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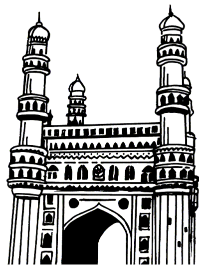

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Ans :

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UNIT I	**Sets, Relation and Function**: Operations and Laws of Sets, Cartesian Products, Binary Relation, Partial Ordering Relation, Equivalence Relation, Image of a Set, Sum and Product of Functions, Bijective functions, Inverse and Composite Function, Size of a Set, Finite and infinite Sets, Countable and uncountable Sets, Cantor's diagonal argument and The Power Set theorem, Schroeder-Bernstein theorem. **Principles of Mathematical Induction**: The Well-Ordering Principle, Recursive definition, The Division algorithm: Prime Numbers, The Greatest Common Divisor: Euclidean Algorithm, The Fundamental Theorem of Arithmetic.

1.1 Sets

Q1. Define Set.

Ans :

Meaning

A set is determined only when we know definitely what objects it contains. There must be no ambiguity or doubt in this regard.

For example, the collection of all tall boys in a college does not define a set; because there will always be some doubt about as to which boys are to be regarded as tall!. For this reason, the objects of a set are required to be "well defined". By this we mean that, given an object, it must be possible for us to decide whether the object belongs to the collection (under consideration) or not. Thus, if we speak of the collection of all boys in a college who are above 170 cms then we are actually speaking of a set; because without any ambiguity we can say whether a given boy is in the set or not.

If the number of elements in a set is finite, then we say that the set is a finite set. Sets having infinitely mans elements are called infinite sets. A set having only one element is called a singleton set.

1.1.1 Size of a Set

Q2. Explain size of set.

Ans :

We can compute the size of a set by explicitly counting its elements.

For example, $|\emptyset| = 0$, $|\{\text{Larry, Moe, Curly}\}| = 3$, and $|\{x \in \mathbb{N} \mid x < 100 \wedge x \text{ is prime}\}| = |\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}| = 25$. But sometimes it is easier to compute sizes by doing arithmetic. We can do this because many operations on sets correspond in a natural way to arithmetic operations on their sizes.

Two sets A and B that have no elements in common are said to be disjoint; in set-theoretic notation, this means $A \cap B = \emptyset$. In this case we have $|A \cup B| = |A| + |B|$. The operation of disjoint union acts like addition for sets.

For example, the disjoint union of 2-element set {0, 1} and the 3-element set {Wakko, Jakko, Dot} is the 5-element set {0, 1,Wakko, Jakko, Dot}.

The size of a Cartesian product is obtained by multiplication: $|A \times B| = |A| \cdot |B|$. An example would be the product of the 2-element set {a, b} with the 3-element set {0, 1, 2}: this gives the 6-element set {(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2)}. Even though Cartesian product is not generally commutative, swapping each pair (a, b) to (b, a) is a bijection, so $|A \times B| = |B \times A|$.

For power set, it is not hard to show that $|P(S)| = 2^{|S|}$. This is a special case of the size of AB, the set of all functions from B to A, which is $|A|^{|B|}$; for the power set we can encode P(S) using 2^S, where 2 is the special set {0, 1}, and a subset T of S is encoded by the function that maps each $x \in S$ to 0 if $x \notin 2$ T and 1 if $x \in T$.

1.1.2 Finite and Infinite Sets

Q3. Define Finite and Infinite Sets.

Ans :

Finite Set

A set is said to be a finite set if it is either void set or the process of counting of elements surely comes to an end is called a finite set.

In a finite set the element can be listed if it has a limited i.e., countable by natural number 1, 2, 3, ... and the process of listing terminates at a certain natural number N.

The number of distinct elements counted in a finite set S is denoted by n(S). The number of elements of a finite set A is called the order or cardinal number of a set A and is symbolically denoted by n(A).

Thus, if the set A be that of the English alphabets, then n(A) = 26; For, it contains 26 elements in it. Again if the set A be the vowels of the English alphabets i.e., A = [a, e, i, o, u] then n(A) = 5.

Note: The element does not occur more than once in a set. A set which is not finite is called an infinite set.

Examples of Finite Set

1. Let

   A = {5, 10, 15, 20, 25, 30}

   Then,

   A is a finite set and n(P) = 6.

2. Let

   B = {Natural numbers less than 25}

   Then,

   B is a finite set and n(P) = 24.

3. Let

   C = {Whole numbers between 5 and 45}

   Then,

   C is a finite set and n(R) = 38.

4. Let

   D = {x : x " Z and x ^ 2 – 81 = 0}

   Then,

   D = {–9, 9} is a finite set and n(S) = 2.

Infinite Set

A set is said to be an infinite set whose elements cannot be listed if it has an unlimited (i.e., uncounted) by the natural number 1, 2, 3, 4, ... n, for any natural number n is called a infinite set.

Examples of Infinite Set :

1. Set of all positive integers which is multiple of 3 is an infinite set.
2. Set of all points in a plane is an infinite set.
3. W = {0, 1, 2, 3, ...} i.e., set of all whole numbers is an infinite set.
4. Set of all points in a line segment is an infinite set.

Thus, from the above discussions we know how to distinguish between the finite sets and infinite sets with examples.

1.1.3 Countable and Uncountable Sets

Q4. Define Countable and uncountable Sets.

Ans :

Countable Sets

The sets N, $\mathbb{N}^2$, and N* all have the property of being countable, which means that they can be put into a bijection with N or one of its subsets. Countability of N* means that anything you can write down using finitely many symbols (even if they are drawn from an infinite but countable alphabet) is countable. This has a lot of applications in computer science: one of them is that the set of all computer programs in any particular programming language is countable.

Uncountable Sets

Exponentiation is different. We can easily show that $2^{\aleph_0} \neq \aleph_0$, or equivalently that there is no bijection between $P(\mathbb{N})$ and $\mathbb{N}$. This is done using Cantor's diagonalization argument, which appears in the proof of the following theorem.

Q5. Explain basic terminology used in set theory.

Ans :

(i) The Null Set

In many discussions, we require the use of the set which contains no object (element) at all. This set is called the empty set or the null set, it is denoted by { }, ϕ, or Φ.

For example, the set of all positive integers less than 10 which are divisible by 11 is the null set. The set of all real numbers whose squares are less than zero is also the null set.

(ii) Equal Sets

Two sets A and B are said to be equal if they have precisely the same elements. Then we write A = B.

Thus, for example, if A = {1, 2, 3, 4} and B = { x | x is a positive integer with $x^2 < 20$}, ten A = B.

(iii) Subsets

Given two sets A and B. we say that A is a subset of B, or that A is contained in B if every element of A is an element of B as well. If A contains an element which is not in B, then A is not a subset of B.

The statement "A is a subset of B" is symbolically written as "$A \subseteq B$". Here, the symbol $\subseteq$ stands for "is a subset of" or "is contained in".

Similarly, the statement "A is not a subset of B" is symbolically written as $A \not\subseteq B$, the symbol $\not\subseteq$ denoting "is not a subset of" or "not contained in".

For example, let

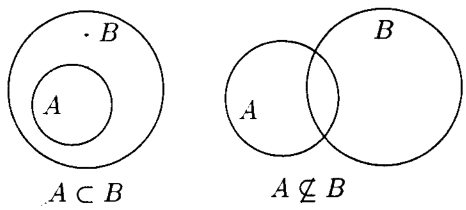
A = {1, 2, 3}, B = {1, 2, 3, 4, 5) and C = {2, 3, 5, 6}

We observe that every element of A is in B, and A contains an element namely 1, which is not in C. Therefore, $A \subseteq B$ and $A \not\subseteq C$.

In the above example, we also observe that the set B, which contains A as a subset, pos-sesses elements that are not in A (namely the elements 4 and 5). In such a situation we say that the set A is properly contained in or is a proper subset of the set B.

Thus, a set A is a proper subset of a set B if (i) $A \subseteq B$, and (ii) B possesses at least one element that is not in A. In this situation we write $A \subset B$. Here, the symbol $\subset$ stands for "is a proper subset of".

(iv) Venn Diagram



(a) (b)

Relationships between sets (like the relationship between a set A and a set B containing A or not containing A) can be depicted in diagrams called Venn diagrams§ for a clearer grasp of the situation. Figures 3.1 (a), (b) represent two such Venn diagrams.

Some Consequences

The following consequences of the definition of a subset of a set are of basic importance:

1. Every set is a subset of itself.
2. Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$.
3. The null set Φ is a subset of every set A.
4. For any sets A , B and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
5. For any sets A, B, C, if A = B and B = C, then A = C.

The statements (1), (2) and (5) are obviously true.

(v) Universal Set

Suppose, in a discussion, all sets that we consider are subsets of a certain set U. This set, U, is called the universal set or the universe of discourse or the universe for that discussion.

For example, in a study concerned with integers the set of all integers is taken as the uni-versal set, and in studies concerned with defective parts of a machine the set of all parts of the machine is taken as the universal set.

The universal set varies from one discussion to the other, and the context indicates the choice of the universal set. The universal set is not unique.

(vi) Power Set

Given a set A, suppose we construct the set consisting of all subsets of A. The set so obtained (constructed) is called the power set of A and is denoted by P(A).

For example, consider the set A = {a, b}. We check that all possible subsets of A are Φ, {a}. {b}, A. Therefore,

P(A) = {Φ, {a}, {b}, A).

As another example, if we consider the set A = {1, 2, 3), the subsets of A are

Φ, {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3}, A.

Therefore,

P(A) = {Φ, UK {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3}, A}.

We observe that, in the first Example, the set A has two elements and the set P{A} has 4 = 2^2 elements, and in the second example, the set A has 3 elements and the set P(A) has 8 = 2^3 elements. This fact illustrates the following important result:

If a finite set A has n elements, then P(A), the power set of A, has 2^n elements.

1.2 Operations and Laws of Sets

Q6. Explain the operations on set theory.

Ans : **(Imp.)**

Union of Sets

Consider two sets A and B. Then the set consisting of all elements that belong to A or B is called the union of A and B and is denoted by A ∪ B. Thus,

A ∪ B = {x | x ∈ A or x ∈ B}

Here, "or" is an inclusive or; that is, the statement "x ∈ A or x ∈ B" actually stands for "x he ngs to A or B or both".

The Venn-diagram for A ∪ B is shown below. Here, ∪ is some universal set which contains A and B as subsets.

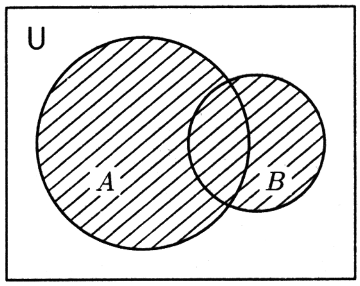


Fig. : A ∪ B (shaded)

It is obvious that A ⊆ A ∪ B and B ⊆ A ∪ B.

Intersection of Sets

Given two sets A and B, the set consisting of all elements that belong to both A and B is called the intersection of A and B and is denoted by A ∩ B. Thus,

A ∩ B = {x | x ∈ A and x ∈ B)

The Venn-diagram for A ∩ B is shown below.

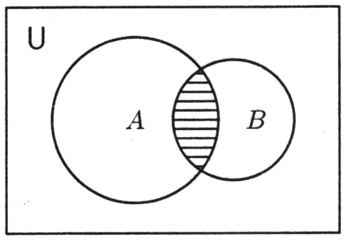


Fig. : A ∩ B (shaded)

It is obvious that

A ∩ B ⊆ A and A ∩ B ⊆ B.

For example,

if A = {1, 2, 3, 4, 5, 6, 7} and B = {4, 5, 8, 9}, then

A ∪ B = {1, 2, 3, 4, 5, 6, 7, 8, 9},

A ∩B = {4, 5},

Two sets A and B are said to be disjoint whenever $A \cap B = \Phi$.

For example, A = {1, 2, 3} end B = {5 ,7} are disjoint sets.

Complement of a Set

Given a universal set $\bigcup$ and a set A contained in $\bigcup$, the set of all elements that belong to $\bigcup$ but not to A is called the complement of A (in $\bigcup$) and is denoted by $\overline{A}^{\,||}$.

Thus,

$\overline{A} = \{x \mid x \in \cup \text{ and } x \notin A)$.

The Venn-diagram for A is shown below:

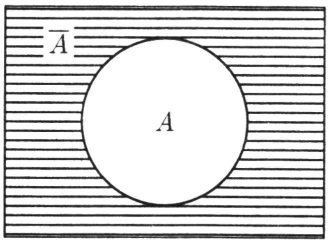


Fig.: $\overline{A}$ (shaded)

For example, if the set of all integers is taken as the universal set and E is the set of all even integers, then the complement of E (in the chosen universal set) is the set of all odd integers.

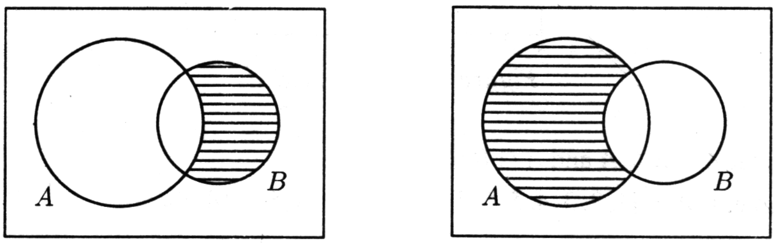
It is obvious that $\overline{\cup} = \Phi$, = $\overline{\Phi} = \cup$ and if $A \subseteq \cup$, then $\overline{A} \subseteq \cup$, and A and $\overline{A}$ are disjoint.

Relative Complement

Given two sets A and B, the set of all elements that belong to B but not to A is called the complement of A relative to B (or relative complement of A in B) and is denoted by B – A; that is

$B - A = \{x \mid x \in B \text{ and } x \notin A)$

The set A – B is defined similarly. Note that B – A is not the same as A – B. See Figures (a), (b) below:



(a) : B – A (Shaded) **(b) : A – B (Shaded)**

Fig. : Relative Complements

For example, if A = {1, 2, 3, 4} and B = {3, 4, 7, 8}, then B – A = {7, 8} and A – B = {1, 2}.

The following results are obvious:

1. For any sets A and B, the sets A – B and B – A are disjoint.
2. If A and B are disjoint, then A – B = A and B – A = B.

3. If $\cup$ is the universal set and $A \subseteq \cup$, then $\cup - A = \bar{A}$.

4. $A = (A \cup B) - (B - A)$, $B = (A \cup B) - (A - B)$

5. $A = (A \cap B) \cup (A - B)$, $B = (A \cap B) \cup (B - A)$

Symmetric Difference

For two sets A and B, the relative complement of $A \cap B$ in $A \cup B$ is called the symmetric difference of A and B and is denoted by $A\Delta B$ **.

Thus,

$$A \Delta B = (A \cup B) - (A \cap B)$$
$$= \{x \mid x \in A \cup B \text{ and } x \notin A \cap B)$$

The Venn diagram for $A \Delta B$ is shown below.

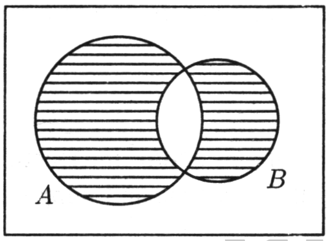


Fig. : A Δ B (shaded)

For example, if A = {1, 2, 3, 4, 5} and B = {4, 5, 7, 8}, then A Δ B = (1, 2, 3, 7, 8)

Q7. Discuss briefly about law of set theory.

Ans :

The operations on sets satisfy certain laws. The following are a few of these laws wherein A, B, C are subsets of a universal set $\cup$.

I. Commutative Laws

1. $A \cup B = B \cup A$
2. $A \cap B = B \cap A$

II. Associative Laws

3. $A \cup (B \cup C) = (A \cup B) \cup C$
4. $A \cap (B \cap C) = (A \cap B) \cap C$

III. Distributive Laws

5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

IV. Idempotent Laws

7. $A \cup A = A$
8. $A \cap A = A$

V. Identity Laws

9. $A \cup \Phi = A$

10. $A \cap U = A$

VI. Law of Double Complement

11. $\overline{\overline{A}} = A$

VII. Inverse Laws

12. $A \cup \overline{A} = \cup$

13. $A \cap \overline{A} = \Phi$

VIII. DeMorgan Laws

14. $\overline{A \cup B} = \overline{A} \cap \overline{B}$

15. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

IX. Domination Laws

16. $A \cup \bigcup = \bigcup$

17. $A \cap \Phi = \Phi$

X. Absorption Laws

18. $A \cup (A \cap B) = A$

19. $A \cap (A \cup B) = A$

1.3 CARTESIAN PRODUCTS

Q8. Explain Cartesian Products of sets.

Ans : **(Imp.)**

Let A and B be two sets. Then the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$, is called the Cartesian Product, or Cross Product or Product Set of A and B (in this order) and is denoted by $A \times B$.

Thus,

$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

It is to be noted that the product set $A \times B$ is not the same as the product set $B \times A$; that is, $A \times B \neq B \times A$, in general. Because,

$B \times A = \{(b, a) \mid b \in B \text{ and } a \in A\}$

and $(a, b) \neq (b, a)$ in general.

For example, if $A = \{1, 0, -1\}$ and $B = \{2, 3\}$.

Then

$A \times B = \{1. 2), (1, 3), (0, 2), (0. 3), (-1, 2), (-1. 3)\}$

$B \times A = \{(2, 1), (2, 0), (2, -1), (3, 1), (3, 0), (3, -1)\}$.

Evidently, $A \times B + B \times A$. It should be noted that $A \times B$ can be defined even when $B = A$.

Thus. we can have the product of a set A with itself, and this product is defined by

$A \times A = \{(a, b) \mid a \in A \text{ and } b \in A\}$.

The product $A \times A$ is also denoted by A^2.

For example, if $A = \{1, 0, -1\}$

We have

$A^2 = A \times A = \{(1, 1), (1, 0), (1, -1), (0, 1), (0, 0), (0, -1), (-1, 1), (-1, 0), (-1, -1)\}$

If a set A has m elements and a set B has n elements, then a can be chosen from A in m ways and with every one of these choices (of a), b can be chosen from B in n ways. Accordingly, (a. b) can be chosen in $m \times n$ ways; this means that $A \times B$ has exactly mn elements. Thus, we have the following result:

If A and B are finite sets, then

$|A \times B| = |A|\ |B|$.

From this result, it follows that

$|B \times A| = |B|\ |A| = |A|\ |B| = |A \times B|$ and $|A \times A| = |A|^2$.

For example, if A has 5 elements and B has 8 elements, then $A \times B$ and $B \times A$ will have $5 \times 8 = 8 \times 5 = 40$ elements each, $A \times A$ will have $5 \times 5 = 25$ elements, and $B \times B$ will have $8 \times 8 = 64$ elements.

The idea of Cartesian product of sets can be extended to any finite number of sets. For any non-empty sets $A_1, A_2, ..., A_k$, the k-foldproduct $A_1 \times A_2 \times ... \times A_k$ is defined as the set of all odered k-tuples (a_1, a_2, a_k), where $a_i \in A_i$, $i = 1, 2, ..., k$. That is,

$A_1 \times A_2 \times ... \times A_k = \{(a_i, a_2, ..., a_k) \mid a_i \in A_i, i = 1, 2, ..., k\}$.

For example, if $A = \{1, 0\}$, $B = \{2, -2\}$, $C = \{0, -1\}$, then

$A \times B \times C = \{(1, 2, 0), (1, 2, -1), (1, -2, 0), (1, -2, -1), (0, 2, 0), (0, 2, -1),$
$(0, -2, 0), (0, -2, -1)\}$

As with the ordered pairs, if $(a_1, a_2, ..., a_k)$, $(b_1, b_2, b_3 ...\ k)$ are k-tuples, then

$(a_1, a_2, ..., a_k) = (b_1, b_2, ..., b_k)$ if and only if $= b_i$ for $i = 1, 2, ..., k$.

A little thinking will indicate that if A_1 has n_1 elements, A_2 has n_2 elements, ... A_k has n_k elements, then $A_1 \times A_2 \times ... \times A_k$ has $n_1 n_2 n_3 ... n_k$ elements.

That is,

$|A_1 \times A_2 \times ..... \times A_k| = |A_1| - |A_2|\ .....\ |A_k|$.

PROBLEMS

9. Find x and y in each of the following cases:

(i) (2x, x + y) = (6, 1)

(ii) (y – 2, 2x + 1) = (x – 1, y + 2)

Sol :

(i) We note that $(2x, x + y) = (6, 1)$ if and only if $2x = 6$ and $x + y = 1$.

These yield $x = 3$ and $y = 1 - x = -2$.

(ii) $(y - 2, 2x + 1) = (x - 1, y + 2)$ if and only if $y - 2 = x - 1$ and $2x + 1 = y + 2$; that is,

$x - y + 1 = 0$ and $2x - y - 1 = 0$.

These yield $x = 2$, $y = 3$.

10. Let A = {1, 3, 5), B = {2, 3}, and C = {4, 6}. Write down the following :

1. A × B **2. B × A**

3. B × C **4. A × C**

5. (A ∪ B) × C **6. A ∪ (B × C)**

7. (A × B) ∪ C **8. A ∩ (B × C)**

9. (A × B) ∪ (B × C) **10. (A × B) ∩ (B × A)**

11. (A × B) ∩ (B × C)

Sol :

By using the definition of the product of sets, we find that

1. A × B = {(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)},
2. B × A = {(2, 1), (2, 3), (2, 5), (3 ,1), (3, 3), (3, 5)},
3. B × C = {(2, 4), (2, 6), (3, 4), (3, 6)},
4. A × C = {(1, 4), (1, 6), (3. 4), (3, 6), (5, 4), (5, 6)},
5. (A ∪ B) × C = {1, 2, 3, 5,) × (4, 6)
   = {(1, 4), (1, 6), (2, 4), (2, 6), (3, 4), (3, 6), (5, 4), (5, 6)}
6. A ∪ (B × C) = {1, 3, 5} ∪ {(2, 4), (2, 6), (3. 4), (3, 6)}
   = {1, 3, 5. (2, 4), (2, 6), (3, 4), (3, 6)}
7. (A × B) ∪ C = {(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3), 4, 6}
8. A ∩ (B × C) = Φ.
9. (A × B) ∪ (B × C) = {(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3), (2, 4), (2, 6), (3, 4), (3, 6)},
10. (A × B) ∩ (B × A) = {(3, 3)}
11. (A × B) ∩ (B × C) = Φ

11. For any set A ⊆ U, prove that

A × Φ = Φ × A = Φ.

Sol :

Suppose A × Φ ≠ Φ.

Then, a × Φ has at least one element (a, b) in it such that a ∈ A and b ∈ Φ.

Now, b ∈ Φ means that Φ is not the null set. This is a contradiction. Therefore, A × Φ = Φ.

Similarly, Φ × A = Φ.

1.4 Binary Relation

Q12. Discuss briefly about Binary Relation.

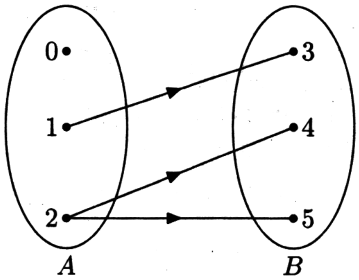
Ans :

Let A and B be two sets. Then a subset of A × B is called a relation from A to B. Thus, if R is a relation from A to B, then R is a (some) set of ordered pairs (a, b) where a ∈ A and b ∈ B, and conversely if R is a set of ordered pairs (a, b) where a ∈ A and b ∈ B, then R is a relation from A to B. If (a, b) ∈ R, we say that "a is related to b by R"; this is denoted by aRb

If R is a relation from A to A, that is, if R is a subset of A × A, we say that R is a binary relation on A.

For example, consider the sets A = {0, 1, 2}, B = {3, 4, 5}.

Let R = {(1, 3), (2, 4), (2, 5)}. Evidently, R is a subset of A × B. As such, R is a relation from A to B. and 1R3, 2R4, 2R5. This relation can be depicted in a digram as shown below, called the arrow diagram:



As another example, consider the sets A = {0, 1, –1} and B = {2, -2}.

Let

R_1 = {(0, 2), (1, 2), (–1, 2)} and R_2 = {(0, –2), (1, –2), (–1, –2)}.

Then R_1 and R_2 are subsets of A × B and are therefore relations from A to B. We observe that R_1 consists of elements (a, b) ∈ A × B for which the relationship a < b holds. Hence, here, "aR_1b" is read as "a is less than b", the symbol R_1 standing for the phrase "is less than". Further, R_2 consists of elements (a, b) ∈ A × B for which the relationship a > b holds. Hence, here, "aR_2b" is read as "a is greater than b", the symbol R_2 standing for the phrase "is greater than". The arrow diagrams of R_1 and R_2 are shown in Figure (a) and (b) respectively.

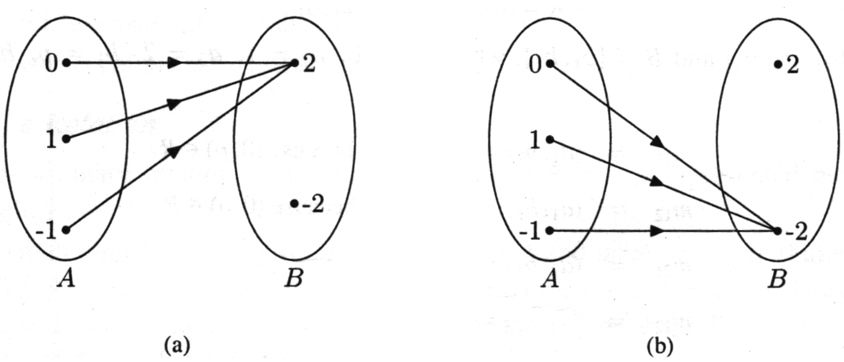


Fig.

As yet another example, we note that if

$$A = \left\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{2}{4},\frac{3}{12}\right\} \text{ then}$$

$$R = \left\{\left(\frac{1}{2},\frac{2}{4}\right), \left(\frac{1}{4},\frac{3}{12}\right)\right\}$$

is a binary relation on A with aRb standing for the statement "a is equal to b".

Q13. Discuss the operations on relations.

Ans : **(Imp.)**

Since a relation is a subset of the Cartesian product of two sets, the set-theoretic operations may be used to construct new relations from given relations.

1. Union and Intersection of Relations

Given the relations R_1 and R_2 from a set A to a set B, the union of R_1 and R_2, denoted by $R_1 \cup R_2$, is defined as a relation from A to B with the property that $(a, b) \in R_1 \cup R_2$ if and only if $(a, b) \in R_1$ or $(a, b) \in R_2$.

Similarly, the intersection of R_1 and R_2, denoted by $R_1 \cap R_2$, is defined as a relation from A to B with the property that $(a, b) \in R_1 \cap R_2$ if and only if $(a, b) \in R_1$ and $(a, b) \in R_2$.

Evidently, $R_1 \cup R_2$ is the union of the sets R_1 and R_2 and $R_1 \cap R_2$ is the intersection of the sets R_1 and R_2 in the universal set $A \times B$.

2. Complement of a Relation

Given a relation fi from a set A to a set B, the complement of R. denoted by R, is defined as a relation from A to fi with the property that $(a, b) \in \bar{R}$ if and only if $(a, b) \notin R$. In other words, $\bar{R}$ is the complement of the set R in the universal set $A \times B$.

3. Converse of a Relation

Given a relation R from a set A to a set B, the converse*** of R, denoted by R^C, is defined as a relation from B to A with the property that $(a, b) \in R^C$ if and only if $(b, a) \in R$.

From the definition of R^C the following results are immediate:

(i) If M_R is the matrix of R, then $[M_R]^T$, the transpose of M_R, is the matrix of R^C

(ii) $(R^C)^C = R$.

Q14. Explain the properties of relations.

Ans : **(Imp.)**

A relation R on a set A is said to be reflexive (or said to have the reflexive property) if $(a, a) \in R$, for all $a \in A$.

In other words, a relation R on a set A is reflexive whenever every element a of A is related to itself by R (i.e., aRa, for all $a \in A$).

It follows that fi is not reflexive if there is some a e A such that $(a. a) \notin R$.

For example, the relation "is less than or equal to" is a reflexive relation on the set of all real numbers. Because, a = a for every real number a.

It is obvious that the relations "is less than" and "is greater than" are not reflexive on the set of all real numbers.

As another example, we observe that if A = {1, 2, 3, 4), then the relation R = {(1, 1), (2, 2), (3, 3)} is not reflexive. Because, $4 \in A$ but $(4, 4) \notin R$.

The following results are easy to see:

1. The matrix of a reflexive relation must have 1's on its main diagonal.
2. At every vertex of the digraph of a reflexive relation there must be a cycle of length 1.
3. On a set A, the relation Δ defined by

$$\Delta_A = [(a, a) \mid a \in A|$$

is reflexive.*** Furthermore, Δ_A is a subset of every reflexive relation on A. The matrix of Δ_A contains 1's on the main diagonal and 0's in all other positions.

1. Irreflexive Relation

A relation on a set A is said to be irreflexive if $(a, a) \notin R$ for any $a \in A$. That is, a relation R is irreflexive if no element of A is related to itself by R.

For example, the relations "is less than" and "is greater than"are irreflexive on the set of all real numbers.

It is to be noted that an irreflexive relation is not the same as a non-reflexive relation. A relation can be neither reflexive nor irreflexive. For example, consider the relation R = {(1, 1), (1, 2)} defined on the set A = {1, 2, 3}. This relation is not reflexive because (2, 2) ∉ R and (3, 3) ∉ R. The relation is not irreflexive because (1, 1) ∉ R.

The following results are obvious:

1. The matrix of an irreflexive relation must have 0's on its main diagonal.
2. The digraph of an irreflexive relation has no cycle of length 1 at any vertex.

2. Symmetric Relation

A relation R on a set is said to be symmetric (or said to have the symmetric property) if (b, a) ∈ R whenever (a, b) ∈ R for all a, b ∈ A.

It follows that R is not symmetric if there exist a, b ∈ A such that (a, b) ∈ R but (b, a) ∉ R.

A relation which is not symmetric is called an asymmetric relation.

For example, if A = {l, 2, 3} and R_1 = {(1, 1), (1, 2), (2, 1)} and R_2 = {(1, 2), (2, 1), (1, 3)} are relations on A, then R_1 is symmetric but R_2 is asymmetric; because (1, 3) ∈ R_2 but (3, 1) ∉ R_2.

It is evident that for the matrix $M_R = [m_{i,j}]$ of a symmetric relation the following property holds:

If $m_{ij} = 1$ then $m_{ij} = 1$, and if $n_{ij} = 0$ then $m_{ji} = 0$.

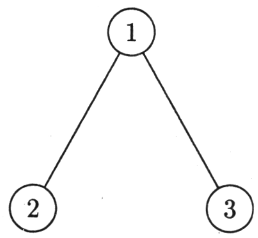
This means that the matrix M_R of a symmetric relation R is such that the $(i, j)^{th}$ element of M_R is equal to the $(j, i)^{th}$ element of M_R. In other words, the matrix of a symmetric relation is a symmetric matrix.

In the digraph of a symmetric relation, if there is an edge from vertex a to a vertex b, then there is an edge from b to a; this means that if two vertices are connected by an edge, they must always be connected in both directions. Because of this, in a digraph of a symmetric relation, the edges are shown without arrows — the arrows are understood both ways. The digraph of a symmetric relation is called the graph of the relation and an edge connecting two vertices a and b is always a bi-directed edge; it is denoted by {a, b}. Two vertices a and fi of a graph which are connected by an edge are called adjacent vertices.

For example, consider the relation

R = {(1, 2), (2, 1), (1, 3), (3, 1)}

on the set A = {1, 2, 3}. Evidently, this relation is a symmetric relation, and its graph is as shown below:



In the above graph, 1 and 2 are adjacent vertices, 1 and 3 are adjacent vertices, but 2 and 3 are not adjacent vertices.

3. Antisymmetric Relation

A relation fi on a set A is said to be antisymmetric (or said to have the antisymmetric property) if whenever (a, b) ∈ R and (b, a) ∈ R then a = b.

It follows that R is not antisymmetric if there exist a, b ∈ A such that (a, b) ∈ R and (b, a) ∈ R but a ≠ b.

For example, the relation "is less than or equal to" on the set of all real numbers is an antisymmetric relation (because if a ≤ b and b ≤ a, then a = b).

It should be emphasized that asymmetric (i.e., not symmetric) and antisymmetric relations are not one and the same. A relation can be both symmetric and antisymmetric. A relation can be neither symmetric nor antisymmetric.

For example, let A = {1, 2, 3} and R_1 = {(1, 1), (2, 2)} and R_2 = {(1, 2), (2, 1), (2, 3)}. We check that R_1 is both symmetric and antisymmetric, and R_2 is neither symmetric nor antisym-metric.

The following results are obvious:

1. If $M_R = [m_{ij}]$ is the matrix of an antisymmetric relation, then, for i ≠ j, we have either $m_{ij} = 0$ or $m_{ji} = 0$.

2. In the digraph of an antisymmetric relation, for two different vertices a and fi, there cannot be a bidirectional edge between a and b.

4. Transitive Relation

A relation fi on a set A is said to be transitive (or said to have the transitive property) if whenever (a, b) $\in$ R and (b, c) $\in$ R then (a, c) $\in$ R, for all a, b, c $\in$ A.

It follows that fi is not transitive if there exist a, b, c $\in$ A such that (a, b) $\in$ R and (b, c) $\in$ R but (a, c) $\notin$ R.

For example, the relations "is less than or equal to" and "is greater than or equal to" are transitive relations on the set of all real numbers. Because, if a $\leq$ b and b $\leq$ c then a $\leq$ c, and if a $\geq$ b and b $\geq$ c then a $\geq$ c, for all real numbers a, b, c.

As another example, if we consider the set A = {1, 2, 3) and the relations R_1 = {(1, 1), (1, 2), (2, 3), (1, 3), (3, 1), (3, 2)} and R_2 = {(1, 2), (2, 3), (1, 3), (3, 1)} on A, then R_1 is transitive but R_2 is not transitive.

The following results are easy to prove:

1. A relation fi on a set A is transitive if and only if its matrix $M_R = [m_{ij;}]$ has the following property:

   If m_{ik} = 1 and m_{ij} = 1. then m_{ij} = 1.

2. A relation R on set A is transitive if and only if $R^n \subseteq R$ for all n $\geq$ 1.

PROBLEMS

15. Let A = {1, 2, 3}. Determine the nature of the following relations on A:

(i) R_1 = {(1, 2), (2, 1), (1, 3), (3, 1)}

(ii) R_2 = {(1. 1), (2, 2), (3, 3), (2, 3)}

(iii) R_3 = {(1, 1), (2, 2), (3, 3)}

(iv) R_4 = {(1, 1), (2, 2), (3, 3), (2, 3), (3,2)}

(v) R_5 = {(1, 1), (2, 3), (3, 3)}

(vi) R_6 = {(2, 3), (3, 4), (2, 4))

(vii) R_7 = {(1, 3), (3, 2)}.

Sol :

By examining all ordered pairs present in the relations given, we find that:

R_1 is symmetric and irreflexive, but neither reflexive nor transitive.

R_2 is reflexive and transitive, but not symmetric.

R_3 and R_4 are both reflexive and symmetric; that is, they are compatibility relations.

R_5 is neither reflexive nor symmetric.

R_6 is transitive and irreflexive, but not symmetric.

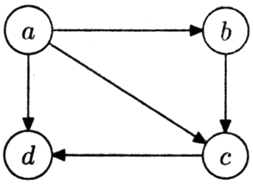
R_7 is irreflexive, but neither transitive nor symmetric.

16. Let A = {1, 2, 3, 4}. Determine the nature of the following relations on 4

(i) R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)}

(ii) R_2 = {(1, 2), (1, 3), (3, 1), (1, 1), (3, 3), (3, 2), (1, 4), (4, 2), (3, 4)}

(iii) R_3 represented by the following digraph:



Sol :

By examining all ordered pairs present in R_1 and R_2, we find that:

(i) R_1 is reflexive, symmetric and transitive, and

(ii) R_2 is transitive.

By examining the edges in the digraph in Figure, we find that the relation R_3 is both asymmetric and antisymmetric.

17. Find the nature of the relations represented by the following matrices:

(a) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Sol :

(a) Here, the given matrix is symmetric (that is, $a_{ji} = a_{ij}$ for i, j = 1, 2, 3). Therefore, the corresponding relation is symmetric.

(b) Here, the given matrix has 1's on its main diagonal and is symmetric. Therefore, the corresponding relation is reflexive and symmetric (i.e., it is a compatibility relation).

(c) Here, the given matrix is not symmetric. Therefore, the corresponding relation is not symmetric. Further, the presence of 1 in the $(1, 4)^{th}$ and $(4, 1)^{th}$ positions of the matrix indicates that the relation is not antisymmetric.

18. Show that the relation R represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

is transitive.

Sol :

Let A = {a, b, c} be the set on which R is defined. Then, by examining the given M_R, we find that

R = {(a, a), (a, b), (a, c), (b, c), (c, c)}

By examining the elements of R, we find that R is transitive.

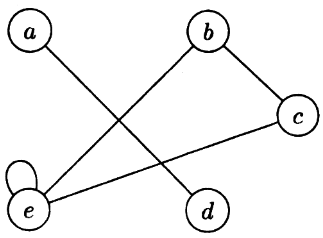
19. Let A = {a, b, c, d, e}, and

R = {(a, d), (d, a), (c, b), (b, c), (c, e), (e, c), (b, e), (e, b), (e, e)}

be a symmetric relation on A. Draw the graph of R.

Sol :

By examining the elements in R, we find that the graph of R is as shown in Figure



20. Consider the set A = {ball, bed, dog, let, egg} and define the relation R on A by

R = {(x, y) | x, y ∈ A and xRy if x and y contain some common letter}.

Verify that R is a compatibility relation which is not transitive. Draw the graph of R.

Sol :

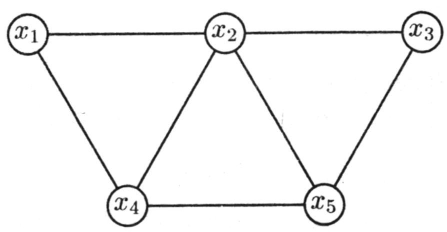
From the definition of R, we readily note that xRx for all x ∈ A, and yRx whenever xRy for all x, y ∈ A. Hence R is reflexive and symmetric; that is fi is a compatibility relation.

Let x_1, x_2, x_3, x_4, x_5 respectively denote the words ball, bed, dog, let, egg that belong to A.

Then, A = $\{x_1, x_2, x_3, x_4, x_5\}$

We note that $(x_1, x_2) \in R$ and $(x_2, x_3) \in R$, but $(x_1, x_3) \notin R$ (- because x_1 and x_3 do not have a common letter). As such, R is not a transitive relation.

The graph of R is as shown in Figure. In this graph, loops are understood but not shown and all edges are bidirectional.



21. On the set Z^+, a relation R is defined by aRb if and only if a divides b (exactly). Prove that R is reflexive, transitive and antisymmetric, but not symmetric.

Sol :

For any a ∈ Z^+, the statement "a divides a" is true. Thus, aRa for all a ∈ Z^+.

Hence R is reflexive.

Next, we note that, for any a, b ∈ Z^+, "a divides b" need not imply that "b divides a" (For instance, 3 divides 6 but 6 does not divide 3). Thus, aRb does not always imply bRa. Hence R is not symmetric.

Further, "a divides b" and "b divides a" imply that a = b. Thus aRb and bRa imply a = b. Therefore, R is antisymmetric.

Lastly, we note that for any a, b, c $\in Z^+$, "a divides b" and "b divides c" imply that "a divides c". Thus aRb and bRc imply aRc. Hence R is transitive.

22. Let S be a nonempty set. On 'P(S), define a relation R by (A, B) $\in$ R if and only if A $\subseteq$ R. Prove that R is reflexive, antisymmetric and transitive, but not symmetric.

Sol :

For any subset A of S (i.e., for any A $\in$ P(S)), we have A $\subseteq$ A. This means that (A, A) $\in$ R. Hence R is reflexive on 'P(S).

Next, we note that for any A, B $\in$ P(S), A $\subseteq$ B and B $\subseteq$ A imply A = B. That is, (A, B) $\in$ R and (B, A) $\in$ R imply A = B. Hence fi is antisymmetric on P(S).

Further, for any A, B, C $\in$ P(S), if A $\subseteq$ B and B $\subseteq$ C, we have A $\subseteq$ C. That is, if (A, B) $\in$ R and (B, C) $\in$ R, then (A, C) $\in$ R. Hence R is transitive on P(S).

Finally, we note that for any A, B = P(S), A $\subseteq$ R does not necessarily imply that B $\subseteq$ A. That is, (A, B) $\in$ R does not always imply that (B, A) $\in$ R. Therefore, R is not symmetric on P(S).

23. Let R be a relation on a set A. Prove the following:

(i) R is reflexive if and only if $\bar{R}$ is irreflexive.

(ii) If R is reflexive, so is R^c.

(iii) If R is symmetric, so are R^c and R.

(iv) IfR is transitive, so is R^c.

Sol :

(i) Suppose R is reflexive. Then (a, a) $\in$ R for every a $\in$ A. Consequently, (a, a) $\notin \bar{R}$ for any a $\in$ A. This means that R is irreflexive. Reversing the steps, we find that if R is irreflexive then fi is reflexive.

(ii) Suppose R is reflexive. Then (a, a) $\in$ R for all a $\in$ R. Consequently, (a, a) $\in$ R as well. Therefore, R^C is reflexive.

(iii) Take any (a, b) $\in R^C$. Then (b, a) $\in$ R. Consequently, (a, b) $\in$ R, because R is symmetric. This implies (b, a) $\in R^C$. Thus, R^C is also symmetric.

Next take any (a, b) $\in \bar{R}$. Then (a, b) $\notin$ R. Consequently, (b, a) $\notin$ R, because R is symmetric. This implies (b, a) $\in$ R. Thus, R is also symmetric.

(iv) Take any (a, b), (b, c) $\in R^C$. Then (b, a), (c, b) $\in$ R. This implies that (c, a) $\in$ R. because R is transitive. Therefore (a, c) $\in R^C$. Thus, R^C is transitive.

24. Let R and S be relations on a set A. Prove the following:

(i) If R and S are reflexive, so are R $\cap$ S and R $\cap$ S.

(ii) If R and S are symmetric, so are R $\cap$ S and R $\cup$ S.

(iii) If R and S are antisymmetric, so is R $\cap$ S.

(iv) If R and S are transitive, so is R $\cap$ S.

Sol :

(i) Suppose R and S are reflexive. Then (a, a) $\in$ R and (a, a) $\in$ S for all a $\in$ A. Consequently, (a, a) $\in$ R $\cap$ S and (a, a) $\in$ R $\cup$ S. Therefore, R $\cap$ S and R $\cup$ S are reflexive.

(ii) Suppose R and S are symmetric. Take any (a, b) $\in$ R $\cap$ S. Then (a, b) $\in$ R and (a, b) $\in$ S. Therefore, (b, a) $\in$ R and (b, a) $\in$ S. Consequently, (b, a) $\in$ R $\cap$ S. Hence R $\cap$ S is symmetric.

Next, take any (x, y) $\in$ R $\cup$ S. Then (x, y) $\in$ R or (x, y) $\in$ S. Therefore, (y, x) $\in$ R or (y, x) $\in$ S. Consequently, (y, x) $\in$ R $\cup$ S. Hence R $\cup$ S is symmetric.

(iii) Suppose R and S are antisymmetric. Take any (a, b), (b, a) $\in$ R $\cap$ S. Then (a, b), (b. a) $\in$ R and (a, b), (b, a) $\in$ S. By the antisymmetry of R (or S), it follows that b = a. Thus. R $\cap$ S is antisymmetric.

(iv) Suppose R and S are transitive. Take (a, b), (b, c) $\in R \cap S$. Then (a, b) $\in$ R, (a, b) $\in$ S, (b, c) $\in$ R, (b, c) $\in$ S. These yield (a, c) $\in$ R and (a, c) $\in$ S, so that (a, c) $\in R \cap S$. Therefore, $R \cap S$ is transitive.

1.5 Partial Ordering Relation

Q25. Discuss about Partial Ordering Relation.

Ans : **(Imp.)**

A relation R on set A is said to be a partial ordering relation or a partial order on A if

(i) R is reflexive,

(ii) R is antisymmetric, and

(iii) R is transitive, on A.

A set A with a partial order R defined on it is called a partially ordered set or an ordered set or a poset, and is denoted by the pair (A, R).

The most familiar partial order is the relation "less than or equal to", denoted by $\leq$, on the set Z of all integers. (Because, this relation is reflexive, antisymmetric and transitive). Thus, (Z, $\leq$) is a poset.

The relation "is greater than or equal to", denoted by $\geq$, is also a partial order on Z; that is, (Z, $\geq$) is also a poset.

The "divisibility relation" on the set Z^+ defined by a divides b (denoted by a/b) for all a. b $\in Z^+$ is a partial order on Z^+.

The "subset relation" $\subseteq$ defined on the power set of a set S is a partial order on S;

Thus, for any set 5, P(S) $\subseteq$) is a poset.

