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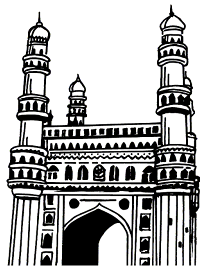

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CONTENTS

# REAL ANALYSIS
## (MATHEMATICS)

### STUDY MANUAL


| | |
|---|---|
| **Important Questions** | **III - VIII** |
| **Unit - I** | **1 - 74** |
| **Unit - II** | **75 - 108** |
| **Unit - III** | **109 - 142** |
| **Unit - IV** | **143 - 188** |


### SOLVED PREVIOUS QUESTION PAPERS


| | |
|---|---|
| **December - 2017** | **189 - 192** |
| **May / June - 2018** | **193 - 196** |
| **Nov. / Dec. - 2018** | **197 - 202** |
| **June / July - 2019** | **203 - 204** |


### SOLVED MODEL PAPERS


| | |
|---|---|
| **Model Paper - I** | **205 - 206** |
| **Model Paper - II** | **207 - 208** |
| **Model Paper - II** | **209 - 210** |

# SYLLABUS

## UNIT - I

**Sequences:** Limits of Sequences - A Discussion about Proofs - Limit Theorems for Sequences Monotone Sequences and Cauchy Sequences - Subsequences - Lim sup's and Lim inf's-Series-Alternating Series and Integral Tests.

## UNIT - II

**Continuity:** Continuous Functions - Properties of Continuous Functions - Uniform Continuity - Limits of Functions

## UNIT - III

**Differentiation :** Basic Properties of the Derivative - The Mean Value Theorem - * L'Hospital Rule - Taylor's Theorem.

## UNIT - IV

**Integration :** The Riemann Integral - Properties of Riemann Integral-Fundamental Theorem of Calculus.

# Contents

## UNIT - I


| Topic | | Page No. |
|---|---|---|
| 1.1 | Sequence | 1 |
| | 1.1.1 Real Sequence | 1 |
| | 1.1.2 Range of a Sequence | 1 |
| | 1.1.3 Constant Sequence | 1 |
| | 1.1.4 Bounded and Unbounded Sequence | 1 |
| 1.2 | Limits of Sequence | 3 |
| | 1.2.1 Convergent Sequence | 3 |
| 1.3 | A Discussion about Proofs | 6 |
| 1.4 | Limit Theorems for Sequences | 17 |
| | 1.4.1 Divergent Sequence | 22 |
| | 1.4.2 Oscillatory Sequence | 22 |
| 1.5 | Monotone Sequences and Cauchy Sequences | 28 |
| | 1.5.1 Cauchy Sequence | 38 |
| 1.6 | Subsequence | 39 |
| 1.7 | Lim sup's and lim Inf's | 45 |
| 1.8 | Series (or) Infinite Series | 49 |
| | 1.8.1 Partial Sums | 49 |
| 1.9 | Alternating Series | 55 |
| | 1.9.1 Leibnitz's Test | 56 |
| | 1.9.2 Absolute and Conditional Convergence | 57 |
| 1.10 | Integral Test | 63 |
| ➢ | Choose the Correct Answers | 69 |
| ➢ | Fill in the Blanks | 73 |

**Topic** **Page No.**

**UNIT - II**

2.1 Definition of Continuous Function ......75
2.2 Properties of Continuous Function ......81
2.3 Uniform Continuity ......87
2.4 Limits of Functions ......98
- Choose the Correct Answers ......106
- Fill in the Blanks ......108

**UNIT - III**

3.1 Basic Properties of the Derivative ......109
3.2 The Mean Value Theorem ......117
3.3 L – Hospital Rule ......128
3.4 Taylor's Theorems ......134
- Choose the Correct Answers ......140
- Fill in the Blanks ......142

**UNIT - IV**

4.1 Integration ......143
    4.1.1 Partition of a Closed Interval ......143
    4.1.2 Norm of a Partition ......143
    4.1.3 Refinement of a Partition ......143
    4.1.4 Upper and Lower Riemann Sums ......143
    4.1.5 Oscillatory Sum ......143
    4.1.6 Lower and Upper Riemann Integrals ......144
4.2 Riemann Integral ......144
4.3 Properties of Riemann Integral ......153
4.4 Fundamental Theorem of Calculus ......167
    4.4.1 Necessary and Sufficient Condition for Integrability ......167
    4.4.2 Another Definition of Riemann Integral ......173
    4.4.3 Primitive (Definition) ......173
4.5 Fundamental Theorem of Integral Calculus ......173
- Choose the Correct Answers ......185
- Fill in the Blanks ......187

# *Important Questions*

## UNIT - I

**1. If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences such that (i) $a_n \leq b_n \leq c_n$ for $n \geq K$ where K is some positive integer and (ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$ = then $\lim_{n\to\infty} b_n = l$.**

*Sol.*

Refer Unit-I, Q.No. 5.

---

**2. Prove that $a^n = 0$ for $|a| < |$**

**(a) $\lim_{n\to\infty} n^{1/n} = 1$**

**(b) $\lim_{n\to\infty} a^{1/n} = 1$ for $a > 0$**

*Sol.*

Refer Unit-I, Q.No. 7.

---

**3. Prove $\lim_{n\to\infty} (a^{1/n}) = 1$ for $a > 0$.**

*Sol.*

Refer Unit-I, Q.No. 8.

---

**4. Let $\{s_n\}$ be sequence in R prove that the $\lim s_n = 0$ iff $\lim |s_n| = 0$.**

*Sol.*

Refer Unit-I, Q.No. 23.

---

**5. If $\{s_n\}$ is converges to s, and $\{t_n\}$ is converges to 't'. Then $\{s_n + t_n\}$ converges to $s + t$ that is $\lim \{s_n + t_n\} = \lim s_n + \lim t_n$.**

*Sol.*

Refer Unit-I, Q.No. 25.

---

**6. Let $t_1 = 1$ and $t_{n+1} = \dfrac{t_n^2 + 2}{2t_n}$ for $n \geq 1$. Assume that $\{t_n\}$ converges and find the limit.**

*Sol.*

Refer Unit-I, Q.No. 33.

---

**7. Prove that $\lim_{n\to\infty} \dfrac{1}{n^p} = 0$ for $p > 0$.**

*Sol.*

Refer Unit-I, Q.No. 36.

**8. All bounded monotone sequence converge.**

**(i) Every monotonically increasing sequence which is bounded above is convergent.**

**(ii) Every monotonically decreasing sequence which is bounded below is convergent.**

**OR**

**State and prove Montone Converge Theorem.**

*Sol.*

Refer Unit-I, Q.No. 40.

**9. Let $(S_n)$ be an increasing sequence of positive number and define $\sigma_n = \frac{1}{n}(S_1 + S_2 + .... + S_n)$ prove $(\sigma_n)$ is an increasing sequence.**

*Sol.*

Refer Unit-I, Q.No. 45.

**10. Let $t_1 = 1$ and $t_{n+1} = \left[1 - \frac{1}{(n+1)^2}\right]$ tn for all $n \geq 1$.**

**(a) Show Lim tn exists.**

**(b) What do you think Limtn is?**

**(c) Use induction to show tn = $\frac{n+1}{2n}$**

**(d) Repeat part (b)**

*Sol.*

Refer Unit-I, Q.No. 47.

**11. Let $S_1 = 1$ and $S_{n+1} = \frac{1}{3}(S_{n+1})$ for $n \geq 1$.**

**(a) Find $S_2$, $S_3$ and $S_4$**

**(b) Use induction to show $S_n > \frac{1}{2}$ for all n.**

**(c) Show $(S_n)$ is a decreasing sequence**

**(d) Show lim $S_n$ exists and find lim $S_n$.**

*Sol.*

Refer Unit-I, Q.No. 48.

**12. If the sequence $\{s_n\}$ converges, then every subsequence converges to the same limit.**

*Sol.*

Refer Unit-I, Q.No. 52.

## UNIT - II

**1. Let f be a real valued function whose domain is a subset of R. Then f is continuous at $x_0$ in dom(f) if and only if for each $e > 0 \exists d > 0 \ni x \in$ dom(f) and $|x - x_0| < d \Rightarrow |f(x) - f(x_0)| < \varepsilon$.**

*Sol.*

Refer Unit-II, Q.No. 1.

**2. Let $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$ Prove that f is continuous at 0.**

*Sol.*

Refer Unit-II, Q.No. 3.

**3. If f and g are real valued functions at $x_0$ then,**

**(1) f + g is continuous at $x_0$**

**(2) fg is continuous at $x_0$**

**(3) f/g is continuous at $x_0$ if $g(x_0) \neq 0$.**

*Sol.*

Refer Unit-II, Q.No. 5.

**4. Let f be a continous on [a, b] and assume f(a) < f(b) then for every k such that f(a) < k < f(b) there exists $c \in [a, b]$ such that f(c) = k.**

*Sol.*

Refer Unit-II, Q.No. 8.

**5. Let f be a continuous function mapping [0, 1] into [0, 1] in other words, dom(f) = [0, 1] and $f(x) \in [0, 1]$ for all $x \in [0, 1]$ show f has fixed point, i.e., a point $x_0 \in [0, 1]$ such that $f(x_0) = x_0$, $x_0$ is left fixed by f.**

*Sol.*

Refer Unit-II, Q.No. 10.

**6. Prove that x = cos(x) for some x in $\left(0, \frac{\pi}{2}\right)$.**

*Sol.*

Refer Unit-II, Q.No. 15.

**7. Let $S \subseteq R$ and suppose there exists a sequence $\{x_n\}$ in S converying to a number $x_0 \notin S$ show there exists an unbounded continuous function on S.**

*Sol.*

Refer Unit-II, Q.No. 16.

**8. Let f and g be continuous function, on [a, b] such that f(a) $\geq$ g(a) and f(b) $\leq$ g(b) prove that $f(x_0) = g(x_0)$ for at lest one $x_0$ in [a, b].**

*Sol.*

Refer Unit-II, Q.No. 17.

**9. Show that f(x) = $\frac{1}{x^2}$ is uniformly continous on [0, $\infty$) where a > 0.**

*Sol.*

Refer Unit-II, Q.No. 21.

**10. Let $f_1$ and $f_2$ be function for which the limits $L_1 = \lim_{x \to a^s} f_1(x)$ and $L_2 = \lim_{x \to a^s} f_2(x)$ exist and are finite. Then**

**(i) $\lim_{x \to a^s} (f_1+f_2)(x)$ exists and equals $L_1 + L_2$**

**(ii) $\lim_{x \to a^s} (f_1 f_2)(x)$ exits and equals $L_1 L_2$**

**(iii) $\lim_{x \to a^s} (f_1 / f_2)(x)$ exits and equals $L_1/L_2$ provides $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in s$**

*Sol.*

Refer Unit-II, Q.No. 45.

## UNIT - III

**1. If f is differentiable at a point 'a'. Then 'f' is continuous at a.**

*Sol.*

Refer Unit-III, Q.No. 1.

**2. Find h'(a) where h(x) = $x^{-m}$ for $x \neq 0$. h(x) = $\frac{f(x)}{g(x)}$ where f(x) = 1 & g(x) = $x^m$ for all x.**

*Sol.*

Refer Unit-III, Q.No. 3.

**3. Determine by using mean value theorem.**

**(a) $x^2$ on [–1, 2]  (b) sin x on [0, $\pi$]  (c) |x| on [–1, 2]**

**(d) $\frac{1}{x}$ on [–1, 1]  (e) $\frac{1}{x}$ on [1, 3]  (f) sgn (x) on [–1, 2]**

*Sol.*

Refer Unit-III, Q.No. 14.

**4. Prove that |cosx – cosy| $\leq$ |x – y| for all x, y $\in$ R.**

*Sol.*

Refer Unit-III, Q.No. 15.

**5. Let a, b $\in$ R. let f(x) = $e^{ax}$ cos(bx) and g(x) = $e^{ax}$ sin(bx)**

**(i) Compute f'(x) and g'(x)**

**(ii) Use (i) to compute f'' and f'''**

*Sol.*

Refer Unit-III, Q.No. 20.

**6. Suppose that f is differentiable on R that i $\leq$ f'(x) $\leq$ 2 for x $\in$ R, and that f(0) = 0 prove that x $\leq$ f(x) 2x for all x > 0.**

*Sol.*

Refer Unit-III, Q.No. 24.

**7. Let f, g are derivable on (a, a + h) such that**

**(i) g'(x) $\neq$ 0 $\forall$ x $\in$ (a, a + h),**

**(ii)** $\lim_{x\to a^+}$ **f(x) = 0 =** $\lim_{x\to a^+}$ **g(x)**

**(a) If** $\lim_{x\to a^+}\frac{f'(x)}{g'(x)} = l$**, a real number the** $\lim_{x\to a^+}\frac{f'(x)}{g'(x)} = l$.

**(b) If** $\lim_{x\to a^+}\frac{f'(x)}{g'(x)} = \pm\infty$ **then** $\lim_{x\to a^+}\frac{f(x)}{g(x)} = \pm\infty$

*Sol.*

Refer Unit-III, Q.No. 26.

**8. State and prove L - Hospital Rule II :**

**(OR)**

**If f, g are derivable in a deleted nbd of 'a'**

$$\lim_{x\to a^+} f(x) = \pm\infty,\ \lim_{x\to a^+} g(x) = \pm\infty \text{ and } \lim_{x\to a^+}\frac{f'(x)}{g'(x)} = l \text{, then } \lim_{x\to a^+}\frac{f(x)}{g(x)} = l$$

*Sol.*

Refer Unit-III, Q.No. 27.

**9. State and prove Binomial Series Theorem :**

**If** $\alpha \in R$ **and |x| < 1 Then**

$$(1 + x)^{\alpha} = 1 + \sum_{k=1}^{\infty}\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}x^k$$

*Sol.*

Refer Unit-III, Q.No. 39.

**10. Expassion of $e^x$.**

*Sol.*

Refer Unit-III, Q.No. 40.

## UNIT - IV

**1. If $f : [ab] \to R$ is a bounded function then $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$.**

*Sol.*

Refer Unit-IV, Q.No. 1.

**2. If, f, g $\in$ R [a b] and $f(x) \ge g(x)$ $\forall$ x $\in$ [a, b] then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$**

*Sol.*

Refer Unit-IV, Q.No. 17.

**3. If f $\in$ R [a b] and m, M are the inf. and sup. of f in [a b] then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$ and $\int_a^b f(x)\,dx = \mu(b-a)$ where $\mu \in [m, M]$.**

*Sol.*

Refer Unit-IV, Q.No. 19.

**4. Prove that $\left|\int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)\,dx\right| \le \dfrac{16\pi^3}{3}$**

*Sol.*

Refer Unit-IV, Q.No. 24.

**5. A bounded function f is integrable on [ab] if and only if for each $\in > 0$, $\exists$ a partition p of [ab]. Such that $U(p, f) - L(p, f) < \in$.**

*Sol.*

Refer Unit-IV, Q.No. 28.

**6. if 'g' is integrable on [a, b] & g is a continuous function on [a, b] which is differentiable on [a, b].**

**Then prove that $\int_a^b g' = g(b) - g(a)$**

*Sol.*

Refer Unit-IV, Q.No. 40.

| UNIT I | **Sequences:** Limits of Sequences - A Discussion about Proofs - Limit Theorems for Sequences Monotone Sequences and Cauchy Sequences - Subsequences - Lim sup's and Lim inf's-Series-Alternating Series and Integral Tests. |
|---|---|

## 1.1 Sequence

A sequence is a funtion whose domain is the set N of all natural numbers where as the range may be any set S.

In other words if A is a non empty set then a function $S : N \rightarrow A$ is called a Sequence.

Sequences are useful in deciding the continuity of a real valued function on a subset of R.

### 1.1.1 Real Sequence

A real sequence is a function whose domain is the set N of all natural numbers and range of subset of the set R of real numbers

i.e., $x : N \rightarrow R$ is a real sequence which is denoted by $\{x_n\}$ or $<x_n>$. Sometimes the sequence x is represented by an argument of the terms in increasing order of the argument n such as $\{x_1, x_2, x_3 \, ..... \, x_n \, .....\}$

### 1.1.2 Range of a Sequence

The set of all distinct terms of a sequence is called its range. If $n \in N$, where N is an infinite set then the number of terms of a sequence is always infinite. But the range of a sequence may be a finite set.

for example if $x_n = (-1)^n$ then $\{x_n\} = \{1, -1, 1, -1 \, .....\}$

$\therefore$ The range of sequence $\{x_n\} = \{-1, 1\}$ which is finite set

### 1.1.3 Constant Sequence

A sequence $\{x_n\}$ defined by $x_n = C \in R \; \forall \, n \in N$ is called a constant sequence. Therefore $\{x_n\} = \{C, C, C, .....\}$ is a constant sequence with range $= \{C\}$, a singleton set.

### 1.1.4 Bounded and Unbounded Sequence

**1. Bounded above sequence**

A sequence $\{a_n\}$ is said to be bounded above if $\exists$ a real number K such that $a_n \leq K \; \forall \, n \in N$.

**2. Bounded Below Sequence**

A sequence $\{a_n\}$ is said to be bounded below if $\exists$ a real number K such that $a_n \geq K \; \forall \, n \in N$.

**3. Bounded Sequence**

A sequence is said to be bounded if it is bounded above as well as bounded below. Thus a sequence $\{a_n\}$ is bounded if $\exists$ two real numbers $K_1$ and $K_2$ such that $K_1 \le K_2$, then $K_1 \le a_n \le K_2 \ \forall \ n \in N$.

**4. A Sequence is said to be unbounded if it is not bounded**

**(i) Unbounded above sequence :** A sequence $\{a_n\}$ is said to be unbounded above if it is not bounded above i.e., for every real number $K_1$, $\exists$ $m \in N \ni a_m > K_1$.

**(ii) Unbounded below sequence :** A sequence $\{a_n\}$ is said to be unbounded below if it is not bounded below, i.e., for every real number $K_2$ $\exists$ $m \in N \ni a_m < K_2$.

**Examples**

(i) Every constant sequence is bounded

(ii) The sequence $<-n>$ is bounded above because $a_n \le -1 \ \forall \ n \in N$ and it is not bounded below

(iii) The sequence $\{a_n\}$ defined by $a_n = (-1)^n$, n is neither bounded above nor bounded below.

**Note:** The sequence $\{a_n\}$ is bounded iff $\exists$ a positive real number $M \ni |a_n| \le M \ \forall \ n \in N$.

**5. Least Upper bound and greatest lower bound of a sequence**

**a) Least upper bound of a sequence**

If a sequence $\{a_n\}$ is bounded above, then $\exists$ a real number $K_1 \ni a_n \le K_1 \ \forall \ n \in N$.

$K_1$ is called an upper bound of the sequence.

If $K_1 < K_2$ then $a_n < K_2 \ \forall \ n \in N$.

$\Rightarrow$ $K_2$ is also an upper bound of the sequence

$\Rightarrow$ Any number $> K_1$ is also an upper bound of the sequence.

Therefore if a sequence is bounded above, it has infinitely many upper bounds of all the upper bounds of the sequence, if K is the least, then K is called the least upper bound (*l*ub) of the sequence or Supremum of the sequence.

**b) Greatest lower bound of a sequence**

If a sequence $\{a_n\}$ is bounded below, then a real number $K_1 \ni K_1 \le a_n$ or $a \ge K_1 \ \forall \ n \in N$.

$\Rightarrow$ $K_1$ is called an lower bound of the sequence.

If $K_2 < K_1$ or $K_1 > K_2$ then $a_n > K_2 \ \forall \ n \in N$.

$\Rightarrow$ $K_2$ is also a lower bound of the sequence

$\Rightarrow$ Any number $< K_1$ is also an lower bound of the sequence.

$\Rightarrow$ If a sequence is bounded below, it has infinitely many lower bounds of all the lower bounds of the sequence, if K is the greatest, then K is called the greatest lower bound (g*l*b) of the sequence or infimum of the sequence.

## 1.2 Limits of Sequence

Let $\{a_n\}$ be a sequence and $l \in R$. The real number $l \in R$ is said to be the limit of the sequence $\{a_n\}$ if to each $\in > 0 \; \exists \; m \in N \ni |a_n - l| < \in \; \forall n \geq m$.

If $l$ is the limit of $\{a_n\}$, then we write $a_n \to l$ as $n \to \infty$

or $\lim\limits_{n\to\infty} a_n = l$

**Note**

$|a_n - l| < \in \; \forall n \geq M$

$\Rightarrow \; -\in < a_n - l < \in \; \forall n \geq m$

$\Rightarrow \; l - \in < a_n < l + \in \; \forall n \geq m$

$\Rightarrow \; a_n \in (l - \in, \; l + \in) \; \forall n \geq m$

### 1.2.1 Convergent Sequence

If $\lim\limits_{n\to\infty} a_n = l$, then we say that the sequence $\{a_n\}$ converges to '$l$'.

i.e., A sequence $\{a_n\}$ is said to converge to a real number '$l$' if given $\in > 0$, $\exists$ a positive integer m $\ni |a_n - l| < \in \; \forall n \geq m$. the real number $l$ is called the limit of the sequence $\{S_n\}$.

**1. Every convergent sequence has a unique limit.**

*Sol.*

If possible, let the sequence $\{a_n\}$ converge to two distinct real numbers $l$ and $l^1$

Let $\in = |l - l^1|$ $\quad \because \; l \neq l^1 \Rightarrow |l - l^1| > 0 \Rightarrow \in > 0$

$\because$ The sequence $\{a_n\}$ converges to '$l$'

$\Rightarrow$ Given $\in > 0 \; \exists$ a postive integer $m_1 \ni |a_n - l| < \in/2 \; \forall n \geq m_1$ ... (1)

again the sequence $\{a_n\}$ converges to $l^1$

$\Rightarrow$ Given $\in > 0 \; \exists$ a postive integer $m_2 \ni |a_n - l^1| < \in/2 \; \forall n \geq m_2$ ... (2)

let $m = \max\{m_1, m_2\}$

$\Rightarrow$ From (1) and (2)

$\forall n \geq m \Rightarrow |a_n - l| < \in/2$ and $|a_n - l^1| < \in/2$

consider

$$|l - l^1| = |l - a_n + a_n - l^1|$$

$$= |a_n - l| + |a_n - l^1|$$

$$< \in/2 + \in/2$$

$\Rightarrow$ $|l - l^1| < 2\in/2$

$\Rightarrow$ $|l - l^1| < \in$

$\Rightarrow$ $|l - l^1| < |l - l^1| \quad \because \in = |l - l^1|$

Which is a contradiction

Hence our assumption is wrong

$\therefore\ l = l^1$

$\Rightarrow$ Every convergent sequence has a unique limit.

**2. Every convergent sequence is bounded.**

*Sol.* **(May/June-18, Imp.)**

Let $\{a_n\}$ be a convergent sequence. Which converges to '$l$'

Let $\in = 1$, $\exists$ a positive integer m $\ni |a_n - l| < 1\ \forall\ n \geq m$

$\Rightarrow$ $l - 1 < a_n < l + 1\ \forall\ n \geq m$

Let $K_1 = \min\{a_1, a_2, .....a_{n-1}, l - 1\}$ and

$K_2 = \max\{a_1, a_2, .....a_{m-1}, l + 1\}$

$\Rightarrow$ $K_1 \leq an \leq K_2\ \forall\ n \geq N$

$\Rightarrow$ Sequence $\{a_n\}$ is a bounded sequence.

## Note

1. Converse of the above theorem need not be true.
2. If a sequence is not bounded, it cannot be convergent.

**3. If $\lim\limits_{n\to\infty} a_n = l \Rightarrow \lim\limits_{n\to\infty} |a_n| = |l|$ but the converse is not true.**

*Sol.*

$\lim\limits_{n\to\infty} a_n = l$

$\Rightarrow$ Given $\in > 0$, $\exists$ a positive integer m such that $|a_n - l| < \in\ \forall\ n \geq m$ ... (1)

$\Rightarrow$ $||a_n| - |l|| < |a_n - l|$ ... (2)

$\therefore$ from (1) and (2) we get

$\Rightarrow$ $||a_n| - |l|| < \in\ \forall\ n \geq m$

$\Rightarrow$ $|a_n| = |l|$

To prove converse need not be true

Let $\{a_n\} = \{(-1)^n\} = \{-1, 1, -1, 1 ......\}$

$\Rightarrow$ $\{a_n\}$ does not converges to any limit

Whereas

$$\{|a_n|\} = \{|(-1)^n|\} = \{1, 1, 1, .....\}$$

Hence proved

**4. If $a_n \geq 0\ \forall\ n \geq N$ and $\lim\limits_{n\to\infty} a_n = l$ then $l \geq 0$.**

*Sol.*

If possible, let $l < 0$

$\because \lim\limits_{n\to\infty} a_n = l$

$\Rightarrow$ Given $\in > 0$, $\exists$, a positive integer $\ni |a_n - l| < \in\ \forall\ n \geq m$

$\Rightarrow$ $l - \in < a_n < l + \in\ \forall\ n \geq m$ ... (1)

$\because l < 0$, let $\in = \frac{-l}{2} > 0$

Substituting $\in$ in (1) then we get

$$l+\frac{l}{2}<a_n< l-\frac{l}{2} \ \forall n \geq m \qquad \text{... (2)}$$

$$\Rightarrow \quad \frac{3l}{2}< a_n \ \frac{l}{2}< \ \forall n \geq m$$

$$\because \ a_n<\frac{l}{2}<0 \ \forall n \geq m$$

Which is a contradiction to the hyp that $a_n \geq 0$. Hence our assumption $l < 0$ is wrong

$\therefore \quad l \geq 0$

**Note**

1. If $\lim\limits_{n\to\infty} a_n = l$ and $\lim\limits_{n\to\infty} b_n = l^1$ then $\lim\limits_{n\to\infty} (a_n + b_n) = l + l^1$ and $\lim\limits_{n\to\infty} (a_n - b_n) = l - l^1$
2. If $\lim\limits_{n\to\infty} a_n = l$ and $\lim\limits_{n\to\infty} b_n = l^1$ then $\lim\limits_{n\to\infty} (a_n b_n) = ll^1$.
3. If $\lim\limits_{n\to\infty} a_n = l$ and $C \in R$ then $\lim\limits_{n\to\infty} Ca_n = Cl$
4. If $b_n \neq 0$ for every n, $l^1 \neq 0$, $\lim\limits_{n\to\infty} a_n = l$ and $\lim\limits_{n\to\infty} b_n = l^1$ then $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\left(\frac{l}{l^1}\right)$
5. If $\{a_n\}$ and $\{b_n\}$ be two convergent sequences then $a_n \leq b_n \ \forall n \geq N \Rightarrow \lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$

---

**5. If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences such that (i) $a_n \leq b_n \leq c_n$ for $n \geq K$ where K is some positive integer and (ii) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = l =$ then $\lim\limits_{n\to\infty} b_n = l$**

**(OR)**

**State and prove Sandwich Theorem or Squeeze Theorem**

*Sol.* **(Imp.)**

Let $\in > 0$

$$\lim_{n\to\infty} a_n = l$$

$$\Rightarrow \ \exists \ m_1 \in z^+ \ni \left|a_n - l\right| < \in \ \forall n \geq m_1$$

$$\Rightarrow l - \in < a_n < l + \in \ \forall n \geq m_1 \qquad \text{... (1)}$$

Simillarly

$$\lim_{n\to\infty} c_n = l$$

$$\Rightarrow \ \exists \ m_2 \in z^+ \ni \left|c_n - l\right| < \in \ \forall n \geq m_1$$

$$\Rightarrow l - \in < c_n < l + \in \ \forall n \geq m_2 \qquad \text{... (2)}$$

Also by hyp we have

$a_n \le b_n \le c_n \ \forall \ n \ge K$ ... (3)

Let m = max $\{m_1, m_2, K\}$

$\therefore \quad l - \in \ < a_n \le b_n \le c_n < l + \in \ \forall n \ge m$

$\Rightarrow l - \in \ < b_n < l + \in \ \forall n \ge m$

$\Rightarrow |b_n - l| < \in \ \forall n \ge m$

$\Rightarrow \lim\limits_{n\to\infty} b_n = l$

**6. If $\{a_n\}, \{b_n\}$ are two sequences such that $|a_n| \le |b_n| \ \forall n \ge K$ where K is a positive integer and $\lim\limits_{n\to\infty} b_n = 0$ then $\lim\limits_{n\to\infty} a_n = 0$.**

*Sol.*

Let $\lim\limits_{n\to\infty} b_n = 0$

$\Rightarrow \lim\limits_{n\to\infty} |b_n| = 0$ ... (1)

$\Rightarrow \lim\limits_{n\to\infty} (-|b_n|) = 0$ ... (2)

$\therefore \quad |a_n| \le |b_n| \ \forall n \ge K$

$\Rightarrow -|b_n| \le a_n \le |b_n| \ \forall n \ge K$

$\therefore$ By Sandwich Theorem and from (1) and (2) we get

$\lim\limits_{n\to\infty} a_n = 0$

## 1.3 A Discussion about Proofs

**7. Prove that $a^n = 0$ for $|a| < |$**

**(a) $\lim\limits_{n\to\infty} n^{1/n} = 1$**

**(b) $\lim\limits_{n\to\infty} a^{1/n} = 1$ for a > 0**

*Sol.* **(Nov./Dec.-18, Dec.-17, Imp.)**

(a) $\lim\limits_{n\to\infty} a^n = 0$ for $|a| < 1$

If a = 0

Then the result is thus

If $|a| < |$ where $a \ne 0$

we have $|a| < \dfrac{1}{1+b}$ ; b > 0

Apply binomial theorem for $(1+b)^n$

$$(1+b)^n = 1 + nb + \frac{n(n-1)}{2}b^2 + ....$$

$$(1+b)^n = 1 + nb > 0$$

$$(1+b)^n > nb \qquad ... (1)$$

To show that $\lim\limits_{n\to\infty} a^n = 0$

That is to find the natural number N ∋

$|a^n - 0| < \varepsilon \ \forall n > N$

consider $|a^n - 0| < \varepsilon$

$|a^n| < \varepsilon$

$$\left(\frac{1}{1+b}\right)^n < \varepsilon$$

$$\frac{1}{nb} < \varepsilon$$

$$nb > \frac{1}{\varepsilon}$$

$$n > \frac{1}{\varepsilon b}$$

Select $N = \dfrac{1}{\varepsilon b}$

for $n > N = n > \dfrac{1}{\varepsilon b} = nb > \dfrac{1}{\varepsilon}$

$$\frac{1}{nb} < \varepsilon$$

$$\frac{1}{(1+b)^n} < \varepsilon$$

$$|a|^n < \varepsilon$$

$\Rightarrow |a^n - 0| < \varepsilon$ for n > N

$\Rightarrow \lim a^n = 0$

(b) To prove $\lim_{n\to\infty} n^{1/n} = 1$

$\lim_{n\to\infty} n^{1/n} - 1 = 0$

here $S_n = n^{1/n} - 1$ $[\because s_n \geq 0]$

Let us consider

$$s_n = n^{1/n} - 1$$

$$1 + s_n = n^{1/n}$$

Add power 'n' on both sides

$$(1 + s_n)^n = (n^{1/n})^n$$

$$(1 + s_n)^n = n \quad \ldots (1)$$

By using binomial Expansion

$$(1+s_n)^n = 1 + ns_n + \frac{n(n-1)}{2} s_n^2 + \ldots.$$

$$(1 + s_n)^n = 1 + n s_n + \frac{n(n-1)}{2} s_n^2 +$$

$$(1 + s_n)^n > \frac{n(n-1)}{2} s_n^2$$

$$(n^{1/n})^n > \frac{n(n-1)}{2} s_n^2$$

$$n > \frac{n(n-1)}{2} s_n^2$$

$$1 > \frac{n-1}{2} s_n^2$$

$$\frac{1}{s_n^2} > \frac{n-1}{2}$$

$$s_n^2 < \frac{2}{n-1}$$

$$s_n < \frac{\sqrt{2}}{\sqrt{n-1}}$$

consider $s_n \geq 0$ and $s_n = \frac{\sqrt{2}}{\sqrt{n-1}}$

$$0 \leq s_n < \frac{\sqrt{2}}{\sqrt{n-1}}$$

$$\lim_{n\to\infty} s_n = 0$$

By sandwich theorem

$$\lim_{n\to\infty} \frac{\sqrt{2}}{\sqrt{n-1}} = \sqrt{2} \lim_{n\to\infty} \frac{1}{\sqrt{n}-1}$$

$$n \to \infty, \frac{1}{n} \to 0$$

$$= \sqrt{2}(0) = 0$$

$$\lim_{n\to\infty} s_n = 0$$

$$\lim_{n\to\infty} n^{1/n} - 1 = 0$$

$$\Rightarrow \lim_{n\to\infty} n^{1/n} = 1$$

**8. Prove $\lim_{n\to\infty} (a^{1/n}) = 1$ for a > 0.**

*Sol.* **(Imp.)**

For a > 0, 0 < a < 1

(i) If $a \geq 1$

Then $n \geq a$

we have $1 \leq a \leq n$

$n^{th}$ root of each term

$1^{1/n} \leq a^{1/n} \leq n^{1/n}$

By sandwich theorem

$\lim 1^{1/n} \leq \lim a^{1/n} \leq \lim n^{1/n}$

$1 \leq \lim a^{1/n} \leq 1$

$\lim a^{1/n} = 1$

(ii) 0 < a < 1

consider a < 1

$\frac{1}{a} > 1$

by case (i)

$$\lim \left(\frac{1}{a}\right)^{1/n} = 1$$

$$\lim \frac{1}{a^{1/n}} = 1$$

$$\therefore \lim a^{1/n} = 1$$

$\because$ If $\{s_n\}$ converges to s then $\left\{\frac{1}{s_n}\right\}$ converges to $\frac{1}{s}$.

---

**9. Prove that $\lim s_n = \frac{1}{4}$ where**

$$s_n = \frac{n^3 + 6n^2 + 7}{4n^3 + 3n - 4}.$$

*Sol.*

Given that,

$$s_n = \frac{n^3 + 6n^2 + 7}{4n^3 + 3n - 4}$$

$$\lim s_n = \lim \left[\frac{n^3 + 6n^2 + 7}{4n^3 + 3n - 4}\right]$$

$$= \lim \left[\frac{n^3\left(1 + \frac{6}{n} + \frac{7}{n^3}\right)}{n^3\left(4 + \frac{3}{n^2} - \frac{4}{n^3}\right)}\right]$$

$$= \frac{1 + 6\lim\frac{1}{n} + 7\lim\frac{1}{n^3}}{4 + 3\lim\frac{1}{n^2} - 4\lim\frac{1}{n^3}}$$

as $n \to \infty$, $\frac{1}{n} \to 0$

$$= \frac{1 + 6(0) + 7(0)}{4 + 3(0) - 4(0)} = \frac{1}{4}$$

$$\lim s_n = \frac{1}{4}$$

**10. Find $\lim \frac{n-5}{n^2+7}$.**

*Proof :*

Given that $s_n = \frac{n-5}{n^2+7}$

$$\lim s_n = \lim \left(\frac{n-5}{n^2+7}\right)$$

$$= \lim \left[\frac{n^2\left[\frac{1}{n} - \frac{5}{n^2}\right]}{n^2\left[1 + \frac{7}{n^2}\right]}\right]$$

$$= \frac{\lim\frac{1}{n} - 5\lim\frac{1}{n^2}}{1 + 7\lim\frac{1}{n^2}}$$

as $n \to \infty$, $\frac{1}{n} \to 0$

$$= \frac{0}{1}$$

$$\therefore \lim \frac{n-5}{n^2+7} = 0$$

---

**11. Show that $\lim\limits_{n\to\infty} \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + ..... + \frac{1}{n}\right) = 0.$**

*Sol.*

Let $a_n = \frac{1}{n}$

then $\lim\limits_{n\to\infty} a_n = 0$

by cauchy's first theorem on limits we have

$$\lim_{n\to\infty} \left[\frac{a_1 + a_2 + ..... + a_n}{n}\right] = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\left[1 + \frac{1}{2} + \frac{1}{3} + ..... + \frac{1}{n}\right] = 0$$

**12. Show that $\lim_{n\to\infty}\frac{1}{n}\left(1+2^{1/2}+3^{1/2}+.....+n^{1/n}\right) = 1.$**

*Sol.*

Let $a_n = n^{1/n}$

then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{1/n} = 1$

By cauchy's first theorem on limits we have

$$\therefore \quad \lim_{n\to\infty}\left[\frac{a_1+a_2+.....+a_n}{n}\right] = 1$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{1}{n}\left[1+2^{1/2}+3^{1/3}+.....+n^{1/2}\right] = 1$$

**13. Using cauchy's first theorem on limits show that**

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+n}}\right] = 1.$$

*Sol.*

Let $a_K = \frac{n}{\sqrt{n^2+K}}$ then $a_n = \frac{n}{\sqrt{n^2+n}}$

$$\Rightarrow \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}} = 1$$

$\therefore$ By cauchy's first theorem on limits we have

$$\lim_{n\to\infty}\left[\frac{a_1+a_2+.....+a_n}{n}\right] = 1$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n}\left[\frac{n}{\sqrt{n^2+1}}+\frac{n}{\sqrt{n^2+2}}+.....+\frac{n}{\sqrt{n^2+n}}\right] = 1$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+.....+\frac{n}{\sqrt{n^2+n}}\right] = 1$$

**14. Prove that $\lim_{n\to\infty}\left(\frac{n^n}{n!}\right)^{1/n} = e.$**

*Sol.*

Let $a_n = \frac{n^n}{n!}$

then $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

consider

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!}\times\frac{n!}{n^n}$$

$$\Rightarrow \frac{(n+1)^n(n+1)}{n!(n+1)}.\frac{n!}{n^n} = \frac{(n+1)^n}{n^n}$$

$$\Rightarrow \quad \frac{n^n(1+1/n)^n}{n^n}$$

$$\Rightarrow \quad (1+1/n)^n$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}(1+1/n)^n = e$$

$\therefore$ By cauchy's second theorem on limits

$$\lim_{n\to\infty}(a_n)^{1/n} = e \Rightarrow \lim_{n\to\infty}\left(\frac{n^n}{n!}\right)^{1/n} = e.$$

**15. Show that $\lim_{n\to\infty}\frac{(n!)^{1/n}}{n} = \frac{1}{e}.$**

*Sol.*

Given $\lim_{n\to\infty}\frac{(n!)^{1/n}}{(n^n)^{1/n}}$

Let $a_n = \frac{n!}{n^n} \Rightarrow a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

Consider

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}}\times\frac{n^n}{n!}$$

$$\Rightarrow \frac{n!(n+1).n^n}{(n+1)^n.(n+1).n!} = \frac{n^n}{n^n[1+1/n]^n} = \frac{1}{(1+1/n)^n}$$

$$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e}$$

$\therefore$ By cauchy's second theorem on limits

$$\lim_{n\to\infty} (a_n)^{1/n} = \frac{1}{e}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left[\frac{n!}{n^n}\right]^{1/n} = \frac{1}{e} \Rightarrow \lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}$$

---

**16. Prove that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ where x is any real number.**

*Sol.*

Let $a_n = \frac{x^n}{n!}$

$\because$ x is any real number, three cases arises they are

(i) $x = 0$

Where $x = 0 \Rightarrow a_n = 0 \ \forall n$

$$\Rightarrow \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

(ii) when $x < 0$ or $x > 0$

$$a_n = \frac{x^n}{n!} \quad a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

consider

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{x^n.x.n!}{n!(n+1)x^n} = \frac{x}{n+1}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1$$

we know that if $\{a_n\}$ is a sequence $\ni a_n \neq 0 \ \forall n$ and $\frac{a_{n+1}}{a_n} \to l$ where $|l| < 1$ then $a_n \to 0$ as $n \to \infty$

$$\therefore \lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

**17. Prove that the sequence is $\left\{\left(\frac{(3n)!}{(n!)^3}\right)^{1/n}\right\}$ convergent**

*Sol.* **(Imp.)**

Let $a_n = \frac{(3n)!}{(n!)^3}$ = then $a_{n+1} = \frac{3(n+1)!}{((n+1)!)^3}$

Consider

$$\frac{a_{n+1}}{a_n} = \frac{(3n+3)!}{[(n+1)!]^3} \times \frac{(n!)^3}{(3n)!}$$

$$= \frac{(3n+3)(3n+2)(3n+1)(3n)!}{[n!(n+1)]^3} \times \frac{(n!)^3}{(3n)!}$$

$$\Rightarrow \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3} = \frac{3(3n+2)(3n+1)}{(n+1)^2}$$

$$\Rightarrow \frac{3.9n^2\left[1+\frac{2}{3n}\right]\left[1+\frac{1}{3n}\right]}{n^2\left[1+\frac{1}{n}\right]^2} = \frac{27\left[1+\frac{2}{3n}\right]\left[1+\frac{1}{3n}\right]}{\left[1+\frac{1}{n}\right]^2}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 27 \lim_{n\to\infty} \frac{\left(1+\frac{2}{3n}\right)\left(1+\frac{1}{3n}\right)}{\left(1+\frac{1}{n}\right)^2}$$

$\Rightarrow \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 27$

By cauchy's second theorem on limits

$\Rightarrow \quad \lim_{n\to\infty} (a_n)^{1/n} = 27$

Let $(a_n)^{1/n} = x_n = \left[\frac{3n!}{(n!)^3}\right]^{1/n}$

$\Rightarrow \quad \lim_{n\to\infty} x_n = 27$

$\Rightarrow \quad \{x_n\}$ is convergent

$\Rightarrow \quad \left[\frac{3n!}{(n!)^3}\right]^{1/n}$ is convergent

**18. Show that $\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+.....\frac{1}{(2n)^2}\right]=0$.**

*Sol.*

Let $a_n = \frac{1}{n^2}+\frac{1}{(n+1)^2}+.....+\frac{1}{(2n)^2}$

We know that

$n^2 < (2n)^2$, $\quad (n+1)^2 < (2n)^2$

$\Rightarrow \frac{1}{n^2} > \frac{1}{(2n)^2} \quad \Rightarrow \frac{1}{(n+1)^2} > \frac{1}{(2n)^2}$ and so on

$$\Rightarrow a_n > \frac{1}{(2n)^2}+\frac{1}{(2n)^2}+.....+\frac{1}{(2n)^2}$$

$$\Rightarrow a_n > \frac{n+1}{(2n)^2}$$

$$\Rightarrow a_n > \frac{n}{4n^2} \qquad \because n+1 > n$$

$$\Rightarrow a_n > \frac{1}{4n} \qquad ... (1)$$

Also

$$a_n = \frac{1}{n^2}+\frac{1}{(n+1)^2}+.....+\frac{1}{(2n)^2}$$

$$\Rightarrow a_n < = \frac{1}{n^2}+\frac{1}{n^2}+.....+\frac{1}{n^2} \qquad \because (n+1)^2 > n^2$$

$$\Rightarrow a_n < \frac{n+1}{n^2} \qquad \Rightarrow \frac{1}{(n+1)^2} < \frac{1}{n^2} \text{ and so on}$$

$$\Rightarrow a_n < \frac{1}{n}+\frac{1}{n^2} \qquad ... (2)$$

$\therefore$ from (1) and (2) we get

$$\frac{1}{4n} < a_n < \frac{1}{n}+\frac{1}{n^2}$$

also as $n \to \infty$, $\frac{1}{4n} \to 0$ and $\left(\frac{1}{n}+\frac{1}{n^2}\right) \to 0$

$\therefore$ by squeeze theorem, we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+.....+\frac{1}{(2n)^2}\right] = 0$$

**19. For each sequence below determine it converges.**

**a)** $a_n = \frac{n}{n+1}$

**b)** $b_n = \frac{n^2+3}{n^2-3}$

**c)** $C_n = 2^{-n}$

**d)** $t^n = 1 + \frac{2}{n}$

**e)** $x_n = 73 + (-1)^n$

**f)** $S_n = (2)^{\frac{1}{n}}$

*Sol.*

**a)** $a_n = \frac{n}{n+1}$.

Given equation is $a_n = \frac{n}{n+1}$

Apply limit on both sides

We get

$$\underset{n\to\infty}{\text{Lt}}\, a_n = \underset{n\to\infty}{\text{Lt}} \frac{n}{n+1}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{n}{n\left(1+\frac{1}{n}\right)}$$

$$= \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{1+\frac{1}{n}}\right)$$

$$= \frac{1}{1+\frac{1}{0}}$$

$$\underset{n\to\infty}{\text{Lt}}\, a_n = 1$$

For each $\in > 0 \;\exists m \in z^+ \;\ni \left|\frac{n}{n+1} - 1\right| < \in$

**b)** $b_n = \frac{n^2+3}{n^2-3}$

Given equation is $b_n = \frac{n^2+3}{n^2-3}$

Apply limit on both sides

We get,

$$\underset{n\to\infty}{\text{Lt}}\, b_n = \underset{n\to\infty}{\text{Lt}} \frac{n^2+3}{n^2-3}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1+\frac{3}{n^2}}{1-\frac{3}{n^2}}$$

$$= \frac{1+\frac{3}{\infty}}{1-\frac{3}{\infty}}$$

$$= \frac{1+0}{1-0}$$

$$\underset{n\to\infty}{\text{Lt}}\, b_n = 1$$

For each $\in > 0 \;\exists m \in z^+ \;\ni \left|\frac{n^2+3}{n^2-3} - 1\right| < \in$

**c)** $C_n = 2^{-n}$

Given equation is $C_n = 2^{-n}$

Apply limit on bothsides we get

$$\underset{n\to\infty}{\text{Lt}}\, C_n = \underset{n\to\infty}{\text{Lt}}\, 2^{-n}$$

This can be written as

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{2^n}$$

$$= \frac{1}{2^\infty} = \frac{1}{\infty} = 0$$

$$\therefore \underset{n\to\infty}{\text{Lt}}\, C_n = 0$$

For each $\in > 0 \;\exists m \in z^+ \;\ni \left|2^{-n} - 0\right| < \in$

**d)** $t^n = 1 + \frac{2}{n}$

Given equation is $t^n = 1 + \frac{2}{n}$

Apply limit on bothsides we get

$$\underset{n\to\infty}{Lt}\, t_n = \underset{n\to\infty}{Lt}\left(1+\frac{2}{n}\right)$$

$$= 1+\frac{2}{\infty} = 1+0$$

$$\underset{n\to\infty}{Lt}\, t_n = 1$$

For each $\in > 0\ \exists m \in z^+ \ni \left|\left(1+\frac{2}{n}\right)-1\right| < \in$

**e)** $x_n = 73 + (-1)^n$

Given equation is $x_n = 73 + (-1)^n$

$x_1 = 73 + (-1)^1 = 73 - 1 = 72$

$x_2 = 73 + (-1)^2 = 73 + 1 = 74$

$x_3 = 73 + (-1)^3 = 73 - 1 = 72$

$x_4 = 73 + (-1)^4 = 73 + 1 = 74$

$x_n = \{72, 74\}$

$x_n$ is not a convergent sequence

$x_n$ is a oscillatory sequence

**f)** $S_n = (2)^{\frac{1}{n}}$

Given that $S_n = (2)^{\frac{1}{n}}$

Apply limit on bothsides, we get

$$\underset{n\to\infty}{Lt}\, S_n = \underset{n\to\infty}{Lt}\,(2)^{1/n}$$

$$= 2^{\frac{1}{\infty}} = (2)^0$$

$$\underset{n\to\infty}{Lt}\, S_n = 1$$

For each $\in > 0\ \exists m \in z^+ \ni \left|(2)^{1/n} - 1\right| < \in$

**20. Determine the limits of the following sequences and then prove your claims.**

**a)** $a_n = \frac{n}{n^2+1}$

**b)** $b_n = \frac{7n-19}{3n+7}$

**c)** $C_n = \frac{4n+3}{7n-5}$

**d)** $d_n = \frac{2n+4}{5n+2}$

**e)** $S_n = \frac{1}{n}\sin n$

*Sol.*

**a)** $a_n = \frac{n}{n^2+1}$

Given that

$$a_n = \frac{n}{n^2+1}$$

$$= \frac{n}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \frac{1}{n}\,\frac{1}{\left(1+\frac{1}{n^2}\right)}$$

Applying limits on both sides

$$\underset{n\to\infty}{Lt}\, a_n = \underset{n\to\infty}{Lt}\,\frac{1}{n}\,\frac{1}{\left(1+\frac{1}{n^2}\right)}$$

$$= \frac{1}{\infty}\cdot\frac{1}{1+\frac{1}{\infty}}$$

$$= 0\cdot\left(\frac{1}{1+0}\right)$$

$$\underset{n\to\infty}{Lt}\, a_n = 0$$

For each $\in > 0\ \exists m \in z^+ \ni \left|\frac{n}{n^2+1} - 0\right| < \in$

**b)** $b_n = \dfrac{7n-19}{3n+7}$

Given that $b_n = \dfrac{7n-19}{3n+7}$

Applying limits on both sides

$$\underset{n\to\infty}{Lt}\ b_n = \underset{n\to\infty}{Lt}\ \frac{7n-19}{3n+7}$$

$$= \underset{n\to\infty}{Lt}\ \frac{7-\frac{19}{n}}{3+\frac{7}{n}}$$

$$= \frac{7-\frac{19}{\infty}}{3+\frac{7}{\infty}}$$

$$\underset{n\to\infty}{Lt}\ b_n = \frac{7}{3}$$

For each $\in > 0\ \exists m \in z^+ \ni \left|\dfrac{7n-19}{3n+7} - \dfrac{7}{3}\right| < \in$

$$\Rightarrow \left|\frac{21n-57-21n-49}{9n+21}\right| < \in$$

(or)

$$\left|\frac{-106}{9n+21}\right| < \in$$

$$\left|\frac{-106}{3(3n+7)}\right| < \in$$

$\because$ 3n + 7 > 0, we can drop the absolute value manipulate, the inequality to solve for n.

$$\frac{106}{3\in} < 3n + 7$$

$$\frac{106}{3\in} - 7 < 3n$$

$$\frac{106}{9\in} - \frac{7}{3} < n$$

So we will put $n = \dfrac{106}{9\in} - \dfrac{7}{3}$

$\therefore$ N to be any number larger than $\dfrac{106}{9\in} - \dfrac{7}{3}$.

**c)** $C_n = \dfrac{4n+3}{7n-5}$

Given that $C_n = \dfrac{4n+3}{7n-5}$

Apply limits on both sides

$$\underset{n\to\infty}{Lt}\ C_n = \underset{n\to\infty}{Lt}\ \frac{4n+3}{7n-5}$$

$$= \underset{n\to\infty}{Lt}\ \frac{4+\frac{3}{n}}{7-\frac{5}{n}}$$

$$= \frac{4+\frac{3}{\infty}}{7-\frac{5}{\infty}}$$

$$\underset{n\to\infty}{Lt}\ C_n = \frac{4+0}{7-0}$$

$$\underset{n\to\infty}{Lt}\ C_n = \frac{4}{7}$$

$\therefore$ For each $\in > 0\ \exists m \in z^+ \ni \left|\dfrac{4n+3}{7n-5} - \dfrac{4}{7}\right| < \in$

$$\left|\frac{28n+21-28n+20}{49n-35}\right| < \in$$

$$\left|\frac{1}{7(7n-5)}\right| < \in$$

7n – 5 > 0, we can drop the absolute value and manipulate the inequality to solve for n.

$$\frac{1}{7\in} < 7n - 5$$

$$\frac{1}{7\in} + 5 < 7n$$

$$\frac{1}{7(7\in)} + \frac{5}{7} < n$$

$$\frac{1}{49\in} + \frac{5}{7} < n$$

$\therefore$ So we will put $N = \dfrac{1}{49\in} + \dfrac{5}{7}$

$\therefore$ N to be any number larger than $\dfrac{1}{49\in} + \dfrac{5}{7}$

**d)** $\mathbf{d_n = \frac{2n+4}{5n+2}}$

Given that $d_n = \frac{2n+4}{5n+2}$

Apply limit on both sides

$$\underset{n\to\infty}{Lt} d_n = \underset{n\to\infty}{Lt} \frac{2n+4}{5n+2} = \underset{n\to\infty}{Lt} \frac{2+\frac{4}{n}}{5+\frac{2}{n}} = \frac{2+\frac{4}{\infty}}{5+\frac{2}{\infty}} = \frac{2+0}{5+0}$$

$$\underset{n\to\infty}{Lt} d_n = \frac{2}{5}$$

For each $\in > 0$ $\exists m \in z^+ \ni \left|\frac{2n+4}{5n+2} - \frac{2}{5}\right| < \in$

$$\left|\frac{10n+20-10n-4}{25n+10}\right| < \in$$

$$\left|\frac{16}{5(5n+2)}\right| < \in$$

5n + 2 > 0, we can drop the absolute values manipulate the inequality to solve for n.

$$\frac{16}{5\in} < 5n + 2$$

$$\frac{16}{5\in} - 2 < 5n$$

$$\frac{16}{25\in} - \frac{2}{5} < n$$

$\therefore$ So we will put $N = \frac{16}{25\in} - \frac{2}{5}$

$\therefore$ N to be any number larger than $\frac{16}{25\in} - \frac{2}{5}$.

