S_2 = 0$ is a radical plane let $S_3 = 0$ be any sphere

Then radical plane of S_1 and S_3 in $S_1 - S_3 = 0$.

$\because$ The radical planes $S_1 - S_2 = 0$ and $S_1 - S_3 = 0$

$\because$ The passes through a line.

33. Find the centres of the two spheres which touch the plane 4x + 3y = 47 at the points (8, 5, 4) and touch the sphere $x^2 + y^2 + z^2 = 1$.

Sol :

Given equation of sphere in $x^2 + y^2 + z^2 = 1$ ... (1)

Equation of plane in $4x + 3y - 47 = 0$ ... (2)

Since the required sphere touches equation (2) at point (8, 5, 4).

Centre of sphere lies on equation (2) at point (8, 5, 4).

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r$$

$$\Rightarrow \frac{x - 8}{4} = \frac{y - 5}{3} = \frac{z - 4}{0} = r$$

$$\frac{x-8}{4} = r$$

$\Rightarrow$ $x = 4r + 8$

$$\frac{y-5}{3} = r$$

$\Rightarrow$ $y = 3r + 5$

$$\frac{z-4}{0} = r$$

$z = 4$

Let c(4r + 8, 3r + 5, 4) be any point on line then, radius of required sphere in cp

$$r = |cp| = \sqrt{(4r+8-8)^2 + (3r+5-5)^2 + (4-4)^2}$$

$$= \sqrt{(4r)^2 + (3r)^2}$$

$$= \sqrt{16r^2 + 9r^2} = \sqrt{25r^2} = 5r$$

Since required sphere touches equation (1)

Then $r_1 \pm r_2 = |C_1 C_2|$

from equation (1)

centre $C_2 = (0, 0, 0)$ and radius $r_2 = 1$

$$5r \pm 1 = \sqrt{(4r+8-0)^2 + (3r+5+0)^2 + (4-0)^2}$$

$$5r \pm 1 = \sqrt{(4r+8)^2 + (3r+5)^2 + 4^2}$$

Squaring on both sides

$(5r \pm 1)^2 = (4r + 8)^2 + (3r + 5)^2 + 16$

$(5r \pm 1)^2 = 16r^2 + 64 + 64r + 9r^2 + 25 + 30r + 16$

$(5r \pm 1)^2 = 25r^2 + 94r + 105$

Case (i)

$(5r \pm 1)^2 = 25r^2 + 94r + 105$

$25r^2 + 1 + 10r = 25r^2 + 94r + 105$

$84r + 104 = 0$

$$r = -\frac{104}{84} = -\frac{26}{21}$$

Case (ii)

$(5r - 1)^2 = 25r^2 + 94r + 105$

$25r^2 + 1 - 10r = 25r^2 + 94r + 105$

$104r + 104 = 0$

$\therefore r = -1$

$$\text{If } r = -\frac{26}{21} \Rightarrow C = \left(4\left(-\frac{26}{21}\right) + 8, 3\left(\frac{-26}{21}\right) + 5, 4\right)$$

$$= \left(\frac{64}{21}, \frac{27}{21}, 4\right)$$

For $r = -1$

$C = (4(-1) + 8, 3(-1) + 5, 4)$

$C = (4, 2, 4)$

$\therefore$ The centres of the two sphere are

$$(4, 2, 4) \text{ and } \left(\frac{64}{21}, \frac{27}{21}, 4\right).$$

Choose the Correct Answer

1. A great circle is the section of a sphere by a plane passing through the ________ of the sphere [c]

   (a) Radii (b) Perpendicular

   (c) Centre (d) None

2. The equation of a sphere whose centre in the point (x_1, y_1, z_1) and radius r is [a]

   (a) $(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 = r^2$ (b) $x_1^2 + y_1^2 + z_1^2 = r^2$

   (c) $x^2 + y^2 + z^2 = r^2$ (d) $(x - x_1)^2 - (y - y_1)^2 - (z - z_1)^2 = r^2$

3. Two lines which are such that the polar of any point on any one passes through the other, are known as ________ [c]

   (a) Parallel lines (b) Straight lines

   (c) Polar lines (d) Tangent

4. The radius of the circle $x^2 + y^2 + z^2 = 25$, $2x + 3y - 4z = 0$ is [b]

   (a) 4 (b) 5

   (c) 3 (d) 2

5. The equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x_1, y_1, z_1) on it is [b]

   (a) $x_1^2 + y_1^2 + z_1^2 = a^2$ (b) $xx_1 + yy_1 + zz_1 = a^2$

   (c) $xx_1 - yy_1 - zz_1 = a^2$ (d) None

6. Centre of sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ [a]

   (a) $(-u, -v, -w)$ (b) (u, v, w)

   (c) $(-u, v, -w)$ (d) None

7. Centre of sphere $x^2 + y^2 + z^2 - 4x + 6y - 8z + 8 = 0$ [a]

   (a) $(-2, -3, -4)$ (b) $(2, 3, 4)$

   (c) $(1, 2, 3)$ (d) $(2, -3, 4)$

8. The radius of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 7 = 0$ [b]

   (a) 49 (b) $\sqrt{7}$

   (c) -7 (d) 5

9. The radius of the circle $x^2 + y^2 + z^2 - 2y - 4z - 20 = 0$, $x + 2y + 2z = 15$ is [c]

   (a) $\sqrt{14}$ (b) $\sqrt{7}$

   (c) 4 (d) 3

10. Centre of sphere for $x^2 + y^2 + z^2 - 6x - 12y - 2z + 20 = 0$ [a]

    (a) (3, 6, 1) (b) (–3, –6, –1)

    (c) (–3, –6, 1) (d) (–3, –6, 1)

Fill in the blanks

1. Radical plane of two spheres is the locus of the point from where the square of the lengths of the tangents to the two spheres are ________

2. the spheres $x^2 + y^2 + z^2 - 2x - 5z + 4 = 0$ and $x^2 + y^2 + z^2 + 6y - 4 = 0$ ________.

3. The equation of the sphere passing through (0, 0, 0), (1, 0, 0), (0, 1, 0), and (0, 0, 1) is ________ .

4. Two spheres touch externally, if the distance between their ________ in equal to the sum of their radii.

5. The curve of intersection of two spheres is ________.

6. The general equations of a sphere contains ________ independent constant.

7. A system of spheres every two members of which have the same radical plane is said to be ________.

8. The radical plane of the two spheres is ________ to the line joining their centres.

9. The spheres $x^2 + y^2 + z^2 - 2x = 3$ and $x^2 + y^2 + z^2 + 6x + 6y + 9 = 0$ touch ________.

10. The section of a sphere by a plane in a ________.

Answers

1 Equal
2. Cut orthogonally
3. $x^2 + y^2 + z^2 - x - y - z = 0$
4. Centres
5. Circle
6. 4
7. Co-axial
8. Perpendicular
9. Extremely
10. Circle

UNIT II	**Cones and Cylinders:** Denition-Condition that the General Equation of second degree Represents a Cone-Cone and a Plane through its Vertex -Intersection of a Line with a Cone- The Right Circular Cone-The Cylinder- The Right Circular Cylinder.

2.1 Cones

Acone is a surface generates by a straight lime which passes through a fixed point and satisfies one more conditions. It may interest a given curve or touch a given surface.

- The fixed point is called the vertex and the given curve the Guiding curve of the cone.
- An Individual straight line on the surface of a cone is called its Generater.

2.1.1 Equation of cone with a conic as Guiding curve

To find the equation of the cone whose vertex is the point (α,β,γ), are whose generators intersects the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$, $z=0$ ... (1)

We have to find the locus of points on lines which pass through the given point (α,β,γ) and interset the given curve.

The equation to any line through (α,β,γ) are $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ ... (2)

This line will be a generater of the cone if and only if it intersects the given curve.

This line meets the plane z=0 in the point $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{-\gamma}{n}$

$$\frac{x-\alpha}{\ell}=\frac{-\gamma}{n}\Rightarrow x-\alpha=\frac{-\gamma\ell}{n}$$

$$x=\alpha-\frac{\gamma\ell}{n}$$

$$\frac{y-\beta}{m}=\frac{-\gamma}{n}$$

$$y-\beta=\frac{-\gamma m}{n}$$

$$y=\beta=\frac{\gamma m}{n}$$

$\therefore\ \left(\alpha-\frac{\gamma\ell}{n},\beta-\frac{\gamma\mu}{n},0\right)$ which will lies on the given conic.

If from equation (1), we get that

$$a\left(\alpha-\frac{\gamma l}{n}\right)^2+2h\left(\alpha-\frac{\gamma l}{n}\right)\left(\beta-\frac{\gamma m}{n}\right)+b\left(\beta-\frac{\gamma m}{n}\right)^2+2g\left(\alpha-\frac{\gamma l}{n}\right)+2f\left(\beta-\frac{\gamma m}{n}\right)+c=0 \quad \ldots(3)$$

from equation (2)

$$=a\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)^2+2h\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)+b\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)^2$$

$$+2g\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)+2f\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)+c=0$$

$$=a\frac{(\alpha z-\alpha\gamma-x\gamma+\alpha\gamma)^2}{(z-r)^2}+2h\frac{(\alpha z-\alpha\gamma-x\gamma+\alpha\gamma)^2}{z-\gamma}\frac{(\beta z-\beta\gamma-\gamma y+\gamma\beta)}{z-\gamma}$$

$$+b\frac{(\beta z-\beta\gamma-\gamma y+\beta\gamma)^2}{(z-\gamma)^2}+2g\frac{(\alpha z-\alpha\gamma-x\gamma+\alpha\gamma)}{z-\gamma}+2f\frac{(\beta z-\beta\gamma-y\gamma+\gamma\beta)}{z-\gamma}+c=0$$

$$=a(\alpha-x\gamma)^2+2h(\alpha z-x\gamma)(\beta z-\gamma y)+b(\beta z-\gamma y)+2g(\alpha z-x\gamma)+2f(\beta z-y\gamma)$$

$$+c(z-\gamma)^2=0$$

Which required equation of the come.

PROBLEMS

1. Find the equation of a come whose vertex is (α,β,γ) and base $y^2 = 4ax$, $z = 0$.

Sol :

Any line through (α,β,γ) is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ ... (1)

If meets the plane $z = 0$

at $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right) = (x, y, z)$ ... (2)

and if it lies on $y^2 = 4ax$, $z = 0$.

Then, $\Rightarrow \left(\beta - \frac{m\gamma}{n}\right)^2 = 4a\left(\alpha - \frac{l\gamma}{n}\right)$

By eliminating l,m,n between (1) & (2).

Then we get

$$\left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2 = 4a\left[\alpha\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]$$

$$\frac{\left(\beta(z-\gamma) - (y-\beta)\gamma\right)^2}{(z-\gamma)^2} = 4a\left[\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]$$

$$\left(\beta z - \cancel{\beta\gamma} - y\gamma + \cancel{\beta\gamma}\right)^2 = 4a\frac{\left[\alpha z - \cancel{\alpha\gamma} - x\gamma + \cancel{\alpha\gamma}\right]}{\cancel{(z-\gamma)}}(z-\gamma)^2$$

$$(\beta z - y\gamma)^2 = 4a[\alpha z - x\gamma](z-\gamma)$$

2. Find the equation of the cone whose vertex is the point (1,1,0) and whose guiding curve is y=0, $x^2 + z^2 = 4$.

Sol :

The equation of the cone whose vertex is (1,1,0) is passing through the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{... (1)}$$

i.e $\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-0}{n}$

and whose guiding curve is y=0.

then by equation (1) we get

$$\frac{x-1}{l} = \frac{-1}{m} = \frac{z}{n}$$

$$\frac{x-1}{l} = \frac{-1}{m}, \frac{z}{n} = \frac{-1}{m}$$

$$x-1 = \frac{-l}{m} \quad z = \frac{-n}{m}$$

$$x = 1 - \frac{l}{m} \quad z = \frac{-n}{m}$$

(x,y,z) = $\left(1-\frac{l}{m}, 0, \frac{-n}{m}\right)$ which points are passing through the $x^2 + z^2 = 4$

$$\therefore \left(1-\frac{l}{m}\right)^2 + \left(\frac{-n}{m}\right)^2 = 4$$

$$\left(\frac{m-l}{m}\right)^2 + \left(\frac{n}{m}\right)^2 = 4$$

$$(m-l)^2 + n^2 = 4m^2 \quad \text{... (2)}$$

eliminating l, m,n between (1) & (2) then we get that

$$((y-1)-(x-1))^2 + z^2 = 4(y-1)^2$$

$$(y-1-x+1)^2 + z^2 = 4(y^2+1-2y)$$

$$(y-x)^2 + z^2 = 4y^2 + 4 - 8y$$

$$y^2 + x^2 - 2xy + z^2 - 4y^2 - 4 + 8y = 0$$

$$x^2 - 3y^2 + z^2 - 2xy + 8y - 4 = 0.$$

3. Find the equation of the cone with vertex (5,4,3) and $3x^2 + 2y^2 = 6$, $y+z=0$ as base.

Sol :

The equation to any line through (α, β, γ) are $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

These the cone with vertex (5, 4, 3)

Then $\frac{x-5}{l} = \frac{y-4}{m} = \frac{z-3}{n}$

Any point on this line has the co-ordinates

$\frac{x-5}{l} = r$ $\quad \frac{y-4}{m} = r$ $\quad \frac{z-3}{n} = r$

$x - 5 = lr$ $\quad y - 4 = mr$ $\quad z-3=nr$

$x = 5 + lr$ $\quad y = 4 + mr$ $\quad z= 3 + nr$

$(x, y, z) = (5 + lr, 4 + mr, 3 + nr)$

This point will lies on the base

i.e., $3x^2 + 2y^2 = 6$

$$3(5+lr)^2 + 2(4+mr)^2 = 6 \quad \text{... (1)}$$

$mr + 4 + nr + 3 = 0$

$r(m + n) + 7 = 0$

$$r(m + n) = -7 \quad \text{... (2)}$$

from (1)

$$3\left[l(m+n)r+5(m+n)^2\right]^2+2\left[m(m+n)r+4(m+n)\right]=(m+n)^2$$

$$7(-7l+5m+5n)^2+2(-7m+4m+4n)^2=6(m+n)^2$$

hence, the locus of (l, m, n) is

$$3[-7(x-5)+5(y-4)+5(z-3)]^2+2[-3(y-4)+4(z-3)]^2$$

$$=6[y-4+z-3]^2$$

$$=3[-7x+35+5y-20+5z-15]^2+2[-3y+12+4z-12]^2$$

$$=6[y+z-7]^2$$

$$=3[-7x+5y+5z]^2+2[-3y+4z]^2=6[y+z-7]^2$$

$$=3\left[49x^2+25y^2+25z^2+(-70xy+25yz+(-70xz)\right]$$

$$+2\left[9y^2+16z^2+(-24yz)\right]=6[y+z-7]^2$$

$$=3\left[49x^2+25y^2+25z^2-70xy+25yz-70xz\right]+18y^2+32z^2-48yz$$

$$=6[y+z-7]^2$$

$$=147x^2+75y^2+75z^2-120xy+75yz-21xz+18y^2+32z^2-48yz$$

$$=6\left[y^2+z^2+49+2yz-14z-14y\right]$$

$$=147x^2+87y^2+101z^2-210xy+90yz-210xz-294=0\cdot$$

4. Find the equation of cone whose vertex (1,2,3) and base is $y^2=4ax$, z=0.

Sol :

The equation of cone whose vertex (1,2,3) is $\dfrac{x-1}{l}=\dfrac{y-2}{m}=\dfrac{z-3}{m}$ ... (1)

It meets the plane z=0 at $\left(\alpha-\dfrac{l\gamma}{n},\beta-\dfrac{m\gamma}{n},0\right)$

i.e $\frac{x-1}{l} = \frac{y-2}{m} = \frac{-3}{n}$

$\frac{x-1}{l} = \frac{-3}{n} \qquad \frac{y-2}{m} = \frac{-3}{n}$

$x-1 = \frac{-3l}{n} \quad y-2 = \frac{-3m}{n}$

$x = 1 - \frac{3l}{n} \quad y = 2 - \frac{3m}{n}$

$\therefore \left(1-\frac{3l}{n}, 2-\frac{3m}{n}, 0\right) = (x, y, z)$

and if it lies on $y^2 = 4ax$

Then $\left(2-\frac{3m}{n}\right)^2 = 4a\left(1-\frac{3\ell}{n}\right)$

$$\left(\frac{2n-3m}{n}\right)^2 = 4a\left(\frac{n-3\ell}{n}\right) \qquad \ldots (2)$$

Eliminating l, m, n from (1) & (2)

$$\left(\frac{2(z-3)-3(y-2)}{z-3}\right)^2 = 4a\left(\frac{(z-3)-3(x-1)}{(z-3)}\right)$$

$$= \frac{[2(z-3)-3(y-2)]^2}{(z-5)^2} = 4a\frac{(z-3-3x+3)}{z-3}$$

$$(2z-6-3y+6)^2 = 4a\frac{(z-3x)}{z-3}(z-3)^2$$

$$(2z-3y)^2 = 4a(z-3x)(z-3)$$

$$4z^2 + 9y^2 - 12zy = (4az-12ax)(z-3)$$

$$4z^2 + 9y^2 - 4az^2 - 12zy + 12az - 12axz - 36ax = 0$$

$$4z^2 + 9y^2 - 4az^2 - 12zy + 12az - 12axz - 36ax = 0$$

$$9y^2 + z^2(4-4a) - 12yz + 12az - 12axz - 36ax = 0$$

5. Find the equation of the cone whose vertex is at the point (–1,1,2) and whose guiding curve is $3x^2 - y^2 = 1$, $z=0$.

Sol :

The equation of line passes through (–1, 1, 2) is $\frac{x+1}{\ell} = \frac{y-1}{m} = \frac{z-2}{n}$ ... (1)

$z = 0$

The equation (1) will be $\frac{x+1}{\ell} = \frac{y-1}{m} = \frac{x-2}{n}$

$$\therefore \frac{x+1}{\ell} = \frac{-2}{n} \qquad \frac{y-1}{m} = \frac{-2}{n}$$

$$x+1 = \frac{-2\ell}{n} \qquad y-1 = \frac{-2m}{n}$$

$$x = 1 - \frac{2\ell}{n} \qquad y = 1 - \frac{2m}{n}$$

$$\therefore (x, y, z) = \left(-1 - \frac{2\ell}{n}, 1 - \frac{2m}{n}, 0\right)$$

Substitute above (x, y, z) in the guiding curve

i.e. $3x^2 - y^2 = 1$

$$\Rightarrow 3\left(-1 - \frac{-2\ell}{n}\right)^2 - \left(1 - \frac{2m}{n}\right)^2 = 1$$

$$3\left(\frac{-n-2\ell}{n}\right)^2 - \left(\frac{n-2m}{n}\right)^2 = 1$$

$$3(-n-2\ell)^2 - (n-2m)^2 = n^2 \quad .....(2)$$

eliminating l, m, n from (1) & (2)

$$3[-(z-2)-2(x+1)]^2[(z-2)-2(y-1)]^2 = (z-2)^2$$

$$3[-z+\cancel{2}-2x-\cancel{2}]^2 - [z-\cancel{2}-2y+\cancel{2}]^2 = z^2+4-4z$$

$$3\left(-(z+2x)^2\right)-(z-2y)^2-z^2-4+4z=0$$

$$-3\left(z^2+4x^2+2xz\right)-\left(z^2+4y^2-4zy\right)-z^2-4+4z=0$$

$$-3z^2+12x^2-6xz-z^2-4y^2+4zy-z^2-4+4z=0$$

$$12x^2-4y^2-5z^2-6xz+4zy+4z-4=0$$

2.1.2 Enveloping Cone of a Sphere

6. Find the equation of the cone where vertex is at the point (α,β,γ) and where generators touch the sphere $x^2+y^2+z^2=a^2$.

Sol :

The equation of sphere is $x^2+y^2+z^2=a^2$ ... (1)

The equation to any line through (α,β,γ) are

$$\frac{x-d}{l}=\frac{y-\beta}{m}=\frac{z-v}{n} \quad \text{... (2)}$$

The points of intersection of line (2) with sphere (1)

$\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ through (α,β,γ) with the given sphere are

$$\frac{x-\alpha}{\ell}=r \qquad \frac{y-\beta}{m}=r \qquad \frac{z-\gamma}{n}=r$$

$$x-\alpha=\ell r \qquad y-\beta=mr \qquad z-\gamma=nr$$

$$x=\alpha+\ell r \qquad y=\beta+mr \qquad z=\gamma+nr$$

$\therefore$ the sphere are $(\ell r+\alpha_1\ mr+\beta,\ nr+\gamma)$

Where the value of r are the roots of the Quadratic equation

$$r^2\left(\ell^2+m^2+n^2\right)+2r\left[\ell(\alpha+u)+m(\beta+v)+n(\gamma+w)\right]$$
$$+\left(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d\right)=0$$

$$r^2\left(\ell^2+m^2+n^2\right)+2r\left[\ell\alpha+m\beta+n\gamma\right]+\left(\alpha^2+\beta^2+\gamma^2-a^2\right)=0$$

If the two roots of the Quadratic equation in r are equal

$$\left(\ell\alpha+m\beta+n\gamma\right)^2=\left(\ell^2+m^2+n^2\right)\left(\alpha^2+\beta^2+\gamma^2-a^2\right) \quad \ldots (3)$$

The above condition for the line (2) to touch the sphere (1)
Eliminating l, m,n between (2) and (3)
Then we get,

$$\left[\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)\right]^2=\left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right]\left(\alpha^2+\beta^2+\gamma^2-a^2\right) \quad \ldots (4)$$

Which is required equation of the cone.