The relations "is less than" and "is greater than" are not partial orders on Z; because, these are not reflexive.

The relation "congruent modulo n" defined on the set of all integers Z is also not a partial order; because this relation is not antisymmetric.

Total Order

Let R be a partial order on a set A. Then R is called a total order (or a linear order) on A if for all x, y $\in$ A, either xRy or yRx. In this case, the poset (A, R) is called a totally ordered set (or a linearly ordered set) or a chain.

For example, the partial order relation "less than or equal to" is a total order on the set R. Because, for any x,y $\in$ R, we have x $\leq$ y or y $\leq$ x. Thus, (R, $\leq$) is a totally ordered set.

If we consider the divisibility relation on the set A = {1, 2, 4, 8}, this relation is a total order on A. The same relation is not a total order on the set A = {1, 2, 4, 6, 8} although it is a partial order on A. (Observe that neither 4 divides 6 nor 6 divides 4).

The subset relation is also not a total order on the power set of an arbitrary set S although it is a partial order; because for any two subsets S_1 and S_2 of S, neither $S_1 \subseteq S_2$ nor $S_2 \subseteq S_1$ can be true. (For example, if S = {1, 2, 3}, S_1 = {1, 2} and S_2 = {1, 3}, then $S_1 \subseteq S_2$ and $S_2 \subseteq S$ but $S_1 \not\subset S_2$ and $S_2 \not\subset S_1$).

From the definition of a total order and the examples given above it is clear that every total order is a partial order, but not every partial order is a total order.

Hasse Diagram

Since a partial order is a relation on a set, we can think of the digraph of a partial order if the set is finite. Since a partial order is reflexive, at every vertex in the digraph of a partial order there would be a loop. In view of this, while drawing the digraph of a partial order, we need not exhibit such loops explicitly; they will be automatically understood (by convention).

If, in the digraph of a partial order, there is an edge from a vertex a to a vertex b and there is an edge from the vertex b to a vertex c, then there should be an edge from a to c (because of transitivity). As such, we need not exhibit an edge from a to c explicitly; it will be automatically understood (by convention).

To simplify the format of the digraph of a partial order, we represent the vertices by dots (bullets) and draw the digraph in such a way that all edges point upward. With this convention, we need not put arrows in the edges.

The digraph of a partial order drawn by adopting the conventions indicated in the above paragraphs is called a poset diagram or the Hasse diagram for the partial order.

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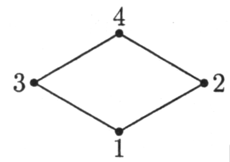
26. Let A = {1, 2, 3, 4}, and R = {(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3),(3, 4), (1,4), (4,4)} Verify that R is a partial order on A. Also, write down the Hasse diagram for R.

Sol :

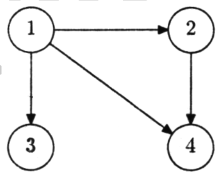
We observe that the given relation R is reflexive and transitive. Further, R does not contain ordered pairs of the form (a, b) and (b, a) with b $\neq$ a. Therefore, R is antisymmetric. As such, fi is a partial order on A.

The Hasse diagram for R must exhibit the relationships between the elements of A as defined by R; if (a, b) $\in$ R, there must be an upward edge from a to R.

By examining the ordered pairs contained in R, we find that the Hasse diagram of R is as shown below:



27. A partial order R on the set A = {1, 2, 3, 4} is represented by the following digraph. Draw the Hasse diagram for R.

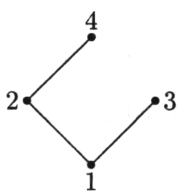


Sol :

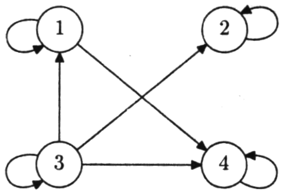
By observing the given digraph, we note that

R = {(1, 2), (1, 3), (1, 4), (2, 4)}.

The Hasse diagram for this R is as shown below:



28. The diagram for a relation on set A = {1, 2, 3, 4} is as shown below :



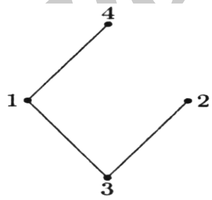
Verify that (A, R) is a poset and find its Hasse diagram.

Sol :

By examining the given digraph, we find that

R = {(1, 1), (1, 4), (2, 2), (3, 3), (3, 1), (3, 2), (3, 4), (4,4)}

We check that R is reflexive, transitive and antisymmetric. Therefore, R is a partial order on A. The Hasse diagram of R is as shown below.



29. If R is a relation on the set A = {1, 2, 3, 4} defined by xRy if x | y, prove that (A, R) is a poset. Draw its Hasse diagram.

Sol :

From the definition of R, we have

R = {(x, y) | x, y $\in$ A and x divides y}

= {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}

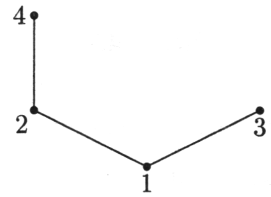
We observe that (a, a) $\in$ R for all a $\in$ A. Hence R is reflexive on A.

We verify that the elements of R are such that if (a, b) $\in$ R and a $\neq$ b, then (b, a) $\notin$ R. Therefore, R is antisymmetric on A.

Further, we check that the elements of R are such that if (a, b) $\in$ R and (b, c) $\in$ R then (a, c) $\in$ R. Therefore, A is transitive on A.

Thus, R is reflexive, antisymmetric and transitive. Hence R is a partial order on A; that is, (A, R) is a poset.

The Hasse diagram for R is as shown below.



30. Let A = {1 ,2, 3, 4, 6, 12}. On A, define the relation R by aRb if and only if a divides b. Prove that R is a partial order on A. Draw the Hasse digram for this relation.

Sol :

From the definition of R, we note that

R = {(a, b) | a, b $\in$ A and a divides b}

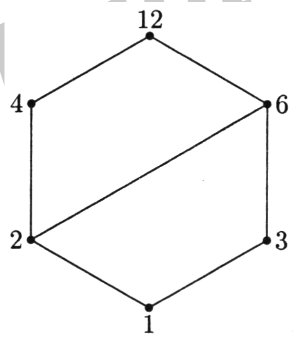
= {(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (1, 12), (2, 2), (2, 4), (2, 6)

(2, 12), (3, 3), (3, 6), (3, 12), (4, 4), (4, 12), (6, 6), (6, 12), (12, 12)}

Evidently, (a, a) $\in$ R for all a $\in$ A. Therefore, R is reflexive.

We check that the elements of R are such that if (a, b) $\in$ R and (b, c) $\in$ R then (a, c) $\in$ R. Therefore, R is transitive. Further, for all a, b $\in$ A, if a divides b and b divides a, then a = b. Hence R is antisymmetric.

Therefore, R is a partial order on A. The Hasse diagram for R is shown in Figure



31. Draw the Hasse diagram representing the positive divisors of 36.

Sol :

The set of all positive divisors of 36 is

D_{36} = {1, 2, 3, 4, 6, 9, 12, 18, 36} (*)

The relation R of divisibility (aRb if and only if a divides b) is a partial order on this set. The Hasse diagram for this partial order is required here. We note that, under R,

1 is related to all elements of D_{36}

2 is related to 2, 4, 6, 12, 18, 36;

3 is related to 3, 6, 9, 12, 18, 36;

4 is related to 4, 12, 36;

6 is related to 6, 12, 18, 36;

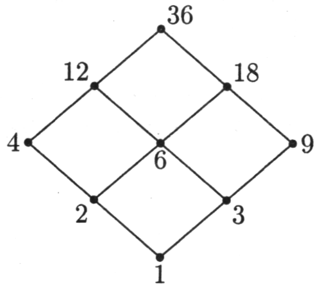
9 is related to 9, 18, 36;

12 is related to 12 and 36;

18 is related to 18 and 36;

36 is related to 36.

The Hasse diagram for R must exhibit all of the above facts. The diagram is as shown below:



32. Show that the set of all positive integers is not totally ordered by the relation of divisibility.

Sol :

For a set A to be totally ordered by a partial order R, we should have aRb or bRa, for every a. b $\in$ A.

If R is the divisibility relation on Z^+, aRb or bRa need not hold for every a, b $\in$ Z^+. For example, if we take a = 2 and b = 3, then a does not divide b and b does not divide a.

Therefore, Z^+ is not totally ordered by the relation of divisibility.

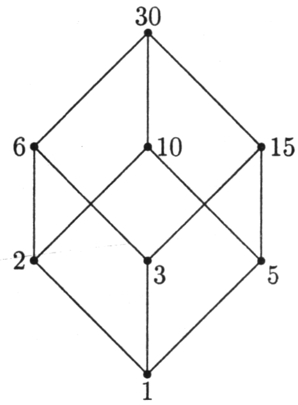
33. In the following cases, consider the partial order of divisibility on the set A. Draw the Hasse diagram for the poset and determine whether the poset is totally ordered or not.

(i) A = {1, 2, 3, 5, 6, 10, 15, 30}

(ii) A = {2, 4, 8, 16, 32}

Sol :

The Hasse diagram for the two cases are as shown below:

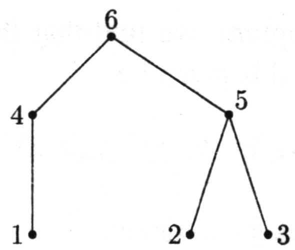


(a)

(b)

By examining the above Hasse diagrams, we find that the given relation is totally ordered in case (ii), but is not totally ordered in case (i).

34. The Hasse diagram of a partial order R on the set A = {1, 2, 3, 4, 5, 6} is as given below. Write down R as a subset of A × A. Construct its digraph.



Sol :

By examining the given Hasse diagram, we note the following:

1R4, 1R6, 2R5, 2R6, 3R5, 3R6, 4R6, 5R6.

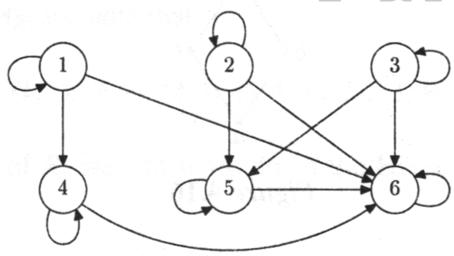
Also, by the convention used in Hasse diagrams,

1R1, 2R2, 3R3, 4R4, 5R5, 6R6.

Therefore,

R = {(1, 1), (1, 4), (1, 6), (2, 2), (2, 5), (2, 6), (3, 3), (3. 5), (3, 6), (4, 4),
(4, 6), (5, 5), (5, 6), (6, 6))

The digraph of this relation is as shown below.



1.6 Equivalence Relation

Q35. Define about Equivalence Relation.

Ans :

A relation R on a set A is said to be an equivalence relation on A if (i) R is reflexive, (ii) R is symmetric, and (iii) R is transitive, on A.

A trivial example of an equivalence relation is the relation "is equal to" on the set of all real numbers. R. An example of a relation which is not an equivalence relation is the relation "is less than " on R.

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36. Let A = {1, 2, 3, 4) and R = {(1, 1), (1, 2), (2.1), (2, 2), (3, 4), (4. 3), (3, 3), (4. 4)} be a relation on A. Verify that R is an equivalence relation.

Sol :

We have to show that R is reflexive, symmetric and transitive.

First, we note that all of (1, 1), (2, 2), (3, 3), (4, 4) belong to R. That is, (a, a) $\in$ R for all a $\in$ A. Therefore, R is a reflexive relation.

Next, we note the following:

(1, 2), (2, 1) $\in$ R and (3, 4), (4, 3) $\in$ R.

That is, if whenever (a, b) $\in$ R then (b, a) $\in$ R for a, b $\in$ A. Therefore, R is a symmetric relation. Lastly, we note that

(1, 2), (2, 1), (1, 1) $\in$ R, (2, 1), (1, 2), (2, 2) $\in$ R,

(4, 3), (3, 4), (4, 4) $\in$ R.

That is, if whenever (a, b) $\in$ R and (b, c) $\in$ R then (a, c) $\in$ R, for a, b, c $\in$ A. Therefore, R is a transitive relation.

Accordingly, R is an equivalence relation.

37. Let A = {1, 2, 3, 4}, and R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3), (1, 3), (4, 1), (4, 4)} be a relation on A. Is R an equivalence relation?

Sol :

Here, we have to check whether or not R is reflexive, symmetric and transitive.

By examining the elements of R, we note the following:

(i) (a, a) $\in$ R for every of a $\in$ R, Therefore R is reflexive.

(ii) (4, 1) $\in$ R, but (1, 4) $\in$ R. Therefore, R is not symmetric.

Since R is not symmetric, R is not an equivalence relation. (We need not check fi for transitivity).

38. If A = $A_1 \cup A_2 \cup A_3$, where A_1 = {1, 2}, A_2 = {2, 3, 4} and A_3 = {5}, define the relation R on A by xRy if any only if x and y are in the same set A_i, i = 1, 2, 3. Is R an equivalence relation ?

Sol :

We note that xRx for every x in A, because x and x belong to the same A_i.

Therefore, R is reflexive.

Further, if x, y $\in$ A_i then y, x $\in$ A; for all x, y in A. Therefore, R is symmetric.

Lastly, we observe that (1, 2) $\in$ R (because 1 and 2 are in the same set, A_1) and (2, 3) $\in$ R (because 2 and 3 are in the same set, A_2) but (1, 3) $\in$ R (because 1 and 3 are not in the same set). Hence R is not transitive.

Accordingly, R is not an equivalence relation. It is just a compatibility relation.

39. A relation R on a set A = {a, b, c} is represented by the following matrix:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determine whether R is an equivalence relation.

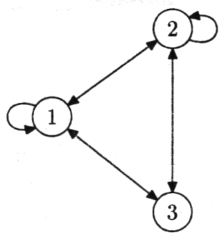
Sol :

By examining the elements of M_R, we find that R = {(a, a), (a, c), (b, b), (c, c)}.

We note that (a, c) $\in$ R but (c, a) $\notin$ R. Therefore, R is not symmetric. Accordingly, R is not an equivalence relation.

(That R is not symmetric can also be seen by the fact that the matrix M_R is not symmetric.)

40. The digraph of a relation R on the set A = {1, 2, 3} is as given below. Determine whether R is an equivalence relation.



Sol :

By examining the digraph, we note that the given relation is symmetric and transitive but not reflexive; observe that (3, 3) $\notin$ R. Therefore, R is not an equivalence relation.

41. Let S be the set of all non-zero integers, and A = S × S. On A, define the relation R by (a, b)R(c, d) if and only if ad = bc. Show that R is an equivalence relation.

Sol :

First, we note that (a, a)R(a, a), because aa = aa for any a $\in$ S. Therefore, R is reflexive on A.

Next, suppose (a, b)R(c, d). Then ad = bc and therefore cb = da. Hence (c, d)R(a, b). Accordingly, R is symmetric on A.

Lastly, suppose that (a,b)R(c,d) and (c,d)R(e, f). Then ad = bc and cf = de, which yield af = be. Hence (a,b)R(e, f). Accordingly, R is transitive on A.

This proves the required result.

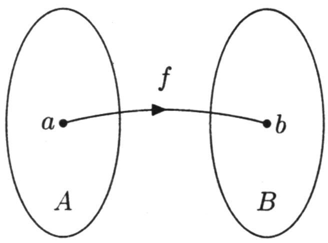
1.7 Functions

Q42. Discuss about functions.

Ans :

Let A and B be two non-empty sets. Then a function (or mapping) f from A to B is a relation from A to B such that for each a in A there is a unique b in B such that (a, b) $\in$ f. Then we write b = f(a). Here, b is called the image of a, and a is called a preimage of b, under f. The element a is also called an argument of the function f, and b = f(a) is then called the value of the function f for the argument a.

A function f from A to B is denoted by f : A $\to$ B. The pictorial representation of f is as shown below.



It has to be emphasized that every function is a relation, but a relation need not be a function. Because, if R is a relation from A to B then an element of A can be related to two different elements of B, under R. This is not the case in respect of a function from A to B; under a function an element of A can be related to only one element of B.

For the function f : A $\to$ B, A is called the domain of f and B is called the co-domain of f.

The subset of B consisting of the images of all elements of A under f is called the range of f and is denoted by f(A).

The following observations are immediate consequences of the definition of a function f : A $\to$ B and other associated definitions given above.

(i) Every a in A belongs to some pair (a, b) $\in$ f, and if (a, b_1) $\in$ f and (a, b_2) $\in$ f, then b_1 = b_2. This means that every element of A has an image in B (under f) and if an element a of A has two images in B, then the two images cannot be different.

(ii) An element b $\in$ B need not have a preimage in A, under f.

(iii) If an element b $\in$ B has a preimage a $\in$ A under f, the preimage need not be unique. In other words, two different elements of A can have the same image in B, under f.

(iv) The statements (a, b) $\in$ f, afb and b = f(a) are equivalent (in the sense that they all carry the same meaning).

(v) If g is a function from A to B (denoted by g : A $\to$ B), then f = g if and only if f(a) = g(a) for every a $\in$ A.

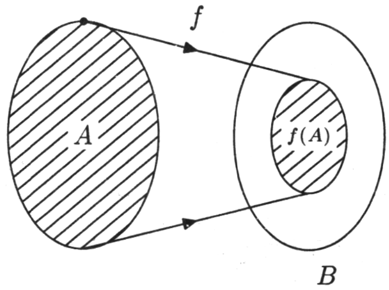
(vi) It is not necessary that B $\neq$ A. That is, one can have a function from A to itself. A function from A to A is called a unary (or monary) operation on A.

For example, if R^+ denotes the set of all positive real numbers, the function f : $R^+ \to R^+$ defined by f(a) = $\frac{1}{a}$ is a unary operation on R^+.

(vii) The range of f : A $\to$ B is given by

$$f(A) = \{f(x) \mid x \in A\}$$

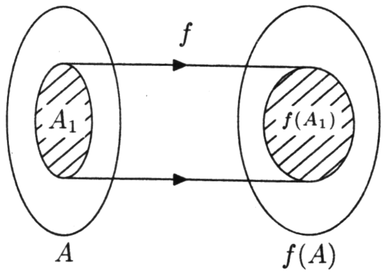
and f(A) is a subset of B.



(viii) For $f : A \rightarrow B$, if $A_1 \subseteq A$ and $f(A_1)$ is defined by

$$f(A_1) = \{f(x) \mid x \in A_1\},$$

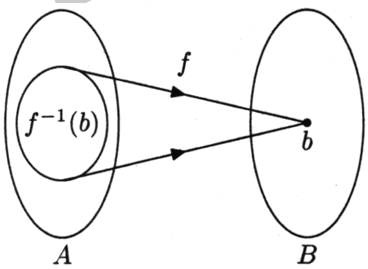
then $f(A_1) \subseteq f(A)$. (Here, $f(A_1)$ is called the image of A_1 under f).



(ix) For $f : A \rightarrow B$, if $b \in B$ and $f^{-1}\{b)$ is defined by

$$f^{-1}(b) = \{x \in A \setminus f(x) = b\},$$

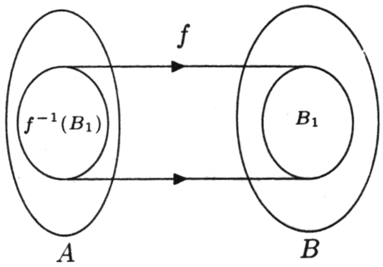
then $f^{-1}(b) \subseteq A$. (Here, $f^{-1}(b)$ is called the preimage set of b under f)



(x) For $f : A \rightarrow B$, if $B_1 \subseteq B$ and $f^{-1}(B_1)$ is defined by

$$f^{-1}(B_1) = \{x \in A \mid f(x) \in B_1\},$$

then $f^{-1}(B_1) \subseteq A$. (Here, $f^{-1}(B_1)$ is called the preimage of B_1 under f).



1.7.1 Types of Functions

1.7.1.1 Sum and Product of Functions, Bijective functions, Inverse and Composite Function

Q43. Discuss various types of functions.

Ans : **(Imp.)**

1. Sum and Product of Functions

The Rule of Sum and Rule of Product are used to decompose difficult counting problems into simple problems.

Sum Rule Principle

Assume some event E can occur in m ways and a second event F can occur in n ways, and suppose both events cannot occur simultaneously. Then E or F can occur in m + n ways. In general, if there are n events and no two events occurs in same time then the event can occur in $n_2 + n_2$ n ways.

Product Rule Principle

Suppose there is an event E which can occur in m ways and, independent of this event, there is a second event F which can occur in n ways. Then combinations of E and F can occur in mn ways. In general, if there are n events occurring independently then all events can occur in the order indicated as $n_1 \times n_2 \times n_3$ n ways.

2. Bijection

From the definitions of onto and one-to-one functions, we note that a function f : A $\rightarrow$ B is both one-to-one and onto if every element of A has a unique image in B and every element of B has a unique preimage in A. A function which is both one-to-one and onto is called a bijective function (or bijection).

A bijective function is also called a one-to-one correspondence.

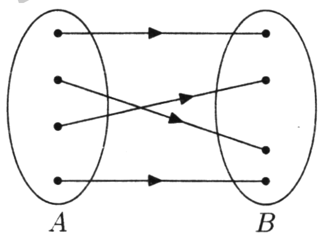
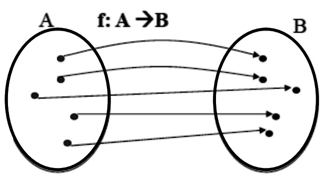


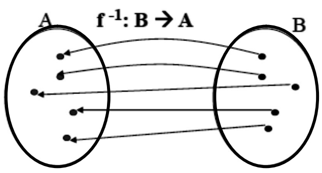
Fig. : One-to-one correspondence.

3. Inverse and Composite Function

Definition: Let f be a bijection from set A to set B. The inverse function of f is the function that assigns to an element b from B the unique element a in A such that f(a) = b. The inverse function of f is denoted by f-1. Hence, f-1 (b) = a, when f(a) = b. If the inverse function of f exists, f is called invertible.



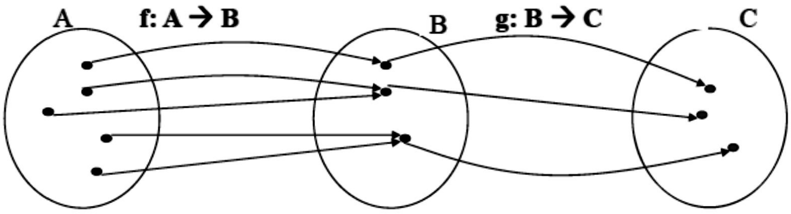
f is bijective



Inverse of f

Composition of Functions

Two functions f: $A \to B$ and g: $B \to C$ can be composed to give a composition g o f. This is a function from A to C defined by (gof)(x) = g(f(x))



PROBLEMS

44. Find the nature of the following functions defined on the set A = (1, 2, 3).

(i) f = {(1, 1), (2, 2), (3, 3)}

(ii) g = {(1, 2), (2, 2), (3, 2))

(iii) h = {(1, 2), (2, 2), (3, 1)}

(iv) p = {(1, 2), (2, 3), (3, 1)}

Sol :

(i) We note that for every $a \in A$, $(a, a) \in f$; that is, a = f(a). Therefore, f is the identity function on A.

(ii) We note that every $a \in A$ has 2 as its image; that is, g(1) = 2, g(2) = 2 and g(3) = 2. Therefore, g is a constant function.

(iii) We note that h is neither the identity function nor a constant function. The range of h is $\{2,1\} \subset A$: the element 3 has no preimage under h. Therefore, h is not onto. We further note that both of 1 and 2 have the same image 2 under h. Therefore, h is not one-to-one.

(iv) We note that every element of A has a unique image and every element of A has a unique preimage, under p. Therefore, p is both one-to-one and onto; it is a bijection. That is, p is a one-to-one correspondence.

45. The functions $f : R \to R$ and $g : R \to R$ are defined by f(x) = 3x + 7 for all $x \in R$, and $g(x) = x(x^3 - 1)$ for all $x \in R$. Verify that f is one-to-one but g is not.

Sol :

For any $x_1, x_2 \in R$, we have

$$f(x_1) = 3x_1 + 7,\ f(x_2) = 3x_2 + 7.$$

Evidently, if $f(x_1) = f(x_2)$ we have $3x_1 + 7 = 3x_2 + 7$ so that $x_1 = x_2$. Therefore, f is an one-to-one function.

Next, we note that g(0) = 0 and g(1) = 0. Thus, for $x_1 = 0$ and $x_2 = 1$ We have $g(x_1) = g(x_2)$ but $x_1 \neq x_2$. Therefore, g is not a one-to-one function.

46. The function $f : (Z \times Z) \to Z$ is defined by f(x, y) = 2x + 3y. Verify that f is onto but not one-to-one.

Sol :

Take any $n \in Z$. We note that

$$n = A_n - 3n = 2(2n) + 3(-n) = f(2n, -n)$$

Thus, every $n \in Z$ has a preimage $(2n, -n) \in Z \times Z$ under f. Therefore, f is an onto function.

Next, we check that $f(0, 2) = 2 \times 0 + 3 \times 2 = 6$ and $f(3, 0) = 2 \times 3 + 3 \times 0 = 6$. Thus, $f(0, 2) = f(3, 0)$, but $(0, 2) \neq (3, 0)$. Therefore, f is not one-to-one.

47. Let $f : Z \to Z$ be defined by $f(a) = a + 1$ for $a \in Z$. Find whether f is one-to-one or onto (or both or neither).

Sol :

Take any $a_1, a_2 \in Z$ with $a_1 \neq a_2$. Then $f(a_1) = a_1 + 1$ and $f(a_2) = a_2 + 1$. Since $a_1 \neq a_2$ it is evident that $f(a_1) \neq f(a_2)$. Thus, different elements of Z have different images under f. Therefore, f is one-to-one.

Take any $b \in Z$. We check that b has $b - 1$ as its preimage under f; because $f(b - 1) = (b - 1) + 1 = b$. Thus, every element of Z has a preimage. Therefore, f is onto.

Thus, f is both one-to-one and onto; that is, f is a one-to-one correspondence.

48. Let A = R and B - {x | x is real and $x \geq 0$}. Is the function $f : A \to B$ defined by $f(a) = a^2$ an onto function? a one-to-one function?

Sol :

Take any $b \in B$. Then b is a non-negative real number. Therefore, its square roots $\pm\sqrt{b}$ exist and are real numbers; that is $\pm\sqrt{b} \in A$. By the definition of f, we note that

$$f(\sqrt{b}) = (\sqrt{b})^2 = b \text{ and } f(-\sqrt{b}) = (-\sqrt{b})^2 = b.$$

Thus, $\pm\sqrt{b}$ are preimages of b under f. Since b is an arbitrary element f B. it follows that every element in B has a (at least one) preimage in A. Hence f is an onto function. Since $b \in B$ has two preimages $\pm\sqrt{b} \in A$ under f, it follows that f is not one-to-one.

Q49. Prove that $g : R \to R$, where $g(x) = 2x - 1$. What is the inverse function g–1 ?

Ans :

Let $g : R \to R$, where $g(x) = 2x - 1$

Approach to determine the inverse:

$y = 2x - 1 \Rightarrow y + 1 = 2x$

$\Rightarrow (y + 1)/2 = x$

Define $g-1(y) = x = (y+1)/2$

Test the correctness of inverse:

$g(3) = 2*3 - 1 = 5$

$g-1\ (5) = (5+1)/2 = 3$

$g(10) = 2*10 - 1 = 19$

$g-1\ (19) = (19+1)/2 = 10.$

1.8 Cantor's Diagonal Argument

Q50. Explain briefly about Cantor's Diagonal Argument.

Ans :

In set theory, Cantor's diagonal argument, also called the diagonalisation argument, the diagonal slash argument, the anti-diagonal argument, the diagonal method, and Cantor's diagonalization proof, was published in 1891 by Georg Cantor as a mathematical proof that there are infinite sets which cannot be put into one-to-one correspondence with the infinite set of natural numbers. Such sets are now known as uncountable sets, and the size of infinite sets is now treated by the theory of cardinal numbers which Cantor began.

The diagonal argument was not Cantor's first proof of the uncountability of the real numbers, which appeared in 1874. However, it demonstrates a general technique that has since been used in a wide range of proofs, including the first of Gödel's incompleteness theorems and Turing's answer to the Entscheidungsproblem. Diagonalization arguments are often also the source of contradictions like Russell's paradox and Richard's paradox.

Uncountable Set

Cantor considered the set T of all infinite sequences of binary digits (i.e. each digit is zero or one). He begins with a constructive proof of the following lemma:

If s_1, s_2, ... , s_n, ... is any enumeration of elements from T, then an element s of T can be constructed that doesn't correspond to any sn in the enumeration.

The proof starts with an enumeration of elements from T, for example

s_1 = (0, 0, 0, 0, 0, 0, 0, ...)

s_2 = (1, 1, 1, 1, 1, 1, 1, ...)

s_3 = (0, 1, 0, 1, 0, 1, 0, ...)

s_4 = (1, 0, 1, 0, 1, 0, 1, ...)

s_5 = (1, 1, 0, 1, 0, 1, 1, ...)

s_6 = (0, 0, 1, 1, 0, 1, 1, ...)

s_7 = (1, 0, 0, 0, 1, 0, 0, ...)

...

Next, a sequence s is constructed by choosing the 1st digit as complementary to the 1st digit of s1 (swapping 0s for 1s and vice versa), the 2nd digit as complementary to the 2nd digit of s2, the 3rd digit as complementary to the 3rd digit of s3, and generally for every n, the nth digit as complementary to the nth digit of sn. For the example above, this yields

s1 = (0, 0, 0, 0, 0, 0, 0, ...)

s_2 = (1, 1, 1, 1, 1, 1, 1, ...)

s_3 = (0, 1, 0, 1, 0, 1, 0, ...)

s_4 = (1, 0, 1, 0, 1, 0, 1, ...)

s_5 = (1, 1, 0, 1, 0, 1, 1, ...)

s_6 = (0, 0, 1, 1, 0, 1, 1, ...)

s_7 = (1, 0, 0, 0, 1, 0, 0, ...)

...

s = (1, 0, 1, 1, 1, 0, 1, ...)

By construction, s is a member of T that differs from each sn, since their nth digits differ (highlighted in the example). Hence, s cannot occur in the enumeration.

Based on this lemma, Cantor then uses a proof by contradiction to show that:

The set T is uncountable.

The proof starts by assuming that T is countable. Then all its elements can be written in an enumeration s_1, s_2, ... , s_n, ... Applying the previous lemma to this enumeration produces a sequence s that is a member of T, but is not in the enumeration. However, if T is enumerated, then every member of T, including this s, is in the enumeration. This contradiction implies that the original assumption is false. Therefore, T is uncountable.

Real numbers

The uncountability of the real numbers was already established by Cantor's first uncountability proof, but it also follows from the above result. To prove this, an injection will be constructed from the set T of infinite binary strings to the set R of real numbers. Since T is uncountable, the image of this function, which is a subset of R, is uncountable. Therefore, R is uncountable. Also, by using a method of construction devised by Cantor, a bijection will be constructed between T and R. Therefore, T and R have the same cardinality, which is called the "cardinality of the continuum" and is usually denoted by c or 2.

An injection from T to R is given by mapping binary strings in T to decimal fractions, such as mapping t = 0111... to the decimal 0.0111.... This function, defined by f(t) = 0.t, is an injection because it maps different strings to different numbers.

Constructing a bijection between T and R is slightly more complicated. Instead of mapping 0111... to the decimal 0.0111..., it can be mapped to the base b number: 0.0111...b. This leads to the family of functions: $f_b(t) = 0.t_b$. The functions $f_b(t)$ are injections, except for $f_2(t)$.

This function will be modified to produce a bijection between T and R.

1.9 THE POWER SET THEOREM

Q51. Explain briefly about Power Set Theorem.

Ans :

A power set includes all the subsets of a given set including the empty set. The power set is denoted by the notation P(S) and the number of elements of the power set is given by 2^n. A power set can be imagined as a place holder of all the subsets of a given set, or, in other words, the subsets of a set are the members or elements of a power set.

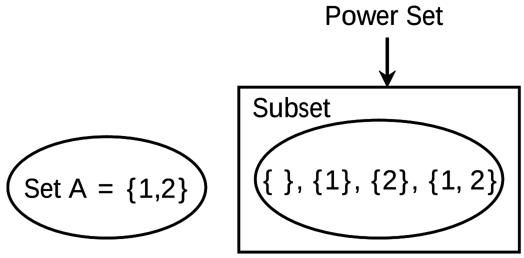
A power set is defined as the set or group of all subsets for any given set, including the empty set, which is denoted by {}, or, ϕ. A set that has 'n' elements has 2^n subsets in all. For example, let Set A = {1, 2, 3}, therefore, the total number of elements in the set is 3. Therefore, there are 2^3 elements in the power set. Let us find the power set of set A.

Set A = {1, 2, 3}

Subsets of set A = {}, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3}

Power set P(A) = { {}, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3} }

Power Set



Cardinality of a Power Set

The cardinality of a set is the total number of elements in the set. A power set contains the list of all the subsets of a set. The total number of subsets for a set of 'n' elements is given by 2^n.

Power Set Properties

A power set is a set that has a list of all the subsets of a given set. The power set which is denoted by P(A) with 'n' elements has the following properties:

- The total number of elements of a set is 2n.
- An empty set is a definite element of a power set.
- The power set of an empty set has only one element.
- The power set of a set with a finite number of elements is finite. For example, if set X = {b, c, d}, the power sets are countable.
- The power set of an infinite set has infinite number of subsets. For example, if Set X has all the multiples of 5 starting from 5, then we can say that Set X has an infinite number of elements. Though there is an infinite number of elements, a power set still exists for set X, in this case, it has infinite number of subsets.
- These the power set exists for both finite and infinite sets.

Power Set Proof

Let us see how a set containing 'n' elements has a power set that has 2^n elements. In other words, the cardinality of a finite set A with 'n' elements is $|P(A)| = 2^n$.

The proof of the power set follows the pattern of mathematical induction. To start with, let us consider the case of a set with no elements or an empty set.

Case 1:

A set with no elements. Let A = {}.

Here, the power set of A, which is denoted by P(A) = { } and the cardinality of the power set of A = |P(A)| = 1, since there is only one element, which is the empty set. Also, by the formula of the cardinality of a power set, there will be 2n power sets, which are equal to 2^0 or 1.

Case 2:

This is an inductive step. It is to be proved that P(n) $\rightarrow$ P(n + 1). This means, if a set that has 'n' elements has 2^n subsets, then a set that has 'n+1' elements will have 2n + 1 subsets.

To prove this, let us assume two sets 'X' and 'Y' with the following elements.

$X = \{a_1, a_2, a_3, a_4, a_n\}$ and

$Y = \{a_1, a_2, a_3, a_4, a_n, a_{n+1}\}$

The cardinality of the two sets 'X' and 'Y' are,

|X| = n , which means there are 2^n subsets for the set 'X'.

|Y| = n + 1

We can write that Y = X U {a_n + 1}, this means, every subset of set 'X' is also a subset of set 'Y'.

A subset of set Y may or may not contain the element a_{n+1}.

If an element of set 'Y' does not contain the element a_{n+1}, then it is clear that it is an element of set 'X'.

Also, if the subset of 'Y' has the element a_{n+1}, this means that the element a_{n+1} is included in any of the 2n subsets of the set 'X'. So we can conclude that, set 'Y' has 2n subsets with the element a_{n+1}. Therefore, set Y has 2n subsets with element a_{n+1} and 2n subsets without the element a_{n+1}. An example of this proof is as follows.

Example:

Let X = {1, 2}

Let Y = {1, 2, 3}

Here, the |X| = 2, so there will be 22 subsets for set X.

and |Y| = 3. We will prove that set Y has 23 subsets.

Subsets of X are = {ϕ}, {1}, {2}, {1, 2}

Subsets of Y are = {ϕ}, {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3}, {1, 2, 3}

Here, '3' is the extra element in set Y that is not in set X. Also, set Y includes 4 subsets that do not include element 3 and 4 other subsets that have element 3. So, in all, for set Y there are 4 subsets without the element '3' and 4 subsets with the element '3'.

1.10 Schroeder - Bernstein Theorem

Q52. Explain about Schroeder - Bernstein Theorem.

Ans : **(Imp.)**

Shows that '$\leq$', as applied to infinite cardinal numbers, has some familiar properties, that is, some properties of '$\leq$' in more familiar settings, like the integers. Another property that we rely on when dealing with Z or Q or R is anti-symmetry: if $x \leq y$ and $y \leq x$ then x = y. It is far from obvious that the ordering of the infinite cardinals obeys this rule, but it does.

Theorem

(Schroeder - Bernstein Theorem) If $\overline{A} \leq \overline{B}$ and $\overline{B} \leq \overline{A}$ then $\overline{A} - \overline{B}$ \thmrdef{thm:sb}

Proof :

We may assume that A and B are disjoint sets. Suppose f : A $\rightarrow$ B and g : B $\rightarrow$ A are both injections; we need to find a bijection h : A $\rightarrow$ B. Observe that if a is in A, there is at most one b_1 in B such that $g(b_1) = a$. There is, in turn, at most one a_1 in A such that $f(a_1) = b_1$. Continuing in this way, we can find a string of "ancestors" of a:

$a = a_0, b_1, a_1, b_2, a_2, b_3, a_3, \ldots$

such that $g(b_n) = a_{n-1}$ and $f(a_n) = b_n$. Call this the lineage of a. Of course, any $b \in B$ also has a lineage. Note that the lineage of $a \in A$ consists of just itself if a is not in the image of g; likewise, an element b?B might have no ancestors other than itself.

The lineage may take three forms: it may be infinite; it may end at some term a_k or b_k, if ak is not in the image of g or b_k is not in the image of f; or it may "wrap around" to the beginning, if $a_k = a$ for some $k > 0$. If a lineage ends with a term a_k, $k \geq 0$, we say it ends in A. Let A_A and B_A be the subsets of A and B, respectively, consisting of those elements whose lineage ends in A.

Claim 1

If $f(a) = b$, then $a \in A_A$ iff $b \in B_A$. To see this, observe that the lineage of b is

$$b, a, b_1, a_1, b_2, a_2, b_3, a_3, ...$$

i.e., to get the lineage of b, just add it to the lineage of a. Now it is clear that if the lineage of a ends in A, so does the lineage of b. Suppose that the lineage of b ends in A. The lineage of b must then include a, and so the lineage of a ends in A also.

Now $\hat{f} : A_A \to B_A$ by $\hat{f}(a) = f(a)$ (i.e., $\hat{f}(a)$ is f restricted to A_A, and with a different codomain).

Claim 2

$\hat{f}$ is a bijection. Since f is an injection, it fillows easily that $\hat{f}$ is an injection. To show $\hat{f}$ is surjective, suppose $b \in B_A$. Since the lineage of b ends in A, b must be in the image of f. So there is an $a \in A$ such that $f(a) = b$. Since $b \in B_A$. by claim 1, $a \in A_A$. Therefore, $\hat{f}(a) = b$ for some a in A_A, and $\hat{f}$ is surjective.

We outline a parallel construction and leave the details for the exercises. A_{A^c} (the complement of A_A in A) and B_{A^c} consist of those elements whose lineage does not end in A.

Claim 1'

If $g(b) = a$, then $b \in B_{A^c}$ iff $a \in A_{A^c}$

Claim 1' allows us to define $\hat{g} : B_{A^c} \to A_{A^c}$, where $\hat{g}(b) = g(b)$ for any $b \in B_{A^c}$.

Claim 2'

$\hat{g}$ is a bijection

The theorem follows from claoms 2 and 2' : define $h : A \to B$ by the formula,

$$h(a) = \begin{cases} \hat{f}(a) & ; \quad \text{if } a \in A_{A,} \\ \hat{g}^{-1}(a) & ; \quad \text{if } a \in A_{A^c} \end{cases}$$

It is straightforward to verify that h is a bijection

It is sometimes tempting to react to a result like this with, "Of course! How could it be otherwise?" This may be due in part to the use of the familiar symbol '$\leq$' – but of course, just using the symbol hardly guarantees that it acts like '$\leq$' in more familiar contexts. Even paying attention to the new meaning, this theorem may seem "obvious." Perhaps the best way to see that it might not be so obvious is to look at a special case, one in which the injections f and g are easy to find, but there does not seem to be any "obvious" bijection.

1.11 Principles of Mathematical Induction

Q53. Explain about Principles of Mathematical Induction.

Ans : **(Imp.)**

Mathematical induction is one of the techniques, which can be used to prove a variety of mathematical statements which are formulated in terms of n, where n is a positive integer.

Let P(n) be given statement involving the natural number n such that

(i) The statement is true for $n = 1$, i.e. P(1) is true.

(ii) If the statement is true for $n = k$ (where k is a particular but arbitrary natural number), then the statement is also true for $n = k + 1$ i.e. truth of P(k) implies that the truth of P(k + 1). Then, P(n) is true for all natural numbers.

1.11.1 The Well – Ordering Principle

Q54. Write about Well – Ordering Principle.

Ans : **(Imp.)**

Statement of the Principle

The well-ordering principle says that the positive integers are well-ordered. An ordered set is said to be well-ordered if each and every nonempty subset has a smallest or least element. So the well-ordering principle is the following statement:

Every nonempty subset SS of the positive integers has a least element.

Note that this property is not true for subsets of the integers (in which there are arbitrarily small negative numbers) or the positive real numbers (in which there are elements arbitrarily close to zero).

An equivalent statement to the well-ordering principle is as follows:

The set of positive integers does not contain any infinite strictly decreasing sequences.

Uses in Proofs

Here are several examples of properties of the integers which can be proved using the well-ordering principle. Note that it is usually used in a proof by contradiction; that is, construct a set S,S, suppose SS is nonempty, obtain a contradiction from the well-ordering principle, and conclude that SS must be empty.

There are no positive integers strictly between 0 and 1.

Let S be the set of integers xx such that $0 < x < 1$. Suppose SS is nonempty; let nbe its smallest element. Multiplying both sides of $n < 1$ by nn gives $n^2 < n$. The square of a positive integer is a positive integer, so n^2n2 is an integer such that $0 < n^2 < n < 1$. This is a contradiction of the minimality of n. Hence S is empty.

Well-ordering proof

For every positive integer n, the number $n^2 + n + 1$ is even.

Proof:

Let Sbe the subset of positive integers n for which $n^2 + n + 1$ is odd. Assume S is nonempty.

Let m be its smallest element.

Then $m - 1 \notin S$, so $(m - 1)^2 + (m - 1) + 1$ is even.

But $(m - 1)^2 + (m - 1) + 1 = m^2 - m + 1 = (m^2 + m + 1) - 2m$, equals $((m - 1)^2 + (m - 1) + 1 + 2m$, which is a sum of two even numbers, which is even.

So $m \in S$; which is a contradiction. Therefore, S is empty, and the result follows.

Ans: $m - 1$ is not necessarily a positive integer, so the third paragraph is wrong

1.11.2 Recursive Definitiions

Q55. What is recursive definition? How it can be used.

Ans :

In mathematics and computer science, a recursive definition, or inductive definition, is used to define the elements in a set in terms of other elements in the set.

An inductive definition of a set describes the elements in a set in terms of other elements in the set. For example, one definition of the set N of natural numbers is:

1. 1 is in N.
2. If an element n is in N then n + 1 is in N.
3. N is the intersection of all sets satisfying (1) and (2).

There are many sets that satisfy (1) and (2) - for example, the set {1, 1.649, 2, 2.649, 3, 3.649, ...} satisfies the definition. However, condition (3) specifies the set of natural numbers by removing the sets with extraneous members. Note that definition assumes that N is contained in a larger set (such as the set of real numbers) in which the operation + is defined.

Properties of recursively defined functions and sets can often be proved by an induction principle that follows the recursive definition. For example, the definition of the natural numbers presented here directly implies the principle of mathematical induction for natural numbers: if a property holds of the natural number 0, and the property holds of n + 1 whenever it holds of n, then the property holds of all natural numbers.

Principle of Recursive Definition

Let A be a set and let a_0 be an element of A. If ρ is a function which assigns to each function f mapping a nonempty section of the positive integers into A, an element of A, then there exists a unique function $h : Z_+ \rightarrow A$ such that

$$h(1) = a_0$$

$$h(i) = \rho(h|_{\{1,2,\ldots,\ i-1\}}) \text{ for } i > 1.$$

Examples of Recursive Definitions

Elementary Functions

Addition is defined recursively based on counting as

$$0 + a = a$$

$$(1 + n) + a = 1 + (n + 1)$$

Multiplication is defined recursively as

$$aa = 0,$$

$$(1 + n)\, a = a + na$$

Exponentiation is defined recursively as

$$a^0 = 1$$

$$a^{n+1} = a\, a^n$$

Binomial coefficients can be defined recursively as

$$\begin{pmatrix} a \\ 0 \end{pmatrix} = 1,$$

$$\begin{pmatrix} 1+a \\ 1+n \end{pmatrix} = \frac{(1+a)\begin{pmatrix} 0 \\ n \end{pmatrix}}{1+n}$$

1.11.3 Division Algorithms, Prime Numbers

Q56. Explain division algorithm for prime numbers with an example.