**e)** $\mathbf{S_n = \frac{1}{n}\sin n}$

Given that $S_n = \frac{1}{n}\sin n$

Apply limit on both sides

$$\underset{n\to\infty}{Lt} S_n = \underset{n\to\infty}{Lt} \frac{1}{n}\sin n$$

$$\underset{n\to\infty}{Lt} S_n = \underset{n\to\infty}{Lt} \sin\infty$$

$$= 0 \sin \infty$$

$$= 0$$

## 1.4 LIMIT THEOREMS FOR SEQUENCES

**21. Show that If $\lim_{n\to\infty} a_n = l$ then $\lim_{n\to\infty}\left[\frac{a_1 + a_2 + .....a_n}{n}\right] = l$.**

*Sol.*

Define the sequence $\{b_n\}$ such that $b_n = a_n - l$

for all $n \in z^+$

$\therefore \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} (a_n - l) \lim_{n\to\infty} a_n - l = l - l = 0$

$\lim_{n\to\infty} b_n = 0$

$\Rightarrow$ for each $\in > 0$ such that $\exists\ r \in z^+$ such that $|b_n - 0| = |b_n| < \in/2\ \forall n \geq r$

$\because \quad \lim_{n\to\infty} b_n = 0 \Rightarrow \{b_n\}$ is bounded $\Rightarrow \exists\ K \in R^+ \ni |b_n| < K\ \forall n \in z^+$

$$\therefore \quad \left|\frac{b_1 + b_2 + .....b_n}{n}\right| = \left|\frac{b_1 + b_2 + .....b_r}{n} + \frac{b_{r+1} + .....b_n}{n}\right|$$

$$\leq \frac{K + K + ..... + K(r\text{ times})}{n} + \frac{\in/2 + \in/2 + ..... + \in/2(n-r)\text{times}}{n}$$

$$\leq \frac{rK}{n} + \frac{(n-r)\in}{2n}$$

$$\leq \frac{rK}{n} + \frac{\in}{2} - \frac{r\in}{2n}$$

$$< \frac{rK}{n} + \frac{\in}{2} \qquad \because \frac{\in}{2} - \frac{r\in}{2n} < \in/2$$

$$\therefore \quad \left|\frac{b_1 + b_2 + .....b_n}{n}\right| < \frac{\in}{2} + \frac{\in}{2} < \in \qquad \text{put } m = \frac{2Kr}{\in}$$

$$\Rightarrow \quad \left|\frac{b_1 + b_2 + .....b_n}{n} - 0\right| < \in\ \forall\ n > m$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[\frac{b_1 + b_2 + ..... + b_n}{n}\right] = 0$$

but we have

$$\frac{b_1 + b_2 + ..... + b_n}{n} = \frac{(a_1 - l) + (a_2 - l) + ..... + (a_n - l)}{n}$$

$$= \frac{a_1 + a_2 + ..... + a_n}{n} - l$$

$$\therefore \quad \lim_{n\to\infty} \frac{b_1+b_2+.....+b_n}{n} = \lim_{\to\infty} \frac{a_1+a_2+.....+a_n}{n} - l$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[\frac{a_1+a_2+.....+a_n}{n}\right] = \lim_{n\to\infty}\left[\frac{b_1+b_2+.....+b_n}{n}\right] + l$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[\frac{a_1+a_2+.....+a_n}{n}\right] = 0 + l = l$$

**Note:**

If $a_n > 0 \ \forall n \in z^+$ and $a_n = l$ then $\lim\limits_{n\to\infty} (a_1, a_2 .....a_n)^{1/n} = l$

---

**22. If $\{a_n\}$ is a sequence such that $a_n > 0 \ \forall n \in z^+$ and $\lim\limits_{n\to\infty} \frac{a_n+1}{a_n} = l$ then $\lim\limits_{n\to\infty} \lim\limits_{n\to\infty} = l$.**

*Sol.*

Let the sequence $\{b_n\}$ defined by $b_1 = a_1$, $b_2 = \frac{a_2}{a_1}$, $b_3 \frac{a_3}{a_2}$ ...... $b_n = \frac{a_n}{a_{n-1}}$ ......

so that $b_1, b_2, b_3......b_n = a_n$

$\because \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l \Rightarrow \lim\limits_{n\to\infty} \frac{a_n}{a_{n-1}} = l \Rightarrow \lim\limits_{n\to\infty} b_n = l$

$\because \ a_n > 0 \ \forall n \Rightarrow b_n > 0 \ \forall n$

$\therefore$ Now we have a sequence $\{b_n\}$ such that $b_n > 0 \ \forall$ n and $\lim\limits_{n\to\infty} b_n = l$

$\Rightarrow (b_1, b^2, b_3.....b_n)^{1/n} = l$

$\Rightarrow \lim\limits_{n\to\infty} (a_n)^{1/n} = l$

---

**23. Let $\{s_n\}$ be sequence in R prove that the lim $s_n = 0$ iff lim $|s_n| = 0$.**

*Sol.* **(Imp.)**

Let $\{s_n\}$ be a sequence in R.

Suppose that lim $s_n$ = l

i.e., for each $\varepsilon > 0 \exists n \ni | s_n - \ell | <\in \ \forall n \geq N$

We know that lim $s_n = 0$

$\Rightarrow$ for each $\varepsilon > 0 \ \exists n \in N \ni |s_n - 0| < \varepsilon \ \forall n \geq N$

$\Leftrightarrow \ |s_n - 0| < \varepsilon$

$\Leftrightarrow \ ||s_n| - 0| < \varepsilon$

$\Leftrightarrow \ \lim|s_n| = 0$

Hence lim $s_n = 0 \Leftrightarrow \lim|s_n| = 0$.

**24. If the sequence $\{s_n\}$ converges to s and $K \in R$ then the sequence $\{ks_n\}$ converges to ks that is, $\lim\{ks_n\} = k \lim s_n$.**

*Sol.*

Given that $\{s_n\}$ is converges to s i.e., $\lim s_n = s$.

$\Rightarrow$ for each $\varepsilon > 0 \exists n \in N \ni |s_n - s| < \frac{\varepsilon}{|k|} \quad \forall n \geq N$ ... (1)

also, $\{ks_n\}$ converges to ks

$\Rightarrow$ for each $\varepsilon > 0 \exists n \in N \ni |ks_n - ks| < \varepsilon$ ... (2)

Required to prove $\lim ks_n = k \lim s_n$

consider $|ks_n - k_s| = |k(s_n - s)|$

$\leq |k| \, |s_n - s|$

$\leq |k| \frac{\varepsilon}{|k|}$ [$\because$ by(1)]

$\Rightarrow |ks_n - ks| < \varepsilon$

for each $\varepsilon > 0 \exists n \in N \ni |ks_n - ks| < \varepsilon \; \forall n \geq N$

$\lim ks_n = ks$

$\lim ks_n = k \lim s_n$ [$\because s = \lim s_n$]

**25. If $\{s_n\}$ is converges to s, and $\{t_n\}$ is converges to 't'. Then $\{s_n + t_n\}$ converges to s + t that is $\lim \{s_n + t_n\} = \lim s_n + \lim t_n$.**

*Sol.* **(Dec.-2017, Imp.)**

Given that, $\{s_n\}$ converges to s i.e., $\lim s_n = s$.

$\Rightarrow$ for each $\varepsilon > 0 \exists n \in N_1 \ni |s_n - s| < \frac{\varepsilon}{2} \; \forall \, n \geq N_1$ ... (1)

also, $\{t_n\}$ converges to t i.e., $\lim t_n = t$

$\Rightarrow$ for each $\varepsilon > 0 \exists n \in N_2 \ni |t_n - t| < \frac{\varepsilon}{2} \; \forall \, n \geq N_2$ ... (2)

Required to prove $\{s_n + t_n\}$ converges to s + t

To prove for each $\varepsilon > 0 \exists n \in N \ni |(s_n + t_n) - (s + t)| \quad \forall n > N$

Let $N = \max \{N_1, N_2\}$

From (1) $\Rightarrow \varepsilon > 0 \exists n \in N \ni |s_n - s| < \frac{\varepsilon}{2} \quad \forall n > N$ ... (3)

From (2) $\Rightarrow \varepsilon > 0 \exists n \in N \ni |t_n - t| < \frac{\varepsilon}{2} \quad \forall n > N$ ... (4)

consider $|(s_n + t_n) - (s + t)| = |s_n + t_n - s - t|$

$$= |(s_n - s) + (t_n - t)|$$

$$= |s_n - s| + |t_n - t|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{[From } (3) \rightarrow (4)]$$

$$= \frac{2\varepsilon}{2} = \varepsilon$$

$$|s_n + t_n - (s + t)| < \varepsilon$$

$\Rightarrow \lim (s_n + t_n) = s + t$

$\Rightarrow \lim (s_n + t_n) = \lim s_n + \lim t_n$

**26. If $\{s_n\}$ is converges to s and $\{t_n\}$ is converges to t, then $\{s_n \ t_n\}$ converges to st i.e., lim $(s_n t_n) = (\lim s_n)(\lim t_n)$.**

*Sol.* **(May/June - 18, Nov./Dec.-18, Imp.)**

$\{s_n\}$ is converges to s

$\Rightarrow \lim s_n = s$

i.e., for each $\varepsilon > 0 \exists n \in N_1 \ni |s_n - s| < \dfrac{\varepsilon}{2|t|+1} \ \forall n \geq N$ ... (1)

$\{t_n\}$ is converges to t

$\Rightarrow \lim t_n = t$

for each $\varepsilon > 0 \exists n \in N_2 \ni |t_n - t| < \dfrac{\varepsilon}{2M} \ \forall n \geq N$ ... (2)

required to prove that lim $\{s_n \ t_n\}$ converges to st.

i.e., to prove for each $\varepsilon > 0 \exists n \in N \ni |s_n t_n - st| < \varepsilon \forall n \geq N$.

Let $N = \max \{N_1, N_2\}$

consider $|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st|$

$$= |(s_n t_n - s_n t) + (s_n t - st)|$$

$$= |s_n(t_n - t) + |t(s_n - s)|$$

$$= |s_n||t_n - t| + |t(s_n - s)|$$

$$\leq |s_n||t_n - t| + |t||s_n - s|$$

$$\leq |s_n| \frac{\varepsilon}{2M} + |t| \frac{\varepsilon}{2|t|t|} \qquad \text{... (3)}$$

To solve above inequality

We know that every convergent sequence is bounded.

Since $\{s_n\}$ is convergent then it is bounded

i.e., $M > 0 \ni |s_n| \leq M \ \ \forall n$ ... (4)

From (3)

$$\Rightarrow \quad |s_n t_n - st| \leq |s_n| \frac{\varepsilon}{2M} + |t| \frac{\varepsilon}{2|t|+1}$$

$$\leq M \frac{\varepsilon}{2M} + |t| \frac{\varepsilon}{2|t|+1} \quad \text{(by 4)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$< \frac{2\varepsilon}{2} = \varepsilon$$

$\therefore |s_n t_n - st| < \varepsilon$

for each $\varepsilon > 0 \exists n \in N \ni |s_n t_n - st| < \varepsilon \ \forall n > N$

$\Rightarrow \lim s_n t_n = st$

$\lim s_n t_n = \lim s_n \lim t_n$

Hence proved.

**27. If $\{s_n\}$ converges to s, if $s_n \neq 0 \ \forall n$ and if $s \neq 0$, then $\left\{\frac{1}{s_n}\right\}$ converges to $\frac{1}{s}$.**

**(Nov./Dec.-18, Imp.)**

*Sol.*

Let $\varepsilon > 0 \ \exists m > 0 \ni |s_n| \geq m \ \forall n$

Since $\lim s_n = s$ there exists N suits that

$n > N \Rightarrow |s - s_n| < \varepsilon . m|s|$

Then $n > N \Rightarrow$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right|$$

$$= \left|\frac{s - s_n}{s_n s}\right| \leq \frac{|s - s_n|}{|s_n||s|}$$

$$< \frac{\varepsilon m|s|}{m|s|}$$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon$$

for each $\varepsilon > 0 \exists n \in N \ni \left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon \ \forall n > N$

$$\therefore \lim \left\{\frac{1}{s_n}\right\} = \frac{1}{s}$$

**28. Suppose that $\{s_n\}$ converges to s and $\{t_n\}$ converges to t. If s $\neq$ 0 and $s_n \neq 0 \ \forall$ n then $\left\{\frac{t_n}{s_n}\right\}$ converges to $\frac{t}{s}$.**

*Sol.*

$\{s_n\}$ is converges to s

By previous theorem $\left\{\frac{1}{s_n}\right\}$ is converges to $\frac{1}{s}$ and also, $\{t_n\}$ converges to t.

$\Rightarrow \quad \lim t_n = t$

Required to prove $\left\{\frac{t_n}{s_n}\right\}$ is converges to $\frac{t}{s}$.

i.e., for each $\varepsilon > 0 \exists n \in N \ni \left|\frac{t_n}{s_n} - \frac{t}{s}\right| < \varepsilon \quad \forall n > N$ or $\lim \frac{t_n}{s_n} = \frac{t}{s}$.

$$\Rightarrow \quad \lim \frac{t_n}{s_n} = \lim \frac{1}{s_n} . t_n$$

$$= \lim \frac{1}{s_n} . \lim t_n$$

$$= \frac{1}{s}.t$$

$$\lim \frac{t_n}{s_n} = \frac{t}{s}.$$



### 1.4.1 Divergent Sequence

(i) A sequence $\{a_n\}$ is said to diverge to $+\infty$ if given any positive real number K, however large $\exists$ a positive integer m such that $a_n > k \quad \forall n \geq m$.

(ii) A sequence $\{a_n\}$ is said to diverge to $-\infty$ if given any positive real number K, however large, $\exists$ a positive integer m such that $a_n < -k \ \forall n \geq m$.

### 1.4.2 Oscillatory Sequence

If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence.

**Note :**

If $\lim\limits_{n \to \infty} a_n = 0$ then sequence $\{a_n\}$ is called as null sequence.

**29. Give a formal proof that lim $[\sqrt{n}+7] = +\infty$.**

*Sol.*

Given that lim $(\sqrt{n}+7) = +\infty$

for each $\varepsilon > 0 \exists N \ni n > N \Rightarrow s_n > M$

$\Rightarrow \sqrt{n} + 7 > M$

$\Rightarrow \sqrt{n} > M - 7$

$\Rightarrow n > (M-7)^2$

we will take $N = (M-7)^2$

Formal proof

Let $M > 0$ and Let $N = (M-7)^2$

Then $n > N \Rightarrow n > (M-7)^2$

hence $\sqrt{n} > M - 7$

$\sqrt{n} + 7 > M$

$\lim(\sqrt{n} + 7) = +\infty$

**30. Let $\{s_n\}$ and $\{t_n\}$ be sequence such that $\lim s_n = +\infty$ and $\lim t_n > 0$ [$\lim t_n$ can be finite or $+\infty$] then $\lim s_n t_n = +\infty$].**

*Sol :*

Given that $\{s_n\}$ is sequence which is diverges to $+\infty$.

i.e., $\lim s_n = +\infty$

for each $\varepsilon > 0 \exists n \ni n > N_1 \Rightarrow s_n > \dfrac{M}{m}$ ... (1)

Let $\{t_n\}$ be sequences, then $\lim t_n > 0$ or $\lim t_n = +\infty$

Let $M > 0$

Select a real number m so that $0 < m < \lim t_n$ $\exists$ $N_2$ such that $n > N_2 \Rightarrow t_n > m$.

Put $N = \max\{N_1, N_2\}$

Then $n > N \Rightarrow s_n t_n = t_n$

$\Rightarrow \dfrac{M}{\cancel{m}} . \cancel{m}$

$s_n t_n > M$

$\Rightarrow \lim s_n t_n = +\infty$

**31. Prove that lim $\dfrac{n^2+3}{n+1} = +\infty$.**

*Sol :*

Observe that $\dfrac{n^2+3}{n+1} = \dfrac{n\left[\frac{1}{n}+\frac{3}{n}\right]}{n\left[1+\frac{1}{n}\right]}$

$= \dfrac{\frac{1}{n}+\frac{3}{n}}{1+\frac{1}{n}}$

$= s_n . t_n$

Where $s_n = \dfrac{1}{n}+\dfrac{3}{n}$ and $t_n = \dfrac{1}{1+\frac{1}{n}}$

$\because$ $\lim s_n t_n = \lim\left(n+\dfrac{3}{n}\right)\left(\dfrac{1}{1+\frac{1}{n}}\right)$

$\lim s_n = \lim\left(n+\dfrac{3}{n}\right)$

$\lim s_n = +\infty$

$\lim t_n = \lim \dfrac{1}{\left(1+\frac{1}{n}\right)} = 1$

$\lim s_n t_n = (+\infty)(1) = +\infty$

$\lim(s_n t_n) = +\infty$

**32. For a sequence $\{s_n\}$ of +ve real number we have $\lim s_n = +\infty$ if and only if $\lim \dfrac{1}{s_n} = 0$.**

*Sol.*

Let $\{s_n\}$ be sequence of +ve real numbers.

for each $M > 0 \ \exists n \in n > N \Rightarrow s_n > M$

Required to prove,

i.e., $\lim s_n = +\infty \Rightarrow \lim \frac{1}{s_n} = 0$ ... (1)

and $\lim \frac{1}{s_n} = 0 \Rightarrow \lim s_n = +\infty$ ... (2)

(i) Suppose $\lim s_n = +\infty$

Let $\varepsilon > 0$ and $M = \frac{1}{\varepsilon}$

since $\lim s_n = +\infty$

$\exists N \ni n > N \Rightarrow s_n > M = \frac{1}{\varepsilon}$

$\because n > N \Rightarrow s_n > \frac{1}{\varepsilon}$

$\varepsilon > \frac{1}{s_n}$

$\because \varepsilon > 0 \Rightarrow \frac{1}{s_n} > 0$

for each $\varepsilon > 0 \exists n > N \Rightarrow \left|\frac{1}{s_n} - 0\right| < \varepsilon$

$\Rightarrow \lim \frac{1}{s_n} = 0$

(ii) Suppose that $\lim \frac{1}{s_n} = 0$

at $M > 0$ and $\varepsilon = \frac{1}{M}$

Then $\varepsilon > 0 \ \exists N \ni n > N \Rightarrow \left|\frac{1}{s_n} - 0\right| < \varepsilon \left(= \frac{1}{M}\right)$

$\frac{1}{s_n} < \frac{1}{M}$

$s_n > 0$ we can write

$n > N \Rightarrow 0 < \frac{1}{s_n} < \frac{1}{M}$

$n > N \Rightarrow M < s_n$

then $\lim s_n = +\infty$.

**33. Let $t_1 = 1$ and $t_{n+1} = \frac{t_n^2 + 2}{2t_n}$ for $n \geq 1$. Assume that $\{t_n\}$ converges and find the limit.**

*Sol.* **(Nov./Dec.-18, Imp.)**

Let $\lim t_n = t$

$\{t_n\}$ is converges to t

$$t_{n+1} = \frac{t_n^2 + 2}{2t_n}$$

$$\lim t_{n+1} = \lim \left(\frac{t_n^2 + 2}{2t_n}\right) = \frac{\lim (t_n^2 + 2)}{\lim (2t_n)}$$

$$\lim (t_n^2 + 2) = \lim t_n^2 + 2$$
$$= \lim t_n . \lim t_n + 2$$
$$= t^2 + 2$$

$$\lim (2t_n) = 2 \lim t_n$$
$$= 2t$$

$$\lim t_{n+1} = \frac{\lim (t_n^2 + 2)}{\lim (2t_n)} = \frac{t^2 + 2}{2t}$$

$$\therefore \lim t_{n+1} = \frac{t^2 + 2}{2t}$$

here $t_1 = 1$

$n \geq 1$

If $n = 1 \Rightarrow t_{1+1} = \frac{t_1^2 + 2}{2t_1}$

$$t_2 = \frac{1^2 + 2}{2(1)}$$

$$t_2 = \frac{3}{2}$$

If $n = 2 \Rightarrow t_{2+1} = \frac{t_2^2 + 2}{2t_2} \Rightarrow \frac{\left(\frac{3}{2}\right)^2 + 2}{2\left(\frac{3}{2}\right)}$

$$= \frac{9+8}{12}$$

$$t_3 = \frac{17}{12}$$

If $n = 3 \Rightarrow t_{3+1} = \frac{t_3^2+2}{2t_3} \Rightarrow \frac{\left(\frac{17}{12}\right)^2+2}{2\left(\frac{17}{12}\right)}$

$$= \frac{6(289+288)}{2448}$$

$$\simeq 1.414257$$

Since $1.4142 \simeq \sqrt{2}$

$\therefore \lim \{t_n\}$ is converges to $\simeq \sqrt{2}$

**34. Suppose that there exists $N_0$ such that $s_n \leq t_n \forall n > N_0$.**

**(a) Prove that if $\lim s_n = +\infty$ then $\lim t_n = +\infty$.**

**(b) Prove that if $\lim t_n = -\infty$ then $\lim s_n = -\infty$**

**(c) Prove that if $\lim s_n$ and $\lim t_n$ exist. Then $\lim s_n \leq \lim t_n$.**

*Sol.*

Given that $\exists N_0 \ni s_n \leq t_n \ \forall n > N_0$

(a) If $\lim s_n = +\infty \Rightarrow \lim t_n = +\infty$

Suppose $\lim s_n = +\infty$

for each $M > 0 \exists\ n > N \Rightarrow s_n > M$

$\exists N_0 \ni s_n \leq t_n \ \forall n > N_0$

$M < s_n \leq t_n$

$M < t_n$

for each $M > 0 \exists\ n \ni t_n > M$

$\Rightarrow \lim t_n = +\infty$

(b) If $\lim t_n = -\infty$ then $\lim s_n = -\infty$

Suppose $\lim t_n = -\infty$

for each $M > 0 \exists\ n>N \Rightarrow t_n < M \ \forall n > N$

$\because \ s_n \leq t_n$

$\Rightarrow s_n \leq t_n < M$

$\Rightarrow s_n < M \qquad \forall n > N$

for each $M > 0 \exists\ n \ni s_n < M \ \forall n > N$

$\Rightarrow \lim s_n = -\infty$

(c) If $\lim s_n$ and $\lim t_n$ exist then $\lim s_n \leq \lim t_n$ from (a) and (b)

The limits one infinite

So, assume $\{t_n\}, \{s_n\}$ converges.

i.e., $t_n - s_n \geq 0 \qquad \forall n > N$

$\lim (t_n - s_n) \geq 0$

$\lim t_n - \lim s_n \geq 0$

$\lim t_n \geq \lim s_n$

$\lim s_n \leq \lim t_n$

**35. Calculate,**

$$\lim_{n\to\infty}\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+....+\frac{1}{3^n}\right).$$

*Sol.* **(June/July-19)**

Given that $\lim_{n\to\infty}\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+....+\frac{1}{3^n}\right)$

Which can written as

$$\Rightarrow \quad 1 + \lim_{n\to\infty}\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+....+\frac{1}{3^n}\right) \quad ... (1)$$

By GP, $S_n = \frac{a}{1-r} \quad r < 1$

Since $a =$ first term

$$r = \frac{t_2}{t_1} = \frac{1}{3} < 1$$

$$\Rightarrow \quad s_n = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

from (1) $1 + \lim_{n\to\infty}\left(\frac{1}{2}\right)$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{3}+\frac{1}{9}+.....+\frac{1}{3^n}\right)=\frac{3}{2}.$$

**36. Prove that $\lim_{n\to\infty}\frac{1}{n^p} = 0$ for $p > 0$.**

*Sol.* **(Imp.)**

To prove that $\lim_{n\to\infty}\frac{1}{n^p}$

Required to prove that, for each $\varepsilon>0\ \exists$ $n\in N \ni \left|\frac{1}{n^p}-0\right|<\varepsilon\ \ \forall n>N$

for $n > N$, $\left|\frac{1}{n^p}-0\right|<\varepsilon$

$$\left|\frac{1}{n^p}\right|<\varepsilon$$

$$\frac{1}{n^p}<\varepsilon$$

$$n^p > \frac{1}{\varepsilon}$$

$$n > \left(\frac{1}{\varepsilon}\right)^{1/p}$$

Selecting $N = \left(\frac{1}{\varepsilon}\right)^{1/p}$

for $n > N$

$$n > \left(\frac{1}{\varepsilon}\right)^{1/p}$$

$$n^p > \frac{1}{\varepsilon}$$

$$\frac{1}{n^p}<\varepsilon$$

$$\left|\frac{1}{n^p}-0\right|<\varepsilon$$

$$\lim_{n\to\infty}\frac{1}{n^p} = 0 \quad p > 0.$$

**37. Assume all $s_n \neq 0$ and that the Limit $L = \lim\left|\frac{s_{n+1}}{s_n}\right|$ exists.**

**(a) Show that if $L < 1$, then $\lim s_n = 0$**

**(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.**

*Sol.*

If $L < 1$ then $\lim s_n = 0$

Suppose that $L < a < 1$

So, $\varepsilon = a - L \Rightarrow L + \varepsilon = a$

Then $\exists$ N' where $n > N' \Rightarrow \left|\left|\frac{s_{n+1}}{s_n}\right| - L\right| < \varepsilon$

Let $N = N' + 1$ then $n \geq N \Rightarrow \left|\frac{s_{n+1}}{s_n}\right| < L + \varepsilon$ (–a)

$$\left|\frac{s_{n+1}}{s_n}\right| < a$$

$$\frac{|s_{n+1}|}{|s_n|} < a \Rightarrow |s_{n+1}| < a\,|s_n|$$

So, clearly $|s_{N+1}| < a|s_N|$ By Induction Now we see that

$$|s_{N+2}| < a\,|s_{N+1}| < a^2\,|s_N|$$

$|s_{N+k}| < a^k\,|s_N|$ for any $k > 0$

Changing variable and $n = N + k$ for $n > N$

we have $|s_n| < a^{n-N}\,|s_N|$

Now, $\lim a^{n-N}|s_N|$

$|s_N|$ is number so that,

limit is $|s_N| \lim a^{n-N}$

since $|a<1|$, $\lim a^n = 0$

since $|s_N| < a^{n-N}|s_N| \quad \forall n \geq N$

By sandwitch theorem $\lim s_n = 0$

(b) Let $t_n = \frac{1}{|s_n|} \Rightarrow \left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{s_n}{s_{n+1}}\right|$

So, we know that $\left|\frac{s_{n+1}}{s_n}\right|$ converges to L that is, $\left|\frac{s_{n+1}}{s_n}\right| \neq 0$.

$L \neq 0$

$\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{s_n}{s_{n+1}}\right|$ converse to $\frac{1}{L}$, L > 1 we know $\frac{1}{L} < 1$

Apply part (a) to conclude that

$\lim t_n = 0$

$\lim |t_n| = 0$

$|s_n|$ are the real number

$\lim \frac{1}{|s_n|} = \lim |t_n| = 0$

$\lim s_n = +\infty$.

**38. Suppose $\lim a_n = a$, $\lim b_n = b$, and $s_n = \frac{a_n^3 + 4an}{b_n^2 + 1}$ prove that $\lim S_n = \frac{a^3 + 4a}{b^2 + 1}$ carefully, using the limit theorems.**

*Sol.*

Given that $\lim a_n = a$, $Lt\, b_n = b$

$$\therefore \quad S_n = \frac{a_n^3 + 4a_n}{b_n^2 + 1} = \frac{a^3 + 4a}{b^2 + 1}$$

First we use by known theorem. If $(S_n)$ converges to s and $(t_n)$ converges to t, then $(S_n t_n)$ converges to $S_t$.

$\Rightarrow \quad Lt\,(S_t t_n) = (L_t S_n)(L_t t_n)$

$$Lt\; a_n^3 = Lt\, a_n \cdot L_t\, a_n^2 \cdot Lt\, a_n^2$$

$$= a\, Lt\, a_n \cdot \lim a_n$$

$$= a \cdot a \cdot a$$

$$= a^3$$

We have that $(S_n + t_n) = Lt\, S_n + Lt\, t_n$

$$\therefore Lt\,(a_n^3 + 4an) = \lim a_n^3 + 4 \cdot Lt\, a_n$$

$$= a^3 + 4a$$

Similarly,

$$Lt\left(b_n^2 + 1\right) = \lim b_n \cdot Lt\, b_n + 1$$

$$= b \cdot b + 1 = b^2 + 1$$

Since $b^2 + 1 \neq 0$ [$\because$ by known theorem

$$\therefore Lt\, S_n = \frac{(a^3 + 4a)}{(b^2 + 1)}$$

$$Lt\, \frac{t_n}{S_n} = Lt\, \frac{1}{S_n} \cdot t_n$$

$\therefore$ Hence the proof.

**39. Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ far $n \geq 1$**

**Show if $a = Lt\, x_n$, then $a = \frac{1}{3}$ or $a = 0$.**

*Sol.*

**a) Let $x_1 = 1$, $n = 1$**

$$x_2 = 3x_1^2 = 3$$

$$n = 2 \Rightarrow x_3 = 3x_2^2 = 3(3)^2 = 27$$

$$\therefore a = Lt\, x_n$$

$$\underset{x \to \infty}{Lt}\, x_n = 3x_{n-1}^2$$

$$\therefore \quad a = \frac{1}{3} \text{ or } a = 0$$

**b) Does $Lt\, x_n$ exist ?**

Yes $Lt\, x_n$ is exist.

We have limit points $a = \frac{1}{3}$ (or) $a = 0$

$\therefore$ $x_n$ has limit point

$\therefore$ $Lt\, x_n$ exist.

**c) Let $a \neq \text{Lim}\, x_n$**

$a > \text{Lim}\ x_n$ (or) $a < \lim x_n$

a is constant

We know that $\text{Lim}\ x_n > a$

We prove $a > \lim_{x \to \infty} x_n$

$a > x_0$

$\therefore$ $a > x_0$

$\therefore$ But which is contradiction

$\therefore$ $a \neq \text{Lim}\, x_n$ is wrong

$\therefore$ $a = \text{Lt}\ x_n$

## 1.5 Monotone Sequences and Cauchy Sequences

(i) A sequence $\{a_n\}$ is said to be monotonically increasing if $a_{n+1} \geq a_n\ \forall\ n \in N$

i.e., $a_1 \leq a_2 \leq a_3 \leq \ldots\ldots \leq a_n \leq a_{n+1} \leq \ldots\ldots$

(ii) A sequence $\{a_n\}$ is said to be monotonically decreasing if $a_{n+1} \leq a_n\ \forall\ n \in N$

i.e., $a_1 \geq a_2 \geq a_3 \geq \ldots\ldots \geq a_n \geq a_{n+1} \geq \ldots\ldots$

(iii) A sequence $\{a_n\}$ is said to be monotonic if it monotonically increasing or monotonically decreasing.

(iv) A sequence $\{a_n\}$ is said to be strictly monotonically increasing if $a_{n+1} > a_n\ \forall\ n \in N$

(v) A sequence $\{a_n\}$ is said to be strictly monotonically decreasing if $a_{n+1} < a_n\ \forall\ n \in N$

(vi) A sequence $\{a_n\}$ is said to be strictly monotonic if it is either strictly monotonically increasing or strictly monotonically decreasing

**Note**

1) Every monotonically increasing sequence which is bounded above converges to its supremum.

2) Every monotonically decreasing sequence which is bounded below converges to its infimum.

**40. All bounded monotone sequence converge.**

**(i) Every monotonically increasing sequence which is bounded above is convergent.**

**(ii) Every monotonically decreasing sequence which is bounded below is convergent.**

**OR**

**State and prove Montone Converge Theorem.**

*Sol.* **(June/July - 19, Dec.-17, Imp.)**

(i) Let $\{s_n\}$ be sequence which is monotonically increasing and bounded above.

To prove that $\{s_n\}$ is convergent.

i.e., to prove that $\{s_n\}$ exists

$\lim\ s_n = \sup\{s_n | n \in N\}$

for each $\varepsilon > 0 \; \exists \, m \in N \ni |s_n - k| < \varepsilon \; \forall \, n \geq m$ let the range of the sequence.

$S = \{s_n : n \in N\}$

Clearly it is non empty and is bounded above energy non empty subset of R which $\{s_n\}$ is bounded above has supremum.

Let sup s = k

where k is least upper bound.

$k - \varepsilon$ is not an upper bound of s

$\exists m \in N \ni \quad s_m > k - \varepsilon$ ... (1)

$\therefore \{s_n\}$ is monotonically increasing sequence $\forall n \geq m \Rightarrow s_n \geq s_m$ ... (2)

from (1) and (2)

$k - \varepsilon < s_m \leq s_n$ ... (3)

k is the supremum of s $\forall n \in N$

$s_n \leq k < k + \varepsilon$ ... (4)

(3), (4)

$k - \varepsilon < s_n < k + \varepsilon$

$\therefore |s_n - k| < \varepsilon$

$\therefore$ Every monotonically increasing sequence which is bounded above is convergent.

(ii) Let $\{s_n\}$ be sequence which is monotonically decreasing and bounded below.

To prove that $\{s_n\}$ is convergent

i.e., to P.T $\lim\{s_n\}$ is exists

$\lim s_n = \text{Inf}\{s_n / n \in N\}$

To prove that for each $\varepsilon > 0 \exists m \in N \ni |s_n - \ell| < \varepsilon \; \forall n \geq m$

Since $\{s_n\}$ is bounded below.

$\{s_n\}$ has intimum $= \ell$

Let inf $= \ell$

Where $\ell$ is a great lower bound $\ell + \varepsilon$ is not a lower bound of s

$\exists m \in N \ni sm < \ell + \varepsilon$ ... (1)

$\therefore \{s_n\}$ is monotonically decreasing sequence.

$\forall n \geq M \Rightarrow s_n \leq s_m$ ... (2)

from (1) and (2) $\Rightarrow s_n \leq s_m < \ell + \varepsilon$ ... (3)

$\ell$ is intimum of s $\forall n \in N$ ... (4)

from (3) and (4)

$\ell - \varepsilon < s_n < \ell + \varepsilon$

$|s_n - \ell| < \varepsilon \; \forall n \in N$

$\therefore$ Every anatomically decreasing sequence which is bounded below is convergent.

**41. If $\{s_n\}$ is an unbounded non decreasing sequence then lim $s_n = +\infty$.**

*Sol.*

Let $\{s_n\}$ be non decreasing sequence but not bounded above.

$\{s_n\}$ is an increasing sequence $\Rightarrow s_n \geq s_m$ for $n > m$.

$\{s_n\}$ is not bounded above

$\Rightarrow \exists m \in z + \ni s_m > M$ where $M > 0$

$s_n \geq s_m > M$ for $n > m$

$s_n > M \quad \forall n > m$

$\{s_n\}$ is diverges to infinity

i.e., $\lim s_n = +\infty$

**42. If $\{s_n\}$ is an unbounded non increasing sequence then lim $s_n = -\infty$.**

*Sol.*

Let $\{s_n\}$ be decreasing sequence and not bounded below.

$\{s_n\}$ is an decreasing sequence $\Rightarrow s_n \leq s_m$ for $n > m$.

$\{s_n\}$ is not bounded below

$\exists m \in z^+ \ni s_m < M$ where $M > 0$

$\therefore \quad s_n \leq s_m < M$ for $n > m$

$s_n < M \quad \forall n > m$

$\{s_n\}$ diverges to $-\infty$

$\lim s_n = -\infty$



**43. Which of the following sequences are increasing decreasing ? Bounded ?**

a) $\frac{1}{n}$

b) $\frac{(-1)^n}{n^2}$

c) $n^5$

d) $\text{Sin}\left(\frac{n\pi}{7}\right)$

e) $(-2)^n$

f) $\frac{n}{3^n}$

*Sol.*

**a)** $\frac{1}{n}$

Let $S_n = \frac{1}{n}$

$$S_{n+1} = \frac{1}{n+1}$$

$$S_n = \frac{1}{n}$$

$$n = 1 \Rightarrow S_1 = \frac{1}{1} = 1;$$

$$S_{1+1} = S_2 = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

$$n = 2 = \frac{1}{2} = 0.5$$

$$S_{2+1} = S_3 = \frac{1}{2+1} = \frac{1}{3} = 0.33$$

$\therefore \quad S_n \geq S_{n+1}$

$\therefore \quad \frac{1}{n}$ is decreasing sequence and bounded.

**b)** $\mathbf{\frac{(-1)^n}{n^2}}$

Let $S_n = \frac{(-1)^n}{n^2}$

$$S_{n+1} = \frac{(-1)^{n+1}}{(n+1)^2}$$

$$n = 1 \Rightarrow S_1 = \frac{(-1)^1}{1^2} = -1;$$

$$n = 1 \Rightarrow S_{1+1} = S_2 = \frac{(-1)^2}{1^2} = 1$$

$n = 2 \Rightarrow S_2 = \frac{(-1)^2}{(2)^2} = \frac{1}{4}$; $\qquad n = 2 \Rightarrow S_{2+1} = S_3 = \frac{(-1)^2}{2^2} = \frac{1}{4}$

$n = 3 \Rightarrow S_3 = \frac{(-1)^3}{(3)^2} = \frac{-1}{9}$; $\qquad n = 3 \Rightarrow S_{3+1} = \frac{(-1)^3}{3^2} = \frac{-1}{9}$

$\therefore \quad S_n = \frac{(-1)^n}{n^2}$ is bouned.

**c) $n^5$**

Let $S_n = n^5$

$S_{n+1} = (n+1)^5$

Put

$n = 1 \Rightarrow S_1 = 1^5 = 1$ ; $n = 1 \Rightarrow S_{1+1} = S_2 = (1+1)^5 = 2^5$

$n = 2 \Rightarrow S_2 = 2^5 =$ ; $n = 2 \Rightarrow S_{2+1} = S_3 = (2+1)^5 = 3^5$

$n = 3 \Rightarrow S_3 = 3^5 =$ ; $n = 3 \Rightarrow S_{3+1} = S_4 = (3+1)^5 = 4^5$

$\therefore \quad S_n \leq S_{n+1}$

This shows increasing

$\therefore \quad n^5$ is increasing sequence.

**d) $\text{Sin}\left(\frac{n\pi}{7}\right)$**

Let $S_n = \text{Sin}\left(\frac{n\pi}{7}\right)$

$S_{n+1} = \text{Sin}\left(\frac{(n+1)\pi}{7}\right)$

Put $n = 1$ in $S_n \Rightarrow S_1 = \text{Sin}\left(\frac{\pi}{7}\right)$; $n = 1$ in $S_{1+1} = S_2 = \text{Sin}\left(\frac{2\pi}{7}\right)$

$n = 2 \Rightarrow S_n = \text{Sin}\left(\frac{2\pi}{7}\right)$; $n = 2 \Rightarrow S_{2+1} = S_3 = \text{Sin}\left(\frac{3\pi}{7}\right)$

$n = 3 \Rightarrow S_3 = \text{Sin}\left(\frac{3\pi}{7}\right)$; $n = 3$ ; $S_{3+1} = S_4 = \text{Sin}\left(\frac{4\pi}{7}\right)$

$\therefore |S_{n+1} - S_n| < \in$

$\in$ is arbitrary

$\therefore \quad \text{Sin}\left(\frac{n\pi}{7}\right)$ is bounded sequence.

**e) $(-2)^n$**

Let $S_n = (-2)^n$

$S_{n+1} = (-2)^{n+1}$

Put $n = 1$ in $S_n$ ; Put $n = 1$ in $S_{n+1}$

$S_1 = (-2)^1 = 2$ ; $S_{1+1} = S_2 = (-2)^{1+1} = 4$

$n = 2 \Rightarrow$ ; $n = 2 \Rightarrow$

$S^2 = (-2)^2 = 4$ ; $S_{2+1} = S_3 = (-2)^3 = -8$

$n = 3 \Rightarrow$ ; $n = 3$

$S_3 = (-2)^3 = -8$ ; $S_{3+1} = S_4 = (-2)^4 = 16$

$n = 4 \Rightarrow$

$S_4 = (-2)^4 = 16$

$\therefore \quad S_1 < S_2$ ; $S_2 < S_3$ ; $S_3 < S_4$

$\therefore$ It is increasing and bounded sequence.

**f)** $\frac{n}{3^n}$

Let $S_n = \frac{n}{3^n}$ ; $S_{n+1} = \frac{n+1}{3^{(n+1)}}$

Put n = 1

$S_1 = \frac{1}{3}; S_{1+1} = S_2 = S_2 = \frac{2}{3^2} = \frac{2}{9}$

n = 2

$S_2 = \frac{2}{9}; S_{2+1} = S_3 = \frac{1}{3^2}$

n = 3

$S_3 = \frac{1}{3^2}; \ S_{3+1} = S_4 = \frac{4}{3^4}$

$\therefore$ $\frac{n}{3^n}$ is decreasing bounded sequence.

**44. Let $(S_n)$ be a sequence such that $|S_{n+1} - S_n| < 2^{-n}$ for all $n \in N$. Prove $(S_n)$ is a Cauchy sequence and hence a convergent sequence.**

*Sol.*

Given $\{S_n\}$ is a sequence $\ni |S_{n+1} - S_n| < 2^{-n}$

Let $\{S_n\}$ is a Cauchy sequence

$\Rightarrow$ $\{S_n\}$ is bounded

$\therefore$ By Balzano weiestrass theorem we know that $\{S_n\}$ has atleast one limit point say $l$.

If possible, Let $l^1$ be another limit point of $\{S_n\}$

Let $\in = |l - l'| > 0$

$\because$ $\{S_n\}$ is a Cauchy sequence, for each

$\in > 0 \ \exists m \in z^+ \ \ni |s_{n+1} - s_n| < 2^{-n} \ \ \forall n \in N, \ n \geq m$ here $\in = 2^{-n}$

$\because$ $l, l'$ are limit points, $\exists$ positive integers $n+1 \geq m, n \geq 0$

$|S_{n+1} - l| < \frac{\in}{3}$ and $|S_n - l| < \frac{\in}{3}$

Consider

$|l - l'| = |l - S_{n+1} + S_{n+1} + S_n - S_n - l|$

$|S_{n+1} - S_n| < 2^{-n}$

$n=1 \Rightarrow |S_{1+1} - S_n| < 2^{-1}$ ; $n=2 \Rightarrow |S_3 - S_2| < 2^{-2}$

$|S_2 - S_1| < \frac{1}{2}$ $|S_3 - S_2| < \frac{1}{4}$

$\therefore$ $|S_{n+1} - S_n| < \in \quad \forall n \in N$ $\therefore$ $|S_{n+1} - S_n| < \in \quad \forall n \in N$

$\in = \frac{1}{2}$ $\in = \frac{1}{4}$

$\therefore$ $S_n$ is bounded $\therefore$ $S_n$ is bounded

$\leq |S_{n+1} - l| + |S_{n+1} - S_n| + |S_n - l|$

$\leq \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3}$

$< \in$

$\therefore$ $|l - l'| < |l - l'|$

$\therefore$ $|S_{n+1} - S_n| < |S_{n+1} - S_n|$

which is contradiction

$\therefore$ Hence our assumption is wrong

$\therefore$ $\{S_n\}$ has a unique limit point '$l$'

$\therefore$ $\{S_n\}$ is bounded and has a unique limit point

$\Rightarrow$ $\{S_n\}$ is convergent.

**45. Let $(S_n)$ be an increasing sequence of positive number and define $\sigma_n = \frac{1}{n}(S_1 + S_2 + .... + S_n)$ prove $(\sigma_n)$ is an increasing sequence.**

*Sol.* **(June/July-19, May/June-18, Dec.-17, Imp.)**

Given $\{S_n\}$ is an increasing sequence of positive number

$\therefore$ $S_n \leq S_{n+1} \quad \forall n \in N \geq +$

$$\sigma_n = \frac{1}{n}(S_1 + S_2 + ... + S_n)$$

$$\sigma_{n+1} = \frac{1}{n+1}(S_1 + S_2 + ... + S_n)$$

$n = 1 \Rightarrow$

$$\sigma_1 = \frac{1}{1}(S_1 + S_2 + ... + S_1)$$

$$= (2S_1 + S_2 + ... + S_0)$$

$$\sigma_{n+1} = \frac{1}{n+1}(S_1 + S_2 + ... + S_n)$$

Put $n = 1$

$$\sigma_{n+1} = \sigma_2 = \frac{1}{1+1}(S_1 + S_2 + ... + S_1)$$

$n = 2$

$\sigma_2 = \frac{1}{2}(S_1 + S_2 + ... + S_2)$

$= \frac{1}{2}(S_1 + 2S_2 + ... + S_1)$

$n = 3$

$\sigma_3 = \frac{1}{3}(S_1 + S_2 + ... + S_3)$

$= \frac{1}{3}(S_1 + S_2 + 2S_3 ... + S_2)$

$\sigma_{1+1} = \sigma_2 = \frac{1}{1+1}(S_1 + S_2 + ... + S_1)$

$\sigma_2 = \frac{1}{2}(2S_1 + S_2 + ... + S_0)$

$n = 2$

$\sigma_{2+1} = \sigma_3 = \frac{1}{2+1}(S_1 + S_2 + ... + S_2)$

$= \frac{1}{3}(S_1 + 2S_2 + ... + S_1)$

$n = 3$

$\sigma_{3+1} = \sigma_4 = \frac{1}{3+1}(S_1 + S_2 + ... + S_3)$

$= \frac{1}{4}(S_1 + S_2 + 2S_3 + ... + S_2)$

$\therefore \sigma_n < \sigma_{n+1}$

$\therefore \sigma_n$ is an increasing sequence.

---

**46. Let $t_1 = 1$ and $t_{n+1} = \left[1 - \frac{1}{4n^2}\right] \times t_n$ for $n \geq 1$**

**a) Show lim $t_n$ exists**

**b) What do you think lim $t_n$ is ?**

*Sol.*

Given $t_1 = 1$ and $t_{n+1} = \left[1 - \frac{1}{4n^2}\right] \cdot t_n$ for $n \geq 1$

a) We will show that Lim $t_n$ is exist

It is enough to show that $\{t_n\}$ is a bounded monatone sequence

First we prove that $\{t_n\}$ is a bounded

$\exists$ two real numbers $k_1$ and $k_2$ $\ni k_1 \leq k_2$

then $k_1 \leq t_n \leq k_2 \quad \forall n \in N$

$\therefore \quad |t_n| < |t_{n+1}|$

$t_n$ is monotone sequence

$\therefore \quad t_n$ is bounded sequence

$\therefore \quad \{t_n\}$ is bounded monotone sequence

$\therefore \quad$ Lim $t_n$ is exist.

b) The answer is not obvious!

It twins out that $\lim t_n$ is a waltes product and has value $\frac{2}{\pi}$ which is about 0.6366

Observe how much easier part (a) is than part (b).

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**47. Let $t_1 = 1$ and $t_{n+1} = \left[1 - \frac{1}{(n+1)^2}\right]$ tn for all $n \geq 1$.**

**(a) Show Lim tn exists.**

**(b) What do you think Limtn is?**

**(c) Use induction to show tn = $\frac{n+1}{2n}$**

**(d) Repeat part (b)**

*Sol.* **(June/July-19, May/June-18, Imp.)**

(a) (b) and (d) same as the above problem.

(c) $t_1 = 1,\ t_{n+1} = \left[1 - \frac{1}{(n+1)^2}\right]$ tn ... (1)

We have to show that $t_n = \frac{n+1}{2n}$

We will prove by induction

$0 < t_{n+1} < t_n < 1$

It is holds for $n \geq 1$

multiply $\frac{n+1}{2n}$ b/s

$0 < t_{n+1} < t_n < \frac{n+1}{2n}$ ... (2)

this holds $t_n < \frac{n+1}{2n}$ for n

Now to show $t_n > \frac{n+1}{2n}$

from (2)

$0 < t_{n+1}$

$t_{n+1} > 0$

$\therefore\ t_{n+1} > \frac{n+1}{2n}$

Thus (2) holds for $n+1$ for n

Hence (2) holds for all n by induction

Thus $\lim t_n = t$ exits.

---

**48. Let $S_1 = 1$ and $S_{n+1} = \frac{1}{3}(S_{n+1})$ for $n \geq 1$.**

**(a) Find $S_2$, $S_3$ and $S_4$**

**(b) Use induction to show $S_n > \frac{1}{2}$ for all n.**

**(c) Show $(S_n)$ is a decreasing sequence**

**(d) Show lim $S_n$ exists and find lim $S_n$.**

*Sol.* **(June/July-19, Imp.)**

(a) Given $S_1 = 1$ and $S_{n+1} = \frac{1}{3}(S_{n+1})$ for $n \geq 1$

$S_1 = 1$

put $n = 1$ in $S_{n+1}$

$S_{1+1} = S_2 + = \frac{1}{3}(S_1 + 1)$

$= \frac{1}{3}(1+1)$

$= \frac{2}{3}$

Put $n = 2$

$S_{2+1} = S_3 + = \frac{1}{3}(S_2 + 1)$

$= \frac{1}{3}\left(\frac{2}{3} + 1\right)$

$= \frac{5}{9} = \frac{5}{3.3} = \frac{5}{3^2}$

Put n = 3

$$S_{3+1} = S_4 + = \frac{1}{3}(S_3+1)$$

$$= \frac{1}{3}\left(\frac{5}{9}+1\right)$$

$$= \frac{14}{3.3.3} = \frac{14}{3^3}$$

(b) $S_1 = 1$ and $S_{n+1} = \frac{1}{3}(S_n + 1)$ for $n \geq 1$

We will prove by induction $S_n > \frac{1}{2} \forall n.$

$$0 < S_{n+1} < S_n < \frac{1}{3}(S_n + 1)$$

It is holds for $n = \frac{1}{2}$

Hence $S_n > \frac{1}{2}$ is holds $n = \frac{1}{2} \forall n$

We prove that $n = n + 1 \forall n$

$$0 < S_{n+1} < S_n < \frac{1}{3}(S_n + 1)$$

$$S_{n+1} < \frac{1}{3}(S_n + 1)$$

$S_{n+1}$ is greater

$\therefore$ $S_{n+1} > \frac{1}{2}$ is holds

$S_{n+1}$ also holds for n + 1 for n

$\therefore$ $0 < S_{n+1} < S_n < \frac{1}{3}(S_n + 1)$ holds for n

$\therefore$ $S_n > \frac{1}{2} \forall n$

(c) Given

$S_1 = 1$ ... (1)

$S_{n+1} = \frac{1}{3}(S_{n+1})$ ... (2)

Put n=1 in equation (2)

$$S_{1+1} = S_2 = \frac{1}{3}(S_1 + 1)$$

$$= \frac{1}{3}(1 + 1)$$

$$= \frac{1}{3}(2) = \frac{2}{3} = 0.66$$

Put n = 2 in equation (2)

$$S_{2+1} = S_3 = \frac{1}{3}(S_2 + 1)$$

$$= \frac{1}{3}\left(\frac{2}{3}+1\right) = \frac{5}{9} = 0.55$$

Put n = 3 in equation (2)

$$S_{3+1} = S_4 = \frac{1}{3}(S_3 + 1)$$

$$= \frac{1}{3}\left(\frac{5}{9}+1\right) = \frac{1}{3}\left(\frac{14}{9}\right)$$

$$= \frac{14}{27} = 0.52$$

$\therefore S_n > S_{n+1}$

$\therefore \{S_n\}$ is decreasing sequence

(d) We will show that Lim $t_n$ is exist

It is enough to show that $\{t_n\}$ is a bounded monatone sequence

First we prove that $\{t_n\}$ is a bounded

$\exists$ two real numbers $k_1$ and $k_2$ $\ni$ $k_1 \leq k_2$

then $k_1 \leq t_n \leq k_2 \quad \forall n \in N$

$\therefore$ $|t_n| < |t_{n+1}|$

$t_n$ is monotone sequence

$\therefore$ $t_n$ is bounded sequence

$\therefore$ $\{t_n\}$ is bounded monotone sequence

$\therefore$ Lim $t_n$ is exist.

### 1.5.1 Cauchy Sequence

**Definition (1)**

A sequence $\{a_n\}$ is said to be a Cauchy sequence if given $\in > 0$, however small, $\exists$ a positive integer m such that $|a_n - a_m| < \in \ \forall\ n \geq m$.

**Definition (2)**

A sequence $\{a_n\}$ is said to be a Cauchy sequence if given $\in > 0$, however small, $\exists$ a positive integer m such that $|a_{m+p} - a_m| < \in \ \forall\ p > 0, p \in m$.

**Definition (3)**

A sequence $\{a_n\}$ is said to be a Cauchy sequence if given $\in > 0$, however small, $\exists$ a positive integer m such that $|a_p - a_q| < \in \ \forall\ p, q \geq m$.