Here $S \equiv x^2+y^2+z^2-a^2$

$S_{11}=\alpha^2+\beta^2+\gamma^2-a^2$

$S_1=\alpha x+\beta y+\gamma z-a^2$

From equation we can rewrite that,

$$\left(\alpha x-\alpha^2+\beta y-\beta^2+\gamma z-\gamma^2\right)^2=(x^2+\alpha^2-2x\alpha+y^2+\beta^2-2y\beta$$

$$+z^2+\alpha^2-2z\alpha)\left(\alpha^2+\beta^2+\gamma^2-a^2\right)$$

$$\left(\alpha x+\beta y+\gamma z-\left(\alpha^2+\beta^2+\gamma^2\right)\right)=\left(x^2+y^2+z^2\right)-2\left(x\alpha+y\beta+z\alpha\right)$$

$$+\left(\alpha^2+\beta^2+\gamma^2\right)\left[\alpha^2+\beta^2+\gamma^2-a^2\right]$$

$$=\left[x\alpha+y\beta+z\gamma-a^2+a^2-\left(\alpha^2+\beta^2+\gamma^2\right)\right]^2=\left(x^2+y^2+z^2-a^2\right)-2\left(\alpha x+y\beta+z\gamma-a^2\right)$$

$$+\left(\alpha^2+\beta^2+\gamma^2-a^2\right)\left(\alpha^2+\beta^2+\gamma^2-a^2\right)$$

$$\left[\left(\alpha x+y\beta+z\gamma-a^2\right)\right]-\left[\alpha^2+\beta^2+\gamma^2-a^2\right]\Big]=\left(x^2+y^2+z^2-a^2\right)$$

$$-2\left(\alpha x+y\beta+z\gamma-a^2\right)+\left(\alpha^2+\beta^2+\gamma^2-a^2\right)\left(\alpha^2+\beta^2+\gamma^2-a^2\right)$$

$$\left(S_1-S_{11}\right)^2=\left(S-2S_1+S_{11}\right)S_{11}$$

$$S_1^2+\cancel{S_{11}^2}-2S_1S_{11}-S_{11}+2\cancel{S_1}S_{11}-\cancel{S_{11}}S_{11}=0$$

$$S_1^2=5.S_{11}\Rightarrow \boxed{S_1^2=S.S_{11}}$$

7. Find the enveloping cone of the sphere $x^2 + y^2 + z^2 - 2x + 4z = 1$ with its vertex are (1, 1, 1).

Sol :

Given that the enveloping cone of the sphere $S = x^2 + y^2 + z^2 - 2x + 4z - 1 = 0$.

equation of the cone is $S.S_{11} = S_1^{\,2}$

S_1 = at (1,1,1) ie S_{11}

$\Rightarrow 1^2 + 1^2 + 1^2 - 2(1) + 4(1) - 1 = 3 - 2 + 4 - 1$

$S_{11} = 4$

$S_1^{\,2} = \left[x(1) + y(1) + z(1) - 6(x+1) + 0(y+1) + 2(z+1) - 1\right]$

$\therefore S.S_{11} = S_1^{\,2}$

$\Rightarrow \left(x^2 + y^2 + z^2 - 2x + 4z - 1\right)(4) = \left[\cancel{x} + y + z - \cancel{x} - 1 + 2z + 2 - 1\right]^2$

$4x^2 + 4y^2 + 4y^2 - 8x + 16z - 4 = (y + 3z)^2$

$4x^2 + 4y^2 + 4z^2 - 8x + 16z - 4 - y^2 - 9z^2 - 6yz = 0$

$4x^2 + 3y^2 - 5z^2 - 8x + 16z - 6yz - 4 = 0$.

2.1.3 Quadratic cones with vertex at origin

Theorem

The equation of a cone where vertex at the Origin is homogeneous.

Proof

The General equation is of the second degree and it represents a cone with its vertex at the origin.

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy + 2ux + 2vy + 2wz + d = 0 \qquad \dots (1)$$

$u = v = w = d = 0$

Let $P(x^1, y^1, z^1)$ be a point on the cone represented by equation (1)

The equation of line $= \dfrac{x-0}{x^1-0} = \dfrac{y-0}{y^1-0} = \dfrac{z-0}{z^1-0} = r$

$$x = x^1 r, \qquad y = y^1 r \qquad z = z^1 r$$

$\therefore (x^1 r,\ y^1 r,\ z^1 r)$ are the general coordinates of a point on the line OP joining the point P to the Origin .

Since the line OP is a generater of cone (rx^1, ry^1, rz^1) lies on if for every value of r implying that the equation

$$r^2(ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y') + 2r(ux' + vy' + wz') + d = 0$$

$$\Rightarrow ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' = 0 \quad \text{... (a)}$$

$$ux' + vy' + wz' = 0 \quad \text{... (b)}$$

$$d = 0 \quad \text{... (c)}$$

from (b),

We see that if u, v, w be not all zero, then the co-ordinates x^1, y^1, z^1 of any point on the cone satisfy as equation of the first degree.

$$ux + vy + wz = 0$$

So that the equation of a cone with its vertex at the origin is necessarily homogeneous.

conversely, we show that every homogeneous equation of the second degree represents a cone with its vertex at the origin.

- If l, m, n be the direction ratios of any generators of the cone.

  $$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0$$

  So that the point (lr, mr, nr) lies on it for every value of r, we have

  $$a\ell^2 + bm^2 + cn^2 + 2fmn + 2gn\ell + 2h\ell n = 0$$

- The general equation of a cone with its vertex at the point (α, β, γ) is

  $$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 + 2f(z-\gamma)(y-\beta) + 2g(x-\alpha)(z-\gamma) + 2h(x-\alpha)(y-\beta) = 0$$

  as can easily verified by transferring the origin to the point (α, β, γ).

8. Find the equation of the cone whose vertex is at the origin and which passes through the curve given by the equation $ax^2 + by^2 + cz^2 = 1$, $lx + my + nz = P$.

Sol :

The curve are $ax^2 + by^2 + cz^2 = (1)^2$ ... (1)

and $lx + my + nz = P$ .... P ... (2)

from (2),

$$\ell x + my + nz = p \Leftrightarrow \frac{\ell x + my + nz}{p} = 1 \quad \text{... (3)}$$

from (1) & (2) we get that

$$ax^2 + by^2 + cz^2 = \left(\frac{\ell x + my + nz}{p}\right)^2$$

$$p^2\left[ax^2 + by^2 + cz^2\right] = (\ell x + my + nz)^2$$

$$ap^2x^2 + bpy^2 + cpz^2 = \ell^2x^2 + m^2y^2 + n^2z^2 + 2\ell xmy + 2mynz + 2\ell xnz$$

$$ap^2x^2 + bp^2y^2 + cp^2z^2 - \ell x^2 - m^2y^2 - n^2z^2 - 2\ell x\ my - 2my\ nz - 2\ell x\ nz = 0$$

$$x^2(ap^2 - \ell^2) + y^2(bp^2 - m^2) + z^2(cp^2 - n^2) - 2(\ell x\ my + 2ny\ nz + \ell x\ nz) = 0$$

9. Show that the equation of the cone where vertex is the origin and base curve z=k, f(x,y)=0 is $f\left(\frac{xk}{z}, \frac{yk}{z}\right) = 0$.

Sol :

Let $f(x, y) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ ... (1)

By making (1) homogenous with the help of z=k.

We get the equation of required cone as

$$ax^2 + by^2 + 2hxy + 2gx\left(\frac{z}{k}\right) + 2fy\left(\frac{z}{k}\right) + c\left(\frac{z}{k}\right)^2 = 0$$

Multiplying by $\frac{k^2}{z^2}$

$$a\left(\frac{xk}{z}\right)^2 + b\left(\frac{yk}{z}\right)^2 + 2h\left(\frac{xk}{z}\right)\left(\frac{yk}{z}\right) + 2g\left(\frac{xk}{z}\right) + 2f\left(\frac{yk}{z}\right) + c = 0$$

$$= f\left(\frac{xk}{z}, \frac{yk}{z}\right) = 0$$

10. Show that the general equation to a cone which passes through the there axes is fyz + gzx + hxy =0 f, g, h being parameters.

Sol :

The general equation of a cone with its vertex at the origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0 \quad \ldots (1)$$

x axis is a generators

its directions cosines (1,0,0)

from (1) $a(1) + b(0) + c(0) + 2f(0)(0) + 2g(1)(0) + 2h(1)(0) = 0$

$a = 0$

by $b = 0$ & $c = 0$

Sub a = 0, b = 0 and c = 0 in (1)

$$0.x^2 + 0.y^2 + 0.z^2 + 2(fyz + gxz + hxy) = 0$$

$$fyz + gxz + hxy = 0$$

11. Show that a cone of second degree can be found to pass through any five concurrent lines.

Sol :

Let origin be the point of concurrence of fine lines which are

$$\frac{x}{\ell_r} = \frac{y}{m_r} = \frac{z}{n_r}, \; r = 1, 2, 3, 4, 5$$

General second degree equation of the cone with vertex at origin is

$$ax^2 + by^2 + c^2 + 2fyz + 2gxz + 2hxy = 0$$

divide by 'a'.

$$x^2 + \frac{b}{a}y^2 + \frac{c}{a}z^2 + \frac{2f}{a}yz + \frac{2g}{a}xz + \frac{2h}{a}xy = 0.$$

$$x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'xz + 2h'xy = 0$$

12. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes is A, B, C. Prove that the equation to the cone generator by lines drawn from 0 to meet the circle ABC is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

Sol :

Points A, B, C are (a, 0,0), (0, b, 0) and (0, 0, c) respectively.

Equation of sphere OABC is $x^2+y^2+z^2-ax-by-cz=0$ ... (1)

The circle ABC is obtained by intersection of given plane with equation of sphere.

Making (1) homogenous with the help of given plane,

The required cone is

$$\left(x^2+y^2+z^2\right)-\left(ax+by+cz\right)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

13. Find the equation to the cone which passes through the three co-ordinate axes as well as the two lines.

$$\frac{x}{1}=\frac{y}{-2}=\frac{z}{3},\frac{x}{3}=\frac{y}{-1}=\frac{z}{1}$$

Sol :

The General equation of the cone passing through the axis is

$$fyz+gzx+hxy=0 \qquad ... (1)$$

and also given the line equations

i.e. $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ ... (2)

and $\frac{x}{3}=\frac{y}{-1}=\frac{z}{1}$ ... (3)

from equation (2) lies on equation (1)

i.e. (x, y, z) = (1, –2, 3)

$$f(-2)(3)+g(3)(1)+h(1)(-2)=0$$

$$-6f+3g-2h=0 \qquad \text{... (4)}$$

from equation (2) lies on equation (1) then

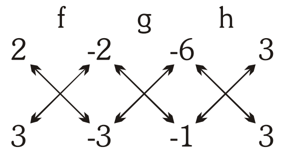
$\therefore$ (x, y, z) = (3, –1, 1)

from (1)

$$f(-1)(1)+g(1)(3)+h(3)(-1)=0$$

$$-f+3g-3h=0 \qquad \text{... (5)}$$

By Solving equations (4) & (5)



$$\frac{f}{-9+6}=\frac{g}{2-18}=\frac{h}{-18+3}$$

$$\frac{f}{-3}=\frac{g}{-16}=\frac{h}{-15}$$

$\therefore$ f = 3, g = 16, h = 15

Sub f, g, h value in equation (1)

then we get that

$$fyz+gyz+hxy=0$$

$$\Rightarrow 3yz+16yz+15xy=0$$

2.2 Condition that the General Equation of the Second Degree Should Represent a Cone

Co- ordinate of the vertex.

Let $f(x,y,z)=ax^2+by^2+cz^2+2fy^2+2gxz+2hxy+2ux+2vy+2wz+d=0$ ... (1)

represent a cone having its vertex at (x',y',z') origin to the vertex (x',y',z').

So, we change x to $x+x'$

y to $y+y'$ and

z to $z+z'$

The transformed equation is

$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy+2\left[x\left(ax^1+hy^1+gz^1+u\right)\right.$$

$$\left.+y\left(hx^1+by^1+fz^1+v\right)+z\left(gx^1+fy^1+cz^1+w\right)\right]+f\left(x^1,y^1,z^1\right)=0 \qquad \dots (2)$$

Cone with its vertex at the origin be homogeneous

$$ax^1+hy^1+gz^1+u=0 \qquad \dots (a)$$

$$hx^1+by^1+fz^1+v=0 \qquad \dots (b)$$

$$gx^1+fy^1+cz^1+w=0 \qquad \dots (c)$$

$$f\left(x^1,y^1,z^1\right)=0 \qquad \dots (d)$$

Also, $f\left(x^1,y^1,z^1\right)\equiv x^1\left(ax^1+hy^1+gz^1+u\right)+y^1\left(bx^1+by^1+fz^1+v\right)$

$$+z^1\left(gx^1+fy^1+cz^1+w\right)+\left(ux^1+vy^1+wz^1+d\right)$$

from (a), (b), (c) and (d) is equivalent to $ux^1+vy^1+wz^1+d=0$ ... (e)

The given equation represents a cone, there exist $\left(x^1,y^1,z^1\right)$ satisfying the equations (a), (b), (c) & (e) these four equations are constantant.

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}=0$$

$\therefore$ The required second degree to represent a cone.

The equation F(x, y, 0)=0 represents a cone it, and only if, the four linear equations.

$F_x=0$, $F_y=0$, $F_z=0$, $F_t=0$ are consistent.

In the case of consistency the vertex is given by any three of these.

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0 \text{ as well as } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$\therefore$ It is required pair of planes.

14. Find the equation to the lines in which the plane 2x +y – z =0 cuts the cone $4x^2 - y^2 + 3z^2 = 0$.

Sol :

Let the equation of line is $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$ be the equation of any one of the two lines in which the given plane meets the given cone

$$2l + m - n = 0 \quad \text{... (1)}$$

$$4l^2 - m^2 + 3n^2 = 0 \quad \text{... (2)}$$

These two equations are now to be solved for l, m,n Eliminating n, we have

$$4l^2 - m^2 + 3n^2 = 0$$

from (1)

$$n = 2l + m$$

Sub n = 2l + m in above equation then we get

$$4\ell^2 - m^2 + 3(2\ell + m)^2 = 0$$

$$4\ell^2 - m^2 + 3\left[4\ell^2 + m^2 + 4\ell m\right] = 0$$

$$16\ell^2 + 2m^2 + 12\ell m = 0$$

$$8\ell^2 + m^2 + 6\ell m = 0$$

$$\Rightarrow 8\ell^2 + 6\ell m + m^2 = 0$$

$$-6\frac{\pm\sqrt{(6)^2 - 4(8)(1)}}{2(8)}$$

$$= \frac{-6 \pm \sqrt{36-32}}{16} = \frac{-6 \pm \sqrt{4}}{16} = \frac{-6 \pm 2}{16} = \frac{-6+2}{16} \text{ or } \frac{-6-2}{16}$$

i.e $\frac{-4}{16}$ or $\frac{-8}{16}$

$\Rightarrow \frac{-1}{4}$ or $\frac{-1}{2}$

from equation (1) we divide by m.

Then we get

$$\frac{2\ell}{m} + \frac{m}{n} - \frac{n}{m} = 0$$

$$\Rightarrow \frac{2\ell}{m} + 1 - \frac{n}{m} = 0$$

$$\frac{\ell}{m} = \frac{-1}{2} \Rightarrow 2\left(\frac{-1}{2}\right) + 1 - \frac{n}{m} = 0$$

$$-1 + 1 - \frac{n}{m} = 0$$

$$\therefore \frac{n}{m} = 0 \qquad \frac{\ell}{-1} = \frac{m}{2}, n = 0$$

If $\frac{\ell}{m} = \frac{-1}{4} \Rightarrow 2\left(\frac{-1}{4}\right) + 1 - \frac{n}{m} = 0$

$$\frac{-1}{2} + 1 - \frac{n}{m} = 0$$

$$\frac{1}{2} - \frac{n}{m} = 0$$

$$\frac{n}{m} = \frac{1}{2}$$

$$\therefore \frac{\ell}{-1} = \frac{m}{4} = \frac{n}{2}$$

Thus, the two required lines are.

$$\frac{x}{-1} = \frac{y}{4} = \frac{z}{2}; \frac{x}{-1} = \frac{y}{-1} = \frac{y}{2}; z = 0$$

15. Find the equation of the line of the line of inter section of the plane and cone.

$$\mathbf{x + 3y - 2z = 0, \quad x^2 + 9y^2 - 4z^2 = 0}$$

Sol :

Let the equation of line is $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$.

be the equation of any one of the two lines in which the given plane meets the given cone so, that we have

$$\ell + 3m - 2n = 0 \quad \text{... (1)}$$

$$\ell^2 + 9m^2 - 4n^2 = 0 \quad \text{.... (2)}$$

These two equation are now to be solved for l, m, n eliminating we have.

from (1) = 2n – 3m

sub in (2) then we get

$$(2n - 3m)^2 + 9m^2 - 4n^2 = 0$$

$$\cancel{4x^2} + 9m^2 - 12mn + 9m^2 - \cancel{4n^2} = 0$$

$$18m^2 - 12mn = 0$$

m(18m – 12n) = 0

m = 0 18m – 12n = 0

$$-3m - 2n = 0$$

3m = 2n

$\Rightarrow 3y = 2z$ and z = 0

from equation (1) eliminating m.

then 3m = 2n – *l*

$$m = \frac{2n - \ell}{3}$$

sub in equation (2) then we get

$$\ell^2 + 9m^2 - 4n^2 = 0$$

$$\ell^2 + 9\left(\frac{2n-\ell}{3}\right)^2 - 4n^2 = 0$$

$$\ell^2 + \frac{\cancel{9}}{\cancel{9}}(2n-\ell)^2 - 4n^2 = 0$$

$$\ell^2 + \cancel{4n^2} + \ell^2 - 4n\ell - \cancel{4n^2} = 0$$

$$2\ell^2 - 4n\ell = 0$$

$$\ell(2\ell - 4n) = 0$$

$l = 0,\ 2l - 4n = 0$

$2l = 4n$

$\Rightarrow l = 2n$

i.e. $x = 2z$.

$\therefore$ The solutions are $x = 2z,\ y = 0$,

$3y = 2z,\ x = 0$.

---

16. The section of a cone whose vertex is P and guiding curve the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ by the plane $z = 0$ is a rectangular hyperbola. S/T the locus of p is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$. ... (1)

Sol :

Let the point P be (α, β, γ).

Then from required cone equation as $a(\alpha z - \gamma x)^2 + b(\beta z - y\gamma)^2 = (z-\gamma)^2$ as from (1)

$$\Rightarrow \frac{1}{a^2}(\alpha z - \gamma x)^2 + \frac{1}{b^2}(\beta z - y\gamma)^2 = (z-\gamma)^2 \quad \text{... (2)}$$

This meets $x = 0$ in a curve then (2) as

$$\frac{1}{a^2}(\alpha z - 0)^2 + \frac{1}{b^2}(\beta z - y\gamma)^2 = (z-\gamma)^2$$

$$\frac{\alpha^2 z^2}{a^2} + \frac{(\beta z - \gamma y)^2}{b^2} = (z-\gamma)^2,\ x = 0 \quad \text{... (3)}$$

from (3)

$$\frac{\alpha^2 z^2}{a^2} + \frac{\beta^2 z^2 + \gamma^2 y^2 - 2\beta z\gamma y}{b^2} - z^2 - \gamma^2 + 2z\gamma = 0$$

If coefficient of y^2 + coefficient of z^2 =0

$$\Rightarrow \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{b^2} - 1 = 0$$

$$\Rightarrow \frac{\alpha^2}{a^2} + \frac{\beta^2 + \gamma^2}{b^2} - 1 = 0$$

$$\Rightarrow \frac{\alpha^2}{a^2} + \frac{\beta^2 + \gamma^2}{b^2} = 1$$

$\therefore$ locus of (α, β, γ) is $\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$

2.3 Cone and a Plane Through it's Vertex Mutually Perpendicular Generators of Cone

Theorem

Show that the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ admits of sets of three mutually perpendicular generators if and only if a + b + c = 0.

Proof :

Let $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{v}$ be generators as the cone ... (1)

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{... (2)}$$

$$\Rightarrow a\lambda^2 + b\mu^2 + cv^2 + 2f\mu v + 2gv\lambda + 2h\lambda\mu = 0 \quad \text{... (3)}$$

Equation of the plane through the origin perpendicular to the line (1) is

$$\lambda x + \mu y + vz = 0 \quad \text{... (4)}$$

If (l, m, n) be the direction cosines of any one of the two generators in which the plane cuts the given cone, we have

$$a\ell^2 + bm^2 + cn^2 + 2fmn + 2gn\ell + 2h\ell m = 0 \qquad \dots (5)$$

and $\ell\lambda + m\mu + n\nu = 0$ ... (6)

Eliminating n between (5) and (6), we obtain, from (6)

$$n\upsilon = -m\mu - \ell\lambda$$

$$n = \frac{-(m\mu + \ell\lambda)}{\upsilon}$$

Sub in (5) when we get

$$a\ell^2 + bm^2 + c\left[\frac{-(m\mu + \ell\lambda)}{\upsilon}\right]^2 + 2fm\left[\frac{-(m\upsilon + \ell\lambda)}{\upsilon}\right] + 2g\left[\frac{-(m\mu + \ell\lambda)}{\upsilon}\right]\ell + 2h\ell m = 0$$

$$v^2a\ell^2 + v^2bm^2 + c\left[m^2\mu^2 + \ell^2\lambda^2 + 2m\mu\,\ell\lambda\right] - 2fm^2\mu v - 2fm\,\ell\lambda v$$

$$-2gm\ell^2 v - 2g\ell^2\lambda v + v^2 2h\ell m = 0$$

$$\ell^2\left[av^2 + c\lambda^2 - 2g\lambda v\right] + 2\ell m\left[c\lambda\mu + hv^2 - g\mu v + f\lambda v\right] + m^2\left[bv^2 + c\mu^2 - 2f\mu v\right] = 0$$

Which being a quadratic in *l*, *m* we see that the plane (4) cuts the given cone in two generators.

hence if, $(\ell_1, m_1, n_1)(\ell_2, m_2, n_2)$ be direction cosines of these two generators, we have $\frac{\ell_1\ell_2}{m_1m_2} = \frac{bv^2 + c\mu^2 - 2f\mu}{av^2 + c\lambda^2 - 2g\lambda v}$

$$\frac{\ell_1\ell_2}{b\nu^2 + c\mu^2 - 2f\mu\gamma} = \frac{m_1m_2}{av^2 + c\lambda^2 - 2g\lambda\gamma}$$

from symmetry, each of there is $= \frac{n_1n_2}{a\mu^2 + b\lambda^2 - 2h\lambda\mu} = k$.