Ans :

The division algorithm is an algorithm in which given 2 integers N and D, it computes their quotient Q and remainder R, where $0 \le R < < |D|$. There are many different algorithms that could be implemented, and we will focus on division by repeated subtraction. This is very similar to thinking of multiplication as repeated addition.

Let's say we have to divide N (dividend) by D (divisor). We will take the following steps:

Step 1: Subtract D from N repeatedly, i.e. N – D – D – D – ... until we get a result that lies between 0 (inclusive) and D (exclusive) and is the smallest non-negative number obtained by repeated subtraction.

Step 2: The resulting number is known as the remainder R, and the number of times that D is subtracted is called the quotient Q.

Let's experiment with the following examples to be familiar with this process:

Example:

Describe the distribution of 7 slices of pizza among 3 people using the concept of repeated subtraction.

We have 7 slices of pizza to be distributed among 3 people. We initially give each person one slice, so we give out 3 slices leaving 7 – 3 = 4. We then give each person another slice, so we give out another 3 slices leaving 4 – 3 = 1. We are now unable to give each person a slice. So, each person has received 2 slices, and there is 1 slice left.

Example:

Divide 21 by 5 and find the remainder and quotient by repeated subtraction.

Subtracting 5 from 21 repeatedly till we get a result between 0 and 5. This gives us

$$21 - 5 = 16$$

$$16 - 5 = 11$$

$$11 - 5 = 6$$

$$6 - 5 = 1$$

At this point, we cannot subtract 5 again. Hence 4 is the quotient (we subtracted 5 from 21 four times) and 1 is the remainder. We say that

$$21 = 5 \times 4 + 1.$$

Example:

Since x has a remainder of 80 when divided by 99, there exists k"Z such that x=99k+80x. Thus, we have:

$$x^2 + 5 = (99k + 80)^2 + 5$$
$$= 99^2k^2 + 2 \times 99 \times 80k + 80^2 + 5$$
$$= 99(99k^2 + 2 \times 80k) + 80^2 + 5$$
$$= 99(99k^2 + 2 \times 80k) + 6405$$
$$= 99(99k^2 + 2 \times 80k) + 64 \times 100 + 5$$
$$= 99(99k^2 + 2 \times 80k) + 64 (99 + 1) + 5$$
$$= 99(99k^2 + 2 \times 80k) + 64 \times 99 + 64 + 5$$
$$= 99(99k^2 + 2 \times 80k + 64) + 64 + 5$$
$$= 99(99k^2 + 2 \times 80k) + 69$$

Then, since $99k^2 + 2 \times 80k$ is an integer, and since $0 \le 69 < 99$, we see that 69 is the unique remainder (as dictated by the Division Algorithm) for x^2+5 when divided by 99.

This solution just uses the starting point of the division algorithm, which is what you used by writing $x = 99k + 80$ for some k. Then, you just take this, and manipulate it into an expression for $x^2 + 5$. The key part is to group everything that has a 99 (i.e. factor 99 out of everything that you can after expanding the brackets and such).

1.11.4 The Greatest Common Divisor, Euclidean Algorithm

Q57, Explain about Euclidean Algorithm for the Greatest Common Divisor.

Ans : **(Imp.)**

Greatest common divisor of given integers and we consider the Euclidean algorithm, which is one of the oldest mathematical algorithms.

Algorithm I : Division Algorithm

Given two integers a and b such that b > 0. There exist unique integers q and r for which

$$a = qb + r, 0 < r < b.$$

Here q is called quotient and r is called remainder. There is a special case, when the Division algorithm yields r = 0.1. In such a situation a = qb for some q.

Definition 1

We say that b divides a (b is a divisor of a or a is a multiple of b) if there exists q such that a = qb.

Notation: b | a.

Definition 2

Let a, b $\in$ Z. A positive integer d is called a common divisor of a and b. if d divides a and d divides b. The largest possible such integer is called the greatest common divisor of a and b.

Notation : gcd(a, b).

Algorithm 2

The Euclidean algorithm. Now we study a method to determine gcd(a, b) of given positive integers a and b. The method also provides solution of the linear Diophantine equation.

$$ax + by = \gcd(a, b).$$

If we apply the Division algorithm to a, b, $a > b$,

Then we have

$$a = qb + r, 0 \leq r < b.$$

If d is a common divisor of a and b, then d divides $r = a - qb$ as well. That is the basic idea of the algorithm. The Euclidean algorithm works as follows. First we apply the Division algorithm for a and b to obtain a quotient q_1 and a remainder r_1. Then we apply the Division algorithm for b and r_1 to get a new quotient q_2 and a new remainder r_2. We continue, we divide r_a by r_2 to obtain q_3 and r_3. We stop if we obtain a zero remainder. Since the procedure producess a decreasing sequence of non-negative integers so must eventually terminate by descent. The last non-zero remainder in the greatest common divisor of a and b.

1.11.5 Fundamental Theorem of Arithmetic

Q58. Write about fundamental theorem of arithmetic.

Ans :

The fundamental theorem of arithmetic (FTA), also called the unique factorization theorem or the unique-prime-factorization theorem, states that every integer greater than 1 either is prime itself or is the product of a unique combination of prime numbers.

Definition

For every integer ne”2, it can be expressed as a product of prime numbers:

$$n = p_1^{a_1} \, p_2^{a_2} \, \ldots \, p_i^{a_i}$$

Existence of a Factorization

The following proof shows that every integer greater than 1 is prime itself or is the product of prime numbers.

Base case: This is clearly true for $n = 2$.

Inductive step: Suppose the statement is true for $n = 2, 3, 4, \ldots, k$.

If $(k + 1)$ is prime, then we are done. Otherwise, $(k + 1)$ has a smallest prime factor, which we denote by p. Let $k + 1 = p \times N$. Since $N < k$, by our inductive hypothesis, N can be written as the product of prime numbers. That means $k + 1 = p \times N$ can also be written as a product of primes. We're done!

Examples

Given the polynomial

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$$

with integer confficients $a_1, a_2, a_3, ..., a_n$ and given that there exist four distinct integers a, b, c and d such that

$$f(a) = f(b) = f(c) = f(d) = 5,$$

show that there is no integer k for which f(k) = 8

Let g(x) – f(x) – 5. Then we must have

$$g(x) - k(z - a)(z - b)(z - c)(z - d)\,h(x)$$

for some h(x) $\in$ z[x], Let k be such that f(k) = 9. Then g(k) = 3 and we get

$$3 = h(x - a)(x - b)(x - c)(x - d)\,h(x).$$

By the fundamental theorem of arithmetic, we can express 3 as a product of at most three different integers (–1, –3, 1). Since, (x – a), (x – b), (x – c) and (x – d) are all distinct, this is an obvious contradiction.

UNIT II	Basic counting techniques-inclusion and exclusion, pigeon-hole principle, permutation and combination.

2.1 Basic Counting Techniques

Q1. Discuss briefly about Basic Counting Techniques with an examples.

Ans :

In many situations of computational work, we employ two basic rules of counting, called the Sum Rule and the Product Rule.

(i) The Sum Rule

Suppose two tasks T_1 and T_j are to be performed. If the task T_1 can be preformed in m afferent ways and the task T_2 can be performed in n different ways and if these two tasks cannot be performed simultaneously, then one of the two tasks (T_1 or T_2) can be performed in m + n ways.

More generally, if T_1, T_2, T_3, ... T_k are k tasks such that no two of these tasks can be performed at the same time and if the task T_1 can be performed in n_i different ways, then one of the k tasks (namely T_1 or T_2 or T_3. .., or T_k) can be performed in $n_1 + n_2 + ... + n_k$ ways.

Example 1

Suppose there are 16 boys and 18 girls in a class and we wish to select one of these students (either a boy or a girl) as the class representative. The number of ways of selecting a boy is 16 and the number of ways of selecting a girl is 18. Therefore, the number of ways of selecting a student (boy or girl) is 16 + 18 = 34.

Example 2

Suppose a library has 12 books on Mathematics, 10 books on Physics, 16 books on Computer Science and 11 books on Electronics. Suppose a student wishes to choose one of these books for study. The number of ways in which he can choose a book is 12 + 10 + 16 + 11 = 49.

Example 3

Suppose T_1 is the task of selecting a prime number less than 10 and T_2 is the task of selecting an even number less than 10. Then T_1 can be performed in 4 ways (– by selecting 2 or 3 or 5 or 7), and T_2 can be performed in 4 ways (– by selecting 2 or 4 or 6 or 8). But, since 2 is both a prime and an even number less than 10, the task T_1 or T_2 can be performed in 4 + 4 – 1 = 7 ways.

(ii) The Product Rule

Suppose that two tasks T_1 and T_2 are to be performed one after the other. If T_1 can be performed in N_1 different ways, and for each of these ways T_2 can be performed in n_2 different ways, then both of the tasks can be performed in n_1n_2 different ways.

More generally, suppose that k tasks $T_1, T_2, T_3, ..., T_k$ are to be performed in a sequence. If T_1 can be performed in n_1 different ways and for each of these ways T_2 can be performed in n_2 different ways, and for each of n_1n_2 different ways of performing T_1 and T_2 in that order. T_3 can be performed in n_3 different ways, and so on, then the sequence of tasks $T_1, T_2, T_3, ..., T_k$ can be performed in $n_1 n_2 n_3, ..., n_k$ different ways.

Example 1

Suppose a person has 3 shirts and 5 ties. Then he has $3 \times 5 = 15$ different ways of choosing a shirt and a tie.

Example 2

Suppose we wish to construct sequences of four symbols in which the first 2 are English letters and the next 2 are single digit numbers. If no letter or digit can be repeated, then the number of different sequences that we can construct is $26 \times 25 \times 10 \times 9 = 58500$. If repetition of letters and digits is allowed then the number of different sequences that we can construct is $26 \times 26 \times 10 \times 10 = 67600$.

Example 3

Suppose a restaurant sells 6 South Indian dishes, 4 North Indian dishes, 3 hot beverages and 2 cold beverages. For breakfast, a student wishes to buy 1 South Indian dish and 1 hot beverage, or 1 North Indian dish and 1 cold beverage. Then he can have the first choice in $6 \times 3 = 18$ ways and he can have the second choice in $4 \times 2 = 8$ ways. The total number of ways he can buy his breakfast items is $18 + 8 = 26$.

PROBLEMS

2. There are four bus routes between the places A and B and three bus routes between the places B and C. Find the number of ways a person can make a round trip from A to A via B if he does not use a route more than once.

Sol :

The person can travel from A to B in four ways and from B to C in three ways, but only in two ways from C to B and only in three ways from B to A if he does not use a route more than once. Therefore, the number of ways he can make the round trip under the given condition is $4 \times 3 \times 2 \times 3 = 72$.

3. Let A be a set with n elements. How many different sequences, each of length r, can be formed using the elements from A if the elements in the sequence may be repeated?

Sol :

Since repetition is allowed, each place in the sequence can be filled in n different ways. Thus, in a sequence of length r, there are n^r ways of filling the r places in the sequence. This means that there are n^r possible sequences (of the required type).

4. (a) Find the number of binary sequences of length n.

(b) Find the number of binary sequences of length n that contain an even number of 1's.

Sol :

(a) A binary sequence of length n contains n positions. Each of these positions can be filled in two ways (with 0 or 1). Therefore, the number of ways of filling n positions is 2". This is precisely the number of binary sequences of length n.

(b) If a binary sequence of length n – 1 has an even number of 1's, we append the digit 0 to it to obtain a binary sequence of length n which contains an even number of l's. If a binary sequence of length n – 1 contains an odd number of 1's, we append the digit 1 to it to obtain a sequence of length n which contains an even number of l's. As such, the number of binary sequences of length n with an even number of 1's is equal to the number of binary sequences of length n – 1, which is 2^{n-1}.

5. A bit is either 0 or 1. A byte is a sequence of 8 bits. Find

(i) the number of bytes

(ii) the number of bytes that begin with 11 and end with 11

(iii) the number of bytes that begin with 11 and do not end with 11

(iv) the number of bytes that begin with 11 or end with 11.

Sol :

(i) Since each byte contains 8 bits and each bit is a 0 or 1 (two choices), the number of bytes is $2^8 = 256$.

(ii) In a byte beginning and ending with 11, there occur 4 open positions. These can be filled in $2^4 = 16$ ways. Therefore, there are 16 bytes which begin and end with 11.

(iii) There occur six open positions in a byte beginning with 11. These positions can be filled in $2^6 = 64$ ways. Thus, there are 64 bytes that begin with 11. Since there are 16 bytes that begin and end with 11, the number of bytes that begin with 11 but do not end with 11 is $64 - 16 = 48$.

(iv) As in (iii), the number of bytes that end with 11 is 64. Also the number of bytes that begin and end with 11 is 16. Therefore, the number of bytes that begin or end with 11 is $64 + 64 - 16 = 112$.

6. A telegraph can transmit two different signals : a dot and a dash. What length of these symbols is needed to encode 26 letters of the English alphabet and the ten digits 0, 1, 2, ..., 9?

Sol :

Since there are two choices for each signal, the number of different sequences of length k of these signals is 2^k. Therefore, the number of nontrivial sequences of length n or less is

$$2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2.$$

To encode 26 letters and 10 digits, we require at least $26+10=36$ sequences of the above type: that is,

$$2^{n+1} - 2 \geq 36$$

The least value of n (positive integer) for which this inequality holds is $n = 5$. Hence, the length of the symbols needed to encode 26 alphabets and 10 digits is at least 5.

7. Find the number of 3-digit even numbers with no repeated digits.

Sol :

Here we consider numbers of the form xyz, where each of x, y, z represents a digit under the given restrictions. Since x y z has to be even, z has to be 0, 2, 4, 6, or 8. If z is 0, then x has 9 choices and if z is 2, 4, 6 or 8 (4 choices) then x has 8 choices. (Note that x cannot be zero). Therefore, z and x can be chosen in $(1 \times 9) + (4 \times 8) = 41$ ways. For each of these ways, y can be chosen in 8 ways. Hence, the desired number is $41 \times 8 = 328$.

8. How many among the first 1,00,000 positive integers contain exactly one 3, one 4 and one 5 in their decimal representation ?

Sol :

The number 1,00,000 does not contain 3 or 4 or 5. Therefore, we have to consider all possible positive integers with 5 places that meet the given conditions. In a 5-place integer the digit 3 can be in any one of the 5 places. Subsequently, the digit 4 can be in any one of the 4 remaining places. Then the digit 5 can be in any one of the 3 remaining places. There are 2 places left and either of these may be filled by 7 digits, (-digits from 0 to 9 other than 3, 4, 5). Thus, there are $5 \times 4 \times 3 \times 7 \times 7 = 2940$ integers of the required type.

9. Find the number of proper divisors of 441000.

Sol :

$441000 = 2^3 \times 3^2 \times 5^3 \times 7^2$. Therefore, every divisor of 441000 must be of the form $d = 2^p \times 3^q \times 3^r \times 7^s$ where $0 \leq p \leq 3$, $0 \leq q \leq 2$, $0 \leq r \leq 3$, $0 \leq s \leq 2$. Thus, for a divisor d, p can be chosen in 4 ways, q in 3 ways, r in 4 ways and s in 3 ways. Accordingly, the number of possible d's is $4 \times 3 \times 4 \times 3 = 144$. Of these, two are not proper divisors. Therefore, the number of proper divisors is $144 - 2 = 142$.

2.2 Permutation and Combination

Q10. Explain the concept of Permutations.

Ans : **(Imp.)**

Suppose that we are given n distinct objects and wish to arrange r of these objects in a line. Since there are n ways of choosing the first object, and, after this is done, n – 1 ways of choosing the second object, ..., and finally n – r + 1 ways of choosing the r^{th} object, it follows by the product rule of counting (stated in the preceding section) that the number of different arrangements, or permutations (as they are commonly called) is $n(n-1)(n-2) \ldots (n-r+1)$. We denote this number by P(n, r)[1] and is referred to as the number of permutations of size r of n objects. Thus (by definition),

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1).$$

Using the factorial notation defined by

$$k! = k(k-1)(k-2) \ldots 2 - 1,$$

for any positive integer k, and 0! = 1, we find that

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 2 . 1}{(n-r)(n-r-1)\ldots 2 . 1}$$

$$= \frac{n!}{(n-r)!}$$

As a particular case of this, we get

$$P(n, n) = n!.$$

That is, the number of different arrangements (permutations) of n distinct objects, taken all at a time, is n!. This is simply called the number of permutations of n distinct objects.

In the above analysis, we have considered the situation where all the objects that are to be arranged are distinct.

Suppose it is required to find the number of permutatuions that can be formed from a collection of n objects of which n_1 are of one type, n_2 are of a second type, ..., n_k are of k^{th} type, with $n_1 + n_2 + ... + n_k = n$. Then, the number of permutations of the n objects is

$$\frac{n!}{n!\ n_2! \ldots n_k!}$$

Proof:

There are n! permutations when all the n objects are different. We must therefore divide n! by $n_1!$ to account for the fact that the n\ objects which are alike will identify $n_1!$ of these permutations (for any given set of positions of the n_1 objects in the permutation). Similarly, we must divide n_1 by $n_2!$, $n_3!$, ..., $n_k!$ which are the numbers of permutations of the corresponding alike objects. Thus, n! divided by all of $n_1!$, $n_2!$, ..., $n_k!$ gives the required number of permutations.

PROBLEMS

11. How many different strings (sequences) of length 4 can be formed using the letters of the word FLOWER?

Sol :

The given word has 6 letters all of which are distinct. Therefore, the required number of strings is

$$P(6, 4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6\times5\times4\times3\times2\times1}{2\times1} = 360.$$

---

12. Find the number of permutations of the letters of the word SUCCESS.

Sol :

The given word has 7 letters, of which 3 are S; 2 are C and 1 each are U and E. Therefore, the required number of permutations is

$$\frac{7!}{3!\ .\ 2!\ .\ 1!\ .\ 1!} = \frac{7\times6\times5\times4\times3\times2\times1}{(3\times2)\times(2\times1)\times1\times1} = 420.$$

---

13. How many 9 letter "words" can be formed by using the letters of the word DIFFICULT?

Sol :

The given word contains 9 letters of which there are 2 F's, 2 I's, and 1 each of D, C, U, L, T. The number of permutations of these letters is the required number of "words". This number is

$$\frac{9!}{2!\ 2!\ 1!\ 1!\ 1!\ 1!\ 1!} = 90720.$$

---

14. Find the number of permutations of the letters of the word MASSASAUGA. In how many of these, all four A's are together? How many of them begin with S ?

Sol :

The given word has 10 letters of which 4 are A, 3 are S and 1 each are M, U and G. Therefore, the required number of permutations is

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25{,}200.$$

If, in a permutation, all A's are to be together, we treat all of A's as one single letter. Then the letters to be permuted read (AAAA), S, S, S, M, U, G and the number of permutations is

$$\frac{7!}{1!\,3!\,1!\,1!\,1!} = 840.$$

For permutations beginning with S, there occur nine open positions to fill, where two are S, four are A, and one each are M, U, G. The number of such permutations is

$$\frac{9!}{2!\,4!\,1!\,1!\,1!} = 7560.$$

15. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

Sol :

Here n must be of the form

$$n = x_1\,x_2\,x_3\,x_4\,x_5\,x_6\,x_7$$

where x_1, x_2, ..., x_7 are the given digits with $x_1 = 5$, 6 or 7. Suppose we take $x_1 = 5$. Then $x_2\,x_3\,x_4\,x_5\,x_6\,x_7$ is an arrangement of the remaining 6 digits which contains two 4's and one each of 3, 5, 6, 7. The number of such arrangements is

$$\frac{6!}{2!\,1!\,1!\,1!\,1!} = 360.$$

Next, suppose we take $x_1 = 6$. Then, $x_2\,x_3\,x_4\,x_5\,x_6\,x_7$ is an arrangement of 6 digits which contains two each of 4 and 5 and one each of 3 and 7. The number of such arrangements is

$$\frac{6!}{1!\,2!\,2!\,1!} = 180.$$

Similarly, if we take $x_1 = 7$, the number of arrangements is

$$\frac{6!}{1!\,2!\,2!\,1!} = 180.$$

Accordingly, by the Sum Rule, the number of n's of the desired type is

$$360 + 180 + 180 = 720.$$

16. In how many ways can 6 men and 6 women be seated in a row (i) if any person may sit next to any other? (ii) if men and women must occupy alternate seats?

Sol :

(i) If any person may sit next to any other, no distinction need be made between men and women in their seating. Accordingly, since there are 12 persons in all, the number of ways they can be seated is

$$12! = 479{,}001{,}600.$$

(ii) When men and women are to occupy alternate seats, the six men can be seated in 6! ways in odd places and the six women can be seated in 6! ways in even places, and corresponding to each arrangement of the men there is an arrangement of the women. Therefore, the number of ways in which the men occupy the odd places and the women the even places is

$$6! \times 6! = 720 \times 720 = 518400.$$

Similarly, the number of ways in which the women occupy the odd places and the men the even places is 518400. Accordingly, the total number of ways is

$$518400 + 518400 = 1,036,800.$$

---

17. Four different mathematics books, five different computer science books and two different control theory books are to be arranged in a shelf. How many different arrangements are possible if

(a) the books in each particular subject must all be together?

(b) only the mathematics books must be together?

Sol :

(a) The mathematics books can be arranged among themselves in 4! different ways, the computer science books in 5! ways, the control theory books in 2! ways, and the three groups in 3! ways. Therefore the number of possible arrangements is

$$4! \times 5! \times 2! \times 3! = 24 \times 120 \times 2 \times 6 = 34,560.$$

(b) Consider the four mathematics books as one single book. Then we have 8 books which can be arranged in 8! ways. In all of these ways the mathematics books are together. But the mathematics books can be arranged among themselves in 4! ways. Hence, the number of arrangements is

$$8! \times 4! = 40320 \times 24 = 967,680.$$

Q18. Explain the concept of combinations.

Ans : **(Imp.)**

Suppose we are interested in selecting (choosing) a set of r objects from a set of n $\geq$ r objects without regard to order. The set of r objects being selected is traditionally called a combination of r objects.

The total number of combinations of r different objects that can be selected from n different objects can be obtained by proceeding in the following way. Suppose this number is equal to C, say; that is, suppose there is a total of C number of combinations of r different objects chosen from n different objects. Take any one of these combinations. The r objects in this combination can be arranged in r! different ways. Since there are C combinations, the total number of permutations is (C . r!). But this is equal to P(n, r). Thus,

$$C \cdot r! = P(n, r), \qquad \text{or} \qquad C = \frac{P(n, r)}{n!}$$

Thus, the total number of combinations of r different objects that can be selected from n different objects is equal to P(n, r)/r!. This number is denoted by C(n, r) or $\binom{n}{r}$. Thus,

$$C(n, r) = \binom{n}{r} = \binom{n}{r} = \frac{n!}{(n-r)!\,r!} \quad \text{for } 0 \leq r \leq n.$$

Replacing r by n – r in this expression, we get

$$C(n, n-r) = \frac{n!}{r!(n-r)!} = C(n, r) \quad \text{for} \quad 0 \leq r \leq n.$$

Consequently, we have

$$C(n, n) = C(n, 0) = 1 \quad \text{and} \quad C(n, 1) = C(n, n-1) = n.$$

For $r > n$, $C(n, r)$ is defined to be equal to zero.

PROBLEMS

19. How many committees of five with a given chairperson can be selected from 12 persons?

Sol :

The chairperson can be chosen in 12 ways, and, following this, the other four on the com-mittee can be chosen in C(11, 4) ways. Therefore, the possible number of such committees is

$$12 \times C(11, 4) = 12 \times \frac{111}{4!\ 7!} = 12 \times 330 = 3960.$$

---

20. Find the number of committees of 5 that can be selected from 7 men and 5 women if the committee is to consist of at least 1 man and at least 1 woman.

Sol :

From the given 12 persons the number of committees of 5 that can be formed is C(12, 5). Among these possible committees, there are C(7, 5) committees consisting of 5 men and 1 = C(5, 5) committee consisting of 5 women. Accordingly, the number of committees containing at least one man and one woman is

$$C(12, 5) - C(7, 5) - 1 = \frac{12!}{7!\ 5!} - \frac{7!}{5!\ 2!} - 1$$

$$= 792 - 21 - 1 = 770.$$

---

21. At a certain college, the housing office has decided to appoint, for each fldor, one male and one female residential advisor. How many different pairs of advisors can be selected for a seven-floor building from 12 male and 15 female candidates?

Sol :

From 12 male candidates, 7 candidates can be selected in C(12, 7) ways. From 15 female candidates, 7 candidates can be selected in C(15, 7) ways. Therefore, the total number of possible pairs of advisors of the required type is

$$C(12, 7) \times C(15, 7) = \frac{12!}{7!\ 5!} \times \frac{15!}{7!\ 8!} = 792 \times 6435 = 5{,}096{,}520.$$

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22. A certain question paper contains two parts A and B each containing 4 questions. How many different ways a student can answer 5 questions by selecting at least 2 questions from each part?

Sol :

The different ways a student can select his 5 questions are

(I) 3 questions from Part A and 2 questions from Part B. This can be done in C(4,3) × C(4,2) = 4 × 6 = 24 ways.

(II) 2 questions from Part A and 3 questions from Part B. This can be done in C(4, 2) × C(4, 3) = 24 ways.

Therefore, the total number of ways a student can answer 5 questions under the given restrictions is 24 + 24 = 48.

23. A certain question paper contains three parts A, B, C with four questions in part A, five questions in part B and six questions in part C. It is required to answer seven questions selecting at least two questions from each part. In how many different ways can a student select his seven questions for answering?

Sol :

The different possible ways in which a student can make a selection are

(I) 2 questions from Part A, 2 from Part B and 3 from Part C.

(II) 2 questions from Part A, 3 from Part B and 2 from Part C.

(III) 3 questions from Part A, 2 from Part B and 2 from Part C.

Now, selection (I) can be made in

C(4, 2) × C(5, 2) × C(6, 3) = 6 × 10 × 20 = 1200 ways,

the selection (II) can be made in

C(4, 2) × C(5, 3) × C(6, 2) = 6 × 10 × 15 = 900 ways,

and the selection (III) can be made in

C(4, 3) × C(5, 2) × C(6, 2) = 4 × 10 × 15 = 600 ways.

Consequently, the total number of possible selections is

1200 + 900 + 600 = 2700.

24. A woman has 11 close relatives and she wishes to invite 5 of them to dinner. In how many ways can she invite them in the following situations:

(i) There is no restriction on the choice.

(ii) Two particular persons will not attend separately.

(iii) Two particular persons will not attend together.

Sol : **(Imp.)**

(i) Since there is no restriction on the choice of invitees, five out of 11 can be invited in

$$C(11, 5) = \frac{11!}{6!\,5!} = 462 \text{ ways.}$$

(ii) Since two particular persons will not attend separately, they should both be invited or not invited. If both of them are invited, then three more invitees are to be selected from the remaining 9 relatives. This can be done in

$$C(9, 3) = \frac{9!}{6!\,3!} = 84 \text{ ways.}$$

If both of them are not invited, then five invitees are to be selected from 9 relatives. This can be done in

$$C(9, 5) = \frac{9!}{5!\ 4!} = 126 \text{ ways.}$$

Therefore, the total number of ways in which the invitees can be selected in this case is

$$84 + 126 = 210.$$

(iii) Since two particular persons (say A and B) will not attend together, only one of them can be invited or none of them can be invited. The number of ways of choosing the invitees with A invited is

$$C(9, 4) = \frac{9!}{5!\ 4!} = 126$$

Similarly the number of ways of choosing the invitees with B invited is 126.

If both A and B are not invited, the number of ways of choosing the iavitees is

$$C(9, 5) = 126.$$

Thus, the total number of ways in which the invitees can be selected in this case is

$$126 + 126 + 126 = 378.$$

25. Find the number of 5-digit positive integers such that in each of them every digit is greater than the digit to the right.

Sol :

A set of 5 distinct digits can be selected in C(10, 5) ways. Once these digits are chosen, there is only one way of arranging them in a decreasing order from left to right. So, the number of 5-digit positive integers of the required type is $1 \times C(10, 5) = C(10, 5)$.

26. From seven consonants and five vowels, how many sets consisting of four different consonants and three different vowels can be formed?

Sol :

The four different consonants can be selected in C(7, 4) different ways and three different vowels can be selected in C(5, 3) ways, and the resulting seven different letters (four consonants and three vowels) can then be arranged among themselves in 7! ways. Therefore, the number of possible sets is

$$C(7, 4) \times C(5, 3) \times 7! = \frac{7!}{4!\ 3!} \times \frac{5!}{3!\ 2!} \times 7! = 35 \times 10 \times 5040 = 1{,}764{,}000.$$

27. Find the number of arrangements of the letters in TALLAHASSEE which have no adjacent A's.

Sol :

Here the number of letters is 11 of which 3 are A's, 2 each are L's, S's, E's, and 1 each are T and H. First, let us disregard the A's. The remaining 8 letters can be arranged in

$$\frac{8!}{2!2!2!1!1!} = 5040 \text{ ways.}$$

In each of these arrangements, there are 9 possible locations for the three A's. These locations can be chosen in C(9, 3) ways. Therefore, by the product rule, the required number of arrangements is

$$5040 \times C(9, 3) = 5040 \times \frac{9!}{3!\ 6!} = 5040 \times 84 = 423{,}360.$$

28. Find the number of ways of seating r out of n persons around a circular table, and the others around another circular table.

Sol :

First, choose a set of r persons for the first table – this can be done in C(n, r) ways. These r persons can be seated around the first table in (r – 1)! ways. The remaining (n – r) persons can be seated around the second table in (n – r – 1)! ways. So, the required number is

$$C(n, r) \times (r-1)! \times (n-r-1)!.$$

29. A party is attended by n persons. If each person in the party shakes hands with all the others in the party, find the number of handshakes.

Sol :

Each handshake is determined by exactly two persons. Therefore, if each person shakes hands with all the other persons, the total number of handshakes is equal to the number of combinations of two persons that can be selected from the n persons. This number is

$$C(n, 2) = \frac{n!}{(n-2)!2!} = \frac{1}{2}n(n-1)$$

30. There are n married couples attending a party. Each person shakes hands with every person other than his or her spouse. Find the total number of handshakes.

Sol :

The number of persons at the party is 2n. These 2n persons fall into C(2n, 2) pairs out of which n pairs are married couples. Thus, the number of pairs who are not married couples is

$$C(2n, 2) - n = \frac{(2n!)}{(2n-2)!2!} - n = \frac{1}{2}\,.\,2n(2n-1) - n = 2n(n-1)$$

This number is identical with the number of handshakes.

31. (a) How many diagonals are there in a regular polygon with n sides?

(b) Which regular polygon has the same number of diagonals as sides?

Sol :

(a) A regular polygon of n sides has n vertices. Any two vertices determine either a side or a diagonal. Thus, the number of sides plus the number of diagonals is C(n, 2). Consequently, the number of diagonals is

$$C(n, 2) - n = \frac{(2n!)}{(2n-2)!2!} - n = \frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-3)$$

(b) If the number of diagonals is the same as the number of sides, we should have

$$\frac{1}{2}\,n(n-3) = n, \text{ or } n^2 - 5n = 0, \text{ or } n(n-5) = 0.$$

Since n > 0, we should have n = 5. Thus, the regular polygon which has the same number of diagonals as sides must have 5 sides; that is, it must be a pentagon.

32. A string of length n is a sequence of the form $x_1x_2x_3 \ldots x_n$, where each x_i is a digit. The sum $x_1 + x_2 + x_3 \ldots + x_n$ is called the weight of the string. If each x_i can be one of 0, 1, or 2, find the number of strings of length n = 10. Of these, find the number of strings whose weight is an even number.

Sol :

There are 10 positions in a string of length 10, and each of these positions can be filled in 3 ways (with 0, 1, 2). Therefore, the number of ways of filling the 10 positions of a string of length 10 is 3^{10}. This means that there are 3^{10} number of strings of length 10 (with 0, 1 or 2 as its digits).

Since each digit in the strings being considered here is 0, 1, or 2, the weight of a string is even only when it contains an even number of 1's. Thus, strings of even weight have zero, two, four, six, eight or ten number of 1's.

If a string has no 1's, then all its places are filled by 0's and 2's. The number of such strings is 2^{10}. If a string has two 1's, it can have two 1's in C(10, 2) number of locations. For each of these locations, the remaining eight locations are filled by 0's and 2's. Therefore, the number of strings having two 1's is C(10, 2) × 2^8.

Similarly, the numbers of strings having four 1's, six 1's and eight 1's are C(10, 4) × 2^6, C(10, 6) × 2^4 and C(10, 8) × 2^2 respectively. Lastly, the number of strings having ten 1's is evidently only one. Accordingly, the number of strings that have even weight is

$$2^{10} + C(10, 2) \times 2^8 + C(10, 4) \times 2^6 + C(10, 6) \times 2^4 + C(10, 8) \times 2^2 + 1.$$

33. Prove the following identities:

(i) C(n + 1, r) = C(n, r – 1) + C(n, r)

(ii) C(m + n, 2) – C(m, 2) – C(n, 2) = mn.

Sol :

We have

(i) $$C(n, r-1) + C(n. r) = \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}$$

$$[6pt] = \frac{n!}{(r-1)!(n-r)!}\left\{\frac{1}{n-r+1} + \frac{1}{r}\right\}$$

$$= \frac{n!}{(r-1)!(n-r)!} \cdot \frac{n-1}{r(n-r+1)}$$

$$= \frac{(n+1)1}{r(n-r+1)!} = C(n+1, r).$$

(ii) $$C(n, 2) + C(n\ 2) + mn = \frac{n!}{(m-2)!-2} + \frac{n!}{(n-2)!-2} + mn$$

$$= \frac{1}{2}\{m(m-1) + n(n-1)\} + mn$$

$$= \frac{1}{2}\{m + n)(m + n - 1) = \frac{(m+n)!}{2(m+n-2)!}$$

$$= C(m + n, 2)$$

2.3 Inclusion and Exclusion

Q34. Describe the basic principles of Inclusion and Exclusion.

Ans : **(Imp.)**

Consider a finite set S containing p number of elements. Here, the number p is called the order, size or the cardinality of the set S and is denoted by o(S), or n(S), or |S|.

For example, if A = {1, 2, 6} and B = {a, b, c, d}, then o(A) = |A| = 3 and o(B) = |B| = 4.

It is obvious that $|\phi| = 0$, and $|S| \geq 1$ for every non-empty finite set S. Further, for any two finite sets A and B, if $A \subseteq B$ then $|A| \leq |B|$ and if $A \subset B$ then $|A| < |B|$.

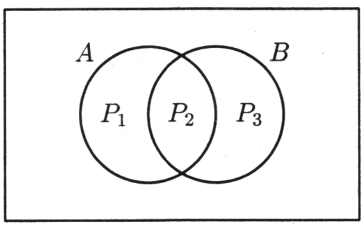
If A is a subset of a finite universal set $\bigcup$, then the number of elements in the complement $\bar{A}$ (of A in $\bigcup$) is given by

$$|\bar{A}| = |\bigcup| - |A|, \qquad \text{..... (1)}$$

Suppose we consider the union of two finite sets A and B and wish to determine the number of elements in $A \cup B$. Since the elements of $A \cup B$ consist of all elements which are in A or B or both A and B, the number of elements in $A \cup B$ is equal to the number of elements in A plus the number of elements in B minus the number of elements (if any) that are common to A and B. That is,

$$|A \cup B| = |A| + |B| - |A \cap B|. \qquad \text{..... (2)}$$

A more explicit (visual) way of obtaining this result is through the use of a Venn diagram.



Consider the Venn diagram shown above. In this diagram, the set A is made up of two parts P_1 and P_2, and the set B is made up of two parts P_2 and P_3, where $P_2 = A \cap B$, and $A \cup B$ is made up of parts P_1, P_2 and P_3. Therefore,

$$|A| = \text{Number of elements in } P_1 + \text{Number of elements in } P_2 = |P_1| + |P_2|$$

Similarly,

$|B| = |P_2| + |P_3|$, $|A \cap B| = |P_2|$ and $|A \cup B| = |P_1| + |P_2| + |P_3|$.

From these, we get

$$|A \cup B| = |P_1| + |P_2| + |P_3| = (|P_1| + |P_2|) + (|P_2| + |P_3|) - |P_3|$$
$$= |A| + |B| - |A \cap B|.$$

Thus, for determining the number of elements in $A \cup B$, we first include all elements in A and all elements in B, and then exclude all elements that are common to A and B.

If $\bigcup$ is a finite universal set of which A and B are subsets, then, by virtue of a De'Morgan law and the expression (1) above, we have

$$|\bar{A} \cap \bar{B}| = [\overline{A \cup B} = |\bigcup| - |A \cup B|$$

With the use of formula (2) above, this becomes

$$|\bar{A} \cap \bar{B}| = |\bigcup| - \{|A| + |B| - |A \cap B|\}$$
$$= |\bigcup| - |A| - |B| + |A \cap B| \qquad ..... (3)$$

Expressions (2) and (3) are equivalent to one another. Either of these is referred to as the Addition Principle (Rule) or the Principle of inclusion-exclusion for two sets.

In the particular case where A and B are disjoint sets so that $A \cap B = \Phi$, the addition rule (2) becomes

$$|A \cup B| = |A| + |B| - |\Phi| = |A| + |B| \qquad ..... (4)$$

This is known as the Principle of disjunctive counting for two sets.

PROBLEMS

35. A computer company requires 30 programmers to handle systems programming jobs and 40 programmers for applications programming. If the company appoints 55 programmers to carry out these jobs, how many of these perform jobs of both types? How many handle only system programming jobs? How many handle only applications programming?

Sol :

Let A denote the set of programmers who handle systems programming job and B the set of programmers who handle applications programming. Then $A \cup B$ is the set of programmers appointed to carry out these jobs. By what is given, we have

$$|A| = 30, |B| = 40, |A \cup B| = 55.$$

Therefore, the addition rule $|A \cup B| = |A| + |B| - |A \cap B|$ gives

$$|A \cap B| = |A| + |B| - |A \cup B| = 30 + 40 - 55 = 15.$$

This means that 15 programmers perform both types of jobs.

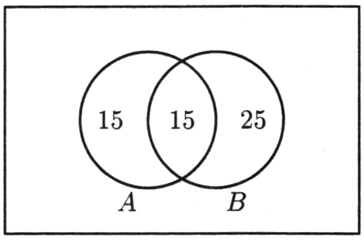
Next, we note that the set of programmers who handle only systems programming is A – B. By examining Figure (b), we observe that $|A - B| = |A| - |A \cap B|$. Accordingly, the number of programmers who handle only systems programming jobs is

$|A - B| = |A| - |A \cap B| = 30 - 15 = 15$

Similarly, the number of programmers who handle only applications programming is

$|B - A| = |B| - |A \cap B| = 40 - 15 = 25.$

These results are illustrated in the following Venn diagram:



36. In a class of 52 students, 30 are studying C++, 28 are studying Pascal and 13 are studying both languages. How many in this class are studying at least one of these languages? How many are studying neither of these languages?

Sol : **(Imp.)**

Let $\cup$ denote the set of all students in the class, A denote the set of students in the class who are studying C++, and B is the set of students in the class who are studying Pascal.

Then, the set of students in the class who are studying both languages is $A \cap B$, the set of students who are studying at least one of the two languages is $A \cup B$ and the set of students who are studying neither of these languages is $(\overline{A \cup B})$.

From what is given, we have

$|\cup| = 52, |A| = 30, |B| = 28, |A \cap B| = 13.$

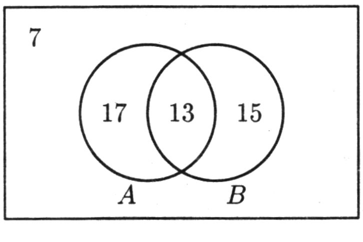
Therefore, by the addition principle,

$|A \cup B| = |A| + |B| - |A \cap B| = 30 + 28 - 13 = 45.$

Also,

$|(\overline{A \cup B})| = |\cup| - |A \cup B| = 52 - 45 = 7$

Thus, 45 students of the class study at least one of the two languages indicated and 7 students of the class study neither of these languages.



Q37. Explain the process of Inclusion and Exclusion for n sets.

Ans : **(Imp.)**

Statement

The principle of inclusion-exclusion as given by expression (2) can be extended to n sets, $n > 2$.

Let $\cup$ be a finite universal set and $A_1, A_2, ..., A_n$ be subsets of $\cup$. Then the Principle of Inclusion-Exclusion for $A_1, A_2, ...,$ An states that

$$|A_1 \cup A_2 \cup A_3 \cup ... \cup An| = \Sigma|A_i| - \Sigma|A_i \cap A_j| + \Sigma|A_i \cap A_j \cap A_k| +$$
$$... + (-1)^{n-1} |A_1 \cap A_2 \cap ... \cap A_n|$$

Proof:

Take any $x \in A_1 \cup A_2 \cup ..... \cup A_n$.

Then x is in m of the sets $A_1, A_2, ..., A_n$

Where

$1 \leq m \leq n$. Without loss of generality, let us assume that $x \in A$, for $1 \leq i \leq m$ and $x \notin A_i$ for $i > m$. Then x will be counted once in each of the terms $|A_i|$, $i = 1, 2, ..., m$. Thus, x will be counted m times in $\Sigma|A_i|$

We note that there are C(m, 2) pairs of sets A_i A_j where x is in both A_i and A_j. As such, x will be counted C(m, 2) times in $\Sigma|A_i \cap A_j|$.

Similarly, x will be counted C(m, 3) times in $\Sigma|A_i \cap A_j \cap A_k|$, and so on.

Continuing in this way, we see that, in the right hand side of expression (5), x is counted

$$m - C(m, 2) + C(m, 3) + ... + (-1)^{m-1} C(m, m)$$

number of times. (Bear in mind that $C(m, n) = 0$ for $n > m$).

We note that

$$m - C(m, 2) + C(m, 3) + ... (-1)^{m-1} C(m, m)$$
$$= 1 - (1 - m + C(m, 2) - C(m, 3) + ... + (-1)^m C(m, m)\}$$
$$= 1 - (1 + (-1))^m, \text{ by binomial theorem}$$
$$= 1.$$

Thus, on the right hand side of expression (5) every element x of $A_1 \cup A_2 \cup ... \cup A_n$ is counted exactly once. This means that the number of elements in $A_1 \cup A_2 \cup ... \cup A_n$ is equal to the right hand side of expression (5). This completes the proof of expression (5).

Corollary :

By virtue of a De'Morgan law, we have

$$\overline{(A_1 \cup A_2 \cup A_3 ... \cup A_n)} = \overline{A_1} \cap \overline{A_2} \cap \overline{A_n}$$

Since $|\bar{A}| = |\cup| - |A|$ for any subset A of $\cup$, this yields

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \dots \cap \overline{A_n}| = |(\overline{A_1 \cup A_2 \cup A_3 \cup \dots A_n})|$$

$$= |\cup| - |(A_1 \cup A_2 \cup \dots \cup A_n)|$$

Using expression (5), this becomes

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \dots \cap \overline{A_n}| = |\cup| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + \dots$$

$$+ (-1)\ |A_1 \cap A_2 \cap \dots \cap A_n| \qquad \dots (6)$$

This is an equivalent version of the Principle of inclusion-exclusion, given by (5). Note that, for n = 2, expressions (5) and (6) reduce to expressions (2) and (3) respectively,

PROBLEMS

38. If A, B, C are finite sets, prove that

$$|A - B - C| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

Sol :

We first note that A – B – C is the set of elements that belong to A, but not to B or C.

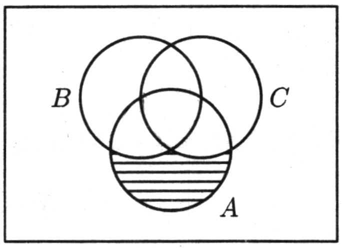


Fig. : A - B - C (shaded)

Therefore,

$$|A - B - C| = |A \cup B \cup C| - |B \cup C|;$$

$$= (|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|)$$

$$- (|B| + |C| - |B \cap C|), \text{ on using addition principle}$$

$$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

39. A survey of 500 television viewers of a sports channel produced the following information: 285 watch cricket, 195 watch hockey, 115 watch football, 45 watch cricket and football, 70 watch cricket and hockey, 50 watch hockey and football and 50 do not watch any of the three kinds of games.

(a) How many viewers in the survey watch all three kinds of games?

(b) How many viewers watch exactly one of the sports?

Sol : **(Imp.)**

Let ∪ denote the set of all viewers included in the survey, A denote the set of viewers who watch cricket, B denote the set of viewers who watch hockey, and C denote the set of viewers who watch football. Then, from what is given, we have

$|U| = 500, \; |A| - 285, \; |B| = 195, \; |C| = 115,$

$|A \cap C| = 45, \; |A \cap B| = 70, \; |B \cap C| = 50.$

$|\overline{A \cup B \cup C}| = 50, \; |A \cup B \cup C| = 500 - 50 = 450.$

Using the addition principle for 3 sets, namely,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

We find that

$$|A \cap B \cap C| = |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |A \cap C|$$
$$= 450 - 285 - 195 - 115 + 70 + 50 + 45 = 20.$$

Thus, the number of viewers who watch all three kinds of games is 20.