**Note :** All the above definitions are equivalent.

**49. Every Convergent Sequence is a Cauchy Sequence.**

*Sol.* **(Dec.-17)**

Let $\{a_n\}$ converges to '$l$'.

$\therefore$ For each $\in > 0$, $\exists$ $m \in z^+ \ni |a_n - l| < \frac{\in}{2}\ \forall\ n \geq m$.

If $p, q \geq m$ then $|a_p - l| < \in/2$,

$|a_q - l| < \in/2$

Consider

$$|a_p - a_q| = |a_p - l + l - a_q|$$
$$\leq |a_p - l| + |a_q - l|$$
$$< \in/2 + \in/2$$
$$< \in$$

$\therefore$ $|a_p - a_q| < \in \ \forall\ p, q \geq m$.

$\Rightarrow$ $\{a_n\}$ is a Cauchy sequence.

**50. If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is bounded.**

*Sol.* **(Dec.-17)**

Let $\{a_n\}$ is a Cauchy sequence

$\Rightarrow$ For $\in = 1$, $\exists$ $m \in z^+ \ni |a_p - a_q| < 1\ \forall\ p, q \geq m$.

$\Rightarrow$ $|a_p - a_m| < 1\ \forall\ p \geq m$

$\Rightarrow$ $a_m - 1 < a_p < a_m + 1\ \forall\ p \geq m$

Let $K_1 = \min. \{a_1, a_2, ....., a_{m-1}, a_m - 1\}$ and

$K_2 = \min. \{a_1, a_2, ....., a_{m-1}, a_m + 1\}$

$\Rightarrow$ $K_1 \leq a_n \leq K_2\ \forall\ n \in z^+$

$\Rightarrow$ $\{a_n\}$ is bounded.

**Note :** Converse of the above theorem need not be true.

**51. If $\{a_n\}$ is a Cauchy sequence then $\{a_n\}$ is convergent.**

*Sol.* **(May/June-18, Dec.-17)**

Let $\{a_n\}$ is a Cauchy sequence

$\Rightarrow$ $\{a_n\}$ is bounded

$\therefore$ By bolzano weierstrass theorem we know that $\{a_n\}$ has atleast one limit point say '$l$'.

If possible, let $l'$ be another limit point of $\{a_n\}$

Let $\in = |l - l'| > 0$

$\because$ $\{a_n\}$ is a Cauchy sequence, for each $\in > 0$, $\exists$ $m \in z^+ \ni |a_p - a_q| < \frac{\in}{3}\ \forall\ p, q \geq m$

$\because$ $l$, $l'$ are limits points, $\exists$ positive integers $p \geq m$, $q \geq m \ni |a_p - l| < \frac{\in}{3}$ and $|a_p - l'| < \frac{\in}{3}$

Consider

$$|l - l'| = |l - a_p + a_p - a_q + a_q - l'|$$
$$\leq |a_p - l| + |a_p - a_q| + |a_q - l'|$$
$$< \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3}$$
$$< \in$$

$\therefore$ $|l - l'| < |l - l'|$

which is a contradiction.

Hence our assumption is wrong

$\therefore$ $\{a_n\}$ has a unique limit point '$l$'.

$\because$ $\{a_n\}$ is bounded and has a unique limit point.

$\Rightarrow$ $\{a_n\}$ is convergent.

## 1.6 Subsequence

If $\{s_n\}$ is a sequence and $\{n_k\}$ is a sequence of positive integer such that $n_1 < n_2 < .... < n_k$. Then the sequence $\{s_{n_r}\}$ is called subsequence of $\{s_n\}$.

**Example**

$s_n = n^2 (-1)^n$

$s_1 = -1, s_2 = 4, s_3 = -9, s_4 = 16$ .... and 4, 16, 36, ... are subsequence of $s_n$.

**52. If the sequence $\{s_n\}$ converges, then every subsequence converges to the same limit.**

*Sol.* **(May/June-18, Nov./Dec.-18, Dec-2017, Imp.)**

Let $\{s_{n_k}\}$ be subsequence of $\{s_n\}$ $n \geq 1$.

To prove

$s_{n_k}$ is converge to $\ell$

$\forall\, \varepsilon > 0 \;\exists\; k \in N \ni$ for any $k \geq K \ni |s_{n_k} - \ell| < \varepsilon$

as we know $s_n$ converges to $\ell$, $\exists$ k such that for any $k \geq k$

$|s_n - \ell| < \varepsilon$

then for any $k \geq K$ we have,

$n_k > n_k \geq k$

$n_k > K$

$|s_{n_k} - \ell| < \varepsilon$

For given $\varepsilon > 0 \;\exists\; k \ni$ for any $k \geq K \Rightarrow |s_{n_k} - \ell| < \varepsilon$

since $\varepsilon > 0$ was arbitrary it holds for any $\varepsilon$

$\forall\, \varepsilon > 0, \exists\, k \ni |s_{n_k} - \ell| < \varepsilon \;\forall k \geq k$

**53. If the sequence $\{s_n\}$ converges to $\ell$ prove that it is subsequence also converges to $\ell$.**

*Sol.* **(Imp.)**

Given sequence $\{s_n\}$ is converges to $\lim\limits_{n\to\infty} s_n = \ell$.

for each $\varepsilon > 0 \exists m \in N$ n $|s_n - \ell| < \varepsilon \;\forall\, n \geq m$ ... (1)

Let k be any natural number

for each $\varepsilon > 0 \exists m \in N \ni |s_k - \ell| < \varepsilon \;\forall\, n \geq m$ ... (2)

To prove that

The subsequence $s_{n_k}$ converges to $\ell$.

*i.e.*, to prove

for each $\varepsilon < 0 \;\exists\; m \in N$ $|s_{n_k} - \ell| < \varepsilon \;\forall\, n_k \geq m$

$\because\; n_k \geq k$

from (1) and (2) $|s_{n_k} - \ell| < \varepsilon \;\forall\, n \geq m$

$\therefore\; \lim s_{n_k} = \ell$

subsequence $\{s_{n_k}\}$ converges to $\ell$.

**54. Every sequence $\{s_n\}$ has a monotonic subsequence.**

*Sol.* **(June/July-19, Dec.-17, Imp.)**

Let $\{s_n\}$ be a sequence to prove that $\{s_n\}$ has a monotone subsequence $\{s_n\}$ is any sequence then three cases arise.

**Case (i) :** $\{s_n\}$ has no peak point

**Case (ii) :** $\{s_n\}$ has finite number of peak point

**Case (iii) :** $\{s_n\}$ has infinite number of peak point

**Case (i)**

$\{s_n\}$ has no peak point

$\because\ 1 \in N$

$(n_1)$ 1 is not a peak point of $\{s_n\}$

$\exists n_2 \in N$ and $n_2 \geq 1$

$\ni s_{n_2} \geq s_{n_1}$

$\therefore\ n_2 \in N$

$n_2$ is not a peak point of $\{s_n\}$

$\exists n_3 \in N$ and $n_3 > n_2 \ni s_{n_3} \geq s_{n_2}$

Repeating the same argument, we get

$$n_1 < n_2 < n_3 < .... \ni s_{n_1} \leq s_{n_2} \leq s_{n_3} <....$$

where $\{s_{n_r}\}$ is a subsequence of $\{s_n\}$

$\therefore$ $\{s_n\}$ has a monotone subsequence.

**Case (ii)**

$\{s_n\}$ has finite number of peak point, let m be the maximum among all the peak point

Let $n_1 > m \in N$

Then $n_1$ is not a peak point of $\{s_n\}$

$\exists n_2 \in N \ni n_2 > n_1$ and $s_{n_2} \geq s_{n_1}$

$\because$ $n_2 \in N$ and $n_2 > n_1 > m$

$\because$ $n_2$ is not a peak point

$\exists n_3 \in N \ni n_3 > n_2$ and $s_{n_3} \geq s_{n_2}$

Repeating the same process than we get

$$n_1 < n_2 < n_3 .... \ni s_{n_1} \leq s_{n_2} \leq s_{n_3} \leq .....$$

where $\{s_{n_r}\}$ is a subsequence of $\{s_n\}$ and it is monotonically increasing sequence.

$\because$ $\{s_n\}$ has monotone subsequence.

**Case (iii)**

$\{s_n\}$ has infinite number of peak points let $n_1$, $n_2$ ... be the infinite number of peak points.

$\ni n_1 < n_2 < n_3 < ....$

$\because$ $n_1$ is a peak point

Then $n_2 > n_1 \Rightarrow s_{n_2} \leq s_{n_1}$

$\because$ $n_2$ is peak point

Then $n_3 > n_2 \Rightarrow s_{n_3} \leq s_{n_2}$

Repeating the above process, we get

$$n_1 < n_2 < n_3 < .... \Rightarrow s_{n_1} \geq s_{n_2} \geq s_{n_3} \geq .....$$

where $\{s_{n_r}\}$ is a subsequence of $\{s_n\}$ and it is monotonically subsequence.

$\therefore$ Every sequence contains monotone subsequence.

**55. State and prove Bolzano Weierstrass theorem**

**OR**

**Every bounded sequence has convergent subsequence.**

*Sol.* **(June/July-19, Nov./Dec.-18, Imp.)**

Let $\{s_n\}$ be a bounded sequence

To prove that $\{s_n\}$ has convergent subsequence

$\because \{s_n\}$ is a sequence.

As we know that every sequence has monotone subsequences.

$\because \{s_n\}$ is bounded and the subsequence of $\{s_{nk}\}$ is also bounded.

Also subsequence of $\{s_n\}$ is either monotonically increasing or monotonically decreasing.

$\therefore$ By monotone convergence theorem

Subsequence $\{s_{n_r}\}$ is convergent

$\therefore$ Every bounded sequence has a convergent.

**56. Find the subsequence limit of $s_n = n^2(-1)^n \; \forall n \in N$.**

*Sol :*

$S_n = n^2(-1)^n \; \forall n$

$s_1 = -1, \quad s^2 = (2)^2 (-1)^2 = 4$

$s_3 = -9, \quad s_4 = 16$

$s_5 = -25, \quad s_6 = 36.....$

The subsequence of even terms on {4, 16, 36...} is diverge to $+\infty$.

The subsequence of odd terms are {–1, –9, –25, ---} is diverges to $-\infty$.

$\therefore$ All subsequence that have a limit diverge to $+\infty$ or $-\infty$.

$S = \{-\infty, +\infty\}$ subsequential limit of $\{s_n\}$.

**57. Let s denote the set of subsequential limit of sequence $\{s_n\}$. Suppose $\{t_n\}$ is a sequence in $S \cap R$ and that $t = \lim t_n$ then $t \in s$.**

*Sol.*

$\{s_{n_k}\}$ is subsequence of $\{s_n\}$ is converges to $t_1 \ni n_1 \ni |s_{n_1} - t_1| < 1$.

Assume that $n_1, n_2, \ldots, n_k$ have been selected, so that $n_1 < n_2 < \ldots. n_k$ ... (1)

it is $j^{th}$ term

$$|s_{n_j} - t_j| < \frac{1}{j} \text{ for } j = 1, 2, \ldots k \quad \ldots (2)$$

If $\{s_{n_k}\}$ is subsequence converges to $t_{k+1}$

$$\exists\, n_{k+1} > n_k \ni \left|s_{n_{k+1}} - t_{k+1}\right| < \frac{1}{k+1}$$

from (1) and (2) hold k + 1

case (i) suppose $t \in R$

i.e., t is not $+\infty$ to $-\infty$

consider $\left|s_{n_k} - t\right| = \left|s_{n_k} - t_k + t_k - t\right|$

$$= \left|(s_{n_k} - t_k) + (t_k - t)\right|$$

$$= \left|s_{n_k} - t_k\right| + \left|t_k - t\right|$$

$$= \frac{1}{k} + \left|t_k - t\right| \quad \ldots (3)$$

$\{t_n\}$ is sequence is convergent

i.e., $\lim t_n = t$

for each $\varepsilon > 0 \, \exists \, N \ni |t_n - t| < \varepsilon$

From (3)

$$|s_{n_k} - t| < \frac{1}{k} + \varepsilon$$

$$|s_{n_k} - t| < \varepsilon \qquad \forall k \in N$$

$$\lim_{n\to\infty} s_{n_k} = t$$

**Case (ii)**

Suppose $t = +\infty$ from equation (4)

$$\left|s_{n_j} - t_j\right| < \frac{1}{j} \qquad j = 1, 2, \ldots k$$

$$\left|s_{n_k} - t_k\right| < \frac{1}{k} \qquad \forall k \in N$$

$$s_{n_k} > t_k - \frac{1}{k} \text{ for } s_{n_k} < \frac{1}{k} + tk$$

$$s_{n_k} < t_k - \frac{1}{k}$$

$\therefore \lim s_{n_k} = +\infty$

---

**58. Let an = 3 + 2(–1)n for $n \in N$.**

**a) List the first eight terms of the sequence $(a_n)$.**

**b) Give a subsequence that is constant {takes a single values specify the selection function $\sigma$.**

*Sol.* **(Imp.)**

**a) First eight terms of the sequence $(a_n)$.**

Given that $a_n = 3 + 2(-1)^n$ for $n \in N$.

Put n = 1 in $a_n$

$a_1 = 3 + 2(-1)^1$

$= 3 - 2 = 1$

$n = 2 \Rightarrow a_2 = 3 + 2(-1)2$

$= 3 + 2$

$a_2 = 5$

Put n = 3 in

$a_3 = 3 + 2(-1)^3$

$= 3 - 2$

$a_3 = 1$

Put n = 4 in

$a_4 = 3 + 2(-1)^4$

$= 3 + 2$

$= 5$

Put n = 5 in

$a_5 = 3 + 2(-1)^5$

$= 3 - 2$

$= 1$

Put n = 6 in

$a_6 = 3 + 2(-1)^6$

$= 3 + 2$

$= 5$

Put n = 7 in

$a_7 = 3 + 2(-1)^7$

$= 3 - 2$

$= 1$

Put n = 8 in

$a_8 = 3 + 2(-1)^8$

$= 3 + 2$

$a_8 = 5$

b) Let a $\sigma(k) = n_k = 2k$

Then $(an_k)$ is the sequence that takes the single value 5.

There are many other possible choice of $\sigma$.

**59. Consider the sequences defined as follows:**

$$a_n = (-1)^n, bn = \frac{1}{n}, C_n = n^2,$$

$$dn = \frac{6n+4}{7n-3}$$

**For each sequence, given an example of a monatone subsequence.**

*Sol.*

Given an = $(-1)^n$ is sequence

Let $a_{nk}$ be subsequence of $a_n$.

$a_n = (-1)^n$ ; $a^{n+1} = (-1)^{n+1}$

an is (–1, 1, –1, 1, –1, 1, – 1, 1....)

$an_k = (-1)^{nk}$ ; $an_{k+1} = (-1)^{n_{k+1}}$

= possitive value

$\therefore$ k = 1 $an_k < an_{k+1}$

$an_1 < an_2$

$\therefore$ $an_{k+1}$ is monotone subsequence

$\therefore$ $(-1)^{nk+1}$ is monotone subsequence

**Example**

$\therefore$ $n_k - 2k$

$an_k = (-1)^{2k}$

$\Rightarrow$ $b_n = \frac{1}{n}$

$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} .... )$

The subsequence is $(bn_k)$ $K \in N$ where $n_k = 2k$ monotone subsequence.

$bn_k = \frac{1}{2k}$

$\Rightarrow$ $c_n = n^2$

The subsequence is (1, 4, 9, 16, 25 ....)

The subsequence is $(cn_k)$ $K \in N$ where $n_k = 2k$ monotone subsequence is

$cn_k = (2k)^2$

$\Rightarrow$ $dn = \frac{6n+4}{7n-3}$

The sequence is $d_1 = \frac{6+4}{7-3} = \frac{10}{4}$

$d_2 = \frac{12+4}{14-3} = \frac{16}{110}$

$d_3 = \frac{18+4}{21-3} = \frac{22}{18} = \frac{11}{9}$

$\therefore$ The sequence is $\left(\frac{5}{2}, \frac{16}{11}, \frac{11}{9} .....\right)$

The subsequence is $(d_{nk})$ $K \in N$ where $n_k = 2k$

$$C_{nk} = \frac{6(2k)+4}{7(2k)-3}$$

$$= \frac{12K+4}{14K-3}$$

**b) Subsequential Limits**

$\Rightarrow a_n = (-1)^n \quad \Rightarrow b_n = \frac{1}{n}$

Subsequence is $bn_k = \frac{1}{\infty}$ subsequence

$an_k = (-1)^{nk}$

$\underset{k\to\infty}{Lt}\ a_{n_\infty} = (-1)^n = (-1)^\infty = 0$

$\Rightarrow C_n = n^2$

$= \infty$

$\Rightarrow dn = \frac{6n+4}{7n-3}$

$dn_k = \frac{6+\frac{4}{nk}}{7-\frac{3}{nk}}$

$dn_k = \frac{6}{7}$

**c) Lim sup and lim inf**

$\Rightarrow$ an $= (-1)^n$ sub sequence is $an_k$

We have that

$\therefore$ Lim sup = Lim inf

Lim sup $an_k$ = Lim inf $an_k$.

$\because n_k \geq k$.

$n = 1$

$a_1 = (-1)$; an = 1, an = – 1

Lt sup $an_k$ = Lt inf $an_k$

$\Rightarrow bn = \frac{1}{n}$

subsequence $bn_k = \frac{1}{n_k}$

Lt sup $bn_k$ = sup $bn_k$ and

Inf $bn_k$ = Lt inf $bn_k$

$\Rightarrow$ Lt sup $\frac{1}{nk}$ = sup $\frac{1}{nk} = \frac{1}{n_k}$

$\Rightarrow$ Lt Inf $bn_k$ = inf $\frac{1}{nk} + \frac{1}{nk}$

$\Rightarrow c_n = n^2$

Subsequence is $cn_k = n_k^2$

Lt sup $cn_k$ = sup $cn_k = n_k^2$

Lt inf $cn_k$ = inf cnk $= n_k^2$

$\Rightarrow d_n = \frac{6n+4}{7n-3}$ subsequence is

$dn_k = \frac{6n_k+4}{7n_k-3}$

$\therefore$ Lt sup $dn_k$ = sup $dn_k = \frac{6n_k+4}{7n_k-3}$

$\therefore$ Lt Inf $dn_k$ = Inf $dn_k = \frac{6n_k+4}{7n_k-3}$

**d) Converge? Diverges to + $\infty$? Diverges to $-\infty$.**

$\Rightarrow$ from (b) condition

an is diverges at $-\infty$

$\Rightarrow b_n$ is converges

$\Rightarrow c_n$ is diverges at $+\infty$

$\Rightarrow d_n$ is converges

**e) Which of the sequences is bounded?**

$\Rightarrow a_n = (-1)^n$

$a_{n+1} = (-1)^{n+1}$

$a_1 = -1$; $a_2 = 1$, $a_3 = -1$

$\therefore$ –1 c an < 1

$a_n$ is bounded sequence

$\Rightarrow b_n = \frac{1}{n}$

$b_{n+1} = \frac{1}{n+1}$

$b_1 = 1$; $b_2 = \frac{1}{2}$; $b_3 = \frac{1}{3}$

$\frac{1}{2} < b_n < 1$

It is not bounded sequence.

$\Rightarrow \quad c_n = n^2 \; ; c_{n+1} + (n+1)^2$

$c_1 = c_2 = 4$

$c_n < c_{n+1} \qquad 1 \le c_n \le 4$

$1 \le c_n \le c_{n+1} \le 4$

$c_n$ is bounded sequence

$\Rightarrow \quad d_n = \frac{6n+4}{7n-3}$

Put $n = 1 \Rightarrow d_1 = \frac{6+4}{7-3} = \frac{10}{4} = \frac{5}{2} = 2.5$

$d_2 = \frac{16}{11} = 1.4$

$\therefore \quad d_n > d_{n+1}$

$\therefore \quad d_n$ is not bounded sequence.

## 1.7 Lim sup's and lim Inf's

Let $\{s_n\}$ be any sequence of real number and let s be the set of subsequential limit of $\{s_n\}$.

$\lim\sup s_n = \lim_{N\to\infty} \sup \{s_n : n > N\} = \text{Sups}$

$\lim\text{Inf } s_n = \lim_{N\to\infty} \text{Inf } \{s_n : n > N\} = \text{Infs}$

**60. If $\{s_n\}$ converges to a positive real number s and $\{t_n\}$ is any sequence then lim sup $s_n t_n$ = s lim sup $t_n$.**

*Sol.* **(Nov./Dec.-18, Imp.)**

For every sequence there exists a montane subsequences.

Let $s_{n_k}$ and $t_{n_k}$ be the monotonic subseqnce of $s_n$ and $t_n$ respectively.

If sequence converges to a limit. Then its subsequence also converges to the same limit

First we show that lim sup $s_n t_n \ge$ s. lim sup $t_n$.

Let lim sup $s_n = s$

lim sup $t_n = \beta$ ... (1)

**Case (i) $\beta$ is finite**

$\lim_{k\to\infty} t_{n_k} = \beta$ ... (2)

$\because$ Sequence converges to limit then subsequence also converges to the same limit.

Similarly $\lim_{k\to\infty} s_{n_k} = s$

Consider sequence $s_n t_n$ such that there exist a monotone subsequence $s_{n_k} t_{n_k}$

$\limsup (s_n t_n) = s\beta$

$\limsup (s_{n_k} t_{n_k}) = s\beta$ ... (3)

Then $\lim (s_{n_k} t_{n_k}) = s\beta$

As $\limsup s_n t_n$ is the largest possible limit of subsequence of $\{s_n t_n\}$.

$\lim_{k\to\infty} \sup (s_n t_n) \geq s\beta$

$\lim_{k\to\infty} \sup (s_n t_n) \geq s. \limsup t_n$ ... (4)

Replace $s_n$ by $\frac{1}{s_n}$ and $t_n$ by $s_n t_n$

$\limsup t_n = \limsup \frac{1}{s_n} (s_n t_n)$

$\geq \frac{1}{s} \limsup s_n t_n$

$s \limsup t_n \geq \limsup s_n t_n$ ... (5)

from (4) and (5)

$\limsup (s_n t_n) = s.\limsup t_n$

**Case (ii)** $\beta = +\infty$

from equation (1) $\limsup t_n = +\infty$

$\lim t_{n_k} = +\infty$

$\limsup (s_n t_n) = s\beta$

$\limsup (s_n t_n) = s(+\infty)$

$\limsup (s_n t_n) = s. \limsup t_n$

**Case (iii)** $\beta = -\infty$

Equation (1) $\limsup t_n = -\infty$

Equation (2) $\lim_{k\to\infty} t_{n_k} = -\infty$

$\limsup (s_n t_n) = s\beta$

$\limsup (s_n t_n) = s(-\infty)$

$\limsup (s_n t_n) = s . \limsup t_n$

All the three cases are holds

$\limsup (s_n t_n) = s \limsup t_n$.

**61. Prove that for sequence of non zero real numbers** $\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$.

*Sol.* **(June./July.-19)**

$\{s_n\}$ be any sequence of non zero real number.

Consider

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \quad \ldots (1)$$

$$\liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \quad \ldots (2)$$

$$\limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right| \quad \ldots (3)$$

The inequality (2) is true for all sequences

Now, required to prove (1) and (3)

Consider inequality (3)

$$\limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

Let, $\limsup |s_n|^{1/n} = \alpha$

$$\limsup \left|\frac{s_{n+1}}{s_n}\right| = L$$

We need to prove that $\alpha \leq L$

Consider M be any positive number such that

$$L < M \quad \ldots (4)$$

i.e., $\limsup \left|\frac{s_{n+1}}{s_n}\right| < M$

$$\lim_{N\to\infty} \sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n > N\right\} < M$$

$$\sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n \geq N\right\} < M$$

$$\left|\frac{s_{n+1}}{s_n}\right| < M \quad \text{for } n \geq N \quad \ldots (5)$$

for n>N

$$|s_n| = \left|\frac{s_n}{s_{n-1}}\right|\left|\frac{s_{n-1}}{s_{n-2}}\right| \ldots\ldots \left|\frac{s_{N-1}}{s_N}\right||s_N|$$

There are n – N fractions

$$n - (N + 1) + 1 = n - N - 1 + 1$$

$$= n - N$$

Then (2) becomes

$|s_n| < M^{n-N}|s_N|$ for $n > N$

$|s_n| < M^n M^{-N}|s_N|$ for $n > N$

As M and L are fixed

Assume $M^{-N}|S_n|$ as a constant value a

$\rightarrow$ $|s_n| < M^n . a \ \ \forall n > N$

$|s_n|^{1/n} < (M^n a)^{1/n}$ for $n > N$

$\Rightarrow$ $|s_n|^{1/n} < M\ a^{1/n}$ for $n > N$

$$\lim_{N\to\infty} |s_n|^{1/n} < M \lim_{N\to\infty} a^{1/n}$$

$$\lim_{N\to\infty} |s_n|^{1/n} < M \ (1)$$

$\lim\sup |s_n|^{1/n} < M$

$\alpha = \lim\sup |s_n|^{1/n} \le L$

$$\lim\sup |s_n|^{1/n} \le \lim\sup \left|\frac{s_{n+1}}{s_n}\right|$$

Similarly

$\lim\inf |s_n|^{1/n} \le \lim\inf |s_n|^{1/n}$

$\therefore$ $\lim\inf \left|\frac{s_{n+1}}{s_n}\right| \le \lim\inf |s_n|^{1/n} \le \lim\sup |s_n|^{1/n} \le \lim\sup \left|\frac{s_{n+1}}{s_n}\right|.$

**62. Prove Lim sup $|S_n| = 0$ Iff $S_n = 0$.**

*Sol.*

Let $(S_n)$ be a sequence

We have Lim sup $|S_n| = 0$

We prove that $S_n = 0$

We know that

Lim sup $S_n$ = Sup Sn

Lim sup $|S_n|$ = sup$|S_n|$

$\Rightarrow$ Lim sup$|S_n| = 0$

Sup$|S_n| = 0$

$|S_n| = 0$

$S_n = 0$

Conversly prove that Lim sup$|S_n| = 0$

we have $S_n = 0$

apply suprimum both sides

Sup $S_n = 0$

Sup $|S_n| = 0$

apply limit on b/s

Lim sup$|S_n| = 0$

$\therefore$ Lim sup $|S_n| = 0 \Leftrightarrow S_n = 0$

**63. Let $(S_n)$ and $(t_n)$ be the following squares that repeat in cycles of four.**

**$(S_n)$ = (0, 1, 2, 1, 0, 1, 2, 1, 2, 1, 0, 1, 2, 1, 0 ....)**

**$(t_n)$ = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2 ....)**

**find a) lim inf $s_n$ + lim inf $t_n$,**

*Sol.*

a) Lim inf $S_n$ + lim Inf $t_n$

= (0, 1, 2, 1, 0, 1, 2, 1,0, 1,2, 1, 0, 1, 2, 1,0....)

= (2, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2 ...)

= 0 – 0 + 2 – 2

= 0

b) Lim inf $(S_n + t_n)$

Lim inf (0 + 2 – 1) = 1

c) Lim inf $S_n$ + lim Sup $t_n$

$\Rightarrow$ Lim inf $S_n = 0$

$\Rightarrow$ Lim inf $t_n = 2$

$\therefore$ lim inf $S_n$ + lim sup $t_n$ = 0 + 2 = 2

d) Lim sup $(S_n + t_n)$

lim sup $(S_n + t_n) = 1 + 2 = 3$

e) Lim sup $S_n$ + Lt sup $t_n$

Lim sup $S_n = 2$

Lim sup $t_n = 2$

$\therefore$ Lim sup $S_n$ = Lim sup $t_n = 2 + 2 = 4$

f) Lim inf $(S_n\ t_n)$

$\therefore$ Lim Inf $(0.1) = 0$

g) Lim sup $(S_n\ t_n)$

$\therefore$ Lim sup $(S_n\ t_n) = 1.2 = 2$.

## 1.8 Series (or) Infinite Series

If $\{u_n\}$ is a sequence of real numbers then $u_1 + u_2 + u_3 + ... u_n + ...$ is called an infinite series. and is denoted by $\sum_{n=1}^{\infty} u_n$ or $\Sigma u_n$.

The numbers $u_1, u_2, u_3, ... u_n, ...$ are called the 1st, 2nd, 3rd, ... nth .. term of the series.

**1. Series of Positive Terms**

If all the terms of the series $\Sigma u_n = u_1 + u_2 + .... + u_n + ...$ are positive i.e., if $u_n > 0\ \forall$ n. Then the series is called a series of positive terms.

**2. Alternating Series**

A series in which the terms are alternatively positive and negative is called an alternating series.

$\therefore$ $\Sigma(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + ... + (-1)^{n-1} u_n + ...$ where $u_n > 0\ \forall$n is an alternating series.

### 1.8.1 Partial Sums

If $\Sigma u_n = u_1 + u_2 + u_3 + ... + u_n + ...$ is an infinite series where the terms may be +ve or –ve then $s_n = u_1 + u_2 + ... + u_n$ is called the nth partial sum of $\Sigma u_n$. Thus the nth partial sum of an infinite series is the sum of its first n terms.

$\because$ $n \in N$, $\{s_n\}$ is a sequence called the sequence of partial sums of the infinite series $\Sigma u_n$.

$\therefore$ To every infinite series $\Sigma u_n$ there corresponds a sequence $\{s_n\}$ of its partial sums.

**Note :**

1. The series $\Sigma u_n$ converges if the sequence $\{s_n\}$ of its partial sums converges.
2. The series $\Sigma u_n$ diverges if the sequence $\{s_n\}$ of its partial sums diverges.
3. The series $\Sigma u_n$ oscillates finitely if the sequence $\{s_n\}$ of its partial sum oscillates finitely.
4. A necessary and sufficient condition for the convergence of an infinite series is if the series $\sum_{n=1}^{\infty} u_n$ converges, then $\lim_{n\to\infty} u_n = 0$.

5. **Geometric Series :** If $|r| < 1$ or $-1 < r < 1$ the series $\sum_{n=0}^{\infty} r^n (r \in R)$ converges to $\frac{1}{1-r}$ and if $|r| \geq 1$ the series $\sum_{n=0}^{\infty} r^n$ diverges.

6. **Auxilary series or p-series test :** The series $\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ..., P \in R$ a) converges if $P > 1$, b) diverges if $0 < p \leq 1$ and c) diverges if $p \leq 0$.

7. **Comparison test of the first type :** Let $\Sigma u_n$ and $\Sigma v_n$ be two positive term series such that $\Sigma v_n$ is convergent and $\exists\ m \in N \ni u_n \leq v_n\ \forall n \geq m$ then $\Sigma u_n$ is convergent.

8. **Comparison Test of the Second Type :** If $\Sigma u_n$ and $\Sigma v_n$ are two series of non negative terms such that $\Sigma v_n$ is divergent and $\exists m \in N \ni u_n \geq v_n \forall n \geq m$ then $\Sigma u_n$ is divergent.

9. **Limit comparison test :** Let $\Sigma u_n$ and $\Sigma v_n$ be two series of positive terms such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l \in R$ then if $l \neq 0$ then the series $\Sigma u_n$, $\Sigma v_n$ either converges or diverges together.

10. **Cauchy's n^th root test :** Let $\Sigma u_n$ be a +ve term series and Let $\lim_{n\to\infty} (u_n)^{1/n} = l$. then the series is

    (i) converges if $l < 1$

    (ii) diverges if $l > 1$ and

    (iii) test fails to decided the nature of the series if $l = 1$.

11. **D'Alemberts ratio test :** If $\Sigma u_n$ is a series of +ve terms $\ni \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then

    a) $\Sigma u_n$ converges if $l < 1$

    b) $\Sigma u_n$ diverges if $l > 1$ and

    c) Test fails to decided the nature of the series if $l = 1$.

12. If $\Sigma u_n$ is a series of +ve terms $\ni \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \infty$ then $\Sigma u_n$ diverges.

---

**64. Determine which of the following series converge. Justify your answers.**

*Sol.*

**a)** $\Sigma \frac{n^4}{2^n}$ **this will prove by ratio test**

Let $a_n = \frac{n^4}{2^n}$

$$a_{n+1} = \frac{(n+1)^4}{2^{n+1}}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^4}{2^{n+1}}}{\frac{n^4}{2^n}} = \frac{\frac{(n+1)^4}{2^n.2}}{\frac{n^4}{2^n}}$$

$$= \frac{(n+1)^4}{2} \times n^4 = \frac{(n+1)^4 n^4}{2}$$

apply limit on b/s

$$\underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| = \underset{n\to\infty}{Lt}\left|\frac{(n+1)^4.n^4}{2}\right|$$

$$\therefore \underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\therefore \Sigma\frac{n^4}{2^n} \text{ is converges}$$

**b)** $\Sigma\frac{2^n}{n!}$

This will prove by ratio test

$$\text{Let } a_n = \frac{2^n}{n!}$$

$$a_{n+1} = \frac{2^{n+1}}{(n+1)!}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2^{n+1}}{(n+1)!} + \frac{n(n-1)!}{2^n}$$

$$= \frac{n(n-1)!}{(n+1)(n)!} = \frac{(n-1)!}{(n+1)!}$$

apply limit on b/s

$$\underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| = \underset{n\to\infty}{Lt}\left|\frac{(n-1)!}{(n+1)!}\right|$$

$$\underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\therefore \Sigma\frac{2^n}{n!} \text{ is converges.}$$

c) $\mathbf{a_n = \frac{n^2}{3^n}}$

This will prove by ratio test

$$\text{Let } a_n = \frac{n^2}{3^n}$$

$$a_{n+1} = \frac{(n+1)^2}{3^{(n+1)}}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{3^{(n+1)}}}{\frac{n^2}{3^n}}$$

$$= \frac{(n+1)^2}{3} \times \frac{3^n}{(n^2)}$$

$$= \frac{(n+1)^2}{3n^2}$$

apply limit on b/s

$$\underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| = \underset{n\to\infty}{Lt}\left|\frac{(n+1)^2}{3n^2}\right|$$

$$\therefore \underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\therefore \frac{\Sigma n^2}{3^n} \text{ is converges.}$$

**d)** $\mathbf{\frac{\Sigma n!}{n^4 + 3}}$

$$\text{Let } a_n = \frac{n!}{n^4+3}$$

$$a_{n+1} = \frac{(n+1)!}{(n+1)^4+3}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^4+3}}{\frac{n!}{n^4+3}}$$

$$= \frac{(n+1)!}{(n+1)^4+3} \times \frac{n^4+3}{n!}$$

$$= \frac{(n^4+3)(n+1)}{(n+1)^4+3}$$

apply limit on b/s

$$\underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| = \underset{n\to\infty}{Lt}\left|\frac{(n^4+3)(n+1)}{(n+1)^4+3}\right|$$

nth terms do not converges to 'O'

$\therefore \quad \Sigma \frac{n!}{n^4+3}$ is diverges.

**e)** $\Sigma \frac{\cos^2 n}{n^2}$

$$a_n = \frac{\cos^2 n}{n^2}, \quad a_{n+1} = \frac{\cos^2(n+1)}{(n+1)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{\cos^2(n+1)}{(n+1)^2}}{\frac{\cos^2 n}{n^2}} \text{ (or)}$$

$$= \frac{\cos^2(n+1)}{(n+1)^2} \cdot \frac{n^2}{\cos^2 n}$$

Compare this with $\frac{1}{n^2}$

$\therefore \frac{1}{n^2}$ is converges

$\therefore \Sigma \frac{1}{n^2}$ is converges

$\therefore \Sigma \frac{\cos^2 n}{n^2}$ is converges

**f)** $\sum_{n=2}^{\infty} \frac{\mathbf{1}}{\mathbf{\log n}} \Rightarrow$ **logn < n**

$$\Rightarrow \frac{1}{\log n} > \frac{1}{n}$$

Compare with $\Sigma \frac{1}{n}$

$$\Rightarrow \frac{1}{\log n} \geq \frac{1}{n} \quad \forall n$$

$\therefore \Sigma \frac{1}{n}$ is diverges [$\therefore$ Comparison test]

$\therefore \frac{1}{\log n}$ is also diverges.

**Second Method**

$\text{Cos } n \leq 1$

$\frac{\text{Cos } n}{n^2} \leq \frac{1}{n^2}$

$\Sigma \frac{\cos^2 n}{n^2} \leq \frac{1}{n^2}$

But $\Sigma \frac{1}{n^2}$ is get by P-test

$\Sigma \frac{\cos^2 n}{n^2}$ is less the convergence

$\therefore$ it is converges, [$\because$ comparison test]

**65. Prove that if $\Sigma a_n$ is a convergent series of nonnegative numbers and P>1, then $\Sigma a_n^P$ converges.**

*Sol.*

$\Sigma a_n$ is a sequence

there exists N such that $a_n < 1$ for $n > N$

$\therefore$ $\Sigma a_n$ is a convergent series of non negative numbers.

Since P>1,

$\Rightarrow a_n^P = a_n\, a_n^{P-1} < a_n$ for $n > N$

$\Rightarrow a_n\, a_n^{P-1} <$ an

$\therefore a_n^P$ is converges series

$\therefore \Sigma a_n^P$ is converges.

**66.** Show that if $\Sigma a_n$ **and** $\Sigma b_n$ **are convergent series of non-negative numbers, then** $\Sigma\sqrt{a_n b_n}$ **converges.**

*Sol.*

Given that $\Sigma a_n$ and $\Sigma b_n$ are convergent series of non negative numbers by the known theorem

$\Rightarrow$ $a_n$ converges and $b_n$ converges

$\Rightarrow$ $a_n + b_n$ also converges

$(a_n + b_n)^{1/2}$ converges

We prove $\sqrt{a_n b_n}$ is converges

$a_n b_n \le (a_n + b_n)^{1/2}$

$\therefore$ $a_n + b_n$ is converges $\sqrt{a_n b_n}$ also converges

$\sqrt{a_n b_n} < \alpha$, $\alpha$ is positive integer

$\therefore$ $\Sigma\sqrt{a_n b_n}$ is converges

**67. We have seen that**

**(a) Calculate** $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ **and** $\sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n$

**(b)** $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$

**(c) Prove** $\sum_{n=1}^{\infty}\frac{n-1}{2^{n+1}} = \frac{1}{2}$

**(d)** $\sum_{n=1}^{\infty}\frac{n}{2^n} = \sum_{k=1}^{\infty}\frac{4k}{2^{k+1}} - \frac{k}{2^k}$

*Sol.* **(Imp.)**

$\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ and $\sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n$

$\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2+....$ and

$\left(-\frac{2}{3}\right)+\left(-\frac{2}{3}\right)^2+\left(\frac{-2}{3}\right)^3+....$

$\frac{2}{3}\left[1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+...\right]$ and

$\left[-\frac{2}{3}+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^3+....\right]$

$\frac{2}{3}\left[\frac{5}{3}+\left(\frac{2}{3}\right)^2+....\right]$ and

$-\frac{2}{3}\left[1+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^3+...\right]$

$\frac{2}{3}[3]$ and $\left(\frac{-2}{3}\right)\frac{3}{5}$

2 and $\left(\frac{-2}{5}\right)$

**b)** $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$

$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}\right]$

$S_n = \sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}\right]$

$S_n = \left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+....+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$

$S_n = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}-\frac{1}{5}+\frac{1}{5}+.....-\frac{1}{n}+\frac{1}{n}-\frac{1}{n+1}$

$\therefore S_n = 1-\frac{1}{n+1}$

$\therefore \underset{n\to\infty}{Lt} S_n = 1 - \frac{1}{\infty + 1} = 1 - 0$

$\therefore \underset{n\to\infty}{Lt} S_n = 1$

**c) Prove** $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$

By Partial fractions

Let $S_n = \sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \sum_{k=1}^{\infty}\left[\frac{k}{2^k} - \frac{k+1}{2^{k+1}}\right]$

$$S_n = \left[\left(\frac{1}{2^1} - \frac{2}{2^2}\right) + \left(\frac{2}{2^2} - \frac{3}{2^3}\right) + \left(\frac{3}{2^3} - \frac{4}{2^4}\right) + .... + \left(\frac{n}{2^n} - \frac{n+1}{2^{n+1}}\right)\right]$$

$$= \frac{1}{2} - \frac{2}{2^2} + \frac{2}{2^2} - \frac{3}{2^3} - \frac{4}{2^4} + ..... - \frac{n}{2^n} + \frac{1}{2^n} - \frac{n+1}{2^{n+1}}$$

$S_n = \frac{1}{2} - \frac{n+1}{2^{n+1}}$

apply limit on b/s

$\underset{n\to\infty}{Lt} S_n = \underset{n\to\infty}{Lt}\left[\frac{1}{2} - \frac{n+1}{2^{n+1}}\right]$

$= \frac{1}{2} - 0$

$\underset{n\to\infty}{Lt} S_n = \frac{1}{2}$

$\therefore \sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$

**d)** $\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{k=1}^{\infty} \frac{4k}{2^{k+1}} - \frac{k}{2^k}$

Let $S_n = \sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{k=1}^{\infty} \frac{4k}{2^{k+1}} - \frac{k}{2^k}$

$S_n = \left[\frac{4}{2^2} - \frac{1}{2} + \frac{8}{4} - \frac{2}{2^2} + \frac{12}{16} .....\right]$

$S_n = 2 - \frac{n}{2^n}$

apply limit on b/s

$\underset{n\to\infty}{Lt} S_n = \underset{n\to\infty}{Lt}\left[2 - \frac{n}{2^n}\right]$

$\therefore \underset{n\to\infty}{Lt} S_n = 2$

$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$

**68. Does series converge? Justify your answer.**

**a)** $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$

**b)** $\sum_{n=2}^{\infty} \frac{\log n}{n}$

**c)** $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n}$

**d)** $\sum_{n=2}^{\infty} \frac{\log_n}{n^2}$

*Sol.*

**a)** $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$

$S_n = \frac{1}{\sqrt{n}\log n}$

$Lt\, S_n = \underset{n\to\infty}{Lt} \frac{1}{\sqrt{n}\log n}$

$\sqrt{n} \log n < n$

$\sqrt{n} \frac{1}{\log n} > \frac{1}{n}$

$\frac{1}{\sqrt{n}\log n} > \frac{1}{n}$

$\therefore$ it is divergence.

**b)** $\sum_{n=2}^{\infty} \frac{\log n}{n}$

$\log n > n$

$$\frac{1}{\log n} < \frac{1}{n}$$

$$\sum_{n=2}^{\infty}\frac{1}{\log n} < \sum_{n=2}^{\infty}\frac{1}{n}$$

$$< \frac{1}{2}$$

$\therefore$ it is convergence sequence.

**c)** $\displaystyle\sum_{n=4}^{\infty}\frac{1}{n(\log n)\,(\log\log n}$

n log n > n

(n log n) log logn) > (log log n)

$$\frac{1}{(n\log n)\,(\log\log n)} < \frac{1}{(\log\log n)n}$$

it is diverges.

**d)** $\displaystyle\sum_{n=2}^{\infty}\frac{\log_n}{n^2}$

log n > n2

$$\frac{1}{\log n} < \frac{1}{n^2}$$

$\therefore$ compare p - test P > 1 2 > 1

$\therefore$ it is convergence.

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**69.** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\log n)P}$ **converges if and only if P >1.**

*Sol.* **(Nov./Dec.-18)**

Given that

$$\sum_{n=2}^{\infty}\frac{1}{n(\log n)P}$$

We have that $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\log n)P}$ is convergence we prove P > 1.

**Necessary Condition**

By P - Test

$$\frac{1}{n\log n} = \frac{1}{n}$$

$$\frac{1}{(n\log n)^p} \le \frac{1}{n^p}\ ;\ p > 1$$

$$\frac{1}{(n\log n)^P} \le \frac{1}{n^p} \qquad [\because \text{P – test}]$$

**Sufficient Condition**

Conversly P > 1

We prove that $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\log n)^P}$ is convergent

this prove by integral test

$$\lim_{n\to\infty}\int_3^n \frac{1}{n(\log n)^P}\,d_n = \underset{n\to\infty}{\text{Lt}}\int_{\log 3}^{\log n}\frac{1}{S_n^p}\,ds_n$$

by using P test where P > 1.

$$\int_3^n \frac{1}{n(\log_n)^p} \le \frac{1}{n}$$

$\therefore$ $\displaystyle\int_3^n \frac{1}{n(\log_n)^p}$ is convergence sequence

$\therefore$ $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\log_n)^p}$ is convergence sequence at P > 1.

## 1.9 Alternating Series

A series whose terms are alternatively positive and negative is called an alternating series.

An alternating series may be written as $u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots$ where each $u_n$ is positive or negative and it is denoted by $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ where $u_n > 0$.

### 1.9.1 Leibnitz's Test

**70. The alternating series $\Sigma(-1)^{n-1}u_n = u_1 - u_2 + u_3 - u_4 + \ldots$ ($u_n > 0 \ \forall n$) converges if (i) $u_n \geq u_{n+1} \forall n$ and (ii) $\lim\limits_{n\to\infty} u_n = 0$.**

**OR**

**State and prove Alternating Series (or) Leibnitz's Test**

*Sol.*

Let $s_n$ denote the $n^{th}$ partial sum of the series $\Sigma(-1)^{n-1}u_n$.

$\Rightarrow \quad s_n = u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1}u_n$

Then $\quad s_{2n} = u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n}$

and $\quad s_{2n+2} = u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}$

Consider

$s_{2n+2} - s_{2n} = u_{2n+1} - u_{2n+2} \geq 0$ by cond .(i)

$\Rightarrow s_{2n+2} \geq s_{2n} \ \forall n$

$\therefore$ The subsequence $\{s_{2n}\}$ of $\{s_n\}$ is an increasing sequence (1)

Now consider

$s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots - (u_{2n-2} - u_{2n-1}) - u_{2n}$

$s_{2n} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + \ldots + (u_{2n-2} - u_{2n-1}) + u_{2n}]$

$s_{2n} = u_1 -$ [a positive number] $\because u_n > 0 \ \forall n$.

$s_{2n} < u_1 \ \forall n$

$\Rightarrow \{s_{2n}\}$ is bounded above ... (2)

$\therefore$ from (1) and (2)

$\{s_{2n}\}$ converges

$\Rightarrow \lim\limits_{n\to\infty} s_{2n} = l$

we have $s_{2n} = u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n}$

$\Rightarrow \quad s_{2n} = s_{2n-1} - u_{2n}$

$\Rightarrow \quad s_{2n-1} = s_{2n} + u_{2n}$

$\Rightarrow \quad \lim\limits_{n\to\infty} s_{2n-1} = \lim\limits_{n\to\infty} s_{2n} + \lim\limits_{n\to\infty} u_{2n}$

$\Rightarrow \quad \lim\limits_{n\to\infty} s_{2n-1} = l + 0$

$\Rightarrow \quad \lim\limits_{n\to\infty} s_{2n-1} = l$

$\Rightarrow$ $\{s_{2n-1}\}$ converges to '$l$'

$\because$ The subsequence of $\{s_n\}$ converges to '$l$'

$\Rightarrow$ The sequence $\{s_n\}$ converges to '$l$'

$\Rightarrow$ The series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ converges.

### 1.9.2 Absolute and Conditional Convergence

A series $\sum_{n=1}^{\infty} u_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |u_n|$ is convergent.

If $\sum_{n=1}^{\infty} u_n$ converges but not absolutely i.e., $\sum_{n=1}^{\infty} |u_n|$ diverges then the series $\sum_{n=1}^{\infty} u_n$ is known as conditionally convergent.

**Note**

Every absolutely convergent series is convergent converse need not be true. i.e., A convergent series need not be absolutely convergent.

**71. If a series $\Sigma a_n$ converges them lim $a_n = 0$.**

*Sol.* **(Dec.-17, Imp.)**

Given $\Sigma a_n$ is convergent

Let $\Sigma a_n$ convergent to A

$s_n = a_1 + a_2 + ..... + a_n$ be the $n^{th}$ partial sum of $\Sigma a_n$.

Let $\lim s_n = A$

$\lim s_{n-1} = A$

$s_n = a_1 + a_2 + .... + a_n$

$s_{n-1} = a_1 + a_2 + .... + a_{n-1}$

Consider,

$s_n - s_{n-1} = a_1 + a_2 + .... + a_n - a_1 - a_2 .... a_{n-1}$

$s_n - s_{n-1} = a_n$

Apply limit on both sides

$\lim (s_n - s_{n-1}) = \lim a_n$

$\lim s_n - \lim s_{n-1} = \lim a_n$

$A - A = \lim a_n$

$\lim a_n = 0$

for each $\varepsilon > 0 \exists N \ni |a_n| < \varepsilon$

then $\lim a_n = 0$

Hence proved

**Note**

Converse of the above theorem is not there i.e., $\lim a_n = 0 \Rightarrow \Sigma a_n$ is convergent.

**72. Absolutely convergent series are convergent.**

*Sol.*

Let $\Sigma a_n$ be an absolutely convergent series.

i.e, $\Sigma|a_n|$ is convergent

To prove that $\Sigma a_n$ is convergent

$\because$ $\Sigma|a_n|$ is convergent

By Cauchy's general principle of convergent we know that

$$\exists m \in z^+ \ni |a_{p+1} + a_{p+2} + .... + a_q| < \varepsilon \quad \forall q \geq p \geq m$$

for each $\varepsilon > 0 \exists m \in N \ni ||a_{p+1}| + |a_{p+2}| + ..... + |a_q| < \varepsilon \forall q \geq p > m$

$|a_{p+1} + a_{p+2} + .... + a_q| \leq ||a_{p+1}| + |a_{p+2}| + ..... + |a_q||$

$|a_{p+1} + a_{p+2} + .... + a_q| < \varepsilon + a, \geq p > m$

$\therefore$ $\Sigma a_n$ is convergent by Cauchy general principle

$\Sigma|a_n|$ is convergent

$\Sigma a_n$ is convergent

**73. Suppose that $\Sigma a_n = A$ and $\Sigma b_n = B$ where A and B are real numbers.**

**(a) $\Sigma(a_n + b_n) = A + B$**

**(b) $\Sigma k a_n = kA \forall k \in R$**

*Sol.*

(a) Given $\Sigma a_n$ converges to A and $\Sigma b_n$ is converges to B.

To prove that $a_n + b_n$ converges to $A + B$

Let $s_n$ be the nth partial sum of $\Sigma a_n$

$$s_n = a_1 + a_2 + .... + a_n$$

Let $t_n$ be the $n^{th}$ partial sum of $\Sigma b_n$

$$t_n = b_1 + b_2 + ..... + b_n$$

$\therefore$ $\Sigma a_n$ is converges to A

$$\lim s_n = A$$

$\Sigma b_n$ is converges to B

$$\lim t_n = B$$

Let $p_n$ be the $n^{th}$ partial sum of $\Sigma(a_n + b_n)$

$$p_n = (a_1 + b_1) + (a_2 + b_2) + .... + (a_n + b_n)$$
$$= (a_1 + a_2 + .... + a_n) + (b_1 + b_2 + .... + b_n)$$
$$= s_n + t_n$$

Consider $\lim p_n = \lim(s_n + t_n)$

$$= \lim s_n + \lim t_n$$

$$\Sigma(a_n + b_n) = A + B$$

$\Sigma(a_n + b_n)$ is converges to $A + B$

(b) Given that $\Sigma a_n$ converges to A

Let $s_n$ be the $n^{th}$ partial sum of $\Sigma a_n$

$\therefore\ s_n = a_1 + a_2 + .... + a_n$

$\therefore\ \Sigma a_n$ is converges to A

$\lim s_n = A$

To prove that $\Sigma k\, a_n$ is converges to kA

let $t_n$ be the $n^{th}$ partial sum of $\Sigma ka_n$

i.e., $t_n = ka_1 + ka_2 + ka_3 + .... + ka_n$

$= k(a_1 + a_2 + ..... + a_n)$

$= k\, s_n$

$\lim t_n = \lim (ks_n)$

$= k \lim s_n$

$\lim t_n = kA$

$\Sigma ka_n = kA$

$\Sigma ka_n$ is converges to kA

**74. Test for convergence, absolute convergence and conditional convergence of the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^p} = 1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + ...$ for p > 0.**

*Sol.*

Let $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\, u_n = \sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^p}$

$\because$ P > 0 and we know that

$(n + 1) > n$

$\Rightarrow (n + 1)^P > n^p$

$\Rightarrow \dfrac{1}{(n+1)^P} < \dfrac{1}{n^P} \Rightarrow u_{n+1} < u_n\ \forall\, n \in N$

also we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n^P} = 0$$

$\therefore$ by Lebnitz's test $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^p}$ is convergent

and $\left|\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^p}\right| = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^p}$ is convergent, if P > 1 and divergent if $P \le 1$.

$\therefore$ The given series is absolutely convergent if P > 1 and conditionally converges if $0 \le P \le 1$.