Thus, we have

$$\ell_1\ell_2 = k\left(b\nu^2 + c\mu^2 - 2f\mu v\right)$$

$$m_1m_2 = k\left(av^2 + c\lambda^2 - 2g\lambda v\right)$$

$$n_1n_2 = k\left(a\mu^2 + b\lambda^2 - 2h\lambda\mu\right)$$

$\therefore \ell_1\ell_2 + m_1m_2 + n_1n_2 = k\left(b\upsilon^2 + c\mu^2 - 2f\mu\upsilon\right) + k\left(a\upsilon^2 + c\lambda^2 - 2gh\gamma\right) + k\left[a\mu^2 + b\lambda^2 - 2h\lambda\mu\right]$

$= k\left[b\upsilon^2 + c\mu^2 - 2f\mu\gamma + a\upsilon^2 + c\lambda^2 - 2gh\gamma + a\mu^2 + b\lambda^2 - 2h\lambda\mu\right]$

$= k\left[a\left(\mu^2 + \upsilon^2\right) + b\left(\gamma^2 + \lambda^2\right) + c\left(\mu^2 + \lambda^2\right) - 2f\mu\gamma - 2gh\gamma - 2h\lambda\mu\right]$

$\ell_1\ell_2 + m_1m_2 + n_1n_2 = k\left(a + b + c\right) + \left(\lambda^2 + \mu^2 + \upsilon^2\right)$

The two generators in which the plane (4) intersets the curve (1) will be at light angles if and only if $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$

$\Rightarrow a + b + c = 0 \cdot$

Theorem

Find the angle between the lines of intersection of the plane $ux + vy + wz = 0$ and $f(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

Proof :

The given plane is ux + vy + wz =0 ... (1)

and the equation of cone is

$f(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ ... (2)

The plane (1) will cut the cone (2) in two lines passing through the origin.

Let one of these lines be $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$ ... (3)

line (3) lies in plane (1).

$\therefore u\ell + vm + wn = 0$ ... (4)

Also, line (3) lies on (2).

$\Rightarrow a\ell^2 + bm^2 + cn^2 + 2fmn + 2gn\ell + 2h\ell m = 0$ ... (5)

from (4) $\Rightarrow wn = -\left(\mu\ell + vm\right)$

$n = -\left(\frac{\mu\ell + vm}{w}\right)$

Sub $n = \frac{-\left(\mu\ell + vm\right)}{w}$ in (5)

$$\Rightarrow a\ell^2 + bm^2 + c\left(-\frac{\mu\ell + vm}{w}\right)^2 + 2fm\left(-\frac{\mu\ell + vm}{w}\right) + 2g\ell\left(-\frac{\mu\ell + vm}{w}\right)^2 + 2h\ell m = 0$$

$$= w^2a\ell^2 + w^2bm^2 + c\mu^2\ell^2 + cv^2m^2 + 2c\mu\ell vm - 2fm\mu\ell w$$

$$-2fm^2vw - 2g\ell^2\mu w - 2g\ell vmw + 2h\ell mw^2 = 0$$

$$\Rightarrow \ell^2\left(aw^2 + c\mu^2 - 2g\mu w\right) + 2\ell m\left(c\mu v - fwu - gvw + hw^2\right) + m^2\left(bw^2 + cv^2 - 2fvw\right) = 0$$

divide by 'm^2'

$$\frac{\ell^2}{m^2}\left(aw^2 + c\mu^2 - 2g\mu w\right) + \frac{2\ell m}{m^2}\left(c\mu v - fwu - gvw - hw^2\right)$$

$$\Rightarrow \frac{m^2}{m^2}\left(bw^2 + cv^2 - 2fvw\right) = 0$$

$$\frac{\ell^2}{m^2}\left(aw^2 + c\mu^2 - 2g\mu w\right) + \frac{2\ell}{m}\left(c\mu v - fwu - gvw + hw^2\right) + \left(bw^2 + cv^2 - 2fvw\right) = 0 \quad \ldots(6)$$

Now, from (6) is Quadratic equation $\frac{\ell}{m}$ and show that plane (1) cuts cone in two lines. direction ratio are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 then we have

$$\frac{\ell_1}{m_1}\frac{\ell_2}{m_2} = \frac{bw^2 + cv^2 - 2fvw}{aw^2 + cu^2 - 2gwu}$$

$$\frac{\ell_1\ell_2}{bw^2 + cv^2 - 2fvw} = \frac{m_1m_2}{aw^2 + cu^2 - 2gw\mu} = \frac{n_1n_2}{bu^2 + av^2 - 2huv}$$

$$\Rightarrow \frac{\ell_1\ell_2 + m_1m_2 + n_1n_2}{(b+c)u^2 + (c+a)v^2 + (a+b)w^2 - 2fvw - 2gwu - 2huv}$$

$$= \frac{\ell_2\ell_1 + m_1m_2 + n_1n_2}{(a+b+c)\left(u^2 + v^2 + w^2\right) - f(u,v,w)}$$

Also sum of the roots of (6) gives

$$\frac{\ell_1}{m_1} + \frac{\ell_2}{m_2} = \frac{-2\left(cuv - fwu - gvw + hw^2\right)}{aw^2 + cu^2 - 2gwu}$$

$$\therefore \quad \frac{\ell_1 m_2 + \ell_2 m_1}{-2\left(cuv - fwu - gvw + hw^2\right)} = \frac{m_1 m_2}{aw^2 + cu^2 - 2guw}$$

$$= \frac{\ell_1 \ell_2}{bw^2 + cv^2 - 2fvw} = \frac{n_1 n_2}{av^2 + bu^2 - 2huv}$$

$$= \frac{\left[\left(\ell_1 m_2 + \ell_2 m_1\right)^2 - 4\ell_1 \ell_2 \; m_1 m_2\right]^{1/2}}{4\left(cuv - fwu - gvw + hw^2\right)^2 - 4\left(bw^2 + cv^2 - 2fvw\right)\left(aw^2 + cu^2 - 2gwh\right)}$$

$$= \frac{\ell_1 m_2 - \ell_2 m_1}{\pm 2wp} \text{ where } p^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$$

$$= \frac{m_1 n_2 - m_2 n_1}{\pm\ 2up} = \frac{n_1 \ell_2 - n_2 \ell_1}{\pm\ 2vp} = \frac{\left[\sum\left(m_1 n_2 - m_2 n_1\right)^2\right]^{1/2}}{\pm\ 2p\left(u^2 + v^2 + w^2\right)^{1/2}}$$

If θ be the angle between lines, then

$$\tan\theta = \frac{\left[\Sigma\left(m_1 n_2 - m_2 n_1\right)^2\right]^{1/2}}{\ell_1 \ell_2 + m_1 m_2 + n_1 n_2}$$

$$\text{or } \tan\theta = \pm\frac{2p\left(u^2 + v^2 + w^2\right)^{1/2}}{(a+b+c)\left(u^2 + v^2 + w^2\right) - f(u, v, w)}$$

17. Prove that the plane ax + by + cz =0 cuts the cone yz + zx + xy =0 in perpendicular lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.

Sol :

The lines of intersection be $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$

The line lies on given cone and plane,

hence $mn + nl + lm = 0$ ... (1)

and $a\ell + bm + cn = 0$ ... (2)

from (2) $cn = -a\ell - bm$

$$n = \frac{-a\ell - bm}{c}$$

Sub n value in (1) then we get

$$m\left(\frac{-a\ell - bm}{c}\right) + \left(\frac{-a\ell - bm}{c}\right)\ell + \ell m = 0$$

$$-am\ell - bm^2 - a\ell^2 - bm\ell + c\ell m = 0$$

$$am\ell + bm^2 + a\ell^2 + bm\ell - c\ell m = 0$$

$$m\ell(a + b - c) + a\ell^2 + bm^2 = 0$$

divide m^2

$$\frac{a\ell^2}{m^2} + \frac{(a + b - c)}{m^2} + \frac{b\cancel{m^2}}{\cancel{m^2}} = 0$$

$$a\left(\frac{\ell}{m}\right)^2 + (a + b - c)\frac{\ell}{m} + b = 0.$$

Let $\frac{\ell_1}{m_1}, \frac{\ell_2}{m_2}$ be the roots,

$$\frac{\ell_1}{m_1} \cdot \frac{\ell_2}{m_2} = \frac{b}{a}$$

$$\frac{\ell_1 \ell_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c}$$

The angle between the lines will be a light angle if

$$\ell_1 \ell_2 + m_1 m_2 + n_1 n_2 = 0 \Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

2.4 Intersection of Line with a Cone

18. Find the point of intersection of the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ and the cone $f(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

Sol :

The equation of line $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ ... (1)

equation of cone is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ ... (2)

from equation (1)

$$\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$$

$\frac{x-\alpha}{\ell}=r$ $\quad \frac{y-\beta}{m}=r$ $\quad \frac{z-\gamma}{n}=r$

$x-\alpha=\ell r$ $\quad y-\beta=mr$ $\quad z-\gamma=rn$

$x=\ell r+\alpha$ $\quad y=mr+\beta$ $\quad z=rn+\gamma$

The point $(\ell r+\alpha, mr+\beta, rn+\gamma)$ which lies on the line (1) for all values of r which lies on the cone (2) for values of r given by the equation

$$a(\ell r+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2+2f(nr+\beta)(r\,n+\gamma)$$
$$+2g(r\,n+\gamma)(\ell r+\alpha)+2h(\ell r+\alpha)(mr+\beta)=0$$

$$\Rightarrow a(\ell^2r^2+\alpha^2+2\ell r\alpha)+b(m^2r^2+\beta^2+2mr\beta)+c(n^2r^2+\gamma^2+2nr\gamma)$$

$$+2f[mr^2n+mr\gamma+nr\beta+\beta\gamma)+2g(\ell nr^2+\ell r\gamma+\alpha nr+\alpha\gamma)+2h(\ell mr^2+\ell r\beta+\alpha mr+\alpha\beta)=0$$

$$\Rightarrow r^2(a\ell^2+bm^2+cn^2+2fmn+2g\ell n+2h\ell m)+$$

$$2r(\ell a\alpha+mb\beta+cn\gamma+fm\gamma+2fn\beta+g\ell\gamma+2\alpha n+h\ell\beta+h\alpha m)+2f\beta\gamma+2g\alpha\gamma+2h\alpha\beta=0$$

$$\Rightarrow r^2(a\ell^2+bm^2+cn^2+2fmn+2gn\ell+2h\ell m)$$

$$=2r[\ell(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+fy)+n(g\alpha+f\beta+c\gamma)]+f(\alpha,\beta,\gamma)=0$$

Let r_1, r_2 be the roots of this quadratic equation in r.

The two points of intersection are $(\ell r_1 + \alpha, mr_1 + \beta, nr_1 + \gamma), (\ell r_2 + \alpha, mr_2 + \beta, nr_2 + \gamma)$.

- The tangent lines and tangent plane at point.

Let $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ ... (1)

be a line through a point (α, β, γ) of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0 \quad \text{... (2)}$$

So that $a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0$

The two points of intersection coincide with (α, β, γ). The second point of intersection will also coincide with (α, β, γ) if the second root of the same equation is also zero.

$$\ell(a\alpha + h\beta + g\gamma) + m(h\gamma + b\beta + fy) + n(g\alpha + f\beta + c\gamma) = 0$$

The line corresponding to the set of values of l, m, n satisfying the relation to the above equation in a tangent line at (α, β, γ) to the cone.

Eliminating l, m, n between (1) & (2) we obtain the locus of all the tangent lines through (α, β, γ).

$$\Rightarrow x(a\alpha + h\beta + g\gamma) + y(h\alpha + b\beta + fy) + z(g\alpha + f\beta + c\gamma)$$

$$= a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma a + 2h\alpha\beta = 0$$

Which is known as tangent plane.

19. Prove that the cones fyz + gzx + hxy =0.

$\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$ are reciprocal.

Sol :

The given equation can be written as $\sqrt{fx} + \sqrt{gy} = -\sqrt{hz}$

Squaring on both side $\left(\sqrt{fx} + \sqrt{gy}\right)^2 = \left(-\sqrt{hz}\right)^2$

$$fx + gy + 2\sqrt{fxgy} = -hz$$

$\left(fx+gy+hz\right)^2 = 4fg\ xy$

$f^2x^2+g^2y^2+h^2z^2+2ghyz+2hfzx+2fg\ xy=0$ ... (1)

equation is homogeneous equation of second degreem hence it represents a Quadratic cone.

The co-ordinate plane x=0 meet (1)

$gy^2+hz^2+2ghyz=0$

$\left(gy+hz\right)^2=0$

Which being perfect square if follows that the plane x =0. Touches.

Similarly y =0, z =0 also touches the cone (1)

Again for the cone (1)

$a = f^2 \quad b = g^2, \quad c = h^2, \quad f = -gh$

$g = hf \quad h = fg$

$A = bc - f^2 = g^2h^2 - (gh)^2$

$= g^2h^2 - g^2h^2$

$= 0$

by B = C = 0

F = gh + a f

= hf (fg) + f² . gh

$= f^2gh + f^2gh$

$= \phi\ 2f^2gh$

by $G = 2g^2\ hf \quad H = 2h^2fg$

$\therefore$ The required equation of the cone Reciprocal to

$Ax^2+By^2+Cz^2+2Fy^2+2Gzx+2Hxy=0$

$2f^2ghyz+2g^2hfzx+2h^2fgxy=0$

$2fh\left(fyz+gzx+hxy\right)=0$

$= fyz+gzx+hxy=0$

20. Prove that the cones $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ are Reciprocal.

Sol :

The Reciprocal cone of $ax^2 + by^2 + cz^2 = 0$

is $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$ .... (1)

where $\Delta = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

$A = \frac{\partial\Delta}{\partial a} = bc$

$B = \frac{\partial\Delta}{\partial b} = ac$

$C = \frac{\partial\Delta}{\partial a} = ab$

$2F = \frac{\partial\Delta}{\partial f}$ $\quad 2G = \frac{\partial\Delta}{\partial g}$ $\quad 2H = \frac{\partial\Delta}{\partial h}$

$F = \frac{1}{2}\frac{\partial\Delta}{\partial y} = 0$ $\quad G = \frac{1}{2}\frac{\partial\Delta}{\partial g} = 0$ $\quad H = \frac{1}{2}\frac{\partial\Delta}{\partial h} = 0$

Sub above values in (1) becomes

$bc\,x^2 + acy^2 + abz^2 + 0 + 0 + 0 = 0$

$bcx^2 + acy^2 + abz^2 = 0$

~~x^2~~ divide abc on each term

$\frac{\cancel{bc}x^2}{a\cancel{b}c} + \frac{\cancel{ac}y^2}{\cancel{a}\,b\,\cancel{c}} + \frac{\cancel{ab}z^2}{\cancel{ab}c} = 0$

$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$

21. Find the conditions that the lines of section of the plane *l*x + my + nz =0 and the cones fyz + gzx + hxy =0, $ax^2 + by^2 + cz^2 = 0$ should be coincident.

Sol :

Any cone through the intersection of two cones is

$$ax^2 + b^2 + cz^2 + \lambda(fyz + gzx + hxy) = 0 \qquad \text{... (1)}$$

Since the lines of section of the given cone with *l*x + my +nz =0 are coincident.
a pair of planes of which one plane is *l*x + my + nz =0

Let other plane be $\ell^1 x + m^1 y + n^1 z = 0$

Then $ax^2 + by^2 + cz^2 + \lambda(fyz + gzx + hxy) = (\ell x + my + nz)(\ell^1 x + m^1 y + n^1 z)$

$$a = \ell\ell^1, \quad b = mm^1, \quad c = nn^1$$

$$\ell^1 = \frac{a}{\ell}, m^1 = \frac{b}{m}, n^1 = \frac{c}{n}$$

$$\lambda f = mn^1 + m^1 n = \frac{cm}{n} + \frac{bn}{m}$$

$$= \frac{cm^2 + bn^2}{mn}$$

$$\lambda g = \frac{an^2 + c\ell^2}{n\ell} = \lambda h = \frac{am^2 + b\ell^2}{\ell m}$$

$$\frac{cm^2 + bn^2}{fmn} = \frac{an^2 + c\ell^2}{gn\ell} = \frac{am^2 + b\ell^2}{h\ell m}$$

2.4.1 Condition for Tangency

22. Find the condition that the plane lx + my + nz =0.

Should touch the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

Sol :

The plane is lx + my +nz = 0 ... (1)

cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ ... (2)

If (α, β, γ) be the point of contact, the tangent plane

$$x(a\alpha + h\beta + g\gamma) + y(h\alpha + b\beta + f\gamma) + z(g\alpha + f\beta + c\gamma) = 0$$

$$\frac{a\alpha + h\beta + g\gamma}{\ell} = \frac{h\alpha + b\beta + f\gamma}{m} = \frac{g\alpha + f\beta + c\gamma}{n} = k$$

hence $a\alpha + h\beta + g\gamma = \ell k$

$$a\alpha + h\beta + g\gamma - \ell k = 0 \qquad \text{... (a)}$$

$$h\alpha + b\beta + fy = mk$$

$$\Rightarrow h\alpha + b\beta + fy - mk = 0 \qquad \text{... (b)}$$

$$g\alpha + f\beta + c\gamma = nk$$

$$\Rightarrow g\alpha + f\beta + c\gamma - nk = 0 \qquad \text{... (c)}$$

Also, since (α, β, γ) lies on the plane (1) we have

$$\begin{vmatrix} a & h & g & \ell \\ h & b & f & m \\ g & f & c & n \\ \ell & m & n & 0 \end{vmatrix} = 0$$

The determinant

$$A\ell^2 + Bm^2 + Cn^2 + 2Fmn + 2Gn\ell + 2H\ell m = 0$$

Where A, B, C, F,G, H are the co-factors of a, b, c, f, g, h

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$A = bc - f^2 \qquad B = ca - g^2 \qquad C = ab - h^2$

$F = gh - af \qquad G = hf - bg \qquad H = fg - ch$

2.4.2 Reciprocal Cones

23. Find the locus of line through the vertex of the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$. Perpendicular to its tangent planes, lx + my + nz =0.

Sol :

The line through the vertex of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{... (1)}$$

and the tangent plane $lx + my + nz = 0$ ... (2)

Let the tangent plane to the cone (1)

$$A\ell^2 + Bm^2 + Cn^2 + 2Fmn + 2Gn\ell + 2H\ell m = 0 \quad \text{... (3)}$$

The line through the vertex perpendicular to the tangent plane (2) is

$$\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n} \quad \text{....... (4)}$$

Eliminating l, m, n between (3) & (4)

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \quad \text{... (5)}$$

lines through the origin perpendicular to the tangent planes to the cone (5)

We have to substitute for A, B, C, F, G, H

The determint $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$

BC = F^2 = a D

CA – G^2 = bD

AB – H^2 = CD

GH – AF = fD, HF – BG = gD

FG – CH = hD

$\therefore\ D = abc + 2fgh + af^2 - bg^2 - ch^2$

The required locus for the cone (5) is $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

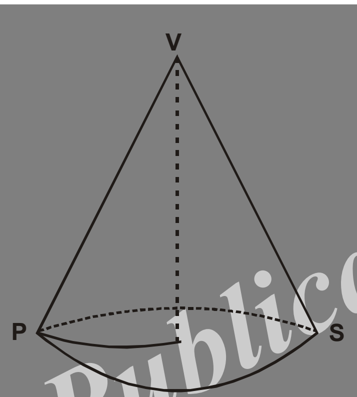
2.5 The Right Circular Cone

The surface generated by a straight line which passes through a fixed point and intersecting a given curve as touching a given surface, in called a cone.

The fixed point is called the vertex and the given curve the guiding curve of the cone.

An individual straight line on the surface of a cone is called a generator.

Thus, a cone is the set of lines called generators through a given point.



24. Find the condition that the plane ux + vy + wz =0 may touch the cone $ax^2 + by^2 + cz^2 = 0$.

Sol :

Equation to the normal to the given plane is $\frac{x}{u} = \frac{y}{v} = \frac{z}{w}$ ... (1)

equation to the Reciprocal cone of $ax^2 + by^2 + cz^2 = 0$ ... (2)

is $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ ... (3)

Now the plane touches the cone (2)

$\Rightarrow$ the normal of the plane lies on cone (2)

$\Rightarrow \frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = 0$ which is required condition.

25. Show that a quadric cone be found to touch any five planes which meet at a point provided no three of them intersect in a line find the equation of the cone which touches the three co-ordinates planes and the planes.

$x + 2y + 3z = 0, 2x + 3y + 4z = 0.$

Sol :

The equation to the cone touching the three axes can be taken as

$$\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0 \quad \text{... (1)}$$

It's Reciprocal cone is $fyz + gzx + hxy = 0$ ... (2)

The planes $x + 2y + 3z = 0$ and $2x + 3y + 4z = 0$

touch cone (1) $\Leftrightarrow$ their normal lies on (2)

(1,2,3) and (2, 3, 4) satisfy (2)

The Reciprocal equation of the cone is $fyz + gzx + hxy = 0$

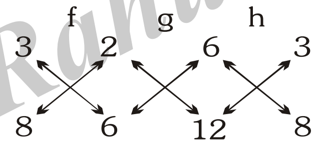
$$(1,2,3) \Rightarrow f(2)(3) + g(3)(1) + h(1)(2) = 0$$

$$6f + 3g + 2h = 0 \quad \text{... (3)}$$

$$(2,3,4) \Rightarrow f(3)(4) + g(4)(2) + h(2)(3) = 0$$

$$12f + 8g + 6h = 0 \quad \text{... (4)}$$

Solving (3) & (4)



$$\frac{f}{18-16} = \frac{g}{24-36} = \frac{h}{48-36}$$

$$\frac{f}{2} = \frac{g}{-12} = \frac{h}{12}$$

$$\frac{f}{1} = \frac{g}{-6} = \frac{h}{6}$$

$$\sqrt{x} + \sqrt{-6y} + \sqrt{6z} = 0$$

$$\Rightarrow (x)^{1/2} + (-6y)^{1/2} + (6z)^{1/2} = 0$$

26. If $x = \frac{1}{2}y = z$ represents one by a set of three mutually perpendicular generators of the cone 11yz + 6zx –14xy =0 find the equation of the other two.

Sol :

The given cone is 11yz +6zx –14xy =0

The plane through the vertex of the cone and perpendicular to the generator.