Let A_1 denote the set of viewers who watch only cricket, B_1 denote the set of viewers who watch only hockey and C_1 denote the set of viewers who watch only football.

Then, $A_1 = A - B - C$, and by virtue of the result proved in Example 3, we have

$$|A_1| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

Accordingly, the number of viewers who watch only cricket is

$$|A_1| = 285 - 70 - 45 + 20 = 190.$$

Similarly, the number of viewers who watch only hockey is

$$|B_1| = |B| - |B \cap A| - |B \cap C| + |B \cap C \cap A|$$
$$= 195 - 70 - 50 + 20 = 95,$$

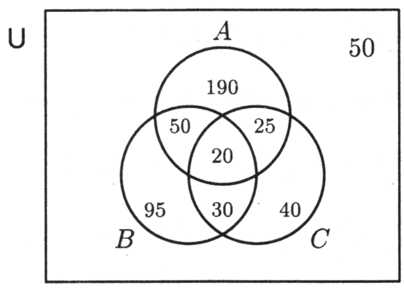
and the number of viewers who watch only football is

$$|C_1| = |C| - |C \cap A| - |C \cap B| + |C \cap A \cap B|$$
$$= 115 - 45 - 50 + 20 = 40.$$

From these, we find that the number of viewers who watch exactly one of the sports is

$$|A_1| + |B_1| + |C_1| = 190 + 95 + 40 = 325.$$

Venn Diagram



40. Thirty cars are assembled in a factory. The options available are a music system, an air conditioner and power windows. It is known that 15 of the cars have music systems, 8 have air conditioners and 6 have power windows. Further, 3 have all options. Determine at least how many cars do not have any option at all.

Sol :

Let $\cup$ denote the set of all cars being considered, and A, B, C respectively denote the sets of cars having music system, air conditioner and power windows respectively. Then, from what is given, we have

$$|\cup| = 30, \ |A| = 15, \ |B| = 8, \ |C| = 6, \ |A \cap B \cap C| = 3 \quad \text{..... (i)}$$

We note that, $A \cup B \cup C$ denotes the set of cars that have at least one of the options, so that $\overline{A \cup B \cup C}$ is the set of cars that do not have any option.

By the addition rule, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| \quad \text{..... (ii)}$$

Since $A \cap B \cap C$ is a subset of $A \cap B$, $B \cap C$ and $C \cap A$, we have

$$|A \cap B| \geq |A \cap B \cap C|, |B \cap C| \geq |A \cap B \cap C|, |C \cap A| \geq |A \cap B \cap C|.$$

Using these in (ii), we get

$$|A \cup B \cup C| \leq |A| + |B| + |C| - |A \cap B \cup C| - |A \cap B \cap C| - |A \cap B \cap C| + |A \cap B \cap C|$$

$$= |A| + |B| + |C| - 2|A \cap B \cap C| = 15 + 8 + 6 - 6 = 23.$$

Consequently,

$$|(\overline{A \cup B \cup C})| = |\cup| - |A \cup B \cup C| \geq 30 - 23 = 7.$$

This shows that at least 7 cars do not have any of the options.

41. A student visits a sports club every day from Monday to Friday after school hours and plays one of the three games: Cricket, Tennis, Football. In how many ways can he play each of the three games at least once during a week (from Monday to Friday) ?

Sol :

On each day, the student has three choices of games. Therefore, the total number of choices of games in a 5-day period is 3^5. Thus, if $\cup$ is the set of all choice of games in a 5-day period, we have $|\cup| = 3^5$.

Let A denote the set of all choices of games which excludes cricket. Then, the number of choices of games in a 5-day period which excludes cricket is $|A| = 2^5$. Similarly, if B is the set of all choices of games which excludes Tennis in a 5-day period and C is the set of all choices of games which excludes Football in a 5-day period, we have $|B| = 2^5$ and $|C| = 2^5$.

Consequently, $A \cap B$ is the set of all choices of games in a 5-day period which excludes cricket and tennis, and $|A \cap B\backslash = 1^5$. Similarly, $|B \cap C| = 1^5$, $|A \cap C| = 1^5$. Also, $A \cap B \cap C$ is the set of all choices of games which excludes all of the three games in the 5-day period, and this set is the null set. Therefore, $|A \cap B \cap C| = 0$.

Further, $A \cup B \cup C$ is the set of all choices of games which excludes at least one of the three games in the 5-day period, and $|A \cup B \cup C|$ is given by

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 2^5 + 2^5 + 2^5 - 1^5 - 1^5 - 1^5 + 0 = 3 \times 2^5 - 3 = 93.$$

Therefore, the number of choices of games in the 5-day period which does not exclude any game is

$$|(\overline{A \cup B \cup C})| = |\cup| - |A \cup B \cup C| = 3^5 - 93 = 243 - 93 = 150.$$

Thus, there are 150 ways for the student to select his daily games so that he plays every iime at least once during a week (from Monday to Friday).

42. Out of 30 students in a hostel 15 study History, 8 study Economics, and 6 study Geography. It is known that 3 students study all these subjects. Show that 7 or more students study none of these subjects.

Sol : **(Imp.)**

Let $\cup$ denote the set of all students in the hostel, and A_1, A_2, A_3 denote the sets of students who study History, Economics and Geography, respectively. Then, from what is given, we have

$$S_1 = \Sigma|A_i| = 15 + 8 + 6 = 29, \text{ and } S_3 = |A_1 \cap A_2 \cap A_3| = 3.$$

The number of students who do not study any of the three subjects is $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |\cup| - \Sigma|A_i| + \Sigma|A_1 \cap A_j| - \Sigma|A_1 \cap A_2 \cap A_3|$$
$$= |\cup| - S_1 + S_2 - S_3$$
$$= 30 - 29 + S_2 - 3 = S_2 - 2 \qquad \text{..... (i)}$$

where $S_2 = \Sigma|A_i \cap A_j|$.

We note that $(A_1 \cap A_2 \cap A_3)$ is a subset of $(A_i \cap A_j)$ for i, j = 1, 2, 3. Therefore, each of $|A_i \cap A_j|$, which are 3 in number, is greater than or equal to $|A_1 \cap A_2 \cap A_3|$. Hence

$$S_2 = \Sigma|A_i \cap A_j| \geq 3|A_1 \cap A_2 \cap A_3| = 9.$$

Using this in (i), we find that

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| \geq 9 - 2 = 7.$$

This proves the required result

43. Find the number of nonnegative integer solutions of the equation

$$\mathbf{x_1 + x_2 + x_3 + x_4 = 18}$$

under the condition $x_1 \leq 7$ for i = 1, 2, 3,4.

Sol :

Let S denote the set of all nonnegative integer solutions of the given equation. The number of such solutions is C(4 + 18 – 1, 18) = C(21, 18), so that

$$|S| = C(21, 18).$$

Let A_1 be the subset of S that contains the nonnegative integer solutions of the given equation under the conditions $x_1 \geq 7$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$. That is,

$A_1 = (x_1 + x_2 + x_3 + x_4) \in S|x_1 > 7|$

Similarly, let

$A_2 = \{(x_1 + x_2 + x_3 + x_4) \in S|x_2 > 7\}$

$A_3 = \{(x_1 + x_2 + x_3 + x_4) \in S|x_3 > 7\}$

$A_4 = \{(x_1 + x_2 + x_3 + x_4) \in S|x_4 > 7\}$

Then the required number of solutions would be $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}|$.

Let us set $y_1 = x_1 - 8$. Then $x_1 > 1$ (i.e., $x_1 \geq 8$) corresponds to $y_1 \geq 0$. When written in terms of y_1, the given equation reads

$y_1 + x_2 + x_3 + x_4 = 10.$

The number of nonnegative integer solutions of this equation is C(4 +10 – 1, 10) = C(13, 10). This is precisely $|A_1|$. Thus, $|A_1|$ = C(13, 10).

Similarly, by symmetry,

$|A_2| = |A_3| = |A_4| = C(13, 10).$

Let us take $y_1 = x_1 - 8$, $y_2 = x_2 - 8$. Then $x_1 > 7$ and $x_2 > 7$ correspond to $y_1 \geq 0$ and $y_2 > 0$. When written in terms of y_1 and y_2, the given equation reads

$y_1 + y_2 + x_3 + x_4 = 2$

The number of nonnegative integer solutions of this equation is C(4 + 2 – 1, 2) = C(5, 2). This is precisely $|A_1 \cap A_2|$. Thus, $|A_1 \cap A_2|$ = C(5, 2).

Similarly, by symmetry,

$|A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4| = C(5, 2).$

In the given equation, more than two x,'s cannot be greater than 7 simultaneously. Hence

$|A_1 \cap A_2 \cap A_3| = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = 0,$

and $|A_1 \cap A_2 \cap A_3 \cap A_4| = 0.$

Accordingly, we find that (using the principle of inclusion-exclusion as given by equation (4))

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = \Sigma|Ai| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$$= C(21, 8) - \binom{4}{1} \times C(13, 15) + \binom{4}{2} \times C(5, 2) - 0 + 0$$

$$= 1330 - (4 \times 286) + (6 \times 30) = 366.$$

This is the required number of solutions.

2.4 Pigeon-Hole Principle

Q44. Explain about Pigeon-Hole Principle.

Ans : **(Imp.)**

In m pigeons occupy npigeonholes and if m > n, then two or more pigeons occupy the same pigeonhole.

This is often restated as follows:

If m pigeons occupy n pigeonholes, where m > n, then at least one pigeonhole must contain two or more pigeons in it.

This statement is known as the Pigeonhole principle. This is one of the simplest and yet fundamental among the principles used in counting.

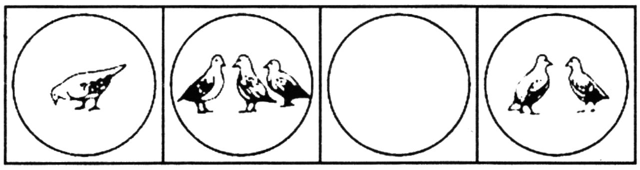


Fig. : The Pigeonhole Principle

A simple illustration of the above principle is that if 6 pigeons occupy 4 pigeonholes, then atleast one pigeonhole must contain two or more pigeons in it.

As a simple application of the principle, we may note that if 8 children are born in the same week, then two or more children are born on the same day of the week.

Generalization

The following is an extension/generalization of the pigeonhole principle.

If m pigeons occupy n pigeonholes, then at least one pigeonhole must contain (p + 1) or more pigeons, where p — [{m – 1)/n]

Proof:

We prove this principle by the method of contradiction.

Assume that the conclusion part of the principle is not true. Then, no pigeonhole contains (p + 1) or more pigeons. This means that every pigeonhole contains p or less number of pigeons. Then:

Total number of pigeons $\leq$ np = n × [(m – 1)/n] $\leq$ n $\left(\frac{m-1}{n}\right)$ = (m – 1).

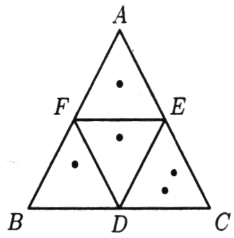
This is a contradiction, because the total number of pigeons is m. Hence our assumption is wrong, and the principle is true.

PROBLEMS

45. ABC is an equilateral triangle whose sides are of length 1 cm each. If we select 5 points inside the triangle, prove that at least two of these points are such that the distance between them is less than 1/2 cm.

Sol :

Consider the triangle DEF formed by the mid-points of the sides BC, CA and AB of the given triangle ABC. Then the triangle ABC is partitioned into four small equilateral triangles (portions), each of which has sides equal to 1/2 cm. Treating each of these four portions as a pigeonhole and five points chosen inside the triangle as pigeons, we find by using the pigeonhole principle that at least one portion must contain two or more points. Evidently, the distance between such points is less than 1/2 cm.



46. How many persons must be chosen in order that at least five of them will have birth days in the same calendar month?

Sol :

Let n be the required number of persons. Since the number of months over which the birthdays are distributed is 12, the least number of persons who have their birthdays in the same month is, by the generalized pigeonhole principle, equal to $\left[\frac{(n-1)}{12}\right] + 1$. This number is 5,

$$\left[\frac{(n-1)}{12}\right] + 1 = 5, \text{ or } n = 49.$$

Thus, the number of persons is 49 (at the least).

47. Find the least number of ways of choosing three different numbers from 1 to 10 so that all choices have the same sum.

Sol :

From the numbers from 1 to 10, we can choose three different numbers in C(10, 3) = 120 ways.

The smallest possible sum that we get from a choice is 1 + 2 + 3 = 6 and the largest sum is 8 + 9 + 10 = 27. Thus, the sums vary from 6 to 27 (both inclusive), and these sums are 22 in number.

Accordingly, here, there are 120 choices (pigeons) and 22 sums (pigeonholes). Therefore, the least number of choices assigned to the same sum is, by the generalized pigeonhole principle,

$$\left|\frac{120-1}{22}\right| + 1 = [6, 4] \approx 6.$$

48. Prove the statement: If m = kn + 1 pigeons (where k $\geq$ 1) occupy n pigeonholes then at least one pigeonhole must contain k + 1 or more pigeons.

Sol :

Assume that the conclusion part of the given statement is false. Then every pigeonhole contains k or less number of pigeons. Then, the total number of pigeons would be nk. This is a contradiction. Hence, the assumption made is wrong, and the given statement is true.

49. A bag contains many red marbles, many white marbles, and many blue marbles. What is the least number of marbles one should take out to be sure of getting at least six marbles of the same color?

Sol :

Let us treat the marbles as pigeons and colors as pigeonholes. Then, the number of pigeon-holes is $n = 3$. Therefore, if $m = 3k + 1$ pigeons, where $k \geq 1$, occupy 3 pigeonholes then at least one pigeonhole must contain $k + 1$ or more pigeons. We note that $k + 1 = 6$ corresponds to $m = 16$. Therefore, the presence of 16 or more pigeons in 3 pigeonholes will ensure that there are 6 or more pigeons in a hole. Thus, 16 is the least number of marbles to be taken out.

50. Suppose $m = (p_1 + p_2 + ... + p_n - n + 1)$ pigeons occupy n pigeonholes $H_1, H_2, ..., H_n$. Prove that some pigeonhole H_j contains p_j or more pigeons.

Sol :

Assume that the conclusion part of the given statement is false. Then every hole H_j contains $P_j - 1$ or less number of pigeons, $j = 1, 2, ... n$. Then the total number of pigeons would be less than or equal to

$$(p_1 - 1) + (p_2 - 1) + ... + (p_n - 1) = \{p_1 + p_2 + ... + p_n - n\} = m - 1.$$

This is a contradiction, because the number of pigeons is equal to m. Hence the assumption made is wrong, and the given statement is true.

UNIT III	**Propositional Logic:** Syntax, Semantics, Validity and Satisfiability, Basic Connectives and Truth Tables, Logical Equivalence: The Laws of Logic, Logical Implication, Rules of Inference, The use of Quantifiers. **Proof Techniques**: Some Terminology, Proof Methods and Strategies, Forward Proof, Proof by Contradiction, Proof by Contraposition, Proof of Necessity and Sufficiency.

3.1 Propositional Logic

Q1. What is propositional Logic ? Define it with an example.

Ans :

Propositional Logic

The rules of mathematical logic specify methods of reasoning mathematical statements. Greek philosopher, Aristotle, was the pioneer of logical reasoning. Logical reasoning provides the theoretical base for many areas of mathematics and consequently computer science. It has many practical applications in computer science like design of computing machines, artificial intelligence, definition of data structures for programming languages etc.

Definition

A proposition is a collection of declarative statements that has either a truth value "true" or a truth value "false". A proposition consists of propositional variables and connectives. We denote the propositional variables by capital letters (A, B, etc). The connectives (   ) connect the propositional variables.

Some examples of Propositions are given below-

- "Man is Mortal", it returns truth value "TRUE"
- "12 + 9 = 3 " 2", it returns truth value "FALSE"

The following is not a Proposition:

"A is less than 2". It is because unless we give a specific value of A, we cannot say whether the statement is true or false.

3.1.1 Syntax, Semantics, Validity and Satisfiability

Q2. Explain briefly about Syntax, Semantics, Validity and Satisfiability.

Ans :

1. Syntax

To define this logic, we will assume a (countably infinite) set of propositions Prop = $\{ P_i \mid i \in N\}$. The formulas of prepositional logic will be strings over the alphabet Prop U{(,), $\rightarrow$, $\perp$ }; here $\rightarrow$ is called implication, and $\perp$ is called false.

Definition

The set of well formed formulas (wff) in proportional logic is the smallest set satisfying flic following properties.

- ⊥ is a wff.
- Any proposition p_i (by itself) is a wff.
- If ϕ and ψ are wffs then $(\varphi \rightarrow \psi)$ is a wff.

(i) Semantics

The semantics of formulas in a logic, are typically defined with respect to a model, which identifies a "world" in which certain facts are true, hi the case of propositional logic, this world or model is a truth valuation or assignment that assigns a truth value (true/false) to every proposition. The truth value truth will be denoted by 1. and the truth value falsity will be denoted by 0.

(ii) Validity and Satisfiability

Two formulas that are syntactically different, could however, be "semantically equivalent". Intuitively, this is when the truth value of each formula in every valuation is the same.

Definition - (Logical Equivalence).

A wff φ is said to be logically equivalent to ψ iff any of the following equivalent conditions hold.

- for every valuation v, v $|= \varphi$ iff v$|= \psi$
- for every valuation v, v$[\varphi]$ = v$[\psi]$,
- $[\varphi] = [\psi]$

  We denote this $\varphi = \psi$

3.2 Basic Connectives and Truth Tables

Q3. What are the called as statements in mathematical logic? Explain various types of statements with its notations.

Ans : **(Imp.)**

Statements (Propositions)

Statements are sentences that claim certain things. Can be either true or false, but not both.

The propositional statements are of the following types:

1. Negation
2. Conjunction
3. Disjunction
4. Conditional
5. Bi conditional

1. Negation (NOT, ~, ¬)

Indicates the opposite, usually employing the word not. The symbol to indicate negation is : ~

Truth table for negation

P	~P
T	F
F	T

~P is true if and only if P is false

Example 1

Original Statement	Negation of statement
Today is Monday. That was fun.	Today is **not** Monday. That was **not** fun.

Example 2

a: The product of two negative numbers is a positive number. True

~a: The product of two negative numbers is not a positive number. False

We can construct a truth table to determine all possible truth values of a statement and its negation.

2. Conjunction (AND, $\wedge$)

In logic, a conjunction is a compound sentence formed by using the word and to join two simple sentences. The symbol for this is $\wedge$. (whenever you see $\wedge$ read 'and') When two simple sentences, p and q, are joined in a conjunction statement, the conjunction is expressed symbolically as $p \wedge q$.

Truth table for Conjunction

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1:

Simple Sentences Conjunction	Compound Sentence
p: Joe eats fries, q: Maria drinks soda, drinks soda.	$p \wedge q$: Joe eats fries, and maria

Example 2:

Given: a: A square is a quadrilateral.

b: Harrison Ford is an American actor.

Problem: Construct a truth table for the conjunction "a and b."

Sol :

a	b	$a \wedge b$
T	T	T
T	F	F
F	T	F
F	F	F

Example 3:

Construct a truth table for each conjunction below:

1. x and y
2. ~x and y
3. ~y and x

Sol :

x	y	$x \wedge y$
T	T	T
T	F	F
F	T	F
F	F	F

x	y	~x	$\sim x \wedge y$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

x	y	~y	$\sim y \wedge x$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

3. Disjunction (inclusive or) (OR, V)

A **disjunction** is a compound statement formed by joining two statements with the connector OR. The disjunction "p or q" is symbolized by $p \vee q$. A disjunction is false if and only if both statements are false; otherwise it is true. The truth values of $p \vee q$ are listed in the truth table below.

Truth table for Disjunction

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P V Q is true iff P is true or Q is true or both are true

P V Q is false iff both P and Q are false

Example 1:

Given:	A. A square is a quadrilateral B. Harrison Fords is an American actor
Problem:	Contract a truth table for the disjunction "a or b"

Sol :

A	B	$a \vee b$
T	T	T
T	F	T
F	T	T
F	F	F

Example 2:

Complete a truth table for each disjunction below.

1. a or b
2. a or not b
3. not a or b

Sol :

a	b	$a \vee b$
T	T	T
T	F	T
F	T	T
F	F	F

a	b	~b	$a \vee \sim b$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

a	b	~a	$\sim a \vee \sim b$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

4. The Conditional Statement (known also as implication ($\rightarrow$))

A conditional statement, symbolized by $p \rightarrow q$, is an if-then statement in which p is a hypothesis and q is a conclusion. The logical connector in a conditional statement is denoted by the symbol . The conditional is defined to be true unless a true hypothesis leads to a false conclusion. A truth table for pq is shown below.

Truth Table,

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The implication $P \rightarrow Q$ is false if P is true however Q is false

In all other cases the implication is true

Example 1:

Given:	a: The sun is made of gas.
	b: 3 is a prime number.
Problem:	Write a $\rightarrow$ b as a sentence. Then construct a truth table for this conditional.

Sol :

The conditional a $\rightarrow$ b represents “If the sun is made of gas, then 3 is a prime number.”

a	b	$a \rightarrow b$
T	T	T
T	F	F
F	T	T
F	F	T

In the following examples, we are given the truth values of the hypothesis and the conclusion and asked to determine the truth value of the conditional.

Example 2:

Given:	r: 8 is an odd number.	false
	s: 9 is composite.	true
Problem:	What is the truth value of $r \rightarrow s$?	

Sol :

Since hypothesis r is false and conclusion s is true, the conditional $r \rightarrow s$ is true.

Example 3:

Given:	r: 8 is an odd number.	false
	s: 9 is composite.	true
Problem:	What is the truth value of $s \rightarrow r$?	

Sol :

Since hypothesis s is true and conclusion r is false, the conditional $s \rightarrow r$ is false.

5. Bi conditional Statement ($\Leftrightarrow$)

A biconditional statement is defined to be true whenever both parts have the same truth value. The biconditional operator is denoted by a double-headed arrow $\leftrightarrow$. The biconditional $p \leftrightarrow q$ represents "p if and only if q," where p is a hypothesis and q is a conclusion. The following is a truth table for biconditional $p \leftrightarrow q$.

Truth Table

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$P \Leftrightarrow Q$ is true iff P and Q have same valume - both are true or both are false

If P and Q have different values, the biconditional is false.

Example 1:

Given:	a: x + 2 = 7
	b: x = 5
Problem:	Write a $\rightarrow$ b as a sentence. Then determine its truth values a $\rightarrow$ b

Sol :

The biconditional a $\leftrightarrow$ b represents the sentence: "x + 2 = 7 if and only if x = 5." When x = 5, both a and b are true. When x = 5, both a and b are false. A biconditional statement is defined to be true whenever both parts have the same truth value. Accordingly, the truth values of ab are listed in the table below.

A	b	a $\leftrightarrow$ b
T	T	T
T	F	F
F	T	F
F	F	T

Q4. Construct the truth table for ~(p$\wedge$q).

Ans :

Whenever we encounter a complex formula like this, we work from the inside out, just as we might do if we had to evaluate an algebraic expression, like - (a + b). Thus, we start with the p and q columns, then construct the p$\wedge$q column, and finally, the ~(p$\wedge$q) column:

p	**q**	**p $\wedge$ q**	**~(p $\wedge$ q)**
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Notice how we get the ~(p$\wedge$q) column from the p$\wedge$q column: we reverse all its the truth values, since that is what negation means.

Q5. Construct the truth table for p$\vee$(p$\vee$q).

Ans :

Since there are two variables, p and q, we again start with the p and q columns. Working from inside the parentheses, we then evaluate p$\wedge$q, and finally take the disjunction of the result with p:

P	Q	$p \vee q$	$p \vee (p \vee q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

Q6. Construct the truth table for $\sim(p \wedge q)(\sim r)$.

Sol :

Here there are three variables: p, q and r. Thus we start with three initial columns showing all eight possibilities:

P	Q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

We now add columns for $p \wedge q$, $\sim(p \wedge q)$ and $\sim r$, and finally $\sim(p \wedge q) \wedge (\sim r)$ according to the instructions for these logical operators. Here is how the table grows as you construct it:

P	Q	R	$p \wedge q$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

and finally,

p	q	r	$p \wedge q$	$\sim r$	$\sim(p \wedge q) \wedge (\sim r)$
T	T	T	T	F	F
T	T	F	T	F	T
T	F	T	F	T	F
T	F	F	F	T	T
F	T	T	F	T	F
F	T	F	F	T	T
F	F	T	F	T	F
F	F	F	F	T	T

Q7. Construct the truth tables of the following compound propositions :

(i) $(p \vee q) \wedge r$ (ii) $p \vee (q \wedge r)$

Sol : **(Imp.)**

Each of the two given combined propositions contains three primitive propositions p, q, r. Each of these three primitive propositions has two possible truth values. Therefore, there exist $2^3 = 8$ sets of possible truth values of p, q, r, and each such set gives a truth value of the compound proposition containing p, q, r. The required truth tables are shown below in a combined form.

The required truth tables are shown below in a combined form.

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$q \wedge r$	$p \vee (q \wedge r)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	1	0	0	0
0	1	1	1	1	1	1
1	0	0	1	0	0	1
1	0	1	1	1	0	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

Observe that the truth values of $(p \vee q) \wedge r$ and $p \vee (q \wedge r)$ are not identical.

Q8. Construct the truth tables of the following compound propositions :

(i) $(p \wedge q) \rightarrow (\neg r)$ (ii) $q \wedge ((\neg r) \rightarrow p)$

Sol :

The required truth tables are shown below in a combined form.

p	q	r	$\neg r$	$(p \wedge q)$	$(p \wedge q) \rightarrow (\neg r)$	$(\neg r) \rightarrow p$	$q \wedge ((\neg r) \rightarrow p)$
0	0	0	1	0	1	0	0
0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	1	1
1	0	0	1	0	1	1	0
1	0	1	0	0	1	1	0
1	1	0	1	1	1	1	1
1	1	1	0	1	0	1	1

Q9. Define the following terms :

1. Tautology

2. Contradiction

Ans :

1. Tautology

A compound proposition which is true for all possible truth values of its components is called a tautology (or a logical truth or a universally valid statement) A tautology is generally denoted by To

2. Contradiction

A compound proposition which is false for all possible truth values of its components is called a contradiction or an absurdity. A contradiction is generally denoted by F_0.

A compound proposition that can be true or false (depending upon the truth values of its components) is called a contingency. In other words, a contringency is a compound proposition which is neither a tautology nor a contradiction.

Q10. Show that the truth values of the following compound propositions are independent of the truth values of their components:

(i) $\{p \wedge (p \rightarrow q)\} \rightarrow q$ (ii) $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

Sol :

Let us first prepare the truth tables for the given compound propositions. These are as shown below.

(i)

p	q	$p \to q$	$r = p \wedge (p \to q)$	$r \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

(ii)

p	q	$u = p \to q$	$\neg p$	$v = (\neg p) \vee q$	$u \leftrightarrow v$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	1	0	1	1

The last columns in both of the above tables show that the truth value of each of the given compound propositions is 1 irrespective of what is the truth values of its components are. This is what we had to show. Furthermore, we have proved that the given compound propositions are tautologies.

Q11. Prove that, for any propositions p and q, the compound proposition

$[(\neg q) \wedge (p \to q)] \to \neg p$ is a tautology.

Sol : **(Imp.)**

Let us prepare the following truth table :

p	q	$p \to q$	$r = [(\neg q) \wedge (p \to q)]$	$r \to \neg p$
0	0	1	1	1
0	1	1	0	1
1	0	0	0	1
1	1	1	0	1

We observe that the proposition $r \to \neg p$, where $r = [(\neg q) \wedge (p \to q)]$, is always true. Therefore, this proposition is a tautology.

Q12. Prove that, for any propositions p,q,r, the compound proposition

$[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.

Sol :

The following truth table proves the required result:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

Q13. Prove that, for any propositions p, q, r, the compound proposition

$\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$ is a tautology.

Sol :

The following truth table proves the required result.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$p \to (q \to r)$	$(p \to q) \to (p \to r)$	$\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	1	0	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	1	1	1
1	0	1	0	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

Q14. Examine whether

$[(p \vee q) \to r] \leftrightarrow [\neg r \to \neg (p \vee q)]$ is a tautology.

Sol :

Let us prepare the following truth table :

p	**q**	**r**	**$p \vee q$**	**$(p \vee q) \rightarrow r$**	**$\neg r$**	**$\neg (p \vee q)$**	**$\neg r \rightarrow \neg (p \vee q)$**
0	0	0	0	1	1	1	1
0	0	1	0	1	0	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	0	0	1
1	0	0	1	0	1	0	0
1	0	1	1	1	0	0	1
1	1	0	1	0	1	0	0
1	1	1	1	1	0	0	1

We observe that $(p \vee q) \rightarrow r$ and $\neg r \rightarrow \neg (p \vee q)$ have the same truth values in all possible situations. Therefore,

$$[(p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg (p \vee q)]$$

has the truth value 1in all possible situations; it is therefore a tautology.

3.3 Logical Equivalence

Q15. What is logical equivalence?

Ans : **(Imp.)**

Logical equivalence is a type of relationship between two statements or sentences in propositional logic or Boolean algebra. The relation translates verbally into "if and only if" and is symbolized by a double-lined, double arrow pointing to the left and right ( ). If A and B represent statements, then $A \Leftrightarrow B$ means "A if and only if B."

The statement $A \Leftrightarrow B$ is exactly the same as

$(A \Leftrightarrow B) * (B \Leftrightarrow A)$

where the asterisk (*) represents the logical AND operation, and the right-pointing, double-lined arrow ($\Leftrightarrow$) represents logical implication.

Logical equivalence works both ways. Thus,

$$(A \Leftrightarrow B) \Leftrightarrow (B \Leftrightarrow A)$$

Definition:

Two propositional expressions P and Q are logically equivalent, if they have same truth tables. We write $P \equiv Q$.

Q16. Show that $P \rightarrow Q$ and $\sim P \rightarrow Q$ are logically equivalent.

Sol:

P	Q	$P \rightarrow Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Q17. For any two propositions p, q, prove that $(p \rightarrow q) \Leftrightarrow (\neg p) \vee q$.

Sol :

We first prepare the following truth table :

p	**q**	$p \rightarrow q$	$\neg p$	$(\neg p) \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

We observe that $p \rightarrow q$ and $(\neg p) \vee q$ have identical truth values for all possible truth values of p and q. Therefore, $(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$.

Q18. Prove that, for any propositions p and q, the compound propositions $p \veebar q$ and $(p \vee q) \wedge (\neg p \vee \neg q)$ are logically equivalent.

Sol : **(Imp.)**

Let us prepare the following truth table.

p	**q**	$p \vee q$	$p \veebar q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
0	0	0	0	1	1	1	0
0	1	1	1	1	0	1	1
1	0	1	1	0	1	1	1
1	1	1	0	0	0	0	0

From columns 4 and 8 of the above truth table, we find that $p \veebar q$ and $(p \vee q) \wedge (\neg p \vee \neg q)$ have identical truth values for all possible truth values of p and q. Therefore,

$$(p \veebar q) \Leftrightarrow \{(p \vee q) \wedge (\neg p \vee \neg q)\}$$

Q19. Prove that, for any three propositions p, q, r.

$$[p \rightarrow (q \wedge r)] \Leftrightarrow [(p \rightarrow q) \wedge (p \rightarrow r)]$$

Sol :

Let us prepare the following truth table.

p	q	r	$(q \wedge r)$	$p \rightarrow (p \wedge r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \wedge (p \rightarrow r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	0	0	1	0
1	1	0	0	0	1	0	0
1	1	1	1	1	1	1	1

Columns 5 and 8 of the above table show that $[p \rightarrow (q \wedge r)]$ and $[(p \rightarrow q)] \wedge (q \rightarrow r)$ have identical truth values in all possible situations. Therefore,

$$[p \rightarrow (q \wedge r)] \Leftrightarrow [(p \rightarrow q)] \wedge (p \rightarrow r)]$$

3.3.1 The Laws of Logic

Q20. Explain about various Laws of Logic.

Ans :

The following results, known as the laws of logic, follow from the definition f logical equivalence. In these laws, T_0 denotes a tautology and F_0 denotes a contradiction.

1. Law of Double Negation

   For any proposition p, $(\neg \neg p) \Leftrightarrow p$

2. Idempotent Laws

   For any proposition p,

   (a) $(p \vee p) \Leftrightarrow p$ (b) $(p \wedge p) \Leftrightarrow p$

3. Identity Laws

   For any proposition p,

   (a) $(p \vee F_0) \Leftrightarrow p$ (b) $(p \wedge T_0) \Leftrightarrow p$

4. Inverse Laws

   For any proposition p,

   (a) $(p \vee \neg p) \Leftrightarrow T_0$ (b) $(p \wedge \neg p) \Leftrightarrow F_0$

5. Domination Laws

   For any proposition p,

   (a) $(p \vee T_0) \Leftrightarrow T_0$ (b) $(p \wedge F_0) \Leftrightarrow F_0$

6. Commutative Laws

   For any two propositions p and q,

   (a) $(p \vee q) \Leftrightarrow (p \vee q)$ (b) $(p \wedge q) \Leftrightarrow (q \wedge p)$

7. Absorption Laws

For any two propositions p and q,

(a) $[p \vee (p \wedge q)] \Leftrightarrow p$ (b) $[p \wedge (p \vee q)] \Leftrightarrow p$

8. DeMorgan Laws

For any two propositions p and q,

(a) $\neg (p \vee q)] \Leftrightarrow \neg p \wedge \neg q$ (b) $\neg (p \wedge q)] \Leftrightarrow \neg p \vee \neg q$

9. Associative Laws

For any three propositions p, q, r,

(a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ (b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

10. Distributive Laws

For any three propositions p, q, r,

(a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ (b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Q21. Prove the following logical equivalences without using truth tables:

(i) $\mathbf{p \vee [p \wedge (p \vee q)] \Leftrightarrow p}$

(ii) $\mathbf{[p \vee q \vee (\neg p \wedge \neg q \wedge r)] \Leftrightarrow (p \vee q \vee r)}$

(iii) $\mathbf{[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q}$

Sol :

(i) We have

$p \vee [p \wedge (p \vee q)] \Leftrightarrow p \vee p$, by an Absorption Law

$\Leftrightarrow p$, by an Idempotent Law

(ii) We have

$\neg p \wedge \neg p \wedge r \Leftrightarrow (\neg p \wedge \neg q) \wedge r$, by Associative Law

$\Leftrightarrow (\neg p \vee q) \wedge r$, by DeMorgan Law

Therefore,

$p \vee q \vee (\neg p \wedge \neg q \wedge r) \Leftrightarrow (p \vee q) \vee [\neg (p \vee q) \wedge r]$

$\Leftrightarrow [(p \vee q) \vee \neg (p \vee q)] \wedge [(p \vee q) \vee r]$, by Distributive Law

$\Leftrightarrow T \wedge (p \vee q \vee r)$, by inverse Law and Associative Law

$\Leftrightarrow p \vee q \vee r$, by commutative and Identity Laws.

(iii) We have

$(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r) \Leftrightarrow \neg (\neg p \vee \neg q) \vee (p \wedge q \wedge r)$, because $(u \rightarrow v) \Leftrightarrow (\neg u \vee v)$

$\Leftrightarrow (p \wedge q) \vee (p \wedge q)] \wedge r]$, by DeMorgan Law and Associative Law

$\Leftrightarrow p \wedge q$, by Absorption Law.

Q22. Prove the following logical equivalences:

(i) $[(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow p \vee q$

(ii) $(p \rightarrow q) \wedge [\neg p \wedge (r \vee \neg q)] \Leftrightarrow \neg (q \vee p)$

Sol : **(Imp.)**

(i) We have

$$(p \vee q) \wedge (p \vee \neg q) \Leftrightarrow p \vee (q \wedge \neg q), \text{ by Distributive Law}$$
$$\Leftrightarrow p \vee F_0. \text{ because } (q \wedge \neg q) \text{ is a contradiction}$$
$$\Leftrightarrow p, \text{ by an Identity Law}$$

Therefore,

$$[(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow p \vee q$$

(ii) We have

$$(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)] \Leftrightarrow (p \rightarrow q) \wedge [\neg q \wedge (\neg q \vee r)]. \text{ by commutative law}$$
$$\Leftrightarrow (p \rightarrow q) \wedge \neg q, \text{ by an absorption law}$$
$$\Leftrightarrow \neg [(p \rightarrow q) \rightarrow q], \text{ because } \neg (u \rightarrow v) \Leftrightarrow (u \wedge \neg v)$$
$$\Leftrightarrow \neg [\neg (p \rightarrow q) \vee q], \text{ because } (u \rightarrow v) \Leftrightarrow (\neg u \vee v)$$
$$\Leftrightarrow \neg [(p \wedge \neg q) \vee q]$$
$$\Leftrightarrow \neg [p \vee (p \wedge \neg q)], \text{ by commutative law}$$
$$\Leftrightarrow \neg [(q \vee p) \wedge (q \vee \neg q)], \text{ by distributive law}$$
$$\Leftrightarrow \neg [(q \vee p) \wedge T_0], \text{ because } q \vee \neg q \text{ is a tautology}$$
$$\Leftrightarrow \neg (q \vee p), \text{ by an identity law.}$$

Q23. Prove the following

(i) $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

(ii) $[\neg p \wedge (\neg q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r.$

Sol : **(Imp.)**

(i) We have

$$p \rightarrow (q \rightarrow r) \Leftrightarrow \neg p \vee (\neg q \vee r)$$
$$\Leftrightarrow (\neg p \vee \neg q) \vee r$$
$$\Leftrightarrow \neg (p \wedge q) \vee r$$
$$\Leftrightarrow (p \wedge q) \rightarrow r$$

(ii) We have

$$[\neg p \vee (\neg q \wedge r)] \Leftrightarrow (\neg p \wedge \neg q) \wedge r$$
$$\Leftrightarrow (\neg (p \vee q)] \wedge r \Leftrightarrow r \wedge [\neg (p \vee q)]$$

and $$(q \wedge r) \vee (p \wedge r) \Leftrightarrow (r \wedge q) \vee (r \wedge p)$$
$$\Leftrightarrow r \wedge (p \vee q)$$
$$\Leftrightarrow r \wedge (p \vee q)$$

Therefore,

$$[\neg p \wedge (\neg q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow \{r \wedge [\neg (p \vee q)]\} \vee \{r \wedge (p \vee q)\}$$
$$\Leftrightarrow r \wedge \{[\neg (p \vee q)]\} \vee (p \vee q)\}$$
$$\Leftrightarrow r \wedge T_0$$
$$\Leftrightarrow r.$$

3.4 Logical Implication

Q24. What is logical implication?

Ans :

Logical implication is a type of relationship between two statements or sentences. The relation translates verbally into "logically implies" or "if/then" and is symbolized by a double-lined arrow pointing toward the right (⇨). If A and B represent statements, then A ⇨ B means "A implies B" or "If A, then B." The word "implies" is used in the strongest possible sense.

As an example of logical implication, suppose the sentences A and B are assigned as follows:

A = The sky is overcast.

B = The sun is not visible.

In this instance, A ðB is a true statement (assuming we are at the surface of the earth, below the cloud layer.) However, the statement B ðA is not necessarily true; it might be a clear night. Logical implication does not work both ways. However, the sense of logical implication is reversed if both statements are negated. That is,

(A ⇨ B) (–B ⇨ –A)

Using the above sentences as examples, we can say that if the sun is visible, then the sky is not overcast. This is always true. In fact, the two statements A ⇨ B and –B ⇨ –A are logically equivalent.

Q25. Prove the following:

(i) $[p \wedge (P \rightarrow q)] \wedge r] \Rightarrow [(p \vee q) \rightarrow r]$

(ii) $([p \vee (q \vee r)] \wedge \neg q) \Rightarrow p \vee r$

Sol : **(Imp.)**

(i) Let us prepare the following truth table.

p	q	r	$p \rightarrow q$	$p \vee q$	$(p \vee q) \rightarrow r$
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1

From the Table, we observe that if p, $P \to q$ and r are all true , then $(p \vee q) \to$ r is true (see the last row). This proves that

$$[p \wedge (p \to q) \wedge r] \Rightarrow [(p \vee q) \to r]$$

(ii) Let us prepare the following truth table.

p	q	r	$p \vee (q \vee r)$	$\neg q$	$[p \vee (q \vee r)] \wedge \neg q$	$p \vee r$
0	0	0	0	1	0	0
0	0	1	1	1	1	1
0	1	0	1	0	0	0
0	1	1	1	0	0	1
1	0	0	1	1	1	1
1	0	1	1	1	1	1
1	1	0	1	0	0	1
1	1	1	1	0	0	1

From rows 2, 5 and 6 of the table we observe that if $[p \vee (q \vee r)] \wedge \neg q$ is true, then, is not false. This proves that

$$\{[p \vee (q \vee r)] \wedge \neg q\} \to p \vee r$$

is a tautology, or equivalently that

$$\{[p \vee (q \vee r) \wedge \neg q\} \Rightarrow p \vee r$$

Q26. Prove that the following are logical implications:

(i) $[(P \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$

(ii) $[(P \to q) \wedge (r \to s) \wedge (\neg p \vee \neg s)] \to (\neg p \vee \neg s)$

Sol :

(i) Suppose $q \vee s$ is false. Then q is false and s is false. Thus, both of q and s have the truth value 0. With these truth values of q and s, let us prepare the following truth table (for all possible truth values of p and r).

q	s	p	r	$p \to q$	$r \to s$	$p \vee r$
0	0	0	0	1	1	0
0	0	0	1	1	0	1
0	0	1	0	0	1	1
0	0	1	1	0	0	1

From the Table, we find that there is no situation where all of $p \to q$, $r \to s$ and $p \vee r$ are true. Thus, when $q \vee s$ is false, $(p \to q) \wedge (r \to s) \wedge (p \vee r)$ cannot be true. This means that the conditional

$$[\{p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$$

is always true; in other words, the conditional is a logical implication.

(ii) Suppose $(\neg p \vee \neg r)$ is false. Then $\neg p$ is false and $\neg r$ is false; that is p is true and r is true. Thus both p and r have the truth value 1. With these truth values of p and r, let us prepare the following truth table for all possible truth values of q and s.

p	r	q	s	$p \to q$	$r \to s$	$\neg q \vee \neg s$
1	1	0	0	0	0	1
1	1	0	1	0	1	1
1	1	1	0	1	0	1
1	1	1	1	1	1	0

From the table, we find that there is no situation where all of $p \to q$, $r \to s$ and $\neg q \vee \neg s$ are true. Thus, when $(\neg p \vee \neg r)$ is false, $(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)$ cannot be true. This means that the conditional

$$[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$$

is always true ; in other words, this conditionalisalogical implication.

3.5 Rules of Inference

Q27. State various Rules of Inference.

Ans : **(Imp.)**

Consider a set of propositions $p_1, p_2, ...p_n$ and a proposition c. Then a compound proposition of the form

$$(p_1 \wedge p_2 \wedge p_3 \wedge ....p_n) \to c$$

is called a conclusion of the argument. Here $p_1, p_2, ...p_n$ are called the premises of the argument and c is called a conclusion of the argument.

Some of these rules are listed below :

1. Rule of Conjunctive Simplification

This rule states, for any two propositions p and q, if $p \wedge q$ is true, then p is true.

i.e., $(p \wedge q) \Rightarrow p$

This rule follows from the definition of conjunction.

2. Rule of Disjunctive Amplication

This rule that, for any two proposiotions p and q, if p is true, then $p \vee q$ is true;

i,e., $p \Rightarrow p \vee q$

This rule follows from the definition of disjunction.

3. Rule of Syllogism

this rule states that, for any three propostions p,q,r, if $p \to q$ is true and $q \to r$ is true, then $p \to r$ is true. i.e.,

$$\{(p \to q) \wedge (q \to r)\} \Rightarrow (p \to r)$$

This rule follows from the tautology $\{(p \to q) \wedge \{q \to r)\} \to (p \to r)$ and is expressed in the following tabular form:

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

4. Modus Pones* (Rule of Detachment)**

This rule states that if p is true and $p \to q$ is true, then q is true; i.e.,

$$\{p \wedge (p \to q)) \Rightarrow q$$

For a proof of this rule, see Example 2 of Section 1.2.4. In tabular form, the rule reads thus:

$$\begin{array}{ll} & p \\ & \underline{p \to q} \\ \therefore & q \end{array}$$

5. Modus Tollens

This rule states that if $p \to q$ is true and q is false, then p is false; i.e.,

$$\{(P \to q) \wedge \neg q\} \Rightarrow \neg p.$$

The rule is expressed in the following tabular form:

$$\begin{array}{ll} & p \to q \\ & \underline{\neg q} \\ \therefore & \neg p \end{array}$$

6. Rule of Disjunctive Syllogism

This rule states that if $p \vee q$ is true and p is false, then q is true; i.e.,

$$\{(p \vee q) \wedge \neg \} \Rightarrow q$$

The rule is expressed in the following tabular form :

$$\begin{array}{ll} & p \vee q \\ & \underline{\neg p} \\ \therefore & q \end{array}$$

3.6 The use of Quantifiers

Q28. What are the various types of qualifiers used in predicate logic? Define them with an example.