**75. Test for convergence, absolute convergence and conditional convergence of the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{\log(n+1)} = \dfrac{1}{\log 2} - \dfrac{1}{\log 3} + \dfrac{1}{\log 4} - ...$**

*Sol.*

Let $u_n = \dfrac{1}{\log(n+1)}\ \forall n \in N$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\log(n+1)} = 0$$

We know that $n + 2 > n + 1\ \forall n \in N$

$\Rightarrow \log(n + 2) > \log(n + 1)$

$\Rightarrow \dfrac{1}{\log(n+2)} < \dfrac{1}{\log(n+1)}$

$= u_{n+1} < u_n\ \forall n \in N$

$\therefore$ by Leibnitz's therom given series is convergent. Now consider,

$\sum\limits_{n=2}^{\infty}|u_n| = \sum\limits_{n=2}^{\infty}\dfrac{1}{\log n}$ which is divergent.

$\Rightarrow \Sigma u_n$ is not absolutely convergent.

$\therefore\ \Sigma u_n$ is conditionally convergent.

**76. Show that the series $\dfrac{\log 2}{2^2} - \dfrac{\log 3}{3^2} + \dfrac{\log 4}{4^2} - ...$ converges.**

*Sol.*

Let $u_n = \dfrac{\log(n+1)}{(n+1)^2}\ \forall n \in N$

$$\because \lim_{n\to\infty}\frac{\log n}{n^2} = 0 \Rightarrow \lim_{n\to\infty} u_n = 0$$

To prove $u_{n+1} < u_n, \ \forall n \in N$

Let $u(x) = \dfrac{\log x}{x^2}$

$\Rightarrow u'(x) = \dfrac{x^2(1/x) - 2x\log x}{x^4}$

$= \dfrac{1 - 2\log x}{x^3} < 0$

$\Rightarrow$ 1 – 2 log x < 0

$\Rightarrow$ log x > 1/2

$\Rightarrow x > e^{1/2} \Rightarrow x > \sqrt{e}$

$\Rightarrow$ u(x) is a decreasing function

$\Rightarrow u_{n+2} \leq u_{n+1} \ \forall n \in N$

$\Rightarrow \dfrac{\log(n+2)}{(n+2)^2} \leq \dfrac{\log(n+1)}{(n+1)^2} \quad \forall n \in N$

$\Rightarrow u_{n+1} < u_n \ \forall n \in N$

$\therefore$ By Leibnitz's therom the given series is convergent.

**77. Test the convergence and absolute convergence of the series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} n}{2n-1}$**

*Sol.*

Let $u_n = \dfrac{n}{2n-1}$

$\therefore \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} n}{2n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$

$\because u_n = \dfrac{n}{2n-1} > 0 \ \forall n \in N$

$\Rightarrow u_{n+1} = \dfrac{n+1}{2n+1}$

Consider

$u_n - u_{n+1} = \dfrac{n}{2n-1} - \dfrac{n+1}{2n+1} = \dfrac{1}{4n^2-1} > 0 \ \forall n \in N$

$\Rightarrow u_n > u_{n+1} \ \forall n \in N$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \dfrac{n}{2n-1}$

$= \lim_{n\to\infty} \dfrac{n}{n(2-1/n)} \lim_{n\to\infty} \dfrac{1}{2-1/n}$

$\Rightarrow \dfrac{1}{2} \neq 0$

$\therefore$ by Leibnitz's therom $\Sigma u_n$ does not converges.

$\Rightarrow$ The given series diverges.

**78. Test for convergence and absolute convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} (\sqrt{n^2+1} - n)$**

*Sol :*

Let the given series be

$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = \sum_{n=1}^{\infty} (-1)^{n+1} (\sqrt{n^2+1} - n)$

$\Rightarrow u_n = \sqrt{n^2+1} - n. \times \dfrac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$

$\Rightarrow u_n = \dfrac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \dfrac{1}{\sqrt{n^2+1}+n} > 0 \ \forall n \in N$

$\Rightarrow u_{n+1} = \dfrac{1}{\sqrt{(n+1)^2+1} + (n+1)} > 0 \forall n \in N$

$\Rightarrow u_n > u_{n+1} \ \forall n \in N$

also $\lim_{n\to\infty} \dfrac{1}{\sqrt{n^2+1}+n} = 0$

$\therefore$ by Leibnitz's test the given series converges.

Now consider

$$|u_n| = \frac{1}{\sqrt{n^2+1}+n}$$

$$= \frac{1}{n\left[\sqrt{1+\frac{1}{n^2}}+1\right]}$$

Let $v_n = \frac{1}{n}$

$$\therefore \quad \frac{|u_n|}{v_n} = \frac{1}{\sqrt{1+\frac{1}{n^2}}+1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{|u_n|}{v_n} \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{2} \neq 0$$

$\therefore$ by comparison test $\Sigma|u_n|$ and $\Sigma v_n$ behave a like

$$\because \quad \Sigma v_n = \Sigma\frac{1}{n} = \Sigma\frac{1}{n^p} \Rightarrow P = 1$$

$\therefore$ by Auxilary series $\Sigma v_n$ diverges

$\Rightarrow$ $\Sigma|u_n|$ also diverges

$\Rightarrow$ The given series is conditionally convergent.

**79. Test for convergence and absolute convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}+\sqrt{a}}$**

*Sol.*

Let $\sum_{n=1}^{\infty}(-1)^{n-1}u_n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}+\sqrt{a}}$

Here $u_n = \frac{1}{\sqrt{n}+\sqrt{a}}$

We know that n + 1 > n

$$\Rightarrow \sqrt{n+1} > \sqrt{n} \Rightarrow \sqrt{n+1}+\sqrt{a} > \sqrt{n}+\sqrt{a}$$

$$\Rightarrow \frac{1}{\sqrt{n+1}+\sqrt{a}} < \frac{1}{\sqrt{n}+\sqrt{a}}$$

$$= u_{n+1} < u_n \ \forall n$$

$$\Rightarrow u_n > u_{n+1} \ \forall n$$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}+\sqrt{a}} = 0$

$\therefore$ by Lebnitz's test, the series is convergent.

Now consider,

$$|u_n| = \frac{1}{\sqrt{n}+\sqrt{a}} \Rightarrow \frac{1}{\sqrt{n}\left[1+\sqrt{a/n}\right]}$$

$$\Rightarrow \quad v_n = \frac{1}{\sqrt{n}}$$

Consider,

$$\frac{|u_n|}{v_n} = \frac{1}{1+\sqrt{\frac{a}{n}}}$$

$$\Rightarrow \lim_{n\to\infty}\frac{|u_n|}{v_n} = \lim_{n\to\infty}\frac{1}{1+\sqrt{\frac{a}{n}}}$$

$$= \frac{1}{1} = 1 \neq 0$$

$\therefore$ By comparision test

$\Sigma|u_n|$ and $\Sigma u_n$ be have a like.

$$\therefore \quad v_n = \Sigma\frac{1}{\sqrt{n}} = \Sigma\frac{1}{n^p}$$

$\Rightarrow$ P = 1/2 < 1

$\therefore$ By Auxilary series $\Sigma v_n$ diverges

$\Rightarrow$ $\Sigma|u_n|$ diverges

$\Rightarrow$ The given series is conditionally convergent.

**80. Test for convergence and absolute convergence of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$**

*Sol.*

Let $\sum_{n=1}^{\infty}(-1)^{n-1}u_n = \sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$

Here $u_n = \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} > 0 \;\forall n \in N$

$\Rightarrow \quad u_{n+1} = \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+1)^2}$

Consider,

$$u_n - u_{n+1} = \frac{1}{n^2} + \frac{1}{(n+1)^2}$$

$$= \frac{4n+4}{n^2(n+2)^2} > 0 \;\forall n \in N$$

$\Rightarrow \quad u_n > u_{n+1} \;\forall n \in N$

Also we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right] = 0$$

$\therefore$ By Lebnitz's test the given series is convergent.

Now consider,

$$|u_n| = \frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{1}{n^2} + \frac{1}{n^2} < \frac{2}{n^2} \;\forall n \in N$$

$\because$ The series $\Sigma \dfrac{2}{n^2} = 2\Sigma \dfrac{1}{n^2}$ is convergent by comparison test.

$\Rightarrow \Sigma |u_n|$ is convergent.

$\therefore$ The given series is absolutely convergent.

**81. Test for convergence and absolute convergence of the series $\sum\limits_{n=1}^{\infty}$ (–1)$^{n-1}$**

**$\dfrac{(-1)^{n-1}\cos^2\alpha}{n\sqrt{n}}$, $\alpha$ is real.**

*Sol.*

Let the given series is,

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos^2 n\alpha}{n\sqrt{n}}$$

Consider,

$$|u_n| = \left|\frac{\cos^2 n\alpha}{n\sqrt{n}}\right|$$

$$= \frac{\cos^2 n\alpha}{n^{3/2}} \le \frac{1}{n^{3/2}} \;\forall n$$

and we know that $\Sigma \dfrac{1}{n^{3/2}}$ is convergent by Auxilary series.

$\therefore$ The series $\Sigma |u_n|$ converges.

$\Rightarrow$ The given series is absolutely convergent.

**82. Show that the series $\sum\limits_{n=1}^{\infty}\left(1 - \cos\dfrac{\pi}{n}\right)$ converges.**

*Sol.*

Let $\sum\limits_{n=1}^{\infty} u_n = \sum\limits_{n=1}^{\infty}\left(1 - \cos\dfrac{\pi}{n}\right) = \sum\limits_{n=1}^{\infty} 2\sin^2\dfrac{\pi}{2n}$

Here $u_n = 2\sin^2\dfrac{\pi}{2n} > 0 \;\forall n$

Let $v_n = \dfrac{1}{n^2}$

$$\therefore \quad \frac{u_n}{v_n} = \frac{2\sin^2\frac{\pi}{2n}}{1/n^2}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\pi^2}{2}\left[\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right]^2$$

$$= \frac{\pi^2}{2}\lim_{n\to\infty}\left[\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right]^2$$

$$= \frac{\pi^2}{2}\times 1 = \frac{\pi^2}{2} \ne 0$$

$\therefore$ By comparison test $\Sigma u_n$ & $\Sigma v_n$ behave a like

$\because \quad \Sigma v_n = \Sigma\dfrac{1}{n^2} = \Sigma\dfrac{1}{n^P}$

Where $P = 2 > 1$

$\therefore$ By auxilary series $\Sigma v_n$ converges

$\Rightarrow \Sigma u_n$ converges.

**83. Test for convergence of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n!}$**

*Sol.*

Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2^n}{n!}$

Consider $|u_n| = \frac{2^n}{n!}$

$\Rightarrow \quad |u_{n+1}| = \frac{2^{n+1}}{(n+1)!}$

$\therefore \quad \lim_{n\to\infty}\frac{|u_n|}{u_{n+1}} = \lim_{n\to\infty}\left[\frac{2^n}{n!}\cdot\frac{(n+1)!}{2^{n+1}}\right]$

$\lim_{n\to\infty}\frac{n+1}{2} = \infty$

$\therefore$ By ratio test, $\Sigma|u_n|$ is convergent

Hence the given series is absolutely convergent.

## 1.10 Integral Test

**84. If for $x \geq 1$, f(x) is a non-negative nonotonically decreasing integrable function of x such that $f(n) = u_n$ for all positive integral values of n, then the series $\sum_{n=1}^{\infty} u_n$ and the improper integral $\int_1^{\infty} f(x)dx$ converges or diverges together.**

*Sol.*

Given f is non-negative on $[1, \infty)$

$\Rightarrow \quad f(x) \geq 0 \quad \forall x \geq 1$

$\Rightarrow \quad \sum_{n=1}^{\infty} f(n)$ is a series of non-negative terms.

$\Rightarrow \quad \sum_{n=1}^{\infty} u_n$ is a series of non-negative terms.

Now let r be any positive integer. Choose a real number x such that $r + 1 \geq x \geq r$.

$\because$ f is monotonically decreasing function of x.

$\Rightarrow \quad f(r + 1) \leq f(x) \leq f(r)$

also f is integrable.

$$\Rightarrow \quad \int_r^{r+1} f(r+1)\,dx \leq \int_r^{r+1} f(x)dx \leq \int_r^{r+1} f(r)dx$$

$$\Rightarrow \quad f(r+1)\int_r^{r+1} dx \leq \int_r^{r+1} f(x)dx \leq f(r)\int_r^{r+1} dx$$

$$\Rightarrow \quad f(r+1)\,[x]_x^{r+1} \leq \int_r^{r+1} f(x)dx \leq f(x)\,[x]_r^{r+1}$$

$$\Rightarrow \quad f(r+1)\,(r+1-r) \leq \int_r^{r+1} f(x)dx \leq f(r)[x]_r^{r+1}$$

$$\Rightarrow \quad f(r+1) \leq \int_r^{r+1} f(x)dx \leq f(r)$$

$$\Rightarrow \quad u_{r+1} \leq \int_r^{r+1} f(r+1)dx \leq u_r$$

$\because f(n) = u_n \forall n \in N$

Putting $r = 1, 2, 3, \ldots, (n-1)$ successively in the above inequality we get,

$$u_2 \leq \int_1^2 f(x)dx \leq u_1$$

$$u_3 \leq \int_2^3 f(x)dx \leq u_2$$

..................

................

$$u_n \leq \int_{n-1}^{n} f(x)dx \leq u_{n-1} \qquad \ldots (1)$$

adding the above inequalities then we get

$$u_1 + u_2 + u_3 + \ldots u_n \leq \int_1^2 f(x)dx + \int_2^3 f(x)dx + \ldots + \int_{n-1}^{n} f(x)dx \leq u_1 + u_2 + \ldots u_{n-1}$$

$$\Rightarrow \quad s_n - u_1 \leq \int_1^n f(x)dx \leq s_n - u_n \quad \because s_n = \sum_{n=1}^{\infty} u_n$$

$\Rightarrow$ $S_n - u_1 \le I_n \le S_n - u_n$ where $I_n = \int_1^n f(x)dx$

$\Rightarrow$ $-u_1 \le I_n - S_n \le -u_n$

$\Rightarrow$ $u_1 \ge S_n - I_n \ge u_n \ge 0$ $\quad \because u_n \ge 0 \forall n \in N$

$\Rightarrow$ $0 \le S_n - I_n \ge u_1$ ... (2)

$\Rightarrow$ The sequence $\{S_n - I_n\}$ is bounded.

Consider,

$$(S_n - I_n) - (S_{n-1} - I_{n-1}) = (S_n - S_{n-1}) - (I_n - I_{n-1})$$

$$= u_n - \left[\int_1^n f(x)dx - \int_1^{n-1} f(x)dx\right]$$

$$= u_n - \left[\int_1^n f(x)dx + \int_{n-1}^1 f(x)dx\right]$$

$$= u_n - \int_1^n f(x)dx$$

$$\le 0 \text{ (from (1))}$$

$$\therefore \; S_n - I_n \le S_{n-1} - I_{n-1}$$

$\Rightarrow$ $\{S_n - I_n\}$ is monotonically increasing

$\because$ Every bounded monotonic sequence converges,

$\Rightarrow$ $\{S_n - I_n\}$ converges

$\therefore$ from (2) we have,

$$0 \le \lim_{n\to\infty} (S_n - I_n) \le u_1$$

$\Rightarrow$ $0 \le \lim_{n\to\infty} S_n - \lim_{n\to\infty} I_n \le u_1$

$\Rightarrow$ $\lim_{n\to\infty} I_n \le \lim_{n\to\infty} S_n$ ... (3)

and $\lim_{n\to\infty} S_n \le u_1 + \lim_{n\to\infty} I_n$ ... (4)

Hence from (3) and (4) we conclude that $\{S_n\}$ and $\{I_n\}$ converges or diverges together and hence $\sum_{n=1}^{\infty} u_1$ and $\int_1^{\infty} f(x)dx$ converges or diverges together.

**85. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(\log n)P}$ .**

*Sol.*

Here, $u_n = \frac{1}{n(\log n)^P}$

**Case (i)**

When $P \le 0$

$$\Rightarrow \quad \frac{1}{n(\log n)P} \ge \frac{1}{n} \forall n \ge 2$$

$\because \sum_{n=2}^{\infty} \frac{1}{n}$ diverges, by comparision test $\sum_{n=2}^{\infty} \frac{1}{n(\log n)P}$ diverges.

**Case (ii)**

When $P > 0$

$\because \{n(\log n)^P\}$ is an increasing sequence

$\Rightarrow \{u_n\}$ is a decreasing sequence.

$\Rightarrow u_n > u_{n+1} > 0 \ \forall n \ge 2$.

$\therefore$ By cauchy's condensation test, the series $\sum_{n=2}^{\infty} u_n$ and $\sum_{n=2}^{\infty} 2^n u_{2n}$ converges or diverges together.

Now $\sum_{n=2}^{\infty} 2^n u_{2n} = \sum_{n=2}^{\infty} 2^n . \frac{1}{2^n (\log 2^n)^P}$

$$= \sum_{n=2}^{\infty} \frac{1}{(\log 2^n)^P}$$

$$= \frac{1}{(\log 2)^P} \sum_{n=2}^{\infty} \frac{1}{n^P}$$

$\because \sum_{n=2}^{\infty} \frac{1}{n^P}$ is convergent if $P > 1$ and diverges if $P \le 1$.

$\therefore \sum_{n=2}^{\infty} 2^n u_{2n}$ is convergent if $P > 1$ and diverges if $P \le 1$

$\Rightarrow \sum_{n=2}^{\infty} u_n$ convergent if $P > 1$ and diverges if $P \le 1$

Hence $\sum_{n=2}^{\infty} u_n$ is convergent if $P > 1$ and diverges if $P \le 1$.

**86. Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$**

*Sol.*

Here $u_n = \frac{1}{n \log n}$

$\because$ {n log n} is an increasing sequence

$\Rightarrow$ $\{u_n\}$ is a decreasing sequence.

$\Rightarrow$ $u_n > u_{n+1} > 0 \ \forall n \geq 2$.

$\therefore$ By cauchy's condensation test the series $\sum_{n=2}^{\infty} u_n$ and $\sum_{n=2}^{\infty} 2^n u_{2^n}$ converges or diverges together.

Now $\sum_{n=2}^{\infty} 2^n u_{2^n} = \sum_{n=2}^{\infty} 2^n . \frac{1}{2^n \log 2^n}$

$= \sum_{n=2}^{\infty} \frac{1}{n \log 2}$

$= \frac{1}{\log 2} \sum_{n=2}^{\infty} \frac{1}{n}$

$\because$ $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent.

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**87. Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{(\log n)^P}$**

*Sol.*

Here $u_n = \frac{1}{(\log n)^P}$

**Case (i)**

When $P = 0 \Rightarrow u_n = 1$

$\because$ $\lim_{n \to \infty} u_n = 1 \neq 0$

$\Rightarrow$ $\sum_{n=2}^{\infty} u_n$ diverges

**Case (ii)**

When $P < 0$, Let $p = -q$ where $q > 0$.

$\because$ $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{(\log n)^{-q}} = \lim_{n \to \infty} (\log n)^q = \infty \neq 0$

$\therefore$ $\sum_{n=2}^{\infty} u_n$ diverges

**Case (iii) When P > 0**

$\because$ $\{(\log n)^P\}$ is an increasing sequence

$\Rightarrow$ $\{u_n\}$ is a decreasing sequence

$\Rightarrow$ $u_n > u_{n+1}\ \forall n \geq 2$

$\therefore$ By cauchy's condensation test, the series $\sum\limits_{n=2}^{\infty} u_n$ and $\sum\limits_{n=2}^{\infty} 2^n u_{2^n}$ converges or diverges together.

Now

$$\sum_{n=2}^{\infty} 2^n u_{2^n} = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{(\log 2^n)^P} = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{(n \log 2)^P}$$

$$= \frac{1}{(\log 2)^P} \sum_{n=2}^{\infty} \frac{2^n}{n^P}$$

Consider,

$$u_n = \frac{2^n}{n^P} \text{ so that } (v_n)^{1/n} = \frac{2}{(n^{1/n})^P}$$

$$\therefore \quad \lim_{n \to \infty} (v_n)^{1/n} = 2 \lim_{n \to \infty} \frac{1}{(n^{1/n})^P} = 2 > 1$$

$\therefore$ By cauchy's $n^{th}$ root test, $\Sigma v_n$ is divergent.

$\Rightarrow$ $\sum\limits_{n=2}^{\infty} 2^n u_{2^n}$ is divergent $\Rightarrow \sum\limits_{n=2}^{\infty} u_n$ is divergent.

$\therefore$ $\sum\limits_{n=2}^{\infty} u_n$ is divergent for all values of P.



**88. Discuss the convergence of the series $\sum\limits_{n=1}^{\infty} \frac{\log n}{n}$**

*Sol :*

Here $u_n = \frac{\log n}{n} \geq 0 \forall n$

Let $f(x) = \frac{\log x}{x}, x > 0$

$$\Rightarrow \quad f'(x) = \frac{x.\frac{1}{x} - \log x}{x^2} = \frac{1 - \log x}{x^2}$$

$\because$ $f'(x) < 0 \Rightarrow 1 - \log x < 0 \Rightarrow \log x > 1$

$\Rightarrow$ $e^{\log x} > e'$

$\Rightarrow$ $x > e$

$\Rightarrow$ $f(x)$ is a decreasing function when $x > e$

$\Rightarrow$ $u_n > u_{n+1} \; \forall n > 2$

$\Rightarrow$ $\{u_n\}$ is a decreasing function of positive terms.

$\therefore$ By cauchy's condensation test, the series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} 2^n . u_{2^n}$ converges or diverges together.

Now $\sum_{n=1}^{\infty} 2^n . u_{2^n} = \sum_{n=1}^{\infty} 2^n . \frac{\log 2^n}{2^n} = \sum_{n=1}^{\infty} n \log 2 = \log 2 \sum_{n=1}^{\infty} n$

$\because$ $\sum_{n=1}^{\infty} n$ is divergent.

$\Rightarrow$ $\sum_{n=1}^{\infty} 2^n . u_{2^n}$ is divergent

$\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ is divergent.

# Choose the Correct Answers

1. $\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+.....+\frac{1}{2n^2}\right]=$ [ a ]

(a) 0 (b) 1

(c) 2 (d) None

2. A sequence $\{a_n\}$ is said to be Cauchy sequence if given $\in > 0$, $\exists$ a positive integer m $\ni$ [ c ]

(a) $|a_p - a_q| > \in \ \forall\ p, q \geq m$ (b) $|a_p - a| < \in \ \forall\ p \geq m$

(c) $|a_p - a_q| < \in \ \forall\ p, q \geq m$ (d) None

3. $\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}=$ [ c ]

(a) e (b) 1

(c) $\frac{1}{e}$ (d) None

4. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=$ [ a ]

(a) e (b) 1

(c) e – 1 (d) e + 1

5. $\lim_{n\to\infty} n^{1/n} =$ [ b ]

(a) $\infty$ (b) 1

(c) 0 (d) $-\infty$

6. The sequence $\left\{\frac{n^2+3n+5}{2n^2+5n+7}\right\}$ converges to [ d ]

(a) 1 (b) 0

(c) –1 (d) $\frac{1}{2}$

7. The sequence $\{(-1)^n . n\}$ oscillates [ b ]

(a) Finitely (b) Infinitely

(d) Diverges (d) Converges

8. $\lim_{n\to\infty} \frac{1}{\sqrt{n!}} =$ [ a ]

(a) 0 (b) 1

(c) –1 (d) 2

9. $\lim_{n\to\infty} r^n = 0$ if [ c ]

(a) $|r| > 1$ (b) $|r| = 1$

(c) $|r| < 1$ (d) $|r| \neq 1$

10. A necessary and sufficient condition for a sequence $\{a_n\}$ to converge to '$l$' is that for each $\in > 0$ there corresponds a $M \in N \ni$ [ d ]

(a) $|a_n - l| = \in \ \forall\ n \geq m$ (b) $|a_n - l| > \in \ \forall\ n \geq m$

(c) $|a_n - l| \neq \in \ \forall\ n \geq m$ (d) $|a_n - l| < \in \ \forall\ n \geq m$

11. Every monotonically decreasing sequence which is bounded above converges to its [ a ]

(a) $l$ u b (b) g $l$ b

(c) ub (d) $l$b

12. Every monotonically increasing sequence which is bounded below converges to its. [ b ]

(a) $l$ u b (b) g $l$ b

(c) ub (d) $l$b

13. If $\{a_n\} = l$ and $\{b_n\} = m$ then $\lim_{n\to\infty} \{a_n + b_n\} =$ [ d ]

(a) $l - m$ (b) $l - 1$

(c) $m + 1$ (d) $l + m$

14. The value of $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ lies between [ a ]

(a) 2 and 3 (b) 2 and 1

(c) 1 and 0 (d) 0 and 1

15. Every bounded monotonic sequence is [ c ]

(a) Divergent (b) Oscillates

(c) Convergent (d) None

16. The series $1 + r + r^2 + r^3 + ....$ is oscillatory if = [ d ]

(a) $r < 1$ (b) $r > 1$

(c) $r = 1$ (d) $r = -1$

17. Infinite series $\Sigma \frac{1}{n^p}$ is convergent if [ b ]

(a) P < 1 (b) P > 1

(c) P = 1 (d) P $\leq$ 1

18. $\Sigma u_n$ is a series of positive terms and $\lim_{n\to\infty}(u_n)^{1/n} > 1$ then the series is [ a ]

(a) Divergent (b) Convergent

(c) Oscillates (d) None

19. Series $\Sigma u_n$ of positive terms is divergent if $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right]$ is [ a ]

(a) < 1 (b) $\geq$ 1

(c) = 1 (d) $\leq$ 1

20. The series $\Sigma \frac{1}{n(\log n)^P}$ is divergent if [ b ]

(a) P > 1 (b) P $\leq$ 1

(c) P < 1 (d) P = 1

21. The series $\Sigma u_n$, where $u_n = \sqrt{n^2+1} - n$ is [ b ]

(a) Convergent (b) Divergent

(c) Oscillates (d) None

22. The series $1 + \frac{3}{1!} + \frac{5}{3!} + \frac{7}{4!} + \ldots$ is [ a ]

(a) Convergent (b) Divergent

(d) Oscillates (d) None

23. The series $\frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \frac{1}{9^p} + \ldots$ converges if [ c ]

(a) p < 1 (b) p = 1

(c) p > 1 (d) None

24. The series $\Sigma \frac{1}{n^{3/4}}$ is [ b ]

(a) Convergent (b) Divergent

(c) Oscillates (d) None

25. If $\Sigma u_n$ converges then $\lim_{n\to\infty} u_n =$ [ a ]

(a) 0 (b) 1

(c) – 1 (d) None

26. The series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$ is [ c ]

(a) Oscillates (b) Divergent

(c) Convergent (d) None

27. $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \infty$, then $\Sigma u_n$ [ a ]

(a) Converges (b) Infinite

(c) Diverges (d) None

28. The series $\sum_{n=2}^{\infty}\frac{1}{n^2 \log n}$ [ c ]

(a) Diverges (b) Oscillates

(c) Converges (d) Infinite

29. The sequence $\left\{\frac{n!}{n^n}\right\}$ converges to [ b ]

(a) 1 (b) 0

(c) 2 (d) – 1

30. A sequence converges to [ a ]

(a) One limit point (b) More than one limit point

(c) Finite limit points (d) None

# Fill in the blanks

1. A function whose domain is the set of natural numbers N and range a subset of real numbers R is called as ________.
2. The set of all distinct terms of a sequence is called its ________.
3. The set of all limit points of a bounded sequence is ________.
4. Limit of a sequene, if it exists then it is ________.
5. $\{0, 1, 2, 0, 1, 2^2, 0, 1, 2^3, 0, 1, 2^4, .....\}$ is an unbounded sequence with exactly two limit pts ________ and ________.
6. Every convergent sequence is ________.
7. The upper and lower bounds of the sequence $\{(-1)^n\}\ \forall\ n \in N$ are ________ and ________.
8. If $a_n \geq 0\ \forall\ n \in N$ and $\lim_{n\to\infty} a_n = l$ then ________.
9. If $\{a_n\}$ and $\{b_n\}$ are two sequences such that $|a_n| \leq |b_n|\ \forall\ n \geq m$ where $m \in N$ and $\lim_{n\to\infty} b_n = 0$ then $\lim_{n\to\infty} a_n =$ ________.
10. Every Cauchy's sequence is ________.
11. Every convergent sequence is a ________.
12. A sequence $\{a_n\}$ is said to be monotonically increasing if ________ $\forall\ n \in N$.
13. A sequence $\{a_n\}$ is said to be monotonically decreasing if ________ $\forall\ n \in N$.
14. The sequence $\{(-1)^n\}$ is neither monotonically ________ nor ________.
15. A sequence which is either monotonically increasing or decreasing is called a ________ sequence.
16. Every bounded sequence has a ________ subsequence.
17. A cauchy sequence of real number is convergent if and only if it has a convergent ________.
18. The smallest limit point of $\{a_n\}$ is called the ________.
19. The greatest limit point of $\{a_n\}$ is called the ________.
20. A bounded sequence $\{a_n\}$ converges to $l$ if and only if ________.
21. The infinite series $\Sigma u_n$ is said to be convergent if the sequence $\{s_n\}$ and its partial sums is ________.
22. The $\lim_{n\to\infty} u_n \neq 0$ then the series ________.
23. If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \infty$ then $\Sigma u_n$ is ________.
24. If $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = l$ then the series $\Sigma u_n$ is convergent if ________.
25. Every absolutely convergent series is ________.

26. A series $\Sigma u_n$ is absolutely convergent if ________ is convergent.
27. A series $\Sigma u_n$ is conditionally convergent if ________ is divergent.
28. $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is called ________ series.
29. If the subsequence converges then ________ converges.
30. The series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} 2^n u_{2n}$ converges or diverges together then the test is known as ________.

## ANSWERS

1. Real sequence.
2. Range
3. Bounded
4. Unique
5. 0 and 1
6. Bounded
7. –1 and 1
8. $l \geq 0$
9. 0
10. Bounded
11. Cauchy sequence
12. $a_{n+1} \geq a_n$
13. $a_{n+1} \leq a_n$
14. Increasing, Decreasing
15. Monotonic
16. Convergent.
17. Subsequence
18. Limit inferior
19. Limit superior
20. $\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf. a_n = l$
21. Convergent
22. Diverges
23. Divergent
24. $l > 1$
25. Convergent
26. $\Sigma |u_n|$
27. $\Sigma |u_n|$
28. Alternating
29. Sequence.
30. Cauchy's condensation test

# UNIT II

**Continuity:** Continuous Functions - Properties of Continuous Functions - Uniform Continuity - Limits of Functions

## 2.1 Definition of Continuous Function

Let f be a real valued function whose domain is a subset of R. Then function f is continuous at $x_0$ in dom(f). If, every sequence $\{x_n\}$ in dom(f) converging to $x_0$. We have $\lim f(x_n) = f(x_0)$. If f is continuous at each point of a set $s \subseteq$ dom(f). Then f is said to be continuous on s. The function f is said to be continuous if it is continuous on dom(f) in other words. f is said to be continuous at $x_0$, If $\varepsilon > 0 \exists \delta > 0 \ni |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \ \forall x \in \text{dom}(f)$.

**1. Let f be a real valued function whose domain is a subset of R. Then f is continuous at $x_0$ in dom(f) if and only if for each $e > 0 \exists d > 0 \ni x \in$ dom(f) and $|x - x_0| < d \Rightarrow |f(x) - f(x_0)| < \varepsilon$.**

*Sol.* **(Imp.)**

Given that f is a real valued function

consider a sequence $\{x_n\}$ in dom(f) such that $\lim x_n = x_0$.

We have to prove that $\lim f(x_n) = f(x_0)$.

Since f is continuous at $x_0$.

for given $\varepsilon > 0 \exists \delta > 0 \ni |x - x_0| < d \Rightarrow |f(x) - f(x_0)| < \varepsilon$ ... (1)

Again, Since $\lim\limits_{n\to\infty} x_n = x_0$

$\exists$ positive integer 'm' $\ni n > m \Rightarrow |x_n - x_0| < \varepsilon$ ... (2)

Setting $x = x_n$ in (1)

We get

$|x_n - x_0| < d \Rightarrow |f(x_n) - f(x_0)| < \varepsilon$ ... (3)

From (2) and (3) gives

$n > m \Rightarrow |f(x_n) - f(x_0)| < \varepsilon$

Hence $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$

Conversely suppose that

Suppose for every sequence $\{x_n\}$ converging to x

We have $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$

Then we have to show that f is continuous at $x_0$.

Let us assume that, f is not continuous at $n_0$ then there exists an $\varepsilon > 0 \exists \delta > 0 \ni |x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \varepsilon \;\forall x \in \text{dom}(f)$.

If we take $\delta = \frac{1}{n}$ we see that for each positive integer n, $\exists$ a $x_n \ni |x - x_0| < \frac{1}{n}$ but $|f(x) - f(x_0)| \geq \varepsilon \;\forall x \in \text{dom}(f)$ fails for each $n \in N$.

So, for each $n \in N \exists\ x_n$ in dom(f) such that $|x_n - x_0| < \frac{1}{n}$ and $|f(x_0) - f(x_n)| \geq \varepsilon$.

Thus we have $\lim x_n = x_0$

but we cannot have $\lim f(x_n) = f(x_0)$

Since $|f(x_0) - f(x_n)| \geq \varepsilon \;\forall$ n.

This shows f cannot be continuous at $x_0$

$\therefore$ Our assumption is wrong.

Hence f is continuous at $x_0$.

**2. Let $f(x) = 2x^2 + 1$ for $x \in R$, Prove f is continuous on R, by.**

**(a) Using the definition**

**(b) Using the $\varepsilon - \delta$ property**

*Sol.*

(a) Suppose that $\lim x_n = n_0$.

Then we have $\lim f(x_n) = \lim(2x_n^2 + 1)$

$= 2 \lim(x_n^2) + 1$

$= 2(x_0^2) + 1$

$= 2x_0^2 + 1$

$= f(x_0)$

$\therefore$ $\text{lin } f(x_n) = f(x_0)$.

(b) Let $x_0$ be in R and Let $\varepsilon > 0$. We have to show $|f(x) - f(x_0)| < \varepsilon$ provided $|x - x_0| < \delta$.

$|f(x) - f(x_0)| = |2x^2 + 1 - (2x_0^2 + 1)|$

$= |2x^2 + 1 - 2x_0^2 - 1|$

$= |2x^2 - 2x_0^2|$

$= 2|x^2 - x_0^2|$

$\leq 2|(x - x_0)(x + x_0)|$

$|f(x) - f(x_0)| \leq 2|x - x_0|\ |x + x_0|$

if $|x - x_0| < 1$ (Say)

then $|x| < |x_0| + 1$

$\Rightarrow |x + x_0| = |x| + |x_0|$

$= |x_0| + 1 + |x_0|$

$= 2|x_0| + 1$

$\therefore \quad |f(x) - f(x_0)| \leq 2|x - x_0|\ (2|x_0| + 1)$

Provided $|x - x_0| < 1$

$\Rightarrow$ To arrange $2|x - x_0|\ (2|x_0| + 1) < \varepsilon$ it suffices to how $|x - x_0| < \dfrac{\varepsilon}{2(2|x_0|+1)}$ and also $|x - x_0| < 1$.

So, $\delta = \min\left\{1, \dfrac{\varepsilon}{2(2|x_0|+1)}\right\}$

$|f(x) - f(x_0)| < 2 \cdot \dfrac{\varepsilon}{2(2|x_0|+1)} 2(|x_0| + 1)$

$|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$.

**3. Let $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$ Prove that f is continuous at 0.**

*Sol.* **(Imp.)**

We have to prove f is continuous at 0.

By definition of continuous we have

for every $\varepsilon > 0 \exists \delta > 0 \ni |x - a| < d \Rightarrow |f(x) - f(0)| < \varepsilon$,

Consider

$|f(x) - f(0)| = |x^2 \sin\left(\frac{1}{x}\right) - 0|$

$= |x^2 \sin \frac{1}{x}|$

$\leq |x_2| \left|\sin \frac{1}{x}\right|$

$\leq |x^2|1 \qquad [\because |\sin x| \leq 1]$

$\leq |x|2$

$\leq x^2 \ \forall x$ .

Let $\delta = \sqrt{\varepsilon}$ Then $|x - 0| < \delta \quad \Rightarrow x^2 < \delta^2 (= (\sqrt{\varepsilon})^2)\ x^2 < \varepsilon$

So, $|x - 0| < d \Rightarrow |f(x) - f(0)| < \varepsilon$

$\therefore$ f is continuous at '0'.

**4. Let f be a real value function with dom(f) $\subseteq$ R. If f is continuous at $x_0$ in dom(f). Then |f| and kf, k $\in$ R are continuous at $x_0$.**

*Sol.*

Consider a sequence $\{x_n\}$ in dom(f) converging to $x_0$.

Since f is continuous at $x_0$.

We have lim $f(x_n) = f(x_0)$

Then we have to prove that (1) kf is continuous at $x_0$.

(2) |f| is continuous at $x_0$.

(1) $k \neq 0$, the result is obvious

If k = 0,

Let $\varepsilon > 0$

Then show that $|kx_n - kx_0| < \varepsilon \, \forall n$

sin lim $x_n = x_0$

There exists N $\ni$ $n > N \Rightarrow |x_n - x_0| < \dfrac{\varepsilon}{|k|}$

Then $n > N \Rightarrow |kx_n - kx_0| < \varepsilon$

$\therefore$ kf is continuous at $x_0$.

(2) To prove |f| is continuous at x0

We need to prove lim $|f(x_n)| = |f(x_0)|$

$\therefore$ f is continues at $x_0$

for each $\varepsilon > 0 \exists \delta > 0 \ni x \in S$, $|x - x_0| < \delta \Rightarrow |f(x_n) - f(x_0)| < \varepsilon$

$\therefore$ $x \in s$, $|x - x_0| < d \Rightarrow ||f(x_n) - |f(x_0)|| \leq |f(x_n) - f(x_0)| < \varepsilon$

$\therefore$ |f| is continuous at $x_0$ i.e, lim$|f(x_n)| = |f(x_0)|$.

**5. If f and g are real valued functions at $x_0$ then,**

**(1) f + g is continuous at $x_0$**

**(2) fg is continuous at $x_0$**

**(3) f/g is continuous at $x_0$ if $g(x_0) \neq 0$.**

*Sol.* **(Imp.)**

Given that f and g are real valued functions at '$x_0$'.

Then prove that (1) f + g is continuous at $x_0$

i.e., to prove that

for $\varepsilon > 0 \exists \delta > 0 \ni |x - x_0| < \delta \Rightarrow |(f + g)(x) - (f + g)(x0)| < e$

$\because$ f is continuous at $x_0$

$\Rightarrow$ for $\varepsilon > 0 \exists \delta_1 > 0 \ni |x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \frac{\varepsilon}{2} \quad \forall x \in S$ ... (1)

$\therefore$ g is continuous at $x_0$

for $\varepsilon > 0 \exists \delta_2 > 0 \ni |x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \frac{\varepsilon}{2} \quad \forall x \in S$ ... (2)

Let $\delta = \min \{\delta_1, \delta_2\}$

Consider $|(f + g)(x) - (f + g)(x_0)| = |f(x) + g(x) - f(x_0) - g(x_0)|$

$= |(f(x) - f(x_0) + (g(x) - g(x_0)|$

$< |f(x) - f(x_0)| + |g(x) - g(x_0)|$

By (1) and (2)

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \frac{2\varepsilon}{2} = \varepsilon$$

$\therefore$ $|(f+g)(x) - (f+g)(x_0)| < \varepsilon$

$\therefore$ f + g is continuous at $x_0$.

(2) f g is continuous at $x_0$

Let $\varepsilon > 0$, since f is continuous at $x_0$

i.e., for each $\varepsilon > 0 \exists \delta_1 > 0 \ni |f(x) - f(x_0)| < \frac{\varepsilon}{2(|g(x_0)| + \varepsilon)}$

$|x - x_0| < \delta_1$ ... (1)

g is continuous at $x_0$

i.e., for each $\varepsilon > 0 \exists \delta_2 > 0 \ni |x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \frac{\varepsilon}{2|f(x_0)| + \varepsilon}$ ... (2)

Also, for $\varepsilon > 0 \exists \delta_3 > 0 \ni$

$x \in s,\ |x - x_0| < \delta_3 \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\Rightarrow ||f(x)| - |f(x_0)|| < \varepsilon \Rightarrow |f(x)| < |f(x_0)| + \varepsilon$ ... (3)

If $\delta = \min \{\delta_1, \delta_2, \delta_3\}$ then (1), (2) (3) holds for $x \in s |x - x_0| < d$

$\therefore$ $|(fg)(x) - (fg)(x_0)| = |f(x) g(x) - f(x_0) g(x_0)| = |f(x) g(x) - f(x) g(x_0) + f(x) g(x_0) - f(x_0) g(x_0)|$

$\leq |f(x) g(x) - f(x) g(x_0)| + |f(x) g(x_0) - f(x_0) g(x_0)|$

$\leq |f(x) (g(x) - g(x_0))| + |g(x_0) (f(x) - f(x_0)|$

$\leq |f(x)|\ |g(x) - g(x_0)| + |g(x_0)|\ |f(x) - f(x_0)|$

$< |f(x_0) + \varepsilon . \frac{\varepsilon}{2|f(x_0)| + \varepsilon} + |g(x_0)| . \frac{\varepsilon}{2(g(x_0)| + \varepsilon)}$

$\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$

$< \varepsilon$ for $x \in s$, $|x - x_0| < \delta$

$\therefore$ fg is continuous at $x_0$.

(3) f/g is continuous at $x_0$

Since f is continuous at $x_0$ and g is continous at $x_0$.

To prove that $\frac{1}{g}$ is continuous at $x_0$.

$x \in s, |x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon \frac{|g(x_0)|^2}{2}$

$||g(x)| - |g(x_0)|| < \varepsilon$

$\Rightarrow |g(x)| > |g(x_0)| - \varepsilon \Rightarrow |g(x)| > |g(x_0)| - \frac{|g(x_0)|}{2} > \frac{|g(x_0)|}{2}$

$\therefore$ for $\varepsilon > 0 \exists \delta > 0 \ni$

$x \in s, |x - x_0| < \delta \Rightarrow \left|\frac{1}{g(x)} - \frac{1}{g(x_0)}\right| = \left|\frac{g(x_0) - g(x)}{g(x)\,g(x_0)}\right| < \frac{|g(x_0) - g(x)|}{|g(x)\,g(x_0)|} < \frac{|g(x) - g(x_0)|}{|g(x)\,g(x_0)|}$

$< \frac{\varepsilon}{\left|\frac{g(x_0)}{2}\right| |g(x_0)|}$

$\therefore \left|\frac{1}{g(x)} - \frac{1}{g(x_0)}\right| < \frac{\varepsilon\left|\frac{g(x_0)}{2}\right|^2}{\left|\frac{g(x_0)}{2}\right| |g(x_0)|} = \varepsilon$

$\therefore \left|\frac{1}{g(x)} - \frac{1}{g(x_0)}\right| < \varepsilon$

$\Rightarrow \frac{1}{g}$ is continuous at $x_0$ and $g \neq 0$.

Since f is continuous at $x_0$ and $\frac{1}{g}$ is also continuous at $x_0$ by (2)

$f \cdot \frac{1}{g} = \frac{f}{g}$ is continuous at $x_0$.

**6. If f is continuous at $x_0$ and g is continuous at $f(x_0)$ then the composite function g of is continuous at $x_0$.**

*Sol.*

Let $y = f(x)$ for $x \in s$ and $b = f(x_0)$

Since g is continuous at $f(x_0) = b$,

for $\varepsilon > 0 \exists \delta_1 > 0 \ni y \in T, |y - b| < \delta_1 \Rightarrow |g(y) - g(b)| < \varepsilon$

Since f is continuous at $x_0$,

for $\delta_1 > 0 \exists \delta > 0 \ni$

$x \in s, |x - x_0| < \delta \Rightarrow |f(x) - f(x0)\} < \delta_1$

i.e., $x \in s, |x - x_0| < \delta \Rightarrow |y - b| < \delta_1, y \in T$

$x \in s, |x - x_0| < \delta \Rightarrow |g(y) - g(b)| < \varepsilon \Rightarrow |g(f(x)) - g(f(x_0)| < \varepsilon$

$\Rightarrow |g\ of(x) - g\ o\ f(x_0)| < \varepsilon$

$\therefore$ g of is continuous at $x_0$ properties of continuous functions.

## 2.2 Properties of Continuous Function

**7. Let f be a continuous real valued function on a closed interval [a, b]. Then f is a bounded function more over f assumes its maximum and minimum values on [a, b], i.e., there exists $x_0$, $y_0$ in [a, b] such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.**

*Sol.*

f is a continous real valued function [a, b] f is a bonded on [a, b].

I.e., if there exists real number M such that $|f(x)| \leq M \ \forall x \in \text{dom}(f)$.

Assume that f is not bounded on [a, b]

Then to each $n \in N$ there correspondence $x_n \in [a, b]$ such that $|\delta(x_n)| > n$.

By Bolzanoweierstrass Theorem,

$\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converge to some real number $x_0$.

The number $x_0$ also must be long to the [a, b]

Since f is continuous at $x_0$.

We have $\lim\limits_{k \to \infty} f(x_{n_k}) = f(x_0)$

But we also have $\lim\limits_{k \to \infty} |f(x_{n_k})| = +\infty$

which is a controdection

$\therefore$ f is bounded.

Now let $M = \sup \{f(x) | x \in [a, b]\}$

For each $n \in N$ there exists $y_n \in [a, b]$

Such that

$M - \frac{1}{n} < f(y_n) \leq M$

$\lim f(y_n) = M$

By Bolzanowierslrass Theorem

There is a subsequence $\{y_{nk}\}$ of $\{y_n\}$ converging to a limit $y_0$ in [a, b].

Since f is continous at $y_0$.

We have $f(y_0) = \lim_{k\to\infty} f(y_k)$.

Since $\{f(y_{nk})\}$ is subsequence of $\{f(y_n)\}$ $\forall n \in N$ .

By theorem [Every sequence $\{s_n\}$ has a monotonic subsequence] shows,

$\lim_{k\to\infty} f(y_{n_k}) = \lim_{n\to\infty} f(y_n) = M$

$\therefore \quad f(y_0) = M$

Thus f assumes its maximum at $g_0$

– f assumes its maximum at some $x_0 \in [a, b]$.

$\Rightarrow$ f assumes its minimum at $x_0$.

**8. State and prove Intermediate value theorem.**

**(OR)**

**If f is a continuous real valued function on an interval I, then f has the intermediate value property on I, whereever a < b and y lies between f(a) and f(b).**

**[i.e., f(a) < y < f(b) or f(b) < y < f(a)] there exists at least one x in (a, b) such that f(x) = y.**

**(OR)**

**Let f be a continous on [a, b] and assume f(a) < f(b) then for every k such that f(a) < k < f(b) there exists c∈[a, b] such that f(c) = k.**

*Sol.* **(Imp.)**

f is continous at a,

for $\varepsilon( = - f(a)) > 0 \ \exists \ d > 0 \ni |x - a| < d \Rightarrow |f(x) - f(a)| < \varepsilon$

Consider

$H = \{x \in [a, b] / f(x) < k\} \neq 0 \Rightarrow c = \sup (H)$

Show that f(c) = k

Suppose $f(c) < k \Leftrightarrow k - f(c) > 0$

We know f is continuous at c so $\forall \varepsilon > k - f(c) > 0 \ \exists \ \delta > 0 \,|\, |f(x) - f(c)| < \varepsilon (= k - f(c))$ when $|x - c| < \delta$.

$\Rightarrow f(x) - f(c) < k - f(c)$

Say $x = c + \delta/2 \Rightarrow f(x) < k \Rightarrow c + \delta/2 \in H$

which is a contradiction the fact c = sup (H) since $\delta > 0$.

Similarly f(c) > k. thus

f(c) = k.

**9. If f is a continuous real valued function on an interval I, then the set f(I) = {f(x) : x ∈ I} is also an interval or a single point.**

*Sol.*

f is a continuous real valued function on I the set,

$J = f(I)$

$y_0, y_1, \in J$ and $y_0 < y < y_1 \Rightarrow y \in J$ ... (1)

If Inf J < Sup J. Then such a set J will be an interval.

We will show inf J<y<sup J $\Rightarrow$ y∈J ... (2)

So, J is an interval with end points inf J and sup J

inf J and sup J may or may not belong to J and they may or may not be finite.

Consider inf J < y < sup J

$\exists\ y_0, y_1$ in J

So that $y_0 < y < y_1$

Thus y∈J by (1).

**10. Let f be a continuous function mapping [0, 1] into [0, 1] in other words, dom(f) = [0, 1] and f(x)∈[0, 1] for all x∈[0, 1] show f has fixed point, i.e., a point $x_0$∈[0, 1] such that $f(x_0) = x_0$, $x_0$ is left fixed by f.**

*Sol.* **(Imp.)**

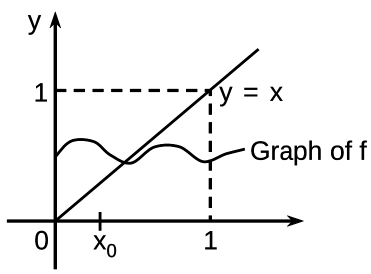
Consider g(x) = f(x) – x

Which is continuous on [0, 1]

Since g(0) = f(0) – 0

= f(0) ≥ 0

g(1) = f(1) – 1 ≤ 1 – 1 = 0




By Intermediate value theorem show $g(x_0) = 0$

for some $x_0 \in [0,1]$

Then obviously we have $f(x_0) = x_0$.

**11. Show that if y > 0 and m ∈ N. Then y has a positive m^th root.**

*Sol.*

The given function is $f(x) = x^m$ is continuous,

$b > 0$ so that $y \le b^m$

Let b = 1 $\Rightarrow$ y ≤ 1

if y > 1 let b = y

Thus f(0) < y ≤ f(b) and the intermediate value theorem $\Rightarrow$ f(x) = y for some x in (0, b),

So, $y = x^m$ and x is an $m^{th}$ root of y.

**12. Let f be a continuous strickly increasing function on some interval I. Then f(I) is an interval J and $f^{-1}$ represents a function with domain J. The function $f^{-1}$ is continous strictly increasing function on J.**

*Sol.*

Let $a < x_1 < x_2 < b$

Then either $f(x_2) > f(x_1)$ or $f(x_2) < f(x_1)$

Suppose that first possibility,

Then we claim f is strickly increasing on (a, b)

Let $a < x_1^1 < x_2^1 < b$ be any other ordered two points in the interval.

Set $x(t) = tx_1^1 + (1 - t)\, x_1$, y(t)

$= tx_2^1 + (1 + t)\, x_2$

Then a < x(t) < y(t) < b for 0 ≤ t < 1

Set g(t) = f(y(t) – f(x(t))

Then g is the composition of continuous function so is continous on [0, 1].

Also g(t) ≠ 0 since f is one to one

So, g(t) cannot change sign by the intermediate value theorem.

Since $g(0) = f(x_2) - f(x_1) > 0$, g(t) > 0

and hence $g(1) = f(x_2^1) - f(x_1^1) > 0$.

**13. Let g be a strictly increasing function on an interval J such that g(J) is an interval I. Then g is continous on J.**

**(or)**

**If f is continous and one to one on an interval then $f^{1}$ is also continous.**

*Sol.*

By previous theorem, f is either strictly increasing or strictly decreasing.

Let $x_0$ be in the interval with $y_0 = f(x_0)$ we must show that $f^{-1}(y) = x_0$.

Let $\varepsilon > 0$ be given

If $x_0 - \varepsilon < x_0 < x_0 + \varepsilon$

Then $f(x_0 - \varepsilon) < f(x_0) < f(x_0 + \varepsilon)$

Choose $\delta = \min (f(x_0) - f(x_0 - \varepsilon), f(x_0 + \varepsilon) f(x_0))$

Then $f(x_0 - \varepsilon) < f(x_0) - \delta$ and $f(x_0) + \delta < f(x_0 + \varepsilon)$

Hence if $f(x_0) \delta < y < f(x_0) + \delta$

then $f(x_0 - \varepsilon) < y < f(x_0 + \varepsilon)$

Since f is strictly increasing.

So, is $f^{-1}$ and therefore $x_0 - \varepsilon < f^{-1}(y) < x_0 + \varepsilon$

i.e, $|f^{-1}(y) - x_0| < \varepsilon$ if $|y - y_0| < \delta$

$\Rightarrow$ $|f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$ if $|y - y_0| < \delta$

$\therefore$ $f^{-1}$ is continous.

**14. Show that if – f assumes its maximum at $x_0 \in [a, b]$. Then f assumes its minimum at $z_0$.**

*Sol.*

Suppose if –f assumes its maximum at $x_0$

i.e., $\forall x \in [a, b]$

We have $-f(x) \leq -f(x_0)$

Thus $f(x) \geq f(x_0) \; \forall \; x \in [a, b]$

Which means exactly f assumes its minimum at $x_0$

**15. Prove that x = cos(x) for some x in $\left(0, \frac{\pi}{2}\right)$.**

*Sol.* **(Imp.)**

Consider the function f(x) = cos(x) – x, which is a continuous function.