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1} \quad \ldots (1)$$

is x + 2y + z =0 the other two generators perpendicular to (1) are the lines of intersection of 11yz + 6 zx – 14xy =0 and x + 2y + z =0

It l, m, n be the direction ratio of one of the common lines

Then $11mn + 6n\ell - 14\ell m = 0 \quad \ldots (2)$

$\ell + 2m + n = 0$

$n = -\ell - 2m$

Sub $n = -\ell - 2m$ in (2)

$11m(-\ell - 2m) + 6(-\ell - 2m)\ell - 14\ell m = 0$

$-37\ell m - 22m^2 - 6\ell^2 = 0$

$6\ell^2 + 37\ell m + 22m^2 = 0$

$(2\ell + 11m)(3\ell + 2m) = 0$

$\Rightarrow 2\ell + 11m = 0$ or $3\ell + 2m = 0$

Solving $\ell + 2m + n = 0$

$\frac{\ell}{-11} = \frac{m}{2} = \frac{n}{7}$

Solving $\ell + 2m + n = 0$

$3\ell + 2m + 0.n = 0$

$\frac{\ell}{-2} = \frac{m}{3} = \frac{n}{-4}$

$\therefore$ The other two perpendicular generator are

$$\frac{x}{-11} = \frac{y}{2} = \frac{z}{7} \text{ and } \frac{x}{2} = \frac{y}{-3} = \frac{z}{4}$$

27. If the plane 2x–y + cz =0 cuts cone yz + zx + xy =0 in perpendicular lines, find the value of c.

Sol :

Given plane of equation is $2x - y + cz = 0$ and cone of equation

$$yz + zx + xy = 0 \quad \ldots (1)$$

contains sets of three mutually perpendicular generators.

$2x - y + cz = 0$ cuts (1) in perpendicular lines.

$\Rightarrow$ The normal of the plane lies on it.

$\Rightarrow$ (2, –1, c) must satisfy the cone equation (1)

$\Rightarrow$ $(-1)\,c + (c)\,(2) + (2)\,(-1) = 0$

$-c + 2c - 2 = 0$

$\therefore$ C = 2.

28. Find the locus of point from which three mutually perpendicular lies can be drawn to intersect the central conic $ax^2 + by^2 = 1, z = 0$

Sol :

Let the point P be (x_1, y_1, z_1)

Any line through P is $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = k \quad \ldots (1)$

Any point on the line is $\frac{x - x_1}{\ell} = k$

$$x = \ell k + x_1$$

$$\frac{y - y_1}{m} = k \Rightarrow z = nk + z_1$$

$$\frac{z - z_1}{n} = k \Rightarrow y = mk + y_1$$

$\therefore (\ell k + x_1, mk + y_1, nk + z_1)$

the point lies on the base curve $ax^2 + bx^2 = 1$, $z = 0$

$\Leftrightarrow a(x_1 + \ell k)^2 + b(mk + y_1) = 1,\; z_1 + nk = 0$

eliminating k. from $z_1 + nk = 0$

$$k = \frac{-z_1}{n}$$

Sub $k = \frac{-z_1}{n}$ in above equation

$$a\left(x_1 - \frac{\ell z_1}{n}\right)^2 + b\left(\frac{-z_1 m}{n} + y_1\right)^2 = 1$$

$$a\left(x_1 - \frac{\ell z_1}{n}\right)^2 + b\left(y_1 - \frac{z_1 m}{n}\right)^2 = 1$$

$$a\left(nx_1 - \ell z_1\right)^2 + b\left(ny_1 - z_1 m\right)^2 = n^2$$

Using (1) to the cone is

$$a\left[x_1\left(z - z_1\right) - z_1\left(x - x_1\right)\right]^2 + b\left[y_1\left(z - z_1\right) - z_1\left(y - y_1\right)\right]^2 = \left(z - z_1\right)^2$$

This contains three mutually perpendicular generators.

$\Rightarrow$ coeff of x_2 + coeff of y^2 + coeff of z^2 =0

$\Rightarrow az_1^{\,2} + bz_1^{\,2} + ax_1^{\,2} + by_1^{\,2} - 1 = 0$

$\therefore$ locus of P is $a\left(x^2 + z^2\right) + b\left(y^2 + z^2\right) = 1$.

29. Find the equation of the right circular cone with its vertex at the origin, axis along z–axis and semi vertical angle α.

Sol :

The equation of the right circular cone with vertex at (0,0,0) and where axis is the z-axis and semi vertical α is $x^2 + y^2 = z^2 \tan^2\alpha$.

Since d.r's of the z-axis are (0,0,1)

l = 0, m = 0, n =1

$\therefore$ The equation to the right circular cone is $\left(x^2 + y^2 + z^2\right)\cos^2\alpha = z^2$

$$\Rightarrow \left(x^2 + y^2\right) = z^2\left(\sec^2\alpha - 1\right)$$

$$\left(x^2 + y^2\right) = z^2\left(\tan^2\alpha\right)$$

30. Find the equation of the right circular cone whose vertex is origin, axis of the line x =t, y = 2t, z = 3t and whose semi-vertical angle is 60^0.

Sol :

Vertex at origin is (0,0,0)

Equation of the axis is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}=t$

$\Rightarrow$ D.r's of the axis (*l*, m, n) = (1,2,3)

equation to the required cone is

$$\Rightarrow\left[(x-0)^2+(y-0)^2+(z-0)^2\right]\left[1^2+2^2+3^2\right]\cos^2 60^0=\left[1(x-0)+2(y-0)+3(z-0)\right]^2$$

$$\Rightarrow\left(x^2+y^2+z^2\right)\left[14\left(\frac{1}{2}\right)^2\right]=(x+2y+3z)^2$$

$$\Rightarrow\left(x^2+y^2+z^2\right)\frac{\overset{7}{\cancel{14}}}{\underset{2}{\cancel{4}}}=(x+2y+3z)^2$$

$$\Rightarrow 7\left(x^2+y^2+z^2\right)=2\left(x^2+4y^2+9z^2+4xy+12yz+6xz\right)$$

$$\Rightarrow 7x^2+7y^2+7z^2-2x^2-8y^2-18z^2-8xy-24yz-12xz=0$$

$$\Rightarrow 5x^2-y^2-11z^2-8xy-24yz-12xz=0\,.$$

31. Find the equation of the cone generated by rotating the line $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$ about the line $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ as axis.

Sol :

Given line pass through the origin $\Rightarrow$ vertex is the origin.

r's of axis are (a, b, c)

Semi vertical angle = angle between the generators and the axis.

$$\Rightarrow \cos\theta=\frac{a\ell+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{\ell^2+m^2+c^2}} \quad \text{... (1)}$$

$\therefore$ Equations to the cone is

$$\left[(x-0)^2+(y-0)^2+(z-0)^2\right](a^2+b^2+c^2)\cos^2\theta=\left[a(x-0)+b(y-0)+c(z-0)^2\right]$$

Using (1) we have

$$(x^2+y^2+z^2)(a\ell+bm+cn)^2=(\ell^2+m^2+n^2)(ax+by+cz)^2$$

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32. Find the equation of the right circular cone which passes through the point (1,1,2) and had its vertex at the origin, axis the line $\frac{x}{2}=\frac{y}{-4}=\frac{z}{3}$.

Sol :

The given vertex at the origin (0,0,0)

Equation to the axis is $\frac{x}{2}=\frac{y}{-4}=\frac{z}{3}$

(l, m, n) = (2, –4, 3)

Let the semi vertical angle be α.

Then the equation to the cone is

$$(x-0)^2+(y-0)^2+(z-0)^2\left[2^2+(-4)^2+3^2\right]\cos^2\alpha=\left[2(x-0)+-4(y=0)+3(z-0)\right]^2$$

$$\Rightarrow x^2+y^2+z^2(4+16+9)\cos^2\alpha=(2x-4y+3z)^2$$

$$\Rightarrow (a^2+y^2+z^2)29\cos^2\alpha=(2x-4y+3z)^2$$

The cone passes through the point (1,1,2)

$$\Leftrightarrow (1+1+2^2)29\cos^2\alpha=(2(1)-4(1)+3(2))^2$$

$$\therefore 6\times 29\cos^2\alpha=16$$

$$\Rightarrow \cos\alpha=\frac{16}{174}=\frac{8}{87}$$

$$(x^2+y^2+z^2)29\left(\frac{8}{87}\right)=(2x-4y+3z)^2$$

$$\Rightarrow 4x^2+40y^2+19z^2-72y^2+36zx-48xy=0$$

33. Find the equation of the right circular cone generated by straight lines drawn from the origin to cut the circle through the three points (1,2,2), (2, 1,-2) and (2, -2, 1).

Sol :

Let A = (1, 2, 2)

B = (2, 1, –2)

C = (2, –2, 1)

The right circular cone generated by straight line, drawn from the origin to cut the circle.

$$OA \Rightarrow \sqrt{(1-0)^2+(2-0)^2+(2-0)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$OB \Rightarrow \sqrt{(2-0)^2+(1-0)^2+(-2-0)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$OC \Rightarrow \sqrt{(2-0)^2+(-2-0)^2+(1-0)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$\therefore$ Clearly OA = OB = OC = 3

$\Rightarrow$ A, B, C lies on the sphere centre O and of radius 3.

Equation to the sphere is $x^2+y^2+z^2=3^2$

Let the plane through A be $a(x-1)+b(y-2)+c(z-2)=0$

It passes through $B(2,1,-2) \Leftrightarrow a-b-4c=0$ ... (1)

$c(2,-2,1) \Leftrightarrow a-4b-c=0$ ... (2)

Solving (1) & (2)

	a		b		c	
-1		-4		1		-1
-4		-1		1		-4

$$\frac{a}{1-16}=\frac{b}{-4+1}=\frac{c}{-4+1} \Rightarrow \frac{a}{-15}=\frac{b}{-3}=\frac{c}{-3}$$

$$\Rightarrow \frac{a}{5}=\frac{b}{1}=\frac{c}{1}$$

Equation to the plane is $5(x-1)+1(y-2)+1(z-2)=0$

$$5x-5+y-2+z-2=0$$

$$5x-y+z-9=0$$

$$5x-y+z=9$$

Now homogenous. The equation of the sphere with the help of plane

$$x^2+y^2+z^2=9\left(\frac{5x+y+z}{9}\right)^2$$

$$x^2+y^2+z^2=\left(\frac{5x+y+z}{9}\right)^2$$

$$9x^2+9y^2+9z^2=25x^2+y^2+z^2+10xy+2yz+10zx$$

$$9x^2-25x^2+9y^2-y^2+9z^2-z^2-10xy-2yz-10xz=0$$

$$-16x^2+8y^2+8z^2-10xy-2yz-10xz=0$$

$$8x^2-4y^2-4z^2+5xy+yz+5xz=0$$

34. Find the equation of the right circular cone whose vertex is $(3,2,1)$ axis the line $\frac{x-3}{4}=\frac{y-2}{1}=\frac{z-1}{3}$ and semi vertices angle 30^o

Sol :

Equation to the axis $\frac{x-3}{4}=\frac{y-z}{1}=\frac{z-1}{3}$

Semi vertical angle 30^0

$\therefore$ Equation to the cone is

$$\left[(x-3)^2+(y-2)^2+(z-1)^2\right](4^2+1^2+3^2)\cos^2 30$$

$$=\left[4(x-3)+1(y-2)+3(z-1)\right]^2$$

$$\Rightarrow (x^2+9-6x+y^2+4-4y+z^2+1-2z)(16+1+9)\cos^2 30$$

$$=(4x-12+y-2+3z-3)^2$$

$$\Rightarrow \left(x^2+y^2+z^2-6x-4y-2z+14\right)26\left(\frac{\sqrt{3}}{2}\right)^2=\left(4x+y+3z-17\right)^2$$

$$\Rightarrow \left(x^2+y^2+z^2-6x-4y-2z+14\right)\overset{13}{\cancel{26}}\left(\frac{3}{\underset{2}{\cancel{4}}}\right)$$

$$=\left(4x+y+3z-17\right)^2$$

$$\Rightarrow \left(x^2+y^2+z^2-6x-4y-2z+14\right)\frac{39}{2}=\left(4x+y+3z-17\right)^2$$

35. If α is the semi vertical angle of the right circular cone which passes through the lines oy, oz, x =y = z show that $\cos\alpha=\left(9-4\sqrt{3}\right)^{-1/2}$.

Sol :

Let (l, m, n) be right circular cone of oy are (0, 1, 0)

r's of oz are (0, 0, 1)

α is the angle between the axis and oy.

$$\cos\alpha=\frac{0.\ell+1m+0.n}{\sqrt{0+1+0}\sqrt{\ell^2+m^2+n^2}}=\frac{m}{\sqrt{\ell^2+m^2+n^2}} \quad \text{... (2)}$$

Also α in the angle between the axis and oz

$$\Rightarrow \cos\alpha=\frac{0.\ell+0.m+1.n}{\sqrt{0+0+1}\sqrt{\ell^2+m^2+n^2}}$$

$$=\frac{n}{\sqrt{\ell^2+m^2+n^2}} \quad \text{... (2)}$$

from (1) & (2)

$$\frac{m}{\sqrt{\ell^2+\cancel{m^2}+n^2}}=\frac{n}{\sqrt{\ell^2+\cancel{m^2}+n^2}}$$

$$\Rightarrow m=n$$

Similarly angle between the axis and $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$ is

$$\cos\alpha = \frac{1.\ell + 1.m + 1.n}{\sqrt{1+1+1}\sqrt{\ell^2+m^2+n^2}} = \frac{\ell+m+n}{\sqrt{3}\sqrt{\ell^2+m^2+n^2}}$$

$$\frac{\ell+m+n}{\sqrt{3\left(\ell^2+m^2+n^2\right)}} \quad \ldots (3)$$

Equating (1) and (3)

$$\frac{m}{\sqrt{\ell^2+m^2+n^2}} = \frac{\ell+m+n}{\sqrt{3\left(\ell^2+m^2+n^2\right)}}$$

$$m = \frac{\ell+m+n}{\sqrt{3}}$$

$$\Rightarrow \sqrt{3}m = \ell+m+n$$

$$\ell+m+n-\sqrt{3}m = 0$$

$$\ell+m\left(1-\sqrt{3}\right)+n = 0$$

$$\ell+m\left(1-\sqrt{3}\right)+m = 0 \quad (\therefore\ m = n)$$

$$= \ell+m-\sqrt{3}m+m = 0$$

$$\ell+2m-\sqrt{3}m = 0$$

$$\ell+m\left(2-\sqrt{3}\right) = 0$$

$$\ell = m\left(\sqrt{3}-2\right)$$

$$\Rightarrow \frac{\ell}{\sqrt{3}-2} = \frac{m}{1} = \frac{n}{1}$$

from (1) $\cos\alpha = \frac{1}{\sqrt{\left(\sqrt{3}-2\right)^2+1+1}} = \frac{1}{\sqrt{\left(\sqrt{3}\right)^2+4-4\sqrt{3}+1+1}}$

$$= \frac{1}{\sqrt{3+4-4\sqrt{3}+2}}$$

$$= \frac{1}{\sqrt{3+4-4\sqrt{3}+2}}$$

$$= \frac{1}{\sqrt{9-4\sqrt{3}}}$$

$$= \frac{1}{\left(9-4\sqrt{3}\right)^{1/2}}$$

$$\cos\alpha = \left(9-4\sqrt{3}\right)^{-1/2}$$

36. Show that the cone whose vertex is the origin and which passes through the curve of intersection of the surface $2x^2 - y^2 + 2z^2 = 3d^2$ and any plane at a distance d, from the origin has three mutually perpendicular generators.

Sol :

Equation to any plane at a distance d from the origin is $\ell x + my + nz = d$ ... (1)

Where l, m, n are the actual dc's of normal to the plane.

Homogenizing the equation of the sphere with that of the plane,

We have $2x^2 - y^2 + 2z^2 = 3d\left(\frac{\ell x + my + nz}{d}\right)^2$

Coefficient of x^2 + coefficient of y^2 + coefficient of z^2

$$\left(2-3\ell^2\right) - 1 - 3m^2 + \left(2-3n^2\right) = 3 - 3\left(\ell^2 + m^2 + n^2\right)$$

$$= 3 - 3\,(1)$$

$$= 0$$

hence plane cuts the cone in three mutually perpendicular generators.

37. If $\frac{x}{1}=\frac{y}{1}=\frac{z}{2}$ be one of a set of three mutually perpendicular generators of the cone $3yz-2zx-2xy=0$ find the equation of other two generators.

Sol :

Let the line perpendicular to given line be $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$ ... (1)

Also $\ell+m+2n=0$ ... (2)

(1) lies on the cone $\Leftrightarrow 3mn-2n\ell-2\ell m=0$ ... (3)

Eliminating l from (2) & (3)

Then, from (2)

$\ell=-m-2n$

Sub in (3)

$\Rightarrow 3mn-2n(-m-2n)-2(-m-2n)m=0$

$3mn+2mn+4n^2+2m^2+4mn=0$

$9mn+4n^2+2m^2=0$

$\Rightarrow 4n^2+9mn+2m^2=0$

$(2m+n)(m+4n)=0$

$\Rightarrow 2m+n=0$ ... (4)

$m+4n=0$ ... (5)

Solving $l+m+2n=0$ from (2)

and from (4) $0.\ell+2m+n=0$

	l		m		n	
$\therefore$ 1		2		1		1
2		1		0		2

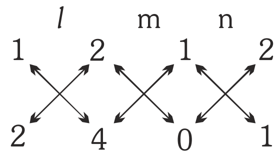
$\frac{\ell}{-3}=\frac{m}{-1}=\frac{n}{2}$

$$\frac{\ell}{3} = \frac{m}{1} = \frac{n}{-2}$$

Now solving $l + m + zn = 0$

and $0.\ell + m + 4n = 0$

from (3) and (5)



$$\frac{\ell}{4-2} = \frac{m}{0-4} = \frac{n}{1-0}$$

$$\frac{\ell}{2} = \frac{m}{-4} = \frac{n}{1}$$

$\therefore$ equation to the generator is $\frac{x}{2} = \frac{y}{-4} = \frac{z}{1}$

38. Show that the mutually perpendicular tangent lines can be drawn to the sphere $x^2 + y^2 + z^2 = r^2$ from any point on the surface $2(x^2 + y^2 + z^2) = 3r^2$.

Sol :

Let (x_1, y_1, z_1) be a point on the sphere $2(x^2 + y^2 + z^2) = 3r^2$

Then $2(x_1^2 + y_1^2 + z_1^2) = 3r^2$

Equation to the enveloping cone $x^2 + y^2 + z^2 = r^2$ with vertex at (x_1, y_1, z_1) is $S_1^2 = S.S_1$

$$\Rightarrow (xx_1 + yy_1 + zz_1 - r^2)^2 = (x^2 + y^2 + z^2 - r^2)(x_1{}^2 + y_1{}^2 + z_1{}^2 - r^2)$$

$$= (x^2 + y^2 + z^2 - r^2)\left(\frac{3}{2}r^2 - r^2\right)$$

$$= \left(x^2 + y^2 + z^2 - r^2\right)\frac{r^2}{2}$$

$$\Rightarrow 2\left(xx_1 + yy_1 + zz_1 - r^2\right)^2 = r^2\left(x^2 + y^2 + z^2 - r^2\right)$$

coefficient of x^2 + coefficient of y^2 + coefficient of z^2

$$= \left(2x_1^2 - r^2\right) + \left(2y_1^2 - r^2\right) + \left(2z_1^2 - r^2\right)$$

$$= 2\left(x_1^2 + y_1^2 + z_1^2\right) - 3r^2 = 2\left(x_1^2 + y_1^2 + z_1^2\right) - 3r^2$$

$$= 0$$

hence the enveloping cones has three mutually perpendicular generators.

$\therefore$ The given sphere has three mutually perpendicular tangent lines.

2.6 The Cylinder

Let P be a point and L be a line through P, with direction ratio l, m, n if S in a surface such that $P \in S \Rightarrow LCS$ then S is called a cylinder. The line L is called a generator of the cylinder.

- A cylinder is a surface generated by a straight line which is always parallel to a fixed line an is subject to one more condition that it may intersect a given curve or touch a given surface.
- The given curve is called the Guiding curve.

2.6.1 Equation of a cylinder

39. Find the equation of the cylinder whose generators intersect the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0$ and are parallel to the line $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$.

Sol :

Let $P(\alpha, \beta, \gamma)$ be a point on the cylinder equation to the generator through P is

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = k$$

Any point on the line is $(\alpha + \ell k, \beta + mk, \gamma + nk)$

This point P lies on the conic

$$\Leftrightarrow a(\alpha + \ell k)^2 + 2h(\alpha + \ell k)(\beta + mk) + b(\beta + mk)^2$$

$$+2g(\alpha + \ell k) + 2f(\beta + mk) + c = 0$$

and $\gamma + nk = 0 \Rightarrow k = \frac{-\gamma}{n}$

eliminating k from (1)

$$a\left(\alpha - \ell\frac{\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{\gamma}{n}k\right)^2\left(\beta - \frac{\gamma}{n}m\right) + b\left(\beta - \frac{\gamma m}{n}\right)^2$$

$$+2g\left(\alpha - \frac{\gamma}{n}\ell\right) + 2f\left(\beta - \frac{\gamma}{n}m\right) + c = 0$$

$$a\left(x - \frac{\ell z}{n}\right)^2 + 2h\left(x - \frac{\ell z}{n}\right)\left(y - \frac{mz}{n}\right) + b\left(y - \frac{mz}{n}\right)^2$$

$$+2g\left(x - \frac{\ell z}{n}\right) + 2f\left(y - \frac{mz}{n}\right) + c = 0$$

$$a(nx - \ell z)^2 + 2h(nx - \ell z)(ny - mz) + b(ny - mz)^2$$

$$+2g(nx - \ell z) + 2f(yn - mz) + n^2c = 0$$

40. Find the equation of the cylinder whose generators are parallel to $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is the ellipse $x^2 + 2y^2 = 1$, $z=3$.

Sol :

Let (α, β, γ) be any point on the surface of the cylinder.

So that the equations of its generators through this point are $\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{z-\gamma}{3}$

This line meets the plane z =3 at the point given by

$$\frac{x-\alpha}{1}=\frac{y-\beta}{-2}=\frac{3-\gamma}{3}$$

$$\frac{x-\alpha}{1}=\frac{3-\gamma}{3}$$

$$x=\frac{3-\gamma}{3}+\alpha$$

and $\frac{y-\beta}{-2}=\frac{3-\gamma}{3}$

$$y=-2\left(\frac{3-\gamma}{3}\right)+\beta$$

$$y=\beta+\frac{2\gamma-6}{3}$$

$$z = 3$$

ie $\left(\alpha+\frac{3-\gamma}{3},\beta+\frac{2\gamma-6}{3},3\right)$

This point will lie on the surface $x^2 + 2y^2 = 1$

$$\left(\alpha+\frac{3-\gamma}{3}\right)^2+2\left(\beta+\frac{2\gamma-6}{3}\right)^2=1$$

$$(3\alpha+3-\gamma)^2+2(3\beta+2\gamma-6)^2=9$$

hence locus of the point (α,β,γ) will be $(3x-z+3)^2+2(3y+2z-6)^2=9$.