Ans : **(Imp.)**

Quantifiers

The variable of predicates is quantified by quantifiers. There are three types of quantifier in predicate logic – Universal Quantifier, Existential Quantifier and Nested Quantifier.

➢ **Universal Quantifier**

Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol $\forall$.

$\forall$ x P(x) is read as for every value of x, P(x) is true.

Example: "Man is mortal" can be transformed into the propositional form $\forall$ x P(x) where P(x) is the predicate which denotes x is mortal and the universe of discourse is all men.

➢ **Existential Quantifier**

Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol $\exists$.

$\exists$ x P(x) is read as for some values of x, P(x) is true.

Example: "Some people are dishonest" can be transformed into the propositional form $\exists$x P(x) where P(x) is the predicate which denotes x is dishonest and the universe of discourse is some people.

➢ **Nested Quantifiers**

If we use a quantifier that appears within the scope of another quantifier, it is called nested quantifier.

Example

➢ $\forall a \; \exists b$ P (x, y) where P (a, b) denotes $a + b = 0$

➢ $\forall a \; \forall b \; \forall c$ P(a, b, c) where P (a, b) denotes $a + (b + c) = (a + b) + c$

Note: $-\forall a \; \exists b$ P (x, y) $\neq$ $\exists a \; \forall b$ P(x, y)

Q29. Use Logical Equivalences to prove that $[(p \wedge (\neg(\neg p) \wedge q)) \vee (p \wedge q)] \rightarrow p$ is a tautology.

Ans :

Proof:

$[(p \wedge \neg(\neg p \vee q)) \vee (p \wedge q)] \rightarrow p \equiv [(p \wedge (\neg(\neg p) \wedge q)) \vee (p \wedge q)] \rightarrow p$ DeMorgan's law

$\equiv [(p \wedge (p \wedge \neg q)) \vee (p \wedge q)] \rightarrow p$ Double Negation law

$\equiv [((p \wedge p) \wedge \neg q) \vee (p \wedge q)] \rightarrow p$ Associative law

$\equiv [(p \wedge \neg q) \vee (p \wedge q)] \rightarrow p$ Idempotent law

$\equiv [p \wedge (\neg q \vee q)] \rightarrow p$ Distributive law

$\equiv [p \wedge (q \vee \neg q)] \rightarrow p$ Commutative law

$\equiv [p \wedge T] \rightarrow p$ Negation law

$\equiv p \rightarrow p$ Identity law

$\equiv \neg p \vee p$ Equivalence of Implication

$\equiv p \vee \neg p$ Commutative law

$\equiv T$ Negation law

Q30. Use Logical Equivalences to prove that $[(p \wedge \neg(\neg p \vee q)) \vee (p \wedge q)] \rightarrow p[(p \wedge \neg(\neg p \vee q)) \wedge (p \wedge q)] \rightarrow p$ is a tautology.

Ans :

implication law... $\neg[(p \wedge \neg(\neg p \vee q)) \vee (p \wedge q)] \vee p \neg[(p \wedge \neg(\neg p \vee q)) \vee (p \wedge q)] \vee p$

demorgans $[\neg(p \wedge \neg(\neg p \vee q)) \wedge \neg(p \wedge q)] \vee p[\neg(p \wedge \neg(\neg p \vee q)) \wedge \neg(p \wedge q)] \vee p$

demorgans $[\neg(p \wedge \neg(\neg p \vee q)) \wedge (\neg p \vee \neg q)] \vee p[\neg(p \wedge \neg(\neg p \vee q)) \wedge (\neg p \vee \neg q)] \vee p$

demorgans $[(\neg p \vee (\neg p \vee q)) \wedge (\neg p \vee \neg q)] \vee p[(\neg p \vee (\neg p \vee q)) \wedge (\neg p \vee \neg q)] \vee p$

distribution... $[(\neg p \vee q) \wedge (\neg p \vee \neg q)] \vee p[(\neg p \vee q) \wedge (\neg p \vee \neg q)] \vee p$

distribution... $[(\neg p \vee (q \wedge \neg q)] \vee p[(\neg p \vee (q \wedge \neg q)] \vee p$

negation... $[\neg p \vee F] \vee p[\neg p \vee F] \vee p$

associativity $(p \vee \neg p) \vee f(p \vee \neg p) \vee f$

domination $T \vee FT \vee F$

T

3.7 Proof Techniques

3.7.1 Proof Methods and Strategies

Q31. Explain about Proof Methods and Strategies

Ans :

The propositions that commonly appear in mathematical discussions are conditionals of the form k $p \rightarrow q$, where p and q are simple or compound propositions which may involve quantifiers as well. Given such a conditional, the process of establishing that the conditional is true by using the rules / laws of logic and other known facts constitutes a proof of the conditional. The process of establishing that a proposition is false constitutes a disproof.

3.7.1.1 Forward Proof

Q32. Discuss about Forward Proof (or) Direct Proof.

Ans :

A direct proof shows that a conditional statement $p \rightarrow q$ is true by showing that p is true, then q must also be true, so that the combination p true and q false never occurs. In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true.

Definition

The integer n is even if there exists an integer k such that n = 2k, and n is odd if there exists an integer k such that n – 2k + 1. (Note that every integer is either even or odd, and no integer is both even and odd.) Two integers have the same parity when both are even or both are odd, they have opposite parity when one is even and the other is odd.

The direct method of proving a conditional $p \to q$ has the following lines of argument:

1. **Hypothesis:** First assume that p is true.
2. **Analysis:** Starting with the hypothesis and employing the rules / laws of logic and other known facts, inter that q is true.
3. **Conclusion:** $p \to q$ is true.

Q33. Give a direct proof of the statement :

"The square of an odd integer is an odd integer"

Sol :

Here, the conditional to be proved is :

"If n is an odd integer, then n^2 is an odd integer".

Assume that n is an odd integer (hypothesis).

Then, $n = 2k + 1$ for some integer k. Consequently.

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1.$

We observe that the right hand side is not divisible by 2. Therefore, n^2 is not divisible by 2. This means that n^2 is an odd integer (conclusion).

The given statement is thus proved by a direct proof.

Q34. Let m and n be integers. Prove that $a^2 = m^2$ if and only if $m = n$ or $m = -n$.

Sol :

Consider the propositions

$p : n^2 = m^2$, $q : m = n$, $r = m = -n$.

We have to prove that $p \leftrightarrow (p \vee r)$ is true. First, assume that $q \vee r$ is true. Then $m = n$ or $m = -n$, so that $m^2 = n^2$, that is, p is true. Thus $(q \vee r) \to p$ is true.

Next assume that $\neg(q \vee r)$ is true: that is, $(\neg q) \wedge (\neg r)$ is true. Then q is false that r is false: that is, $m \neq n$ and $m \neq -n$. Then, $m^2 \neq n^2$; that is, $\neg p$ is true. This proves that $\neg(q \vee r) \to \neg p$ is true. Accordingly, the statement $p \to (q \vee r)$ is true.

Thus, we have proved that both of $(q \vee r) \to p$ and $p \to (q \vee r)$ are true. Hence $p \leftrightarrow (q \vee r)$ is true.

3.7.2 Proof by Contradiction

Q35. Discuss about Proof by Contradiction.

Ans : **(Imp.)**

The indirect method of proof is equivalent to what is known as the Proof by Contradiction. The lines of argument in this method of proof of the statement $p \to q$ are as follows:

1. **Hypothesis:** Assume that $p \to q$ is false. That is, assume that p is true and q is false.
2. **Analysis:** Starting with the hypothesis that q is false and employing the rules of logic and other known facts, infer that p is false. This contradicts the assumption that p is true.
3. **Conclusion:** Because of the contradiction arrived in the analysis, we infer that $p \to q$ is true.

Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \to q$ is true. Because q is false, but $\neg p \to q$ is true, we can conclude that $\neg p$ is false, which means that p is true. How can we find a contradiction q that might help us prove that p is true in this way? Because the statement $r \wedge \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \to (r \wedge \neg r)$ is true for some proposition r. Proofs of this type are called proofs by contradiction.

Q36. Provide a proof by contradiction of the following statement:

For every integer n, if n^2 is odd, then n is odd.

Sol :

Let n be any integer. Then the given statement reads $p \to q$, where

p : n^2 is odd and q : n is odd.

Assume that $p \to q$ is false; that is, assume that p is true and q is false. Now, q is false means: n is even, so that n = 2k for some integer k. This yields $n^2 = (2k)^2 = 4k^2$ from which it is evident that n^2 is even; that is, p is false. This contradicts the assumption that p is true. In view of this contradiction, we infer that the given conditional $p \to q$ is true (for any integer n).

Q37. Prove the statement:

"The square of an even integer is an even integer" by the method of contradiction.

Sol :

Here, the statement to be proved can be put in the form $p \to q$, where

p : n is an even integer, and q : n^2 is an even integer.

Assume that $p \to q$ is false; that is assume that p is true and q is false. Since q is false, $\neg q$ is true; that is, n^2 is not an even integer. Therefore, $n^2 = n \times n$ is not divisible by 2. This implies that n is not divisible by 2. That is, n is not an even integer. This means that p is false, which contradicts the assumption that p is true.

In view of this contradiction, we infer that the given proposition $p \to q$ is true.

Q38. Prove that if m is an even integer, then m + 1 is an odd integer.

Sol :

Here, the given statement is $p \to q$, where

p : in is even, q : m + 7 is odd.

Assume that $p \to q$ is false; that is, assume that p is true and q is false. Since q is false, m + 1 is even. Hence, m + 7 = 2k for some integer k. This yields

$$m = 2k - 7 = (2k - 8) + 1 = 2(k - 4) + 1$$

which shows that m is odd. This means that p is false, which contradicts the assumption that p is true. In view of this contradiction, we infer that the given statement $p \to q$ is true.

Q39. Prove that, for all real numbers x and y, if $x + y \geq 100$, then $x \geq 50$ or $y \geq 50$.

Sol :

Take any two real numbers x and y. Then the statement to be proved reads $p \to (q \vee r)$ where

$p \equiv p(x,y) : x + y \geq 100$, $q \equiv q(x) : x \geq 50$, $r \equiv r(y) : y \geq 50$.

Assume that p is true and $q \vee r$ is false. Since $q \vee r$ is false, q is false and r is false. This means that $x < 50$ and $y < 50$. This yields $x + y < 100$. Thus, p is false. This contradicts the assumption that p is true.

Hence, we infer that the given statement $p \to q$ is true.

3.7.3 Proof by Contraposition

Q40. Discuss about Proof by Contraposition.

Ans :

Proofs by contraposition make use of the fact that the conditional statement $p \to q$ is equivalent to its contrapositive, $\neg q \to \neg p$. This means that the conditional statement $p \to q$ can be proved by showing that its contrapositive, $\neg q \to \neg p$, is true. In a proof by contraposition of $p \to p$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow the proof by contraposition can succeed when we cannot easily find a direct proof.

3.7.4 Proof of Necessity, Sufficiency

Q41. Discuss about Proof of Necessity. and Sufficiency.

Ans : **(Imp.)**

(i) Proof of Necessity and sufficiecy

In logic and mathematics, necessity and sufficiency are terms used to describe a conditional or implicational relationship between two statements. For example, in the conditional statement: "If P then Q", Q is necessary for P, because the truth of Q is guaranteed by the truth of P (equivalently, it is impossible to have P without Q). Similarly, P is sufficient for Q, because P being true always implies that Q is true, but P not being true does not always imply that Q is not true.

(ii) Necessity

The assertion that Q is necessary for P is colloquially equivalent to "P cannot be true unless Q is true" or "if Q is false, then P is false". By contraposition, this is the same thing as "whenever P is true, so is Q".

The logical relation between P and Q is expressed as "if P, then Q" and denoted "$P \Rightarrow Q$" (P implies Q). It may also be expressed as any of "P only if Q", "Q, if P", "Q whenever P", and "Q when P". One often finds, in mathematical prose for instance, several necessary conditions that, taken together, constitute a sufficient condition, i.e., individually necessary and jointly sufficient.

(iii) Sufficiency

If P is sufficient for Q, then knowing P to be true is adequate grounds to conclude that Q is true; however, knowing P to be false does not meet a minimal need to conclude that Q is false.

The logical relation is, as before, expressed as "if P, then Q" or "$P \Rightarrow Q$". This can also be expressed as "P only if Q", "P implies Q" or several other variants. It may be the case that several sufficient conditions, when taken together, constitute a single necessary condition (i.e., individually sufficient and jointly necessary).

UNIT IV

Algebraic Structures and Morphism: Algebraic Structures with one Binary Operation, Semi Groups, Monoids, Groups, Congruence Relation and Quotient Structures, Free and Cyclic Monoids and Groups, Permutation Groups, Substructures, Normal Subgroups, Algebraic Structures with two Binary Operation, Rings, Integral Domain and Fields. Boolean Algebra and Boolean Ring, Identities of Boolean Algebra, Duality, Representation of Boolean Function, Disjunctive and Conjunctive Normal Form

4.1 Algebraic Structures

Q1. Define Algebraic Structures.

Ans :

Let S be a nonempty set on which one or more n-ary operations are defined. Then a system consisting of S and some n-ary operations on S is called an algebraic system or simply an algebra. That is: if $*_1, *_2. *_3$ ....., $*_k$ are some n-ary operations on S, then the system < S, $*_1, *_2$, ..., $*_k$ > is called an algebraic system (or an algebra).

Since n-ary operations define a structure on the elements of a set S upon which the opera-tions are defined, an algebraic system is called an algebraic structure.

For example, since +, × are binary operations on Z, the system < Z, +, × > is an algebraic structure. Similarly, < P(S), $\cup$, $\cap$ > is an algebraic structure. Here, the symbols have their usual meanings.

Identity

In S, suppose there is an element e_1 such that $e_1 * x = x$ for every x in S. Then e_1 is called a left identity in S with respect to $*$. Similarly, if there is an element e_r in S such that $x * e_r = x$ for every x in S, then e_r is called a right identity in S with respect to $*$.

Remark: If $*$ is commutative, then a left identity (if any) w.r.t. $*$ is a right identity as well. Even if $*$ is not commutative, left and right identities (if such exist) are identical and unique. (For a proof, see Theorem 1 below). This unique element is called the identity element in S w.r.t. $*$ and is denoted by e.

Inverse

In S, suppose there is the identity element e w.r.t. $*$. An element a in S is said to be left invertible if there exists an element xi in S such that $x_1 * a = e$. In such a situation, x_1 is called a left inverse of the element a. Similarly, a is said to be right invertible if there exists an element x_r in S such that $a * x_r = e$ and xy is called a right inverse of a. If an element a is both left invertible and right invertible, then we say that a is invertible.

4.1.1 Algebraic Structures with one Binary Operation

Q2. Explain Algebraic Structures with one Binary Operation.

Ans :

Algebric Structures with one Binary Operation

A non-empty set G equipped with one or more binary operations is said to be an algebraic structure. Suppose * is a binary operation on G. Then (G, *) is an algebraic structure. (N,*), (1, +), (1, -) are all the algebraic structure. Here, (R, +,.) is an algebraic structure equipped with two operations.

Binary Operation on A Set

Suppose G is a non-empty set. The G X G = {(a,b): a E G, b E G)}. If f: G X G $\rightarrow$ G then f is called a binary operation on a set G. The image of the ordered pair (a,b) under the function f is denoted by afb.

A binary operation on asset G is sometimes also said to be the binary composition in the set G. If * is a binary composition in G then, a * b E G, a, b E G. Therefore g is closed with respect to the composition denoted by *.

4.2 Semi Groups and Monoids

Q3. Define the following terms :

(i) Semi Groups

(ii) Monoids

Ans :

(i) Semi Groups

An algebraic system < S, ∗> consisting of a non-empty set S and an associative binary operation * defined on S is called a semigroup under that operation.

If the binary operation is commutative as well (in addition to being associative), the semi-group is called a commutative semigroup or an abelian semigroup.

When there is no ambiguity or when the operation ∗ is understood, the semigroup < 5, ∗ > is denoted by just S.

(ii) Monoids

Let < 5, ∗ > be a semigroup. This semigroup is called a monoid if S contains the identity element e w.r.t. ∗.

Evidently, every monoid is a semigroup, but a semigroup need not be a monoid.

4.3 Groups

Q4. Define Groups. Explain properties of Groups.

(OR)

Give examples of groups and non-groups.

Ans : **(Imp.)**

(i) The set of integers Z, with '+' as binary operation forms a group

(ii) The set of rational numbers Q, and real number R are groups under ordinary addition, with identity '0' and the inverse of a is '– a' for a $\in$ Q.

(iii) The set of integers Z under subtraction is not a group since '–'is not associative, i.e., a – (b – c) $\neq$ (a – b) – c, for a, b,$\in$ Z.

(iv) The set of integers Z under multiplication is not a group since inverse does not exist for the integers other than 1 and - 1.

(v) The set Q* of non zero rational numbers is a group under ordinary multiplication, with 1 being multiplicative identity and inverse of a is $\frac{1}{a}$ for a $\in$ Q*.

(vi) The set S of positive irrational numbers, under multiplication is not a group since, $\sqrt{2}.\sqrt{2}$ = 2 which is not an irrational number.

(vii) The set of all 2 × 2 matrices of the form R $= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big/ a, b, c, d \in R\right\}$ with usual addition of matrices forms a group, since addition of matrices is associative. The identity is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and inverse of = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ = $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$.

(viii) The set of all 2 × 2 matrices with real entries, is not a group under the matrix multiplication, since inverse of A does not exists if |A| = 0.

Q5. State and prove uniqueness or the identity tneorem in a group.

Ans :

Stement

In a group G, there is only one identity element.

Proof :

Let, e and e' are identities in a group G, then,

$ae = ea = a \ \forall \ a \in G$

$e'a = ae' = a \ \forall \ a \in G$

Substituting a = e' in equation (1),

$\Rightarrow \quad e'.e = e.e' = e'$

Substituting a = e in equation (2),

$\Rightarrow \quad e'.e = e.e' = e$

Comparing equations (3) and (4),

$e = e'$

Therefore, there exists a unique identity element in a group.

Q6. State and prove cancellation laws.

Ans :

Statement

Let a, b, c be the elements of a group G and * be a binary operation on G then,

(i) $ba = ca \Rightarrow b = c$ (Right cancellation law)

(ii) $ab = ac \Rightarrow b = c$ (Left cancellation law)

Proof:

(i) Right cancellation law ($ba = ca \Rightarrow b=c$)

Let a' be an inverse of a,

$\Rightarrow \quad (ba)a' = (ca)a'$

$\Rightarrow \quad b(aa') = c(aa')$ [$\because$ Associativity in G]

$\Rightarrow \quad b(e) = c(e)$ [$\because$ ad` = a'a = e]

$\therefore \quad b = c$ [$\because$ e is the identity in G]

(ii) Left cancellation law ($ab = ac \Rightarrow b = c$)

Let a' be the inverse of a,

$\Rightarrow \quad a'(ab) = a'(ac)$

$\Rightarrow \quad (a'a)b = (a'a)c$ [$\because$ Associativity in G]

$\Rightarrow \quad eb = ec$ [$\because$ aa' = a'a = e]

$b = c$ [$\because$ e is the identity in G]

Q7. Prove that the set of all positive rational numbers of the form $3^m.6^n$, where m, n $\in$ Z is a group under multiplication.

Ans :

Let, a, b, c $\in$ G

$a = 3^p 6^q$, $b = 3^r 6^s$, $c = 3^t 6^u$

Where p, q, r, s, t, u $\in$ Z

(i) Closure

$ab = (3^p 6^q)(3^r 6^s)$

$= 3^P(6^q 3^r)(6^s)$ [$\because$ Associativity in Q^+]

$= 3^P(3^r 6^q)(6^s)$ [$\because$ Commutativity in Q^+]

$= (3^P 3^r)(6^q 6^s)$

$= (3^{P+r} 6^{q+s})$ [$\because$ p + q, q + s $\in$ Z]

$\Rightarrow$ ab $\in$ G

(ii) Associativity

$a(bc) = (3^p 6^q)[(3^r 6^s)(3^t 6^u)]$

$= (3^{P+r}.6^{q+s})(3^r.6^s)$

$= (a.b)(c)$ [$\because$ p + r, q + s $\in$ Z]

$\therefore$ a(bc) = (ab) (c) $\forall$ a, b, c $\in$ G

(iii) Identity

$\Rightarrow$ $e = 3^\circ 6^\circ \in G$ [$\because$ 0 $\in$ Z]

$ae = 3^p 6^q 3^\circ 6^\circ = 3^p 6^q = a$

$ea = 3^\circ 6^\circ 3^p 6^q = 3^p 6^q = a$

$\Rightarrow$ ae = ea = a

$\therefore$ $e = 3^\circ 6^\circ$ is the identity element in G.

(iv) Inverse

P, q $\in$ Z,

$\Rightarrow$ –p,–q $\in$ Z

$a = 3^p 6^q \in G$

$\Rightarrow$ $b = 3^{-p} 6^{-q} \in G$

$\Rightarrow$ $ab = (3^p 6^q)(3^{-p} 6^{-q})$

$= (3^{p-p}.6^{q-q})$

$= (3^\circ 6^\circ)$

$= e$

$ba = (3^{-p} 6^{-q})(3^p 6^q)$

$= (3^{p-p} 6^{q-q})$

$= (3^\circ 6^\circ)$

$= e$

$\Rightarrow$ ab = ba = e

$\therefore$ G is a group under multiplication.

Q8. Prove that if G is a group with the property that, square of every element is the identity, then G is abelian.

Ans :

Let, a $\in$ G

The square of every element is the identity.

i.e., $a^2 = e$, where e is the identity in G.

$\Rightarrow$ a.a = e

$\Rightarrow$ $a^{-1} = a \; \forall \; a \in G$

Similarly,

Let, b $\in$ G

$\Rightarrow$ $b^2 = e$

$\Rightarrow$ b. b = e

$\Rightarrow$ $b^{-1} = b \; \forall \; b \in G$

Consider,

$ab = (ab)^{-1} = b^{-1} a^{-1} = ba \; \forall \; a,b \in G$

$\Rightarrow$ ab = ba

Commutative property is satisfied.

$\therefore$ G is abelian

Q9. Find the inverse of $A = \begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ in the general linear group : GL(2, Z_{11}).

Ans :

Given,

$A = \begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$

The general linear group can be represented as,

$GL(2, Z_{11}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle/ a, b, c, d \in Z_{11} \text{ and } ad - bc \neq 0 \pmod{11} \right\}$

Inverse of a matrix 'A' is given by,

$A^{-1} = \frac{1}{|A|} \text{ adj } A$

$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$|A| = ad - bc$

$|A| = ad\text{-}bc$

$= 2 \times 5 - 6 \times 3 \bmod 11$

$= -8 \bmod 11$

$= 3 \bmod 11$ **[– 8 = – 8 + 11 = 3 in Z_{11}]**

$\frac{1}{|A|}$ is the multiplicative inverse of $|A|$

$\Rightarrow \frac{1}{|A|}$ = Multiplicative inverse of 3 mod 11

$\Rightarrow \frac{1}{|A|} = 4 \bmod 11$ [$\because$ 3 x 4 = 1 mod 11]

Consider,

$\text{Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \begin{bmatrix} 5 & -6 \\ -3 & 2 \end{bmatrix} \bmod 11$

$= \begin{bmatrix} 5 & 5 \\ 8 & 2 \end{bmatrix} \bmod 11$ [$\because$ –6 = –6 + 11 = 5 mod 11, –3 = – 3 + 11 = 8 mod 11]

$\Rightarrow A^{-1} = 4 \begin{bmatrix} 5 & 5 \\ 8 & 2 \end{bmatrix} \bmod 11$

$$= \begin{bmatrix} 20 & 20 \\ 32 & 8 \end{bmatrix} \text{ mod } 11$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 9 & 9 \\ 10 & 8 \end{bmatrix}$$

Q10. If a group contains elements a and b such that |a| = 4, |b| = 2 and a^3b = ba. Find |ab|.

Ans :

Let, G be the group

a, b ∈ G

$|a| = 4 \Rightarrow a^4 = e$; $|b| = 2 => b^2 = e$

$a^3b = ba$ ... (1)

Consider,

$(ab)^2 = (ab)(ab)$

= a(ba)b

= $a(a^3b)b$ [∵ From equation (1)]

= a^4b^2

= ee

= e

$\Rightarrow (ab)^2 = e$

∴ |ab| = 2.

Q11. If R be the set of all real numbers except 0.Define * on R by a * b = |a|b. Is (R,*) a group. Justify your answer.

Ans :

Given that,

$*$ is the operation defined on R such that,

$a * b = |a|\, b \ \forall\ a,b \in R$

Where,

R is the set of all real numbers except '0'.

(i) Closure:

For a, b∈R

$\Rightarrow a * b = |a|\, b \in R$

∴ (R, $*$) is closed under $*$.

(ii) Associativity:

For a, b, c ∈ R

$(a * b) * c = (|a| b) * c$

= ||a|b|c

= |ab|c

$a * (b * c) = a * (|b|c)$

= |a| |b| c = \ab\c

$(a * b) * c = a * (b * c)$

Therefore, $*$ is associative.

(iii) Identity:

For a ∈ R

$a * e = a$ [∵ e is an identity element]

$\Rightarrow |a|e = a$

If a > 0, then = |a| a

$\Rightarrow |a| e = a$

$\Rightarrow$ a e = a

$\Rightarrow$ e = 1

If a < 0 then |a| = –a

$\Rightarrow$ |a| e = a

$\Rightarrow$ –ae = a

$\Rightarrow$ e = –1

But, identity element (e) always has unique value in a group.

∴ (R, $*$) is not a group

4.4 Congruence Relation and Quotient Structures

Q12. Discuss about Congruence Relation and Quotient Structures.

Ans : **(Imp.)**

Congruence Relation and Quotient Structures

Let S be a semi group and let ~ be an equivalence relation on S. Recall that the equivalence relations ~ induces a partition of S into equivalence classes. Also, [a] denotes the equivalence class containing the element a " S, and that the collection of equivalence classes is denoted by

S/ ~. Suppose that the equivalence relation ~ on S has the following property :

If a ~ a' and b ~ b', then ab ~ a'b'.

Then ~ is called a congruence relation on S. Furthermore, we can now define an operation on the equivalence classes by

[a] * [b] = [a * b] or, simply, [a] [b] = [ab]

Furthermore, this operation on S/ ~ is associative; hence S/ ~ is a semigroup. We state this result formally.

4.5 Free and Cyclic Monoids and Groups

Q13. Discuss about free and cyclic groups.

Ans :

Free and Cyclic

A group is called a free group if no relation exists between its group generators other than the relationship between an element and its inverse required as one of the defining properties of a group.

For example, the additive group of integers is free with a single generator, namely 1 and its inverse, –1. An example of an element of the free group on two generators is $ab^2 a^{-1}$, which is not equal to b^2. The fundamental group of the figure eight serves as another good example of a free group with two generators, since either loop can be traversed, but the two paths do not commute. Moreover, no (nontrivial) path involving more than one loop will ever be homotopic to the identity.

Cyclic Group

A cyclic group is a group that can be generated by a single element. Every element of a cyclic group is a power of some specific element which is called a generator. A cyclic group can be generated by a generator g', such that every other element of the group can be written as a power of the generator 'g'.

4.6 Permutation Groups

Q14. Define Permutation.

An :

A permutation of a set A is defined as a function f : A $\rightarrow$ A which is both one-to-one and onto.

Q15. Prove that every permutation of a finite set can be expressed as a cycle or as a product of disjoint cycles.

Ans : **(Imp.)**

Let A be a set given as

A = { 1,2,3,4,...... n}

Let, α be a permutation on set A.

Let, a_1 be an element of A

i.e., $a_1 \in A$

The element a_2 is obtained as,

$a_2 = \alpha(a_1)$

$$\begin{bmatrix} \because \alpha(a_{ik}) = a_{ik+1} \\ \text{Here}, ik = 1 \Rightarrow ik + 1 = 2 \end{bmatrix} \quad ....(1)$$

Similarly,

$\alpha_3 = \alpha[a_2]$

$= \alpha[\alpha\ (a_1)]$

$= \alpha^2 a_1$ [$\because$ From equation (1)]

$\alpha_4 = \alpha^3 a_1$

The obtained sequence will be of the form,

$a_1, \alpha(a_1), \alpha^2(a_1), \alpha^3(a_1), ...$

The above sequence is a finite sequence as the set A is a finite set.

$\Rightarrow$ a = $\alpha^m(a_1)$ for some $m \leq n$. Consider the following cases.

Case 1

If m = n, then there is no repetition

$a_1 = \alpha^\circ(a_1)$

$= a_1$

$a_2 = (\alpha)^1(a_1) = \alpha a_1$

$a_3 = (\alpha)^2(a_1) = \alpha^2 a_1$

$\Rightarrow$ $a = (a_1, a_2, a_3, ...a_n)$ ....(2)

Equation (2) represents a single cycle.

Hence, a permutation of a finite set can be expressed as a cycle.

Case 2

If m < n, then there must be a repetition

i.e., If $\alpha^i(a_1) = \alpha^j(a_1)$ for some i < j

Then $a_1 = \alpha^m(a_1)$

Where,

$m = j - i$

$a_1 = \alpha°, a_1\ a_2 = \alpha^1\ a_1, a_3 = \alpha^2 a_1, ... a_m = a^n a_1$

The sequence obtained is,

$\alpha_1 = (a_,, a_2, a_3, ... a_m)$ ....(3)

Equation (3) represents a cycle.

Let b, be an element of a which is not present in first cycle i.e., α_1

$b_2 = \alpha(b_1)$ [$\because$ From equation (1)]

$b_3 = \alpha^2(b_2)$

The sequence obtained i.e.,

$b_1, b_2, b_3, ...$ is a finite sequence

$\Rightarrow$ $b_1 = a^k(b_1)$ for some k.

The second cycle and first cycle does not contain common elements as they are disjoint cycles.

If $\alpha^i(a_1) = \alpha^j(b_1)$ for some i and j

$$\frac{\alpha^i}{\alpha^j} a_1 = b_1.$$

$\Rightarrow$ $\alpha^{i-J}\ a_1 = b_1$

$\Rightarrow$ $a_1 = b_1$ is a contradiction

The second cycle is,

$\alpha_2 = (b_1, b_2, b_3, ...b_k)$ ....(4)

Similarly, the third cycle will be of the form,

$\alpha_3 = (c_1, c_2, c_3, ... c_s)$ ....(5)

The process is continued till the elements of A get exhausted.

Multiplying equations (3), (4) and (5),

$\alpha_1, \alpha_2, \alpha_3 = (a_1, a_2, a_3, ... a_m)\ (b_1, b_2, b_3, ...\ b_k)\ (c_1, c_2, c_3, ... c_s)$

If $\alpha_1, \alpha_2, \alpha_3 = \alpha$ then,

$$\alpha = (a_1, a_2, a_3, ... a_m)\ (b_1, b_2, b_3, ...b_k)\ (c_1, c_2, c_3, ... c_s)$$

(Cycle-1) (Cycle-2) (Cycle-3)

$\therefore \quad \alpha = (a_1, a_2, a_3, ... a_m)\ (b_1, b_2, b_3, ... b_k)\ (c_1, c_2, c_3, ... c_s)$

It can be seen from equation (6) that the permutation of A is a product of disjoint cycles.

If there are 'n' number of disjoint cycles then,

$\alpha = (a_1, a_2, a_3, ... a_m)\ (b_1, b_2, b_3, ... b_k)\ (c_1, c_2, c_3, ...c_s)\ ......\ (d_1, d_2, d_3, ... d_n)$

Hence, every permutation of a finite set can be expressed as a product of disjoint cycles.

Q16. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix}$ then compute α^{-1} and $\beta\alpha$.

Ans :

Given permutations are,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix}$$

The value of α^{-1} can be obtained as,

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}$$

The value of $\beta\ \alpha$ a can be obtained as,

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}$$

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 4 & 5 \end{pmatrix}$$

Q17. If $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$ are two permutations in S_6. Then compute $\sigma\tau^2$ and $\sigma\tau\sigma^{-1}$.

Ans :

Given permutations are,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}, \ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

The value of $\sigma\tau^2$ can be obtained as,

$$\tau^2 = \tau.\tau$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 5 & 6 \end{pmatrix} \qquad [\because \ \tau.\tau(l) = \tau(\tau(l)) = \tau(2) = 4]$$

$$\therefore \tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 5 & 6 \end{pmatrix}$$

$$\sigma\tau^2 = \sigma.\tau^2$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 3 & 6 & 2 \end{pmatrix}$$

$$\therefore \quad \sigma\tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 3 & 6 & 2 \end{pmatrix}$$

The value of $\sigma\tau\sigma^{-1}$ can be obtained as,

$$\tau\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 2 & 1 & 3 & 6 \end{pmatrix}$$

$$\therefore \quad \tau\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 2 & 1 & 3 & 6 \end{pmatrix}$$

$$\sigma\tau\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 2 & 1 & 3 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 3 & 4 & 2 \end{pmatrix}$$

$$\therefore \quad \sigma\tau\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 3 & 4 & 2 \end{pmatrix}$$

Q18. If $\sigma \in S_6$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$ then find σ^{2012}.

Ans :

Given permutation is,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

The value of σ^{2012} can be obtained as,

$$\sigma^2 = \sigma.\sigma$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 5 & 6 \end{pmatrix} \qquad [\because \sigma\sigma(1) = \sigma(\sigma(1)) = \sigma(2) = 4]$$

$$\therefore \quad \sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 5 & 6 \end{pmatrix}$$

$$\therefore \quad \sigma^3 = \sigma^2.\sigma$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{pmatrix}$$

$$\sigma^4 = \sigma^3.\sigma$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$= I$$

$$\therefore \quad \sigma^4 = I$$

$$\sigma^{2012} = (\sigma^4)^{503}$$

$$= (I)^{503}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$\therefore \quad \sigma^{2012} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

4.7 Substructures

Q19. Explain briefly about Substructures.

Ans :

Sub Structures

Let H be a subset of a group G. Then H is called a subgroup of G if H itself is a group under the operation of G. Simple criteria to determine subgroups follow.

Proposition: A subset H of a group G is a subgroup of G if:

(i) The identity element $e \in H$,

(ii) H is closed under the operation of G, i.e. if a, $b \in H$, then $ab \in H$.

(iii) H is closed under inverses, that is, if $a \in H$, then $a^{-1} \in H$.

Every group G has the subgroups {e} and G itself. Any other subgroup of G is called a nontrivial subgroup.

4.8 Normal Subgroups

Q20. Define normal subgroup. Prove that every subgroup of an abelian group is normal.

Ans : **(Imp.)**

Normal Subgroup

A subgroup 'H' of a group 'G' is said to be a normal subgroup of 'G', if for all elements $h \in H$ and for all elements of $g \in G$, the element $ghg^{-1} \in H$.

Every Subgroup of an Abelian Group is Normal

'G' is an abelian group.

Let H, be subgroup of G

i.e., $H \leq G$

Then,

$\forall\ g \in G,\ \forall\ h \in H,$

$ghg^{-1} = gg^{-1}h$ [$\because$ 'G' is abelian]

$\Rightarrow$ $ghg^{-1} = eh$

Where

'e' is the identity in G

$\Rightarrow$ $ghg^{-1} = h$

$\Rightarrow$ $ghg^{-1} \in H\ \forall\ g \in G$

$\therefore$ H is normal subgroup of 'G'.

Q21. A Subgroup H of G is normal in G if and only if × H x^{-1} = H $\forall$ x $\in$ H.

(OR)

Define normal subgroup. Prove that a sub group H of group G is normal in G if and only if x H $x^{-1} \subseteq$ H $\forall$ x $\in$ G.

Ans : **(Imp.)**

Let G be a group and H be the subgroup of G.

Let an element x belongs to G.

$\Rightarrow$ $x \in G$

From the definition of normal subgroup, i.e., $xhx^{-1} \in H$ for all $x \in G$ and $x \in H$

Consider,

$xHx^{-1} = \{xhx^{-1} : x \in G, h \in H) \subseteq H$

Consider,

$x^{-1}Hx = \{x^{-1}hx : x \in G, h \in H) \subseteq H$

Hence,

$x^{-1}Hx \subseteq H$

$\Rightarrow$ $xx^{-1}\, Hx \subseteq xH$

$\Rightarrow$ $(xx^{-1})\, Hx \subseteq xH$

$\Rightarrow$ $(e)\, Hx \subseteq xH$ [$\because$ $xx^{-1} = e$]

$\Rightarrow$ $Hx \subseteq xH$ [$\because$ $e = 1$]

$\Rightarrow$ $Hxx^{-1} \subseteq xHx^{-1}$

$\Rightarrow$ $H(xx^{-1}) \subseteq xHr^{-1}$

$\Rightarrow$ $He \subseteq xHx^{-1}$

$\Rightarrow$ $H \subseteq xHx–1$ ... (2)

Comparing equations (1) and (2),

$xHx^{-1} = H$

Hence, a subgroup H of a group G is a normal subgroup of G if $xHx^{-1} = H$ for every x e G.

Q22. Show that intersection of two normal sub-groups is again a normal subgroup.

Ans :

The intersection of two subgroups is again a subgroup,

$H \cap K \le G$

Where, H, K are two normal subgroups of G

$\Rightarrow$ $H \Delta G$ and $K \Delta G$

Let,

$h \in H \cap K$ and $g \in G$

Since

$h \in H \cap K$

$h \in H$ and $h \in K$

$\Rightarrow$ $ghg^{-1} \in H$

[$\because$ $H \triangleleft G, h \in H, g \in G$]

$\Rightarrow$ $ghg^{-1} \in K$

[$\because$ $k \triangleleft G, h \in K, g \in G$]

$\Rightarrow$ $ghg^{-1} \in H \cap K$

$\Rightarrow$ $H \cap K \Delta G$

$\therefore$ $H \cap K$ is a normal subgroup of G.

Therefore, intersection of two normal subgroups is again a normal subgroup.

Q23. If M and N are two sub-groups of group G and N is normal in G, then prove that M $\cap$ N is normal in M.

Ans :

Given that,

M,N are subgroups of G

N is normal in G

Since,

M, N are subgrous of G

$\Rightarrow$ $M \cap N$ is also a subgroups of M

Let,

$a \in M$

$\Rightarrow$ $a \in G$

Let,

$b \in M \cap N$

$\Rightarrow$ $b \in M$ and $b \in N$

Since,

$a \in G, b \in N$

$\Rightarrow$ $aba^{-1} \in N$ [$\because$ $N \triangleleft G$]

.....(1)

And, $b \in M$, $a \in M$

$\Rightarrow$ $a^{-1} \in M$

$\Rightarrow$ $aba^{-1} \in M$ .....(2)

From equations (1) and (2),

$aba^{-1} \in M \cap N$

$\therefore$ $M \cap N$ is a normal subgroup of M.

Q24. If H and K are normal subgroups of a group G such that H $\cap$ K = {e}, then prove that hk = kh for all h $\in$ H and k $\in$ k.

Ans :

Given that,

H and K are normal sub groups of a group G.

$H \cap K = e$

Let, $k \in K$

$\Rightarrow k^{-1} \in K$

k is normal in G and $h \in G$

$\Rightarrow$ $hk^{-1}h^{-1} \in K$

From closure property,

$khK^{-1}h^{-1} \in K$ [$\because$ $k \in K$]

.....(1)

H is normal in G

$\Rightarrow$ $khK^{-1} \in H$

From closure property

$khK^{-1}h^{-1} \in H$ $[\because h^{-1} \in H]$ .... (2)

From equations (1) and (2)

$khK^{-1}h^{-1} \in H \cap K$

$\Rightarrow$ $khK^{-1}h^{-1} = e$ $[\because H \cap K = e]$

$\Rightarrow$ $khK^{-1} = eh$

$\Rightarrow$ $eh = h$

$\Rightarrow$ $hk = kh$

$\therefore$ $hk = kh \ \forall h \in H$ and $k \in K$

Q25. Prove that a subgroups H of a group G is a normal subgroup if and only if every left coset of H in G is a right coset of H in G.

Ans :

(i) Let, G be a group and H be the subgroup of G,

Let, x be an element in G

$\Rightarrow$ $x \in G$

From the property of normal subgroup i.e., H is a normal subgroup of G if and only if,

$xHx^{-1} = H$ for every $x \in G$ .... (1)

Multiplying on both sides by x,

$(xHx^{-1})\, x = (H)x$

$\Rightarrow$ $xHx^{-1}x = Hx$

$\Rightarrow$ $xH(x^{-1}x) = Hx$

$\Rightarrow$ $xH(e) = Hx$ $[\because x^{-1}x = e]$

$\Rightarrow$ $xH = Hx$ for some $x \in G$ ... (2)

Here, xH is the left coset and Hx is the right coset of H in G.

It can be seen from equation (1) that a left coset of H in G is equal to right coset of H in G.

$\therefore$ H is a normal subgroup of G.

(ii) Let, G be a group and H be its subgroup

Let the elements x, y belongs to group G

$\Rightarrow$ $x, y \in G$

Let x H be the left coset and Hy be the right coset of H in G.

Let us assume that every left coset of H in G is a right coset of H in G.

$\Rightarrow$ $xH = Hy$ for some $x,y \in G$

$\Rightarrow$ $x \in xH = Hy$

$\Rightarrow$ $x \in Hy$

From the property of cosets i.e., aH = bH if and only if $a \in bH$. Here, a = x, b = y

$\therefore$ $Hx = Hy$

$\Rightarrow$ $xH = Hx$ for all $x \in G$

Multiplying on both sides by x^{-1}.

$\Rightarrow$ $(xH)x^{-1} = (Hx)x^{-1}$.

$\Rightarrow$ $xHx^{-1} = H(xx^{-1})$

$\Rightarrow$ $xHx^{-1} = He$

$\Rightarrow$ $xHx^{-1} = H$

$\Rightarrow$ $H \Delta G$ $[\because$ From equation (1)]

i.e., H is a normal subgroup of G when the left coset of H in G is a right coset of H in G.

Hence, a subgroup of H of a group G is a normal subgroup if and only if every left coset of H in G is a right coset of H in G.

4.9 Algebraic Structures with two Binary Operation, Rings, Integral Domain and Fields

Q26. Define a ring.

Ans :

A ring (R, + ,.) is a set 'R' combined with two binary operations 'addition' and 'multiplication' which are defined on R such that it satisfies the following conditions.

* (R, +) is an abelian group
* (R, .) is semigroup
* It satisfies both left right distributive laws.

i.e., a.(b + c) = (a.b) + (a.c)

(a + b) .c = (a.c) + (b.c) $\forall$ a, b, c $\in$ R.

Q27. Give any four examples of a ring.

Ans :

(i) The set R = {0} is a ring. It is also called as Zero ring or the null ring.

(ii) The ser of real numbers R is a commutative ring with unity with respect to addition and multiplication of real numbers.

(iii) The set of intgers Z is a commutative ring with unity with respect to addition and multuplication of real numbers.

(iv) The set 2Z if even integers is a commutative ring without unity (1 $\notin$ 2Z) with respect to addition and multiplication of real numbers.

Q28. Define units of a ring and find all the units of Z_{14}.

Ans :

Units of a Ring

In a ring R with unity, an element u $\in$ R is said to be unit of R if it has multiplicative inverse.

Units of Z_{14}

Z_{14} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}

In Z_{14}, the unity element '1' is also a unit.

Since, 14 is an even, the even number in Z_{14} cannot be a unit.

3.5 = 5.3 = 15 = 1 mod 14

9.11 = 11.9 = 99 = 1 mod 14

13.13 = 169 = 1 mod 14

$\therefore$ The units of Z_{14} are 3, 5, 9, 11, 13.