Since both cos(x) and x are continous

$$\text{If } x = 0$$

$$f(0) = \cos 0 - 0$$

$$f(0) = 1$$

If $x = \frac{\pi}{2}$

$$f(\pi/2) = \cos\frac{\pi}{2} - \frac{\pi}{2}$$

$$= 0 - \frac{\pi}{2}$$

$$f(\pi/2) = \frac{\pi}{2}$$

Thus, by the intermediate value theorem, we have that there is come $c \in \left(0, \pi/2\right)$ such that $f(c) = 0$.

This means exactly that cos (x) = x has a solution in this interval.

**16. Let $S \subseteq R$ and suppose there exists a sequence $\{x_n\}$ in S converying to a number $x_0 \notin S$ show there exists an unbounded continuous function on S.**

*Sol.* **(Imp.)**

Let $f : S \to R$ be given by $f(x) = \frac{1}{x - x_0}$ $x_0 \notin S$

f is bounded.

Let M > 0 be given

Choosing $\varepsilon = \frac{1}{M}$

Since $x_n \to x_0$ there exists n for which

$$|x_n - x_0| < \varepsilon = \frac{1}{M} \Rightarrow \frac{1}{|x_n - x_0|} > \frac{1}{\varepsilon} \ (= M)$$

$$\text{So, then } |f(x_n)| = \frac{1}{|x_n - x_0|} = \frac{1}{\varepsilon}$$

$$> M$$

$$\therefore |f(x_n)| > M$$



**17. Let f and g be continuous function, on [a, b] such that f(a) $\geq$ g(a) and f(b) $\leq$ g(b) prove that $f(x_0) = g(x_0)$ for at lest one $x_0$ in [a, b].**

*Sol.* **(Imp.)**

Given that f and g are continuous function on [a, b]

let h = f – g So, h is also continuous function.

We have $h(a) \geq 0$ and $h(b) \leq 0$

$f(a) - g(a) \geq 0$ $\quad$ $f(b) - g(b) \leq 0$

$f(a) \geq g(a)$ $\quad$ $f(b) \leq (b)$

By intermediate value theorem, there exists $x_0 \in [a, b]$ for which $h(x_0) = 0$

i.e., $f(x_0) - g(x_0) = 0$

$\therefore \quad f(x_0) = g(x_0)$.

**18. Prove that a polynomial function f of odd degree has at least one real root.**

*Sol.*

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + .... + a_1 x + a_0$

Where $a_n \neq 0$ and n is odd.

Multiplying f by a non - zero constant does not change its roots.

So, without loss of generality $a_n = 1$

Consider $\frac{f(x)}{x^n} = a_n + \frac{a_{n-1}}{x} + .... + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}$

$\frac{f(x)}{x^n} = 1 + \frac{a_{n-1}}{x} + .... + \frac{a_0}{x^n}$

for each $1 \leq k \leq n$

there exists $R_k > 0$ for which $|x| > R_k$

$\Rightarrow \left|\frac{a_{n-k}}{x^k}\right| < \frac{1}{2n}$

Taking $R = \max \{R_1, ...... R_n\}$

The triangle inequality gives that

$$\left|\frac{a_{n-1}}{x} + ....... + \frac{a_n}{x^2}\right| < \frac{1}{2} \text{ for } |x| > R$$

So, $\frac{f(x)}{x^n} > 1 - \frac{1}{2} > 0$

In particular f(x) has the same sign as $x^n$ for x sufficiently large in magnitude so, when x is large and positive.

f(x) is positive, and when x is large and negative,

f(x) is negative,

Since polynomial functions are continuous

By intermediate value theorem

$f(x) = 0.$

**19. Let $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and let f(0) = 0 show that f has the intermediate value property on R.**

*Sol.*

Let a < b be given

If $a \leq 0 < b$ or $a < 0 \leq b$

Then $\sin\left(\frac{1}{x}\right)$ attains all values between - 1 and 1.

any y between f(a) and f(b) is attained between a and b.

If $0 < a < b$ or $a < b < 0$

Then since $\sin\left(\frac{1}{x}\right)$ is itself continuous on the domains $\{x > 0\}$ and $x < 0$.

f has the intermediate value property.

## 2.3 Uniform Continuity

Let f be a real valued function defined on a set $S \subseteq R$. Then f is uniformly continous on S. if for each $\varepsilon > 0 \exists\ \delta > 0 \ni x, y \in s$ and

$$|x - y| < \delta \Rightarrow |f(x) - f(y) < \varepsilon|$$

f is uniformly continuous if f is uniformly continous on dom(f).

**20. Verify f is continous on set $S \subseteq$ dom(f) if an only if for each $x_0 \in s$ and $\varepsilon > 0$ there is $\delta > 0$ so that $x \in$ dom(f) and $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$ for the function $f(x) = \frac{1}{x^2}$ on $(0, \infty)$.**

*Sol.*

Given that,

$$f(x) = \frac{1}{x^2} \text{ on } (0, \infty)$$

Let $x_0 > 0$ and $\varepsilon > 0$

We have to show $|f(x) - f(x_0)| < \varepsilon$ for $|x - x_0| < \delta$

Consider

$$f(x) - f(x_0) = \frac{1}{x^2} - \frac{1}{x_0^2}$$

$$= \frac{x_0^2 - x^2}{x^2 x_0^2}$$

$$f(x) - f(x_0) = \frac{(x_0 - x)(x_0 + x)}{x^2\, x_0^2}$$

$$\text{Choose } \delta = \frac{x_0}{2}$$

$$\Rightarrow \quad |x - x_0| < \frac{x_0}{2} \text{ then we have } |x| > \frac{x_0}{2}$$

$$|x| < \frac{3x_0}{2} \text{ and } |x_0 + x| < \frac{5x_0}{2}$$

$$|f(x) - f(x_0)| < \frac{|x_0 - x|\frac{5x_0}{2}}{\left(\frac{x_0^2}{2}\right)^2 \left(x_0^2\right)} = \frac{10|x_0 - x|}{x_0^3}$$

Thus if we set $\delta = \min\left\{\frac{x_0}{2}, \frac{x_0^3\varepsilon}{10}\right\}$

$\Rightarrow |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

---

**21. Show that f(x) = $\frac{1}{x^2}$ is uniformly continous on [0, $\infty$) where a > 0.**

*Sol.* **(Imp.)**

Given that f(x) $= \frac{1}{x^2}$ on [0, $\infty$) where a > 0

let $\varepsilon > 0$

We have to show that

$\exists\delta > 0 \ni |x - y| < d \Rightarrow |f(x) - f(y)| < \varepsilon \,\forall\, x, y \geq a$ ... (1)

Consider $f(x) - f(y) = \frac{1}{x^2} - \frac{1}{y^2}$

$$= \frac{y^2 - x^2}{x^2y^2}$$

$$= \frac{(y-x)(y+x)}{x^2y^2}$$

If we can show $\frac{y+x}{x^2y^2}$ is bounded on [a, $\infty$) by a constant M, then we will take $\delta = \frac{\varepsilon}{M}$.

$$\frac{y+x}{x^2y^2} = \frac{y}{x^2y^2} + \frac{x}{x^2y^2}$$

$$x = \frac{1}{x^2y} + \frac{1}{xy^2}$$

$$\leq \frac{1}{a^2.a} + \frac{1}{a.a^2} \qquad x, y \geq a.$$

$$\leq \frac{1}{a^3} + \frac{1}{a^3}$$

$$\frac{y+x}{x^2y^2} \leq \frac{2}{a^3}(= M)$$

$\therefore \quad \delta = \varepsilon \dfrac{a^3}{2}$

$x \geq a,\ y \geq a$ and $|x - y| < \delta \Rightarrow |f(x) - f(y)| = \dfrac{|y \cdot x||(y+x)|}{x^2y^2} < \delta\left(\dfrac{1}{x^2y} + \dfrac{1}{xy^2}\right) \leq \dfrac{2\delta}{a^3} = \varepsilon$

$\therefore\ |f(x) - f(y)| < \varepsilon\ \forall x, y \geq a$

$\therefore$ f is uniformly continous on $[a, \infty)$.

**22. The function f(x) = $\dfrac{1}{x^2}$ is not uniformly continuous on the set $(0, \infty)$ or even on the set (0, 1).**

*Sol.*

Given function is $f(x) = \dfrac{1}{x^2}$

We will show that f is not uniformly continuous let $\varepsilon = 1$.

i.e., for each $\delta > 0\ \exists$ x, y in (0, 1) such that $|x - y| < \delta$ and yet $|f(x) - f(y)| \geq 1$ ... (1)

To show that (1) it suffices to take $y = x + \dfrac{\delta}{2}$

$$\Rightarrow |f(x) - f\left(x + \frac{\delta}{2}\right)| \geq 1 \qquad ... (2)$$

Consider $f(x) - f\left(x + \dfrac{\delta}{2}\right) \geq 1$

$$\Rightarrow \quad 1 \leq \frac{1}{x^2} - \frac{1}{(x+\delta/2)^2}$$

$$\Rightarrow \quad 1 \leq \frac{\left(x + \delta/2\right)^2 - x^2}{x^2\left(x + \delta/2\right)^2}$$

$$\Rightarrow \quad 1 \leq \frac{\left(\left(x + \frac{\delta}{2}\right) + x\right)\left(\left(x + \frac{\delta}{2}\right) - x\right)}{x^2\left(x + \frac{\delta}{2}\right)^2}$$

$$\Rightarrow \quad 1 \leq \frac{\left(2x + \frac{\delta}{2}\right)\left(\frac{\delta}{2}\right)}{x^2\left(x + \frac{\delta}{2}\right)^2}$$

$$\Rightarrow \quad 1 \leq \frac{\delta\left(2x+\frac{\delta}{2}\right)}{2x^2\left(x+\frac{\delta}{2}\right)^2} \qquad \ldots (3)$$

It is sufficient to prove (1) for $\delta < \frac{1}{2}$

Let $x = \delta$

$$\text{by (3)} \Rightarrow \frac{\delta(2.\delta+\frac{\delta}{2})}{2\delta^2\left(\delta+\frac{\delta}{2}\right)^2}$$

$$= \frac{\frac{5\delta^2}{2}}{2\delta^2\left(\frac{3\delta}{2}\right)^2} = \frac{5\delta^2\not{2}/2}{\frac{9\delta^4}{2}} = \frac{5\delta^2}{9\delta^4} > \frac{5}{9\left(\frac{1}{2}\right)^2} = \frac{20}{9} > 1$$

i.e., If $0 < \delta < \frac{1}{2}$ then $|f(d) - f\left(\delta+\frac{\delta}{2}\right)| > |$

So, (1) hold with $x = \delta$ and $y = \delta + \frac{\delta}{2}$

**23. Is the function f(x) = $x^2$ Uniformly continuous on [–7, 7]?**

*Sol.*

Given that $f(x) = x^2$

To check the given function is uniformly continous on [–7, 7].

i.e., to check by definition,

for each $\varepsilon > 0 \; \exists \, \delta > 0 \ni |x - y| < d \Rightarrow |f(x) - f(x)| < \varepsilon$

Consider $|f(x) - f(y)| = |x^2 - y^2| = |x - y| \, |x + y|$

Since $|x + y| \leq |7 + 7|$ for x, y in [–7, 7]

$|x + y| \leq 14$

$\therefore |f(x) - f(y)| \leq 14|x - y|$ for x, y in [–7, 7]

Choose $\delta = \frac{\varepsilon}{14}$

$\Rightarrow |f(x) - f(y)| < 14 \frac{\varepsilon}{14}$

$|f(x) - f(y)| < \varepsilon$ x, y $\in$[–7, 7]

$\therefore |x - y| < \delta \Rightarrow |d(x) - f(y) < e$

$\therefore$ f is uniformly continous on [–7, 7].

**24. If f is continuous on a closed interval [a, b] then f is uniformly continous on [a, b].**

*Sol.*

f is continuous on a closed interval [a, b] we have to prove, f is uniformly continous on [a, b] i.e., to prove.

for any $\varepsilon > 0 \; \exists \; \delta > 0 \ni |f(x_1) - f(x_2)| < \varepsilon$ for any arbitrary points $x_1, x_2$ of [a, b] such that $|x_1, x_2| < \delta$

Let $\varepsilon > 0$,

$\because$ f is continous on [a, b]

$\Rightarrow$ for $\varepsilon > 0$, we can divide [a, b] into a finite number (say n) of sub intervals.

i.e., $a = t_0 < t_1 < ..... < t_n = b$

$a = [t_0, t_1], [t_1, t_2] .... [t_{r-1}, t_r] [t_r, t_{r+1}] .... [t_{n-1}, t_n] = b$

Such that $|f(x_1) - f(x_2)| < \frac{\varepsilon}{2}$ for $x_1, x_2$ belonging to the same sub interval,

Let $\delta = \frac{1}{2} \min \{|t_r - t_{r-1}| > 0, 1 \leq r \leq n\}$

Let $x_1, x_2$ be any two points of [a, b] such that $|x_1 - x_2| < \delta$.

Then $x_1, x_2$ either belong to the same sub interval or to two consecutive sub intervals with a common end point.

Case (1) let $x_1, x_2$ belong to the same subinterval

We have $|f(x_1) - f(x_2)| < \frac{\varepsilon}{2} < \varepsilon$ for $|x_1 - x_2| < \delta$.

Case (2) let $x_1, x_2$ belong to two consecutive sub interval with a common end point.

Say $t_r$.

We have $|f(x_1) - f(t_r)| < \frac{\varepsilon}{2}$ and $|f(t_r) - f(x_2)| < \frac{\varepsilon}{2}$

$$\begin{aligned} \therefore |f(x_1) - f(x_2)| &= |(f(x_1) - f(t_r) + f(t_r) - f(x_2)| \\ &= |(f(x_1) - f(t_r)) + (f(t_r) - f(x_2))| \\ &= |f(x_1) - f(t_r)| + |f(t_r) - f(x_2)| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \text{ for } |x_1 - x_2| < \delta \end{aligned}$$

$\therefore$ Thus in either case,

We have for any e > 0 there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ for any arbitrary points $x_1, x_2$ of [a, b] such that $|x_1 - x_2| < \delta$.

$\therefore$ f is Uniformly continous in [a, b]

**25. If $f : s \to R$ is uniformly continuous, then f is continuous, in S.**

*Sol.* **(Imp.)**

Suppose that f is uniformly continous on s.

$\Rightarrow$ for $\varepsilon > 0 \exists \delta > 0 \ni |f(x_1) - f(x_2)| < \varepsilon$ for $x_1$, $x_2$ being any pair of arbitrary point of such that $|x_1 - x_2| < \delta$.

Let $C \in S$

On taking $x_1 = x$ and $x_2 = C$ we have for $\varepsilon > 0 \exists \delta > 0 \ni |f(x) - f(c)| < \varepsilon$ for $|x - c| < \delta$

$\Rightarrow$ f is continous at any point 'C' of S, Since C is arbitrary:

f is continous at every point of S,

$\therefore$ f is continuous in S.

**26. Prove that $f : R \to R$ given by $f(x) = x^2$ is a continous function on R but not Uniformly continous on R.**

*Sol.* **(Imp.)**

Clearly f is continuous on R,

Now we show that f is not uniformly continous on R

Given $\varepsilon > 0$,

We prove that there is no single $\delta$ that serves for every $x \in R$ in the condition of continuity.

To see this let us assume that there exists such a number $\delta > 0$.

Then for $x_1$ and $x_2 = x_1 + \frac{\delta}{2}$

$$|x_1 - x_2| = \frac{\delta}{2} < \delta$$

$$\therefore |f(x_1) - f(x_2)| = \left|x_1^2 - x_2^2\right| = |x_1 - x_2|\ |x_1 + x_2|$$

$$= \frac{\delta}{2}\ x_1 + x_1 + \frac{\delta}{2}$$

$$= \frac{\delta}{2}\left|2x_1 + \frac{\delta}{2}\right|$$

$$= x_1\delta + \frac{\delta^2}{4}$$

$$= < \varepsilon \text{ if } x_1 > 0$$

Since $\frac{\delta^2}{4} > 0$

we must have $x_1\delta < \varepsilon \forall x_1 \in R,\ x_1 > 0$

But this is impossible.

$\therefore$ $\delta$ depends on $\varepsilon$ and $x_1$ and hence the function f is not uniformly continous on R.

**27. A real valued function f on (a, b) is uniformly continous on (a, b) if and only if it can be extended to a continous function on [a, b].**

*Sol.* **(Imp.)**

Suppose that f is uniformly continous on (a, b)

we have to prove f is continous function on [a, b].

$\because$ f is uniformly continous on (a, b)

for $\varepsilon > 0 \exists \delta > 0 \ni |f(x_1) - f(x_2)| < \varepsilon$ for $x_1$, $x_2$ being any pair of arbitrary points of S such that $|x_1 - x_2| < \delta$.

Let $C \in S$

on taking $x_1 = x$ and $x_2 = C$

We have

for $\varepsilon > 0\ \exists \delta > 0 \ni |f(x) - f(c)| < \varepsilon$ for $|x - c| < \delta$

$\Rightarrow$ f is continous at any point 'c' of s

Since C is any arbitrary

f is continous at every point of S.

$\therefore$ f is continous in [a, b]

Conversely suppose that

f is continous in [a, b] then prove that f is uniformly continous.

$\because$ f is continous on [a, b]

We have to prove that

f is uniformly continuous

i.e., to prove that

for any $\varepsilon > 0 \exists \delta > 0 \ni |f(x_1) - f(x_2)| < \varepsilon$ for any arbitrary point $x_1$, $x_2$ of $[a, b] \ni |x_1 - x_2| < \delta$

Let $\varepsilon > 0$

f is continuous on [a, b]

$\Rightarrow$ for $\varepsilon > 0$, we can divide [a, b] into finite sub intervals (say n)

$a = [t_0, t_1], [t_1, t_2] \ldots [t_{r-1}, t_r], [t_r, t_{r+1}] \ldots [t_{n-1}, t_n] = b$

Such that $|f(x_1) - f(x_2)| < \frac{\varepsilon}{2}$ for $x_1$, $x_2$ belonging to the same sub interval.

Let $\delta = \frac{1}{2} \min \{|t_r - t_{r-1}| > 0, 0 \leq r \leq n\}$

Let $x_1$, $x_2$ be any two points of [a, b] such that $|x_1 - x_2| < \delta$.

Then $x_1$, $x_2$ either belong to the same sub interval or to two consecutive sub interval with a common end point.

**Case (i)**

Let $x_1$, $x_2$ belong to the same sub-interval we have, $|f(x_1) - f(x_2)| < \frac{\varepsilon}{2} < \varepsilon$ for $|x_1 - x_2| < \delta$.

**Case (ii)**

Let $x_1$, $x_2$ belong to two consecutive sub Intervals with a common end point say $t_r$, we have,

$$|f(x_1) - f(t_r)| < \frac{\varepsilon}{2} \text{ and } |f(t_r) - f(x_2)| < \frac{\varepsilon}{2}$$

$$\therefore |f(x_1) - f(x_2)| = |f(x_1) - f(t_r)| + |f(t_r) - f(x_2)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$< \varepsilon \text{ for } |x_1 - x_2| < \delta$$

$\therefore$ Thus in either case, we have for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$|f(x_1) - f(x_2)| < \varepsilon$ for any arbitrary points $x_1$, $x_2$ of [a, b] such that $|x_1 - x_2| < \delta$

$\therefore$ f is uniformly continous.

**28. Show that the function f defined by $f(x) = x^3$ is uniformly continous in [–2, 2].**

*Sol.*

Given that $f(x) = x^3$

Let $x_1, x_2 \in [-2, 2]$ then $|x_1| \le 2$, $|x_2| \le 2$

$$|f(x_2) - f(x_1)| = |x_1^3 - x_2^3|$$

$$= |(x_2 - x_1)(x_1^2 + x_2^2 + x_1 x_2)|$$

$$= |(x_1 - x_1)| \, [|x_1|^2 + |x_2|^2 + |x_1||x_2|]$$

$$= |x_2 - x_1| \, |2^2 + 2^2 + 2, 2|$$

$$= 12|x_2 - x_1|$$

$\therefore$ $|f(x_2) - f(x_1)| < \varepsilon$ whenever $|x_2 - x_1| < \dfrac{\varepsilon}{12}$

$\therefore$ Given $\varepsilon > 0 \, \exists \, \delta = \dfrac{\varepsilon}{12}$ such that $|f(x_2) - f(x_1)|$ whenever $|x_1 - x_1| < \delta$ for every $x_1, x_2 \in [-2, 2]$

$\therefore$ f(x) is uniformly continous in [–2, 2].

**29. If f is uniformly continous on an aggregate s and $\{s_n\}$ is a Cauchy sequence in s, then prove that $\{f(s_n)\}$ is also Cauchy sequence.**

*Sol.*

f is uniformly continuous on s

$\Rightarrow$ given $\varepsilon > 0 \; \exists \; \delta > 0$ such that $x_1, x_2 \in S$,

$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$ ... (1)

$\{s_n\}$ is a Cauchy sequence

$\Rightarrow$ for $\delta > 0$ there exists positive integer 'm' such that $|s_p - s_q| < \delta \ \forall \ p, q \geq m$

But $s_p, s_q \ \varepsilon \{s_n\} \Rightarrow s_p . s_q \in s$

By (1), for each $\varepsilon > 0$ there exists a positive integer 'm' such that $|f(s_p) - f(s_q)| < \varepsilon \forall p, q \geq m$.

$\therefore$ $\{f(s_n)\}$ is also a Cauchy sequences.

**30. Show $f(x) = \frac{1}{x^2}$ is not uniformly continous on (0, 1).**

*Sol.*

Let $s_n = \frac{1}{n}$ for $n \in N$

$$\lim_{n\to\infty} s_n = -\lim_{n\to\infty} \frac{1}{n} = 0$$

which is convergent and we know that every convergent sequence are Cauchy sequence.

$\therefore$ $\{s_n\}$ is a Cauchy sequence

Since $f(s_n) = n_2$

$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} n^2$ which is not a convergent

$\therefore$ $f(s_n)$ is not a cauchy sequence.

$\therefore$ f cannot be uniformly continous on (0, 1).

$\therefore$ f(x) is not a uniformly continous.

**31. Let f be a continuous function on an interval I [I may be bounded or unbounded] Let I° be the interval obtained by removing from I any end points that happen to be in I. If f is differentiable on I° and if f¹ is bounded on I°, then f is uniformly continous on I.**

*Sol.*

Given that f is continous function on I, here 'I' may be bounded or unbounded

Let M be bounded for f' on I

suppose that $|f'(x)| \leq M$ on I

Let $\varepsilon > 0$ be given and set $\delta = \frac{\varepsilon}{M}$

We show that this $\varepsilon - \delta$ pair satisfy the definition of Uniform continuity.

Let $x, y \in I$ such that $|x - y| < \delta\left(= \frac{\varepsilon}{M}\right)$

by mean value theorem

There exists $C \in (x, y)$ such that $f'(c) = \frac{f(x) - f(y)}{x - y}$

But then

$$|f(x) - f(y)| = |f'(c)|\ |x - y|$$
$$< M\delta$$
$$< M\frac{\varepsilon}{M}$$
$$< \varepsilon$$
$$|f(x) - f(y)| < \varepsilon$$

f is uniformly continous on I.

**32. Show $f(x) = \frac{1}{x^2}$ is uniformly continous on $[0, \infty)$.**

*Sol.*

Let $a > 0$,

Consider $f(x) = \frac{1}{x^2}$

Since $f'(x) = \frac{-2}{x^3}$

$$|f'(x)| = \frac{2}{a^3} \text{ on } [a, \infty)$$

We have to show that –f is uniformly continous there exists $\delta > 0 \ni |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

$$\forall x, y \geq a$$

Consider $f(x) - f(y) = \frac{1}{x^2} - \frac{1}{y^2} = \frac{y^2 - x^2}{x^2y^2}$

$$= \frac{(y - x)(y - x)}{x^2y^2}$$

If we can show $\frac{x + y}{x^2y^2}$ is bounded on $[a, \infty)$ by constant M, then we will take $\delta = \frac{\varepsilon}{M}$.

$$\frac{y + x}{x^2y^2} = \frac{1}{x^2y} + \frac{1}{xy^2}$$

$$\leq \frac{1}{a^3} + \frac{1}{a^3}$$

$$\leq \frac{2}{a^3} (= M)$$

$$\therefore \delta = \varepsilon.\frac{a^3}{2}$$

$x \geq a$, $y \geq a$ and $|x - y| < \delta \Rightarrow |f(x) - f(y)|$

$$= \frac{|y - x||y + x|}{x^2\ y^2}$$

$$< \delta\frac{2}{a^3}$$

$$= \varepsilon.$$

$\therefore$ $|f(x) - f(y)| < \varepsilon\ \forall\ x, y \geq a$

f is Uniformly continous on $[a, \infty)$.

**33. Prove $f(x) = 3x + 11$ on R is uniformly continous.**

*Sol.*

Given $f(x) = 3x + 11$ on R,

$\varepsilon > 0$, Let $\delta = \frac{\varepsilon}{3}$

then $|x - y| < \delta \left(= \frac{\varepsilon}{3}\right) \Rightarrow |f(x) - f(y)| < \varepsilon$

Consider,

$$|f(x) - f(y)| = |3x + 11 - (3y + 11)|$$
$$= |3x + 11 - 3y - 11|$$
$$= |3x - 3y|$$
$$= 3|xy|$$
$$< 3 . \frac{\varepsilon}{3}$$
$$= \varepsilon$$

$\therefore$ $|f(x) - f(y)| < \varepsilon$

$\therefore$ f is uniformly continous.

**34. Prove $f(x) = x^2$ on [0, 3] is uniformly continuous.**

*Sol.*

Given that $f(x) = x^2$ on [0, 3]

To prove f is uniformly continous

i.e., to prove $\varepsilon > 0 \exists \delta > 0 \ni |x - y| < d \Rightarrow |f(x) - f(y)| < \varepsilon$

$\varepsilon > 0$, Let $\delta\left(= \frac{\varepsilon}{6}\right) > 0$, then $|x - y| < \delta\left(= \frac{\varepsilon}{6}\right)$

Consider $|f(x) - f(y)| = |x^2 - y^2|$

$= |(x - y)(x + y)|$

$= |x - y||x + y|$

$< \frac{\varepsilon}{6}|3 + 3| \qquad \forall x \geq 3, y \geq 3$

$< \frac{\varepsilon}{6} 6 = \varepsilon$

$\therefore \quad |f(x) - f(y)| < \varepsilon$ on [0, 3]

$\therefore \quad$ f is uniformly continous on [0, 3].

**35. Prove $f(x) = \frac{1}{x}$ on $\left[\frac{1}{2}, \infty\right)$ is uniformly continuous.**

*Sol.*

Given that $f(x) = \frac{1}{x}$

Prove that f(x) is uniformly continous

i.e, to prove.

$\varepsilon > 0 \exists \delta > 0 \ni |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Let $\varepsilon > 0$, $\delta\left(= \frac{\varepsilon}{4}\right) > 0$

Then $|x - y| < \delta\left(= \frac{\varepsilon}{4}\right)$

Consider $|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right|$

$= \left|\frac{y - x}{xy}\right|$

$\leq \left|\frac{y - x}{xy}\right|$

$\leq \frac{|x - y|}{\frac{1}{2} \cdot \frac{1}{2}}$

$< \frac{\frac{\varepsilon}{4}}{\frac{1}{4}} = \varepsilon$

$\therefore \quad |f(x) - f(y)| < \varepsilon \quad$ on $\left[\frac{1}{2}, \infty\right)$

$\therefore \quad$ f is uniformly continous.

**36. Check $f(x) = \frac{1}{x^3}$ on (0, 1] is uniformly continous or not?**

*Sol.*

Let $s_n = \frac{1}{n}$

Since $s_n$ is convergent

[i.e., $\lim s_n = \text{lin} \frac{1}{n} = 0$ which is convergent]

and we know that every convergent sequence are Cauchy sequence.

$\therefore \{s_n\}$ is a cauchy sequence.

But $f(s_n) = n^3$ and $n^3$ is not cauchy sequence since it is diverges to $+\infty$.

$\therefore$ f cannot be uniformly continous on (0, 1].

**37. Show that $f(x) = x^3$ on [0, 1] is uniformly continous.**

*Sol.*

Given that $f(x) = x^3$

To show that f(x) is uniformly continuous.

i.e., show that for each $\varepsilon > 0 \exists \delta > 0 \ni |x - y| < \delta \Rightarrow \left(\frac{\varepsilon}{3}\right) \Rightarrow$

$|f(x) - f(y)| < \varepsilon$

Consider $|f(x) - f(y)| = |x^3 - y^3|$

$= |(x - y)(x^2 + y^2 + xy)|$

$= |x - y|\,|x^2 + y^2 + xy|$

$= |x - y|\,||x|^2 + |y|^2 + |xy||$

$= \delta\,|1 + 1 + 1| = \delta 3$

$= \frac{\varepsilon}{3} 3$

$|f(x) - f(y)| < \varepsilon$.

$\therefore$ f is uniformly continuous on [0, 1]

**38. Which of the following continous functions are uniformly continuous on the specified set? Justify your answer.**

**(a) $f(x) = x^3$ on R**

**(b) $f(x) = x^3$ on (0, 1)**

*Sol.*

**(a) Given that $f(x) = x^3$ on R**

**Claim**

f is not uniformly continous on R.

In particular, for $\varepsilon = 1$ any $\delta > 0 \exists, x, y \in R$

Such that $|x - y| < d$ and $|x^3 - y^3| \geq 1$

To find x and y,

Let's first simplify things by looking for positive x's.

and letting $y = x + \frac{\delta}{2}$

Then $|x^3 - y^3| = |x^3 - \left(x + \frac{\delta}{2}\right)^3|$

$$= x^3 - x^3 + \frac{3}{2}x\delta^2 + \frac{\delta^3}{8} + \frac{3}{4}x^2\delta$$

$$= \frac{3}{2}x^2\delta + \frac{3}{4}x\delta^2 + \frac{\delta^3}{8}$$

$$> \frac{3}{2} - x^2\delta$$

This is equal to 1 if $x = \sqrt{\frac{2}{3\delta}}$ formally, for any $\delta > 0$, let $x = \sqrt{\frac{2}{3\delta}}$ and let $y = x + \frac{\delta}{2}$.

Then $|x - y| = \frac{\delta}{2} < \delta$ and $|x^3 - y^3| > \frac{3\delta}{2}$ $x^2 = |(=\varepsilon)$

So, f is not uniformly continous.

## 2.4 Limits of Functions

Definition : Let s be a subset of R, let 'a' be a real number or symbol '$\infty$' or '$-\infty$' i.e., the limit of some sequence in s, and let L be a real number, we write $\lim_{x \to a^s} f(n) = L$ if f is a function defined on s and for every sequence $\{x_n\}$ in s with limit a, we have $\lim_{x \to a^s} f(x_n) = L$.

The expression $\lim_{x \to a^s} f(x)$ is read "limit, as x tends to a along s, of f(x).

Various standard limit concepts for functions.

1. For $a \in R$ and a function f we write $\lim_{x \to a^+} f(x) = L$ provided $\lim_{x \to a^s} f(x) = L$ for some open interval s = (a, b) $\lim_{x \to a^+} f(x)$ is the right hand limit of – f at a.
2. For $a \in R$ and a function f we write $\lim_{x \to a^+} f(x) = L$ provided $\lim_{x \to a^s} f(x) = L$ for some open interval s = (c, a) $\lim_{x \to a^-} f(x)$ is called left hand limit of f at a.
3. For a function f, we write $\lim_{x \to \infty} f(x) = L$ provided $\lim_{x \to \infty^s} f(x) = L$ for some interval s = (c, $\infty$) like wise we write $\lim_{x \to -\infty} f(x) = L$ provided $\lim_{x \to -\infty^s} f(x) = L$ for some interval s = ($-\infty$, b).

**39. Find**

**(a) $\lim_{x \to 4} x^3$ (b) $\lim_{x \to 2} \frac{1}{x}$**

*Sol.*

**(a) $\lim_{x \to 4} x^3$**

Given that

$f(x) = x^3 \Rightarrow \lim_{x \to 4} x^3 = 4^3$

$\therefore \lim_{x \to 4} x^3 = 64.$

**(b)** $\lim_{x\to 2}\frac{1}{x}$

Given that

$$f(x) = \frac{1}{x}$$

$$\lim_{x\to 2} f(x) = \frac{1}{2}$$

$$\therefore \quad \lim_{x\to 2}\frac{1}{x} = \frac{1}{2}$$

**40. Final $\lim_{x\to 2}\frac{x^2-4}{x-2}$.**

*Sol.*

Given that $\lim_{x\to 2}\frac{x^2-4}{x-2}$

$$f(x) = \frac{x^2-4}{x-2}$$

Rewrite the function as

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2}$$

$$= x + 2 - \text{for } x \neq 2.$$

None it is clear that $\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2} x + 2$

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = 4$$

**41. Find $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$.**

*Sol.*

Given that $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$

$$f(x) = \frac{\sqrt{x}-1}{x-1}$$

We multiply numerator and denominator by $\sqrt{x}$ + 1, then we obtain.

$$\frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{x-1}\times\frac{\sqrt{x}+1}{x+1} = \frac{(\sqrt{x})^2-1}{(x-1)(\sqrt{x}+1)}$$

$$= \frac{(x-1)}{(x-1)(\sqrt{x}+1)}$$

$$= \frac{1}{\sqrt{x}+1}$$

Now it is clear that,

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1}$$

$$= \frac{1}{\sqrt{1}+1}$$

$$\therefore \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \frac{1}{2}.$$

**42. If $f(x) = \frac{1}{(x-2)^3}$ for $x\neq 2$. Then prove that (i) $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$, (ii) $\lim_{x\to 2^+} f(x) = +\infty$ and $\lim_{x\to 2^+} f(x) = -\infty$.**

*Sol.* **(Imp.)**

To verify $\lim_{x\to\infty} f(x) = 0$

We consider sequence $\{x_n\}$,

Such that $\lim_{n\to\infty} x_n = +\infty$

$$f(x) = \frac{1}{(x-2)^3}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{1}{(x-2)^3} = 0$$

This show's that,

$\lim_{x\to\infty} f(x) = 0$. For (2, $\infty$) Now to show that $\lim_{x\to\infty}(x_n - 2)^{-3} = +\infty$ ... (1)

$$\lim_{x\to\infty}\frac{1}{(\infty+2)^3} = 0$$

Here $\varepsilon > 0$ for large n, we need $|x-2|^{-3} < \varepsilon$ or $\varepsilon^{-1} < |x_n - 2|^3$ if $x_n > \varepsilon^{-1/2} + 2$

$$\lim_{n\to\infty} x_n = +\infty$$

There exists N so that $n > N \Rightarrow x_n > \varepsilon^{-1/3} + 2$

i.e., $n > N \Rightarrow |x_n - 2|^{-3} < \varepsilon$.

$$\therefore \lim_{n\to\infty} f(x_n) = 0$$

**43. Find the limit $\lim_{x\to b}\frac{\sqrt{x}-\sqrt{b}}{x-b}$, b > 0.**

*Sol.* **(Imp.)**

Given that $\lim_{x\to b}\frac{\sqrt{x}-\sqrt{b}}{x-b} \Rightarrow f(x) = \frac{\sqrt{x}-\sqrt{b}}{x-b}$

Multiply and divide by $\sqrt{x}+\sqrt{b}$

we obtain,

$$f(x) = \frac{\sqrt{x}-\sqrt{b}}{x-b} \times \frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}+\sqrt{b}} \Rightarrow \frac{\left(\sqrt{x}\right)^2 - \left(\sqrt{b}\right)^2}{(x-b)\left(\sqrt{x}+\sqrt{b}\right)} = \frac{\cancel{(x-b)}}{\cancel{(x-b)}\left(\sqrt{x}+\sqrt{b}\right)} = \frac{1}{\sqrt{x}+\sqrt{b}}$$

$$\therefore \lim_{x\to b} f(x) = \lim_{x\to b}\frac{1}{\sqrt{x}+\sqrt{b}}$$

$$= \frac{1}{2\sqrt{b}}$$

Let $g(x) = \frac{\sqrt{x}-\sqrt{b}}{x-b}$

Note that $(0, \infty)$ is an open interval containing b and $(0, \infty) - \{b\} \subset$ dom (g)

If $\{x_n\}$ is a sequence in $(0, \infty) - \{b\}$

and $\lim x_n = b$. Then $g(x_n) = \frac{\sqrt{x_n}-\sqrt{b}}{x_n - b} = \frac{1}{\sqrt{x_n}+\sqrt{b}}\ \forall\ n \in N$

(Since $x_n \neq b$, for any n)

Since $\lim x_n = b \Rightarrow \lim \sqrt{x_n} = \sqrt{b}$

Since lim $\sqrt{b} = \sqrt{b}$

and lim $\sqrt{x_n} = \sqrt{b}$ .

$\Rightarrow$ lim $\sqrt{x_n} + \sqrt{b} = 2\sqrt{b}$

The reciprocal limit law then implies that

$$g(x_n) = \lim \frac{1}{\sqrt{x_n} + \sqrt{b}} = \frac{1}{2\sqrt{b}}$$

we have shown that whenever $\{x_n\}$ is sequence in $(0, \infty) - \{b\}$ such that lim $x_n$ = b then

$$\lim \frac{\sqrt{x_n} - \sqrt{b}}{x_n - b} = \frac{1}{2\sqrt{b}}$$

So, $\lim\limits_{x \to b} \frac{\sqrt{x} - \sqrt{b}}{x - b} = \frac{1}{2\sqrt{b}}$

**44. Prove that if $\lim\limits_{x \to a} f(x) = 3$ and $\lim\limits_{x \to a} g(x) = 2$**

**Then (a) $\lim\limits_{x \to a} [3f(x) + g(x)^2] = 13$**

**(b) $\lim\limits_{x \to a} \frac{1}{g(x)} = \frac{1}{2}$**

**(c) $\lim\limits_{x \to a} \sqrt{3f(x) + 8g(x)} = 5$**

*Sol :* **(Imp.)**



Given that $\lim\limits_{x \to a} f(x) = 3$

and $\lim\limits_{x \to a} g(x) = 2$ ... (1)

(a) To prove that $\lim\limits_{x \to a} [3 f(x) + g(x)^2] = 13$

Consider R.H.S i.e., $\lim\limits_{x \to a} [3 f(x) + g(x)^2]$

$\Rightarrow \lim\limits_{x \to a} [3f(x)] + \lim\limits_{x \to a} g(x)^2$

$\Rightarrow 3 \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)^2$

$= 3(3) + (2)^2 \Rightarrow 13$ (by (1) & (2)

$\therefore \lim\limits_{x \to a} [3f(x) + g(x)^2] = 3$

(b) To prove that $\lim\limits_{x\to a} \frac{1}{g(x)} = \frac{1}{2}$

consider $\lim\limits_{x\to a} \frac{1}{g(x)} = \frac{1}{\lim\limits_{x\to a} g(x)}$

$= \frac{1}{2}$ by (2)

$\therefore \lim\limits_{x\to a} \frac{1}{g(x)} = \frac{1}{2}$

(c) To prove that $\lim\limits_{x\to a} \sqrt{3f(x)+8g(x)} = 5$

Consider

$\lim\limits_{x\to a} \sqrt{3f(x)+8g(x)} = \sqrt{3\lim\limits_{x\to a} f(x)+8\lim\limits_{x\to a} g(x)}$

$= \sqrt{3(3)+8(2)}$

$= \sqrt{9+16}$

$= \sqrt{25}$

$\therefore \lim\limits_{x\to a} \sqrt{3f(x)+8g(x)} = 5$



**45. Let $f_1$ and $f_2$ be function for which the limits $L_1 = \lim\limits_{x\to a^s} f_1(x)$ and $L_2 = \lim\limits_{x\to a^s} f_2(x)$ exist and are finite. Then**

**(i) $\lim\limits_{x\to a^s} (f_1+f_2)(x)$ exists and equals $L_1 + L_2$**

**(ii) $\lim\limits_{x\to a^s} (f_1 f_2)(x)$ exits and equals $L_1 L_2$**

**(iii) $\lim\limits_{x\to a^s} (f_1 / f_2)(x)$ exits and equals $L_1/L_2$ provides $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in s$**

*Sol.* **(Imp.)**

(i) Given that $f_1$ and $f_2$ are defined on s.

a is the limit of some sequence in s.

clearly the function $f_1 + f_2$ and $f_1 f_2$ are defined on s and so, is $f_1/f_2$ if $f_2(x) \neq 0$ for $x \in s$.

consider a sequence $\{x_n\}$ in s with limit a.

By given hypothesis we have

$L_1 = \lim\limits_{n\to\infty} f_1(x_n)$ ... (1)

and $L_2 = \lim_{n\to\infty} f_2(x_n)$ ... (2)

Let $\varepsilon > 0$, we have to show that

$|f_1 + f_2 - (L_1 + L_2)| < \varepsilon$ for large

by (1) $\Rightarrow$ for each $\varepsilon > 0 \exists\ n \in N_1 \ni |f_1(x_n) - L_1| < \frac{\varepsilon}{2}\ \forall n < N_1$ ... (3)

by (2) $\Rightarrow$ for each $\varepsilon > 0 \exists\ n \in N_2\ |f_2(x_n) - L_2| < \frac{\varepsilon}{2}\ \forall n > N_2$ ...(4)

$N = \max\{N_1, N_2\}$

Consider

$$\begin{aligned} |f_1 + f_2 - (L_1 + L_2)| &= |f_1 + f_2 - L_1 - L_2| \\ &= |(f_1 - L_1) + (f_2 - L_2)| \\ &= |f_1(x) - L_1| + |f_2(x_n) - L_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

$\therefore \quad |f_1 + f_2 - (L_1 + L_2)| < \varepsilon \quad \forall n > N$

$\therefore \quad \lim_{x\to\infty}(f_1 + f_2) = L_1 + L_2$

(ii) To show that $\lim_{x\to\infty}(f_1 f_2)(x) = L_1 L_2$

i.e., to show that $|f_1 f_2 - L_1 L_2| < \varepsilon\ \forall n > N$

consider
$$\begin{aligned} |f_1 f_2 - L_1 L_2| &= |f_1 f_2 - f_1 L_1 + f_1 L_1 - L_1 L_2| \\ &= |(f_1 f_2 - f_1 L_2) + (f_1 L_2 - L_1 L_2)| \\ &\leq |f_1(f_2 - L_2)| + |L_2(f_1 - L_1)| \\ &\leq |f_1|\,|f_2 - L_2| + |L_2|\,|f_1 - L_1| \end{aligned}$$ ...(1)

There is a constant $M > 0$ such that $|f_1| \leq M\ \forall n$

Since $\lim f_2 = L_2$ there exists $N_1$ such that $n > N_1 \Rightarrow |f_2 - L_2| < \frac{\varepsilon}{2M}$

Also, since $\lim f_2 = L_2$ there exists $N_2$ such that $n > N_2 \Rightarrow |f_1 - L_1| < \frac{\varepsilon}{2(1L_2 1 + 1)}$

Now if $N = \max\{N_1, N_2\}$ Then $n > N$ implies

by equation (1) we can write

$|f_1f_2 - L_1L_2| \le |f_1|\,|f_2 - L_2| + |L_2|\,|f_1 - L_1|$

$$< \cancel{M}\frac{\varepsilon}{2\cancel{M}} + |L_2|\frac{\varepsilon}{2(|L_2|+1)}$$

$$< \frac{\varepsilon}{2} + \cancel{|L_2|}\frac{\varepsilon}{2(\cancel{|L_2|}+1)}$$

$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$

$|f_1f_2 - L_1L_2| < \varepsilon$

$\therefore \quad \lim_{n\to\infty} f_1f_2 = L_1L_2$

(iii) To prove that $\lim_{n\to\infty}\frac{f_1}{f_2} = \frac{L_1}{L_2}$

first we will prove $\frac{1}{f_2}$ converges to $\frac{1}{L_2}$

$\therefore$ Let $\varepsilon > 0$ there exists M > 0 such that $|f_2| \ge M \;\; \forall n$.

Since $\lim f_2$ there exists N such that

$n > N \Rightarrow |L_2 - f_2| < \varepsilon . M|L_2|$

Then $n > N \Rightarrow \left|\frac{1}{f_2} - \frac{1}{L_2}\right| = \left|\frac{L_2 - f_2}{f_2L_2}\right|$

$$\le \frac{|L_2 - f_2|}{|f_2||L_2|}$$

$$< \frac{\varepsilon \cancel{M}|\cancel{L_2}|}{M|\cancel{L_2}|}$$

$$\left|\frac{1}{f_2} - \frac{1}{L_2}\right| < \varepsilon$$

$\therefore \quad \lim_{n\to\infty}\frac{1}{f_2} = \frac{1}{L_2}$

Now, $\lim\frac{f_1}{f_2} = \lim f_1, \lim\frac{1}{L_2}$

$$= L_1 \cdot \frac{1}{L_2}$$

$$= \frac{L_1}{L_2}$$

$\therefore \quad \lim\frac{f_1}{f_2} = \frac{L_1}{L_2}$

**46. Let f be a function defined on a subject S of R, Let a be a Real number that is the limit of some sequence in S and let L be a real numbers then $\lim_{x\to a} f(x) = L$ if and only if for each $\varepsilon > 0 \; \exists \; \delta > 0$ such that $x \in S$ and $|x - a| < \delta$ implies $|f(x) - L| < \varepsilon$**

*Sol.*

Given that f is function on 'S' and $S \subseteq R$, where R is Real numbers.

Required to prove $\lim_{x\to a} f(x) = L$ If and only if for each

$\varepsilon > 0 \; \exists \; \delta > 0 \ni x \in s$ and $|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ ...(1)

To show that $\lim_{x\to\infty} f(x_n) = L$

(i) $\lim_{x\to a} f(x) = L \Rightarrow$ for each $\varepsilon > 0 \; \exists \; \delta > 0$

$|f(x) - L| < \varepsilon$ whenever $x \in s$, $0 < |x - a| < \delta$

$x \in s, \; a - \delta < x < a$

$\Rightarrow x \in s, 0 < a - x < \delta$

$\Rightarrow x \in s, 0 < |x - a| < \delta$

$\Rightarrow |f(x) - L| < \varepsilon$

$\therefore \quad \lim_{x\to a^-} f(x) = l$

$x \in s, a < x < a + \delta$

$\Rightarrow x \in s, 0 < |x - a| < \delta$

$\Rightarrow |f(x) - L| < \varepsilon$

$\therefore \quad \lim_{x\to a^+} f(x) = L$

$\therefore$ for each $\varepsilon > 0 \exists\ \delta > 0 \exists\ x \in s$, $|x - a| < \delta$

$\Rightarrow \quad |f(x) - L| < \varepsilon$

Let $\lim_{x \to a^-} f(x) = L$, $\lim_{x \to a^+} f(x) = L$

and let $\varepsilon > 0$

$\lim_{x \to a^-} f(x) = l \rightarrow$ there exists $\delta_1 > 0 \exists$

$|f(x) - l| < \varepsilon$ whenever $x \in s$, $a - \delta_1 < x < a$

$\lim_{x \to a^+} f(x) = l \Rightarrow$ there exists $\delta_2 > 0$ such that

$f(x) - l| < \varepsilon$ whenever $x \in s$, $a < x < a + \delta_2$

If we take $\delta = \min\{\delta_1, \delta_2\}$

Then $x \in s$, $0 < |x - a| < \delta$

$\Rightarrow \quad x \in s$, $0 < a - x < \delta$ or $0 < x - a < \delta$

$\Rightarrow \quad x \in s$, $a - \delta < x < a$ or $a < x < a + \delta$

$\Rightarrow \quad x \in s$, $a - \delta_1 < x < a$ or $a < x < a + \delta_2$

$\Rightarrow \quad |f(x) - l| < \varepsilon$

$\therefore \quad \lim_{x \to a} f(x) = L$

**47. Find the limit of f(x), where f(x) = $\dfrac{x^2 - a^2}{x - a}$.**

*Sol.*

Let $f : R - \{a\} \rightarrow R$

clearly 'a' is a limit point of $R - \{a\}$

$$f(x) = \frac{x^2 - a^2}{x - a}$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{x^2 - a^2}{x - a}$$

$$= \lim_{x \to a} \frac{(x - a)(x + a)}{(x - a)}$$

$$= \lim_{x \to a} (x + a)$$

$$= a + 9$$

$$= 2a$$

$$\therefore \lim_{x \to a} f(x) = 2a$$

i.e., $\lim_{x \to a} \dfrac{x^2 - a^2}{x - a} = 2a$

**48. Find the limit of f(x) = $\dfrac{x^3 - a^3}{x - a}$**

*Sol.*

Given that, $f(x) = \dfrac{x^3 - a^3}{x - a}$

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{x^3 - a^3}{x - a}$$

$$= \lim_{x \to a} \frac{(x - a)(x^2 + ax + a^2)}{x - a}$$

$$= a^2 + a.a + a^2$$

$$= a^2 + a^2 + a^2$$

$$\lim_{x \to a} f(x) = 3a^2$$

i.e., $\lim_{x \to a} \dfrac{x^3 - a^3}{x - a} = 3a^2$

# Choose the Correct Answer

1. If f and g are real valued function then min (f, g) = [ c ]

   (a) max (–f, –g)
   (b) $\frac{1}{2}(a+b)+\frac{1}{2}(a-b)$
   (c) $\frac{1}{2}(f+g)-\frac{1}{2}|f-g|$
   (d) None

2. $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$ = [ a ]

   (a) $\frac{1}{2}$
   (b) $-\frac{1}{2}$
   (c) 1
   (d) 0

3. The domain of g of is [ c ]

   (a) R
   (b) dom (f) $\cap$ dom (g)
   (c) {x $\in$ dom (f) ; f(x) x $\in$ dom (g)}
   (d) {x $\in$ dom (f) $\cap$ x $\in$ dom (g)}

4. If $f(x) = (1 + 3x)^{1/x}$ is continuous at x = 0 then f(0) = [ c ]

   (a) e
   (b) $e^2$
   (c) $e^3$
   (d) 0

5. $\lim_{n\to\infty}\frac{\sin n^0}{n}$ = [ b ]

   (a) 1
   (b) $\frac{\pi}{180}$
   (c) $\frac{180}{\pi}$
   (d) None

6. If f and g are real valued function then max (f, g) (x) = [ a ]

   (a) max {f(x), g(x)}
   (b) $\frac{f(x)}{g(x)}$
   (c) f(x) g(x)
   (d) f(x) – g(x)

7. If $f(x) = \frac{1 - \cos ax}{x \sin x}$ is continous at x = 0 where $f(0) = \frac{1}{2}$ then [ c ]

(a) a = 1 (b) a = –1

(c) a = ± 1 (d) None

8. $f(x) = \frac{\sin x}{x}$ is always [ c ]

(a) Continuous (b) Discontinuous

(c) Continuous if f(0) = 1 (d) None

9. $f(x) = x^2$, is continuous at $x_0$ = [ d ]

(a) 4 (b) 1

(c) 0 (d) 2

10. Limit of $f(x) = \frac{x^3 - a^3}{x - a}$ [ d ]

(a) 3 (b) $3a^2$

(c) 0 (d) None

# Fill in the blanks

1. If f is uniformly continous on [a, b] then f is __________ on [a, b].
2. A function continous in one open interval __________ uniformly continuous in that interval.
3. If |f| is continuous at ‘a’ then f is need not to be __________ at ‘a’.
4. The domain of $\frac{f}{g}$ is the set __________.
5. A function f is continuous in dom (f) = S if and only if __________ continuous.
6. The domain of $\frac{x^2-4}{x-2}$ is __________.
7. The function $f(x) = x^2$ is uniformly continuous on __________.
8. Mean value theorem is f’(x) = __________.
9. The set on which f is defined is called the __________ of f.
10. The natural domain of $f(x) = \sqrt{4-x^2}$ is __________.

**Answers**

1. Continuous
2. Need not to be
3. Continuous
4. Dom (f) $\cap \{x \in \text{dom}(g) : g(x) \neq 0\}$
5. Uniformly Continuous
6. $(-\infty, 2) \cup (2, \infty)$
7. [–7, 7]
8. $\frac{f(b)-f(a)}{b-a}$
9. Domain
10. $\{x \in R : x \neq 0\}$

| UNIT III | **Differentiation :** Basic Properties of the Derivative - The Mean Value Theorem - * L'Hospital Rule - Taylor's Theorem. |
|---|---|

## 3.1 Basic Properties of the Derivative

**Definition 1.**

Let 'f' be a real valued function defined on an open internal containing a point 'a' we say that f is differentiable at a or that f has a derivative at 'a'

if the limit, $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists and is finite

i.e., f is differentiable at 'a' we can write f'(a)

$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

**Definition 2.**

Let 's' be an aggregate and f : S $\to$ R be a function, let C $\in$ S, be a limit point of S and $l \in$ R, f is said to be derivable at 'C' if for a given $\varepsilon > 0$ there exists $\delta > 0$.