41. Find the equation to the cylinder whose generators are parallel to $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and guiding curve is $x^2 + y^2 = 16$, z =0.

Sol :

Given the generators as (1, 2, 3)

Let $p(x_1,y_1,z_1)$ be a point on the cylinder

$\therefore$ Equation to the generator be $\frac{x - x_1}{1} = \frac{y - y_1}{2} = \frac{z - z_1}{3} = r$

Any point on the line is $\frac{x - x_1}{1} = r \Rightarrow x = r + x_1$

$\frac{y - y_1}{2} = r \Rightarrow y = 2r + y_1$

$\frac{z - z_1}{3} = r \Rightarrow z = 2r + z_1$

$\therefore (r + x_1, 2r + y_1, 2r + z_1)$

This point lies on the curve $x^2 + y^2 = 16,\ z=0$

$\Leftrightarrow (r + x_1)^2 + (y_1 + 2r)^2 = 16,\ 3r + z_1 = 0$

$$r = \frac{-z_1}{3}$$

Sub $r = \frac{-z_1}{3}$ in $(x_1 + r)^2 + (y_1 + 2r)^2 = 16$

$$\Rightarrow \left(x_1 - \frac{z_1}{3}\right)^2 + \left(y_1 - 2\frac{z_1}{3}\right)^2 = 16$$

$$(3x_1 - z_1)^2 + (3y_1 - 2z_1)^2 = 9 \times 16$$

$9{x_1}^2 + {z_1}^2 - 6x_1z_1 + 9{y_1}^2 + 4{z_1}^2 - 12y_1z_1 = 144$

$\therefore$ The locus of P is the cylinder

$\Rightarrow 9x^2 + 9y^2 + z^2 - 6xz - 12yz - 144 = 0$

42. Find the equation of the cylinder whose generators are parallel to z-axis and guiding curve is given by $ax^2+by^2+cz^2 = 1$, $lx+my+nz=P$.

Sol :

The equation of the Required cylinder is obtained by eliminating z between the equation $ax^2 + by^2 + cz^2 = 1$

and $lx + my + nz = P$

$nz = p - lx - my$

$$z = \frac{p - \ell x - my}{n}$$

Sub z value in $ax^2 + by^2 + cz^2 = 1$.

$$\Rightarrow ax^2 + by^2 + c\left(\frac{p - \ell x - my}{n}\right)^2 = 1$$

$$n^2ax^2 + n^2by^2 + c(p - \ell x - my)^2 = n^2$$

$$\therefore\ ax^2n^2 + by^2n^2 + c(p - \ell x - my)^2 = n^2$$

which is required cone.

43. Find the equation of the cylinder whose generators intersects the curve $ax^2 + by^2 = 2z$, $lx + my + nz = p$ and are parallel to the z-axis.

Sol :

The Equation of the cylinder whose generators, intersects the curve

$$ax^2 + by^2 = 2z \quad \text{... (1)}$$

$$lx + my + nz = P \quad \text{... (2)}$$

Since the axis of the cylinder is parallel to z-axis by eliminating z.

from (2) $\ell x + my + nz = P$

$$nz = p - \ell x - my$$

$$z = \frac{p - \ell x - my}{n}$$

Sub in (1) then we get

$$ax^2 + by^2 = 2\left(\frac{p - \ell x - my}{n}\right)$$

$$n\,ax^2 + n\,by^2 = 2p - 2\ell x - 2my$$

$$n(ax^2 + by^2) - 2p + 2\ell x + 2my = 0$$

2.6.2 Enveloping Cylinder

The set of parallel tangent lines with direction ratios (*l*, m, n) to a surface form a cylinder called the enveloping cylinder in the direction of (l, m, n).

Theorem

The equation to the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = a^2$ in the direction of (l, m, n) is $(\ell x + my + nz)^2 = (\ell^2 + m^2 + n^2)(x^2 + y^2 + z^2 - a^2)$**.**

Proof :

Let $p(x_1, y_1, z_1)$ be a point on the tangent line of the given surface.

$\therefore$ The equation to the line through P with (*l*, m, n) be

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \text{... (1)}$$

The point $(x_1 + \ell r, y_1 + mr, z_1 + nr)$ of the line lies on the surface.

$$x^2 + y^2 + z^2 - a^2 = 0 \quad \text{... (2)}$$

$$(x_1 + \ell r)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 - a^2 = 0$$

$$x_1^2 + \ell^2 r^2 + 2x_1 \ell r + y_1^2 + m^2 r^2 + 2mry_1 + z_1^2 + n^2 r^2 + 2z_1 nr - a^2 = 0$$

$$(x_1^2 + y_1^2 + z_1^2 - a^2) + r^2(\ell^2 + m^2 + n^2) + 2r(x_1 \ell + my_1 + nz_1) = 0$$

The line (1) is a tangent line to (2)

$$\Rightarrow (\ell x_1 + my_1 + nz_1)^2 - (\ell^2 + m^2 + n^2)(x_1^2 + y_1^2 + z_1^2 - a^2) = 0$$

$\therefore$ The locus of P is type enveloping cylinder

$$(\ell x + my + nz)^2 = (\ell^2 + m^2 + n^2)(x^2 + y^2 + z^2 - a^2)$$

2.7 The Right Circular Cylinder

Let (l, m, n) be the direction number of the normal to the plane π containing a circle c Let L be a normal line to the plane π and passing through a point P.

It S is the surface such that $P \in C \Rightarrow L \subset S$ then S called the right circular cylinder the normal through the centre of the circle is called the axis of the cylinder and the radius of the circle is called the radius of the cylinder.

- The equation to the right circular cylinder with radius or and axis and the line $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is

$$\left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2\right](\ell^2+m^2+n^2)=\left[\ell(x-a)+m(y-\beta)+n(z-r)\right]^2$$

44. Find the right cylinder whose radius is 2 and axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$.

Sol :

Let $P(x_1,y_1,z_1)$ be any point on the cylinder the length of the perpendicular from $P(x_1,y_1,z_1)$ to the given line must be equal to the radius.

$$2^2(2^2+1^2+2^2)=[2(y_1-2)-1(z_1-3)]^2+[2(z_1-3)-2(x_1-1)]^2$$
$$+[1(x_1-1)-2(y_1-2)]^2$$

$$=4[4+1+4]=(2y_1-4-z_1+3)^2+(2z_1-6-2x_1+2)^2+(x_1-1-2y_1+4)^2$$

$$36=(2y_1-z_1-1)^2+(2z_1-2x_1-4)^2+(x_1-2y_1+3)^2$$

$\therefore$ The required equation of the locus of $P(x_1,y_1,z_1)$ is

$$(2y-z-1)^2+(2z-2x-4)^2+(x-2y+3)^2=36$$

$$\Rightarrow 4y^2+z^2+1-4yz+2z-2y+4z^2+4x^2+4x^2+16-8zx+16x$$
$$-16z+x^2+4y^2+9-4xy-12y+6x=36$$

$$5x^2+8y^2+5z^2-4xy-4yz-8zx+2zx-16y-14z-10=0.$$

---

45. The axis of a right circular cylinder of radius 2 is $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$ shows that its equation is $10x^2+5y^2+13z^2-12xy-6yz-47x-8x+30y-74z+59=0$

Sol :

Equation to the axis of the cylinder $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$

$\therefore$ Equation to the circular cylinder of radius 2 is

$$\left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2\right](\ell^2+m^2+n^2)$$

$$=\left[\ell(n-\alpha)+m(y-\beta)+n(z-\gamma)\right]^2$$

$$=\left[(x-1)^2+(y-0)^2+(z-3)^2-2^2\right](2^2+3^2+1^2)$$

$$=\left[2(x-1)+3(y-0)+1(z-3)\right]^2$$

$$(x^2+1-2x+y^2+z^2+9-6z-4)(4+9+1)$$

$$=(2x-2+3y+z-3)^2$$

$$(x^2+y^2+z^2-2x-6z+6)(14)=(2x+3y+z-5)^2$$

$$14x^2+14y^2+14z^2-28x+84z+84$$

$$=4x^2+9y^2+z^2+12xy+6yz+4xz+25$$

$$14x^2+14y^2+14z^2-28x-84z+84-4x^2-9y^2-z^2-12xy$$

$$-6yz-4xz-25=0$$

$$\Rightarrow 10x^2+5y^2+13z^2-12xy-6yz-4xz-8x+30y-74z+59=0$$

46. Find the equation of a circular cylinder whose guiding curve $x^2+y^2+z^2=9, x-y+z=3$.

Sol :

We know that,

The radius circular cylinder is equal to the radius of the guiding curve and the axis of the cylinder is a line passing through the centre of the circle and.

Hence of the sphere and perpendicular to the plane of the circle.

Hence, the radius of the sphere 3 Length of the perpendicular from the centre O (0, 0, 0) to the given plane.

$$= \frac{-3}{\sqrt{1+1+1}} = \frac{-3}{\sqrt{3}} \Rightarrow \frac{-3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{-\cancel{3}\sqrt{3}}{\cancel{3}} = -\sqrt{3}$$

$\therefore$ Radius of the circle $\sqrt{3^2 - 3} = \sqrt{6}$

The axis of the cylinder passes through (0,0,0) and is perpendicular to the plane $x - y + z = 3$

hence the equations are $\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$

Therefore, the equation of the cylinder is

$$\left(\frac{1}{\sqrt{3}}\right)^2 \left\{ \begin{vmatrix} y & z \\ -1 & 1 \end{vmatrix}^2 + \begin{vmatrix} z & x \\ 1 & 1 \end{vmatrix}^2 + \begin{vmatrix} x & y \\ 1 & -1 \end{vmatrix}^2 \right\} = \left(\sqrt{6}\right)^2$$

$$(y+z)^2 + (z-x)^2 + (-x-y)^2 = 18$$

$$y^2 + z^2 + 2yz + z^2 + x^2 - 2zx + x^2 + y^2 + 2xy = 18.$$

$$2x^2 + 2y^2 + 2z^2 + 2yz - 2zx + 2xy = 18.$$

$$x^2 + y^2 + z^2 + yz - zx + xy - 9 = 0.$$

47. Obtain the equation of the right circular cylinder described on the circle as guiding circle (1, 0, 0), (0, 1, 0),(0, 0,1).

Sol :

Circle through the three points.

The circle ABC can be taken as the intersection of sphere OABC and plane ABC, O being the origin.

$\therefore$ The equation of the circle ABC are

$$x^2 + y^2 + z^2 = x - y - z = 0 \quad \text{... (1)}$$

$$x + y + z = 1 \quad \text{... (2)}$$

and the axis of the cylinder will be perpendicular tot he plane (1)

Hence , the generators of the cylinder will have the direction ratios 1, 1, 1.

Let (α, β, γ) be any point on the cylinder .

Then the generator through (α, β, γ) is $\frac{x-\alpha}{1} = \frac{y-\beta}{1} = \frac{z-\gamma}{1} = r$

$\therefore \quad x = \alpha + r$

$y - \beta = r \Rightarrow y = r + \beta$

$z - \gamma = r \Rightarrow z = \gamma + r$

$\therefore (r+\alpha, r+\beta, r+\gamma)$ substituting this points in (1) and (2) then we get

$$(\alpha + r)^2 + (r+\beta)^2 + (r+\gamma)^2 - (r+\alpha) - (r+\beta) - (r+\gamma) = 0$$

$$r = \frac{(1-\alpha-\beta-\gamma)}{3} \quad \text{... (3)}$$

Eliminating r between (2) & (3) , we get

$$(\alpha^2 + \beta^2 + \gamma^2) - (\alpha\beta + \beta\gamma + \gamma\alpha) - 1 = 0$$

Generating it,

We get the equation as cone as $x^2 + y^2 + z^2 - xy - zx - yz - 1 = 0$.

48. Find the equation of the cone whose vertex is at the origin and the direction cosines of whose generators satisfy the relation $3l^2 - 4m^2 + 5n^2 = 0$ ·

Sol :

If *l*, m, n are the generators dying on the cone.

Then the equation to the generators passing through the origin O(0,0,0)

$$\Rightarrow \frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

$$\Rightarrow \frac{x - 0}{\ell} = \frac{y - 0}{m} = \frac{z - 0}{n}$$

$$\Rightarrow \frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$$

eliminating *l*, m, n from $3\ell^2 - 4m^2 + 5n^2 = 0$

The equation of the cone is $3x^2 - 4y^2 + 5z^2 = 0$

Choose the Correct Answer

1. If a right circular cone has three mutually perpendicular generators then semi-vertical angle is ______. [d]

   (a) $\tan^{-1} 2$ (b) $\pi/2$

   (c) $\pi/4$ (d) $\tan^{-1}\sqrt{2}$

2. The equation of a right cone with vertex at the origin, axis the x-axis and semi vertical angle θ is $y^2 + z^2 =$ ______. [c]

   (a) $y^2 \sin^2\theta$ (b) $z^2 \cos^2\theta$

   (c) $x^2 \tan^2\theta$ (d) $x^2 \sin^2\theta$

3. The straight line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ is a generator of the cone [d]

   (a) $x^2 + 2y^2 + 3z^2 = 0$ (b) $x^2 - 2y^2 - 3z^2 = 0$

   (c) $x2 + 2y^2 - 3z^2 = 0$ (d) $x^2 - 2y^2 + 3z^2 = 0$

4. The general equation to a cone which touches the coordinate planes is [b]

   (a) $a^2 + b^2 + c^2 - 2ab - 2bc - 2ca = 0$

   (b) $a^2x^2 + b^2y^2 + c^2z^2 = 2bcyz + 2cazx + 2abxy$

   (c) $ax^2 + by^2 + c^2 = bcyz + cazx + abxy$

   (d) None

5. All the generators of cylinder are [c]

   (a) Straight lines (b) Cone

   (c) Parallel straight line (d) None

6. The angle between the lines in which the planes x + y + z =0 cuts the cone $ayz + bzx + cxy = 0$ will be $\pi / 2$ if [d]

(a) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ (b) a – b – c =0

(c) $\frac{1}{a}-\frac{1}{b}-\frac{1}{c}=0$ (d) a + b + c =0

7. The enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ with generators perpendicular to x-axis meets the plane z =0 in [b]

(a) Sphere (b) Ellipse

(c) Straight line (d) None

8. The cone 5yz – 8zx – 3xy =0 has ______ mutually perpendicular generators. [b]

(a) 4 (b) 3

(c) 5 (d) 10

9. The locus of the lines through the vertex of a cone normal to the tangenet planes is called. [b]

(a) Enveloping cone (b) Reciprocal cone

(c) Right circular cone (d) None

10. The equation 3yz – 4zx + 5xy =0 represents a cone passing through each of the co-ordinate axes. [c]

(a) 4 (b) 5

(c) 3 (d) 2

Fill in the blanks

1. A cone of second degree can be found to pass through ______ concurrent lines.
2. The equation of the enveloping cone can be written as ______ .
3. Guiding curve of a right circular cylinder is ______ .
4. The line which generates the surface of the cylinder is called ______ .
5. Any line on the surface of a cylinder is called its ______ .
6. The equation $yz + zx + xy = 0$ represents ______ .
7. The General equation of a sphere ______ .
8. The locus of the lines through the vertex of a cone normal to the tangent planes is called ______ .
9. The tangent plane at any point of a cone passes through its ______ .
10. The equation of the cone with vertex at the origin is a ______ second degree equation in x, y and z.

ANSWERS

1. 5
2. $SS_1 = T$
3. Circle
4. Generator
5. Generators
6. A cone
7. $(x-x_1)(x-x_2)+(y-y_2)(y-y_2)+(z-z_1)(z-z_2)=0$
8. Reciprocal cone
9. Vertex
10. Homogenous

UNIT III

The Conicoid: The General Equation of the Second Degree- Intersection of Line with a Conicoid- Plane of contact- Enveloping Cone and Cylinder.

3.1 The General Equation of the Degree

Definition

Let S = $ax^2 + by^2 + Cz^2 + 2fyz + 2yzx + ahxy + 2ux + 2vy + 2wz + d = 0$ define a locus in space of the same locus cannot be defined by a first degree equation then the surface S is called a conicoid of a quadric.

➢ The general equation of the second degree can by transformation of co-ordinate axes. be reduced to any one of the following form.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Ellipsoid.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^1}=-1$ Imaginary Ellipsoid.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Hyperboloid of one sheet.

$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Hyperboloid of two sheets.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Elliptic Cylinder

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Hyperbolic Cylinder

$y^2 = 4ax$	Parabolic Cylinder
$y^2 = a^2$	Two real parallel planes
$y^2 = -a^2$	Two imaginary planes
$y^2 = 0$	Two coincident planes
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$	Elliptic paraboloid
$\frac{x^2}{z^2} - \frac{y^2}{b^2} = \frac{2z}{c}$	Hyperbolic paraboloid

3.2 Intersection of Line with a Conicoid

1. Find the points of intersection of the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ with the central conicoid $ax^2 + by^2 + cz^2 = 1$.

Sol :

The points of intersection of the line is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots (1)$$

Central conicoid is $ax^2 + by^2 + cz^2 = 1$ ... (2)

from (1)

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$$

$\therefore \quad \frac{x-\alpha}{l} = r \qquad \frac{y-\beta}{m} = r \qquad \frac{z-\gamma}{n} = r$

$x - \alpha = lr \qquad y - \beta = mr \qquad z - \gamma = nr$

$x = lr + \alpha \qquad y = \beta + mr \qquad z = nr + \gamma$

$\therefore$ A point $(lr + d, mr + \beta, nr + \gamma)$ on the line (1) also lie one the surface (2)

$\therefore$ from (2)

$a(lr + \alpha)^2 + b(mr + \beta)^2 + c(nr + \gamma)^2 = 1$

$\Leftrightarrow$ $a(l^2 r^2 + \alpha^2 + 2l r\alpha) + b(m^2 r^2 + \beta^2 + 2mr\beta) + c(n^2 r^2 + \gamma^2 + 2nr\gamma) = 1$

$\Leftrightarrow$ $a l^2 \gamma^2 + a\alpha^2 + 2l r\alpha a + bm^2 r^2 + b\beta^2 + 2bmr\beta + cn^2 r^2 + c\gamma^2 + 2cnr\gamma = 1$

$\Leftrightarrow$ $r^2(al^2 + bm^2 + cn^2) + 2r(a\alpha l + b\beta m + c\gamma n) + (a\alpha^2 + b\beta^2 + c\gamma^2) = 1$

$\Leftrightarrow$ $r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0$ ...(3)

Let r_1, r_2 be the two roots of (3)

Which we suppose to be real\

Then $(lr_1 + \alpha, mr_1 + \beta, nr_1 + \gamma)$, $(lr_2 + \alpha, mr_2 + \beta, nr_2 + \gamma)$

2. Find the points of intersection of the line $-\frac{1}{3}(x+5) = (y-4) = \frac{1}{7}(z-11)$ with the conicoid $12x^2 - 17y^2 + 7z^2 = 7$.

Sol :

The points of intersection of the line

$$-\frac{(x+5)}{3} = \frac{(y-4)}{1} = \frac{z-11}{7} \quad \text{... (1)}$$

conicoid $12x^2 - 17y^2 + 7z^2 = 7$ ... (2)

from (1) $-\frac{(x+5)}{3} = \frac{y-4}{1} = \frac{z-11}{7} = r$

then $\frac{-x-5}{3} = r \Rightarrow -x-5 = 3r$

$\Rightarrow -x = 3r + 5 \Rightarrow x = -5 - 3r$

$\frac{y-4}{1} = r \Rightarrow y = r + 4$

$\frac{z-11}{7} = r \Rightarrow z - 11 = 7r$

$\Rightarrow z = 7r + 11$

$\therefore$ The point $(-5-3r, r + 4, 7r + 11)$

Sub in (2) then we get

$12x^2 - 17y^2 + 7z^2 = 7$

$\Rightarrow$ $12(-5-3r)^2 - 17(r + 4)^2 + 7(7r+11)^2 = 7$

$\Rightarrow$ $12(-25 -9r^2 -30r) - 17(r^2 + 16 + 8r) + 7(49r^2 + 121 + 154r - 7) = 0$

$\Rightarrow$ $-300 - 108r^2 - 360r - 17r^2 - 272 - 136r + 34\ 3r^2 + 847 + 107\ 8r - 7 = 0$

$\Rightarrow$ $218r^2 + 582r + 268 = 0$

$109r^2 + 291r + 134 = 0$

When we sow

we get the point (1, 2, –3), (–2, 3, 4).