Q29. In a ring R, the elements a, b, c R then , prove that,

(i) a. 0 = 0. a = 0

(ii) a(–b) = –(ab) = (–a)b

(iii) (–a)(–b) = ab

(iv) a(b – c) = ab – ac

(v) (b – c)a = ba – ca

Ans :

Given,

R is a ring

a, b, c $\in$ R

(i) a.0 = 0. a = 0

Consider,

a.0 = a.(0 + 0)

= a.0 + a.0 [$\because$ 0 + 0 = 0]

$\Rightarrow$ $0 + a.0 = a.0 + a.0$ [$\because$ From left distributive law]

Applying right cancellation law, [$\because$ R is a group]

$\Rightarrow$ $0 = a.0$ .....(1)

Similarly,

$0.a = (0 + 0) .a$ [$\because$ $0 + 0 = 0$]

$= 0.a + 0.a$ [$\because$ From right distributive law]

$\Rightarrow$ $0.a + 0 = 0.a + 0.a$

Applying left cancellation law,

$\Rightarrow$ $0 = 0.a$ .....(2)

From equations (1) and (2),

$a.0 = 0.a = 0$

(ii) $a(–b) = –(ab) = (–a)b$

Consider,

$b + (-b) = 0$

Multiplying with 'a' on both sides,

$\Rightarrow$ $a.(b + (–b)) = a.0$ [$\because$ $a.0 = 0$]

$\Rightarrow$ $a.b + a(–b) = 0$ [$\because$ From left distributive law]

$\Rightarrow$ $a(–b) = –(ab)$ [$\because$ $a + b = 0 \Rightarrow a = –b$] .....(1)

Consider, $(–a) + a = 0$

Multiplying with 'b' on both sides,

$\Rightarrow$ $((–a) + a).b = 0.b$

$\Rightarrow$ $(–a)b + ab = 0$ [$\because$ From right distributive law]

$\Rightarrow$ $(–a)b = –(ab)$

From equations (1) and (2),

$a(–b) = – (ab) = (–a) b$

(iii) $(–a)(–b) = ab$

$(–a)(–b) = –[a(–b)]$

$= –[–(ab)] = ab$

(iv) $a(b – c) = ab – ac$

$a(b – c) = a(b + (–c))$

$= ab + a(–c)$ [$\because$ From left distributive law]

$= ab – ac$

$\therefore$ $a(b – c) = ab – ac$

(v) $(b - c)a = ba - ca$

$(b - c)a = (b+(-c))a$

$= ba + (-c)a$ [∵ From right distributive law]

$= ba - ca$

$\therefore$ $(b - c)\,a = ba - ca.$

Q30. If a, b are any two elements of a ring R prove that,

(i) $-(-a) = a$

(ii) $-(a + b) = -a - b$

(iii) $-(a - b) = -a - b.$

Ans : **(Imp.)**

Given,

R is a ring

$a,b \in R$

(i) $-(-a) = a$

[∵ R is a group with respect to addition]

$-(-a) = a$

(ii) $-(a + b) = -a - b$

$-(a + b) = (-b) + (-a)$

[∵ R is a group with respect to addition]

But, addition in R is commutative

$\therefore$ $(-b) + (-a) = (-a) + (-b)$

$= -a - b$

$\therefore$ $-(a + b) = -a - b$

(iii) $-(a - b) = -a + b$

$-(a - b) = -[a + (-b)]$

$= -a + [-(-b)]$

$= -a + b$

$\therefore$ $-(a - b) = -a + b.$

Q31. If R is a ring such that $a^2 = a$ $\forall$ a $\in$ R then prove that,

(i) $a + a = 0$ $\forall$ $a \in R$

(ii) $a + b = 0 \Rightarrow a = b$

(iii) R is a commutative ring i.e., $ab = ba$ $\forall$ a, b $\in$ R.

Ans :

Given,

R is a ring

$a \in R$

$a^2 = a \ \forall a \in R$

(i) $a + a = 0 \ \forall a \in R$

$a \in R$

$\Rightarrow (a + a) \in R$

$\Rightarrow (a + a)^2 = (a + a)$ $[\because a^2 = a]$

$\Rightarrow (a + a)(a + a) = (a + a)$

$\Rightarrow a(a + a) + a(a + a) = (a + a)$

$\Rightarrow (a^2 + a^2)(a^2 + a^2) = a + a$

$\Rightarrow [(a) + (a))] + ((a) + (a))] = (a + a)$ $[\because a^2 = a]$

$\Rightarrow (a + a)(a + a) = (a + a) + 0$

$\Rightarrow a + a = 0$

$\therefore a + a = 0 \ \forall a \in R$

(ii) $a + b = 0 \Rightarrow a = b$

$a + b = 0$

$\Rightarrow a + b = a + a$ $[\because a + a = 0]$

$\Rightarrow b = a$

$\Rightarrow a = b$

$\therefore a + b = 0 \Rightarrow a = b$

(iii) R is a commutative ring i.e., $ab = ba \ \forall a, b \in R$

$a^2 = a$

$a + a = 0$

$a + b = 0$

$(a = b)^2 = a + b$

$\Rightarrow (a + b)(a + b) = a + b$

$\Rightarrow (a + b) a + (a + b) b = a + b$

$\Rightarrow a^2 + ba + ab + b^2 = a + b$

$\Rightarrow (a) + ba + ab + (b) = a + b$

$\Rightarrow (ba + ab) + (a + b) = (a + b) + 0$

$\Rightarrow ba + ab = 0$ $[\because a + b = 0 \Rightarrow b = 0]$

$ba + ab$

$\therefore$ R is a commutative ring.

Q32. If R is ring with unity (1), then show that (–1)a = –a = a (–1) $\forall$ a $\in$ R and (–1)(–1) = 1.

Ans :

Given,

R is a ring with unity

Let a be an element of R i.e., a $\in$ R

Then, a .1 = 1.a = a

Consider,

0.a = 0

$\Rightarrow$ ((–1) + 1). a = 0

$\Rightarrow$ (–1)a + (1)a = 0

$\Rightarrow$ (–1)a + a = 0

$\Rightarrow$ (–1)a = –a

Consider,

a.0 = 0

$\Rightarrow$ a((–1) + (1)) = 0

$\Rightarrow$ a(–1) + a(1) = 0

$\Rightarrow$ a(–1) + a = 0

$\Rightarrow$ a(–1) = – a

$\therefore$ From equations (1) and (2),

$\therefore$ (–1)a = a(–1) = – a

For, a = –1

$\therefore$ (–1)(–1) = – (–1) = 1

Q33. Define an integral domain. Prove that every field ia an integral domain.

(OR)

Define integral domain and field. Prove that every field is an intergral domain.

(OR)

Define an integral domain. Prove that every field is an integral domain. Give an example to show that converse need not be true.

Ans : **(Imp.)**

Integral Domain

A commutative ring R with unity containing atleast two elements and having no zero divisors is called an integral domain.

Field

A field can be defined as a non - zero commutative ring that has a multiplicative inverse for every non Zero element.

Every Field is an Integral Domain

Let F be a field

$\Rightarrow$ F is a commutative ring with unity

F is an integral domain if it has no zero divisors.

Let, a, b F, where a $\neq$ 0, ab = 0

$\Rightarrow$ a^{-1} exists [$\because$ a $\neq$ 0]

Since, a $\in$ F, there exists a^{-1} $\in$ F such that,

a^{-1} (a.b) = a–1 0 = 0

$\Rightarrow$ (a^{-1}. a)b = 0

$\Rightarrow$ 1.b = 0

$\Rightarrow$ b = 0

Similarly, if a,b $\subset$ F, b $\neq$ 0 ,ab = 0

$\Rightarrow$ b^{-1} exists.

Since,

b $\in$ F, there exists b^{-1} $\in$ F such that,

(a. b) b^{-1} = 0.b^{-1} = 0

$\Rightarrow$ a.(b b^{-1}) = 0

$\Rightarrow$ a. 1 = 0

$\Rightarrow$ a = 0

$\therefore$ If a,b $\subset$ F and a. b = 0, then either a = 0 or b = 0

$\Rightarrow$ F has no zero divisors.

$\therefore$ F is an integral domain.

Every Integral Domain is Not a Field

The ring of integers is an integral domain but not a filed because only 1 and –1 have inverses.

Q34. Show that every finite integral domain is a field.

(OR)

Prove that every finite integral domain is a field.

(OR)

Prove that every finite integral domain is a field

(OR)

Show that every finite integral domain is a field.

Ans :

Every Finite Integral Domain is a Field

Let, R be a finite integral domain,

$\Rightarrow$ R = $\{a_1, a_2, ...a_n\}$

Let,

$a \neq 0 \in R$

$\Rightarrow$ $aR = \{a\,a_1, a\,a_2, ...,a\,a_n\}$

The elements of aR are distinct.

Let, $aa_i = aa_j$ for $i \neq j$

$\Rightarrow$ $a(a_i - a_j) = 0$

Since,

$a \neq 0$ and R is without divisors

$\Rightarrow$ $a_i - a_j = 0$

$\Rightarrow$ $a_i = a_j$

This is a contradiction.

$\therefore$ Every element of R must be identical with exactly one element of aR

i.e., $\{a\,a_1, a\,a_2, ..., aa_n\}$

$\Rightarrow$ $a = a_k\,a$ for some k

If R is commutative, then

$a = a_k\,a = a\,a_k$

Now,

Let, $z \in B$, then,

$z.a_k = (a_j\,a)\,a_k$

$= a_j\,(a.\ a_k)$

$= a_j\,.a$

$= z$

$\therefore$ $a_k = 1$

$\therefore$ R has multiplicative identity.

Now,

For $1 \in R$ there exists $a_1 \in R$ such that,

$1 = a_1\,a = a\,a_1$

$\Rightarrow$ $a_i = a^{-1}$

$\Rightarrow$ Every non-zero element in R has multiplicative inverse.

$\therefore$ R is a field.

Q35. Show that the characteristic of an integral domain is zero or a prime.

(OR)

Prove that the characteristic of an integral domain in either zero or prime.

(OR)

Show that characteristic of an integral domain is zero (or) prime number.

(OR)

Show that the characteristic of an integral domain is either zero or a prime.

(OR)

Define characteristic of an integral domain. Show that characteristic of an integral domain is either zero or prime.

(OR)

Define characteristic of a ring ; and prove that "the characteristic of an integral domain is 0 or prime".

(OR)

Define characteristics of a ring R with unity. Show that the characteristics of an integral domain is either zero or a prime.

Ans : **(Imp.)**

Characteristics of a Ring R with Unity

The characteristic of a ring R is defined as the smallest positive integer 'n' such that $na = 0\ \forall\ a \in R$.

Characteristics of an Integral Domain

Let (R, +,.) be an integral domain and 'a' be an element of ring R. The characteristic of an integral domain (R, +,.) is defined as the order of element 'a'.

Characteristic of an Integral Domain is Either Zero or a Prime

The integral domain (R, +,.) with characteristic of R = n where $n \neq 0$ and also 'n' is a non-prime.

$\Rightarrow$ n = s.t

Where,

I<s, t<n ...(1)

$\forall\ a \neq 0$ in R $\exists$ a.a = $a^2 \in$ R and $a^2 \neq 0$ [$\because$ R is an integral domain]

$\Rightarrow$ n . $a^2 = 0$

$\Rightarrow$ n.a^2 = 0

$\Rightarrow$ (st).a^2 = 0 [$\because$ From equation (1)]

$\Rightarrow$ (st). (a.a) = 0

$\Rightarrow$ (sa) (ta) = 0

sa = 0 or ta = 0

If sa = 0 then for any $x \in R$,

(sa)x = 0 $\Leftrightarrow$ (as) x = 0

$\Rightarrow$ (as) x = 0

$\Rightarrow$ a(sx) = 0

$\therefore$ sx = 0 [$\because$ a # 0]

$\therefore$ This is a contradition for 1 < s< n.

If the characteristic of R = n,

For an integral domain R then 'n' has no factor 's' with 1 < s < n.

Therefore, 'n' is a prime.

Q36. If D is an integral domain, then prove that D[x] is an integral domain.

Ans : **(Imp.)**

Given that,

D is an integral domain.

From definition of integral domain,

It should have a commutative ring with unity and has no zero divisors.

Since D[x] is a ring

If D is commutative

$\Rightarrow$ D[x] is also commutative.

If D has unit element 1

$\Rightarrow$ f(x) = 1 is the unity element of D[x]

We prove D[x] has no zero divisors.

Let f(x), g(x) be non-zero polynomials in D[x],

Where,

$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0, a_n \neq 0$

$g(x) = b_m x^m + b_{m-1} x^{m-1} + ... + b_0, b_m \neq 0$

Since D is an integral domain,

$\Rightarrow a_n b_m \neq 0.$

$\Rightarrow f(x) \neq 0, g(x) \neq 0$

$\Rightarrow f(x) g(x) \neq 0.$

Thus, D[x] has no zero divisors.

$\therefore$ If D is an integral domain

$\Rightarrow$ D[x] is also an integral domain.

Q37. Let 'D' be an integral domain. Then show that there exists a field 'F' that contains a subring 1 isomorphic to D.

Ans :

Given,

D is an integral domain

Let S= {(a, b)\a,b $\in$ D,b$\neq$d}

Defined by (a, b) = (c, d) if ad = bc

Let F = {(a, b) ~{c,d) \ ad = bc, a, b,c,d$\in$ S} is a set of equivalence classes of S under the relation $\equiv$.

Let (x, y) = x/y define addition of multiplication operation on F

$$\frac{a}{b} + \frac{c}{d} = \frac{(ad+bc)}{(bd)} \text{ and } \frac{a}{b} + \frac{c}{d} = \frac{(ac)}{(bd)}$$

Well Defined

Let $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$

$\Rightarrow$ ab' = a'b and cd' = c'd ...(1)

Consider,

(ad + bc)b'd' = adb'd' + beb'd'

= (ab')dd'+ (cd')bb'

= (a'b)dd' + (c'd)bb' [$\because$ From equation (1)]

= (a'd')bd + (c'b')bd

= (a'd'+b'c')bd

$\therefore$ (ad + bc)b'd' = (a'd' + b'c)bd

Then from definition,

$$\frac{(ad+bc)}{(bd)} = \frac{(a'd'+b'c')}{(b'd)}$$

$\therefore$ Addition is well-defined.

Consider,

acb'd' = (ab')(cd')

= (a'b)(c'd) [$\because$ From equation (1)]

= (a'c')(bd)

$\therefore$ acb'd' = (a'c)(bd)

Then from definition,

$$\frac{(ac)}{(bd)} = \frac{(a'c')}{(b'd')}$$

i.e., $\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{a'}{b'}\right)\left(\frac{c'}{d'}\right)$

$\therefore$ Multiplication is well-defined.

To Prove F is Field

Let 1 be the unity of D.

$\Rightarrow \frac{0}{1}$ is additive identity of F

The additive inverse of $\frac{a}{b} = -\frac{a}{b}$ and multiplication inverse of $\frac{a}{b} = \frac{b}{a}$

Addition is Associative

Let a, b, c, d, e,f $\in$ F

Consider,

[(a,b) + (c, d) + (e,f)]

$$= \left(\frac{a}{b}+\frac{c}{d}\right) + \left(\frac{e}{f}\right) = \frac{(ad+bc)}{(bd)} + \left(\frac{e}{f}\right)$$

$$= \left[\frac{(ad+bc)f+(bd)(e)}{(bd)f}\right] = [adf + bcf + bce, bdf]$$

$\therefore$ [(a, b) + (c, d)] + (e,f) = [adf+ bcf+ bde, bdf]

Consider,

$$(a, b) + [(c, d) + (e, f)] = \left(\frac{a}{b}\right) + \left(\frac{c}{d}+\frac{e}{f}\right) = \left(\frac{a}{b}\right) + \frac{(cf+ed)}{(df)}$$

$$= \frac{adf + b(cf + ed)}{b(df)}$$

= (adf + bcf + bed, bdf)

$\therefore$ (a, b) + [(c, d) + (e, f)] = (adf + bcf + bed, bdf)

$\Rightarrow$ [(a, b) + (c, d)] + (e, f) = (a, b) + [(c, d**)** + (e, f)]

$\therefore$ Associativity of addition is satisfied.

Addition is Commutative

Let $a,b,c,d \in F$

Consider,

$$(a, b) + (c, d) = \left(\frac{a}{b} + \frac{c}{d}\right)$$

$$= \frac{(ad + bc)}{(bd)}$$

= (ad + be, bd)

= (bc + ad, db)

$$= \frac{(ad + bc)}{(bd)} = \left(\frac{c}{d}\right) + \left(\frac{a}{b}\right)$$

$\Rightarrow$ (a, b) + (c, d) = (c, d) + (a, b)

$\therefore$ Addition under commutative.

Additive Identity

Let (0, 1) be the additive identity.

Consider,

$$(a, b) + (0,1) = \left(\frac{a}{b}\right) + \left(\frac{0}{1}\right)$$

$$= \frac{(a.1 + b.0)}{b}$$

= (a + 0, b)

= (a, b)

$\therefore$ (a, b) + (0, 1) = (a, b)

$\Rightarrow$ Additive identity exists.

Inverse

Consider,

$$[a, b] + [-a, b] = \left(\frac{a}{b}\right) + \left(\frac{-a}{b}\right) = \frac{(ab - ad)}{b^2}$$

$= (0, b^2)$

Since $0.1 = b^2.0$

$= 0.1$

$\therefore \quad [a,b] + [-a, b] = [0, 1]$

Inverse exists.

Similarly multiplication properties can be proved.

To Prove ϕ : D $\rightarrow$ F is Ring Homomorphism

Let $x \rightarrow \frac{x}{1}$

Then for any $y \in F$

$\Rightarrow \quad y \rightarrow \frac{y}{1}$

$\Rightarrow \quad \frac{(x+y)}{1} = \frac{x}{1} + \frac{y}{1}$

Also,

$$\frac{xy}{1} = \left(\frac{x}{1}\right)\left(\frac{y}{1}\right)$$

$\therefore \quad \phi : D \rightarrow F$ is a ring homomorphism.

Hence, it is a ring isomorphism from D to ϕ(D)

4.10 Boolean Algebra and Boolean Ring

Q38. Define Boolean Algebra. Explain the operations of Boolean Algebra.

Ans : **(Imp.)**

Introduction

- The binary operations performed by any digital circuit with the set of elements 0 and 1, are called logical operations or logic functions. The algebra used to symbolically represent the logic function is called Boolean algebra. It is a two state algebra invented by George Boole in 1854.
- Thus, a Boolean algebra is a system of mathematics logic for the analysis and designing of digital systems.
- A variable or function of variables in Boolean algebra can assume only two values, either a 'O' o' a '1'. Hence, (unlike another algebra) there are no fractions, no negative numbers, no square roots cube roots, no logarithms etc.

Logic Operations

- In Boolean algebra, all the algebraic functions performed is logical. These actually repres logical operations. The AND, OR and NOT are the basic operations that are performed in Boolean algebra.
- In addition to these operations, there are some derived operation such asNAND. NOR, EX- OR EX-NOR that are also performed in Boolean algebra.

4.10.1 Identities of Boolean Algebra

Q39. Explain various Identities of Boolean Algebra.

Ans : **(Imp.)**

The Boolean algebra is governed by certain well developed rules and laws.

1. Commutative Laws

The commutative law allows change in position of AND or OR variables. There are two commutative laws.

(i) A + B = B + A

Thus, the order in which the variables are ORed is immaterial.

(ii) A – B = B . A

Thus, the order in which the variables are ANDed is immaterial.

This law can be extended to any number of variables.

2. Associative Laws

The associative law allows grouping of variables. There are two associative laws

(i) (A + B) + C = A + (B + C)

Thus, the way the variables are grouped and ORed is immaterial.

(ii) (A . B) . C = A . (B . C)

Thus, the way the variables are grouped and ANDed is immaterial.

This law can be extended to any number of variables.

3. Distributive Laws

The distributive law allows factoring or multiplying out of expressions. There are two distributive laws.

(i) A(B + C) = AB + AC

(ii) A + BC = (A + B) (A + C)

This law is applicable for single variable as well as a combination of variables.

4. Idempotence Laws

Idempotence means the same value. There are two Idempotence laws

(i) A . A = A

i.e. ANDing of a variable with itself is equal to that variable only.

(ii) A + A = A

i.e. ORing of a variable with itself is equal to that variable only.

5. Absorption Laws

There are two absorption laws

(i) A + AB = A(1 + B) = A

(ii) A(A + B) = A

6. Involutionary Law

This law states that, for any variable 'A'

$\overline{\overline{A}} = (A')' = A$

4.10.2 Theorems

4.10.2.1 Duality

Q40. Explain Duality Theorem.

Ans :

It is one of the elegant theorems proved in advance mathematics.

"Dual expression" is equivalent to write a negative logic of the given Boolean relation. For this we have to

(i) Change each OR sign by an AND sign and vice-versa.

(ii) Complement any '0' or '1 ' appearing in expression.

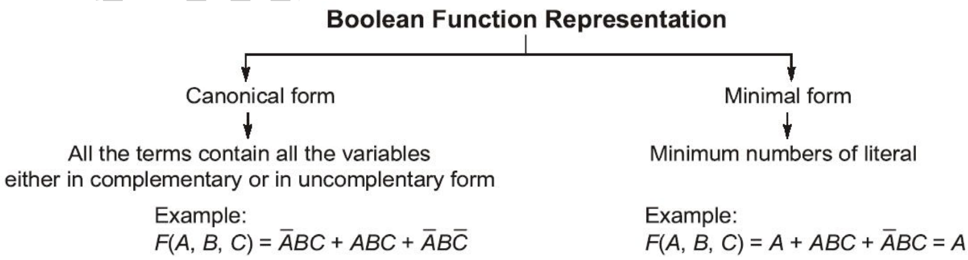
(iii) Keep literals/variables as it is.

4.10.3 Representation of Boolean Function

Q41. Describe the Representation of Boolean Function.

Ans : **(Imp.)**

A function of 'n' Boolean variables denoted by $f(A_1, A_2, ..., A_n)$ is another variable of algebra and takes one of the two possible values either 0 or 1. The various ways of representing a given function are discussed below:



The term 'literal' means a binary variable either in complementary or in uncomplimentary for.

1. Minterms and Maxterms

- n-binary variables have 2^n possible combinations.
- Minterm is a product term, it contains all the variables either complementary or uncomplimentary form for that combination the function output must be '1 '.
- Maxterm is a sum term, it contains all the variables either complementary or uncomplimentary form for that combination the function output must be '0'.

2. Sum of Product (SOP) Form

- The SOP expression usually takes the form of two or more variables ANDed together. Each product term may be minterm or implicant.

$Y = \bar{A}BC + A\bar{B} + AC$

$Y = A\bar{B} + B\bar{C}$

- This form is also called the "disjunctive normal form".
- The SOP expression is used most often because it tends itself nicely to the development of truth tables and timing diagrams.
- SOP circuits can also be constructed easily by using a special combinational logic gates called the "AND-OR-INVERTER" gate.
- SOP forms are used to write logical expression for the output becoming Logic T.

Input (3-Variables)			**Output (*Y*)**
A	*B*	*C*	*Y*
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

∴ Notation of SOP expression is,

$f(A, B, C) = \Sigma m(3, 5, 6, 7)$

∴ $Y = m_3 + m_5 + m_6 + m_7$

Also, $Y = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$

3. Product of Sum (POS) Form

- The POS expression usually takes the form of two or more order variables within parentheses, ANDed with two or more such terms.

$Y = (A + \bar{B} + C) . (B\bar{C} + D)$

- This form is also called the "Conjunctive normal form".
- Each individual term in standard POS form is called Maxterm.
- POS forms are used to write logical expression for output be coming Logic 'O'.

From the above truth table, we get

$F(A, B, C) = \Pi W(0, 1, 2, 4)$

∴ $Y = M_0 \times M_1 \times M_2 \times M_4$

Also, $Y = (A + B + C) (A + B + \bar{C}) (A + \bar{B} + C) (\bar{A} + B + C)$

- We also conclude that From the above truth table, and from above equations

If, $Y = \Sigma m(3, 5, 6, 7)$

Then, $Y = \Pi m\ (0, 1, 2, 4)$

4. Standard Sum of Product Form

- In this form the function is the sum of number of product terms where each product term contains all the variables of the function, either in complemented or uncomplemented form.
- It is also called canonical SOP form or expanded SOP form.
- The function $[Y = A + B\overline{C}]$ can be represented in canonical form as:

$$Y = A + B\overline{C} = A(B + \overline{B})(C + \overline{C}) + B\overline{C}(A + \overline{A})$$

$$= ABC + AB\overline{C} + A\overline{B}C + A\overline{B}\,\overline{C} + AB\overline{C} + \overline{A}B\overline{C}$$

$$Y = ABC + A\overline{B}C + A\overline{B}\,\overline{C} + AB\overline{C} + \overline{A}B\overline{C}$$

5. Standard Product of Sum Form

- This form is also called canonical POS form or expanded POS form.

$$Y = (B + \overline{C}) . (A + \overline{B})$$

- Then, the canonical form of the given function

$$Y = (B + C + A\overline{A})(A + \overline{B} + C\overline{C}) = (B + \overline{C} + A)(B + \overline{C} + \overline{A})$$

$$(A + \overline{B} + C)(A + \overline{B} + \overline{C})$$

6. Truth Table Form

A truth table is a tabular form representation of all possible combinations of given function.

$Y = \overline{A}B + \overline{B}C$

Then

	A	B	C	Y
0	0	0	0	0
1	0	0	1	1
2	0	1	0	1
3	0	1	1	1
4	1	0	0	0
5	1	0	1	1
6	1	1	0	0
7	1	1	1	0

7. Dual Form

- Dual form is used to convert positive logic to the negative logic and vice-versa.
- In positive logic system, higher voltage is taken as logic T and in negative logic system, higher voltage is taken as logic '0'. For example.

(i) For (positive) logic

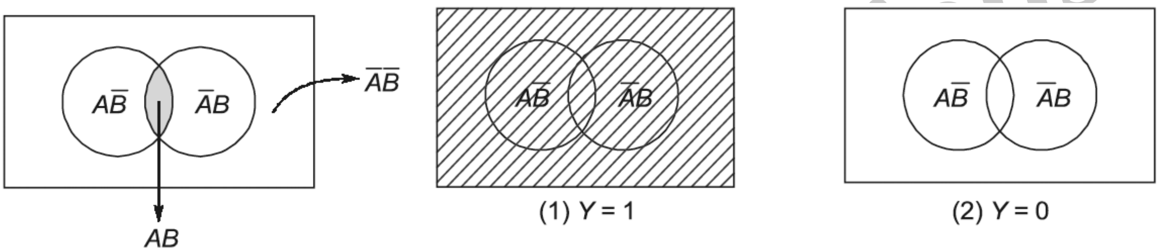
Logic '1' = 0 V; Logic '0' = –5 V

(ii) For (negative) logic

Logic '1' = –0.8 V; Logic '0' = –1.7 V

8. Venn Diagram Form

- A Boolean algebra can be represented by a Venn diagram in which each variable is considered as a set.
- The AND operation is considered as an intersection and the OR operation is considered as a union.



(1) $Y = 1$ (2) $Y = 0$

4.10.4 Conjunctive Normal Form

Q42. What is conjunctive normal form? Explain its construction.

Ans : **(Imp.)**

In Boolean logic, a formula is in conjunctive normal form (CNF) or clausal normal form if it is a conjunction of clauses, where a clause is a disjunction of literals; otherwise put, it is an AND of ORs. As a normal form, it is useful in automated theorem proving. It is similar to the product of sums form used in circuit theory.

All conjunctions of literals and all disjunctions of literals are in CNF, as they can be seen as conjunctions of one-literal clauses and conjunctions of a single clause, respectively. As in the disjunctive normal form (DNF), the only propositional connectives a formula in CNF can contain are and, or, and not. The not operator can only be used as part of a literal, which means that it can only precede a propositional variable or a predicate symbol.

Example 1:

The conjunctive normal form of

Examples

$(A \vee B) \wedge (A \vee C) \wedge (B \vee C \vee D)(A \vee B) \wedge (A \vee C) \wedge (B \vee C \vee D)$

$(P \cup Q)) \cap (Q \cup R)(P \cup Q)) \cap (Q \cup R)$

Q43. Explain the process of conversion into CNF.

Ans :

Conversion into CNF

Every propositional formula can be converted into an equivalent formula that is in CNF. This transformation is based on rules about logical equivalences: the double negative law, De Morgan's laws, and the distributive law.

Since all logical formulae can be converted into an equivalent formula in conjunctive normal form, proofs are often based on the assumption that all formulae are CNF. However, in some cases this conversion to CNF can lead to an exponential explosion of the formula. For example, translating the following non-CNF formula into CNF produces a formula with 2^n clauses.

$$(X_1 \wedge Y_1) \vee (X_2 \wedge Y_2) \vee \ldots \vee (X_n \wedge Y_n).$$

In particular, the generated formula is:

$$(X_1 \vee X_2 \vee \ldots \vee X_n) + (Y_1 \vee X_2 \vee \ldots \vee X_n) \wedge (X_1 \vee Y_2 \vee \ldots \vee X_n) \wedge (Y_1 \vee Y_2 \vee \ldots \vee X_n)$$

Step 1 : Eliminate

Using the rule

$$A \longrightarrow B = \neg A \vee B$$

We may eliminate all occurrences of $\rightarrow$

Example :

$$p \longrightarrow ((q \longrightarrow r) \vee \neg s) \equiv p \longrightarrow ((\neg q \vee r) \vee \neg s)$$
$$\equiv \neg p \vee ((\neg q \vee r) \vee \neg s)$$

Step 2 : Puch Negations Down

Using DeMorgan's Laws and the double negation rule

$$\neg (A \vee B) \equiv \neg A \wedge \neg B$$
$$\neg (A \wedge B) \equiv \neg A \vee \neg B$$
$$\neg \neg A \equiv A$$

We puch negations down towards the atoms until we obtain a formula.

That is formed from literals using only $\wedge$ and $\vee$.

$$\neg (\neg p \wedge (q \vee \neg (r \wedge s)))$$
$$\equiv \neg \neg p \vee \neg (q \vee \neg (r \wedge s)))$$
$$\equiv p \vee (\neg q \vee \neg \neg (r \wedge s))$$
$$\equiv p \vee (\neg q \vee (r \wedge s))$$

Step 3: Use distribution to convert to CNF

Using the distribution rules

$$A \vee (B_1 \wedge \ldots \wedge B_n) \equiv (A \vee B_1) \wedge \ldots \wedge (A \vee B_n)$$
$$(B_1 \wedge \ldots \wedge B_n) \vee A \equiv (B_1 \vee A) \wedge \ldots \wedge (B_n \vee A)$$

We obtain a CNF formula,

Example :

$$(p \vee q) \vee (p \wedge \neg q)$$
$$\equiv ((p \wedge q) \vee p) \wedge ((p \wedge q) \vee \neg q)$$
$$\equiv ((p \vee q) \wedge (q \vee p)) \wedge ((p \vee \neg q) \wedge (q \vee \neg q))$$

Example

$$((p \wedge q) \vee (r \wedge s)) \vee (\neg q \wedge (p \vee t))$$
$$\equiv ((p \wedge q) \vee r) \wedge ((p \wedge q) \vee s)) \vee (\neg q \wedge (p \vee t))$$
$$\equiv ((p \vee r) \wedge (q \vee r) \wedge (p \vee s) \wedge (q \vee s)) \vee (\neg q \wedge (p \vee t))$$
$$\equiv ((p \vee r) \vee (\neg q \wedge (p \vee t)) \wedge$$
$$((q \vee r) \vee (\neg q \wedge (p \vee t)) \wedge$$
$$((p \vee s) \vee (\neg q \wedge (p \vee t)) \wedge$$
$$((q \vee s) \vee (\neg q \wedge (p \vee t))$$
$$\equiv (p \vee r \vee \neg q) \wedge (p \vee r \vee p \vee t) \wedge$$
$$(q \vee r \vee \neg q) \wedge (q \vee r \vee p \vee t) \wedge$$
$$(p \vee s) \vee \neg q) \wedge (p \vee s \vee p \vee t) \wedge$$
$$(q \vee s \vee \neg q) \wedge (q \vee s \vee p \vee t)$$

Q44. Convert the following normal form into CNF

¬((¬p → ¬q)’”¬r

Ans :

For example, converting to conjunctive normal form:

$$\neg((\neg p \rightarrow \neg q) \wedge \neg r \equiv \neg((\neg \neg p(\vee \neg q) \wedge \neg r)$$
$$\equiv \neg((p(\vee \neg q) \wedge \neg r)$$
$$\equiv \neg(p \vee \neg q) \vee \neg \neg r$$
$$\equiv \neg(p \vee \neg q) \vee r$$
$$\equiv (\neg p \wedge \neg \neg q) \vee r$$
$$\equiv (\neg p \wedge q) \vee r$$
$$\equiv (\neg p \vee r) \wedge (q \vee r)$$

Q45. Convert the following formulas into CNF using truth tables.

(a) $(p \Leftrightarrow q) \Rightarrow (\neg p \wedge r)$

(b) $(p \Leftrightarrow q) \Rightarrow r$

(c) $p \Leftrightarrow q$

Ans :

Formula : $(p \Leftrightarrow q) \Rightarrow r(\neg p \wedge r)$

Truth Table

p	q	r	$p \Leftrightarrow q$	$\neg p \wedge r$	$(p \Leftrightarrow q) \Rightarrow (\neg p \wedge r)$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	0	1
0	1	1	0	1	1
1	0	0	0	0	1
1	0	1	0	0	1
1	1	0	1	0	0
1	1	1	1	0	0

Convert the following formulas into CNF using equivalent transformations method

(a) $(p \Rightarrow q) \Leftrightarrow (p \Rightarrow r)$

(b) $(p \vee q) \wedge (p \wedge r)$

(c) $(p \Rightarrow q) \Rightarrow r$

(d) $p \Leftrightarrow q$

Sol :

Formula : $(p \Rightarrow q) \Leftrightarrow (p \Rightarrow r)$

Transformations

$((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \wedge ((p \Rightarrow r) \Rightarrow (p \Rightarrow q))$

$(\neg(p \Rightarrow q) \vee (p \Rightarrow r)) \wedge (\neg(p \Rightarrow r) \vee (p \Rightarrow q))$

$((p \wedge \neg q) \vee (\neg p \vee r)) \wedge ((p \wedge \neg r) \vee (\neg p \vee q))$

$((p \vee \neg q) \wedge (\neg q \vee \neg p)) \vee r) \wedge ((p \vee \neg p) \wedge (\neg r \vee \neg p)) \vee q)$

$((\neg q \vee \neg p \vee r) \wedge (\neg r \vee \neg p \vee q)$

4.10.5 Disjunctive Normal Form

Q46. What is Disjunctive normal form. Explain with an example.

Ans : **(Imp.)**

Disjunctive Normal Forms

A logical formula is considered to be in DNF if and only if it is a disjunction of one or more conjunctions of one or more literals. A DNF formula is in full disjunctive normal form if each of its variables appears exactly once in every clause. As in conjunctive normal form (CNF), the only propositional operators in DNF are and, or, and not. The not operator can only be used as part of a literal, which means that it can only precede a propositional variable.

The following is a formal grammar for DNF:

1. disjunction $\rightarrow$ (conjunction $\vee$ disjunction)
2. disjunction $\rightarrow$ conjunction
3. conjunction $\rightarrow$ (literal $\wedge$ conjunction)
4. conjunction $\rightarrow$ literal
5. literal $\rightarrow$ $\neg$variable
6. literal $\rightarrow$ variable

Where variable is any variable.

Example:

The disjunctive normal form of

$(A \wedge B) \vee (A \wedge C) \vee (B \wedge C \wedge D)$ $(A \wedge B) \vee (A \wedge C) \vee (B \wedge C \wedge D)$

$(P \cap Q) \cup (Q \cap R)$ $(P \cap Q) \cup (Q \cap R)$

Method to construct DNF

- Construct a truth table for the proposition.
- Use the rows of the truth table where the proposition is True to construct minterms
- If the variable is true, use the propositional variable in the minterm
- If a variable is false, use the negation of the variable in the minterm
- Connect the mintermswith $\vee$'s.

Conversion to DNF

Converting a formula to DNF involves using logical equivalences, such as the double negative elimination, De Morgan's laws, and the distributive law.

All logical formulas can be converted into an equivalent disjunctive normal form. However, in some cases conversion to DNF can lead to an exponential explosion of the formula. For example, the DNF of a logical formula of the following form has 2^n terms

$$(X_1 \vee Y_1) \wedge (X_2 \vee Y_2) \wedge ... \wedge (X_n \vee Y_n)$$

Q47. Show DNF of p⊕q is (p∧¬q)∨(¬p∧q)

Ans :

Truth table

P	Q	p⊕q	(p∧¬q) ∨ (¬p∧q)
T	T	F	F
T	*F*	*T*	*T*
F	*T*	*T*	*T*
F	F	F	F

48. Construct the DNF of (p $\vee$ q)$\rightarrow\neg$r

Ans :

P	Q	r	(Pv q)	~r	(p V q)→~ r
T	T	T	T	F	F
T	*T*	*F*	*T*	*T*	*T*
T	F	T	T	F	F
T	*F*	*F*	*T*	*T*	*T*
F	T	T	T	F	F
F	*T*	*F*	*T*	*T*	*T*
F	*F*	*T*	*F*	*F*	*T*
F	*F*	*F*	*F*	*T*	*T*

There are fives sets of inptu that make the statement true. Therefor there are five mintersm.

p	Q	r	(p V q)	~ r	(P v q)→ ~ r
T	T	T	T	F	F
T	*T*	*F*	*T*	*T*	*T*
T	F	T	T	F	F
T	*F*	*F*	*T*	*T*	*T*
F	T	T	T	F	F
F	*T*	*F*	*T*	*T*	*T*
F	*F*	*T*	*F*	*F*	*T*
F	*F*	*F*	*F*	*T*	*T*

From the truth table we can set up the DNF

$(p \vee q) \rightarrow \sim r \Leftrightarrow (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg pq \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$

Q49. Convert $((p \wedge q) \to r) \wedge (\neg(p \wedge q) \to r)((p \wedge q) \to r) \wedge (\neg(p \wedge q) \to r)$ to DNF.

Ans :

$((p \wedge q) \to r) \wedge (\neg(p \wedge q) \to r)((p \wedge q) \to r) \wedge (\neg(p \wedge q) \to r)$

$(\neg(p \wedge q)(\vee r) \wedge ((p \wedge q)(\vee r)(\neg(p \wedge q)(\vee r) \wedge ((p \wedge q)(\vee r)$

$((\neg p(\vee \neg q)(\vee r) \wedge ((p \wedge q)(\vee r)((\neg p(\vee \neg q)(\vee r) \wedge ((p \wedge q)(\vee r)$

Q50. Construct DNF $p \to q$ is $(p\wedge q) \vee (\neg p\wedge q) \vee (\neg p\wedge\neg q)$

Ans :

The DNF of $p \to q$ is $(p\wedge q) \vee (\neg p \wedge q) \vee (\neg p\wedge\neg q)$.

Then, applying DeMorgan's Law, we get that this is equivalent to

$\neg[\neg(p\wedge q) \wedge\neg(\neg p\wedge q) \wedge\neg(\neg p\wedge\neg q)]$.

Now can we write an equivalent statement to $p \to q$ that uses only disjunctions and negations?

$p \to q$

$\Leftrightarrow$	$\neg[\neg(p\wedge q) \wedge\neg(\neg p\wedge q) \wedge\neg(\neg p\wedge\neg q)]$	From Before
$\Leftrightarrow$	$\neg[(\neg p\vee\neg q) \wedge (\neg\neg p\vee\neg q) \wedge (\neg\neg p \vee\neg\neg q)]$	DeMorgan
$\Leftrightarrow$	$\neg[(\neg p\vee\neg q) \wedge (p\vee\neg q) \wedge (p\vee q)]$	Doub. Neg.
	$[\neg(\neg p\vee\neg q) \vee\neg(p\vee\neg q) \vee\neg(p\vee q)]$	DeMorgan

UNIT V	**Graphs and Trees**: Graphs and their properties, Degree, Connectivity, Path, Cycle, Sub Graph, Isomorphism, Eulerian and Hamiltonian Walks, Graph Colouring, Colouring maps and Planar Graphs, Colouring Vertices, Colouring Edges, List Colouring, Perfect Graph, definition properties and Example, rooted trees, trees and soring, weighted trees and prefix codes, Bi-connected component and Articulation Points, Shortest distances.

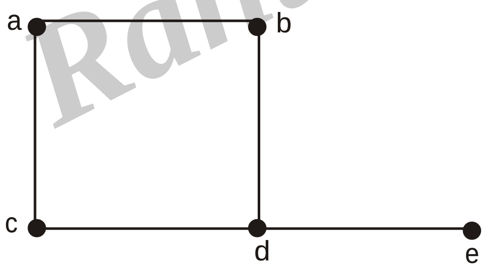
5.1 GRAPHS

Q1. What is a Graph ?

Ans :

A graph is a pictorial representation of a set of objects where some pairs of objects are connected by links. The interconnected objects are represented by points termed as **vertices,** and the links that connect the vertices are called **edges**.

Formally, a graph is a pair of sets **(V, E),** where **V** is the set of vertices and **E** is the set of edges, connecting the pairs of vertices. Take a look at the following graph –



In the above graph,

V = {a, b, c, d, e}

E = {ab, ac, bd, cd, de}

Q2. Define the following terms

a) Even and Odd Vertex

b) Degree of Vertex

c) degree of graph

d) undirected graph

Ans :

Graph Terminology

(a) Even and Odd Vertex " If the degree of a vertex is even, the vertex is called an even vertex and if the degree of a vertex is odd, the vertex is called an odd vertex.

(b) Degree of a Vertex "The degree of a vertex V of a graph G (denoted by deg (V)) is the number of edges incident with the vertex V.

Vertex	Degree	Even / Odd
A	2	Even
B	2	Even
C	3	Odd
D	1	Odd

(c) Degree of a Graph "The degree of a graph is the largest vertex degree of that graph. For the above graph the degree of the graph is 3.

(d) Undirected Graphs - Two vertices *u*, *v* in *V* are *adjacent* or *neighbours* if there is an edge e between *u* and *v*. The edge e *connects* *u* and *v*. The vertices *u* and *v* are endpoints of e.

Q3. Prove that a graph has an even number of vertices of odd degree.

Ans :

Theorem: A graph has an even number of vertices of odd degree.

Proof :

Let V1 = vertices of odd degree

V2= vertices of even degree

The sum must be even. But

- odd times odd = odd
- odd times even = even
- even times even = even
- even plus odd = odd

It doesn't matter whether V2 has odd or even cardinality.

V1 cannot have odd cardinality.

Q4. What is directed graph ?

Ans :

Directed Graphs

Let <u, v>be an edge in G. Then u is an initial vertex and is adjacent to v and v is a terminal vertex and is adjacent from u.

Definition

The in degree of a vertex v, denoted deg-(v) is the number of edges which terminate at v. Similarly, the out degree of v, denoted deg+(v), is the number of edges which initiate at v.

Theorem

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$$

Q5. Define the following terms

(a) complete graphs

(b) cycle

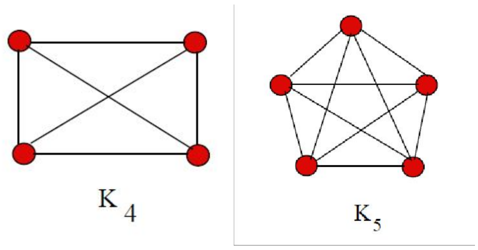
(c) wheel

Ans :

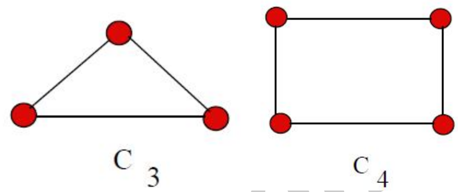
- **Complete graphs** - Kn: the simple graph with - n vertices exactly one edge between every pair of distinct vertices.

  Maximum redundancy in local area networks and processor connection in parallel machines.

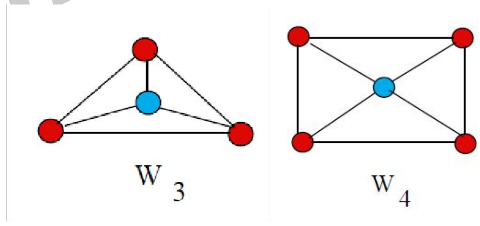
Examples :



- **Cycles** - Cn is an n vertex graph which is a cycle. Local area networks are sometimes configured this way called *Ring* networks.



- **Wheels**: Add one additional vertex to the cycle Cn and add an edge from each vertex to the new vertex to produce Wn. Provides redundancy in local area networks.



Q6. Discuss about various types of graphs.

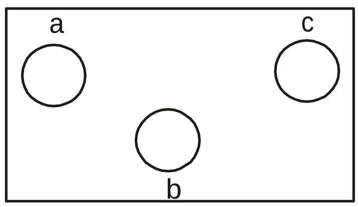
Ans :

Types of Graphs

There are different types of graphs, which we will learn in the following section

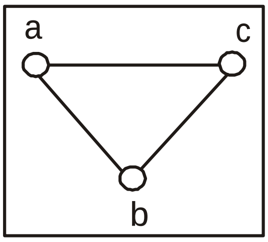
1. Null Graph

A null graph has no edges. The null graph of n vertices is denoted by N_n



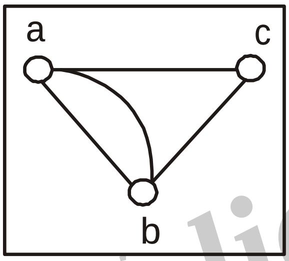
2. Simple Graph

A graph is called simple graph/strict graph if the graph is undirected and does not contain any loops or multiple edges.



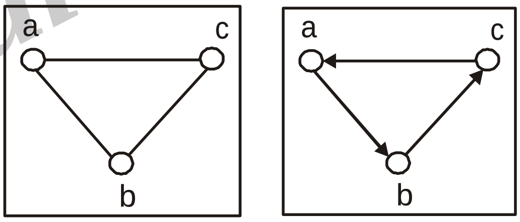
3. Multi-Graph

If in a graph multiple edges between the same set of vertices are allowed, it is called Multigraph.