Such that 0 |x – c| < $\delta \Rightarrow \left|\frac{f(x)-f(x)}{x-c}\right| < \varepsilon$.

The number '*l*' is called the derivative of '*l*' at c and denoted by f'(c).

**1. If f is differentiable at a point 'a'. Then 'f' is continuous at a.**

*Sol.* **(Imp.)**

let f : [a, b] $\to$ R, at $\in$ [a, b]

let, c $\in$ (a, b)

f is derivable at c $\Rightarrow \lim_{n\to c}\frac{f(x)-f(c)}{x-c} = f'(c)$

for $x \neq c$ . f(x) – f(c) = $\left[\frac{f(x)-f(c)}{x-c}\right](x-c)$

$\lim_{x\to c}[f(x)-f(c)] = \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right]\lim_{x\to c}(x-c)$

= f'(c).0

= 0

$\Rightarrow \lim_{x\to c} f(x) - \lim_{x\to c} f(c) = 0$

$\Rightarrow \lim_{x\to c} f(x) = f(c)$

$\therefore$ f is continuous at c $\in$ (a, b)

Let c = a

f is derivable at a $\Rightarrow \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a} = Rf'(a)$

$\lim_{x\to a^+}[f(x)-f(a)] = \lim_{x\to a^+}\left[\frac{f(x)-f(a)}{x-a}\right]\lim_{x\to a^+}(x-a)$

= Rf'(a).0

= 0

$\therefore \lim_{x\to a^+} f(x) = f(a)$

$\Rightarrow$ f is right continuous at 'a'.

Similarly, we can prove that f is left continuous at b.

**2. Let f and of be functions that are differentiable at the points each of the functions cf [c a constant], f+g, fg and f/g is also differentiable at a, except f/g if g(a) = 0 since f/g is not defined at a in this case.**

**The formulas are**

1. **(cf)'(a) = c f'(a)**
2. **(f + g)'(a) = f'(a) + g'(a)**

**3. (fg)'(a) = f(a)g'(a) + f'(a)g(a)**

**4. (f/g)'(a) = [g(a)f'(a) – f(a) g'(a)]/g²(a)**

**if g(a) $\neq$ 0.**

*Sol.* **(Imp.)**

Given, that f & g are functions, which are differentiable at 'a'.

Let f is differentiable at 'a'.

Then $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ ...(1)

Similarly 'g' is differentiable at 'a'

Then $g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$ ...(2)

By definition of (cf) (x) = cf(x). for all $x \in$ dom(f)

$$(cf)'(a) = \lim_{x \to a} \frac{(cf)(x) - (cf)(a)}{x - a}$$

$$= \lim_{x \to a} c \cdot \frac{f(x) - f(a)}{x - a}$$

$$= c \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= c.f'(a)$$

$\therefore$ (cf)' (a) = cf'(a)

2. f & g are differentiable at 'a'

Then $(f + g)'(a) = \lim_{x \to a} \frac{(f+g)(x) - (f+g)(a)}{x - a}$

$$\Rightarrow \frac{(f+g)(x) - (f+g)(a)}{x - a} = \frac{f(x) + g(x) - f(a) - g(a)}{x - a} = \frac{f(x) - f(a) + g(x) - g(a)}{x - a} \quad \frac{(f+g)(x) - (f+g)(a)}{x - a}$$

$$= \frac{f(x) - f(a)}{x - a} + \frac{g(x) - g(a)}{x - a}$$

Apply limit as $x \to a$

$$\lim_{x \to a} \frac{(f+g)(x) - (f+g)(a)}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

$(f + g)'(a) = f'(a) + g'(a)$

3. Observe that

$$\frac{fg(x)-fg(a)}{x-a} = f(x)\frac{g(x)-g(a)}{x-a} + g(a)\frac{f(x)-f(a)}{x-a}$$

for $x \in dom(fg)$, $x \neq a$.

we take the limit as $x \to a$ and note that $\lim_{x\to a} f(x) = f(a)$.

$\therefore$ $(fg)'(a) = f(a)\, g'(a) + g(a)f'(a)$

4. Since $g(a) \neq 0$. and g is continuous at a, There exists an open interval I consisting a such that $g(x) \neq 0$. for $x \in I$.

for $x \in I$ we can write

$$(f/g)(x) - (f/g)(a) = \frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}$$

$$= \frac{f(x)g(a)-f(a)g(x)}{g(x)g(a)}$$

$$= \frac{f(x)g(a)-g(a)f(a)+g(a)f(a)-f(a)g(x)}{g(x)g(a)}$$

So, $$\frac{(f/g)(x)-(f/g)(a)}{x-a} = \left\{g(a)\frac{f(x)-f(a)}{x-a} - f(a)\frac{g(x)-g(a)}{x-a}\right\}\frac{1}{g(x)g(x)}$$

for $x \in I$, $x \neq a$

Now, take the limit as $x \to a$ to obtain

$$\lim_{x\to a}\frac{1}{g(x)g(a)} = \frac{1}{g^2(a)}$$

**3. Find h'(a) where $h(x) = x^{-m}$ for $x \neq 0$. $h(x) = \frac{f(x)}{g(x)}$ where $f(x) = 1$ & $g(x) = x^m$ for all x.**

*Sol.* **(Imp.)**

Let m be the positive integer

$h(x) = x^{-m}$

By the Quotient Rule

$$h'(a) = \frac{g(a)f'(a)-f(a)g'(a)}{g^2(a)}$$

Since $g(x) = x^m$ & $f(x) = 1$

$x = a \Rightarrow g(a) = a^m$ & $g(a) = 1$

$h' = g'(a) = ma^{m-1}$ & $f'(a) = 0$

$\therefore \quad h'(a) = \dfrac{a^m \cdot 0 - 1 \cdot ma^{m-1}}{(a^m)^2}$

$= \dfrac{-ma^{m-1}}{a^2m} \Rightarrow \dfrac{-ma^m \cdot a^{-1}}{a^{2m}}$

$= \dfrac{-ma^{-1}}{a^m}$

$= \dfrac{-m}{a^{m+1}}$

$h'(a) = -ma^{-m-1}$

for $a \neq 0$.

**4. State and prove Chain Rule**

**(OR)**

**If f is differentiable at a and g is differentiable at f(a), then the composite function gof is differentiable at a and (gof)'(a) = g'(f(a)).f'(a).**

*Sol.* **(Imp.)**

Let f(x) = y for $x \in [a, b]$

and f(c) = d for $c \in [a, b]$

Since I is the range of f, $f(c) \in I$

define $h : I \to R$

So that $h(y) = \begin{cases} \dfrac{g(y)-g(d)}{y-d}, & y \neq d \\ g'(d), & y = d \end{cases}$

Since g is deriable at f(c) = d,

$g'(d) = \lim\limits_{y \to d} \dfrac{g(y)-g(d)}{y-d}$

$= \lim\limits_{y \to d} h(y)$

from the definition of $h : I \to R$.

g(y) – g(d) = h(y)(y – d) for $y \neq d$

for $x \neq c$, $\dfrac{(gof)(x)-(gof)(c)}{x-c}$

$= \dfrac{g(f(x)-g(f(c)}{x-c}$

$$= \frac{g(y)-g(d)}{x-c}$$

$$= \frac{h(y)(y-d)}{x-c}$$

$$= h(f(x)\frac{f(x)-f(c)}{x-c}$$

f is differentiable at c $\Rightarrow \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$

$= f'(c)$

f is continuous at $c_i$ h is continuous at f(c)

$= d \Rightarrow$ hof is continuous at c

$$\Rightarrow \lim_{x\to c}(hof)\ (x) = h(f(c))$$

$$\therefore \lim_{x\to c}\frac{(gof)(x)-(gof)(c)}{x-c} = \lim_{x\to c}\left[h(f(x))\cdot\frac{f(x)-f(c)}{x-c}\right]$$

$= h(f(c).f'(c)$

$= h(d).f'(c)$

$= g'(d)\ f'(c)$

$= g'(f(c)).f'(c)$

$$\therefore \lim_{x\to c}\frac{(gof)(x)-(gof)(c)}{x-c} = g'(f(c))\ f'(c)$$

**5. Show that f(x) = sin x is derivable at every a $\in$ R.**

*Sol.*

Given that f(x) = sin x

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{\sin x-\sin a}{x-a}$$

$$= \lim_{x\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a}$$

$= \cos a.1$

$= \cos a.$

$\therefore$ f(x) = sin x is derivable at $a \in R$

and f'(a) = cos a.

Since $a \in R$ is orbitrary

$f'(x) = \cos x \ \forall x \in R$ .

**6. Discuss the differentiability of f(x) = |x – a| in R.**

*Sol.*

Let $C \in R$ and $c < a$

Then $c - a < 0$

There exists a deleted nbd of 'c'

such that $x \in c$ delected nbd $\Rightarrow$ $x < a$

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c}\frac{|x-a|-|c-a|}{x-c}$$

$$= \lim_{x\to c}\frac{-(x-a)-\{-(c-a)\}}{x-c}$$

$$= \lim_{x\to c}\frac{c-x}{x-c}$$

$$= \lim_{x\to c}(-1) = -1$$

$\therefore$ f(x) is derivable at $c(< a) \in R$.

and $f'(c) = -1$

Let $C \in R$ and $c > a$, then $c - a > 0$

There exists a delected nbd of 'c' such that $x \in c$ deleted nbd $\Rightarrow$ $x > a$.

$$\lim_{x\to c}\frac{f(x)-f(a)}{x-c} = \lim_{x\to c}\frac{(x-a)-(c-a)}{x-c}$$

$$= \lim_{x\to c}\frac{x-c}{x-c} = \lim_{x\to c} 1 = 1$$

$\therefore$ f(x) is derivable at $c\ (> a) \in R$.

and $f'(c) = 1$

Let $C \in R$ and $c = a$.

Then $f(c) = c - a = 0$

for $x \in c$ - left hand $\Rightarrow$ $x < a$ so that $Lf'(c) = -1$

for $x \in c$- right hand $\Rightarrow$ $x > a$ so that $Rf'(c) = 1$

$\therefore$ f(x) is not deriable at $c(= a) \in R$

Hence f(x) is derivable in $R - \{a\}$

**7. Discuss the derivability of f(x) = |x| + |x – a| in R.**

*Sol.* **(Imp.)**

We have $f(x) = 1 - 2x,\quad x < 0$

$f(x) = 1 \quad 0 \le x \le 1$

$f(x) = 2x - 1 \quad x > 1$

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{1-2x-1}{x}$$

$$= \lim_{x\to 0^-}(-2)$$

$$= -2$$

$$= Lf'(0)$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{1-1}{x} = \lim_{x\to 0^+} 0$$

$$= 0 = Rf'(0)$$

$\therefore$ $Lf'(0) \neq Rf'(0)$

and hence f'(0) does not exist.

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-}\frac{1-1}{x-1} = \lim_{x\to 1^-} 0 = 0 = Lf'(1)$$

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{2x-1-1}{x-1}$$

$$= \lim_{x\to 1^+}\frac{2(x-1)}{x-1}$$

$$= 2 = Rf'(1)$$

$\therefore$ $Lf'(1) = Rf'(1)$

and hence f'(1) does not exists.

$\therefore$ f is derivable at every R – {0, 1}

Also, $f'(x) = -2$ for $x < 0$;

$f'(x) = 0$ for $0 < x < 1$

$f'(x) = 2$ for $x > 1$

**8. Let $f(x) = x \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$**

**(a) Observe that f is continuous at x = 0**

**(b) Is f differentiable at x = 0 ? Justify your answer.**

*Sol.* **(Imp.)**

(a) Given that $f(x) = x\sin\frac{1}{x}$ $\quad x \neq 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\frac{1}{x}$$

Since $\lim_{x\to 0} x = 0$ and $\sin\left(\frac{1}{x}\right)$ is bounded in a deleted nbd of '0'

$\therefore \quad \lim_{x\to 0} f(x) = f(0)$

$\Rightarrow$ f is continuous at the origin

(b) $$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x\sin\left(\frac{1}{x}\right)-0}{x}$$

$$= \lim_{x\to 0}\sin\frac{1}{x}$$

does not exists

$\therefore$ f is not derivable at $x = 0$

---

**9. State and prove Rolle's Theorem**

**(OR)**

**f : [a, b] $\to$ R is such (i) f is continuous on [a, b] (ii) f is derivable on (a, b) and (iii) f(a) = f(b). The there exists c $\in$ (a,b) such that f'(c) = 0.**

*Sol.*

f is continuous on [a, b]

$\Rightarrow$ f is bounded on [a, b] and attains the inf and sup

$\Rightarrow$ There exists $\alpha, \beta \in [a, b]$ such that

$f(\alpha) = m = \inf f$

$f(\beta) = M = \sup f$ in [a, b]

**case (i)**

Let $m = M$, Then $f(x) = m \ \forall m \in [a, b]$

$\therefore$ f is constant function in [a, b]

and here $f'(x) = 0$ for every $x \in [a, b]$

Thus the theorem is true

**case (ii)**

Let $m \neq M$

Since $f(a) = f(b)$ and $m \neq M$

we have either $M \neq f(a)$ and hence $M \neq f(b)$ or $M \neq f(a)$

and hence $M \neq f(b)$

let us suppose that $M \neq f(a)$, $M \neq f(b)$

$f(\beta) = M \neq f(a) \Rightarrow \beta \neq a$

$f(\beta) = M \neq f(b) \Rightarrow \beta \neq b$

$\therefore \quad \alpha < \beta < b$ or $\beta \in (a, b)$

f is derivable on (a, b) & $\beta \in (a, b)$

$\Rightarrow$ f is derivable at $\beta$

Now, we prove that $f'(\beta) = 0$

If possible, let $f'(\beta) < 0$

$\therefore$ There exists $\delta_1 > 0$ such that $f(x) > f(\beta) = M \quad \forall x \in (\beta - \delta_1, \beta)$ C[a, b]

This is a contradiction as M is supremum.

Similarly, we can prove that $f'(\beta) \not> 0$

Hence $f'(\beta) = 0$

$\therefore$ There exists $\beta \in (a, b)$ such that $f'(\beta) = 0$

## 3.2 The Mean Value Theorem

**10. State and prove Mean value theorem**

**(OR)**

**Let f be continuous function on [a, b] that is differentiable at (a, b). Then there exist [at least one] $c \in [a, b]$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.**

*Sol.*

Define the function $\phi : [a, b] \to R$ such that

$\phi(x) = f(x) + kx$ where $k \in R$ is given by

$\phi(a) = \phi(b)$

$\phi(a) = \phi(b) \Rightarrow f(a) + ka = f(b) + kb$

$f(a) - f(b) = kb - ka$

$-(f(b) - f(a)) = k(b - a)$

$$-k = \frac{f(b) - f(a)}{b - a}$$

$k \in R$ x is continuous on $R \Rightarrow$ kx is continuous and derivable on R.

f is continuous on [a, b] and kx is continuous on R $\Rightarrow \phi$ is continuous on [a, b]

f is derivable on (a, b) and kx derivable on R

$\Rightarrow$ $\phi$ is derivable on (a, b)

$\therefore$ Further from the definition of $\phi$, $\phi(a) = \phi(b)$

$\therefore$ The function $\phi$ satisfies all the conditions of Rolle's theorem

$\therefore$ There exists $c \in (a, b)$ such that $\phi'(c) = 0$

Since $\phi(x) = f(x) + kx \quad \forall x \in [a, b]$

$\phi'(x) = f'(x) + k \quad \forall x \in (a, b)$

$\therefore \quad \phi'(c) = f'(c) + k$ for $c \in (a, b)$

and $\phi'(c) = 0 \Rightarrow f'(c) = -k$

$$\therefore \quad f'(c) = \frac{f(b) - f(a)}{b - a}$$

**11. If f : [a, b] $\rightarrow$ R is such that**

**(i) f is continuous on [a, b]**

**(ii) f is differentiable on (a, b)**

**(iii) f'(x) = 0 for all x $\in$ (a, b) then f is constant function on [a, b]**

*Sol.*

Let $x_1, x_2 \in [a, b]$ and $x_1 < x_2$

Then $[x_1, x_2] \subset [a, b]$

$\therefore$ f satisfies all the condition lagrange's theorem on $[x_1, x_2]$

There exists $C \in (x_1, x_2)$ such that

$f(x_2) - f(x_1) = (x_2 - x_1)\, f'(c)$

$= (x_2 - x_1).0 \quad [\because f'(c) = 0$ by (iii)]

$= 0$

$\therefore \quad f(x_2) - f(x_1) = 0 \Rightarrow f(x_2) = f(x_1)$ for $x_1, x_2 \in (a, b)$

$\Rightarrow$ f is constant function on (a, b)

Since f is continuous on [a, b]

f is constant function on [a, b]

**Note :**

If f : [a, b] $\rightarrow$ R, g : [a, b] $\rightarrow$ R

Satisfy the condition of kagrange's theorem and $f'(x) = g'(x) \ \forall x \in (a, b)$. Then f and g differ by a real numbers (constant) i.e., $f(x) = g(x) + c$ for some $C \in R$.

**Definition**

Let f be a red valued function defined on interval I. We say that f is strictly increasing on I if

$x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

strictly decreasing on I if

$x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Increasing on I if

$x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

Decreasing on I if

$x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

**12. If f is differentiable function on an interval (a, b). Then**

**1. $f'(x) \geq 0$ $\forall x \in (a, b)$, Then f is increasing on (a, b).**

**2. $f'(x) \leq 0$ $\forall x \in (a, b)$, Then f is decreasing on (a, b).**

*Sol.*

Let $x_1, x_2 \in (a, b)$ and $x_1 < x_2$. Then $[x_1, x_2] \subset (a, b)$

f is derivable on (a, b) $\Rightarrow$ f is continuous on (a, b)

Since $[x_1, x_2] \subset (a, b)$,

f satisfies the continuous of lagrange's theorem on $[x_1, x_2]$

$\therefore$ There exists $C \in (x_1, x_2) \subset (a, b)$ such that

$f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$

**Case (1) :**

Let $f'(x) \geq 0$ $\forall x \in (a, b)$

Then $f'(c) \geq 0$ as $C \in (a, b)$

$f'(c) = 0 \Rightarrow f(x_2) = f(x_1)$ or $f'(c) = 0 \Rightarrow f(x_2) > f(x_1)$ $(\because x_2 > x_1)$

$\therefore$ for all $x_1, x_2 \in (a, b)$, $x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$

$\Rightarrow$ f is increasing on (a, b)

**Case (2) :**

Let $f'(x) \leq 0$ $\forall x \in (a, b)$

Then $f'(c) \leq 0$ as $c \in (a, b)$

$f'(c) = 0 \Rightarrow f(x_2) = f(x_1)$ and

$f'(c) < 0 \Rightarrow f(x_2) < f(x_1)$ $(\because x_2 > x_1)$

$\therefore$ for all $x_1, x_2 \in (a, b)$, $x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1)$

$\Rightarrow$ f is monotonically decreasing on (a, b)

**13. If f is derivable at $c \in [a, b]$, $f'(c) \neq 0$ and $f^{-1}$ is continuous at f(c). Then $f^{-1}$ is derivable at $f(c)(f^{-1})'(f(c)) = \dfrac{1}{f'(c)}$.**

*Sol.*

Since $f : [a, b] \to [\alpha, \beta]$ is a bijection

$f^{-1} = g$ is also bijection from $[\alpha, \beta]$ to [a, b]

Let $g = f(x)$ for $x \in [a, b]$ and $d = f(c)$ for $c \in [a, b]$

Since $f^{-1} = g$, $x = f^{-1} = g(y)$

$f^{-1}$ is continuous at f(c)

$\Rightarrow$ g is continuous at d $\Rightarrow \lim_{y \to d} g(y) = g(d)$

$\Rightarrow$ $x \to c$ as $y \to d$ and $x \neq c$ if $y \neq d$

$$\lim_{y \to d} \frac{g(y) - g(d)}{y - d} = \lim_{x \to c} \frac{x - c}{f(x) - f(c)} = \frac{1}{\lim_{x \to c} \frac{f(x) - f(c)}{x - c}} = \frac{1}{f'(c)}$$

$\therefore$ g is derivable at d

i.e., $f^{-1}$ derivable at f(c)

Also, $(f^{-1})'(f(c)) = g'(d) = \lim_{y \to d} \frac{g(y) - g(d)}{y - d} = \frac{1}{f'(c)}$

**14. Determine by using mean value theorem.**

**(a) $x^2$ on [–1, 2]** **(b) sin x on [0, $\pi$]** **(c) |x| on [–1, 2]**

**(d) $\frac{1}{x}$ on [–1, 1]** **(e) $\frac{1}{x}$ on [1, 3]** **(f) sgn (x) on [–1, 2]**

*Sol.* **(Imp.)**

**(a) $x^2$ on [–1, 2]**

yes, let $f(x) = x^2$ with dom(f) = [–1, 2]

Then $f'(x) = 2x$.

Further more, we have $f(-1) = 1$ & $f(2) = 4$

and so, $\frac{f(2) - f(-1)}{2 - (-1)} = \frac{4 - 1}{2 + 1} = 1$

Now we request have to let $f'(x) = 2x = 1$

which implies $x = \frac{1}{2}$

**(b) sinx on [0, $\pi$]**

*Sol.*

yes, let f(x) = sinx with dom(f) = [0, $\pi$]

Then f'(x) = cosx

further more, we have $f(0) = 0 = f(\pi)$

and so, $\frac{f(\pi) - f(0)}{\pi - 0} = 0$

Now, we let $f'(x) = \cos x = 0$

$\Rightarrow x = \frac{\pi}{2}$

**(c) |x| on [–1, 2]**

*Sol.*

No, Notice that

$$f'(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$$

But $\dfrac{f(2) - f(-1)}{2-(-1)} = \dfrac{2-1}{2+1} = \dfrac{1}{3}$

Which is different than f'(x) for every $x \in (-1, 2)$

The hypothesis that fails is the following

f(x) is not differentiable on 0.

In effect let f(x) = |x|, with dom(f) = [–1, 2]

Then

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{|x|-0}{x-0} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{|x|-0}{x-0} = \lim_{x\to 0^+} \frac{x}{x} = 1$$

and there

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} \neq \frac{f(x)-f(0)}{x-0}$$

**(d) $\frac{1}{x}$ on [–1, 1]**

*Sol.*

No, Infact, we have $f'(x) = \dfrac{-1}{x^2}$

however, $\dfrac{f(1) - f(-1)}{f-(-1)} = \dfrac{1-(-1)}{1+1} = \dfrac{2}{2} = 1$

and there is no $x \in (-1, 1)$

such that f'(x) = 1

The hypothesis that fails is this f is

discontinuous at x = 0

Since $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$

and $\lim\limits_{x\to 0^+} \dfrac{1}{x} = \infty$

**(e) $\frac{1}{x}$ on [1, 3]**

*Sol.*

yes, let $f(x) = \frac{1}{x}$, with dom(f) = [1, 3]

Then $f'(x) = -\frac{1}{x^2}$

more over f(1) = 1 and $f(3) = \frac{1}{3}$ and hence

$$\frac{f(3)-f(1)}{3-1} = \frac{\frac{1}{3}-1}{3-1} = \frac{-1}{3}$$

Now we put $f'(x) = \frac{-1}{x^2} = \frac{-1}{3}$ which results in $x = \sqrt{3}$

**(f) sgn (x) on [–1, 2]**

*Sol.*

No, sinc $\text{sgn}(x) = \frac{-x}{|x|}$ for $x \neq 0$

and sgn(0) = 0 we have f'(x) = 0

for $x \neq 0$. while f'(x) is not defined for x = 0 on the other hand

$$\frac{\text{sgn}(2)-\text{sgn}(-2)}{2-(-2)} = \frac{1-(-1)}{2-(-2)} = \frac{1}{2}$$

The hypothesis that fails is this

sgn is discontinuous at x = 0

Since $\lim_{x\to 0^-} \text{sgn}(x) = -1$ and

$\lim_{x\to 0^+} \text{sgn}(x) = 1$

**15. Prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in R$.**

*Sol.* **(Imp.)**

Let us begin with a trivial case

If x = y then

$|\cos x - \cos y| = 0 \leq |0| = |x - x| = |x - y|$

So, clearly the inequality holds for this case.

In what follows, we assume $x \neq y$.

let $f(x) = \cos x$.

Since f is differentiable on R. if it is differentiable only interval $(x, y) \subset R$.

By mean value theorem

These is $v \in (x, y)$ such that

$$f'(v) = \frac{f(x) - f(y)}{x - y}$$

we know that $f'(x) = -\sin x$ .

So the equation above becomes,

$$-\sin v = \frac{\cos x - \cos y}{x - y} \qquad \text{...(1)}$$

Taking the absolute value on both side of equation (1)

$$|-\sin v| = \left|\frac{\cos x - \cos y}{x - y}\right|$$

$$|\sin v| = \frac{|\cos x - \cos y|}{|x - y|} \qquad \text{...(2)}$$

But $|\sin x| \leq |$ for all $x \in R$,

This fast and equation (2) implies

$$\frac{|\cos x - \cos y|}{|x - y|} \leq 1$$

or. equivalenty

$|\cos x - \cos y| \leq |x - y|$

which is a desired result



**16.** Show that $ex \leq$ **$e^x$ for all $x \in$ R**

*Sol.* **(Imp.)**

Let $f(x) = e^x - ex$ then

$f'(x) = e^x - e$

If $x > 1$, $f'(x) > 0$

Since f is strictly increasing

If $x < 1$, $f'(x) < 0$.

as f is strictly decreasing.

and If $x = 0$, $f'(x) = 0$

as f is strictly decreasing for $x < 1$, strictly increasing for $x > 1$ and f is continuous on R

f(1) is minimum for f,

But $f(1) = e - e = 0$

$\therefore$ $f(x) = e^x - ex \geq 0$. for all $x \in R$.

which implies $ex \leq e^x$.

**17. Show that $\sin x \leq x$ for all $x \geq 0$**

*Sol.*

Let $f(x) = x - \sin x$

Then $f'(x) = 1 - \cos x$

Notice that for all $x \geq 0$, $1 - \cos x \geq 0$

$\therefore$ f is increasing on $[0, \infty)$

Since $f(0) = 0 - \sin(0) = 0$

If follows that $f(x) = x - \sin x \geq 0$ $\forall x \geq 0$

Hence $\sin x \leq x$ for all $x \geq 0$.

**18. Suppose that f is twice differentiable on an open interval I and that $f''(x) = 0$ $\forall x \in I$. Show that f has the form $f(x) = ax + b$ be for suitable constants a and b.**

*Sol :*

If $f''(x) = 0$,

as we know that, let f be a differentiate function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$

Then f is constant function on (a, b)

f'(x) is constant function

$f'(x) = a$, where $a \in I$.

Let g be a function on I such that

$g(x) = ax$,

Then g is differentiable and $g'(x) = a = f'(x)$

By corollary

$\Rightarrow$ $f(x) = g(x) + b = a(x) + b$ for

Since constant $b \in I$.

**19. Suppose f is three times differentiable on an open interval I and that f''' = 0. on I. What form does f have ? prove your claim**

*Sol :*

We claim that $f(x) = \frac{a}{2}x^2 + bx + c$

for constants a, b, $c \in I$

In effect, if $f'''(x) = 0$

f''' is constant function defined by $f''(x) = a$.

for some $a \in I$.

Let g be a function on I such that

$g(x) = ax$

Then g is differentiable and $g'(x) = a = f''(x)$

$\Rightarrow$ $f'(x) = g(x) + b = ax + b$ for some constant $b \in I$.

Finally let h be a function on I.

Definitely by $h(x) = \frac{a}{2}x^2 + bx$

Then h is differentiable on I.

and $h'(x) = ax + b = f'(x)$

$\Rightarrow$ $f(x) = h(x) + c = \frac{a}{2}x^2 + bx + c$

for some constant $c \in I$.

Hence the claim is true.



**20. Let a, $b \in R$. let $f(x) = e^{ax} \cos(bx)$ and $g(x) = e^{ax} \sin(bx)$**

**(i) Compute f'(x) and g'(x)**

**(ii) Use (i) to compute f'' and f'''**

*Sol.* **(Imp.)**

(i) We have $f(x) = e^{ax} \cos(bx)$

$g(x) = e^{ax} \sin(bx)$

$f'(x) = -be^{ax} \sin(bx) + ae^{ax} \cos(bx)$

and $g'(x) = be^{ax} \cos(bx) + ae^{ax} \sin(bx)$

(ii) We have

$f''(x) = -b^2e^{ax} \cos(bx) - abe^{ax} \sin(bx) - abe^{ax} \sin(bx) + a^2e^{ax} \cos(bx)$

$= (a^2 - b^2) e^{ax} \cos(bx) - 2abe^{ax} \sin(bx)$

$f'''(x) = -b(a^2 - b^2) e^{ax} \sin(bx) + a(a^2 - b^2) e^{ax} \cos(bx) - 2ab^2 e^{ax} \cos bx - 2a^2be^{ax} \text{ain}(bx)$

$= (a^2 - b^2) e^{ax} (a \cos (bx) - b \sin(bx) - 2abe^{ax} (b \cos(bx) - a \sin(bx))$

**21. (i) Show that x < tan x for all $x \in \left(0, \frac{\pi}{2}\right)$.**

*Sol.*

Let f(x) = tan x – x

Then $f'(x) = \sec^2 x - 1 > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$

Therefore f is strictly increasing on $\left(0, \frac{\pi}{2}\right)$

That is,

$f(x_1) < f(x_2)$ whenever $0 < x_1 < x_2 < \frac{\pi}{2}$

Now let $x_1 \to 0$

Since $f(x_1)$ is decreasing as $x_1 \to 0$,

$0 = f(0) = \lim_{x_1 \to 0} f(x_1) < f(x_2)$

That is f(x) > 0 for all $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ x < tan x.

**22. Show that $\frac{x}{\sin x}$ is a strictly increasing function on $\left(0, \frac{\pi}{2}\right)$.**

*Sol.* **(Imp.)**

If $f(x) = \frac{x}{\sin x}$

Then $f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$

Since sin x > x cos x

So, f′(x) > 0

$\therefore$ f is strictly increasing on $\left(0, \frac{\pi}{2}\right)$

**23. Show that $x \le \frac{\pi}{2} \sin x$ for $x \in \left[0, \frac{\pi}{2}\right]$.**

*Sol.*

Equality holds at the end point $0, \frac{\pi}{2}$ and $\frac{x}{\sin x}$ is increasing on $\left(0, \frac{\pi}{2}\right)$ [by (ii)]

Hence if $0 < x < y < \frac{\pi}{2}$

we have

$$\frac{x}{\sin x} < \frac{y}{\sin y} \text{ and } \frac{x}{\sin x} < \lim_{y \to \frac{\pi}{2}} \frac{y}{\sin y}$$

$$= \frac{\frac{\pi}{2}}{1} = \frac{\pi}{2}$$

**24. Suppose that f is differentiable on R that i $\leq$ f'(x) $\leq$ 2 for x$\in$R, and that f(0) = 0 prove that x$\leq$ f(x) 2x for all x > 0.**

*Sol.* **(Imp.)**

let g(x) = 2x – f(x)

So that g'(x) = 2 – f'(x) > 0

$\therefore$ g is increasing on R

Since g(0) = 0, g(x) $\geq$0, for x $\geq$ 0

Thus, f(x) $\leq$ 2x for x $\geq$ 0.

Let h(x) = f(x) – x

So that h'(x) = f'(x) – 1 $\geq$ 0

$\therefore$ h is increasing on R.

Since h(0) = 0, h(x) $\geq$ 0 for x $\geq$ 0

Thus x $\leq$ f(x) for all x $\geq$ 0.

**25. Let f be a differentiable function on an interval (a, b) then (i) f is strictly decreasing if f'(x) < 0 for all x$\in$(a, b).**

**(i) f is increasing if f'(x) $\geq$ 0 for all x$\in$ (a, b)**

**(iii) f is decreasing if f'(x) $\leq$ 0 all x $\in$(a, b)**

*Sol.* **(Imp.)**

Given that f is differentiable function on an interval (a, b)

(i) If $a < x_1 < x_2 < b$ then

$$\frac{[f(x_2) - f(x_1)]}{x_2 - x_1} = f'(c) < 0 \text{ for some } c \in (x_1, x_2)$$

$\therefore$ $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$

$\Rightarrow f(x_2) - f(x_1) < 0$

$\Rightarrow f(x_1) > f(x_2)$

(ii) If $a < x_1 < x_2 < b$

Then, $\dfrac{[f(x_2) - f(x_1)]}{x_2 - x_1} = f'(c) \geq 0$ for some $c \in (a, b)$

Therefore, $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$

$\Rightarrow f(x_2) - f(x_1) \geq 0$

$\Rightarrow f(x_1) \leq f(x_2)$

(iii) if $a < x_1 < x_2 < b$

Then, $\dfrac{[f(x_2) - f(x_1)]}{x_2 - x_1} = f'(c) \leq 0$ for some $c \in (x)$

Therefore, $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$

$\Rightarrow f(x_1) - f(x_1) \leq 0$

$\Rightarrow f(x_1) \geq f(x_2)$

## 3.3 L – Hospital Rule

If $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m((\neq 0)$

Then by Quotient theorem of limits we have $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$. However, if $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$.

Then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ takes the form $\dfrac{0}{0}$.

In this case lim(f/g) is said to be indeterminate. Depending on that particular functions f, g the limit may be a real number or may not exist.

Also, if $\lim\limits_{x \to 0} f(x) = \infty$ and $\lim\limits_{x \to 0} g(x) = \infty$

Then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ takes the form $\dfrac{\infty}{\infty}$.

The forms $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ taken by the above limits are called indeterminate forms.

**26. State and prove L - Hospital Rule I**

**(OR)**

**Let f, g are derivable on (a, a + h) such that**

**(i) $g'(x) \neq 0 \ \forall \ x \in (a, a + h)$,**

**(ii) $\lim\limits_{x \to a^+} f(x) = 0 = \lim\limits_{x \to a^+} g(x)$**

**(a) If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = l$, a real number the $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = l$.**

**(b) If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \pm \infty$ then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \pm \infty$**

*Sol.* **(Imp.)**

Let $a < \alpha < \beta < a + h$

$g'(x) \neq 0, \ \forall x \in (a, a + h) \Rightarrow g(\alpha) \neq g(\beta)$

Using Cauchy mean value theorem,

for f, g in $[\alpha, \beta]$

we have that there exists $u \in (\alpha, \beta)$

Such that $\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(u)}{g'(u)}$ ...(1)

**Case (i)**

$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = l \Rightarrow$ given $\varepsilon > 0$

There exists $\delta > 0$ such that

$\left| \frac{f'(x)}{g'(x)} - l \right| < \varepsilon$ for $a < x < a + \delta < a + h$

$\Rightarrow \quad l - \varepsilon < \frac{f'(u)}{g'(u)} < l + \varepsilon$

for $a < u < a + \delta$

$\Rightarrow \quad l - \varepsilon < \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} < l + \varepsilon$ for $a < \alpha < \beta < a + \delta$

keeping $\beta$ fixed

proceeding to the limit as $\alpha \to a^+$ to the above inequality

we have,

$l - \varepsilon < \frac{f(\beta)}{g(\beta)} < l + \varepsilon$ for $a < \beta < a + \delta$

Since $\varepsilon > 0$ is arbitrary $\lim_{x \to a^+} \frac{f(x)}{g(x)} = l$

**Case (ii)**

$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \infty \quad \Rightarrow \quad$ for $G > 0$

There exists $\delta > 0$ such that

$$\frac{f'(x)}{g'(x)} > G \text{ for } a < x < a + \delta$$

$$\Rightarrow \quad \frac{f'(u)}{g'(u)} > G \text{ for } a < u < a < a + \delta$$

$$\Rightarrow \quad \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} \;.\; G \text{ for } a < \alpha < \beta < a + \delta$$

keeping $\beta$ fixed, proceeding to the limit as $\alpha \to a+$, we have $\frac{f(\beta)}{g(\beta)} > G$ for $a < \beta < a + \delta$

Since $G > 0$ is arbitrary

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = +\alpha$$

The argument is similar for

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = -\alpha$$

**27. State and prove L - Hospital Rule II :**

**(OR)**

**If f, g are derivable in a deleted nbd of 'a'**

$$\lim_{x \to a^+} f(x) = \pm\infty, \; \lim_{x \to a^+} g(x) = \pm\infty \text{ and } \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = l \text{, then } \lim_{x \to a^+} \frac{f(x)}{g(x)} = l$$

*Sol.* **(Imp.)**

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = l \Rightarrow \text{ for a given } \varepsilon > 0$$

There exists

$\delta > 0$ such that $\left| \frac{f'(y)}{g'(y)} - l \right| < \frac{\varepsilon}{3}$

whenever $a < y < a + \delta$

Let $a + \left(\frac{\delta}{2}\right) = x_0$ so that $a < x < x_0 < a + \delta$

clearly f, g are continuous on $[x, x_0]$ and derivable on $(x, x_0)$

Also, $g'(t) \neq 0, \; \forall \; t \in (x, x_0)$

By Cauchy mean value theorem

There exists $y \in (x, x_0)$

Such that $\frac{f'(y)}{g'(y)} = \frac{f(x_0)-f(x)}{g(x_0)-g(x)}$

$$= \frac{f(x)}{g(x)}\left\{\frac{1-\frac{f(x_0)}{f(x)}}{1-\frac{g(x_0)}{g(x)}}\right\}$$

$$\Rightarrow \quad \frac{f(x)}{g(x)} = \frac{f'(y)}{g'(y)}\left\{\frac{1-\frac{g(x_0)}{g(x)}}{1-\frac{f(x_0)}{f(x)}}\right\}$$

But $\lim_{x\to a^+}\frac{g(x_0)}{g(x)} = g(x_0)\lim_{x\to a^+}\frac{1}{g(x)}$

$= 0$

and $\lim_{x\to a^+}\frac{f(x_0)}{f(x)} = 0$

$$\therefore \lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(y)}{g'(y)} \times \lim_{x\to a^+}\left\{\frac{1-\frac{g(x_0)}{g(x)}}{1-\frac{f(x_0)}{f(x)}}\right\}$$

$= l \times 1 = l$

**28. Calculate $\lim_{x\to 0}\frac{\sin x}{x}$ by using L'Hospital Rule.**

*Sol.* **(Imp.)**

Given $\lim_{x\to 0}\frac{\sin x}{x}$

Note that $f(x) = \sin x$ and $g(x) = x$

$$\lim_{x\to 0}\frac{\sin x-\sin 0}{x-c} = \lim_{x\to 0}\frac{\sin x}{x} = 1$$

**29. Calculate $\lim_{x\to 0}\frac{\cos x-1}{x^2}$ By L'Hospitals Rule.**

*Sol.*

Given that $\lim_{x\to 0}\frac{\cos x-1}{x^2}$

$f(x) = \cos x - 1 \quad \Rightarrow \quad f'(x) = -\sin x$

$g(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$

$$\Rightarrow \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{-\sin x}{2x}$$

$$= \lim_{x\to 0}\frac{-1}{2}\frac{\sin x}{x}$$

$$= \frac{-1}{2}\lim_{x\to 0}\frac{\sin x}{x}$$

$$= \frac{-1}{2}(1)$$

$$= \frac{-1}{2}$$

$$\therefore \lim_{x\to 0}\frac{\cos x-1}{x^2} = \frac{-1}{2}$$

**30. Find the limit for $\lim_{x\to 0}\frac{1-\cos x}{x^2}$**

*Sol.*

Given that $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{1-\cos x}{x^2}$

Hence $f(x) = 1 - \cos x$

$g(x) = x^2$

$f(0) = 1 - \cos(1) = 1 - 1 = 0$

$g(0) = 0^2 = 0$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \left(\frac{0}{0}\text{form}\right)$$

f(x), g(x) are derivable in a nbd of '0' and $f'(x) = \sin x$

$g'(x) = 2x$

again $f'(0) = 0$, $g'(0) = 0$

$\therefore \quad \lim_{x\to 0}\frac{f'(x)}{g'(x)}$ is in $\frac{0}{0}$ form.

$f'(x), g'(x)$ are differentiable in nbd of '0' and $f''(x) = \cos x$, $g''(x) = 2$

$$= \lim_{x\to 0}\frac{f''(x)}{g''(x)} = \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$$

---

**31. Find the limit for $\lim_{x\to 0}\frac{\tan x - x}{x^3}$.**

*Sol.* **(Imp.)**

Given that $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\tan x - x}{x^3}$

Here, $f(x) = \tan x - x$

$g(x) = x^3$

$f(0) = \tan(0) - 0 = 0$

$g(0) = 0$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \left(\frac{0}{0}\text{form}\right)$$

f(x), g(x) are differentiable in a nbd of '0' and

$f'(x) = \sec^2 x - 1$

$g'(x) = 3x^2$

$f'(0) = \sec^2(0) - 1 = 0$

$g'(0) = 0$

$$\lim_{x\to 0^+}\frac{f(x)}{y(x)} = \lim_{x\to 0^+}\frac{\tan^2 x}{3x^2}$$

$$= \frac{1}{3}\lim_{x\to 0^+}\left(\frac{\tan x}{x}\right)^2$$

$$= \frac{1}{3}(1)^2$$

$$= \frac{1}{3}$$

**32. Find the limit for $\lim_{x\to 0}\frac{e^{2x} - \cos x}{x}$.**

*Sol.*

Given that $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{e^{2x} - \cos x}{x}$

$f(x) = e^{2x} - \cos x \Rightarrow f'(x) = 2e^{2x} - (-\sin x)$

$g(x) = x \Rightarrow g'(x) = 1$

$f(0) = e^{2(0)} - \cos(0)$

$= 1 - 1 = 0$

$g(0) = 0$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{0}{0}\text{(form)}$$

and $\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \frac{2e^{2x} + \sin x}{1}$

$f'(0) = 2e^{2(0)} + \sin(0) = 2(1) + 0 = 2$

$g'(0) = 1$

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \frac{2}{y} = 2$$

---

**33. Find the limit for $\lim_{x\to 0}\frac{x^3}{e^{2x}}$.**

*Sol.*

Given that $\lim_{x\to 0}\frac{x^3}{e^{2x}}$

$f(x) = x^3 \Rightarrow f(0) = 0$

$g(x) = e^{2x} \Rightarrow g(0) = e^{2(0)} = e^0 = 1$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{0}{1} = 0$$

---

**34. Find $\lim_{x\to 0}\frac{x^3}{\sin x - x}$.**

*Sol.*

Given that $\lim_{x\to 0}\frac{x^3}{\sin x - x}$

$f(x) = x^3 \quad \Rightarrow f(0) = 0$

$g(x) = \sin x - x \quad \Rightarrow g(0) = \sin(0) - 0 = 0$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \left(\frac{0}{0}\text{form}\right)$$

f(x), g(x) are derivable in a nbd of '0'

and $f'(x) = = 3x^2$

$g'(x) = \cos x - 1$

again, $f'(0) = 3(0)^2 = 0$

$g'(0) = \cos(0) - 1 = 1 - 1 = 0$

f'(x), g'(x) are differentiable is a nbd of '0'

and $f''(x) = 2x$

$g''(x) = -\sin x$

again $f''(0) = 2(0) = 0$

$g''(0) = -\sin(0) = 0$

f''(x), g''(x) are differentiable in a nbd of

and $f'''(x) = 2$

$g'''(x) = -\cos x$

again $f'''(0) = 2$

$g'''(0) = -\cos(0) = -1$

$$\lim_{x\to 0}\frac{f'''(x)}{g'''(x)} = \frac{2}{-1} = -2.$$

---

**35. Find limit $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x$.**

*Sol.* **(Imp.)**

The limit $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x$ is indeterminate of the form $1^\infty$.

Since $\left(1-\frac{1}{x}\right)^x = e^{x\log\left(1-\frac{1}{x}\right)}$

evaluate

$$\lim_{x\to\infty} x\log\left(1-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\log\left(1-\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\left(1-\frac{1}{x}\right)^{-1} x^{-2}}{-x^{-2}}$$

$$= \lim_{x\to\infty} -\left(1-\frac{1}{x}\right)^{-1} = -1$$

We have $= \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x} = e^{-1}$

## 3.4 Taylor's Theorem

Let f be a function defined on some open interval containing 0. If f possess derivatives of all orders at 0, then the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ is called the taylor's series for f about 0.

The remainder $R_n(x)$ is defined by

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k$$

for any x,

$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ if and only if

$$\lim_{x\to\infty} R_n(x) = 0$$

**36. State and prove Taylor's Theorem**

**Let [a, b] $\to$ R such that**

**(i) f and its sucessive derivative f', f'',....$f^{(n)}$ ($n \in N$) are continuous on [a, b] and**

**(ii) $f^{(n+1)}$ exist on (a, b). If $x_0 \in$ [a, b]**

**Then for any $x \in$ [a, b]. There exits a point 'c' between x and $x_0$ such that**

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \ldots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!} f(x+1)(c)$$

*Sol.*

for a given $x_0, x \in [a, b]$

let I = $[x_0, x]$ or $[x, x_0]$ according a $x_0 < x$ or $x_0 > x$.

Define F : I $\to$ R as

$$F(t) = f(x) - f(t) - \frac{(x-t)}{1!} f'(t) .... \frac{(x-t)^n}{n!} f^{(n)}(t) - A\left(\frac{x-t}{x-x_0}\right)^{n+1} \quad \forall\, t \in I \qquad ...(1)$$

where A is a real numbers choosen that $F(x_0) = F(x)$.

$F(x_0) = F(x) \Rightarrow f(x) - f(x_0) - \frac{(x-x_0)}{1!} f'(x_0) ... \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) - A = 0$ ...(2)

from (1) & (2) f, f', f''...$f^{(n)}$ are continuous on [a, b] $\Rightarrow$ f, f', f'''...$f^{(n)}$ are continuous on $I \subset [a, b]$

$f^{(n+1)}$ exist on [a, b] $\Rightarrow$ $f^{(n+1)}$ exists on I

further,

The polynomial in t,

namely, $(x - t), (x - t)....(x - t)^n$ and $\left(\frac{x-1}{x-x_0}\right)^{n+1}$ are continuous and derivable on I

$\therefore$ F(t) is continuous and derivable on I further $F(x_0) = F(x)$.

By Rolle's Theorem,

There exits 'c' between x and $x_0$ such that F' (c) = 0

But for $t \in I$, $F'(t) = -f'(t) - \{(-1) f'(t) + (x-t) f''(t)\} .... \left\{\frac{-n(x-t)^{n-1}}{n!} f^{(n)}(t) + \frac{(x-t)^n}{n!} f^{(n+1)}(t)\right\}$

$$-\frac{A(-1)(n+1)(x-t)^n}{(x-x_0)^{n+1}}$$

$$\Rightarrow F'(t) = \frac{-(x-t)^n}{n!} f^{(n+1)}(t) + \frac{A(n+1)(x-t)^n}{(x-x_0)^{n+1}}$$

$$F(c) = 0 \Rightarrow \frac{-(x-c)^n}{n!} f^{(n+1)}(c) + \frac{A(n+1)(x-c)^n}{(x-x_0)^{n+1}} = 0$$

$$\Rightarrow A = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

from (2)

$$f(x) = f(x_0) + (x-x_0) f'(x_0) + ... + \frac{(x-x_0)^n}{n!} f^{(n)} x_0 + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n)}(c)$$

**Notation :**

We denote $p_n(x) = f(x_0) + (x-x_0) f'(x_0) + ... + \frac{(x-x_0)}{n!} f^{(n)}(x_0)$ and

$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$, where 'C' is a point between x and $x_0$

Then $f(x) = P_n(x) + R_n(x) \Rightarrow f(x) - P_n(x) = R_n(x)$

$P_n(x)$ is called the $n^{th}$ Taylor polynomial for f at $x_0$.

$R_n(x)$ is called the Lagranges form of Remainder.

**37. Let f be defined on (a, b) where a < 0 < b, and suppose the $n^{th}$ derivative $f^{(n)}$ exists and is continuous on (a, b) then for $x \in$ (a, b) we have**

$$R_n(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt$$

*Sol.*

for n = 1, equation (1) assets

$$R_1(x) = f(x) - f(0) = \int_0^x f'(t)\, dt$$

for $n \geq 2$

we repeatedly apply integration by parts

i.e., we use mathematical induction assume (1) holds for some n.

$n \geq 1$

we evaluate the integral in (1) using $u(t) = f^{(n)}(t)$, $v'(t) = \frac{(x-t)^{n-1}}{(n-1)!}$

So that $u'(t) = f^{(n+1)}(t)$ and $v(t) = -\frac{(x-t)^n}{n!}$

we obtain

$$R_n(x) = u(x)v(x) - u(0)\, v(0) - \int_0^x v(t)\, u'(t)\, dt$$

$$= f^{(n)}(x).0 + f^{(n)}(0)\frac{x^n}{n!} + \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}\, dt \qquad ...(2)$$

Hence from (2) we see that (1) holds for n + 1.

**38. If f is defined on (a, b) then for each x in (a, b) different from 0 there is some y between 0 and x such that**

$$R_n(x) = \frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y)x$$ **. This form of $R_n$ is known as cauchy's form of the remainder.**

*Sol.*

Suppose x < 0

The case x > 0

The intermediate value theorem for integrals show that

$$\int_x^0 \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt = [0-x]\frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}\, y \qquad ...(2)$$

for some y in (x, 0)

Since the integral in (2) equals $-R_n(x)$ and formula (1) holds.

The Binomial theorem tells us that $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)...(n-k+1)}{k!} \text{ for } 1 \le k \le n$$

Let $a = x$ and $b = 1$

Then

$$(1 + x)^n = 1 + \sum_{k=1}^{n} \frac{n(n-1)...(n-k+1)}{k!} x^k$$

**39. State and prove Binomial Series Theorem :**

**If $\alpha \in R$ and $|x| < 1$ Then**

$$(1 + x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} x^k$$

*Sol.* **(Imp.)**

for $k = 1, 2, 3 ...$

let $a_k = \dfrac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}$ ...(1)

If $\alpha$ is a non negative integer then $a_k = 0$ for $k > \alpha$.

and (1) holds for all x as noted in our discussion prior to this theorem.

Hence forth we assume $\alpha$ is not a

non negative integer so that $a_k \neq 0$

for all k

Since,

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{\alpha - k}{k+1}\right| = 1$$

The series in (1) has radius of convengence.