3. Tangent lines and tangent plane at a point, let the equation to the conicoid be

$$\mathbf{S \equiv ax^2 + by^2 + cz^2 - 1 = 0}$$

Sol :

Let the equation to a line passing through $P(x_1, y_1, z_1)$ be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \ldots (2)$$

$P(x_1, y_1, z_1) \in$ conicoid $\Rightarrow$ $ax_1^2 + by_1^2 + cz_1^2 = 1$

The points from (1)

$$\frac{x - x_1}{\ell} = r \quad \Rightarrow \quad x = lr + x_1$$

$$\frac{y - y_1}{m} = r \quad \Rightarrow \quad y = mr + y_1$$

$$\frac{z - z_1}{n} = r \quad \Rightarrow \quad z = nr + z_1$$

$\therefore$ $(lr + x_1, mr + y_1, nr + z_1)$ of the line lies in

$ax^2 + by^2 + cz^2 - 1 = 0$

$\Rightarrow \quad a(lr + x_1)^2 + b(mr + y_1)^2 + c(nr + z_1)^2 - 1 = 0$

$$a\left(l^2r^2 + x_1^2 + 2lrx_1\right) + b\left(r^2m^2 + y_1^2 + 2mry_1\right) + \left(n^2r^2 + z_1^2 + 2nrz_1\right) - 1 = 0$$

$$l^2\left(al^2 + bm^2 + cn^2\right)2r\left(alx_1 + bmy_1 + cnz_1\right) + \left(ax_1^2 + by_1^2 + cz_1^2 - 1\right) = 0$$

The line (1) is tangent line to the conicoid

$$\Rightarrow \quad \left(alx_1 + bmy_1 + cnz_1\right)^2 - \left(a\ell^2 + bm^2 + cn^2\right)\left(ax_1^2 + by_1^2 + cz_1^2 - 1\right) = 0$$

$\therefore \quad ax_1^2 + b_1^2 + cz_1^2 = 0$

hence $(alx_1 + bmy_1 + cnz_1)^2 - (al^2 + bm^2 + cn^2)\,(0) = 0$

$\Rightarrow \quad (alx_1 + bmy_1 + cnz_1)^2 = 0$

$\Rightarrow \quad alx_1 + bmy_1 + cnz_1 = 0$

hence the locus of the tangent line in the plane

$\Rightarrow \quad ax_1(x - x_1) + by_1(y - y_1) + cz_1(z - z_1) = 0$

$\Rightarrow \quad ax\,x_1 - ax_1x_1 + by_1y - by_1y_1 + cz_1z - cz_1z_1 = 0$

$\Rightarrow \quad axx_1 - ax_1^2 + by_1y - by_1^2 + cz_1z - cz_1^2 = 0$

$\Rightarrow \quad axx_1 + by_1y + cz_1z - \left(ax_1^2 + by_1^2 + cz_1^2\right) = 0$

as $\left(ax_1^2 + by_1^2 + ca_1^2\right) = 1$

$\Rightarrow \quad ax_1x + by_1y + cz_1z - 1 = 0$

$ax\,x_1 + by\,y_1 + cz_1\,z = 1$

$\therefore$ The tangent plane at (x, y, z) to the conicoid

$ax.x + by.y + cz.z = 1$

$\Rightarrow \quad ax^2 + by^2 + cz^2 = 1$

3.2.1 Condition of Tangency

Theorem

The necessary and sufficient condition that the plane $lx + my + nz = p$,ay be a tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ is $P^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}$.

Proof :

Necessary condition

Let the given plane $lx + my + nz = p$ ... (1)

be the tangent plane at $p(x_1, y_1, z_1)$ on the conicoid $ax^2 + by^2 + cz^2 = 1$

But the equation of the plane it to the coincoid is

$$ax\,x_1 + by\,y_1 + cz\,z_1 = 1 \quad \text{... (2)}$$

from (1) & (2) represent the same plane

$$\frac{l}{ax_1} = \frac{m}{by_1} = \frac{n}{cz_1} = \frac{p}{1}$$

$$\frac{l}{ax_1} = p \qquad \frac{m}{by_1} = p \qquad \frac{n}{cz_1} = \frac{p}{1}$$

$$\frac{ax_1}{l} = \frac{1}{p} \qquad \frac{by_1}{m} = \frac{1}{p} \qquad \frac{cz_1}{n} = \frac{1}{p}$$

$$\Rightarrow \quad x_1 = \frac{l}{ap} \qquad y_1 = \frac{m}{bp} \qquad z_1 = \frac{n}{cp} \qquad p \neq 0$$

$\because$ P lies on the coincoid $\Rightarrow ax_1^2 + by_1^2 + cz_1^2 = 1$

$$\Rightarrow \quad a\left(\frac{l}{ap}\right)^2 + b\left(\frac{m}{bp}\right)^2 + c\left(\frac{n}{cp}\right) = 1$$

$$\Rightarrow \quad \not{a}\frac{l^2}{a^2p^2} + \not{b}\frac{m^2}{b^2p^2} + \not{c}\frac{n^2}{c^2p^2} = 1$$

$$\frac{l^2}{ap^2}+\frac{m^2}{bp^2}+\frac{n^2}{cp^2}=1$$

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2>0$$

The point of contact is $\left(\frac{l}{ap},\frac{m}{bp},\frac{n}{cp}\right)$

Sufficiency of the condition

Let $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2>0$

$$\therefore \quad a\left(\frac{l}{ap}\right)^2+b\left(\frac{m}{bp}\right)^2+c\left(\frac{n}{cp}\right)^2=1$$

$\Rightarrow \left(\frac{l}{ap},\frac{m}{bp},\frac{n}{cp}\right)$ is a point on the conicoid.

Now the tangent plane at this point to the conicoid

$$a\left(\frac{l}{ap}\right)x+b\left(\frac{m}{bp}\right)y+c\left(\frac{n}{cp}\right)z=1$$

$\Rightarrow \quad lx + my + nz = 1$

The equation of the tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ can be taken in the form

$$lx+my+nz=\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

3.2.2 Director sphere

Let the conicoid S have a set of three mutually perpendicular tangent planes and P be their common point. The locus of P is a sphere called the direction sphere of S.

Theorem

Let the conicoid $ax^2 + by^2 + cz^2 = 1$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>0$ have a set of three mutually perpendicular tangent planes and P be their common point. The locus of P is the sphere $x^2 + y^2 + z^2 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

Proof :

Let a set of three mutually perpendicular tangent planes to the coincoid be

$$l_1x + m_1y + n_1z = P_1$$
$$l_2x + m_2y + n_2z = P_2$$
$$l_3x + m_3y + n_3z = P_3$$

where $P_i = \sqrt{\left(\frac{l_i^2}{a} + \frac{m_i^2}{b} + \frac{n_i^2}{c}\right)}$ $\quad i = 1, 2, 3$

Let (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) be taken to the direction cosines of the normals of the respective planes.

Then, we know that

(l_1, l_2, l_3), (m_1, m_2, m_3) and (n_1, n_2, n_3) form the direction cosines of the mutually perpendicular directions.

$\therefore$ We have $l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2$ and

$$l_1m_1 + l_1m_2 + l_3m_3 = m_1n_1 + m_2n_2 + m_3n_3$$
$$= n_1l_1 + n_2l_2 + n_3l_3 = 0$$

Let $p(x_1, y_1, z_1)$ be the common point of the planes,

Then $l_1 x_1 + m_1y_1 + n_1z_1 = p_1$

$$l_2x_2 + m_2y_2 + n_2z_2 = p_2$$
$$l_3x_2 + m_2y_3 + n_3z_3 = p_3$$

Squaring and adding these three conditions

We get

$$x_1^2 + y_1^2 + z_1^2 = p_1^2 + p_2^2 + p_3^2 = \frac{\sum l_i^2}{a} + \frac{\sum m_i^2}{b} + \frac{\sum n_i^2}{c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

$\therefore$ The locus of P is the sphere

$x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ is called the director sphere of the conicoid.

4. Find the equation to the tangent planes to $7x^2 - 3y^2 - z^2 + 21 = 0$ which pass through the line $7x - 6y + 9 = 0, z = 3$.

Sol :

The equation to the tangent plane

$$7x^2 - 3y^2 - z^2 + 21 = 0 \quad \ldots (1)$$

the plane $7x - 6y + 9 + kz - 3k$

$$7x - 6y + kz = 3k - 9$$

from (1)

$$7x^2 - 3y^2 - z^2 + 21 = 0$$

divide '21'

$$\frac{7x^2}{21} - \frac{3y^2}{21} - \frac{z^2}{21} + \frac{21}{21} = 0$$

$$\frac{x^2}{3} - \frac{y^2}{7} - \frac{z^2}{21} + 1 = 0$$

$$\frac{x^2}{3} + \frac{y^2}{7} + \frac{z^2}{21} = 1$$

If and only if $\dfrac{7^2}{-1/3} + \dfrac{(-6)^2}{-1/7} + \dfrac{k^2}{-1/21} = (3k-9)^2$

$\Leftrightarrow \quad 2k^2 + 9k + 4 = 0$

$$2k^2 + k + 8k + 4 = 0$$

$$k(2k+1) + 4(2k+1) = 0$$

$$(k+4)(2k+1) = 0$$

$$k = -4, \frac{-1}{2}$$

The required planes are

$$7x - 6y - 4z + 21 = 0, \; 7x - 6y - \frac{1}{2}z + \frac{21}{2} = 0.$$

5. Obtain the tangent planes to the ellipsoid, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Which are parallel to the plane. $lx + my + nz = 0$.

Sol :

If 2r is the distance between two parallel tangent planes to the ellipsoid,

Prove that line through the origin and perpendicular to the plane lies on the cone

$$x^2(a^2 - r^2) + y^2(b^2 - r^2) + z^2(c^2 - r^2) = 0$$

The two tangent parallel to the plane

$\sum lx = 0$ are $\sum lx = \pm\sqrt{\sum a^2 l^2}$

The distance between these parallel planes which is twice the distance of either from the origin is

$$2\sqrt{\sum a^2 l^2} / \sqrt{\sum l^2}$$

Thus, we have

$$\frac{2\sqrt{\sum a^2 l^2}}{\sqrt{\sum l^2}} = 2r$$

$$\Rightarrow \quad \sum\left(a^2 - l^2\right)l^2 = 0$$

hence the locus of the line

$$\frac{l}{x} = \frac{y}{m} = \frac{z}{n}$$

which is perpendicular to the plane $lx + my + nz = 0$

is $\sum\left(a^2 - r^2\right)x^2 = 0$

i.e., $x^2\left(a^2 - r^2\right) + y^2\left(b^2 - r^2\right) + z^2\left(c^2 - r^2\right) = 0$

6. If the line of intersection of two perpendicular tangent planes to the ellipsoid where equation referred to rectangular axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ passes through the fixed point (0, 0, k) show that it lies of the cone

Sol :

Equation of any plane through (0, 0, k) is

$$lx + my + nz = nk \quad \text{... (1)}$$

If it touches the given ellipsoid then

$$a^2l^2 + b^2m^2 + c^2n^2 = n^2k^2 \quad \text{... (2)}$$

Any line through (0, 0, k) is

$$\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z-k}{\gamma} \quad \text{... (3)}$$

The plane (1) will contain (3) if

$$l\lambda + m\mu + nr = 0 \quad \text{... (4)}$$

Eliminating l between (2) and (4)

We get

from (4) $\quad l\lambda = -m\mu + n\gamma$

$$l = -\frac{m\mu + n\gamma}{\lambda}$$

Sub in (2)

$$a^2\left(-\frac{m\mu+n\gamma}{\lambda}\right)^2 + b^2m^2 + c^2n^2 - n^2k^2 = 0$$

or

$$\left(a^2\mu^2 + b^2\lambda^2\right)\left(\frac{m}{n}\right)^2 + 2a^2\mu\gamma\left(\frac{m}{n}\right) + \left(a^2\gamma^2 + c^2\lambda^2 - k^2\lambda^2\right) = 0$$

Let its two roots be $\frac{m_1}{n_1}, \frac{m_2}{n_2}$

$$\frac{m_1}{n_1}\cdot\frac{m_2}{n_2}=\frac{a^2\upsilon^2+c^2\lambda^2-k^2\lambda^2}{a^2\mu^2+b^2\lambda^2}$$

or

$$\frac{m_1m_2}{a^2\gamma^2+c^2\lambda^2-k^2\lambda^2}=\frac{n_1n_2}{a^2\mu^2+b^2\lambda^2}$$

Similarly eliminating m, we get

$$\frac{l_1l_2}{b^2\upsilon^2+c^2\mu^2-k^2\mu^2}=\frac{n_1n_2}{a^2\mu^2+b^2\lambda^2}$$

$$\frac{l_1l_2}{b^2\upsilon^2+c^2\mu^2-k^2\mu^2}=\frac{m_1m_2}{a^2\upsilon^2+c^2\lambda^2-k^2\lambda^2}=\frac{n_1n_2}{a^2\mu^2+b^2\lambda^2}$$

$$=\frac{l_1l_2+m_1m_2+n_1n_2}{\left(b^2+c^2-k^2\right)\lambda^2+\left(c^2+a^2-k^2\right)\mu^2+\left(a^2+b^2\right)\upsilon^2}$$

Since the planes are perpendicular to each other hence

$$\left(b^2+c^2-k^2\right)\lambda^2+\left(a^2+c^2-k^2\right)\mu^2+\left(a^2+b^2\right)\upsilon^2=0$$

the locus of (3) is

$$\left(b^2+c^2-k^2\right)x^2+\left(c^2+a^2-k^2\right)y^2+\left(a^2+b^2\right)(z-k)^2=0$$

7. Show that the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$ fined the point of contract.

Sol :

The plane is $3x + 12y - 6z - 17 = 0$ ... (1)

touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$ ... (2)

divided 17 to the equation (2) then we get

$$\frac{3x^2}{17}-\frac{6y^2}{17}+\frac{9z^2}{17}+\frac{17}{17}=0$$

$$-\frac{3}{17}x^2 + \frac{6}{17}y^2 - \frac{9}{17}z^2 = 1$$

equation (1) touches the equation (2)

$$\left(-\frac{17}{3}\right)(3)^2 + \left(\frac{17}{6}\right)(12)^2 + \left(-\frac{17}{9}\right)(-6)^2 = (17)^2$$

$$(-17)(3) + (17)(24) + (-17)(4) = (17)^2$$

$$17(-3 + 24 - 4) = (17)^2$$

$$(17)\,[17] = (17)^2$$

$$\Rightarrow \quad (17)^2 = (17)^2$$

Hence the plane touches the conicoid point of contract is $\left(\frac{l^2}{ap}, \frac{m^2}{ap}, \frac{n^2}{ap}\right)$

$$\Rightarrow \quad \left\{(3)^2\left(\frac{17}{-3}\right)\frac{1}{17}, (12)\left(\frac{17}{6}\right)\left(\frac{1}{17}\right), (-6)\left(\frac{-17}{9}\right)\left(\frac{1}{17}\right)\right\}$$

$$= \left\{9\left(\frac{-17}{3}\right)\frac{1}{17}, 12\left(\frac{17}{6}\right)\left(\frac{1}{17}\right), (-6)\left(\frac{-17}{9}\right)\left(\frac{1}{17}\right)\right\}$$

$$= \left(-3, 2, \frac{2}{3}\right)$$

$\therefore$ The point of contract is $\left(-3, 2, \frac{2}{3}\right)$

3.2.3 Normal

Theorem

The normal at any point of a quadric is the line through the point perpendicular to the tangent plane thereat.

Proof :

The equation of the tangent plane at (α, β, γ) to the surface

$$ax^2 + by^2 + cz^2 = 1 \qquad \ldots (1)$$

$$a\,\alpha x + b\,\beta y + c\gamma z = 1 \qquad \ldots (2)$$

The equation to the normal as (α, β, γ)

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma} \quad \ldots (3)$$

So that aα, bβ, cγ are the direction ratios of the normal.

If P, is the length of the perpendicular from the origin to the tangent plane (2)

We have $\frac{1}{a^2\alpha^2+b^2\beta^2+c^2\gamma^2}=p^2$

$$(a\alpha p)^2+(b\beta p)^2+(c\gamma p)^2=1$$

aαp, bβp, cγp are the actual direction cosines of the normal at (α, β, γ)

3.2.4 Cubic Curve through the Feet of Normals

The feet of the six normals from a given point to a central quadric are the intersections of the quadric with a certain cubic curve.

Consider parametric equations are

$$x=\frac{f}{1+ar}, \qquad y=\frac{g}{1+br}, \qquad z=\frac{h}{1+cr}$$

to r is the parameter

The points (x, y, z) this curve, arising from those of the values of r which are the roots of the equation are the six feet of the normals from the point (f, g, h).

The point of intersection of curve with any plane.

$$Ax + By + Cz + D = 0$$

are given by $\frac{Af}{1+ar}+\frac{Bg}{1+br}+\frac{Ch}{1+cr}+D=0$

3.2.5 Quadric cone through six concurrent normals

Theorem

The six normals drawn from any point to a central quadric are the generator of a quadric cone.

Proof:

The lines drawn from (f, g, h) intersect the cubic curve generate a quadric cone.

If any line $\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}$

through (f, g, h) intersection the cubic curve we have

$$\frac{\frac{f}{1+ar}-f}{l}=\frac{\frac{g}{1+br}-g}{m}=\frac{\frac{h}{1+cr}-h}{n}$$

$$\frac{f-f-far}{l(1+ar)}=\frac{g-g-gbr}{m(1+br)}=\frac{h-h-hcr}{n(1+cr)}$$

$$\frac{-raf/l}{1+ar}=\frac{-rgf/m}{1+br}=\frac{-rhcn}{1+cr}$$

$$\frac{af/l}{1+ar}=\frac{bg/m}{1+br}=\frac{ch/n}{1+cr}$$

where eliminating r, we get

$$\frac{at}{l}(b-a)+\frac{bg}{m}(c-a)+\frac{ch}{n}(a-b)=0$$

which is the condition for the line intersect the cubic curve.

Eliminating l, m, n between the equation of the line then we get

$$\frac{af(b-a)}{x-f}+\frac{bg(c-a)}{y-g}+\frac{ch(a-b)}{z-h}=0$$

which is represents a cone of the second degree generater by lines from (f, g, h) to intersects the cubic curve.

3.3 Plane of Contact

Theorem

Find the equation of the plane of contact of the point (x_1, y_1, z_1) with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$.

Proof :

Let (α, β, γ) be any point on the conicoid $ax^2 + by^2 + cz^2 = 1$ ... (1)

Then,. the equation of the tangent plane to the conicoid (1) at the point (α, β, γ) is $a\alpha x + b\beta y + c\gamma z = 1$ ... (2)

It passes through the given point (x_1, y_1, z_1)

then $a\alpha x_1 + b\beta y_1 + c\gamma z_1 = 1$ ... (3)

$\therefore$ locus of the point (α, β, γ) is

$$axx_1 + byy_1 + czz_1 = 0$$

which is required equation of the plane of contact.

8. Any three mutually perpendicular lines drawn through a fixed point c meet the conicoid $ax^2 + by^2 + cz^2 = 1$.

Sol :

Let $C = (\alpha, \beta, \gamma)$

The equation to any line through C be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

Let (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) be direction cosines of the three perpendicular lines through C.

The equation to the line through C and with d.c's (l_1, m_1, n_1)

$$\frac{x-\alpha}{l_1} = \frac{y-\beta}{m_1} = \frac{z-\gamma}{n_1} (= r) \quad \text{... (1)}$$

Any point on the line $(l_1 r + \alpha, m_1 r + \beta, n_1 r + \gamma)$.

If this point lies on the given conicoid $ax^2 + by^2 + cz^2 = 1$

then, $a(l_1 r + \alpha)^2 + b(m_1 r + \beta)^2 c(n_1 r + \gamma)^2 = 1$

$$\Leftrightarrow \quad r^2\left(al_1 + bm_1^2 + cn_1^2\right) + 2r\left(a\alpha l_1 + b\beta m_1 + n\gamma m_1\right) + \left(a\alpha^2 + b\beta^2 + c\gamma^2 - 1\right) = 0$$

The line meets the coincoid in P_1 and P_2

$\therefore$ The roots of this quadratic equation are the values CP_1 & CP_2

$$CP_1 \cdot CP_2 = \frac{a\alpha^2 + b\beta^2 + c\gamma^2 - 1}{al_1^2 + bm_1^2 + cm_1^2} \Rightarrow \frac{1}{CP_1 \cdot CP_2} = \frac{al_1^2 + bm_1^2 + cn_1^2}{a\alpha^2 + b\beta^2 + c\gamma^2 - 1}$$

Similarly, the lines $CQ_1.CQ_2$, CR_1R_2 meet the conicoids

where $\frac{1}{CQ_1 \cdot CQ_2} = \frac{al_2^2 + bm_2^2 + cn_2^2}{a\alpha^2 + b\beta^2 + \gamma^2 - 1}$; $\frac{1}{CR_1 \cdot CR_2} = \frac{al_3^2 + bm_3^2 + cn_3^2}{a\alpha^2 + b\beta^2 + c\gamma^2 - 1}$

Hence $\quad \frac{1}{CP_1 \cdot CP_2} + \frac{1}{CQ_1 \cdot CQ_2} + \frac{1}{CR_1 \cdot CR_2}$

$$\frac{a}{\Delta}\left(\ell_1^2 + \ell_2^2 + \ell_3^2\right) + \frac{b}{\Delta}\left(m_1^2 + m_2^2 + m_3^2\right) + \frac{c}{\Delta}\left(n_1^2 + n_2^2 + n_3^2\right)$$

$$\Delta = a\alpha^2 + b\beta^2 + c\gamma^2 - 1$$

$\because$ CP_1P_2, CQ_1Q_2, CR_1R_2 are there mutually perpendicular lines

$\Rightarrow \quad l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2 = 1$

3.4 Enveloping Cone

The locus of tangent line drawn from a given point to a given surface in called of the enveloping cone as the tangent cone of the surface. The given point is called the vertex of the enveloping cone.

Theorem

Find the equation of the enveloping cone of the surface $ax^2 + by^2 + cz^2 = 1$ with vertex (α, β, γ).

Proof :

Any line through $P(\alpha, \beta, \gamma)$ is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \qquad \dots (1)$$

Any point on this line is

$Q\ (\alpha + lr, \beta + mr, \gamma + nr)$

The given surface is $ax^2 + by^2 + cz^2 = 1$ ... (2)

Let the line (1) meet the surface (2) in Q.