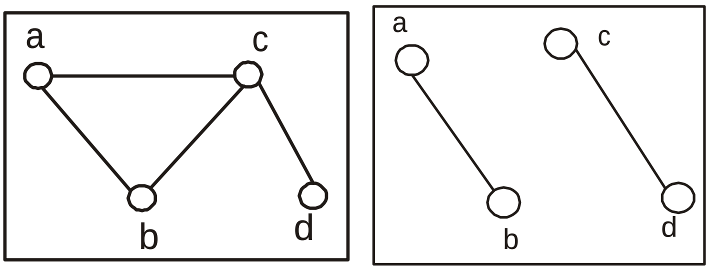
4. Directed and Undirected Graph

A graph G = (V, E) is called a directed graph if the edge set is made of ordered vertex pair and a graph is called undirected if the edge set is made of unordered vertex pair.



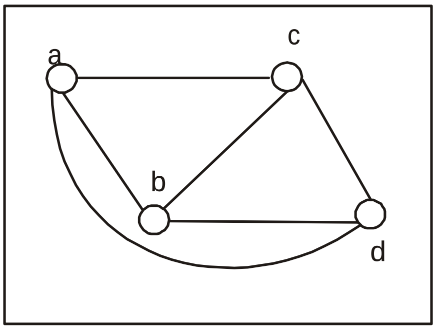
5. Connected and Disconnected Graph

A graph is connected if any two vertices of the graph are connected by a path and a graph is disconnected if at least two vertices of the graph are not connected by a path. If a graph G is unconnected, then every maximal connected subgraph of G is called a connected component of the graph G.



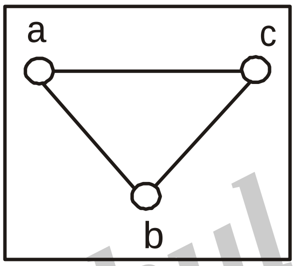
6. Regular Graph

A graph is regular if all the vertices of the graph have the same degree. In a regular graph G of degree r, the degree of each vertex of G is r.



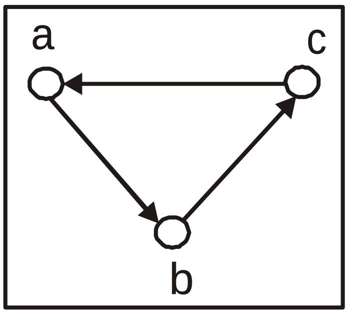
7. Complete Graph

A graph is called complete graph if every two vertices pair are joined by exactly one edge. The complete graph with n vertices is denoted by K_n



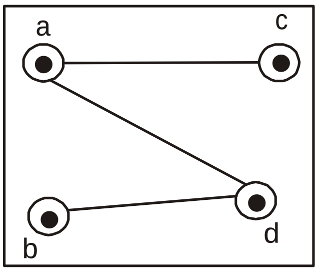
8. Cycle Graph

If a graph consists of a single cycle, it is called cycle graph. The cycle graph with n vertices is denoted by C_n



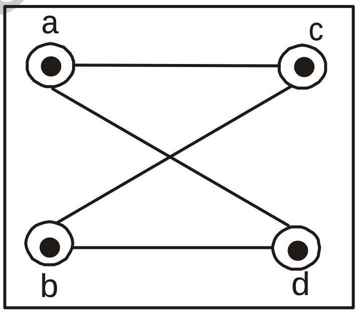
9. Bipartite Graph

If the vertex-set of a graph G can be split into two sets in such a way that each edge of the graph joins a vertex in first set to a vertex in second set, then the graph G is called a bipartite graph. A graph G is bipartite if and only if all closed walks in G are of even length or all cycles in G are of even length.



10. Complete Bipartite Graph

A complete bipartite graph is a bipartite graph in which each vertex in the first set is joined to every single vertex in the second set. The complete bipartite graph is denoted by $K_{r,\ s}$ where the graph G contains x vertices in the first set and y vertices in the second set.



5.1.1 Properties

Q7. Explain the basic properties of graph theory.

Ans :

1. Distance between Two Vertices

Distance is basically the number of edges in a shortest path between vertex X and vertex Y. If there are many paths connecting two vertices, then the shortest path is considered as the distance between the two vertices. Distance between two vertices is denoted by d(X, Y).

2. Eccentricity of a Vertex

Eccentricity of a vertex is the maximum distance between a vertex to all other vertices. It is denoted by e(V). To count the eccentricity of vertex, we have to find the distance from a vertex to all other vertices and the highest distance is the eccentricity of that particular vertex.

3. Radius of connected Graph

The radius of a connected graph is the minimum eccentricity from all the vertices. In other words, the minimum among all the distances between a vertex to all other vertices is called as the radius of the graph. It is denoted by r(G).

4. Diameter of a Graph

Diameter of a graph is the maximum eccentricity from all the vertices. In other words, the maximum among all the distances between a vertex to all other vertices is considered as the diameter of the graph G. It is denoted by d(G).

5. Central Point

If the eccentricity of the graph is equal to its radius, then it is known as central point of the graph.

Or

If r(V) = e(V), then V is the central point of the graph G.

6. Centre

The set of all the central point of the graph is known as centre of the graph.

7. Circumference

The total number of edges in the longest cycle of graph G is known as the circumference of G.

8. Girth

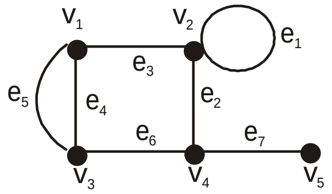
The total number of edges in the shortest cycle of graph G is known as girth. It is denoted by g(G).

5.1.2 Degree, Connectivity, Path, Cycle, Sub Graph

Q8. Explain briefly about degrees and connectivity.

Ans :

When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be incident with each other. Two non parallel edges are said to be adjacent if they are incident on a common vertex. The number of edges incident on a vertex v_i, with self-loops counted twice, is called the degree (also called valency), $d(v_i)$, of the vertex v_i. A graph in which all vertices are of equal degree is called regular graph.



The edges e_2, e_6 and e_7 are incident with vertex v_4.

The edges e_2 and e_7 are adjacent.

The edges e_2 and e_4 are not adjacent.

The vertices v_4 and v_5 are adjacent.

The vertices v_1 and v_5 are not adjacent.

$d(v_1) = d(v_3) = d(v_4) = 3.\ d(v_2) = 4.\ d(v_5) = 1.$

Total degree $= d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$

$= 3 + 4 + 3 + 3 + 1 = 14 =$ Twice the number of edges.

Connectivity is a basic concept of graph theory. It defines whether a graph is connected or disconnected. Without connectivity, it is not possible to traverse a graph from one vertex to another vertex. A graph is said to be connected graph if there is a path between every pair of vertex. From every vertex to any other vertex there must be some path to traverse. This is called the connectivity of a graph. A graph is said to be disconnected, if there exists multiple disconnected vertices and edges. Graph connectivity theories are essential in network applications, routing transportation networks, network tolerance etc.

Q9. State and prove "the number of vertices of odd degree in a graph is always even.

Ans :

Theorem

The number of vertices of odd degree in a graph is always even.

Proof:

Let us now consider a graph G with e edges and n vertices $v_1, v_2, ..., v_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. That is,

$$\sum_{i=1}^{n} d(vi) = 2e$$

If we consider the vertices with odd and even degrees separately, the quantity in the left side of the above equation can be expressed as the sum of two sums, each taken over vertices of even and odd degrees, respectively, as follows:

$$\sum_{i=1}^{n} d(v_i) \sum_{even} d(v_j) + \sum_{odd} + d(v_k)$$

Since the left-hand side in the above equation is even, and the first expression on the right-hand side is even (being a sum of even numbers), the second expression must also be even:

$$\sum_{odd} + d(v_k) = \text{an even number}$$

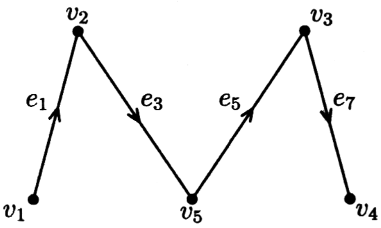
Because in the above equation each d(vk) is odd, the total number of terms in the sum must be even to make the sum an even number. Hence the theorem.

Q10. Discuss about path and cycle in graph theory.

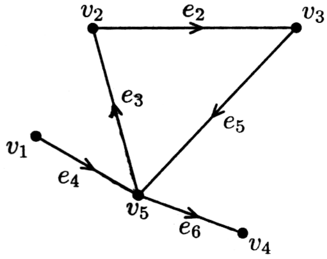
Ans : **(Imp.)**

A trail in which no vertex appears more than once is called a path.

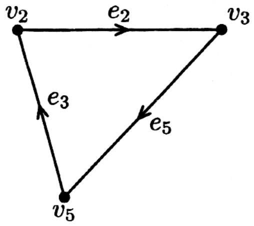
A circuit in which the terminal vertex does not appear as an internal vertex (also) and no internal vertex is repeated is called a cycle.



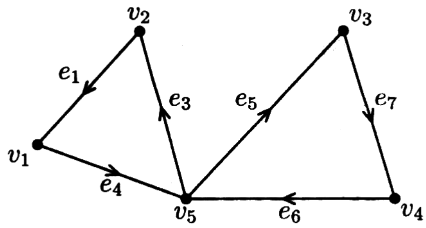
(a) : path



(b) : Not a path



(c) : cycle



(d) : Not a cycle

For example, in Figure., the trail $v_1e_1v_2e_3v_5e_5v_3e_7v_4$ (shown separately in Figure (a) is a path whereas the trail $v_1e_4v_5e_3v_2e_2v_3e_5v_5e_6v_4$ (shown separately in Figure(b)) not a path (because in this trail, v_5 appears twice). Also, in the same Figure, the circuit $v_2e_2v_3e_5v_5e_3v_2$ (shown separately in Figure(c) is a cycle whereas the circuit $v_2e_1v_1e_4v_5e_5v_3e_7v_4e_6v_5e_3v_2$ (shown separately in Figure(d) is not a cycle (because, in this circuit, v_5 appears twice).

The following facts are to be emphasized.

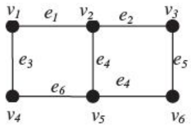
1. A walk can be open or closed. In a walk (closed or open), a vertex and/or an edge can appear more than once.
2. A trail is an open walk in which a vertex can appear more than once but an edge cannot appear more than once.
3. A circuit is a closed walk in which a vertex can appear more than once but an edge cannot appear more than once.
4. A path is an open walk in which neither a vertex nor an edge can appear more than once. Every path is a trail; but a trail need not be a path.
5. A cycle is a closed walk in which neither a vertex nor an edge can appear more than once. Every cycle is a circuit; but, a circuit need not be a cycle.
6. If a cycle contains only one edge, it has to be a loop.
7. Two parallel edges (when they occur) form a cycle.
8. In a simple graph, a cycle must have at least three edges. (A cycle formed by three edges is called a triangle.

Q11. Explain about sub group.

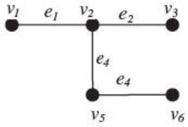
Ans :

A graph G' is said to be a subgraph of a graph G, if all the vertices and all the edges of G' are in G, and each edge of G' has the same end vertices in G' as in G.

Graph G:



Subgraph G' of G:



A subgraph can be thought of as being contained in (or a part of) another graph. The symbol from set theory, g $\subset$ G, is used in stating "g is a subgraph of G".

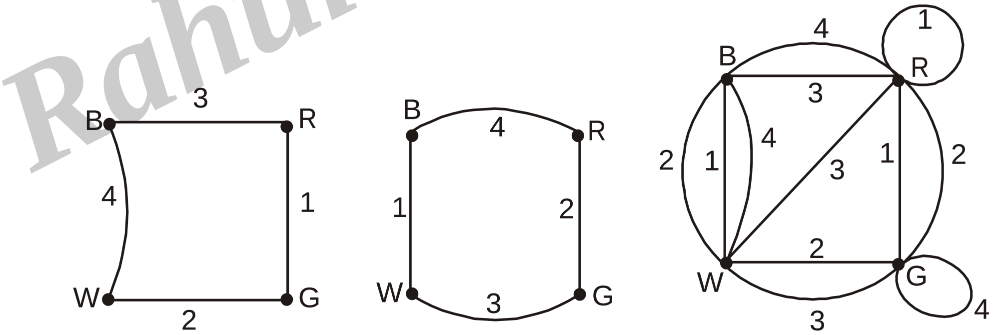
The following observations can be made immediately:

1. Every graph is its own subgraph.
2. A subgraph of a subgraph of G is a subgraph of G.
3. A single vertex in a graph C is a subgraph of G.
4. A single edge in G, together with its end vertices, is also a subgraph of G.

Edge-Disjoint Subgraphs

Two (or more) subgraphs g_1, and g_2 of a graph G are said to be edge disjoint if g_1, and g_2 do not have any edges in common.

For example, the following two graphs are edge-disjoint sub-graphs of the graph G.



Note that although edge-disjoint graphs do not have any edge in common, they may have vertices in common. Sub-graphs that do not even have vertices in common are said to be vertex disjoint. (Obviously, graphs that have no vertices in common cannot possibly have edges in common.)

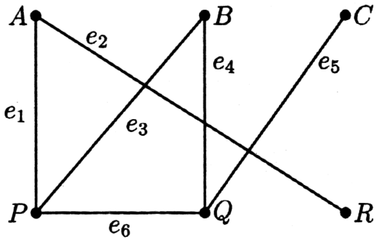
<u>PROBLEMS</u>

12. If G is a simple graph in which every vertex has degree at least k, prove that G contains a path of length at least k.

Sol :

Consider a path p in G which has a maximum number of vertices. Let u be an end vertex of P. Then every neighbor of u belongs to p. Since u has at least k neighbors (because its degree is at least k by what is given) and since G is simple, p must therefore have at least k vertices other than u. Thus, p is a path of length at least k.

13. Find all the cycles in the graph shown belows:



Sol :

There are no cycles beginning and ending with the vertices A, C and R. The cycles beginning and ending with the vertices B, P, Q are

$$Be_3Pe_6Qe_4B, \quad Pe_6Qe_4Be_3P, \quad Qe_4Be_3Pe_6Q.$$

But all of these represent one and the same cycle. Thus, there is only one cycle in the grap.

14. Prove the following:

(1) A path with n vertices is of length n – 1.

(2) If a cycle has n vertices, it has n edges.

(3) The degree of every vertex in a cycle is two.

Sol :

(1) In a path, every vertex except the last vertex is followed by precisely one edge. Therefore, if a path has n vertices, it must have n – 1 edges. Its length is therefore n – 1.

(2) In a cycle, every vertex is followed by precisely one edge. Therefore, if a cycle has n vertices, it must have n edges.

(3) In a cycle, exactly two edges are incident on every vertex (– one edge through which we enter the vertex and one edge through which we leave the vertex). Therefore, the degree of every vertex in a cycle is two.

15. Show that, for all positive integers k $\geq$ 2, there exists a simple cubic graph of order 2k.

Sol :

Consider the cycle $v_1, v_2, \ldots, v_{2k,} v_1$. To this cycle let us add the edges

$$\{v_1, v_{k+1}\}, \{v_2, v_{k+2}\}, \{v_2, v_{k+2}\}, \ldots\ldots., \{v_k, v_{2k}\}.$$

Then the resulting graph is of order 2k and the degree of each vertex in this graph is 3. This proves the existence of a graph of the desired type.

16. Let G be a cycle on n vertices. Prove that G is self-complementary if and only if n = 5.

Sol :

Let G be a cycle of order n = 5 with edges {a,b}, {b, c}, {c, d}, {d,e}, {e,a}. Then $\bar{G}$ is the cycle with edges {a, c}, {c,e}, {e,b}, {b,d}, {d, a}. It is easy to check that G and $\bar{G}$ are isomorphic. Therefore, G is self-complementary.

Conversely, suppose G is a cycle on n vertices and G is self-complementary. Then n(n–1) =4. This yields n = 5.

5.2 Isomorphism

Q17. Explain about Isomorphism.

Ans : **(Imp.)**

Consider two graphs G = (V, E) and G' = (V', E'). Suppose there exists a function f : V $\rightarrow$ V' such that

(i) f is one-to-one and onto, and

(ii) for all vertices A, B of G, the edge {A, B} $\in$ E if and only if the edge {f(A), f(B)} $\in$ E'. Then f is called an isomorphism between G and G'. and we say that G and G' are isomorphic graphs.

In other words, two graphs G and G' are said to be isomorphic (to each other) it their is a one-to-one correspondence between their vertices and between their edges such that the f adjacency of vertices is preserved. Such graphs will have the same structure, differing only in the way their vertices and edges are labeled or only in the way they ate represented geometrically. For many purposes, we regard them as essentially the same.

When G and G' are isomorphic, we write G $\cong$ G'.

When a vertex A of G corresponds to the vertex A' = f(A) of G' under an one-to-one correspondence f : G $\rightarrow$ G', we write A $\leftrightarrow$ A'. Similarly, we write {A, B} $\leftrightarrow$ {A'. B'} to mean that the edge AB of G and the edge A'B' of G' correspond to each other.

For example, look at the graphs shown below:

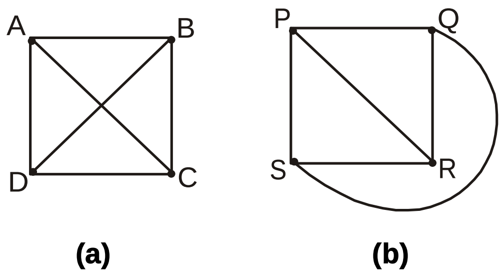


Fig.:

Consider the following one-to-one correspondence between the vertices of these two graphs

A $\leftrightarrow$ P, B $\leftrightarrow$ Q, C $\leftrightarrow$ R, D $\leftrightarrow$ S.

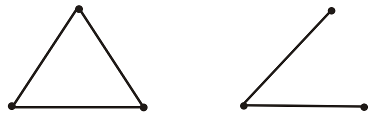
Under this correspondence, the edges in the two graphs correspond with each other: as indicated below:

{A,B) $\leftrightarrow$ {P, Q}, {A, C} $\leftrightarrow$ {P, R}, {A, D} $\leftrightarrow$ {P, S},

{B, C} $\leftrightarrow$ {Q, R}, {B, D} $\leftrightarrow$ {Q, S}, {C, D} $\leftrightarrow$ {R, S}

We check that the above-indicated one-to-one correspondence between the vertices/edges of the two graphs preserves the adjacency of the vertices. The existence of this correspondence proves that the two graphs are isomorphic. (Note that both the graphs represent the complete graph K_4.)

Next, consider the graphs shown in Figures.



(a) **(b)**

Fig.:

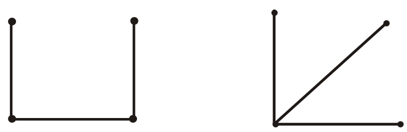
We observe that the two graphs have the same number of vertices but different number of edges. Therefore, although there can exist one-to-one correspondence between the vertices, there cannot be a one-to-one correspondence between the edges. The two graphs are therefore not isomorphic.

From the definition of isomorphism of graphs, it follows that if two graphs are isomorphic, then they must have:

1. The same number of vertices.
2. The same number of edges.
3. An equal number of vertices with a given degree.

These conditions are necessary but not sufficient. This means that two graphs for which these conditions hold need not be isomorphic.

In particular, two graphs of the same order and the same size need not be isomorphic. To see this, consider the graphs shown in Figures.



(a) **(b)**

Fig.:

We note that both graphs are of order 4 and size 3. But the two graphs are not isomorphic. Observe that there are two pendant vertices in the first graph whereas there are three pendant vertices in the second graph. As such, under any one-to-one correspondence between the vertices and the edges of the two graphs, the adjacency of vertices is not preserved.

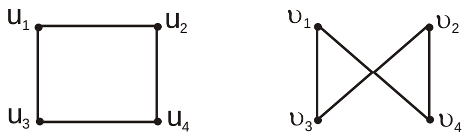
It is not hard to realize that every two complete graphs with the same number of vertices, n, are isomorphic. For this reason, we speak of the complete graph of n vertices, and all complete graphs with n vertices are denoted by K_n.

Similarly, any two complete bipartite graphs with bipartites containing r and .v vertices are isomorphic. For this reason, all complete bipartite graphs with bipartites containing r and s vertices are denoted by $K_{r,s}$. Given two graphs G and G', there is no set procedure for proving or disproving that they are isomorphic.

It is only by carefully examining the nature of vertices and edges of both G and G' that one can find whether or not they are isomorphic. If G and G' are not isomorphic, it is relatively easy to find it out. If G and G’ are isomorphic, the work involved in proving it is quite hard — it gets harder as the orders and sizes of G and G’ get larger.

PROBLEMS

18. Prove that the two graphs shown below are isomorphic.



(a) **(b)**

Fig.:

Sol :

We first observe that both graphs have four vertices and four edges. Consider the following one-to-one correspondence between the vertices of the graphs:

$$u_1 \leftrightarrow v_1, \quad u_2 \leftrightarrow v_4, \quad u_3 \leftrightarrow v_3, \quad u_4 \leftrightarrow v_2,$$

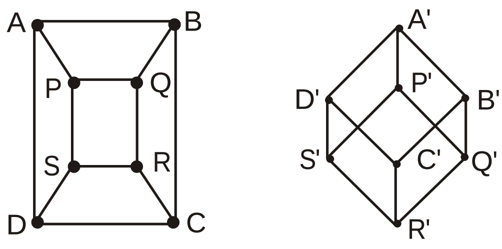
This correspondence gives the following correspondence between the edges:

$$\{u_1 \leftrightarrow u_2\} \leftrightarrow \{v_1, v_4\}, \quad \{u_1 \leftrightarrow u_3\} \leftrightarrow \{v_1, v_3\}, \quad \{u_2 \leftrightarrow u_4\} \leftrightarrow \{v_4, v_2\}, \quad \{u_3 \leftrightarrow u_4\} \leftrightarrow \{v_3, v_2\}$$

These represent one-to-one correspondence between the edges of the two graphs under which the adjacent vertices in the first graph correspond to adjacent vertices in the second graph and vice-versa.

Accordingly, the two graphs are isomorphic.

19. Verify that the two graphs shown in figure are isomorphic.



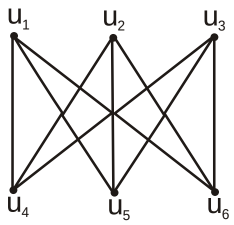
(a) **(b)**

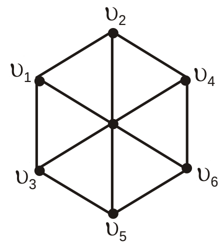
Sol :

Let us consider the one-to-one correspondence between the vertices of the two graphs under which the vertices A, B, C, D, P, Q, R, S of the first graph correspond to the vertices A', B', C', D', P', Q', R', S' respectively, and vice-versa. In this correspondence, the edges determined by the corresponding vertices correspond so that the adjacency of vertices is retained. As such, the two graphs are isomorphic.

We note that the first graph is the graph Q_3 . The second graph is just another drawing of Q_3.

20. Show that the following two graphs are isomorphic:





(a) **(b)**

Sol :

We first note that both the graphs have six vertices each of degree three, and nine edges.

Bearing the edges in the two graphs in mind, consider the correspondence between the edges as shown below:

$\{u_1, u_4\} \leftrightarrow \{v_1, v_2),$ $\{u_1, u_5\} \leftrightarrow \{v_1, v_3\},$ $\{u_1, u_6\} \leftrightarrow \{v_1, v_6\}$

$\{u_2, u_5\} \leftrightarrow \{v_4, v_3),$ $\{u_2, u_4\} \leftrightarrow \{v_4, v_2\},$ $\{u_2, u_6\} \leftrightarrow \{v_4, v_6\}$

$\{u_3, u_6\} \leftrightarrow \{v_5, v_6),$ $\{u_3, u_4\} \leftrightarrow \{v_5, v_2\},$ $\{u_3, u_5\} \leftrightarrow \{v_5, v_3\}$

These yield the following correspondence between the vertices:

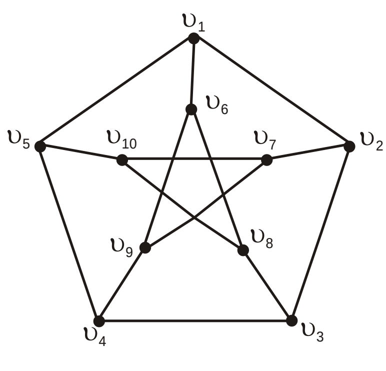
$u_1 \leftrightarrow v_1,$ $u_2 \leftrightarrow v_4,$ $u_3 \leftrightarrow v_5$

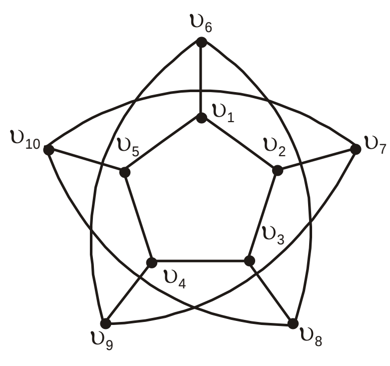
$u_4 \leftrightarrow v_2,$ $u_5 \leftrightarrow v_3,$ $u_6 \leftrightarrow v_6$

We observe that the above correspondences between the edges and the vertices are one-to-one correspondences and that these preserve the adjacency of vertices. In view of the existence of these correspondences, we infer that the two graphs are isomorphic.

We note that the first graph is the complete bipartite graph $K_{3.3}$. The second graph is just another drawing of $K_{3.3}$.

21. Show that the following two graphs are isomorphic.





(a) **(b)**

Sol :

We first note that each of the two graphs is 3-regular (cubic) and has 10 vertices. Consider the correspondence between the vertices as shown below:

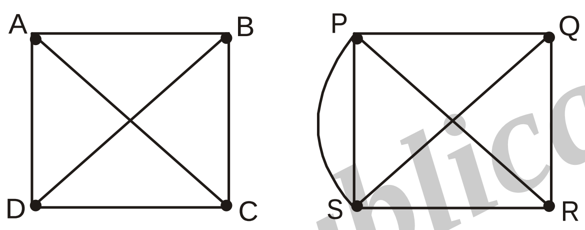
$$v_i \leftrightarrow u_i \text{, for } i = 1, 2, 3, ..., 10$$

This correspondence has been arrived at after closely examining the structures of the two graphs.

We check that the above mentioned correspondence yields one-to-one correspondence between die edges in the two graphs with the property that adjacent vertices in the first graph correspond to the adjacent vertices in the second graph and vice-versa. The two graphs are therefore isomorphic.

We note that the first graph is the Petersen graph: see Figure. The second graph is just another drawing of the Petersen graph.

22. Show that the following graphs are not isomorphic.

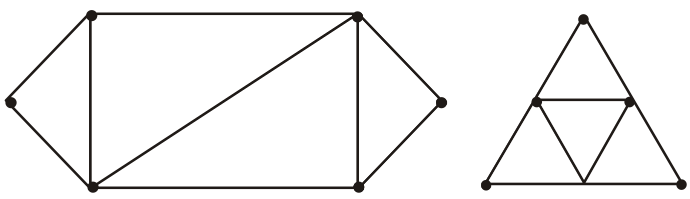


(a) **(b)**

Sol :

We observe that the first graph has 4 vertices and 6 edges and the second graph has 4 vertices and 7 edges. As such, one-to-one correspondence between the edges is not possible. Hence the two graphs are not isomorphic.

23. Show that the following graphs are not isomorphic.



(a) **(b)**

Sol :

We note that each of the two graphs has 6 vertices and 9 edges. But, the first graph has 2 vertices of degree 4 whereas the second graph has 3 vertices of degree 4. Therefore, there can-not be any one-to-one correspondence between the vertices and between the edges of the two graphs which preserves the adjacency of vertices. As such, the two graphs are *not* isomorphic.

5.3 Eulerian Circuit

Q24. Discuss about Eulerian Circuit.

Ans : **(Imp.)**

Consider a connected graph G. If there is a circuit in G that contains all the edges of G, then that circuit is called an Euler circuit (or Eulerian line, or Euler tour) in G. If there is a trail in G that contains all the edges of G, then that trail is called an Euler trail (or unicursal line) in G.

In a trail and a circuit no edge can appear more than once but a vertex can appear more than once. This property is carried to Euler trails and Euler circuits also.

Since Euler circuits and Euler trails include all the edges, they automatically should include all vertices as well.

A connected graph that contains an Euler circuit is called an Euler graph or Eulerian graph. A connected graph that contains an Euler trail is called a semi-Euler graph (or a semi- Eulerian graph or unicursal graph).

For example, in the graph shown in Figure., the closed walk

$$Pe_1\, Qe_2\, Re_3\, Pe_4\, Se_5\, Re_6\, Te_7 P$$

is an Euler circuit. Therefore, this graph is an Euler graph.

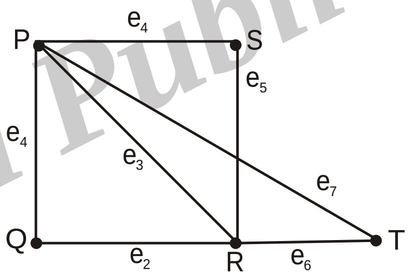


Fig.:

Consider the graph shown in Figure 9.113. We observe that, in this graph, every sequence of edges which starts and ends with the same vertex and which includes all edges will contain at least one repeated edge. Thus, the graph has no Euler circuits. Hence this graph is not an Euler graph.

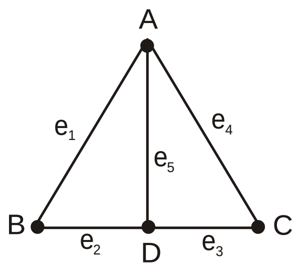


Fig.:

It may be seen that the trail $Ae_1Be_2De_3Ce_4Ae_5D$ in the graph in Figure. is an Euler trail. This graph is therefore a semi-Euler graph.

25. A connected graph G has an Euler circuit (that is, G is an Euler graph) if and only if all vertices of G are of even degree.

Proof :

First, suppose that G has an Euler circuit. While tracing this circuit we observe that every time the circuit meets a vertex v it goes through two edges incident on v (— with the one through which we enter v and the other through which we depart from v). This is true for all vertices that belong to the circuit. Since the circuit contains all edges, it meets all the vertices at least once. Therefore, the degree of every vertex is a multiple, of two (i.e., every vertex is of even degree).

Conversely, suppose that all the vertices of G are of even degree. Now, we construct a circuit starting at an arbitrary vertex v and going through the edges of G such that no-edge is traced more than once. Since every vertex is of even degree, we can depart from every vertex we enter, and the tracing cannot stop at any vertex other than v. In this way, we obtain a circuit q having v as the initial and final vertex. If this circuit contains all the edges in G, then the circuit is an Euler circuit. If not, let us consider the subgraph H obtained by removing from G all edges that belong to q. The degrees of vertices in this subgraph are also even. Since G is connected, the circuit q and the subgraph H must have at least one vertex, say v', in common. Starting from v', we can construct a circuit q' in H as was done in G. The two circuits q and q' together constitute a circuit which starts and ends at the vertex v and has more edges than q. If this circuit contains all the edges in G, then the circuit is an Euler circuit. Otherwise, we repeat the process until we get a circuit that starts from v and ends at v and which contains all edges in G. In this way, we obtain an Euler circuit in G.

This completes the proof of the theorem.

26. A connected graph G has an Euler circuit (that is, G is an Euler graph) if and only if G can be decomposed into edge-disjoint cycles.

Proof :

First, suppose that G can be decomposed (partitioned) into edge-disjoint cycles. Since the degree of every vertex in a cycle is two, it follows that every vertex in G is of even degree. Therefore, by Theorem 1, G has an Euler circuit.

Conversely, suppose G has an Euler circuit Then, by Theorem 1, every vertex in G is of even degree. Now, consider a vertex v_1 in G. Since v_1 is of even degree, there are at least two edges incident on v_1. Choose one of these edges, and let v_2 be the other end vertex of this (chosen) edge.

Then is also of even degree, and therefore there must be at least one other edge incident on v_2. Choose one of such edges, and let v_3 be the other end vertex of the edge. Proceeding like this, we eventually arrive at a vertex that has previously been traversed, thus forming a cycle C_1. Let us remove C_1 from G. All vertices in the resulting graph must also be of even degree, and in this graph we can construct a cycle C_1 as was done in G. Remove this cycle C_2 and proceed as above. The process ends when no edges are left, in this way we get a sequence of cycles whose union is G and whose intersection is a null graph. Thus, G has been decomposed into edge-disjoint cycles.

5.4 Hamiltonian Walks

Q27. Write about Hamiltonian Walks.

Ans : **(Imp.)**

Hamilton cycles and Hamilton paths

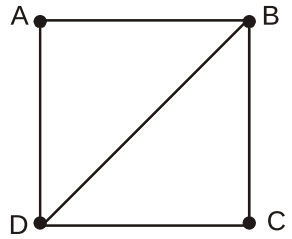
Let G be a connected graph. If there is a cycle in G that contains all the vertices of G, then that cycle is called a Hamilton cycle in G.

A Hamilton cycle (when it exists) in a graph of n vertices consists of exactly n edges. Because, a cycle with n vertices has n edges.

By definition, a Hamilton cycle (when it exists) in a graph G must include all vertices in This does not mean that it should include all edges of G.

A graph that contains a Hamilton cycle is called a Hamilton graph (or Haniiln graph).

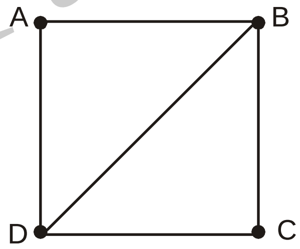
For example, in the graph shown in Figure, the cycle shown in thick lines is a Hamilton cycle. (Observe that this cycle does not include the edge BD). The graph is therefore a Hamilton graph.



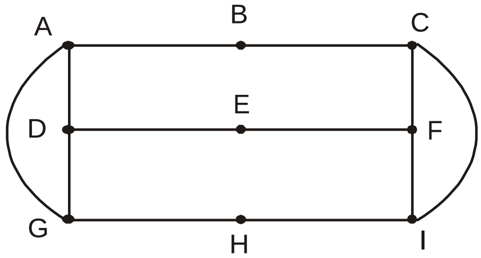
It is easy to see that in the hyper-cube Q_3 shown in Figure, the cycle ABCDS RQPA is a Hamilton cycle. (Observe that this cycle does not include all the edges). Therefore, a Hamilton graph.

A path (if any) in a connected graph which includes every vertex (but not necessarily edge) of the graph is called a Hamilton/Hamiltonian path in the graph.

For example, in the graph shown in Figure, the path shown in thick lines is a Hamilton path.



In the graph shown in Figure 9.122, the path ABCFEDGHI is a Hamilton path. We check feat this graph does not contain a Hamilton cycle.



Since a Hamilton path in a graph G meets every vertex of G, the length of a Hamilton path (if any) in a connected graph of n vertices is n – 1. (Recall that a path with n vertices has n – 1 edges).

PROBLEMS

28. Prove that the complete graph K_n, $n \geq 3$, is a Hamilton graph.

Sol :

In K_n, the degree of every vertex is n –1. If $n \geq 3$, we have n – 2 > 0, or 2n – 2 > n, or (n – 1)> n/2.

Thus, in K_n, where $n \geq 3$, the degree of every vertex is greater than n/2.

---

29. Show that every simple k-regular graph with 2k – 1 vertices is Hamiltonian.

Sol :

In a k-regular graph, the degree of every vertex is k, and

$$k > k - \frac{1}{2} = \frac{1}{2}(2k - 1) = \frac{1}{2}$$

Where n = 2k – 1 is the number of vertices.

---

30. Let G be a simple graph with n vertices and m edges where m is at least 3. If $m \geq \frac{1}{2}(n-1)(n-2)+2$, prove that G is Hamiltonian. Is the converse true?

Sol :

Let u and v be any two non-adjacent vertices in G. Let x and y be their respective degrees. If we delete u, v from G, we get a subgraph with n – 2 vertices. If this subgraph has q edges, then $q \leq \frac{1}{2}(n-2)(n-3)$.

Since u and v are nonadjacent, m = q + x + y. Thus,

$$x + y = m - q \geq \left\{\frac{1}{2}(n-1)(n-2)+2\right\} - \left\{\frac{1}{2}(n-2)(n-3)\right\}$$

$$= n$$

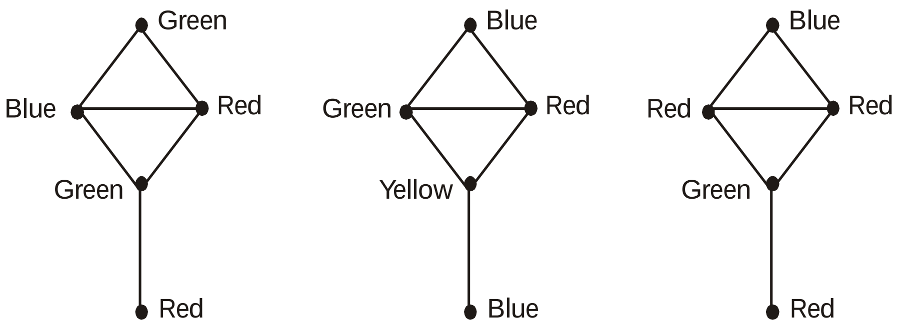
5.5 Graph Colouring

Q31. Discuss about Graph Colouring.

Ans : **(Imp.)**

Graph Coloring

Given a planar or non-planar graph G, if we assign colors (colours) to its vertices in such a way that no two adjacent vertices have (receive) the same color, then we say that the graph G is properly colored. In other words, proper coloring of a graph means assigning colors to its vertices such that adjacent vertices have different colors.



In Figure, the first two graphs are properly colored whereas the third graph is not poperly colored.

By examining the first two graphs in Figure 10.23 which are properly colored, we note the following:

1. A graph can have more than one proper coloring.
2. Two non-adjacent vertices in a properly colored graph can have the same color.

Chromatic number

A graph G is said to be k-colorable if we can properly color it with k (number of) colors.

A graph G which is k-colorable but not (k – 1) - colorable is called a k-chromatic graph.

In other words, a k - chromatic graph is a graph that can be properly colored with k colors but not with less than k colors.

If a graph G is k-chromatic, then k is called the chromatic number of G.

Thus, the chromatic number of a graph is the minimum number of colors with which the graph can be properly colored.

The chromatic number of a graph G is usually denoted by X(G).

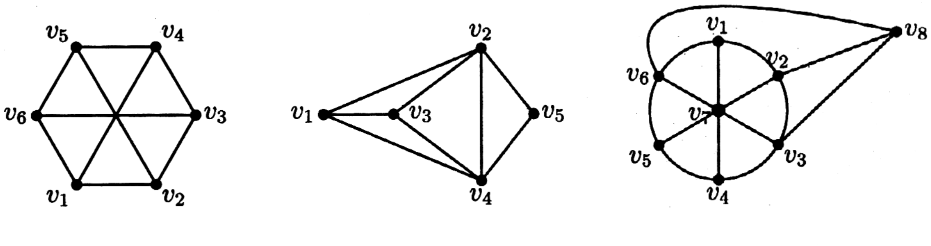
Some Results

The following results are direct consequences of the definition of the chromatic number.

1. A graph consisting of only isolated vertices (i.e. Null graph) is 1-chromatic. (Because no two vertices of such a graph are adjacent and therefore we can assign the same color to all vertices).
2. A graph with one or more edges is at least 2-chromatic. (Because, such a graph has at least one pair of adjacent vertices which should have different colors).
3. If a graph G contains a graph G_1 as a subgraph, then $X(G) \geq X(G_1)$
4. If G is a graph of n vertices, then $x(G) \leq n$.
5. $X(K_n) = n$ for all $n \geq 1$. (Because, in K_n, every two vertices are adjacent, and as such all the n vertices should have different colors).
6. If a graph G contains K_n as a subgraph, then $X(G) \geq n$.

PROBLEMS

32. Find the chromatic number of each of the following graphs.



(a) (b) (c)

Sol :

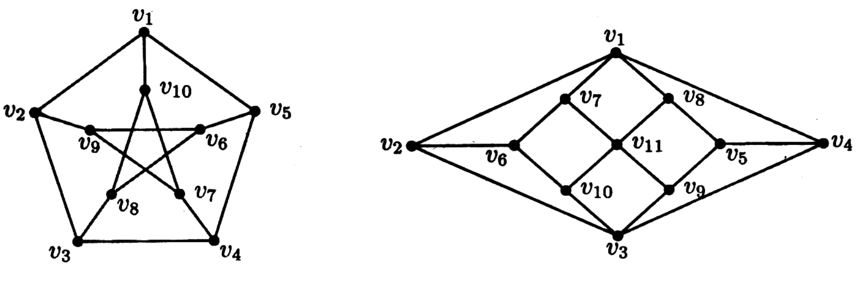
(i) For the graph (a), let us assign a color a to the vertex v_1. Then, for a proper coloring, we have to assign a different color to its neighbors v_2, v_4, v_6. Since v_2, v_4, v_6 are mutually, nonadjacent vertices, they can have the same color, say β (which is different from α). Since v_3,v_5 are not adjacent to v_1, these can have the same color as v_1, namely α .

Thus, the graph can be property colored with at least two colors, with the vertices v_1,v_3, v_5 having one color α , and v_2, v_4, v_6 having a different color β. Hence, the chromatic number of the graph is 2.

(ii) For the graph (b), let us assign the color $\propto$ to the vertex v_1. Then, for a proper coloring, its neighbors v_2,v_3 and v_4 cannot have the color α , but v_5 can have the color α . Furthermore, v_2, v_3, v_4 must have different colors, say β,γ,δ Thus, at least four colors are required for a proper coloring of the graph. Hence the chromatic number of the graph is 4.

(iii) For the graph (c), we can assign the same color, say α , to the non-adjacent vertices v_1, v_3, v_5. Then the vertices v_2, v_4, v_6 can be assigned the same color other than α . Suppose we assign a color, β to v_2, v_4, v_6. Consequently v_7 and v_8 can be assigned the same color which is different from both a and β. Thus, a minimum of three colors are needed for a proper coloring of the graph. Hence its chromatic number is 3.

33. Find the chromatic numbers of the following graphs :



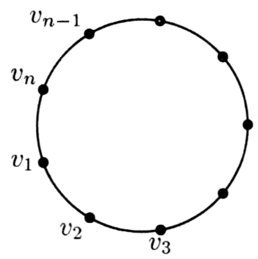
(a) (b)

Sol :

i) We note that the graph (a) is the Petersen graph. By observing the graph, we note that the vertices v_1, v_3, v_6 and v_7 can be assigned the same color, say α . Then the vertices v_2, v_4, v_8 and v_{10} can be assigned the same color, β (other than α). Now, the vertices v_5 and v_9 have to be assigned colors other than α and β; they can have the same color γ. Thus, a minimum of three colors are required for a proper coloring this graph. Hence, the chromatic number of this graph is 3.

ii) By observing the graph (b) (– this graph is called the Herschel graph), we note that the vertices v_1, v_3, v_5, v_6 and v_{11} can be assigned the same color α and all the remaining vertices: v_2, v_4, v_7, v_8, v_9 and v_{10} can be assigned the same color β (other than α). Thus, two colors are sufficient (-one color is not sufficient) for proper coloring of the graph. Hence its chromatic number is 2.

34. Prove that a graph of order n($\geq$ 2) consisting of a single cycle is 2-chromatic if n is even, and 3-chromatic if n is odd.



Sol :

Obviously, the graph cannot be properly colored with a single color. Assign two colors alternatively to the vertices, starting with v_1. That is, the odd vertices, v_1, v_3, v_5 etc., will have a color α and the even vertices, v_2, v_4, v_6 etc., will have a different color β.

Suppose n is even. Then the vertex v_n is an even vertex and therefore will have the color β, and the graph gets properly colored. Therefore. the graph is 2-chromatic.

Suppose n is odd. Then the vertex v_n is an odd vertex and therefore will have the color α, and the graph is not properly colored. To make it properly colored, it is enough if v_n is assigned a third color γ. Thus, in this case, the graph is 3-chromatic.

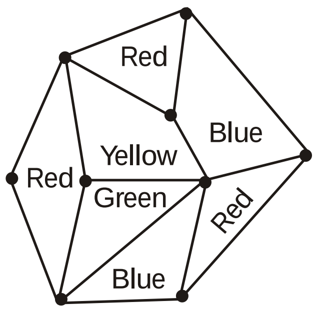
5.6 Colouring Maps and Planar Graphs

Q35. Discuss about map colouring.

Ans :

A plane representation (drawing) of a planar graph divides a plane into a number of parts called regions (or faces) of which only one is exterior. We say that . these regions are properly colored if no two adjacent regions have the same color. By adjacent regions we mean regions which have a common edge between them. Two regions having one or more common vertices are not regarded as adjacent regions. A proper coloring of regions is called map coloring in view of the fact that an atlas is always colored in such a way that countries with common boundaries have different colors.