Likewise $\Sigma k a_k x^{k-1} = 0$ convenges for $|x| < 1$

Hence $\lim_{x\to\infty} n a_n x^{n-1} = 0$ for $|x| < 1$

let $f(x) = (1 + x)^\alpha$ for $|x| < 1$ for $n = 1, 2 ...$

we have

$f^{(n)}(x) = \alpha(\alpha - 1) ...(\alpha - n + 1) (1 + x)^{\alpha - n}$

$= n!\, a_n (1 + x)^{\alpha - n}$

Thus $f^{(n)}(0) = n!\, a_n$ for all $n \geq 1$ and the series in(1) is the taylor series for f

$$R_n(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} n!\, a_n (1 + t)^{\alpha - n}\, dt$$

$$= \int_0^x n a_n \left[\frac{x-t}{1+t}\right]^{n-1} (1 + t)^{\alpha - 1}\, dt \qquad ...(3)$$

for $|x| < 1$

It is easy to show that

$$\left|\frac{x-t}{x+t}\right| \leq |x| \text{ if } -1 < x \leq t \leq 0 \text{ or } 0 \leq t \leq x < 1$$

To see this, note that $t = xy$ for some $y \in [0, 1]$, So

$$\left|\frac{x-t}{1+t}\right| = \left|\frac{x-xy}{1+xy}\right| = |x| \left|\frac{1-y}{1+xy}\right| \leq |x|$$

Since $1 + xy \geq 1 - y$

Thus the integrade in (3) is bounded by $n|a_n|.|x|^{n-1}.(1 + t)^{\alpha-1}$

$\therefore \quad |R_n(x)| \leq n|a_n|.|x|^{n-1} \int_{-|x|}^{|x|} (1 + t)^{\alpha - 1}\, dt$

Applying (2), we now see that $\lim\limits_{x \to \infty} R_n(x)$

for $|x| < 1$ equation (1) holds good

**40. Expassion of $e^x$.**

*Sol.* **(Imp.)**

domain of $e^x$ is R.

let $f(x) = e^x \; \forall x \in R$

we know that $f^{(n)}(x) = e^x$

$\therefore \quad f^{(n)}(0) = e^0 = 1 \quad \forall n \in N$

Further $f^{(n)}(x) = e^x \quad \forall x \in R$ and $r \in N$

$\therefore \quad$ f has continuous derivative of every order on [–h, h]

Lagrange's form of remainder

$R_n(x) = \frac{x^n}{n!} f^{(n)}(\theta x)$ where $0 < \theta < 1$

$= \frac{x^n}{n!} e^{\theta x}$; $0 < \theta < 1$

But $\lim_{x \to \infty} \frac{x^n}{n!} = 0 \quad \forall x \in R$

$\therefore \lim_{n \to \infty} R_n(n) = \lim_{n \to \infty} \frac{x^n}{n!} = \lim_{n \to \infty} e^{\theta x}$

$= 0.e^{\theta x}$

$= 0$

$\therefore$ $f(x) = e^x$ has maclarian series expassion $\forall x \in [-h, h]$

$\therefore$ for all $x \in R$, $e^x = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + .... + \frac{x^n}{n!} f^{(n)}(0) + ...$

$= 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

# Choose the Correct Answer

1. f(x) is strictly increasing at x = a then [ a ]

   (a) f'(a) > 0 (b) f'(a) < 0

   (c) f'(a) ≥ 0 (d) f'(a) = 0

2. If $\lim_{h\to 0^+} \frac{f(a+b)-f(a)}{h} = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$ then at x = a, f(x) [ d ]

   (a) is continuous (b) exists

   (c) is a constant (d) is differentiable

3. $\frac{d}{dx}\tan^{-1}\left[\frac{2x}{1-x^2}\right] =$ [ b ]

   (a) 2 (b) $\frac{2}{1+x^2}$

   (c) $\frac{1}{1+x^2}$ (d) none

4. f(x) = tanx is differentiable at every point in[ b ]

   (a) R (b) $R - \left\{(2n+1)\frac{\pi}{2} / n \in z\right\}$

   (c) $R - \left\{\frac{n\pi}{n \in z}\right\}$ (d) $R^+$

5. The derivative of x|x| for x ∈ R is [ c ]

   (a) 2x (b) – 2x

   (c) 2|x| (d) none

6. If f and g are functions that are differentiable at point ‘a’ (f + g)' (a) = [ b ]

   (a) f'(a) g'(a) (b) f(a) + g'(a)

   (c) f'(a) – g'(a) (d) $\frac{f'(a)}{g'(a)}$

7. The function $e^x$ on R is [ b ]

   (a) increasing (b) strictly increasing

   (c) strictly decreasing (d) continuous

8. The function cosx on $[0, \pi]$ is [ b ]

(a) increasing (b) continuous

(c) differentiable (d) strictly decreasing

9. If $f(x) = x^3$. Then $f'(x) =$ [ a ]

(a) $3x^2$ (b) $3x$

(c) 3 (d) none

10. $\lim_{x \to a} x^k =$ [ c ]

(a) k (b) a

(c) $a^k$ (d) $k^2$

# Fill in the blanks

1. $\lim_{x\to\infty} x \tan\frac{1}{x}$ = ______ .

2. If f is continuous on [a, b] and differentiable on (a, b) and $f'(x) = 0$ for $a < x < b$ then f is ______ on [a, b].

3. If $\lim_{x\to a} f(x).g(x)$ exits then both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ = ______ .

4. If $f(x) = x^2 - 4x - 2$ then f(x) is increasing on ______ and decreasing on ______ .

5. In Taylor's Theorem, Lagrange's form of remainder is ______ .

6. The $\lim_{x\to 0^+} x^x$ is of the indeterminate form ______ .

7. The derivative of $f(x) = x + 2$ at $x = a$ is ______ .

8. The domain of 'f' is set of points at which f is ______ .

9. If f is differentiable at a point as then f is ______ at 'a'.

10. If f is differentiable function on an interval (a, b) then strictly increasing if ______ .



## Answers

1. 1
2. constant
3. need not exist
4. $(2, \infty)$, $(-\infty, 2)$
5. $\frac{1}{3}$
6. $e^{-1}$
7. 1
8. differentiable
9. continuous
10. $f'(x) > 0$

# UNIT IV

**Integration :** The Riemann Integral - Properties of Riemann Integral-Fundamental Theorem of Calculus.

## 4.1 Integration

### 4.1.1 Partition of a Closed Interval

Let I = [ab] be a finite closed interval. If $a = x_0 < x_1 < x_2 < ..... < x_n = b$, then the finite set $p = \{x_0, x_1, x_2, ....., x_n\}$ is called a partition of [ab].

The n + 1 points $x_0, x_1, x_2, ....., x_n$ are called partition points of p.

The n sub-intervals $[x_0, x_1], [x_1, x_2], ..... [x_{r-1}, x_r] ..... [x_{n-1}, x_n]$ are called the segments of the partition p and the union of these n subintervals is equal to the closed interval [a, b].

The $r^{th}$ subinterval $[x_{r-1}, x_r]$ is denoted by $I_r$ and its length $= x_r - x_{r-1}$ is denoted by $\delta_r$.

A closed interval [ab] can be partitioned in infinitely many ways. The set of all partitions of [ab] is denoted by $\phi$[ab].

### 4.1.2 Norm of a Partition

The maximum of the lengths of the sub intervals of a partition p is called the norm of the partition p and is denoted by $\|p\|$.

Thus norm $p = \|p\| = \max.\{\delta_1, \delta_2, ....., \delta_r, ..... \delta_n\}$

where $\delta_1, \delta_2, ....., \delta_r, ..... \delta_n$ are the length of the n-subintervals

### 4.1.3 Refinement of a Partition

If $P_1$, $P_2$ be two partitions of [ab] and $P_1 \subset P_2$, then the partition $P_2$ is called a refinement of partition $P_1$ on [ab] (or) $P_2$ is finer than $P_1$

Thus, if $P_2$ is finer than $P_1$, then every point of $P_1$ is a point of $P_2$ and $P_2$ has some more points.

If $P_1, P_2 \in \phi[ab]$ and $P_1 \subset P_2$ then $\|P_2\| \leq \|P_1\|$

**Note :**

$$\sum_{r=1}^{n} \delta_r = \delta_1 + \delta_2 + ..... + \delta_n = (x_1 - x_0) + (x_2 - x_1) + ..... + (x_n - x_{n-1}) = x_n - x_0 = b - a.$$

### 4.1.4 Upper and Lower Riemann Sums

Let $f : [ab] \to R$ be a bounded function and $P = \{a = x_0, x_1, x_2, ..... x_n = b\}$ be a partition of [ab].

Since f is bounded on [ab], f is also bounded on each of the subintervals.

Let M and m be the supremum and infimum of in [ab] and $M_r$, $m_r$ be the supremum and infimum of f in the $r^{th}$ subinterval. $I_r = [x_{r-r}, x_r]$ $\forall$ r = 1, 2, 3, ....., n. The sums $M_1\delta_1 + M_2\delta_2 + ..... + M_r\delta_r + ..... + M_n\delta_n + = \sum_{r=1}^{n} M_r\delta_r$ is called the upper Riemann Sum and is denoted by U(P, f) and read as upper Riemann sum for the f w.r.t partition P.

Similarly, the sums $m_1\delta_1 + m_2\delta_2 + ..... + m_r\delta_r + ..... + m_n\delta_n = \sum_{r=1}^{n} m_r\delta_r$ is called the lower Riemann sum and is denoted by L(P, f) and read as lower Riemann sum for the function function and w.r.t partition P.

### 4.1.5 Oscillatory Sum

Let $f : [ab] \to R$ be a bounded function and $P = \{a = x_0, x_1, x_2, ....., x_n = b\}$ be a partition of [ab].

Let $m_r$ and $M_r$ be the infimum and supremum of on $I_r = [x_{r-1}, x_r]$ $\forall$ r = 1, 2, 3, ....., n then U(p,

$$U(p, f) - L(p, f) = \sum_{r=1}^{n} M_r\delta_r - \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} (M_r - m_r)\,\delta_r$$

is called the oscillatory sum of f w.r.t partition P and is denoted by W(p, f).

$\therefore$ W(p, f) = U(p, f) – L(p, t)

$$= \sum_{r=1}^{n} (M_r - m_r)\,\delta_r.$$

**Note :**

1. If $f : [a, b] \to R$ be a bounded function and $p \in \phi$ [ab] then
   (i) $U(p, f) \geq L(p, f)$
   (ii) $U(p, -f) = -L(p, f)$
   (iii) $L(p, -f) = -U(p, f)$

### 4.1.6 Lower and Upper Riemann Integrals

Let $f : [ab] \to R$ be a bounded function and $p = \{a = x_0, x_1, x_2, \ldots.., x_n = b\}$ be a partition of [ab].

Then the lower Riemann integral of f on [ab] is defined as sup $\{L(p, f) \mid p \in \phi$ [ab]$\}$ and is denoted by $\underline{\int}_a^b f(x)\,dx$ i.e., $\underline{\int}_a^b f(x)\,dx = \sup\{L(p, f) \mid p \in \phi\ [ab]\}$.

Similarly, the upper Riemann integral of f on [ab] is defined as infimum $\{U(p, f) \mid p \in \phi$ [ab]$\}$ and is denoted by $\underline{\int}_a^b f(x)\,dx$.

i.e., $\underline{\int}_a^b f(x)\,dx = \text{infimum}\{U(p, f) \mid p \in \phi\ [ab]\}$.

**Note :**

Let $f : [ab] \to R$ be a bounded function then for every $p \in \phi$ [ab] we have $m(b - a) \leq L(p, f) \leq U(p, f) \leq M(b - a)$ where m and M are infimum and supremum of f on [ab].

Since $L(p, f) \leq M(b - a)$

$$\Rightarrow \underline{\int}_a^b f(x)\,dx = \sup\{L(p, f) \mid p \in [ab]\} \leq M(b - a)$$

Since $U(p, f) \geq m(b - a)$

$$\Rightarrow \overline{\int}_a^b f(x)\,dx = \text{infimum}\{U(p, f) \mid p \in [ab]\} \geq m(b - a)$$

## 4.2 Riemann Integral

Let $f : [ab] \to R$ be a bounded function and $p = \{a = x_0, x_1, \ldots.., x_n = b\}$ be a partition of [ab].

If $\underline{\int}_a^b f(x)\,dx =$ then f is said to be Riemann integral on [ab].

i.e., $\underline{\int}_a^b f(x)\,dx = \overline{\int}_a^b f(x)\,dx = \int_a^b f(x)\,dx$

**1. If $f : [ab] \to R$ is a bounded function then $\underline{\int}_a^b f(x)\,dx \leq \overline{\int}_a^b f(x)\,dx$.**

*Sol.* **(Dec.-17)**

Let $P_1, P_2 \in \phi$ [ab].

$\Rightarrow$ $L(p_1, f) \leq U(p_2, f)$ which is true for each $p_1 \in \phi$ [ab]

$\therefore$ The set of lower sums has an upper bound $U(p_2, f)$ we know that

$$\underline{\int}_a^b f(x)\,dx = \sup\{L(p, f) \mid p_1 \in \phi\ [ab]\}$$

But supremum $\leq$ Any upper bound

$$\therefore \underline{\int}_a^b f(x)\,dx \leq U(p_2\ f)$$

$$\Rightarrow U(p_2, f) \geq \underline{\int}_a^b f(x)\,dx \quad \forall\ p_2 \in \phi\ [ab]$$

$\therefore \quad \underline{\int_a^b} f(x)\,dx$ is a lower bounded of the set of all upper sums.

$\because \quad \overline{\int_a^b} f(x)\,dx = \text{ infimum } \{U(p_2, f) \mid p_2 \in \phi\,[ab]\}$

But any lower bound $\leq$ Infimum, we get

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$$

**Note :**

By definition of lower and upper Riemann integral $\forall\ p \in \phi\,[ab]$, $L(p, f) \leq \underline{\int_a^b} f(x)\,dx$ and $\overline{\int_a^b} f(x)\,dx \leq U(p, f)$

$$\therefore \quad L(p, f) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U(p, f)\ \forall\ p \in \phi\,[ab]$$

Also $m(b - a) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq M(b - a)$.

**2. A constant Function is Riemann Integrable on [ab].**

*Sol.*

Let $f(x) = K\ \forall\ x \in [ab]$

Where K is a constant function.

Clearly f is bounded on [a, b] and infimum f = K and Sup. f = K

Let $P = \{a = x_0, x_1, x_2, ....., x_n = b\}$ be a partition on [ab].

Let $m_r$, $M_r$ be the infimum and sup. of f on $I_r = [x_{r-1}, x_r]$

$\because \quad f(x) = K\ \forall\ x \in [a\,b],\ m_r = M_r = K$

Now consider

$$L(p, f) = \sum_{r=1}^{n} M_r\,\delta_r = K \sum_{r=1}^{n} \delta_r = K(b - a)$$

and $$U(p, f) = \sum_{r=1}^{n} M_r\,\delta_r = K \sum_{r=1}^{n} \delta_r = K(b - a)$$

$\Rightarrow \quad L(p, f) = U(p, f) = K\,(b - a)$ which is a constant.

Consider

$$\underline{\int_a^b} f(x)\,dx = \text{Sup.}\,\{L(p, f) \mid p \in \phi\,[ab]\}$$

$$= K(b - a)$$

Similarly

$$\overline{\int_a^b} f(x)\,dx = \text{infimum}\,\{U(p, f) \mid p \in \phi\,[ab]\}$$

$$= K(b - a)$$

$$\therefore \quad \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = K(b - a)$$

$\therefore$ f is Riemann integrable on [a b].

---

**3. If $f \in R\,[a\,b]$ and m, M are the infimum and Supremum of f on [a b], then $m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$.**

*Sol.*

Let $f \in R\,[a\,b]$

$$\Rightarrow \quad \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \qquad \ldots (1)$$

Let $p = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition on [ab] and $m_2$, $M_2$ be the infimum and supremum of f on $[x_{r-1}, x_r]$.

Then we have

$$m \leq m_r \leq M_r \leq M \quad \forall\ r = 1, 2, 3, \ldots n$$

$$\Rightarrow \quad m\delta_r \leq m_r\delta_r \leq M_r\delta_r \leq M\delta_r \quad \forall\ r = 1, 2, 3, \ldots n$$

Adding these n inequalities

$$\Rightarrow \quad \sum_{r=1}^{n} m\,\delta_r \leq \sum_{r=1}^{n} m_r\,\delta_r \leq \sum_{r=1}^{n} M_r\,\delta_r \leq \sum_{r=1}^{n} M\delta_r$$

$$\Rightarrow \quad m(b - a) \leq L(p, f) \leq U(p, f) \leq M(b - a)$$

$$\because \quad \underline{\int_a^b} f(x)\,dx = \sup\,\{L(p, f) \mid p \in \phi\,[ab]\}$$

and $\underline{\int_a^b} f(x)\,dx \geq L(p, f)$

Similarly

$$\overline{\int_a^b} f(x)\,dx = \text{infimum}\,\{U(p, f) | p \in \phi\,[ab]\}$$

and $\overline{\int_a^b} f(x)\,dx \leq U(p, f)$

$$\therefore \quad m(b-a) \leq L(p, f) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U(p, f) \leq M(b-a)$$

$$\Rightarrow \quad m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

$\because$ $f \in R\,[a\ b]$, from (1)

**4. If 'f' is a bounded function on [a, b]. Then prove that $L(f) \leq U(f)$**

*Sol.* **(Dec.-17)**

$P, Q, \in [a, b]$

Since $L(f, p) \leq U(f, Q)$

Keeping P fixed,

The set {L(f, p)/ p is partition of [a, b] / has on upper bound U(f, Q)

also sup{L(f, p)/ p is a partion of [a, b]} = L(f)

Since sup $\leq$ any upper bound.

$L(f) \leq U(f, Q)$

$\Rightarrow$ Now, the set {U(f, Q)/ Q is partition of [a, b]} = U(f)

lower bound $\leq$ Inf

$L(f, p) \leq U(f)$

we know that $L(f, p) \leq U(f, Q)$

$L(f) \leq U(f)$

**5. Prove that every monotonic function on [a, b] is integrable.**

*Sol.* **(May /June-18)**

Given that 'f' is monotonic on [a, b]

Suppose that f is monotic increasing on [a, b]

$\Rightarrow$ $a \leq x \leq b$ $\Rightarrow$ $f(a) \leq f(x) \leq f(b)$

$\Rightarrow$ f(x) is bounded function

$\Rightarrow$ f is bounded on [a, b]

Suppose $f(a) \leq f(b)$

$\varepsilon > 0$, let $P = \{a = t_0 < t_1 < ...< t_{k-1} < t_k < ...< t_n = b|$ be a partition of [a, b], where mesh p.

$$\text{mesh } P < \frac{\varepsilon}{f(b) - f(a)} \qquad ...(1)$$

To prove that f is integrable

we consider

$$U(f, p) - L(f, p) = \sum_{k=1}^{n} M(f, [k_{k-1}, t_k]) (t_k - t_{k-1}) - \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) (t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} [M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) (t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] (t_k - t_{k-1}) \qquad \text{f is increasing f}$$

$$< \sum_{k=1}^{n} f(t_k) - f(t_{k-1}). \frac{\varepsilon}{f(b) - f(a)} \qquad \text{mesh(p)} < \frac{\varepsilon}{f(b) \cdot f(a)}$$

$$< \frac{\varepsilon}{f(b) - f(a)} \sum_{k=1}^{n} f(t_k) - f(t_{k-1})$$

$$< \frac{\varepsilon}{f(b) - f(a)} f(b) - f(a)$$

$= \varepsilon \Rightarrow U(f, p) - L(f, p) < \varepsilon$

f is integrable

**6. Prove that every continuous function defined on [a, b] is integrable.**

*Sol.* **(May /June-18)**

Since f(x) is continuous function on [a, b]

$\Rightarrow$ f(x) is uniformly continuous.

By def: $\forall \varepsilon > 0 \exists \ \delta > 0 \exists |f(x) - f(y)| < \dfrac{\varepsilon}{b - a}$ whenever $|x - y| < \delta$ ...(1)

Let $P = \{a = t_0 < t_1 <...< t_{k-1} < t_k ... < t_n = b\}$ be a partition of [a, b] with mesh $||p|| < \delta$

i.e., $\max (t_k - t_{k-1}) < \delta$

$\therefore$ Since 'f' is continuous on $[t_{k-1}, t_k]$

'f' is continuous on $[t_{k-1}, t_k]$

$\Rightarrow$ f attains its sup & inf in $[t_{k-1}, t_k]$

$\exists\ M_k.m_k \in [t_{k-1}, t_k]$

Sup of f $M_k = f(M_k) = M(f, [t_{k-1}, t_k])$ ...(2)

inf of f $m_k = f(m_k) = m(f,[t_{k-1}, t_k])$

Now, to prove that 'f' is integrable

consider U(f, p) – L (f, p)

$$= \sum_{k=1}^{n}[M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])]\ (t_k - t_{k-1})$$

$$= \sum_{k=1}^{n}[f(M_k) - f(m_k)]\ (t_k - t_{k-1})$$

$$\leq \sum_{k=1}^{n}|f(M_k) - f(m_k)|\ (t_k - t_{k-1})$$

$$< \sum_{k=1}^{n} \frac{\varepsilon}{b-a}(t_k - t_{k-1})$$

$$< \frac{\varepsilon}{b-a}\sum_{k=1}^{n}(t_k - t_{k-1}) \quad \Rightarrow \quad < \frac{\varepsilon}{\cancel{b-a}}\cancel{(b-a)}$$

U(f, p) – L(f, p) < $\varepsilon$

$\therefore$ f is integrable on [a, b]



**7. By the definition Let f: [a, b] $\rightarrow$ R is a bounded function.**

*Sol.*

(i) $U(p, f) < \int_a^b f(x)dx + \in$ and

(ii) $L(p, f) > \int_a^b f(x)dx - \in$ for each $p \in \phi$ [a, b] with $||P|| < \delta$.

Given f(x) = x for rational x = and f(x) = 0 for international x interval [0, b].

$$\int_0^b f(x)\,dx = \text{Infimum } \{U(P, F)|\ p \in \phi\ [0, b]\}$$

$\therefore$ For each $\in > 0, \exists$ a partition $P_1 = \{0 = x_0, x_1, x_2 .... x_n = b\} \ni$

$$U(P, f) < \int_0^b f(x)\,dx + \frac{\in}{2} \qquad ...(1)$$

The partition $P_1$ has (P – 1) points excluding the end points 0 and b choose

$$\delta > 0 \ni 2k(P-1)\delta = \frac{\in}{2} \qquad \text{... (2)}$$

Let P be any partition with $||P|| < \delta$. Thus P may contain some an none of the partition points.

$x_r$, r = 1, 2 .... P – 1 belonging to $P_1$

If $P_2 = P \cup P_1$ then $P_2$ is finer than P and contains.

At the most (P – 1) additional points

$$\therefore \quad U(P, f) - 2k(P-1)\delta \le U(P_2, f) \le$$

$$U(P_1, f) < \int_0^{\bar{b}} f(x)\,dx + \frac{\in}{2} \qquad [\because \text{ from (1)}]$$

$$\Rightarrow \quad U(p, f) < 2k(P-1)\delta + \int_0^{\bar{b}} f(x)\,dx + \frac{\in}{2}$$

$$\Rightarrow \quad U(p, f) < \frac{\in}{2} + \int_0^{\bar{b}} f(x)\,dx + \frac{\in}{2} \qquad [\because \text{ form (2)}]$$

$$\Rightarrow \quad U(p, f) < \int_0^{\bar{b}} f(x)\,dx + \in \text{ for any partition P with } ||P|| < \delta$$

i) $\displaystyle\int_{\bar{0}}^{b} f(x)\,dx = \text{Sup}\{L(P, f)/P \in \phi[0, b]\}$

For each $\in > 0, \exists$a partition

$$P_1 = \{0, x_0, x_1, \ldots x_n = b\} \ni$$

$$L(P_1, f) > \int_{\bar{0}}^{b} f(x)\,dx - \frac{\ni}{2} \qquad \text{... (3)}$$

The partition $P_1$ has (P – 1) points excluding the end points 0 and b choose $\delta > 0 \ni 2k(P-K)$ $\delta = \in/2$ ... (4)

Let P be any partition with $||P|| < \delta$. Thus P may contain some or none of the partition $x_r$, r = 1, 2, 3, ... P – 1 belonging to $P_1$.

If $P_2 = P \cup P_1$ thus $P_2$ is finer than P and contains at most P – 1 additional points.

$\therefore \quad L(p, f) + 2k(P - 1)\delta \geq L(P_2, f) \geq L(P_1, f) > \int_0^b f(x)\,dx - \frac{\in}{2} \quad [\because \text{ from}(3)]$

$\Rightarrow \quad L(p, f) > \int_0^b f(x)\,dx - \frac{\in}{2} - \frac{\in}{2} \quad [\because \text{ from}(4)]$

$\Rightarrow \quad L(p, f) > \int_0^b f(x)\,dx - \in$ for any partition P with $||P|| < \delta$.

b) If f integrable on [0, b]?

$\Rightarrow \quad \int_0^b f(x)\,dx$

$\Rightarrow \quad [x]_0^b \Rightarrow 0 - b = -b$

Integrable.

**8. Given that f is a bounded function on [a, b] their exist sequence ($U_n$) and ($L_n$) upper and lower darboux.**

*Sol.*

Suppose first that f is Darboux integral on [a, b] in the sense that

For each $\in > 0$ and let $\delta > 0$ be chosen so that

$$\left| s - \int_a^b f \right| < \in \quad \ldots (1)$$

For every ricmann sum

$$\int = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$$

associated with a partition P having (P) < $\delta$.

Clearly we have L(f, P) $\leq \int \leq$ U(f, P), so (1) follows from the inequalities.

$$U(f, p) < L(f, P) + \in \leq L(f) \in = \int_a^b f + \in$$

and

$$L(f, P) > U(f, P) - \in \geq U(f) - \in = \int_a^b f - \in$$

Hence f is integrable

$\exists (U_n)$ and $(L_n)$ upper and lower darboux sum

$\therefore \quad Lt(U_n - L_n) = 0$

$$\therefore \int_a^b f(x)dx = \underset{n\to\infty}{Lt} (U_n - L_n) = 0$$

$$\int_a^b f(x)dx = \underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} L_n$$

**9. A function f on [a, b] is called a step function $\exists$ a partition P = {a = $u_0$ < $u_1$ < ... $u_m$ = b} of [a, b] such that f is constant on each interval $(u_{j-1}, u_j)$.**

**Say $f(x) = c_j$ for x in $(u_{j-1}, u_j)$**

**(a) Partition P = {a = $u_0$ < $u_1$ < ... $u_m$ = b}**

*Sol.*

Show that step function is f is integrable

$$\therefore \int_a^b f(x)dx = \int_a^b f(x)dx \text{ is increasing}$$

$$\int_a^b f(x)dx < \infty$$

$\therefore$ f is integrable

**(b) $\int_0^4 P(x)dx$**

*Sol.*

$$\int_0^4 P(x)dx = \int_a^b P(x)dx = [x]_0^4 = 4A$$

$$= 4A + 6B$$

A & B two $\int_0^1 + \int_1^2 + \int_2^3 + \int_3^4$

Constant function 1 + 1 + 1 + 1 + 2 = 6B

## 4.3 Properties of Riemann Integral

**10. If $f \in R[ab]$ then $-f \in R[ab]$ and $\int_a^b (-f)(x)\,dx = -\int_a^b f(x)\,dx$.**

*Sol.*

Let $f \in R[ab]$

$$\Rightarrow \quad \underline{\int}_a^b f(x)\,dx = \overline{\int}_a^b f(x)\,dx = \int_a^b f(x)\,dx \quad \ldots (1)$$

Let $p = \{a = x_0, x_1, x_2, \ldots.., x_n = b\}$ be a partition of [ab].

Let $m_r, M_r$ be the infimum and sup. of f on $I_r = [x_{r-1}, x_r]$

$\because$ f is bounded on [ab], –f is also bounded on [ab]

$\therefore$ Infimum $(-f) = -\sup f = -M_r \;\forall\; I_r$ where $r = 1$ to n.

and $\sup(-f) = -\text{Infimum } f = -m_r \;\forall\; I_r$ where $r = 1$ to n.

$$\therefore \quad U(p, f) = \sum_{r=1}^{n} (-m_r)\,\delta_r = -\sum_{2=1}^{n} m_r\,\delta_r = -L(p, f) \text{ and}$$

$$L(p, f) = \sum_{2=1}^{n} (-M_r)\,\delta_r = -\sum_{2=1}^{n} M_r\,\delta_r = -U(p, f)$$

$$\therefore \quad \overline{\int}_a^b (-f)(x)\,dx$$

$$= \inf.\{U(p, -f) \mid p \in \phi[ab]\}$$

$$= \inf.\{-L(p, f) \mid p \in \phi[ab]\}$$

$$= -\sup\{L(p, f) \mid p \in \phi[ab]\}$$

$$= \underline{\int}_a^b f(x)\,dx$$

$$= -\int_a^b f(x)\,dx \quad \ldots (2) \quad \text{from (1)}$$

Similarly

$$\underline{\int}_a^b (-f)(x)\,dx$$

$$= -\sup\{L(p, -f) \mid p \in \phi[ab]\}$$

$= \sup \{-U(p, -f) \mid p \in \phi [ab]\}$

$= -\inf \{U(p, -f) \mid p \in \phi [ab]\}$

$= -\int_{\bar{a}}^{b} f(x)\, dx$

$= -\int_{a}^{b} f(x)\, dx$ ... (3) from (1)

$\therefore$ from (2) and (3) we get

$$\int_{a}^{\bar{b}} (-f)(x)\, dx = \int_{\bar{a}}^{b} (-f)(x)\, dx = -\int_{a}^{b} f(x)\, dx.$$

Hence $(-f) \in R\, [ab]$ and $\int_{a}^{b} (-f)(x)\, dx$

$$= -\int_{a}^{b} f(x)\, dx$$

**11. If $f \in R\,[a\, b]$ and $K \in R$, then $K f \in [a\, b]$ and $\int_{a}^{b} (K f)(x)\, dx = K \int_{a}^{b} f(x)\, dx$.**

*Sol.*

Let $f \in R\, [a\, b]$

$\Rightarrow \int_{\bar{a}}^{b} f(x)\, dx = \int_{a}^{\bar{b}} f(x)\, dx = \int_{a}^{b} f(x)\, dx)$ ... (1)

Since $K \in R \Rightarrow K \geq 0$ and $K < 0$.

**Case (i)**

Let $K \geq 0$

Let $p = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be a partition of $[a\, b]$

Let inf. $f = m_r$ and sup. of $f = M_r \;\forall\; I_r$ where $r = 1$ to $n$.

$\because$ f is bounded on $[a\, b]$

$\Rightarrow$ Kf is bounded on $[a\, b]$

$\therefore$ Inf. $(K f) = K$ inf. $f = Km_r \;\forall\; r = 1$ to $n$.

and sup. $(K f) = K$ sup. $f = KM_r \;\forall\; r = 1$ to $n$.

Consider

$$U(p, K f) = \sum_{r=1}^{n} (KM_r)\, \delta_r = K\, U(p, f)$$

and

$$L(p, K f) = \sum_{r=1}^{n} (Km_r)\, \delta_r = K\, L(p, f)$$

$\therefore \int_{a}^{\bar{b}} (K f)(x)\, dx$

$= \inf. \{U(p, K f) \mid p \in \phi [ab]\}$

$= K \inf. \{U(p, f) \mid p \in \phi [ab]\}$

$= K \int_{\bar{a}}^{b} f(x)\, dx$

$= K \int_{a}^{b} f(x)\, dx$ ... (2) from (1)

Similarly,

$\int_{\bar{a}}^{b} (K f)(x)\, dx$

$= \sup. \{L(p, K f) \mid p \in \phi [ab]\}$

$= K \sup. \{L(p, f) \mid p \in \phi [ab]\}$

$= K \int_{\bar{a}}^{b} f(x)\, dx$

$= K \int_{a}^{b} f(x)\, dx$ ... (3) from (1)

$\therefore$ From (2) and (3) we get.

$$\int_{\bar{a}}^{b} (K f)(x)\, dx = \int_{a}^{\bar{b}} (K f)(x)\, dx = K\int_{a}^{b} f(x)\, dx$$

$\Rightarrow K f \in R\ [a\, b]$ and $\int_{a}^{b} (K f)(x)\, dx$

$$= K\int_{a}^{b} f(x)\, dx.$$

**Case (ii)**

Let $K < 0$, put $K = -l$ where $l > 0$

$\Rightarrow K f = f(-l)$

$\because$ $f \in R[a\ b] \Rightarrow -f \in R[a\ b]$

By Case (i)

$l > 0 \Rightarrow -f \in R[a\ b] \Rightarrow l(-f) \in R[a\ b]$

$\Rightarrow$ $K f \in R[a\ b]$

Also $\int_a^b (K f)(x)\,dx = \int_a^b l(-f)(x)\,dx = l\int_a^b f(x)\,dx$

$\Rightarrow$ $l(-1)\int_a^b f(x)\,dx = -l\int_a^b f(x)\,dx = K\int_a^b f(x)\,dx$

**12. If $f \in R[a\ b]$ then $|f| \in R[a\ b]$**

*Sol.*

Let $f \in R[a\ b]$

$\Rightarrow$ for a given $\in > 0$, $\exists$ a partition

$p = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$

$\ni 0 \le U(p, f) - L(p, f) < \in$ ... (1)

$\because$ f is bounded on [a, b]

$\Rightarrow$ $|f(x)| < K\ \forall\ K \in R^+$ and $x \in [a, b]$

$\Rightarrow$ $|f|$ is bounded on [a, b]

Let $m_r$, $M_r$ be the inf. and sup. of f on $I_r$ and $m'_r$, $M'_r$ be the inf. and sup. of $|f|$ on $I_r$

Now for each $\alpha, \beta \in I_r$,

$|f(\alpha) - f(\beta)| = ||f(\alpha) - |f(\beta)|| \le |f(\alpha) - f(\beta)|$

$\therefore$ $M'_r - m'_r \le M_r - m_r\ \forall\ r = 1$ to n.

Now

$U(|p|, f) - L(|p|, f)$

$= \sum_{r=1}^{n} (M'_r - m'_r)\,\delta_r$

$\le \sum_{r=1}^{n} (M_r - m_r)\,\delta_r$

$\le U(p, f) - L(p, f)$

$< \in$ from (1)

$\therefore$ $U(|p|, f) - L(|p|, f) < \in$

$\Rightarrow$ $|f| \in R[a\ b]$.

**13. If f, g $\in$ R [a b], then f + g $\in$ R [a b] and $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.**

*Sol.*

Let f, g are bounded on [a b]

$\Rightarrow$ f + g is bounded on [a b]

Let $\in > 0$

$f \in R[a\ b] \Rightarrow \exists\ \delta_1 > 0 \ni U(p_1, f) - L(p_1, f) < \frac{\in}{2}$ with $\|p\| < \delta_1$ ... (1)

and $g \in R[a\ b] \Rightarrow \exists\ \delta_2 > 0 \ni U(p_2, f) - L(p_2, f) < \frac{\in}{2}$ with $\|p_2\| < \delta_2$ ... (2)

Let $p = p_1 \cup p_2$

Then $\|p\| \le \|p_1\|$ or $\|p_2\|$

$\Rightarrow \|p\| < \delta_1$ and $\|p\| < \delta_2$

$\therefore$ (1) and (2) conditions holds for the partition p we know that

$$W(p, f+g) = U(p, f+g) - L(p, f+g)$$
$$\le \{U(p, f) - L(p, f)\} + \{U(p, g) - L(p, g)\}$$
$$< \frac{\in}{2} + \frac{\in}{2}$$
$$< \in$$

$\therefore$ For each $\in > 0$, $\exists\ \delta = \max\{\delta_1, \delta_2\} \ni 0 \le W(p, f+g) < \in$ with $\|p\| < \delta$.

$\therefore$ $f + g \in R[a\ b]$

$\because$ $f \in R[a\ b] \Rightarrow \int_a^b f(x)\,dx$

$$\Rightarrow \lim_{\|p\| \to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r$$

Similarly

$g \in R[a\ b] \Rightarrow \int_a^b g(x)\,dx$

$$\Rightarrow \lim_{\|p\|\to 0} \sum_{r=1}^{n} g(\xi_r)\,\delta_r$$

$$\because \lim_{\|p\|\to 0} \sum_{r=1}^{n} (f + g)(\xi_r)\,\delta_r$$

$$= \lim_{\|p\|\to 0} \sum_{r=1}^{n} \{f(\xi_r) + g(\xi_r)\}\,\delta_r$$

$$\Rightarrow \lim_{\|p\|\to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r + \lim_{\|p\|\to 0} \sum_{r=1}^{n} g(\xi_r)\,\delta_r$$

$$\therefore \int_a^b (f + g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

**14. If $f \in R\,[a\ b]$ then $f^2 \in R\,[a\ b]$**

*Sol.*

Let $f \in R\,[a\ b]$

$\Rightarrow$ $|f| \in R\,[a\ b]$

$\because$ f is bounded on [a b]

$\Rightarrow$ $|f|$ is bounded on [a b]

$\Rightarrow$ $|f|^2 = f^2$ is bounded on [a b]

$\because$ $f^2 = |f|^2 \Rightarrow f \geq 0$

Let sup. f in [a b] = M > 0

Let $\in > 0$ and $f \in R\,[a\ b]$

$\Rightarrow$ $\exists$ a $p \in \phi\,[a\ b] \ni$

$$\sum_{r=1}^{n} (M_r - m_r)\,\delta_r = U(p, f) - L(p, f) < \frac{\in}{2M} \qquad \ldots (1)$$

Let inf. $(f^2) = m_r^2$ and sup. $(f^2) = M_r^2$ in $I_r$ $\forall$ r = 1 to n.

$\therefore$ $U(p, f^2) - L(p, f^2)$

$$= \sum_{r=1}^{n} (M_r^2 - m_r^2)\,\delta_r$$

$$= \sum_{r=1}^{n} (M_r - m_r)(M_r - m_r)\,\delta_r$$

$$\leq \sum_{r=1}^{n} (M_r - m_r)(M - M)\,\delta_r$$

$\leq 2M \sum_{r=1}^{n} (M_r - m_r)\, \delta_r$

$< 2M \dfrac{\in}{2M}$

$\therefore$ $U(p, f^2) - L(p, f^2) < \in$

$\therefore$ For each $\in > 0$ we can find $p \in \phi$ [a b] $\ni$

$U(p, f^2) - L(p, f^2) < \in$

$\Rightarrow$ $f^2$ is integrable on [a, b].

**15. If $f \in R$ [a b] and a < c < b then $f \in R$ [a c], $f \in R$ [c, b] and $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.**

*Sol.*

Let $f \in R$ [a b]

$\Rightarrow$ f is bounded on [a, b]

$\Rightarrow$ f is bounded on [a, c] and [c, b]

$\because$ a < c < b.

$\because$ $f \in R$ [a b], for a given $\in > 0$, $\exists$ a partition p of [a b] such that $U(p, f) - L(p, f) < \in$

Let $p' = p \cup \{C\}$ then $L(p, f) \leq L(p', f) \leq U(p', f) \leq U(p, f)$

$\Rightarrow$ $U(p', f) - L(p', f) \leq U(p, f) - L(p, f) < \in$ ... (1)

Let $p_1$, $p_2$ denote the set of points of p' on [a, c], [c, b] respectively, then $p_1$, $p_2$ are partitions on [a, c] and [c, b] respectively and $p' = p_1 \cup p_2$.

$\therefore$ $U(p', f) = U(p_1, f) + U(p_2, f)$ and ... (2)

$L(p', f) = L(p_1, f) + L(p_2, f)$ ... (3)

Subtracting (3) from (2), we get

$U(p', f) - L(p', f) = [U(p_1, f) - L(p_1, f)] + [(U(p_2, f) - L(p_2, f)]$

$\Rightarrow$ $U(p', f) - L(p', f) < \in$ from (1)

$\because$ For partitions $p_1$, $p_2$ of [a, c] and [c, b] respectively $U(p_1, f) - L(p_1, f) < \in$ and $U(p_2, f) - L(p_2, f) < \in$

Hence $f \in R$ [a, c] and $f \in R$ [c, b]

Now consider

$U(p', f) = U(p_1, f) + U(p_2, f)$

$\Rightarrow$ $\inf. U(p', f) = \inf. U(p_1, f) + \inf. U(p_2, f)$

$$\Rightarrow \quad \overline{\int_a^b} f(x)\,dx = \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$\because \quad f \in R\,[a\ b],\ f \in R\,[a, c]$ and $f \in R\,[c, b]$

---

**16. If $f \in R\,[a\ b]$ and $f(x) \geq 0\ \forall\ x \in [a, b]$ then $\int_a^b f(x)\,dx \geq 0$.**

*Sol.*

Let m, M be the inf. and sup. of f in [a b].

$\because \quad f(x) \geq 0\ \forall\ x \in [a, b] \Rightarrow m \geq 0$

$\therefore \quad$ For $p \in \phi\,[a\ b],\ L(p, f) \geq m(b - a)$

$\Rightarrow \quad L(p, f) \geq 0$

$$\Rightarrow \quad \underline{\int_a^b} f(x)\,dx = \sup\{L(p, f) \mid p \in \phi\,[ab]\} \geq 0$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx \geq 0$$

---

**17. If, $f, g \in R\,[a\ b]$ and $f(x) \geq g(x)\ \forall\ x \in [a, b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$**

*Sol.* **(May/June-18, June/July-19)**

$f, g \in R\,[a, b] \Rightarrow f - g \in R\,[a\ b]$

$\forall\ x \in [a, b],\ f(x) \geq g(x)$

$\Rightarrow \quad f(x) - g(x) \geq 0\ \forall\ x \in [a\ b]$

$\Rightarrow \quad (f - g)(x) \geq 0$

Consider

$$\int_a^b (f - g)(x)\,dx = \int_a^b [f(x) - g(x)]\,dx \geq 0$$

$$\Rightarrow \quad \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0$$

$$\Rightarrow \quad \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

**18. If $f \in R[a\ b]$ then $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$**

*Sol.*

Let $f \in R[a\ b]$

$\Rightarrow\ |f| \in R[a\ b]$

$\Rightarrow\ -|f| \in R[a\ b]$

$\Rightarrow\ -|f|(x) \leq f(x) \leq |f|(x)\ \forall\ x \in [a\ b]$

$$\Rightarrow \int_a^b -|f|(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f|(x)\,dx$$

$$\Rightarrow \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

---

**19. If $f \in R[a\ b]$ and m, M are the inf. and sup. of f in [a b] then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$ and $\int_a^b f(x)\,dx = \mu(b-a)$ where $\mu \in [m, M]$.**

*Sol.* **(June/July-19), (Imp.)**

Let $g : [a, b] \to R$ and $h : [a\ b] \to R$ be defined by $g(x) = m$ and $h(x) = M\ \forall\ x \in [a, b]$

$\because$ m, M are inf. and sup. of f in [a, b]

$\Rightarrow\ m \leq f(x) \leq M\ \forall\ x \in [a\ b]$

$\therefore\ g(x) \leq f(x) \leq h(x)\ \forall\ x \in [a\ b]$

$$\Rightarrow \int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b h(x)\,dx$$

$$\Rightarrow \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$$\Rightarrow m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Now $\exists$ a real number $\mu \in [m, M] \ni \int_a^b f(x)\,dx = \mu(b-a)$

**20. If $f \in R[a\ b]$ and $|f(x)| \leq K\ \forall\ x \in [a, b]$ where $K \in R^+$, then $\left|\int_a^b f(x)\,dx\right| \leq K(b-a)$.**

*Sol.*

Given $|f(x)| \leq K\ \forall\ x \in [a, b]$

$\Rightarrow\ -K \leq f(x) \leq K\ \forall\ x \in [a, b]$

If m, M are the inf. and sup. of f in [a, b] then $-K \leq m \leq f(x) \leq M \leq K\ \forall\ x \in [a, b]$

Bu we have $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

$\therefore\ -K(b-a) \leq m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \leq K(b-a)$

$\Rightarrow\ -K(b-a) \leq \int_a^b f(x)\,dx \leq K(b-a)$

$\Rightarrow\ \left|\int_a^b f(x)\,dx\right| \leq K(b-a)$

**21. If f and g are integrable on [a, b] and if $f(x) \leq g(x)$ for $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.**

*Sol.*

Given that $f(x) \leq g(x) \quad \forall x \in [a,b]$

$f(x) - g(x) \leq 0$

$g(x) - f(x) \geq 0$

$(g - f) \geq 0$

Given that f is integrable on [a, b] & g is integrable on [a, b]

i.e., $U(f) = L(f) = \int_a^b f(x)dx$ & $U(g) = L(g) = \int_a^b g(x)dx$

if $(g - f) \geq 0 \Rightarrow \int_a^b (g-f)(x)\,dx \geq 0$

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx \geq 0$$

$$\int_a^b g(x)\,dx \geq \int_a^b f(x)\,dx$$

i.e., $$\int_a^b f \leq \int_a^b g$$

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**22. If f is integrable on [a, b]; then |f| is integrable and** $\left|\int_a^b f\right| \leq \int_a^b |f|$

*Sol.* **(May/June-18)**

Given that f is integrable in [a, b]

Since $|f| \leq f \leq |f|$

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$$

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

Now, to to Show that |f| is integrable on [a, b]

i.e., to show that $U(|f|, p) - L(|f|, p) < \varepsilon$

Since $|(f(x)| - |f(y)| \leq |f(x) - f(y)|$

taking supremum of on both sides

$M(|f(x)|, [t_{k-1}, t_k]) - m\ |f|, [t_{k-1}, t_k])$

$\leq M((f, [t_{k-1}, t_k]) - m\ (f, [t_{k-1}, t_k])$

multiply $(t_k - t_{k-1})$

$[M(|f|, [t_{k-1}, t_k]) - m(|f|, [t_{k-1}, t_k])]\ (t_k - t_{k-1}) \leq [M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])]\ (t_k - t_{k-1})$

Now, taking $\Sigma$ on both side.

$$\sum_{k=1}^{n} M(|f|, [t_{k-1}, t_k])\ (t_k - t_{k-1}) - \sum_{k=1}^{n} m(|f|, [t_{k-1}, t_k])\ (t_k - t_{k-1}) \leq \sum_{k=1}^{n}(f, [t_{k-1}, t_k]\ (t_k - t_{k-1})$$

$$- \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])\ (t_k - t_{k-1})$$

$U(|f|, p) - L\ (|f|, p) \leq U\ (f, p) - L(f, p)$

$U(|f|, p) - L(|f| - p) < \varepsilon$

$\therefore$ |f| is integrable on [a, b]

### 4.3.1 Darboux's Theorem

**23. If $f : [a\ b] \to R$ is a bounded function, then for each $\in\ > 0$, $\exists\ \delta > 0$ such that**

(i) $U(p, f) < \overline{\int_a^b} f(x)\,dx + \in$ and

(ii) $L(p, f) > \underline{\int_a^b} f(x)\,dx - \in$ for each $p \in \phi\,[a\ b]$ with $\|p\| < \delta$.

*Sol.*

Let f is bounded on [a, b], then $\exists$ a real number $K > 0 \ni |f(x)| \leq K\ \forall\ x \in [a\ b]$.

(i) By definition we have

$$\overline{\int_a^b} f(x)\,dx = \text{infimum}\,\{U(p, f) \mid p \in \phi\,[ab]\}$$

$\therefore$ For each $\in\ > 0$, $\exists$ a partition $p_1 = \{a = x_0, x_1, x_2, \ldots, x_n = b\} \ni U(p, f) < \overline{\int_a^b} f(x)\,dx + \frac{\in}{2}$ ... (1)

The partition $p_1$ has (p – 1) points excluding the end points a and b. Choose

$$\delta > 0 \ni 2K\,(p-1)\,\delta = \frac{\in}{2} \qquad \text{... (2)}$$

Let p be any partition with $\|p\| < \delta$. Thus p may contain some or none of the partition points $x_r$, r = 1, 2, ..., p – 1 belonging to $p_1$.

If $p_2 = p \cup p_1$ then $p_2$ is finer than p and contains.

At the most (p – 1) additional points

$\therefore$ $U(p, f) - 2K\,(p-1)\,\delta \leq U(p_2, f) \leq U(p_1, f) < \overline{\int_a^b} f(x)\,dx + \frac{\in}{2}$ from (1)

$\Rightarrow$ $U(p, f) < 2K\,(p-1)\,\delta + \overline{\int_a^b} f(x)\,dx + \frac{\in}{2}$

$\Rightarrow$ $U(p, f) < \frac{\in}{2} + \overline{\int_a^b} f(x)\,dx + \frac{\in}{2}$ from (2)

$\Rightarrow$ $U(p, f) < \overline{\int_a^b} f(x)\,dx + \in$ for any partition p with $\|p\| < \delta$.

(ii) By definition we have

$$\int_a^b f(x)\,dx = \sup\{L(p, f) \mid p \in \phi[ab]\}$$

$\therefore$ For each $\in > 0$, $\exists$ a partition $p_1 = \{a = x_0, x_1, x_2, ....., x_n = b\} \ni L(p_1, f) > \int_a^b f(x)\,dx - \frac{\in}{2}$ ... (3)

The partition $p_1$ has (p – 1) points excluding the end points a and b choose $\delta > 0 \ni 2K(p-1)$

$\delta = \frac{\in}{2}$ ... (4)

Let p be any partition with $\|p\| < \delta$. Thus p may contain some or none of the partition points $x_r$, r = 1, 2, 3, ..... p – 1 belonging to $p_1$.

If $p_2 = p \cup p_1$ thus $p_2$ is finer than p and contains atmost p – 1 additional points.

$\therefore$ $L(p, f) + 2K(p-1)\delta \geq L(p_2, f) \geq L(p_1, f) > \int_a^b f(x)\,dx - \frac{\in}{2}$ from (3)

$\Rightarrow$ $L(p, f) > \int_a^b f(x)\,dx - \frac{\in}{2} - 2K(p-1)\delta$

$\Rightarrow$ $L(p, f) > \int_a^b f(x)\,dx - \frac{\in}{2} - \frac{\in}{2}$ from (4)

$\Rightarrow$ $L(p, f) > \int_a^b f(x)\,dx - \in$ for any partition p with $\|p\| < \delta$.

---

**24. Prove that $\left|\int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)\,dx\right| \leq \frac{16\pi^3}{3}$**

*Sol.* **(May/June-18, Nov/Dec.-18), (Imp.)**

$$\leq \int_{-2\pi}^{2\pi} |x^2 \sin^8(e^x)|\,dx$$

$$\leq \left[\frac{x^3}{3}\cos^8(e^x\right]_{-2\pi}^{2\pi} - \int_{-2\pi}^{2\pi} \sin^8(e^x).2x\,dx$$

$$\leq \left[\frac{(+2\pi)^3}{3} + \frac{(2\pi)^3}{3}\right]\cos^8(e^{2\pi} - e^{-2\pi}] \_$$

$$\left[\cancel{2}\frac{x^2}{\cancel{2}}\cos^8(e^x) - 2\sin^8 e^x\right]_{-2\pi}^{2\pi}$$

$$\le \frac{8\pi^3}{3} + \frac{8\pi^3}{3}(i) - [(2\pi)^2 + (2\pi)^2 \cos^8(2\pi - 2\pi) - 2\sin^8(e^{2\pi/2\pi})$$

$$\le \frac{8\pi^3}{3} + \frac{8\pi^3}{3} - 0$$

$$\le \frac{16\pi^3}{3}$$

---

**25. Let f be a bounded function on [a, b], so that there exists B > 0 such that $|f(x)| \le B$ for all $x \in [a, b]$**

**(a) For any set $S \le [a, b]$ and $x_0, y_0 \in S$,**

*Sol.*

We have $f(x_0)^2 - f(y_0)^2 \le$

$\le |f(x_0) + f(y_0)| \,.\, |f(x_0) - f(y_0)|$

$\le 2B|f(x_0) - f(y_0)|$

$\le 2B[M(f,S) - m(f, S)]$

It follows that $M(f^2, S) - m(f^2, S) \le$

$\le 2B[M(f, S) - m(f, S)$

$\therefore$ $U(f^2, P) - L[f^2, P] \le 2B\,[U(f, P) - L(f, P)]$

**(b) Suppose that f is integrable and consider $\in > 0 \;\exists$ partitions $P_1$ and $P_2$ of [a, b] satisfying.**

*Sol.*

$L(f_1 P_1) > L(f) = \frac{\in}{2}$ and $U(f_1 P_2) < U(f) + \frac{\in}{2}$

For $P = P_1 \cup P_2$

$U(f_1 P) - L(f_1 P) \le U(f_1 P_2) - L(f_1 P_1)$

$< U(f) + \frac{\in}{2} - \left[L(f) - \frac{\in}{2}\right]$

$< U(f) - L(f) + \in$

$U(f) = L(f)$

$\therefore$ f is integrable

$\therefore$ f is integrable

$f^2 = f.f$

$$\int_a^b f^2 dx = \int_a^b (f.f)\,dx$$

$$\therefore \quad \int_a^b f dx < \infty \qquad [\because \text{ using part (a)}]$$

$$\therefore \quad \int_a^b f^2 dx < \infty$$

$\therefore$ $f^2$ also integrable on [a, b]

**26. (a)** $\mathbf{\underset{x \to 0}{Lt} \frac{1}{x} \int_0^x e^{t^2} dt}$

*Sol.*

$$I = \int_0^x e^{-(-t^2)} dt$$

$$I^2 = \int_0^x e^{-t^2} dt$$

$$I^2 = \int_0^x \int_0^x e^{-t^2} e^{-y^2} dy\,dt - (t^2 + y^2) = r^2$$

$$= \int_0^x \int_0^x e^{-r^2} r.d\theta.dr$$

$$x \int_0^x e^{-r^2} r.dr = x\left[\frac{-e^{-r^2}}{2}\right]_0^x$$

$$I^2 = x\left[\frac{-e^{-x^2}}{2}\right] \text{ substitute given}$$

$$\underset{x \to \infty}{Lt} \frac{1}{x}.x\left[\frac{-e^{-x^2}}{2}\right]$$

$$= \frac{-1}{2}$$

**(b)** $\mathbf{\underset{h \to 0}{Lt} \frac{1}{h} \int_3^{3+h} e^{t^2} dt}$

*Sol.*

Consider $\int_3^{3+h} e^{t^2} dt$

Let $I = \int_3^{3+h} e^{-(-t^2)} dt$

$$I^2 = \int_3^{3+h} e^{-(-t^2)} dt \int_3^{3+h} e^{-y^2} dy$$

$$= \int_3^{3+h} \int_3^{3+h} e^{-t^2-y^2} dy\,dt$$

$$\Rightarrow \int_3^{3+h} \int_3^{3+h} e^{-r^2}.r.d\theta.dr$$

$$\Rightarrow \int_3^{3+h} e^{-r^2} r.dr$$

$$h\left[-\frac{e^{-r^2}}{2}\right]_3^{3+h}$$

Substitute in given equation

$$\underset{h \to 0}{Lt} \frac{1}{h}.h\left[-\frac{e^{-(3+h)^2}}{2} + \frac{e^{-(3)^2}}{2}\right]$$

$$\underset{h \to 0}{Lt} \left[-\frac{e^{-(3+h)^2}}{2} + \frac{e^{-(3)^2}}{2}\right]$$

$$\left[\frac{e^{-3^2}}{2} - \frac{e^{-(3)^2}}{2}\right]$$

$$= 0$$

**27. Let f(x) & g(x) is continuous real valued function on [a, b].**

*Sol.*

For each $\in > 0 \exists\, m \in z^+ \ni |f(x) - g(x)| < \varepsilon$

f(x) is continous and g(x) is continous

$\therefore$ f(x) g(x) also continuous function

$$\therefore \int_a^b f(x)g(x)\,dx = 0$$

For every continuous function g on [a, b]

$$\therefore \int_a^b f(x)g(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = 0$$

$\therefore$ f(x) = 0 for all x in [a, b]

## 4.4 Fundamental Theorem of Calculus

### 4.4.1 Necessary and Sufficient Condition for Integrability

**28. A bounded function f is integrable on [ab] if and only if for each $\in > 0$, $\exists$ a partition p of [ab]. Such that U(p, f) – L(p, f) < $\in$.**

*Sol.* **(Nov/Dec.-18, June/July-19, Dec.-17)**

Let f be Riemann integrable on [ab].

$$\Rightarrow \int_{\underline{a}}^{b} f(x)\,dx = \int_a^{\overline{b}} f(x)\,dx = \int_a^b f(x)\,dx \quad \text{... (1)}$$

Let $\in > 0$

By Darboux's theorem, $\exists\ \delta > 0 \ni U(p, f) < \int_a^{\overline{b}} f(x)\,dx + \frac{\in}{2}$ ... (2)

and $L(p, f) > \int_{\underline{a}}^{b} f(x)\,dx - \frac{\in}{2}$ ... (3)

For each $p \in \phi$ [a b] with $\|p\| < \delta$.