Then, the coordinates of Q will satisfy equation (2), we get

$$a(\alpha + lr)^2 + b(\beta + mr)^2 + c(\gamma + nr)^2 = 1$$

$$\Rightarrow r^2(al^2 + bm^2 + cn^2) + 2r(a\alpha l + b\beta m + c\gamma n) + a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0 \qquad \dots (3)$$

equation (3) is quadratic in r, so, it will gives two values of r corresponding to two points in which line (1) meets the surface (2)

If line (1) is a tangent to the surface (2), then these two points must coincide, the condition for which that the roots of (3) must be equal

$$\therefore \quad \{2(a\alpha l + b\beta m + c\gamma n)\}^2 = 4(al^2 + bm^2 + cn^2)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$$

$$\Rightarrow \quad (a\alpha l + b\beta m + c\gamma n)^2 = (al^2 + bm^2 + cn^2)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$$

$\therefore$ locus of (1) is

$$\{a\alpha(x-a) + b\beta(y-\beta) + c\gamma(z-\gamma)\}^2$$

$$= \{a(x-a)^2 + b(y-\beta)^2 + c(z-\gamma)^2\}(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$$

$$\Rightarrow \quad \left\{(a\alpha x + b\beta y + c\gamma z) - \left(a\alpha^2 + b\beta^2 + c\gamma^2\right)\right\}^2$$

$$= \left\{\left(ax^2 + by^2 + cz^2\right) + \left(a\alpha^2 + b\beta^2 + c\gamma^2\right) - 2(a\alpha x + b\beta y + c\gamma z)\left(a\alpha^2 + b\beta^2 + c\gamma^2 - 1\right)\right\} \quad \dots (4)$$

Let $S \equiv ax^2 + by^2 + cz^2 - 1$

$S_1 \equiv a\alpha^2 + b\beta^2 + c\gamma^2 - 1$

$T = a\alpha x + b\beta y + c\gamma z - 1$

Then (4) can be written as

$$\{(T+1) - (S_1+1)\}^2 = \{(S+1) + (S_1+1) - 2(T+1)\}S_1$$

$$\Rightarrow \quad (T + \cancel{1} - S_1 - \cancel{1})^2 = \{S + \cancel{1} + S_1 + \cancel{1} - 2T - \cancel{2}\}S_1$$

$$(T - S_1)^2 = (S + S_1 - 2T)S_1$$

$$T^2 + S_1^2 - 2TS_1 = SS_1 + S_1 \cdot S_1 - 2TS_1$$

$$T^2 + \cancel{S_1^2} - 2\cancel{T}S_1 - SS_1 - \cancel{S_1^2} + 2\cancel{T}S_1 = 0$$

$$T^2 - SS_1 = 0$$

$$SS_1 = T^2$$

$$\Rightarrow \quad \left(ax^2 + by^2 + cz^2 - 1\right)\left(a\alpha^2 + b\beta^2 + c\gamma^2 - 1\right)$$

$$= (a\alpha x + b\beta y + c\gamma z - 1)^2$$

which is the required of the enveloping cone of the surface $ax^2 + by^2 + cz^2 = 1$ with vertex (α, β, γ).

---

9. If the section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ where vertex is P by the plane z = 0 is a rectangular hyperbola, show that the locus of P is $\frac{x^2 + y^2}{a^2 + b^2} + \frac{z^2}{c^2} = 1$.

Sol :

The enveloping cone of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

whose vertex is P by the plane z = 0 is a rectangle hyperbola.

here $\quad S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

$$S_1 \equiv \frac{\alpha^2}{a^2}+\frac{\beta}{b^2}+\frac{\gamma^2}{c^2}-1$$

$$T = \frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1$$

$\therefore$ Equation of the enveloping cone of the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

With vertex at $P(\alpha, \beta, \gamma) = 1$ ... (1)

$$SS_1 = T^2$$

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{a^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1\right), z=0 \quad \text{... (3)}$$

Equation (3) represents a conic is the plane z = 0

If (3) represents a rectangular hyperbola

from (3)

$$\left[\left(\frac{x^2}{a^2}\right)\left(\frac{\alpha^2}{a^2}\right)+\left(\frac{x^2}{a^2}\right)\left(\frac{\beta^2}{b^2}\right)+\left(\frac{x^2}{a^2}\right)\left(\frac{\gamma^2}{c^2}\right)-\frac{x^2}{a^2}\right]+\left[\left(\frac{y^2}{b^2}\right)\left(\frac{\alpha^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)\left(\frac{\beta^2}{b^2}\right)+\left(\frac{y^2}{a^2}\right)\left(\frac{\gamma^2}{c^2}\right)-\frac{y^2}{b^2}\right]$$

$$-\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right]=\left[\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1\right]$$

$\Rightarrow$ Coefficient of x^2 + Coeffient of $y^2 = 0$

$$\Rightarrow \frac{\alpha^2}{a^2a^2}+\frac{\beta^2}{a^2b^2}+\frac{\gamma^2}{a^2c^2}-\frac{1}{a^2}+\frac{\alpha^2}{b^2a^2}+\frac{\beta^2}{b^2b^2}+\frac{\gamma^2}{b^2c^2}-\frac{1}{b^2}=0$$

$$\Rightarrow \left[\frac{1}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right]+\frac{1}{b^2}\left[\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right]=0$$

$$\Rightarrow \frac{\alpha^2+\beta^2}{a^2b^2}+\frac{\gamma^2}{c^2} \quad \frac{a^2+b^2}{a^2b^2}=\frac{a^2+b^2}{a^2b^2}$$

$$\Rightarrow \frac{\alpha^2+\beta^2}{a^2+b^2}+\frac{\gamma^2}{c^2}=1$$

$\therefore$ locus of $p(\alpha, \beta, \gamma)$ is $\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1$

3.5 Enveloping Cylinder

The locus of tangent lines drawn to a given surface is a fixed direction is called the enveloping cylinder of that surface.

3.5.1 Equation of Enveloping Cylinder

Theorem

Find the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ with its generators parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

Proof :

Let (α, β, γ) be a point on the enveloping. Cylinder, So that the equation of the generator through it are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$$

$x = lr + \alpha$, $\quad y = mr + \beta \quad z = nr + \gamma$

$\therefore$ $(lr + \alpha,\ mr + \beta,\ nr + \gamma)$

$a(lr + \alpha)^2 + b(mr + \beta)^2 + c(nr + \gamma)^2 = 1$

$a(l^2 r^2 + \alpha^2 + 2lrd) + b(m^2r^2 + \beta^2 + 2mr\,\beta) + c(n^2l^2 + \gamma^2 + 2nr\gamma) = 1$

$al^2r^2 + a\alpha^2 + 2alr\alpha + bm^2r^2 + b\beta^2 + 2bmr\beta + cn^2r^2 + cr^2 + 2ncnr\gamma - 1 = 0$

$r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + Cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0$

$(al\alpha + bm\beta + cn\gamma)^2 = (al^2 + bm^2 + cn^2)\,(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$

Thus, the locus of (α, β, γ) is the surface

$$\left(ax^2 + by^2 + cz^2 - 1\right)\left(al^2 + bm^2 + cn^2\right) = \left(alx + bmy + cnz\right)^2$$

which is required equation of the enveloping cylinder.

The generator of the enveloping cylinder touch the quadraic at point where it is meet by the plane $alx + bmy + cnz = 0$

which is known as the plane of contact

10. Find the equations to the tangent planes to the surface $4x^2 - 5y^2 + 7z^2 + 13 = 0$ parallel to the plane 4x + 20y – 21z =0 find their point of contact also.

Sol :

Given that the equation of coincoid is $4x^2 - 5y^2 + 7z^2 + 13 = 0$

$\Rightarrow$ divide by –'13'

$$\left(\frac{4}{-13}\right)x^2 - \left(\frac{5}{-13}\right)y^2 + \left(\frac{7}{-13}\right)z^2 + \left(\frac{13}{-13}\right) = 0$$

$$\left(\frac{-4}{13}\right)x^2 + \left(\frac{5}{13}\right)y^2 + \left(\frac{-7}{13}\right)z^2 = 1 \qquad \text{... (1)}$$

and equation of plane is $4x + 20y - 21z = 0$ ... (2)

Comparing equation (1) with $ax^2 + by^2 + cz^2 = 1$

$$a = \frac{-4}{13} \qquad b = \frac{5}{13} \qquad c = \frac{-7}{13}$$

Comparing equation (2) with $\ell x + my + nz = p$

$l = 4 \quad m = 20 \quad n = -21$

The equation of any tangent plane is given by $\ell x + my + nz = \pm\sqrt{\frac{\ell^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$

Substitute the corresponding values in above equation.

$$4x + 20y - 21z = \pm\sqrt{\frac{(4)^2}{\left(-4/13\right)} + \frac{(20)^2}{\left(5/13\right)} + \frac{(-21)^2}{\left(-7/13\right)}}$$

$$4x + 20y - 21z = \pm\sqrt{\frac{16 \times 13}{(-4)} + \frac{400 \times 13}{5} + \frac{441 \times 13}{(-7)}}$$

$$4x + 20y - 21z = \pm 13$$

$\therefore$ The required tangent planes are

$$4x + 20y - 21z = \pm 3$$

Let $P(x_1, y_1, z_1)$ be the point of contact.

Then the equation of tangent plane at the point $P(x_1, y_1, z_1)$ to coincoid $4x^2 - 5y^2 + 7z^2 + 13 = 0$ is $4xx_1 - 5yy_1 + 7zz_1 + 13 = 0$

Comparing equation (1) with equation (2)

$$\frac{4x_1}{4} = \frac{-5y_1}{20} = \frac{7z_1}{-21}$$

$$\frac{x_1}{1} = \frac{y_1}{-4} = \frac{z_1}{-3}$$

The point of contact is (1, -4, –3)

$\therefore$ The point of contact of equation (3) $(\pm 1, \pm 4, \pm 3)$

11. Find the locus of the perpendicular from the origin to the tangent planes to the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which cut off from its axes intercepts the sum of whose reciprocals is equal to a constant $\frac{1}{k}$.

Sol :

Given that equation of ellip soid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ... (1)

The equation of any tangent plane to equation (1)

$$\ell x+my+nz=\sqrt{a^2\ell^2+b^2m^2+c^2n^2} \quad \text{... (2)}$$

The equation of perpendicular from the centre (0, 0, 0) of equation (1) to equation (2)

$$\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$$

Since equation (2) makes the intercepts on the co-ordinates axes.

$$\ell x=\sqrt{a^2\ell^2+b^2m^2+c^2n^2}$$

$$x=\frac{1}{\ell}\sqrt{a^2\ell^2+b^2m^2+c^2n^2}$$

$$my=\sqrt{a^2\ell^2+b^2m^2+c^2n^2}$$

$$y=\frac{1}{m}\sqrt{a^2\ell^2+b^2m^2+c^2n^2}$$

and $nz=\sqrt{a^2\ell^2+b^2m^2+c^2n^2}$

$$z=\frac{1}{n}\sqrt{a^2\ell^2+b^2m^2+c^2n^2}$$

The sum of reciprocals of intercepts is given by $\frac{1}{k}$

i.e. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{k}$

$$\Rightarrow \frac{\ell}{\sqrt{a^2\ell^2+b^2m^2+c^2n^2}}+\frac{m}{\sqrt{a^2\ell^2+b^2m^2+c^2n^2}}+\frac{n}{\sqrt{a^2\ell^2+b^2m^2+c^2n^2}}=\frac{1}{k}$$

$$\Rightarrow \frac{\ell+m+n}{\sqrt{a^2\ell^2+b^2m^2+c^2n^2}}=\frac{1}{k}$$

$$k(\ell+m+n)=\sqrt{a^2\ell^2+b^2m^2+c^2n^2}$$

Squaring on both sides

$$k^2(\ell+m+n)^2 = a^2\ell^2 + b^2m^2 + c^2n^2$$

$\therefore$ The required locus is $a^2x^2 + b^2y^2 + c^2z^2 = k^2(x+y+z)^2$

12. P(1, 3, 2) is a point on the coincoid $x^2 - 2y^2 + 3z^2 + 5 = 0$ find the locus of the mid - points of chords drawn parallel to OP.

Sol :

Given that equation of coincoid is $x^2 - 2y^2 + 3z^2 + 5 = 0$ ... (1)

and point is P (1,3, 2).

Comparing equations (1) with $ax^2 + by^2 + cz^2 = 0$

$a = 1, b = -2, \quad c = 3$

The equation of coincoid whose centre is the point (1, 3, 2) and passes through the point (x^1, y^1, z^1) is given by $a\alpha(x^2 - \alpha) + b\beta(y^1 - \beta) + c\gamma(z^1 - \gamma) = 0$

$$\Rightarrow 1(1)(x^1 - 1) + (-2)(3)(y^1 - 3) + 3(2)(z^1 - 2) = 0$$

$$\Rightarrow (x^1 - 1) - 6(y^1 - 3) + 6(z^1 - 2) = 0$$

$$x^1 - 6y^1 + 6z^1 + 5 = 0 \quad \text{... (2)}$$

The equation of line parallel to equation (2) is $x^1 - 6y^1 + 6z^1 + k = 0$ ... (3)

Since, it passes through origin $\Rightarrow k = 0$

Substitute the value of k =0 in equation (3)

$$x^1 - 6y^1 + 6z^1 + 0 = 0$$

$$\Rightarrow x^1 - 6y^1 + 6z^1 = 0$$

$\therefore$ The locus of the mid - points of chords drawn parallel to OP is $x - 6y + 6z = 0$.

Choose the Correct Answer

1. How many normals can be drawn from any point to a conicoid. [b]

   (a) 2 (b) 6

   (c) 8 (d) 4

2. Condition that the plane $\ell x + my + nz = p$ should touch the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is [a]

   (a) $a^2\ell^2 + b^2m^2 + c^2n^2 = p^2$ (b) $\frac{\ell^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}+p^2$

   (c) $\frac{\ell^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}+p^2$ (d) $\frac{\ell^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0$

3. The surface represented by the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is [d]

   (a) Ellipsoid (b) Hyper boloid of one sheet

   (c) Paraboloid (d) Hyperboloid of two sheet

4. Straight line which intersects a central collicoid in two coincodent points is called a [c]

   (a) Polar line (b) Diameter

   (c) Tangent line (d) Chard of contact

5. If the plane $x + 2y - 2z = 4$ touches the paraboloid $3x^2 + 4x^2 = 24z$ then the point of contact is [c]

   (a) (1, 2, 3) (b) (1, 2, 4)

   (c) (2, 3, 2) (d) (3, 2, 1)

6. The locus of the centres of section of central coincoid which pass through a given line is a [c]

   (a) Paraboloid (b) Pair of straight line

   (c) Conic (d) Circle

7. Number of normals that can be drawn through a given point to a paraboloid is [c]

(a) 9 (b) 3

(c) 5 (d) 6

8. The plane 2x + 3y + 4z =3 touches the conicoid $2x^2+3y^2+4z^2=1$ at [c]

(a) (2, 3, 4) (b) $\left(\frac{2}{3},1,\frac{4}{3}\right)$

(c) $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ (d) (1, 1, 1)

9. A plane which bisects a system of parallel chords of a central conicoid is called [d]

(a) Polar plane (b) Tangent plane

(c) Plane of contact (d) Diametral plane

10. Equation of the tangent plane to the surface $3x^2+y^2+z^2=21$ at the point (2, 3, 0) is [d]

(a) x + 2y = 7 (b) 6x + 3y = 21

(c) 3x + 2y = 21 (d) 2x + y = 7

Fill in the blanks

1. $a\alpha x + b\beta y = c(z+\gamma)$ is the equation of the polar plane of the point (α, β, γ) with respect to the paraboloid ________.
2. The locus of the point of intersection of three mutually perpendicular tangent planes to the paraboloid ________ is a sphere.
3. The equation of the locus of the mid point of system of parallel chords with directions cosines l, m, n of the conicoid $ax^2 + by^2 + cz^2 = 1$ is ________.
4. The equation of an ellipsoid is ________.
5. Two diametral planes of the paraboloid $ax^2 + by^2 = 2cz$ are called ________ each bisects the chords parallel to the other.
6. The General equation of a hyperbolic paraboloid is $\frac{x^2}{a^2} - \frac{b^2}{b^2} =$ ________.
7. The pole of the plane $\ell x + my + nz = p$ with respect to the conicoid is ________.
8. $ax^2 + by^2 + cz^2 = 1$ is the standard equation of the ________.
9. The central conicoid $ax^2 + by^2 + cz^2 = 1$ is an ellipsoid of the constant a, b, c are all ________.
10. The centre of the central conicoid $ax^2 + by^2 + cz^2 = 1$ is at the ________.

Answers

1. $ax^2 + by^2 = 2cz$
2. $ax^2 + by^2 = 2cz$
3. $alx + bmy + cnz = 0$
4. $ax^2 + by^2 + cz^2 = 1$
5. Conjugate
6. $\frac{2z}{c}$
7. $\left(\ell/ap, m/bp, n/cp\right)$
8. Conicoid
9. Positive
10. Origin

Proactical Problems

UNIT - I

ANSWERS

1. Find the equation of the sphere through the four points (4, –1, 2), (0, –2, 3), (1, –5, –1), (2, 0, 1). **(Unit-I, Q.No. 5)**

2. Find the equation of the sphere through the four points (0, 0, 0), (–a, b, c), (a, –b, c), (a, –b, c), (a, b, –c). **(Unit-I, Q.No. 9)**

3. Find the centre and the radius of the circle $x + 2y + 2z = 15$, $x^2 + y^2 + z^2 - 2y - 4z = 11$. **(Unit-I, Q.No. 16)**

4. Show that the following points are concyclic: **(Unit-I, Q.No. 17)**

   (i) (5, 0, 2), (2, –6, 0), (7, –3, 8), (4, –9, 6).

   (ii) (–8, 5, 2), (–5, 2, 2), (–7, 6, 6) (–4, 3, 6).

5. Find the centres of the two spheres which touch the plane $4x + 3y = 47$ at the points (8, 5, 4) and which touch the sphere $x^2 + y^2 + z^2 = 1$. **(Unit-I, Q.No. 33)**

6. Show that the spheres $x^2 + y^2 + z^2 = 25$, $x^2 + y^2 + z^2 - 24x - 40y - 18z + 225 = 0$ touch externally and find the point of the contact. **(Unit-I, Q.No. 28)**

7. Find the equation of the sphere that passes through the two points (0, 3, 0), (–2, –1, –4) and cuts orthogonally the two sphere $x^2+y^2+z^2+x-3z-2 = 0$, $2(x^2 + y^2 + z^2) + x + 3y + 4 = 0$. **(Unit-I, Q.No. 29)**

8. Find the limiting points of the coaxial system of spheres. $x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 + \lambda(2x - 3y + 4z) = 0$. **(Unit-I, Q.No. 30)**

9. Find the eqution to the two spheres of the coaxial systems $x^2 + y^2 + z^2 - 5 + \lambda(2x + y + 3z - 3) = 0$ which touch the plane, $3x + 4y = 15$. **(Unit-I, Q.No. 31)**

10. Show that the radical plane of the sphere of a coaxial system and of the given any sphere pass through a line. **(Unit-I, Q.No. 32)**

UNIT - II

ANSWERS

1. Find the equation of the cone whose vertex is the point (1,1,0) and whose guiding curve is y = 0, $x^2 + z^2 = 4$. **(Unit-II, Q.No. 2)**

2. The section of a cone whose vertex is P and guiding curve the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, z = 0 by the plane z = 0 is a rectangular hyperbola. S/T the locus of p is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$. **(Unit-II, Q.No. 16)**

3. Find the enveloping cone of the sphere $x^2 + y^2 + z^2 - 2x + 4z = 1$ with its vertex are (1, 1, 1). **(Unit-II, Q.No. 7)**

4. Find the equation of the quadric cone whose vertex is at the origin and which passes through the curve given by the equation $ax^2 + by^2 + cz^2 = 1$, lx + my + nz = P. **(Unit-II, Q.No. 8)**

5. Find the equation of the cone whose vertex is at the origin and the direction cosines of whose generators satisfy the relation $3\ell^2 - 4m^2 + 5n^2 = 0$. **(Unit-II, Q.No. 48)**

6. Find the equation of the cylinder whose generators are parallel to $x = \frac{1}{2}y = \frac{1}{3}z$ and whose guiding curve is the ellipse $x^2 + 2y^2 = 1$, z = 3. **(Unit-II, Q.No. 40)**

7. Find the right cylinder whose radius is 2 and axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$. **(Unit-II, Q.No. 44)**

8. The axis of a right circular cylinder of radius 2 is $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$ shows that its equation is $10x^2+5y^2+13z^2-12xy-6yz-47x-8x+30y-74z+59=0$. **(Unit-II, Q.No. 45)**

9. Find the equation of a circular cylinder whose guiding curve $x^2 + y^2 + z^2 = 9, x - y + z = 3$. **(Unit-II, Q.No. 46)**

10. Obtain the equation of the right circular cylinder described on the circle as guiding circle (1, 0, 0), (0, 1, 0),(0, 0,1). **(Unit-II, Q.No. 47)**

UNIT - III

ANSWERS

1. Find the points of intersection of the line $-\frac{1}{3}(x+5) = (y-4) = \frac{1}{7}(z-11)$ with the conicoid $12x^2 - 17y^2 + 7z^2 = 7$. **(Unit-III, Q.No. 2)**

2. Find the equation to the tangent planes to $7x^2 - 3y^2 - z^2 + 21 = 0$ which pass through the line $7x - 6y + 9 = 0, z = 3$. **(Unit-III, Q.No. 4)**

3. Obtain the tangent planes to the ellipsoid, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which are parallel to the plane. $lx + my + nz = 0$. **(Unit-III, Q.No. 5)**

4. Show that the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$ find the point of contract. **(Unit-III, Q.No. 7)**

5. Find the equations to the tangent planes to the surface $4x^2 - 5y^2 + 7z^2 + 13 = 0$ parallel to the plane $4x + 20y - 21z = 0$ find their point of contact also. **(Unit-III, Q.No. 10)**

6. Find the locus of the perpendicular from the origin to the tangent planes to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which cut off from its axes intercepts the sum of whose reciprocals is equal to a constant $\frac{1}{k}$. **(Unit-III, Q.No. 11)**

7. If the section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ where vertex is P by the plane $z = 0$ is a rectangular hyperbola, show that the locus of P is $\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1$. **(Unit-III, Q.No. 9)**

8. Any three mutually perpendicular lines drawn through a fixed point c meet the conicoid $ax^2 + by^2 + cz^2 = 1$. **(Unit-III, Q.No. 8)**

9. P(1, 3, 2) is a point on the coincoid $x^2 - 2y^2 + 3z^2 + 5 = 0$ find the locus of the mid - points of chords drawn parallel to OP. **(Unit-III, Q.No. 12)**

FACULTY OF SCIENCE

B.Sc III Year, V-Semester Examination

Model Paper - I

SOLID GEOMETRY

Time : 3 Hours] [Max. Marks : 60

PART - A (5 × 3 = 15 Marks)

Answer any Five of the following

ANSWERS

1. Find the equation to the sphere through the points (0, 0, 0), (0, 1 –1), (–1, 2, 0), (1, 2, 3). **(Unit-I, Q.No. 4)**
2. Obtain the equation of the sphere passing through the three points (1, 0, 0), (0, 0) and has its radius as small as possible. **(Unit-I, Q.No. 15)**
3. Show that a cone of second degree can be found to pass through any five concurrent lines. **(Unit-II, Q.No. 11)**
4. Show that a quadric cone be found to touch any five planes which meet at a point provided no three of them intersect in a line find the equation of the cone which touches the three co-ordinates planes and the planes $x + 2y + 3z = 0$, $2x + 3y + 4z = 0$. **(Unit-II, Q.No. 25)**
5. If the plane $2x - y + cz = 0$ cuts cone $yz + zx + xy = 0$ in perpendicular lines, find the value of c. **(Unit-II, Q.No. 27)**
6. Find the equation of the plane of contact of the point (x_1, y_1, z_1) with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$. **(Unit-III, Theorem: 3.3)**
7. If the line of intersection of two perpendicular tangent planes to the ellipsoid where equation referred to rectangular axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ passes through the fixed point (0, 0, k) show that it lies of the cone. **(Unit-III, Q.No. 6)**
8. If the section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ where vertex is P by the plane $z = 0$ is a rectangular hyperbola, show that the locus of P is $\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1$. **(Unit-III, Q.No. 9)**

PART - B (3 × 15 = 45 Marks)

Answer all the following three questions

Each question carries fifteen marks

9. (a) Find the value of a for which the plane $x + y + z = a\sqrt{3}$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$. **(Unit-I, Q.No. 22)**

OR

(b) If $S_1 = 0, S_2 = 0$ be two spheres, then the equation $S_1 + \lambda S_2 = 0$. λ being the parameter, represents a system of sphere such that any two members of the systems have the same radical plane. **(Unit-I, Q.No. 27)**

10. (a) (i) Find the point of intersection of the line $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ and the cone $f(x, y, z) \equiv ax^2 + by^2 + c^2 + 2fyz + 2gzx + 2hxy = 0$. **(Unit-II, Q.No. 18)**

(ii) Prove that the cones $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ are Reciprocal. **(Unit-II, Q.No. 20)**

OR

(b) Find the equation to the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$. **(Unit-II, Q.No. 14)**

11. (a) The normal at any point of a quadric is the line through the point perpendicular to the tangent plane thereat. **(Unit-III, Theorem: 3.2.3)**

OR

(b) Show that the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$ fined the point of contract. **(Unit-III, Q.No. 7)**

FACULTY OF SCIENCE

B.Sc III Year, V-Semester Examination

Model Paper - II

SOLID GEOMETRY

Time : 3 Hours] [Max. Marks : 60

PART - A (5 × 3 = 15 Marks)

Answer any Five of the following

ANSWERS

1. Show that the plane lx + my + nz = p will touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ $(ul + vm + wn + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + m^2 - d)$. **(Unit-I, Q.No. 19)**

2. Find the centre and the radius of the circle $x + 2y + 2z = 15$, $x^2 + y^2 + z^2 - 2y - 4z = 11$. **(Unit-I, Q.No. 16)**

3. Find the condition that the plane $ux + vy + wz = 0$ may touch the cone $ax^2 + by^2 + cz^2 = 0$. **(Unit-II, Q.No. 24)**

4. If $x = \frac{1}{2}y = z$ represents one by a set of three mutually perpendicular generators of the cone $11yz + 6zx - 14xy = 0$ find the equation of the other two. **(Unit-II, Q.No. 26)**

5. Find the equation of the right circular cone with its vertex at the origin, axis along z–axis and semi vertical angle α. **(Unit-II, Q.No. 29)**

6. Find the points of intersection of the line **(Unit-III, Q.No. 2)**

$$-\frac{1}{3}(x+5) = (y-4) = \frac{1}{7}(z-11)$$

with the conicoid $12x^2 - 17y^2 + 7z^2 = 7$.