The following Figure illustrates a proper coloring of regions of a planar graph.



From the way the dual of a planar graph is defined (constructed), it follows that a proper coloring of the regions of a planar graph G is equivalent to a proper coloring of the vertices of its dual G*, and vice-versa. Thus, the problem of map coloring reduces to the problem of vertex coloring of planar graphs.

Q36. Analyze the proof of five colour theorem.

Ans :

Statement

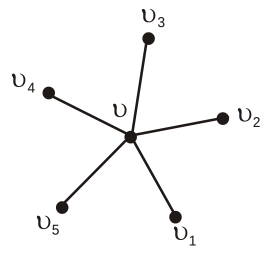
The vertices of every connected simple planar graph can he properly colored with five colors.

Proof:

Let n be the number of vertices in a connected, simple planar graph. If $n \leq 5$. then the theorem is trivially true. Assume that the theorem is true for all graphs with $n \leq k$. Consider a graph G with k + 1 vertices. Then, by virtue of Euler's theorem, G contains a vertex v of degree at most 5[§§]. If we consider the graph H = G – v, obtained by deleting v from G. then H has k vertices. Therefore, by the assumption made, H is 5-colorable.

Since the degree of v is at most 5, v has at most 5 neighbors in G. Suppose v has 4 or less number of neighbors. Then the neighbors can be colored with at most four different colors and v can be colored with the fifth color, all drawn from the colors used in H. Thus, a proper coloring of G can be done by using the five colors with which H can be colored. Thus, G is 5-colorable.

Next, suppose that v has 5 neighbors, say v_1, v_2, v_3, v_4, v_5. Let us arrange them around v in anti-clockwise order as in Figure. If the vertices v_1, v_2, v_3 ...., v_5 are all mutually adjacent, then they constitute which is non-planar. This is not possible, because, being a planar graph. G cannot contain a non-planar graph as a subgraph. Therefore, at least two of v_i, v_2,..., V5, say v_1 and v_3, are non-adjacent.



Now, construct a graph G' by merging the edges v_3 v and vv_1. The graph G' will have 1+1) – 2 = k – 1 vertices (with v_3, vv_1 as the merged vertex). This graph is, therefore, S-colorable. Let us assign a color α_1 to the merged vertex v_3 vv_1, a color α_2 to α_4, a color α_5 to α_4 and a color α_5 to v_5. With this scheme of coloring of v_1, v_2, v_3, v_4, v_5 and with the use of just one more color α_3 assigned to other appropriate vertices, the graph G' gets properly colored, low, unravel the merged vertex v_3 vv_1 and assign the color a to both v_3 and v_1 and the color v_3 to v, without disturbing the colors of other vertices. This will produce a proper coloring of with colors α_1, α_2, α_3, α_4, α_5. Thus, G is 5-colorable in this case also (where the degree of is 5).

We have proved that a graph with n = k + 1 vertices is 5-colorable if a graph with $n \leq k$ Vertices is 5-colorable. Hence, by induction, it follows that a graph with n vertices, where n is my positive integer, is 5-colorable.

This completes the proof of the theorem.

5.7 Colouring Vertices, Colouring Edges

Q37. Define the following terms :

(a) Colouring Vertices

(b) Colouring Edges

Ans :

(a) Colouring Vertices

In general, given any graph G, a coloring of the vertices is called (not surprisingly) a vertex coloring. If the vertex coloring has the property that adjacent vertices are colored differently, then the coloring is called proper. Every graph has a proper vertex coloring. For example, you could color every vertex with a different color. But often you can do better. The smallest number of colors needed to get a proper vertex coloring is called the chromatic number of the graph, written x(G).

(b) Colouring Edges

The chromatic number of a graph tells us about coloring vertices, but we could also ask about colouring edges. Just like with vertex coloring, we might insist that edges that are adjacent must be coloured differently. Here we are thinking of two edges as being adjacent if they are incident to the same vertex. The least number of colours required to properly colour the edges of a graph G is called the Chromatic Index of G, written x'(G).

5.7.1 Perfect Graph

Q38. Define about perfect graph.

Ans :

To define perfect graphs first we need to review several graph parameters. Given a graph G = (V, E), X(G) denotes the minimum number of colors required to properly color all vertices of G and ω(G) denotes the size of the largest clique in G.Since each vertex of a clique should get a distinct color, X(G) $\geq$ ω(G). In this lecture we consider a family of graphs in which the inequality is tight.

5.8 Trees

5.8.1 Definition, Properties, Examples

Q39. Define tree. Discuss various properties of trees.

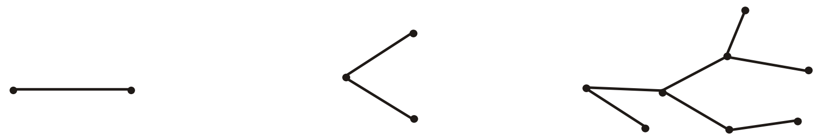
Ans :

A graph G is called a tree if it is connected and has no cycles.

It immediately follows that a tree has tc be a simple graph; because loops and parallel edges form cycles.

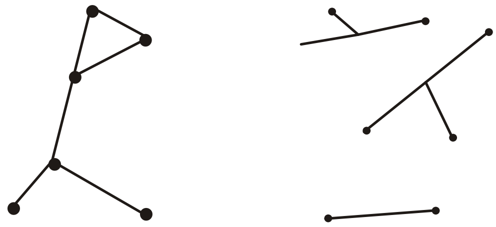
The graphs shown in Figure are all trees. We observe that each of these trees possesses at least two pendant vertices.

A pendant vertex of a tree is also called a leaf.



(a) (b) (c)

Fig.:



(a) (b)

Fig.:

The graphs shown in Figure are no, trees. Observe ,ha, ,he firs, of these contains a cycle whereas the second is not connected. However, each component of the second (disconnected) graph is a tree. Such a graph is called a forest.

A graph which is a tree is usually denoted by T (instead of G) to emphasize the structure.

The following theorems contain some basic properties of trees.

40. In a tree, there is one and only one path between every pair of vertices.

Proof:

Let T be a tree. Then T is a connected simple graph. Since T is connected, there must be at least one path between every two vertices. If there are two paths between a pair of vertices of T, the union of the paths will become a cycle, and T cannot be a tree. Thus, between every pair of vertices in a tree there must exist one and only one path.

41. If in a graph G there is one and only one path between every pair of vertices, then G is a tree.

Proof:

Since there is a path between every pair of vertices in G, it is obvious that G is connected. Since there is only one path between every pair of vertices, G cannot have a cycle. Because, if there is a cycle, then there exist two paths between two vertices on the cycle. Thus, G is a connected graph containing no cycles. This means that G is a tree.

The above two theorems may be combined together and put in the following form:

A graph G is a tree if and only if there is one and only one path between every pair of vertices in G.

42. A tree with n vertices has n – 1 edges

Proof:

We prove the theorem by induction on n.

The theorem is obvious for n =1, n = 2 and n – 3; see the first three trees in Figure. Assume that the theorem holds for all trees with fewer than k vertices, where k is a specified positive integer.

Consider a tree T with k vertices. In T, let e be an edge with end vertices u and v. Since T is a tree, there exists no other edge or path between u and v. Therefore, deletion of e from T will disconnect the graph and T – e consists of exactly two components, say T_2 and T_3. Since T does not contain any cycle, the components T_2 and T_3 too do not contain any cycles. Hence, T_1 and T_2 are trees in their own right. Both of these trees have fewer than k vertices each, and therefore, according to the assumption made, the theorem holds for these trees; that is, each of T_1 and T_2 contains one less edge than the number of vertices in it. Therefore, since the total number of vertices in T_1 and T_3 (taken together) is k, the total number of edges in T_1 and T_2. (taken together) is k – 2. But T_1 and T_2 taken together is T – e. Thus, T – e consists of k – 2 edges. Consequently, T has exactly k – 1 edges.

Thus, if the theorem is true for a tree with n < k vertices, it is true for a tree with n = k vertices. Hence, by induction, the theorem is time for all positive integers n.

43. Any connected graph with n vertices and n – J edges is a tree.

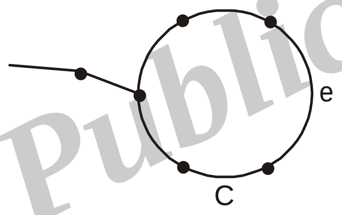


Fig.:

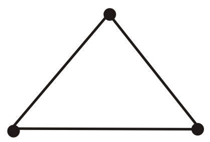
Proof:

Let G be a connected graph with n vertices and n – 1 edges. Assume that G is not a tree. Then G contains a cycle, say C. Let e be an edge in C. The graph G will not become disconnected if e is deleted. Thus, G – e is a connected graph. But, on the other hand, G – e has n vertices and n – 2 edges; therefore, it cannot be connected. This is a contradiction. Hence, G must not have a cycle; this means that G must be a tree.

This completes the proof of the theorem.

Remark:

A graph with n vertices and n – 1 edges need not be a tree: see the graph shown below (which is disconnected and has 4 vertices and 3 edges).



A graph with n vertices is a tree if and only if it is connected and has n – 1 edges.

44. A connected graph G is a tree if and only if adding an edge between any two vertices in G creates exactly one cycle in G.

Proof:

Suppose a connected graph G is a tree. Then G has no cycles and there is exactly one path between any two vertices, u, v. If we add an edge between u and v, then an additional path is created between u and v and the two paths constitute a cycle. Since G had no cycles earlier, this is the only cycle which G now possesses.

Conversely, suppose G is connected and adding an edge between any two vertices u and v in G creates exactly one cycle in G. This implies that, before adding this edge, exactly one path was there between u and v. This implies that G is a tree.

---

45. A connected graph is a tree if and only if it is minimally connected.

Proof:

Suppose G is a connected graph which is not a tree. Then G contains a cycle C. The I removal of any one edge e from this cycle will not make the graph disconnected. Therefore. G is not minimally connected. Thus, if a connected graph is not a tree then it is not minimally connected. This is equivalent to saying that if a connected graph is minimally connected then it is a tree (contrapositive).

Conversely, suppose G is a connected graph which is not minimally connected. Then there exists an edge e in G such that G – e is connected. Therefore, e must be in some cycle in G. This implies that G is not a tree. Thus, if a connected graph is not minimally connected then it is not a tree. This is equivalent to saying that if a connected graph is a tree, then it is minimally connected (contrapositive).

PROBLEMS

46. (a) Show that the complete graph, K_n is not a tree when $n > 2$.

(b) Show that the complete bipartite graph $K_{r,s}$ is not a tree when $r \geq 2$.

Sol :

(a) If v_1, v_2, v_3 are any three vertices of K_n, $n > 2$, then the closed walk $v_1\, v_2\, v_3\, v_1$ is a cycle in K_n. Since K_n has a cycle, it cannot be a tree.

(b) Let v_1 and v_2 be any two vertices in the first bipartite and v'_1, v'_2 be any two vertices in the other bipartite of $K_{r,s}$, $s \geq r > 1$. Then, the closed walk $v_1\, v'_1\, v_2\, v'_2\, v_1$ is a cycle in $K_{r,s}$. Since $K_{r,s}$ has a cycle, it cannot be a tree.

---

47. Prove that a graph with a vertices, n – 1 edges and no cycles is connected.

Sol :

Consider a graph G which has n vertices, n – 1 edges and no cycles. Suppose G is not connected. Let the components of G be H_i, $i = 1, 2, \ldots k$. If H_i has n_i vertices, we have $n_1 + n_2 + \ldots + n_k = n$. Since G has no cycles, H_is also do not have cycles. Further, they are all connected graphs. Therefore, they are trees. Consequently, each H_i must have $n_i - 1$ edges. Therefore, the total number of edges in these H_is is

$$(n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1) = n - k.$$

This must be equal to the total number of edges in G; that is $n - k = n - 1$. This is not possible, since $k > 1$. Therefore, G must be connected.

48. Let F be a forest with k components (trees). If n is the number of vertices and m is the number of edges in F, prove that n = m + k.

Sol :

Let $H_1, H_2, ..., H_k$ be the components of F. Since each of these is a tree, if n_i is the number of vertices in H_i and m_i is the number of edges in H_i, we have

$$m_i = n_i - 1, \quad \text{or} \quad i = 1, 2, ..l., k.$$

This gives

$$m_1 + m_2 + ... + m_k = (n_1 - 1) + (n_2 + 1) + ... + (n_k - 1) = n_1 + n_2 + ... + n_k - k.$$

But

$$m_1 + m_2 + ... + m_k = m \quad \text{and} \quad n_1 + n_2 + ... + n_k = n.$$

Therefore,

$$m = n - k, \text{ or } n = m + k.$$

49. Prove that, in a tree with two or more vertices, there are at least two leaves (pendant vertices).

Sol :

Consider a tree T with n vertices, $n \geq 2$. Then it has $n - 1$ edges. Therefore, the sum of the degrees of the n vertices must be equal to $2(n - 1)$. Thus, if $d_1, d_2, ..., d_n$ are the degrees of vertices of T, we have

$$d_1 + d_2 + ... + d_n = 2(n - 1) = 2n - 2.$$

If each of $d_1, d_2, ..., d_n$ is ≥ 2, then their sum must be at least 2n. Since this is not true, at least once of the d's is less than 2. Thus, there is a d which is equal to 1. (Since T is connected, no d can be zero). Without loss of generality, let us take this to be d_1. Then

$$d_2 + d_3 + ... + d_n = (2n - 2) - 1 = 2n - 3.$$

This is possible only if at least one of $d_2, d_3, ..., d_n$ is equal to 1. So, there is at least one more d which is equal to 1. Thus, in T, there are at least two vertices with degree 1; that is, there are at least two pendant vertices (leaves).

50. Show that if a tree has exactly two pendant vertices, the degree of every other vertex is two.

Sol :

Let n be the number of vertices in a trees T. Suppose it has exactly two pendant vertices (so that their degrees are 1 each). Let $d_1, d_2, ..., d_{n-2}$ be the degrees of the other vertices. Then, since T has exactly $n - 1$ edges, we have

$$1 + 1 + d_1 + d_2 + ... + d_{n-2} = 2(n - 1)$$

$$d_1 + d_2 + ... + d_{n-2} = 2n - 4 = 2(n - 2)$$

The left hand side of this condition has $n - 2$ terms, and none of these is one or zero. Therefore, this condition holds only if each of the d_is is equal to two.

51. Show that, in a tree, if the degree of every non-pendant vertex is 3, the number of vertices in the tree is even.

Sol :

Let n be the number of vertices in a tree T. Of these, let ke be the number of pendant vertices. Then, if each non-pendant vertex is of degree 3, the sum of the degree of vertices is k + 3(n – k). This must be equal to 2(n – 1). Thus,

$$k + 3(n - k) = 2(n - 1), \quad \text{or} \quad n = 2(k - 1).$$

This shows that n is even.

52. Suppose that a tree T has two vertices of degree 2, four vertices of degree 3 and three vertices of degree 4. Find the number of pendant vertices in T.

Sol :

Let N be the number of pendant vertices in T. It is given that T has two vertices of degree 2, four vertices of degree 3 and three vertices of degree 4. Therefore,

Total number of vertices = N + 2 + 4 + 3 = N + 9.

Sum of the degrees of vertices = N + (2 × 2) + (4 × 3) + (3 × 4) = N + 28.

Since T has N + 9 vertices, it has N + 9 – 1 = N + 8 edges. Therefore, by handshaking property, we have N + 28 = 2(N + 8) which gives N = 12. Thus, the given tree has 12 pendant vertices.

53. Suppose that a tree T has N_1 vertices of degree 1. N_2 vertices of degree 2, N_3 vertices of degree 3, ..., N_k vertices of degree k. Prove that

$$N_1 = 2 + N_3 + 2N_4 + 3N_5 + ... + (k - 2) N_k.$$

Sol :

From what is given, we note that, in T,

Total number of vertices = $N_1 + N_2 + ... + N_k$, and

Sum of the degrees of vertices = $N_1 + 2N_2 + 3N_3 + ... + kN_k$.

Therefore, the total number of edges in T is $N_1 + N_2 + ... + N_k - 1$, and the handshaking property gives.

$$N_1 + 2N_2 + 3N_3 + 4N_4 + 5N_5 + ... + kN_k = 2(N_1 + N_2 + ... + N_k - 1)$$

Rearranging terms, this gives

$$N_3 + 2N_4 + 3N_5 + ... + (k - 2) N_k = N_1 - 2.$$

This is the required result.

5.9 Rooted Trees

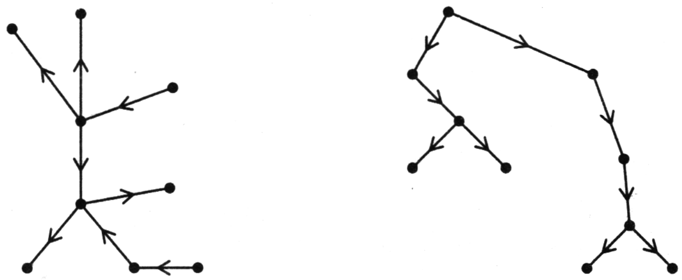
Q54. Explain briefly about Rooted Trees.

Ans :

Let D be a directed graph and G be its underlying graph. We say that D is a directed tree whenever G is a tree. Thus, a directed tree is a directed graph whose underlying graph is a tree.

A directed tree T is called a rooted tree if (i) T contains a unique vertex, called the root, whose in-degree is equal to 0, and (ii) the in-degrees of all other vertices of Tare equal to 1.

Figures 10(a) and 7(b) depict two directed trees. The first of these is not a rooted tree whereas the second is a rooted tree.



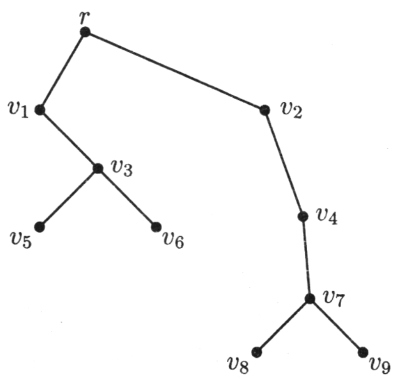
(a) (b)

Figure

In a rooted tree, we denote the root by r and draw the (diagram of the) tree downward from an upper level to a lower level, so that the arrows can be dropped. Then the root r will be at the uppermost level and all other vertices will be at lower levels.

A vertex v of a rooted tree is said to be at the k-th level or has level number k if the path from r to v is of length k. If v_1 and v_2 are two vertices such that v_1 has a lower level number than v_2 and there is a path from v_1 to v_2, then we say that v\ is an ancestor of v_2, or that v_2 is a descendant of v_1. In particular, if v_1 and v_2 are such that v_1 has a lower level number than v_2 and there is an edge (- directed edge, actually) from v_1 to v_2, then v_1 is called the parent of v_2, or v_2 is called the child of v_1. Two vertices with a common parent are referred to as siblings. A vertex whose out-degree is 0 is called a leaf. A vertex which is not a leaf is called an internal vertex.

For example, suppose we redraw the directed tree of Figure 7(a) as shown below without arrows (- which are understood) and with vertices labeled.



Figure

In this rooted tree,

1. v_1 and v_2 are at the first level, v_3, v_4 are at the second level, v_5, v_6, v_7 are at the third level. and v_8 and v_9 are at the fourth level.

2. v_i is the ancestor of v_3, v_5, v_6 (or v_3, v_5, v_6 are the descendants of v_1), and v_2 is the ancestor of v_4, v_7, v_8, v_9 (or v_4, v_7, v_8, v_9 are the descendants of v_2).
3. v_1 is the parent of v_3 (or v_3 is a child of v_1).
4. v_5 and v_6 are siblings, and v_8 and v_9 are siblings.
5. v_5, v_6, v_8, v_9 are leaves, and all other vertices are internal vertices.

5.10 Trees and Sorting

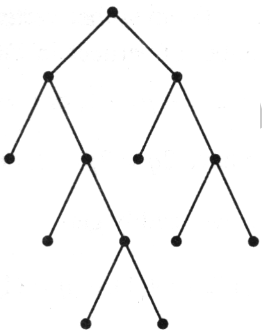
Q55. Define binary tree.

Ans :

A rooted tree T is called a binary rooted tree, or just a binary tree if every internal vertex of T is of out-degree 1 or 2; that is if every vertex has at most two children.

A rooted tree T is called a complete binary tree if every internal vertex of T is of out-degree 2; that is if every internal vertex has two children.

The rooted tree shown in Figure is a binary tree; it is not a complete binary tree. The rooted tree shown in Figure is a complete binary tree.



Figure

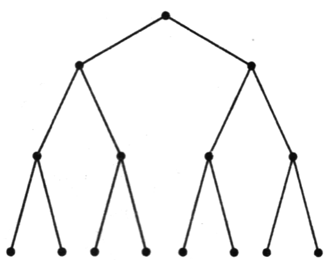
Balanced Tree

If T is a rooted tree and h is the largest level number achieved by a leaf of T, then T is said to have height h. A rooted tree of height h is said to be balanced if the level number of every leaf is h or h – 1.

The tree is of height 4 and is balanced too.

Full Binary Tree

Let T be a complete binary tree of height h. Then T is called a full binary tree if all the leaves in T are at level h.



Q56. What is tree sort? Explain algorithm of tree sort.

Ans :

Tree sort is a sorting algorithm that is based on Binary Search Tree data structure. It first creates a binary search tree from the elements of the input list or array and then performs an in-order traversal on the created binary search tree to get the elements in sorted order.

Algorithm:

Step 1: Take the elements input in an array.

Step 2: Create a Binary search tree by inserting data items from the array into the binary search tree.

Step 3: Perform in-order traversal on the tree to get the elements in sorted order.

5.11 Weighted Trees and Prefix Codes

Q57. Write about Prefix Codes.

Ans :

Weighted Graph

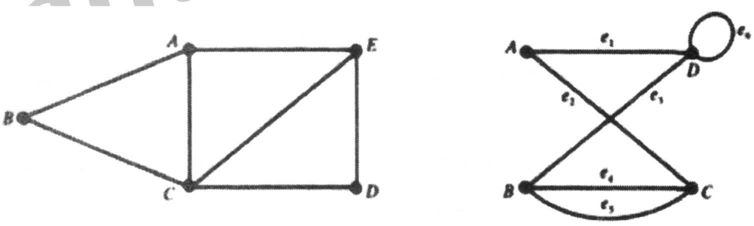
If the weight is assigned to each edge of the graph then it is called as Weighted or Labeled graph.

The definition of a graph may be generalized by permitting the following:

Multiple edges: Distinct edges e and e' are called multiple edges if they connect the same end points, that is, if e = [u, v] and e' = [u, v].

Loops: An edge e is called a loop if it has identical endpoints, that is, if e = [u, u].

Finite Graph: A multi graph M is said to be finite if it has a finite number of nodes and a finite number of edges.



(a) Graph. (b) Multigraph.

Prefix codes

A prefix code tree is a rooted tree such that:

- Each edge is labeled by a bit
- Each leaf denoted by a character.
- The codeword for the character is based on the labels on root-to-leaf path.

$A \to 0$, $B \to 1$, $C \to 00$, $D \to 01$, $E \to 010$

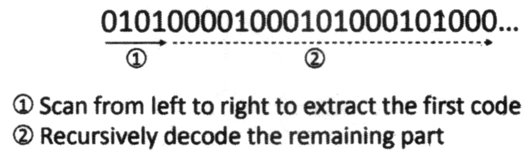
Suppose the encoded text is : 0101

We cannot tell if the original is

ABAB or ABD or DAB or DD or EB The problem comes from

one codeword is a prefix of another

- To avoid the problem, we generally want that each codeword is NOT a prefb
- Such an encoding scheme is called a prefix code, or prefix-free code
- For a text encoded by a prefix code, we can easily decode it in the following



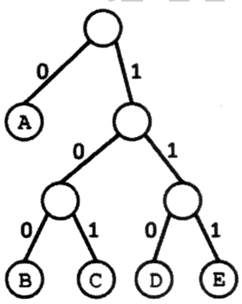
Prefix Code Tree

Naturally, a prefix code scheme corresponds to a prefix code tree

- The tree is a rooted, with

1. each edge is labeled by a bit;
2. each leaf $\rightarrow$ a character;
3. labels on root-to-leaf path $\rightarrow$ codword for the character.

E.g., A $\rightarrow$ 0, B $\rightarrow$ 100, C $\rightarrow$ 101, D $\rightarrow$ 110, E $\rightarrow$ 111



5.12 Bi-connected Component and Articulation Points, Shortest Distances

Q58. Explain about Bi-connected Component and Articulation Points.

Ans : **(Imp.)**

The operations that we have implemented thus far are simple extensions of depth first and breadth first search. The next operation we implement is more complex and requires the introduction of additional terminology. We begin by assuming that G is an undirected connected graph.

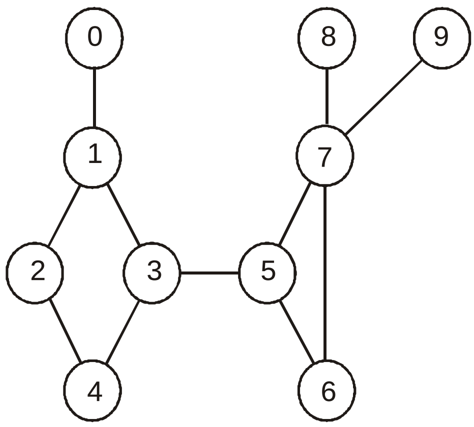
An articulation point is a vertex v of G such that the deletion of v, together with all edges incident on v, produces a graph, G', that has at least two connected components. For example, the connected graph of Figure 1.1 has four articulation points, vertices 1,3,5, and 7.

A biconnected graph is a connected graph that has no articulation points. In many graph applications, articulation points are undesirable. For instance, suppose that the graph of Figure 1.1(a) represents a communication network. In such graphs, the vertices represent communication stations and the edges represent communication links. Now suppose that one of the stations that is an articulation point fails. The result is a loss of communication not just to and from that single station, but also between certain other pairs of stations.

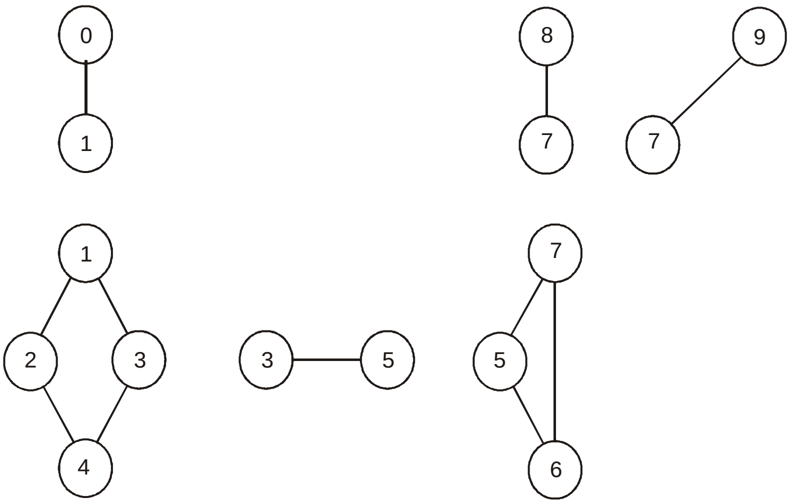
A biconnected component of a connected undirected graph is a maximal biconnected subgraph, H, of G. By maximal, we mean that G contains no other subgraph that is both biconnected and properly contains H. For example, the graph of Figure 1.1(a) contains the six biconnected components shown in Figure 1.1(b). The biconnected graph however, contains just one biconnected component: the whole graph. It is easy to verify that two biconnected components of the same graph have no more than one vertex in common. This means that no edge can be in two or more biconnected components of a graph. Hence, the biconnected components of G partition the edges of G.

We can find the biconnected components of a connected undirected graph, G, by using any depth first spanning tree of G. For example, the function call dfs (3) applied to the graph of Figure 1.1(a) produces the spanning tree of Figure 1.2(a). We have redrawn the tree in Figure 1.2(b) to better reveal its tree structure. The numbers outside the vertices in either figure give the sequence in which the vertices are visited during the depth first search. We call this number the depth first number, or dfn, of the vertex. For example, dfn (3) = 0, dfn (0) = 4, and dfn (9) = 8. Notice that vertex 3, which is an ancestor of both vertices 0 and 9, has a lower dfn than either of these vertices. Generally, if u and v are two vertices, and u is an ancestor of v in the depth first spanning tree, then dfn (u) < dfn (v).

The broken lines in Figure 1.2(b) represent nontree edges. A nontree edge (u, v) is a back edge iff either u is an ancestor of v or v is an ancestor of u. From the definition of depth first search, it follows that all nontree edges are back edges. This means that the root of a depth first spanning tree is an articulation point iff it has at least two children. In addition, any other vertex u is an articulation point iff it has at least one child w such



(a) Connected graph

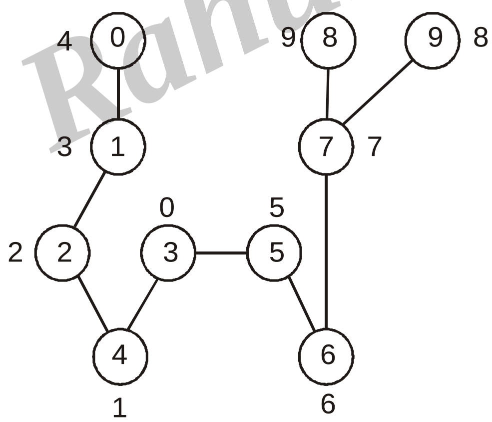


(b) Biconnected component

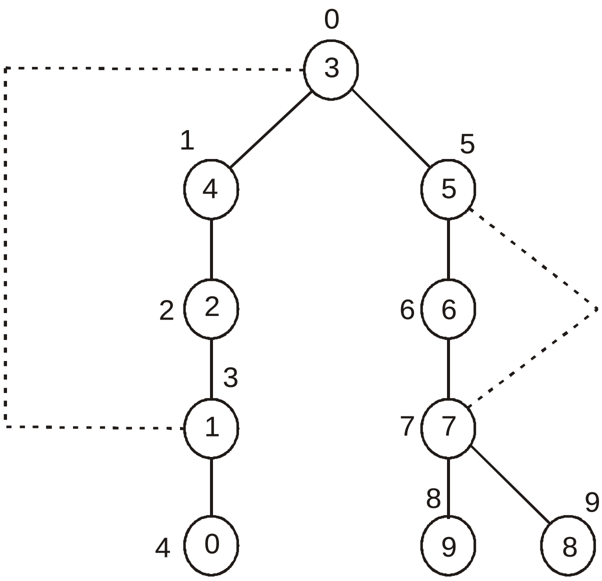
Fig 1.1 : A connected graph and its biconnected components that we cannot reach an ancestor of u using a path that consists of only w, descendants of w, and a single back edge. These observations lead us to define a value, low, for each vertex of G such that low (u) is the lowest depth first number that we can reach from u using a path of descendants followed by at most one back edge:

$$\text{low}(u) = \min\{\text{dfn}(u), \min\{\text{low}(w) \mid w \text{ is a child of } u\}, \min\{\text{dfn}(w) \mid (u, w) \text{ is a back edge}\}\}$$

Therefore, we can say that u is an articulation point iff u is either the root of the spanning tree and has two or more children, or u is not the root and u has a child w such that low (w) $\geq$ dftz (u). Figure 1.3 shows the dftz and low values for each vertex of the spanning tree of Figure 1.2(b). From this table we can conclude that vertex 1 is an



(a) depth first spanning tree



(b)

Fig.1.2 Depth first spanning free of Figure 1.1 (a) articulation point since it has a child 0 such that low (0) = 4 ≥ dfn (1) = 3. Vertex 7 is also an articulation point since low (8) = 9 ≥ dfn (7) = 7, as is vertex 5 since (6) = 5 ≥ dfn(5) = 5. Finally, we note that the root, vertex 3, is an articulation point because it has more than one child.

Vertex	0	1	2	3	4	5	6	7	8	9
dfn	4	3	2	0	1	5	6	7	9	8
low	4	3	0	0	0	5	5	7	9	8

Q59. Explain about Calculation of Shortest Distances Points.

Ans : **(Imp.)**

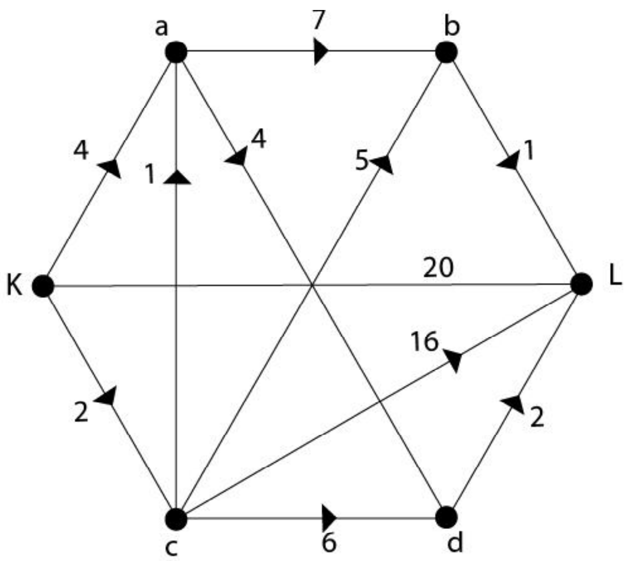
This algorithm maintains a set of vertices whose shortest paths from source is already known. The graph is represented by its cost adjacency matrix, where cost is the weight of the edge. In the cost adjacency matrix of the graph, all the diagonal values are zero. If there is no path from source vertex Vs to any other vertex Vi then it is represented by +8.In this algorithm, we have assumed all weights are positive.

1. Initially, there is no vertex in sets.
2. Include the source vertex Vs in S.Determine all the paths from Vs to all other vertices without going through any other vertex.
3. Now, include that vertex in S which is nearest to Vs and find the shortest paths to all the vertices through this vertex and update the values.
4. Repeat the step until n-1 vertices are not included in S if there are n vertices in the graph.

After completion of the process, we got the shortest paths to all the vertices from the source vertex.

Example:

Find the shortest paths between K and L in the graph shown in fig using Dijkstra's Algorithm.



Sol :

Step1: Include the vertex K is S and determine all the direct paths from K to all other vertices without going through any other vertex.

S	K	a	b	c	d	L
K	0	4(K)	∞	2(K)	∞	20(K)

Step2: Include the vertex in S which is nearest to K and determine shortest paths to all vertices through this vertex and update the values. The closest vertex is c.

S	K	a	b	c	d	L
K	0	3(K,c)	7(K,c)	2(K)	8(K,c)	18(K,c)

Step3: The vertex which is 2nd nearest to K is 9, included in S.

S	K	a	b	c	d	L
K	0	3(K,c)	7(K,c)	2(K)	7(K,c,a)	18(K,c)

Step 4: The vertex which is 3rd nearest to K is b, included in S.

S	K	a	b	c	d	L
K	0	3(K,c)	7(K,c)	2(K)	7(K,c,a)	8(K,c,b)

Step 5: The vertex which is next nearest to K is d, is included in S.

S	K	a	b	c	d	L
K(c, a, b, d)	0	3(K,c)	7(K,c)	2(K)	7(K, c, a)	8(K, c, b)

Since, n-1 vertices included in S. Hence we have found the shortest distance from K to all other vertices. Thus, the shortest distance between K and L is 8 and the shortest path is K, c, b, L.

FACULTY OF INFORMATICS

M.C.A. I Year I Semester Examination

Model Paper - I

DISCRETE MATHEMATICS

Time : 3 Hours] **Max. Marks : 70**

Answer all the questions according to the internal choice

Max. Marks (5 × 14 = 70)

ANSWERS

1. (a) Explain the operations on set theory. **(Unit-I, Q.No.6)**

   (b) Let A = {1, 3, 5), B = {2, 3}, and C = {4, 6}. Write down the following :

1.	A × B	2.	B × A
3.	B × C	4.	A × C
5.	$(A \cup B) \times C$	6.	$A \cup (B \times C)$
7.	$(A \times B) \cup C$	8.	$A \cap (B \times C)$
9.	$(A \times B) \cup (B \times C)$	10.	$(A \times B) \cap (B \times A)$
11.	$(A \times B) \cap (B \times C)$		

(Unit-I, Q.No.10)

OR

2. (a) Define about Equivalence Relation. **(Unit-I, Q.No.35)**

   (b) Explain about Schroeder - Bernstein Theorem. **(Unit-I, Q.No.52)**

OR

3. (a) A telegraph can transmit two different signals : a dot and a dash. What length of these symbols is needed to encode 26 letters of the English alphabet and the ten digits 0, 1, 2, ..., 9? **(Unit-II, Q.No.6)**

   (b) Find the number of permutations of the letters of the word SUCCESS. **(Unit-II, Q.No.12)**

OR

4. (a) Explain the concept of Permutations. **(Unit-II, Q.No.10)**

   (b) Describe the basic principles of Inclusion and Exclusion. **(Unit-II, Q.No.34)**

OR

5. (a) What are the called as statements in mathematical logic? Explain various types of statements with its notations. **(Unit-III, Q.No.3)**

   (b) Prove that, for any propositions p, q, r, the compound proposition

   $\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$ is a tautology. **(Unit-III, Q.No.13)**

OR

6. (a) Construct the truth tables of the following compound propositions :

   (i) $(p \vee q) \wedge r$ (ii) $p \vee (q \wedge r)$ **(Unit-III, Q.No.7)**

   (b) What is logical equivalence? **(Unit-III, Q.No.15)**

OR

7. (a) Define Algebraic Structures. **(Unit-IV, Q.No.1)**

   (b) Explain various Identities of Boolean Algebra. **(Unit-IV, Q.No.39)**

OR

8. (a) If a, b are any two elements of a ring R prove that,

   (i) $-(-a) = a$

   (ii) $-(a + b) = -a - b$

   (iii) $-(a - b) = -a - b$ **(Unit-IV, Q.No.30)**

   (b) Discuss about free and cyclic groups. **(Unit-IV, Q.No.13)**

OR

9. (a) Show that the following two graphs are isomorphic: **(Unit-V, Q.No.20)**

   (b) Discuss about Eulerian Circuit. **(Unit-V, Q.No.24)**

OR

10. (a) Explain about Calculation of Shortest Distances Points. **(Unit-V, Q.No.59)**

    (b) Explain about Isomorphism. **(Unit-V, Q.No.17)**

FACULTY OF INFORMATICS

M.C.A. I Year I Semester Examination

Model Paper - II

DISCRETE MATHEMATICS

Time : 3 Hours] **Max. Marks : 70**

Answer all the questions according to the internal choice

Max. Marks (5 × 14 = 70)

ANSWERS

1. (a) Discuss the operations on relations. **(Unit-I, Q.No.13)**

   (b) For any set $A \subseteq U$, prove that
   $A \times \Phi = \Phi \times A = \Phi$. **(Unit-I, Q.No.11)**

OR

2. (a) Discuss about Partial Ordering Relation. **(Unit-I, Q.No.25)**

   (b) Explain briefly about Cantor's Diagonal Argument. **(Unit-I, Q.No.50)**

OR

3. (a) Prove the following identities:

   (i) $C(n + 1, r) = C(n, r - 1) + C(n, r)$

   (ii) $C(m + n, 2) - C(m, 2) - C(n, 2) = mn$. **(Unit-II, Q.No.33)**

   (b) Explain about Pigeon-Hole Principle. **(Unit-II, Q.No.44)**

OR

4. (a) A telegraph can transmit two different signals : a dot and a dash. What length of these symbols is needed to encode 26 letters of the English alphabet and the ten digits 0, 1, 2, ..., 9? **(Unit-II, Q.No.6)**

   (b) A student visits a sports club every day from Monday to Friday after school hours and plays one of the three games: Cricket, Tennis, Football. In how many ways can he play each of the three games at least once during a week (from Monday to Friday) ? **(Unit-II, Q.No.41)**

OR

5. (a) Construct the truth table for $\sim(p \wedge q)$. **(Unit-III, Q.No.4)**

   (b) Construct the truth tables of the following compound propositions :

   (i) $(p \wedge q) \rightarrow (\neg r)$ (ii) $q \wedge ((\neg r) \rightarrow p)$ **(Unit-III, Q.No.8)**

OR

6. (a) Prove that, for any propositions p and q, the compound propositions $p \vee q$ and $(p \vee q) \wedge (\neg p \vee \neg q)$ are logically equivalent. **(Unit-III, Q.No.18)**

(b) Discuss about Proof of Necessity. and Sufficiency. **(Unit-III, Q.No.41)**

OR

7. (a) Define Groups. Explain properties of Groups. **(Unit-IV, Q.No.4)**

(b) Prove that a subgroups H of a group G is a normal subgroup if and only if every left coset of H in G is a right coset of H in G. **(Unit-IV, Q.No.25)**

OR

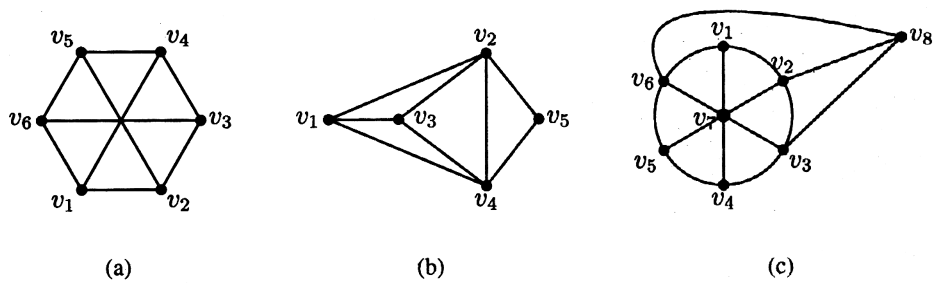
8. (a) Define normal subgroup. Prove that every subgroup of an abelian group is normal. **(Unit-IV, Q.No.20)**

(b) Show DNF of $p \oplus q$ is $(p \wedge \neg q) \vee (\neg p \wedge q)$ **(Unit-IV, Q.No.47)**

OR

9. (a) Write about Hamiltonian Walks. **(Unit-V, Q.No.27)**

(b) Find the chromatic number of each of the following graphs.



(Unit-V, Q.No.32)

OR

10. (a) A connected graph G has an Euler circuit (that is, G is an Euler graph) if and only if G can be decomposed into edge-disjoint cycles. **(Unit-V, Q.No.26)**

(b) Write about Prefix Codes. **(Unit-V, Q.No.57)**

FACULTY OF INFORMATICS

M.C.A. I Year I Semester Examination

Model Paper - III

DISCRETE MATHEMATICS

Time : 3 Hours] **Max. Marks : 70**

Answer all the questions according to the internal choice

Max. Marks (5 × 14 = 70)

ANSWERS

1. (a) Explain the properties of relations. **(Unit-I, Q.No.14)**

   (b) Discuss various types of functions. **(Unit-I, Q.No.43)**

OR

2. (a) Discuss briefly about Binary Relation. **(Unit-I, Q.No.12)**

   (b) Explain about Principles of Mathe-matical Induction. **(Unit-I, Q.No.53)**

OR

3. (a) Explain the concept of combinations. **(Unit-II, Q.No.18)**

   (b) Explain the process of Inclusion and Exclusion for n sets. **(Unit-II, Q.No.37)**

OR

4. (a) How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000? **(Unit-II, Q.No.15)**

   (b) A bag contains many red marbles, many white marbles, and many blue marbles. What is the least number of marbles one should take out to be sure of getting at least six marbles of the same color? **(Unit-II, Q.No.49)**

OR

5. (a) Construct the truth table for $p \vee (p \vee q)$. **(Unit-III, Q.No.5)**

   (b) What is logical implication? **(Unit-III, Q.No.24)**

OR

6. (a) Show that the truth values of the following compound propositions are independent of the truth values of their components:

   (i) $\{p \wedge (p \rightarrow q)\} \rightarrow q$ (ii) $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ **(Unit-III, Q.No.10)**

   (b) Provide a proof by contradiction of the following statement:

   For every integer n, if n^2 is odd, then n is odd. **(Unit-III, Q.No.36)**

OR

7. (a) Discuss about Congruence Relation and Quotient Structures. **(Unit-IV, Q.No.12)**

   (b) Give any four examples of a ring. **(Unit-IV, Q.No.27)**

OR

8. (a) Explain briefly about Substructures. **(Unit-IV, Q.No.19)**

(b) Convert the following formulas into CNF using truth tables.

(a) $(p \Leftrightarrow q) \Rightarrow (\neg p \wedge r)$

(b) $(p \Leftrightarrow q) \Rightarrow r$

(c) $p \Leftrightarrow q$ **(Unit-IV, Q.No.45)**

OR

9. (a) Analyze the proof of five colour theorem. **(Unit-V, Q.No.36)**

(b) (a) Show that the complete graph, K_n is not a tree when $n > 2$.

(b) Show that the complete bipartite graph $K_{r, s}$ is not a tree when $r \geq 2$.

(Unit-V, Q.No.46)

OR

10. (a) Explain about Calculation of Shortest Distances Points. **(Unit-V, Q.No.59)**

(b) Define tree. Discuss various properties of trees. **(Unit-V, Q.No.39)**