7. Tangent lines and tangent plane at a point, let the equation to the conicoid be $S \equiv ax^2 + by^2 + cz^2 - 1 = 0$ **(Unit-III, Q.No. 3)**

8. The six normals drawn from any point to a central quadric are the generator of a quadric cone. **(Unit-III, Theorem: 3.2.5)**

PART - B (3 × 15 = 45 Marks)

Answer all the following three questions

Each question carries fifteen marks

9. (a) Obtain the equation of the sphere passing through the three points (1, 0, 0), (0, 1, 0), (0, 0) and has its radius as small as possible. **(Unit-I, Q.No. 15)**

OR

(b) Show that the equation of the sphere passing through the three points (3, 0, 2), (–1, 1, 1) (2, –5, 4) and having its centre on the plane $2x + 3y + 4z = 6$ is $x^2 + y^2 + z^2 + 4y - 6z = 1$. **(Unit-I, Q.No. 14)**

10. (a) Find the point of intersection of the line **(Unit-II, Q.No. 18)**

$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ and the cone $f(x,y,z) \equiv ax^2+by^2+cz^2$.

$+2fyz+2gzx+2hxy=0$

OR

(b) (i) Find the conditions that the lines of section of the plane $lx + my + nz = 0$ and the cones $fyz + gzx + hxy = 0$, $ax^2+by^2+cz^2=0$ should be coincident. **(Unit-II, Q.No. 21)**

(ii) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes is A, B, C. Prove that the equation to the cone generator by lines drawn from 0 to meet the circle ABC is $yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+$ $xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$ **(Unit-II, Q.No. 12)**

11. (a) Let the conicoid $ax^2 + by^2 + cz^2 = 1$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>0$ have a set of three mutually perpendicular tangent planes and P be their common point. The locus of P is the sphere $x^2 + y^2 + z^2 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. **(Unit-III, Theorem: 3.2.2)**

(b) Find the equation of the enveloping cone of the surface $ax^2 + by^2 + cz^2 = 1$ with vertex (α, β, γ). **(Unit-III, Theorem: 3.4)**

FACULTY OF SCIENCE

B.Sc III Year, V-Semester Examination

Model Paper - III

SOLID GEOMETRY

Time : 3 Hours] [Max. Marks : 60

PART - A (5 × 3 = 15 Marks)

Answer any Five of the following

ANSWERS

1. If the tangent plane tot he sphere $x^2 + y^2 + z^2 = r^2$ makes intercepts a, b, c on the coordinate axes, show that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}$. **(Unit-I, Q.No. 21)**

2. If $S_1 = 0$, $S_2 = 0$ be two spheres, then the equation $S_1 + \lambda S_2 = 0$. λ being the parameter, represents a system of sphere such that any two members of the systems have the same radical plane. **(Unit-I, Q.No. 27)**

3. Find the value of a for which the plane $x + y + z = a\sqrt{3}$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$. **(Unit-I, Q.No. 22)**

4. Find the equation of the cone generated by rotating the line $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$ about the line $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ as axis. **(Unit-II, Q.No. 31)**

5. Find the equation of the right circular cone which passes through the point (1,1,2) and had its vertex at the origin, axis the line $\frac{x}{2}=\frac{y}{-4}=\frac{z}{3}$. **(Unit-II, Q.No. 32)**

6. The six normals drawn from any point to a central quadric are the generator of a quadric cone. **(Unit-III, Theorem: 3.2.5)**

7. Any three mutually perpendicular lines drawn through a fixed point c meet the conicoid $ax^2 + by^2 + cz^2 = 1$. **(Unit-III, Q.No. 8)**

8. Find the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ with its generators parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$. **(Unit-III, Theorem: 3.5.1)**

PART - B (3 × 15 = 45 Marks)

Answer all the following three questions

Each question carries fifteen marks

9. (a) Equation to the radical plane of spheres $S = 0, S^1 = 0$ in $S - S^1 = 0$. **(Unit-I, Theorem: 1.7)**

OR

(b) Show that the spheres $x^2 + y^2 + z^2 = 64$ and $x^2 + y^2 + z^2 - 12x + 4y - 6z + 48 = 0$ touch internally and find their point of contact. **(Unit-I, Q.No. 20)**

10. (a) If α is the semi vertical angle of the right circular cone which passes through the lines oy, oz, $x = y = z$ show that $\cos\alpha = \left(9 - 4\sqrt{3}\right)^{-1/2}$. **(Unit-II, Q.No. 35)**

OR

(b) Find the equation of the cylinder whose generators are parallel to $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is the ellipse $x^2 + 2y^2 = 1, z = 3$. **(Unit-II, Q.No. 40)**

11. (a) Let the conicoid $ax^2 + by^2 + cz^2 = 1, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 0$ have a set of three mutually perpendicular tangent planes and P be their common point. The locus of P is the sphere $x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. **(Unit-III, Theorem: 3.2.2)**

OR

(b) If the section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ where vertex is P by the plane $z = 0$ is a rectangular hyperbola, show that the locus of P is $\frac{x^2 + y^2}{a^2 + b^2} + \frac{z^2}{c^2} = 1$. **(Unit-III, Q.No. 9)**

FACULTY OF SCIENCE

B.Sc. V - Semester (CBCS) Examination

June / July - 2019

SOLID GEOMETRY

PAPER - VI (A) : (MATHEMATICS)

Time : 3 Hrs] [Max. Marks : 60

PART - A (5 × 3 = 15 Marks)
[Short Answer Type]

Note : Answer any five of the following questions

1. Find the equation of the sphere through the four points.

   (0, 0, 0), (–a, b, c), (a, –b, c), (a, b, –c)

2. Find the equation of the tangent plane to the sphere

   $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 00$

3. Find the equation of the cone whose vertex is at the origin and which passes through the curve by the equations.

   $ax^2 + by^2 + cz^2 = 1,\ lx + my + nz = p.$

4. Show that the general equation of a cone which touches the three coordinate planes is $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$.

5. Find the equations to the tangent planes to $7x^2 - 3y^2 - z^2 + 21 = 0$ which passes throught he line $7x - 6y + 9 = 0,\ z = 3$.

6. Find the pole of the plane $lx + my + nz = p$ with respect to the quadratic $ax^2 + by^2 + by^2 + cz^2 = 1$.

7. Find the sphere having the circle x2 + y2 + z2 + 10y – 4z – 8 = 0, x + y + z = 3 as the great circle.

8. Find the equation to the cone which passes through the three coordinate axes as well as the two lines.

   $$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3},\ \frac{x}{31} = \frac{y}{-1} = \frac{z}{1}$$

PART - B (45 Marks)
[Essay Answer Type]

Note : Answer all from the following questions.

9. (a) Find the equation of the sphere that passes through the two pints (0, 3, 0), (–2, –1, –4) and cuts orthogonally the two spheres $x^2 + y^2 + z^2 + x - 3z - 2 = 0$, $2(x^2 + y^2 + z^2) + x + 3y + 4 = 0$.

OR

(b) Find the equations to the two speheres of the co-axial system

$$x^2 + y^2 + z^2 - 5 + \lambda(2x + y + 3z - 3) = 0$$

Which touch the plane $3x + 4y = 15$.

10. (a) Find the angle between the lines of intersection of $10x + 7y - 6z = 0$ and $20x^2 + 7y^2 - 108z^2 = 0$.

OR

(b) Prove that the tangent planes to the cone $x^2 - y^2 + 2z^2 - 3yz + 4zx - 5xy = 0$ are perpendicular to the generators of the cone.

$17x^2 + 8y^2 + 29z^2 + 28yz - 46zx - 16xy = 0$.

11. (a) Show that the enveloping cylinders of the ellipsoid $ax^2 + by^2 + cz^2 = 1$ with generators perpendicular to z-axis meet the plane $z = 0$ in parabolas.

(b) Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the conicoid $ax^2 + by^2 + cz^2 = 1$.

FACULTY OF SCIENCE

B.Sc. V - Semester (CBCS) Examination

November / December - 2018

SOLID GEOMETRY

PAPER - V : (MATHEMATICS)

Time : 3 Hrs] [Max. Marks : 60

PART - A (5 × 3 = 15 Marks)

[Short Answer Type]

Note : Answer any five of the following questions

ANSWERS

1. Find the centre and radius of the sphere

$$2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 3 = 0$$

Sol :

The given sphere equation is

$$2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 3 = 0$$

To get into general form, divid by '2'

$$x^2 + y^2 + z^2 - x + 2y + z + \frac{3}{2} = 0 \quad ...(1)$$

The sphere equation

$$x^2+y^2 + z^2 - x + 2ux + 2vy + 2wz + d = 0 \quad ...(2)$$

Compare (1) with (2)

$$2u = -1 \qquad 2v = 2 \qquad 2w = 1 \qquad d = \frac{3}{2}$$

$$u = \frac{-1}{2} \qquad v = 1 \qquad w = \frac{1}{2} \qquad d = \frac{3}{2}$$

Centre $C = (-u, -v_1 -w)$

$$= \left(\frac{1}{2}, -1, \frac{-1}{2}\right)$$

Radius $r = \sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + (-1)^2 + \left(\frac{-1}{2}\right)^2 - \left(\frac{3}{2}\right)}$$

$$= \sqrt{\frac{1}{4} + 1 + \frac{1}{4} - \frac{3}{2}}$$

$$= \sqrt{\frac{3}{2} - \frac{3}{2}}$$

$= 0$

2. Find the limiting points of the co-axial system define by the sphere.

$$x^2 + y^2 + z^2 + 3x - 3y + 6 = 0 \text{ and } x^2 + y^2 + z^2 - 6y - 6z + 6 = 0.$$

Sol :

Given sphere are

$s_1 = x^2 + y^2 + z^2 + 3x - 3y + 6 = 0$

$s_2 = x^2 + y^2 + z^2 - 6y - 6z + 6 = 0$

The redical plane of two sphere is

$s_1 - s_2 = 0$

$x^2 + y^2 + z^2 + 3x - 3y + 6 - x^2 - y^2 - z^2 + 6y + 6z - 6 = 0$

$3x + 3y + 6z = 0$

$x + y + 2z = 0$

$u = x + y + 2z$

The equation of coaxial system of sphere are $s_1 + \lambda u = 0$

$x^2 + y^2 + z^2 + 3x + 3y + 6 + \lambda (x + y + 2z) = 0$

$x^2 + y^2 + z^2 + x (3 + \lambda) + y (-3 + \lambda) + 2\lambda z + 6 = 0$

Centre of the above sphere is

$$C = \left[\frac{-(3+\lambda)}{2}, \frac{-(\lambda-3)}{2}, -\lambda\right] \text{ and } d = 6$$

for limiting points radius $r = 0$

$$\sqrt{\left(\frac{\lambda+3}{2}\right)^2+\left(\frac{\lambda-3}{2}\right)^2+\lambda^2-6}=0$$

squaring on both sides

$$\frac{(\lambda+3)^2}{4}+\frac{(\lambda-3)^2}{4}+\lambda^2-6=0$$

$$\lambda^2+9+6\lambda+\lambda^2+9-6\lambda+4\lambda^2-24=0$$

$$6\lambda^2-6=0$$

$$6\lambda^2=6$$

$$\lambda^2=\frac{6}{6},\ \lambda=\pm 1$$

Sub λ in centre, we get

$$\left[\frac{-(3+1)}{2},\frac{-(1-3)}{2},-1\right]\left[\frac{-(3-1)}{2},\frac{-(-1-3)}{2},1\right]$$

$$\left[\frac{-4}{2},\frac{2}{2},-1\right]\left[\frac{-2}{2},\frac{4}{2},1\right]$$

$(-2, 1, -1)\ (-1, 2, 1)$

The limting points of the coaxial system is $[-2, 1, -1]$ & $[-1, 2, 1]$

3. Find the equation of the cone whose vertex is the point $(1, 1, 0)$ and whose guiding curve is $y = 0, x^2 + z^2 = 4$. **(Unit - II, Q.No.2)**

4. Show that the general equation to a cone which passes through the three coordinate axes is $fyz + gzx + hxy = 0$ where f, g, h are parameters. **(Unit - II, Q.No.71)**

5. Find the points of intersection of the line $\frac{x+5}{-3}=\frac{y-4}{1}=\frac{z-11}{7}$ with the conicoid $12x^2 - 17y^2 + 7z^2 = 7$. **(Unit - III, Q.No.2)**

6. Find the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ with its generations parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ **(Unit - III, Q.No.3.5.1-Theorem)**

7. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9,\ 2x + 3y + 4z = 5$ then point (1, 2, 3). **(Unit - I, Q.No.12)**

8. Find the enveloping cone of the sphere $x^2 + y^2 + z^2 - 2x + 4z - 1 = 0$ with its vertex at (1, 1, 1). **(Unit - I, Q.No.7)**

PART - B (45 Marks)

[Essay Answer Type]

Note : Answer all from the following questions.

9. (a) Show that the two circles

$$2(x^2 + y^2 + z^2) + 8x - 13y + 17z - 17 = 0 = 2x + y - 3z + 1$$

and $x^2 + y^2 + z^2 + 3x - 4y + 3z = 0 = x - y + 2z - 4$

lie on some sphere and find its equation.

Sol :

Given two circles are

$$2(x^2 + y^2 + z^2) + 8x - 13y + 17z - 17 = 0 = 2x + y - 3z + 1 \quad ...(1)$$

$$x^2 + y^2 + z^2 + 3x - 4y + 3z = 0 = x - y + 2z - 4 \quad ...(2)$$

from (1)

$$x^2 + y^2 + z^2 + 4x - \frac{13}{2}y + \frac{17}{2}z - \frac{17}{2} = 0$$

equation of the sphere through the circle (1)

$$(x^2 + y^2 + z^2 + 4x - \frac{13}{2}y + \frac{17}{2}z - \frac{17}{2}) + \lambda_1 (2x + y - 3z - 1) = 0$$

$$x^2 + y^2 + z^2 + 4x - \frac{13}{2}y + \frac{17}{2}z - \frac{17}{2} + 2\lambda_1 x + \lambda_1 y - 3z\lambda_1 - \lambda_1 = 0$$

$$x^2 + y^2 + z^2 + (4 + 2\lambda_1)x + (\lambda_1 - \frac{13}{2})y + z(\frac{17}{2} - 3\lambda_1) - \frac{17}{2} - \lambda_1 = 0 \quad ...(3)$$

equation of the sphere through the circle (2)

$$(x^2 + y^2 + z^2 + 3x - 4y + 3z) + \lambda_2 (x - y + 2z - 4) = 0$$

$$x^2 + y^2 + z^2 + (3 + \lambda_2)x + (-4 - \lambda_2)y + (3 + 2\lambda_2)z - 4\lambda_2 = 0 \quad ...(4)$$

The equation (3) & (4) will represents the same sphere

$$4 + 2\lambda_1 = 3 + \lambda_2$$

$$\Rightarrow 2\lambda_1 - \lambda_2 + 1 = 0 \quad ...(i)$$

$$\lambda_1 - \frac{13}{2} = -4 - \lambda_2$$

$$\Rightarrow \lambda_1 + \lambda_2 - \frac{5}{2} = 0 \quad ...(ii)$$

$$\frac{17}{2} - 3\lambda_1 = 3 + 2\lambda_2$$

$$\Rightarrow -3\lambda_1 - 2\lambda_2 + \frac{11}{2} = 0 \quad ...(iii)$$

$$-\frac{17}{2} + \lambda_1 = -4\lambda_2$$

$$\Rightarrow \lambda_1 + 4\lambda_2 - \frac{17}{2} = 0 \quad ...(iv)$$

solving (i) & (ii)

	λ_1	λ_2	1
-1	1	2	-1
1	$\frac{-5}{2}$	1	1

$$\frac{\lambda_1}{\frac{5}{2}-1} = \frac{\lambda_2}{1+5} = \frac{1}{2+1}$$

$$\frac{\lambda_1}{\frac{3}{2}} = \frac{\lambda_2}{6} = \frac{1}{3}$$

$$\frac{\lambda_1}{\frac{3}{2}} = \frac{1}{3} \Rightarrow \lambda_1 = \frac{1}{2}$$

$$\frac{\lambda_2}{6} = \frac{1}{3}$$

$$\lambda_1 = 2$$

sub λ_1 in equation (3) we get

$$\left(x^2 + y^2 + z^2 + 4x - \frac{13}{2}y + \frac{17}{2}z - \frac{17}{2}\right) + \frac{1}{2}(2x + y - 3z - 1) = 0$$

$$x^2 + y^2 + z^2 + 4x - \frac{13}{2}y + \frac{17}{2}z - \frac{17}{2} + \frac{2x}{2} + \frac{1}{2}y - \frac{3}{2}z - \frac{1}{2} = 0$$

$$x^2 + y^2 + z^2 + 5x - 6y + 7z - 8 = 0$$

substiting λ_2 in equation (4) we get

$$x^2 + y^2 + z^2 + 3x - 4y + 3z + 2(x - y + 2z - 4) = 0$$

$$x^2 + y^2 + z^2 + 5x - 6y + 8z - 8 = 0$$

OR

(b) Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is $\frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$ **(Unit - I, Q.No.24)**

10. (a) Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$. **(Unit - II, Q.No.14)**

OR

(b) Find the equation of the right circular cone with its vertex at the origin, axis along z-axis and semi vertical angle α. **(Unit - II, Q.No.96)**

11. (a) Find the equations to the two planes which contain the line given by $7x + 10y - 30 = 0$, $5y - 3z = 0$ and touch the ellipsoid $7x^2 + 5y^2 + 3z^2 = 60$.

Sol :

The equation in of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{7x^2}{60} + \frac{5y^2}{60} + \frac{3z^2}{60} = 1$$

The standard equation of ellipsoid $ax^2 + by^2 + cz^2 = 1$

Now the equation of any plane through the given line is

$$7x + 10y - 30 - \lambda (5y - 3z) = 0$$

$$7x + (10 + 5\lambda) y - 3\lambda z - 30 = 0$$

Comparing this with $lx + my + nz = p$

$l = 7, \quad m = 10 + 5\lambda \quad n = -3\lambda$

$p = 30$

The condition for tangency is,

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

$$\frac{\frac{49}{7}}{60} + \frac{\frac{(10+5\lambda)^2}{5}}{60} + \frac{\frac{(-3\lambda)^2}{3}}{60} = (30)^2$$

$2\lambda^2 + 5\lambda + 3 = 0$

$2\lambda^2 + 2\lambda + 3\lambda + 3 = 0$

$2l\,(\lambda + 1) + 3\,(\lambda + 1) = 0$

$\lambda = -1$, and $2\lambda = -3$

$\lambda = -\frac{3}{2}$

If $\lambda = -1$

$7x + 10y - 30 - 1\,(5y - 3z) = 0$

$7x + 10y - 5y + 3z - 30 = 0$

$7x + 5y + 3z - 30 = 0$

If $\lambda = -\frac{3}{2}$

$7x + 10y - 30 - \frac{3}{2}\,[5y - 3z] = 0$

$14x + 20y - 60 - 15y + 9z = 0$

$14x - 5y - 9z - 60 = 0$

OR

(b) Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the conicoid $ax^2 + by^2 + cz^2 = 1$. **(Unit - III, Q.No.8)**