$\therefore$ From (1) and (2) and from (1) and (3) we get.

$$U(p, f) < \int_a^b f(x)\,dx + \frac{\in}{2} \text{ and } L(p, f) > \int_a^b f(x)\,dx - \frac{\in}{2}$$

$$\Rightarrow \quad U(p, f) < \int_a^b f(x)\,dx + \frac{\in}{2} \text{ and } \int_a^b f(x)\,dx < L(p, f) + \frac{\in}{2}$$

$$\Rightarrow \quad U(p, f) < L(p, f) + \frac{\in}{2} + \frac{\in}{2}$$

$$\Rightarrow \quad U(p, f) - L(p, f) < \in$$

Also we have $U(p, f) - L(p, f) \geq 0$

$$\Rightarrow \quad 0 \leq U(p, f) - L(p, f) < \in$$

**Conversely :**

Let for each $\in > 0$, $\exists$ a partition p of [a, b] $\exists$ $0 \leq U(p, f) - L(p, f) < \in$

By definition we have

$$\overline{\int_a^b} f(x)\,dx = \text{infimum } \{U(p, f) \mid p \in \phi\,[ab]\}$$

$$\Rightarrow \quad \overline{\int_a^b} f(x)\,dx \leq U(p, f) \qquad \text{... (4)}$$

Similarly

$$\underline{\int_a^b} f(x)\,dx = \text{Sup. } \{L(p, f) \mid p \in \phi\,[ab]\}$$

$$\Rightarrow \quad \underline{\int_a^b} f(x)\,dx \geq L(p, f)$$

$$\Rightarrow \quad -\underline{\int_a^b} f(x)\,dx \leq -L(p, f) \qquad \text{... (5)}$$

Adding (4) and (5)

$$\Rightarrow \quad \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq U(p, f) - L(p, f) < \in$$

Also we have

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \geq 0$$

$\Rightarrow \quad 0 \le \int_a^{\bar{b}} f(x)\,dx - \int_{\bar{a}}^{b} f(x)\,dx < \in$

$\because \quad \in > 0$ is arbitrary

$\Rightarrow \quad \int_a^{\bar{b}} f(x)\,dx - \int_{\bar{a}}^{b} f(x)\,dx = 0$

$\Rightarrow \quad \int_a^{\bar{b}} f(x)\,dx = \int_{\bar{a}}^{b} f(x)\,dx$

$\Rightarrow$ f is Riemann integrable on [a b].

---

**29. Show that f(x) = 3x + 1 is integrable on [1, 2] and $\int_1^2 (3x+1)\,dx = \frac{11}{2}$**

*Sol.*

Let f(x) = 3x + 1 is bounded on [0, 2]

Consider the partition

$$p = \left\{1,\ 1+\frac{1}{n},\ 1+\frac{2}{n},\ \ldots\ 1+\frac{r}{n},\ \ldots\ 2\right\}$$

Let $r^{th}$ subinterval $I_r = \left[1+\frac{r-1}{n},\ 1+\frac{r}{n}\right]$ and length of each subinterval $\delta_r = \frac{1}{n}$.

$\because \quad$ f(x) = 3x + 1 is increasing on [1, 2].

$\therefore \quad M_r$ = sup. of f in $I_r = 3\left(1+\frac{r}{n}\right) + 1 = 4 + \frac{3r}{n}$ and $m_r$ = inf. of f in

$$I_r = 3\left(1+\frac{r-1}{n}\right) + 1 = 4 + \frac{3(r-1)}{n}$$

$$\therefore \quad U(p, f) = \sum_{r=1}^{n} M_r\,\delta_r = \sum_{r=1}^{n}\left(4+\frac{3r}{n}\right)\frac{1}{n}$$

$$= \sum_{r=1}^{n}\frac{4}{n} + \frac{3}{n^2}\sum_{r=1}^{n} r$$

$$= \frac{4}{n}(n) + \frac{3}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= 4 + \frac{3}{2n^2}\,n^2\left(1+\frac{1}{n}\right)$$

$$= 4 + \frac{3}{2}\left(1+\frac{1}{n}\right)$$

Similarly

$$L(p, f) = \sum_{r=1}^{n} m_r \, \delta_r$$

$$= \sum_{r=1}^{n} \left(4 + \frac{3(r-1)}{n}\right) \frac{1}{n}$$

$$= \frac{4}{n} \sum_{r=1}^{n} (1) + \frac{3}{n^2} \sum_{r=1}^{n} (r-1)$$

$$= \frac{4}{n}(n) + \frac{3}{n^2} \frac{(n-1)\, n}{2}$$

$$= 4 + \frac{3n^2}{2n^2}\left(1 - \frac{1}{n}\right)$$

$$= 4 + \frac{3}{2}\left(1 - \frac{1}{n}\right)$$

$$\therefore \underline{\int_1^2} f(x)\, dx = \lim_{n\to\infty} L(p, f)$$

$$= \lim_{n\to\infty} \left[4 + \frac{3}{2}\left(1 - \frac{1}{n}\right)\right]$$

$$= 4 + \frac{3}{2} = \frac{11}{2}$$

Similarly

$$\overline{\int_1^2} f(x)\, dx = \lim_{n\to\infty} U(p, f)$$

$$= \lim_{n\to\infty} \left[4 + \frac{3}{2}\left(1 - \frac{1}{n}\right)\right]$$

$$= 4 + \frac{3}{2} = \frac{11}{2}$$

$$\Rightarrow \underline{\int_1^2} f(x)\, dx = \overline{\int_1^2} f(x)\, dx = \frac{11}{2}$$

$\Rightarrow$ $f(x) = 3x + 1$ is integrable on $[1, 2]$

and $\int_1^2 (3x + 1)\, dx = \frac{11}{2}$

**30. Prove that $f(x) = x^2$ is integrable on $[0, a]$ and $\int_0^a x^2\, dx = \frac{a^3}{3}$.**

*Sol.*

Let $f(x) = x^2$ is bounded on $[0, a]$

Consider the partition

$$p = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots\ldots, \frac{ra}{n}, \ldots\ldots a\right\}$$

Let $r^{th}$ subinterval $I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$

Length of each subinterval $= \delta_r = \frac{a}{n}$

$\because$ $f(x) = x^2$ is an increasing function in $[0,a]$

Let $M_r$ = sup. of f in $I_r = \left(\frac{ra}{n}\right)^2 = \frac{r^2 a^2}{n^2}$

and $m_r$ = infimum of f in

$$I_r = \left(\frac{(r-1)\, a}{n}\right)^2 = \frac{(r-1)^2\, a^2}{n^2}$$

Now

$$U(p, f) = \sum_{r=1}^{n} M_r \, \delta_r = \sum_{r=1}^{n} \frac{r^2 a^2}{n^2} \cdot \frac{a}{n}$$

$$= \sum_{r=1}^{n} \frac{a^3 r^2}{n^3} = \frac{a^3}{n^3} \sum_{r=1}^{n} r^2$$

$$= \frac{a^3}{6n^3} \times \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{a^3}{6n^3}\, n^3 \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

$$= \frac{a^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

Similarly

$$L(p, f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} \frac{(r-1)^2 a^2}{n^2} \times \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} (r-1)^2$$

$$= \frac{a^3}{n^3} \times \frac{(n-1)\,n(2n-1)}{6}$$

$$= \frac{a^3}{6n^3}\, n^3 \left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

Consider

$$\underline{\int_0^a} f(x)\,dx = \lim_{n\to\infty} L(p, f)$$

$$= \lim_{n\to\infty} \frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{2a^3}{6} = \frac{a^3}{3}$$

Similarly

$$\int_0^{\bar{a}} f(x)\,dx = \lim_{n\to\infty} U(p, f) = \lim_{n\to\infty} \frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{2a^3}{6} = \frac{a^3}{3}$$

$$\therefore \quad \int_0^{a} f(x)\,dx = \int_0^{\bar{a}} f(x)\,dx = \frac{a^3}{3}$$

$\Rightarrow$ $f(x) = x^2$ is integrable on [0, 4] and $\int_0^a x^2\,dx = \frac{a^3}{3}$

**31. Prove that f(x) = sinx is integrable on $\left[0, \Pi/2\right]$ and $\int_0^{\Pi/2} \sin x\,dx = 1$.**

*Sol.*

Let $f(x) = \sin x$ is bounded on $\left[0, \Pi/2\right]$

Consider the partition

$$p = \left\{0, \Pi/2n, 2\Pi/2n, 3\Pi/2n \ldots.. 2\Pi/2n \ldots.. n\Pi/2n\right\}$$

Let $r^{th}$ subinterval $I_r = \left[\frac{(r-1)\Pi}{2n}, \frac{r\Pi}{2n}\right]$

and Length of each subinterval $\delta_r = \frac{\Pi}{2n}$

$\because$ f(x) = sinx is increasing in $\left[0, \Pi/2\right]$

Let $M_r$ = sup. of f in $I_r = \sin\frac{r\Pi}{2n}$

and $m_r$ = Infimum of f in $I_r = \sin\frac{(r-1)\Pi}{2n}$

Now consider

$$U(p, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \sin\frac{r\Pi}{2n} \times \frac{\Pi}{2n} = \frac{\Pi}{2n}\left[\sin\frac{\Pi}{2n} + \sin\frac{2\Pi}{2n} + ..... + \sin\frac{n\Pi}{2}\right]$$

We know that sina + sin(a + d) + ..... + sin (a + (n – 1) d) = $\dfrac{\sin\left(a + \frac{(n-1)}{2}d\right)\sin\frac{nd}{2}}{\sin\left(\frac{d}{2}\right)}$

$$\therefore \quad U(p, f) = \frac{\Pi}{2n}\left[\frac{\sin\left(\frac{\Pi}{2n} + \frac{n-1}{2}.\frac{\Pi}{2n}\right)\sin\frac{n\Pi}{4n}}{\sin\left(\frac{\Pi}{4n}\right)}\right]$$

$$= \frac{\frac{\Pi}{2n}\left[\sin\frac{(n+1)\Pi}{4n}.\sin\frac{\Pi}{4}\right]}{\sin\left(\frac{\Pi}{4n}\right)}$$

$$= \frac{\frac{\Pi}{2\sqrt{2}n}\left\{\sin\frac{\Pi}{4}\cos\frac{\Pi}{4n} + \cos\frac{\Pi}{4}\sin\frac{\Pi}{4n}\right\}}{\sin\left\{\frac{\Pi}{4n}\right\}}$$

$$= \frac{\Pi}{2\sqrt{2}n} \cdot \frac{1}{\sqrt{2}}\left(\cos\frac{\Pi}{4n} + 1\right) = \frac{\Pi}{4n}\left(\cot\frac{\Pi}{4n} + 1\right)$$

Similarly we can prove that $L(p, f) = \frac{\Pi}{4n}\left(\cot\frac{\Pi}{4n} - 1\right)$

$$\therefore \int_0^{\Pi/2} f(x)\,dx = \lim_{n\to 0} L(p, f) = \lim_{n\to 0} \frac{\left(\Pi/4n\right)}{\tan\left(\Pi/4n\right)} - \lim_{n\to 0} \frac{\Pi}{4n}$$

$$= 1 - 0 = 1 \qquad \left(\because \lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1\right)$$

Similarly

$$\int_0^{\Pi/2} f(x)\,dx = \lim_{n\to 0} U(p, f) = 1$$

$$\therefore \int_0^{\Pi/2} f(x)\,dx = \int_0^{\bar{\Pi}/2} f(x)\,dx = 1$$

$\therefore$ f(x) = sinx is integrable on $\left[0, \Pi/2\right]$ and $\int_0^{\Pi/2} \sin x\,dx = 1$.

### 4.4.2 Another Definition of Riemann Integral

Let f : [ab] $\to$ R be a function and p = {a = $x_0$, $x_1$, $x_2$, ....., $x_n$ = b} be a partition of [ab]. Let {$\xi_1$, $\xi_2$, .....,$\xi_n$} $\subset$ [ab] be such $x_{r-1} \leq \xi_r \leq x_r$ $\forall$ r = 1, 2, 3, ..... n. The function f is said to be Riemann integrable over [ab], if to each $\in > 0$, $\exists$ $\delta > 0$ and a number I such that $\left|\sum_{r=1}^{n} f(\xi_r)\,\delta_r - I\right| < \in$ for p $\in \phi$ [ab] with $\|p\| < \delta$ and $\xi_r \in [x_{r-1}, x_r]$. The number I is the Riemann integral of f over [a, b] $\Rightarrow \int_a^b f(x)\,dx = \lim_{n\to 0}\left[\sum_{r=1}^{n} f(\xi_r)\,\delta_r\right]$.

### 4.4.3 Primitive (Definition)

If f $\in$ R [a b] and if $\exists$ $\phi$ : [a b] $\to$ R $\ni$ $\phi'(x) = f(x)$ $\forall$ x $\in$ [a, b] then $\phi$ is called a primitive or antiderivature of f.

## 4.5 Fundamental Theorem of Integral Calculus

**32. If f $\in$ R [a b] and $\phi$ is a primitive of $\phi$ then $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$.**

*Sol.*

$\phi$ is a primitive of f on [a, b]

$\Rightarrow \quad \phi'(x) = f(x) \;\forall\; x \in [a, b]$ ... (1)

Consider the partition $p = \{a = x_0, x_1, x_2, ....., x_n = b\}$ of [a b]

$\because \quad f \in R\,[a, b]$

$\Rightarrow \quad x_{r-1} \le \xi_r \le x_r \;\forall\; r = 1, 2, ..... n$

$\Rightarrow \quad \lim_{\|p\|\to 0} \sum_{r=1}^{n} f(\xi_r)\, \delta_r = \int_a^b f(x)\, dx$ ... (2)

$\because \quad \phi$ is derivable on [a b]

$\Rightarrow \quad \phi$ is continuous and derivable on $[x_{r-1}, x_r] \;\forall\; r = 1$ to n

$\therefore \quad$ By lagrange's mean value theorem we have

$$\phi'(\xi_r) = \frac{\phi(x_r) - \phi(x_{r-1})}{x_r - x_{r-1}} \;\forall\; \xi_r \in (x_{r-1}, x_r),\ r = 1 \text{ to } n.$$

$\Rightarrow \quad \phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\, \phi'(\xi_r) \;\forall\; r = 1$ to n.

Adding these n equalities we get

$$\sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \sum_{r=1}^{n} \phi'(\xi_r)\, \delta_r$$

$\Rightarrow \quad \sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \sum_{r=1}^{n} f(\xi_r)\, \delta_r$ from (1)

$\Rightarrow \quad \sum_{r=1}^{n} f(\xi_r)\, \delta_r = \phi(x_1) - \phi(x_0) + \phi(x_2)$

$\quad - \phi(x_1) + ..... + \phi(x_n) - \phi(x_{n-1})$

$\Rightarrow \quad \sum_{r=1}^{n} f(\xi_r)\, \delta_r = \phi(x_n) - \phi(x_0)$

$\therefore \quad \lim_{\|p\|\to 0} \sum_{r=1}^{n} f(\xi_r)\, \delta_r = \lim_{\|p\|\to 0} [\phi(x_n) - \phi(x_0)]$

$\Rightarrow \quad \int_a^b f(x)\, dx = \phi(b) - \phi(a)$ from (2)

**33. Show that $\int_0^1 x^4\, dx = \frac{1}{5}$**

*Sol.*

Let $f(x) = x^4$ is continuous on R

$\Rightarrow \quad$ Continuous on [0, 1]

$\Rightarrow \int_0^1 x^4\,dx$ exists.

Let $\phi(x) = \frac{x^5}{5}$ defined on [0, 1]

Clearly $\phi$ is derivable on [0, 1] and

$\phi'(x) = x^4 = f(x)\ \forall\ x \in [0, 1]$

$\therefore$ $\phi$ is primitive of f on [0, 1]

$\therefore$ By fundamental theorem

$$\int_0^1 x^4\,dx = \phi(1) - \phi(0) = \frac{1}{5} - 0 = \frac{1}{5}$$

$$\Rightarrow \int_0^1 x^4\,dx = \frac{1}{5}$$

**34. Show that $\int_a^b \cos x\,dx = \sin b - \sin a$.**

*Sol.*

Let f(x) = cosx is continuous on R.

$\Rightarrow$ f(x) is continuous on [a, b]

$\Rightarrow \int_a^b \cos x\,dx$ exists.

Let $\phi(x) = \sin x$ defined on [a, b]

$\Rightarrow$ $\phi$ is derivable on [a, b] and

$\phi'(x) = \cos x = f(x)$

$\Rightarrow$ $\phi$ is primitive of f on [a, b]

$\therefore$ By fundamental theorem

$$\int_a^b \cos x\,dx = \phi(b) - \phi(a)$$

$$\Rightarrow \int_a^b \cos x\,dx = \sin b - \sin a$$

**35. Prove that $\int_a^{\bar{\bar{b}}} e^x\,dx = e^b - e^a$.**

*Sol.*

Let $f(x) = e^x$ is continuous on R

$\Rightarrow$ f(x) is continuous on [a, b]

$\Rightarrow \int_a^b e^x\,dx$ exists

Let $\phi(x) = e^x$ defined on [a, b] and $\phi'(x) = e^x = f(x)$

$\Rightarrow$ $\phi$ is primitive of f on [a, b]

$$\therefore \int_a^b e^x\,dx = \phi(b) - \phi(a)$$

$$\Rightarrow \int_a^b e^x\,dx = e^b - e^a$$

**36. Evaluate $\int_0^{\Pi/4} (\sec^4 x - \tan^4 x)\,dx$**

*Sol.*

Let $f(x) = \sec^4 x - \tan^4 x$

$= (\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x)$

$= (1)(\sec^2 x + \tan^2 x)$

$f(x) = 2\sec^2 x - 1$

$\because$ $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x - \tan^2 = 1$

which is continuous on $\left[0, \Pi/4\right]$ and

Hence $\int_0^{\Pi/4} f(x)\,dx$ exists.

Let $\phi(x) = 2\tan x - x$ defines on $\left[0, \Pi/4\right]$

and $\phi'(x) = 2\sec^2 x - 1 = f(x)$

$\Rightarrow$ $\phi$ is primitive of f on $\left[0, \Pi/4\right]$

$\therefore$ By fundamental theorem

$$\int_0^{\Pi/4} (\sec^4 x - \tan^4 x)\, dx$$

$$= \phi\left(\Pi/4\right) - \phi(0)$$

$$= \left(2 - \Pi/4\right) - 0$$

$$\therefore \int_0^{\Pi/4} (\sec^4 x - \tan^4 x)\, dx = 2 - \Pi/4$$

---

**37.** $f(x) = \frac{1}{2^n}\ ;\ \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}\ \forall\ n = 0, 1, 2, \ldots, f(0) = 0$

*Sol.*

Let $f \in R\,[0, 1]$

We have

$$\int_{(1/2)^n}^{1} f(x)\,dx = \int_{(1/2)^n}^{(1/2)^{n-1}} f(x)\,dx + \int_{(1/2)^{n-1}}^{(1/2)^{n-2}} f(x)\,dx$$

$$+ \ldots + \int_{(1/2)^n}^{1} f(x)\,dx$$

$$\Rightarrow \int_{(1/2)^n}^{(1/2)^{n-1}} \frac{1}{2^{n-1}}\,dx + \int_{(1/2)^{n-1}}^{(1/2)^{n-2}} \frac{1}{2^{n-2}}\,dx + \ldots + \int_{1/2}^{1} 1\,dx$$

$$\Rightarrow \frac{1}{2^{n-1}}\left[\frac{1}{2^{n-1}} - \frac{1}{2^n}\right] + \frac{1}{2^{n-1}}$$

$$\left[\frac{1}{2^{n-2}} - \frac{1}{2^{n-1}}\right] + \ldots + \left[1 - \frac{1}{2}\right]$$

$$\Rightarrow \frac{1}{2^{n-1}}\left[\frac{1}{2^n}\right] + \frac{1}{2^{n-2}}\left[\frac{1}{2^{n-1}}\right] + \ldots + \left[\frac{1}{2}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{4^{n-1}} + \frac{1}{4^{n-2}} + \ldots + 1\right]$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{1-\left(\frac{1}{4}\right)^n}{1-\frac{1}{4}}\right]$$

which is a geometric series

with CR $\frac{1}{4} < 1$

$$\Rightarrow \quad \frac{2}{3}\left(1-\left(\frac{1}{4}\right)^n\right)$$

$$\therefore \quad \int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_{(1/2)^n}^{1} f(x)\,dx$$

$$= \lim_{n\to\infty} \frac{2}{3}\left(1-\frac{1}{4^n}\right) = \frac{2}{3}$$

---

**38. Show that $\int_a^b x^n\,dx = \frac{1}{n+1}(b^{n+1} - a^{n+1})$ where $n \in N$.**

*Sol.*

Let $f(x) = x^n$ is continuous on R

$\Rightarrow$ f(x) is continuous on [a, b]

and $\int_a^b x^n\,dx$ exists.

Let $\phi(x) = \frac{x^{n+1}}{n+1}$ defined on [a, b]

$\Rightarrow$ $\phi(x)$ is derivable on [a, b] and

$\phi'(x) = x^n = f(x) \ \forall \ x \in [a, b]$

$\Rightarrow$ $\phi$ is a primitive of f on [a, b]

$\therefore$ By fundamental theorem

$$\int_a^b x^n\,dx = \phi(b) - \phi(a) = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

$$\therefore \quad \int_a^b x^n\,dx = \frac{1}{n+1}(b^{n+1} - a^{n+1}) \ \forall \ n \in N$$

**39. Intermediate value theorem for integrability**

**Statement : If 'f' is continuous on [a, b]. Then prove that for atleast x in [a, b],**

$$f(x) = \frac{1}{b-a}\int f(x)dx$$

*Sol.* **(May/June-18)**

Let $M = \text{Sup}\{f(x) / x \in [a, b]\}$

$m = \inf\{f(x) / x \in [a, b]\}$

$\Rightarrow m \le M$

**Case (i)**

Let m – M

f(x) = k,

i.e., a constant function

$$\text{R.H.S} = \frac{1}{b-a}\int f(x)dx - \frac{1}{b-a}\int_a^b kdx = \frac{1}{b-a}[kx]_a^b$$

$$= \frac{1}{b-a}(b-a)k$$

$$= k$$

$$= f(x)$$

**Case (ii)**

m < M

$\therefore$ f is continuous on [a, b]

it attains its sup & inf on [a, b]

$\Rightarrow \exists, x_0, y_0 \in [a, b] \ni f(x_0) = M, f(y_0) = m$ ...(1)

$\Rightarrow m < f(x) < M$

Integrating throughout with respect to 'x' between the limits a & b

$$\int_a^b mdx < \int_a^b f(x)dx < \int_a^b Mdx$$

$$m(b-a) < \int_a^b fdx < M(b-a)$$

$$m < \frac{1}{b-a}\int_a^b fdx < M$$

$$f(y_0) < \frac{1}{b-a}\int_a^b fdx < f(x_0)$$

$$\frac{1}{b-a}\int_a^b f = f(x) \qquad x \in [x_0, y_0]$$

$$\frac{1}{b-a}\int_a^b f = f(x)$$

**40. if ‘g’ is integrable on [a, b] & g is a continuous function on [a, b] which is differentiable on [a, b].**

**Then prove that $\int_a^b g' = g(b) - g(a)$**

*Sol.* **(Imp.)**

Since g' is integrable on [a, b]

by cauchy criteria

$U(g', p) - L(g'.p) < \varepsilon$

where $P = \{a = t_0 < t_k \ldots < t_{k-1} < t_k \ldots < t_n = b\}$ is partition of [a, b]

since g is continuous & differentiable

$\Rightarrow$ g is continuous in $[t_{k-1}, t_k]$

g is differentiable in $[t_{k-1}, t_k]$

By Legrange’s mean value theorem

$$x \in (t_{k-1}, t_k), \exists \ \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}} = g' x_k$$

$$g(t_k) - g(t_{k-1}) = g'(x_k)(t_k - t_{k-1})$$

$$\sum_{k=1}^{n} [g(t_k) - g(t_{k-1}) = \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1})$$

$$g(b) - g(a) = \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}) \quad \ldots(1)$$

$\therefore$ $L(f, p) \leq L(f) \leq U(f) \leq U(f, p)$

$\Rightarrow$ $L(g', p) \leq L(g') \leq U(g') \leq U(g', p)$ ...(2)

$\therefore$ g' is integrable $\Rightarrow$ by differentiable, $L(g') = U(g') = \int_a^b g'$

$$(2) \Rightarrow \quad L(g', (p)) \leq \int_a^b g' \leq U(g', p) \quad \ldots(3)$$

$$m(g', [t_{k-1}, t_k]) \leq g'(x_k) \leq M(g', [t_{k-1}, t_k])$$

multiply $(t_k - t_{k-1})$ & taking $\sum_{k=1}^{n}$

$$\sum_{k=1}^{n} m(g', [t_{k-1}, t_k])(t_k - t_{k-1}) \leq \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}) \leq \sum_{k=1}^{n} M(g', [t_{k-1}, t_k])(t_k - t_{k-1})$$

$$L(g', p) \leq \sum_{k=1}^{n} g(t_k) - g(t_{k-1}) \leq U(g', p)$$

$$L(g', p) \leq g(b) - g(a) \leq U(g', p) \qquad ...(4)$$

Using (3) & (4)

$$\left[\int_a^b g' - (g(b) - g(a))\right] < \varepsilon$$

$\varepsilon > 0$ is arbitrary

$$\int_a^b g' = g(b) - g(a)$$

**41. If U and V are continuous function on [a, b] that are differentiable on (a, b) and if U' and V' are integrable on [a, b] then $\int_a^b U(x)\, V'(x)\, dx + \int_a^b U'(x)\, V(x) dx = U(b)\, V(b) - U(a)V(a)$**

*Sol.* **(Imp.)**

Suppose that g(x) = u(x) v(x)

u & v are differentiable

Since every differentiable function is continuous

$\because$ U, V are continuous on [a, b]

Since every continuous function is integrable

i.e., U & V are Integrable on [a, b]

Since $g(x) = U(x)\, V(x)$

$g'(x) = U(x)\, V'(x) + V(x)U'(x)$

$U', V' \in [a, b]$

By Fundamental Theorem of Integral Calculus

$$\int_a^b g'(x)dx = g(b) - g(a)$$

$$\int_a^b g'(x)dx = \int_a^b [U(x)\, V'(x) + U'(x)V(x)]dx$$

$$[g(x)]_a^b = \int_a^b U(x)\, V'(x)\, dx + \int_a^b U'(x)V(x)dx$$

$$[U(x)V(x)]_a^b = \int_a^b U(x)V'(x)dx + \int_a^b U'(x)\, V(x)dx$$

$$U(b)V(b) - U(a)V(a) = \int_a^b U(x)V'(x)dx + \int_a^b U'(x)V(x)dx$$

**42. Fundamental Theorem of Integrate Calculus - II**

**Let f be an integrable function on [a, b] for x in [a, b]**

**Let** $f(x) = \int_a^x f(t)dt$**, then F is continuous in [a, b]**

**if f is continuous at** $x_0$ **in (a, b) then F is differentiable at** $x_0$ **and** $f'(x_0) = f(x_0)$

*Sol.*

Select $B > 0$ such that $|f(x)| \leq B \ \forall x \in [a, b]$

If $x, y \in [a, b]$ where $|x - y| < \frac{\varepsilon}{B}$ then

$$|F(y) - F(x)| = |\int_x^y f(t)dt| \leq \int_x^y |f(t)|dt$$

$$\leq \int_x^y B\, dt$$

$$= B(y - x)$$

$$< B\frac{\varepsilon}{B}$$

$|F(g) - F(x)| < \varepsilon$

$\Rightarrow$ F is uniformly continuous on [a, b]

$\Rightarrow$ F is continuous on [a, b]

Suppose f is continuous at $x_0$ in (a, b)

Then $\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0}\int_{x_0}^x f(t)dt$ where $x \neq x_0$

$$f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x f(x_0)\, dt$$

**43. Let f be a function defined on [a, b] if a < c < b and f in integrable on [a, b] and [c, b] then f in integrable on [a, b] and** $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b (x)dx$

*Sol.* **(June/July-19)**

$f \in R[a, b] \Rightarrow$ f is bounded on [a, b]

$\Rightarrow$ f is bounded on [a, b] & [c, b]

Since $f \in R[a, b]$

for a given $\varepsilon > 0$, $\exists$ a partition O of [a, b] $\ni$ $U(f, p) - L(f, p) < \varepsilon$

[p C p' C p' in refinement of p

Then $L(f, p) \leq L(f, p') \leq U(f, p') \leq U(f, p)$

$\Rightarrow$ $U(f, p') - L(f, p') < \varepsilon$ ...(1)

Let $P_1, P_2$ denote the set of points of p' on [a, b] & [c, b] respectively.

Then $P_1, P_2$ are partition on [a, b] & [c, b] respectively.

$P' = P_1 \cup P_2$

$\therefore$ $U(f, P') = U(f, p_1) + U(f, p_2)$

$L(f, p') = L(f, p_1) + L(f, p_2)$

$U(f, p') - L(f, p') = U(f, p_1) + U(f, p_2) - (L(f, p_2) + L(f, p_2) < \varepsilon$

$= [U(f, p_1) - L(f, p_1)] + [U(f, p_2) - L(f, p_2)] < \varepsilon$ by (1)

Since each of $[U(f, p_1) - L(f, p_1)]$ and $[U(f, p_2) - L(f, p_2)]$ are non negative, each of these is less then $\varepsilon$.

i.e., $U(f, p_1) - L(f, p_1) < \varepsilon$ and $U(f, p_2) - L(f, p_2) < \varepsilon$

for partition on [a, b] & [c, b] respectively.

Hence $f \in R[a, c]$ and $f \in R[c, b]$

Now $U(f, p') = U(f, p_1) + U(f, p_2)$

$\inf U(f, p') = \inf U(f, p_1) + \inf U(f, p_2)$

$$\int_a^{\bar{b}} f(x)dx = \int_a^{\bar{c}} f(x)dx + \int_c^{\bar{b}} f(x)dx$$

$$\int_a^{b} f(x)dx = \int_a^{c} f(x)dx + \int_a^{b} f(x)dx$$

Since $f \in R[a, b]$, $f \in [a, c]$ and $f \in R[c, b]$

**44 Let u be a differentiable function on an open interval J such that U is continuous and let I be can open interval such that $u(x) \in I \ \forall x \in J$. If f is continuous on I, then $f_0$ u is continuous on J and $\int_a^b fou(x)u'(x)dx = \int_{u(a)}^{u(b)} f(u)dx. \ \forall a,b \in J$**

*Sol.*

Let $F(x) = \int_a^x f(u)du$

Since f is continuous on I

$\Rightarrow$ F is differentiable in J with $F'(u) = f(u) \ \forall u \in T$

Let $g(x) = Fou(x) \Rightarrow g'(x) = [F(ux)]'$

$= F'(u(x))u'(x)$

$g'(x) = f(u(x))u'(x)$

U' is continuous on I and fou is continuous on J

By fundamental theorem on integral continous

$$\int_a^b g'(x)dx = g(b) - g(a)$$

$$\int_a^b f(u(x)u'(x)dx = \int_a^b (fou)'dx \Rightarrow \int_a^b g'(x)dx$$

$$= g(b) - g(a)$$

$$\Rightarrow F(U(b)) - F(U(a))$$

$$= \int_c^{U(b)} f(u)du - \int_c^{U(a)} f(u)du$$

$$= \int_c^{U(b)} f(u)du + \int_{U(a)}^{c} f(u)du$$

$$\therefore \int_a^b fou(x)u'(x)dx = \int_{U(a)}^{U(b)} f(u)du$$

**45. If f is bounded function on [a, b] and if 'p' and 'Q' are two partitions of [a, b] such that P $\subseteq$ Q then prove that L(f, p) $\leq$ L(f, Q) $\leq$ U(f, Q) $\leq$ U(f, p)**

*Sol.* **(Nov/Dec.-18)**

We have prove that

$L(f, p) \leq L((f, Q) \leq U(f, Q) \leq U(f, p)$

To prove that

i.e., $L(f,p) \leq L(f, Q)$ ...(1)

$L(f, Q) \leq U(f, Q)$ ...(2)

$L(f, Q) \leq U(f, p)$ ...(3)

here (2) i.e., $L(f, Q) \leq U(f, Q)$ is obivous

Now we prove (1) $L(f, p) \leq L(f, Q)$

let $p = \{a = t_0 < t_1 <....<t_{k-1} < t_k < ....<t_n = b\}$ be partition on [a, b]

let Q be the partition which continuous one more point (say U) more then of P because $P \subseteq Q$

i.e., $Q = \{Q = t_0 < t_1 < ...<t_{k-1} < U < t_k <...<t_n = b\}$ be the partition of [a, b]

By differentiable,

$$L(f, p) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])\,(t_k - t_{k-1})$$

$$= m(f, [t_0, t_1])\,\cancel{(t_1 - t_0)} + m(f, \cancel{[t_1 - t_0]})\,(t_1 - t_2) + \ldots + m(f, [t_{k-2}, t_{k-1}])\,(t_{k-1} - t_{k-2})$$

$$m(f(t_{k-1}, t_k])\,(t_k - t_{k-1}) + \ldots + m(f, [t_{n-1}, t_n])\,(t_n - t_{n-1})$$

$$L(f, (Q) = m[f, [t_0 - t_1])\,(t_1 - t_0) + m[f, \cancel{[t_1, t_2]})\,(t_2 - t_1) + \ldots + m[f, [t_{k-2}, t_{k-1}])\,\cancel{(t_{k-1} - t_{k-2})}$$

$$+ m[f, [t_{k-1}, t_k])\,(U - t_{k-1}) + \ldots + m(f, \cancel{[t_{n-1}, t_n]})\,(t_n - t_{n-1})$$

$$L(f, Q) - L(f, P) = m(f, [t_{k-1}, U])\,(U - t_{k-1}) + m[f, [U, t_k])\,(t_k - t_u) - m(f,[t_{k-1}, t_k])\,(t_k - t_{k-1}) \quad ..(1)$$

Since $U \in [t_{k-1}, t_k]$

$[t_{k-1}, U] \subset [t_{k-1,}\, t_k]$

$m(f, [t_{k-1}, U]) \geq m(f, [f_{k-1}, t_k])$

Similarly $[U, t_k] \subset [t_{k-1}, t_k]$

$\Rightarrow \quad m(f, [U, t_k]) \geq m(f, [t_{k-1}, t_k])$ by note

Consider

$m(f, [t_{k-1}, t_k])\,(t_k - t_{k-1})$

$= m(f, [t_{k-1}, t_k])\,(t_k - U + U - t_{k-1})$

$= m(f, [t_{k-1}, t_k])\,(t_k - U) + m(f, [t_{k-1}, t_k])\,(U - t_{k-1})$

$\leq m(f, [U, t_k])\,(t_k - U) + m(f, [t_{k-1}, U])\,(U - t_{k-1})$ by (2)

$\therefore$ we have

$m(f, [t_{k-1}, t_k])\,(t_k - t_{k-1}) \leq m(f, [U, t_k])\,(t_k - U) + m(f, [t_{k-1}, U])\,(U - t_{k-1}) \quad \ldots(3)$

$\therefore$ (1) $\Rightarrow m(f, [t_{k-1}, t_k])\,(U - t_{k-1}) + m(f, [U, tk])\,(t_k - U) - m(f, [t_{k-1}, t_k])\,(t_k - t_{k-1}) \geq 0$ by (3)

$L(f, Q) - L(f, p) \geq 0$

$\Rightarrow \quad L(f, p) \leq L(f, Q)$

Similarly we can prove that $U(f, Q) \leq U(f, p)$

$L(f, p) \leq L(f, Q) \leq L(f, Q) \leq U(f, Q)$

$L(f, Q) - L\,(f, p) \leq 0$

$L(f, Q) \leq L(f, Q)$

$L(f, Q) - L(f, p) \geq m(f, \cancel{[t_{k-1}, t_k]})\,(t_k - t_{k-1}) - m(f, \cancel{[t_{k-1}, t_k]})\,(t_k - t_{k-1})$

$L(f, Q) - L(f, P) \geq 0$

$L(f, Q) \geq L(f, p)$

$L(f, p) \leq L(f, Q)$

# Choose the Correct Answer

1. $U(p, f) =$ [ a ]

   (a) $\sum_{r=1}^{n} M_r \delta_r$  (b) $\sum_{r=1}^{n} m_r \delta_r$

   (c) $\sum_{r=1}^{n} p \delta_r$  (d) None

2. $L(p, f) =$ [ b ]

   (a) $\sum_{r=1}^{n} M_r \delta_r$  (b) $\sum_{r=1}^{n} m_r \delta_r$

   (c) $\sum_{r=1}^{n} \delta_r$  (d) $\|p\|$

3. $\int_{\underline{a}}^{b} f(x)\, dx =$ [ d ]

   (a) Inf. $\{L(p, f) \mid p \in \phi [ab]\}$  (b) Inf $\{U(p, f) \mid p \in \phi [ab]\}$

   (c) Sup. $\{U(p, f) \mid p \in \phi [ab]\}$  (d) Sup. $\{L(p, f) \mid p \in \phi [ab]\}$

4. $\int_{a}^{\overline{b}} f(x)\, dx =$ [ b ]

   (a) Inf. $\{L(p, f) \mid p \in \phi [ab]\}$  (b) Inf $\{U(p, f) \mid p \in \phi [ab]\}$

   (c) Sup. $\{U(p, f) \mid p \in \phi [ab]\}$  (d) Sup. $\{L(p, f) \mid p \in \phi [ab]\}$

5. Necessary and sufficient condition for integrability is [ a ]

   (a) $U(p, f) - L(p, f) < \in$  (b) $L(p, f) - U(p, f) < \in$

   (c) $U(p, f) < L(p, f)$  (d) $U(p, f) - L(p, f)$

6. If $f \in R [a\ b]$ then $\int_{a}^{b} f(x)\, dx =$ [ a ]

   (a) $\lim_{\|p\| \to 0} \sum_{r=1}^{n} f(\xi_r)\, \delta_r$  (b) $\lim_{\|p\| \to 0} f(\xi_r)\, \delta_r$

   (c) $\sum_{r=1}^{n} f(\xi_r)\, \delta_r$  (d) None

7. $f(x) = \begin{cases} 0, & x \text{ is rational} \\ -1, & x \text{ is irrational} \end{cases}$ then $\int_{a}^{\overline{b}} f(x)\, dx =$ [ a ]

   (a) 0  (b) –a

   (c) a – b  (d) –1

8. f(x) is defined in (0, 1) as $f(x) = \frac{1}{n}$ for $\frac{1}{n} \geq x > \frac{1}{n+1}$ and f(0) = 0. Then f(x) in (0, 1) is [ b ]

(a) R - Integrable (b) Not R - Integrable

(c) Totally discontinuous (d) None of these

9. If $f : [a\ b] \to R$ is bounded function and $P_1, P_2 \in [a\ b] \ni P_1 \subset P_2$ then [ c ]

(a) $U(p_1, f) \leq U(p_2, f)$ (b) $L(p_1, f) \geq L(p_2, f)$

(c) $W(p_1, f) \geq W(P_2, f)$ (d) None

10. If f is bounded on [a b] and p be a partition of [a b] then L(p, f) is [ c ]

(a) $\leq m\ (b - a)$ (b) $\geq m\ (b - a)$

(c) $\leq M\ (b - a)$ (d) $\geq M\ (b - a)$

11. A bounded function f is R – integrable on [a b] iff [ a ]

(a) $\int_{\underline{a}}^{b} f(x)\,dx = \int_{a}^{\bar{b}} f(x)\,dx$ (b) $\int_{a}^{b} f(x)\,dx$

(c) f is continuous (d) None

12. If $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{n^2}{n^2 + r^2} = \frac{\Pi}{K}$ then K = [ c ]

(a) 0 (b) 5

(c) 4 (d) 1

13. A bounded function f is R - integrable on [a b] and M, m are bounds of f(x) on [a b] then $\int_{a}^{b} f(x)\,dx$ lies between [ d ]

(a) m (b – a) and M (b + a) (b) m (b + a) and M (b – a)

(c) m (b + a) and M (b + a) (d) m (b – a) and M (b – a)

14. For $f(x) = x^2$, the lower R - integral on [2, 4] is [ b ]

(a) $\frac{1}{3}$ (b) $\frac{56}{3}$

(c) $\frac{1}{2}$ (d) 0

15. The set of ordered pairs $p = \{(I_1, t_1)\ (I_2, t_2)\ .....\ (I_r, t_r)\ .....\ (I_n, t_n)\}$ is called [ b ]

(a) Sub intervals of partition (b) Tagged partition of I

(c) Partition of [a b] (d) None

# Fill in the blanks

1. If f be a bounded function defined on [a, b] and $p_1$, $p_2$ be two partitions of [a. b] such that $p_2$ is refinement of $p_1$ then ________.

2. For Riemann integrability condition of continuity is ________.

3. If f is Riemann integrable on [a, b] then $\left| \int_a^b f(x)\,dx \right| \leq$ ________.

4. If the function f(x) is bounded and integrable on [a, b] such that $f(x) \geq 0 \ \forall \ x \in [a, b]$ where $b \in a$ then $\int_a^b f(x)\,dx$ is ________.

5. If $f(x) = x \ \forall \ x \in [0, 3]$ and $p = \{0, 1, 2, 3\}$ be a partition of p then L(p, f) and U(p, f) are ________ and ________.

6. Length of the $r^{th}$ subinterval $I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$ is ________.

7. If $p = \{x_0, x_1, x_2, ....., x_n\}$ is a partition of [a b] then the n + 1 points are called as ________.

8. $\|p\| =$ ________.

9. $\sum_{r=1}^{n} \delta_r =$ ________.

10. For every partition p of [a b] $L(p, f) \leq$ ________.

11. U(p, f) and L(p, f) are known as ________ and ________.

12. If $P_1, P_2 \in f$ [a b] and $p_1 < p_2$, then the partition $p_2$ is called as a ________ of $p_1$.

13. If f is bounded on [a b] then M and m are known as ________ and ________ of f in [a b].

14. If $f : [a\ b] \to R$ is a bounded function and $p \in \phi$ [a b] then U(p, f) – L(p, f) is called the ________ of f w.r.t partition p.

15. If $\int_{\bar{a}}^{b} f(x)\,dx = \int_{a}^{\bar{b}} f(x)\,dx = \int_a^b f(x)\,dx$ then f is known as ________.

## ANSWERS

1. $L(p_2, f) \geq U(p_1, f)$
2. Sufficient
3. $\int_a^b |f(x)|\, dx$
4. $\geq 0$
5. 3 and 6
6. $\frac{a}{n}$
7. Partition points
8. max $\{\delta_1, \delta_2, ....., \delta_n\}$
9. b – a
10. U(p, f)
11. Upper and Lower Riemann sums
12. Refinement
13. Supremum and Infimum
14. Oscillatory Sum
15. Riemann Integrable

# FACULTY OF SCIENCE

**B.Sc. III - Semester, (CBCS) Examination**

**DECEMBER - 2017**

MATHEMATICS (REAL ANALISYS)

Time : 3 Hours] [Max. Marks : 80

**PART - A (5 × 4 = 20 Marks)**

*Answer any Five of the following questions.*

**ANSWERS**

1. Prove that $\lim_{n\to\infty}\left(n^{\frac{1}{n}}\right)=1.$ **(Unit-I, Q.No. 7)**
2. Prove that every convergent sequence is a Cauchy sequence. **(Unit-I, Q.No. 49)**
3. Let $\{s_n\}$ be a sequence converging to s. Then prove **(Unit-I, Q.No. 45)**

   that $\lim_{n\to\infty}\sigma_n = s$, where $\sigma_n = \frac{1}{n}(s_1+s_2+....+s_n)$.
4. If a series $\Sigma a_n$ converges, then show that $\lim_{n\to\infty} a_n = 0$. **(Unit-I, Q.No. 71)**
5. Find the radius of convergence of $\sum_{n=1}^{\infty}\left(\frac{3^n}{n\cdot 4^n}\right)x^n$. **(Out of Syllabus)**
6. Let $\{f_n\}$ be a sequence of continuous functions on [a, b] and suppose **(Out of Syllabus)**

   that $f_n \to f$ uniformly on [a,b]. Then prove that $\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx$.
7. If f is a bounded function on [a, b] and if P and Q are partitions **(Unit-IV, Q.No. 1)**

   of [a, b] then prove that $L(f,P)\le U(f,Q)$.
8. Let f(x) = x for rational x and f(x) = 0 for irrational x. Calculate the upper and lower Darboux integrals for f on the interval [a, b].

*Ans :*

Given $f(x) = x$ for rational x

$f(x) = 0$ for irrational x

By Daraboux theorem of lower integral.

$$L(P, F) > \underline{\int_a^b} f(x)dx - \in \qquad ...(1)$$

Darboux theorem of upper integral

$$U(P, f) \le \overline{\int_a^b} f(x)dx + \in \qquad ...(2)$$

$\therefore$ The upper and lower integrals far f(x) = x for rational x and f(x) = 0 for irrational 'x' on the internal [0, 1]

## PART - B (4 × 15 = 60 Marks)
## [Essay Answer Type]

*Note : Answer* ***ALL*** *the questions.*

9. a) i) If $\{s_n\}$ converges to $s$ and $\{t_n\}$ converges to $t$ then prove that $s_n + t_n$ converges to $s+t$. **(Unit-I, Q.No. 25)**

ii) Prove that a bounded monotone sequence converges. **(Unit-I, Q.No. 40)**

(OR)

b) i) Prove that every Cauchy sequence is bounded. **(Unit-IV, Q.No. 50)**

ii) Prove that every Cauchy sequence of real numbers is convergent. **(Unit-IV, Q.No. 51)**

10. a) Let $\{s_n\}$ be a sequence, $t \in \text{IR}$. Then prove that there is a subsequence of $\{s_n\}$ converging to $t$ if and only if the set $\{n \in \text{IN} : |s_n - t| < \in\}$ is infinite for each $\in > 0$.

*Ans :*

Let $\{S_n\}$ be a sequence, $t \in \text{IR}$

and Let $\{S_n\}$ be a sequence

We shall prove that $\{S_n\}$ is converges to $t$

i.e. $|S_n - t| < \in$

Suppose the set $\{n \in N : S_n = t\}$ is infinite

Then there are subsequences $(S_{n_k})_{k \in N}$ $\exists\, S_{n_k} = t \;\forall\, k$

Subsequence of $\{s_n\}$ converging to $t$.

We assume $\{n \in N : S_n = t\}$ is finite

then $\{n \in N : 0 < |S_n - t| < \in\}$ is infinite far $\in > 0$

$\because$ $\{n \in N : t - \in < S_n - t\} \cup \{n \in N : t < S_n < t + \in\}$, as $\in \to 0$, we have

$\{n \in N : t - \in < S_n - t\}$ is infinite for all $\in > 0$ ...(1)

$\{n \in N : t < S_n < t + \in\}$ is infinite for all $\in > 0$ ...(2)

Both (1) & (2) finite

Now we will show subsequence $\{S_{n_k}\} k \in N$

$t - 1 < S_{n'} < t$ and

$$\text{Max}\left\{S_{n_{k-1}}, t - \frac{1}{k}\right\} \leq S_{n_k} < t \quad \text{far} \quad k \geq 2 \qquad ...(3)$$

We assume $n_1, n_2 \,.....\, n_{k-1}$ satisfying

This will give us an infinite inceasing sequence $\{n_K\}\ k \in N$ and subsequence

$\{S_n k\}$ of $\{S_n\}$ satisfied (3)

We have $S_{nk-1} \le S_{nk} \nabla k$

$\{S_{n_k}\}$ is monitonically increasing

$\because$ (3) $\Rightarrow$ $t - \frac{1}{k} \le S_{n_k} < t\ \forall k$

$\lim_{k} S_{n_k} = t$

$n_1 < n_2 < .... < n_{k-1}$ ...(4)

max $\left\{sn_{j-1}, t - \frac{1}{j}\right\} \le Sn_j < t$ far $j = 2 .... k - 1$ ...(5)

Using (1) with $\in = \max\left\{S_{n_{k-1}}, t - \frac{1}{k}\right\}$

We can choose $n_k > n_{k-1}$ satisfies (5) for $j = k$,

So that (3) holds for $k$

The sequence $\{n_k\}\ k \in N$ for $|S_n - t| < \in$

Hence the proof.

(OR)

b) i) If the sequence $\{s_n\}$ converges, then prove that every subsequence converges to the same limit.

*Ans :*

Let the sequence $\{S_n\}$ converges t $l$ and

Let the subsequence $\{S_{2n}\}$ of sequence $\{S_n\}$

$\because$ $\{S_n\}$ converges to $l$

$\Rightarrow$ Given $\in > 0\ \exists$ a positive integer $m \to |s_n - l| < \in\ \forall n \ge m$ ...(1)

We can find a natural number $2n_0 \ge m$

If $2n \ge 2n_0$ then $2n \ge m$

$\therefore$ from equation (1) we get,

$|Sn - l| < \in \forall 2n \ge m$

$\Rightarrow$ $\{S_n\}$ converges to $l$

$\therefore$ Every subsequence converges to the same limit.

ii) Prove that every sequence has monotone subsequence. **(Unit-I, Q.No. 54)**

11. a) i) Find the radius of convergence of the series $\sum_{n=1}^{\infty} x^{n!}$. **(Out of Syllabus)**

ii) Prove that the uniform limit of continuous functions is continuous. **(Out of Syllabus)**

(OR)

b) i) State and prove Weierstrass M-test. **(Unit - III, Page No. 71)**

ii) Show that if the series $\Sigma g_n$ converges uniformly on a set s, **(Out of Syllabus)**

then $\lim_{n\to\infty} \sup\{|g_n(x)| : x \in s\} = 0$.

12. a) Define Riemann integral $\int_a^b f(x)dx$. If f is a bounded function on [a, b] then prove that $L(f) \le U(f)$. **(Unit - IV, Q.No. 4)**

(OR)

b) Prove that a bounded function f on [a, b] is integrable if and only if for each $\in > 0$ there exists a partition P of [a,b] such that $U(f, P) - L(f, P) < \in$. **(Unit - IV, Q.No. 28)**

# FACULTY OF SCIENCE

**B.Sc. III - Semester (CBCS) Examination**

**MAY / JUNE - 2018**

## MATHEMATICS

## REAL ANALYSIS

Time : 3 Hours] [Max. Marks : 80

**ANSWERS**

**PART - A (5 × 4 = 20 Marks)**

*Answer any Five of the following questions*

1. Let $\{s_n\}$ be a sequence of non-negative real numbers converging to $s$. Prove that $n \lim\limits_{n \to \infty} \sqrt{s_n} = \sqrt{s}$.

*Sol :*

**Case(i) :**

Given $\{s_n\}$ be the sequence & $s_n \geq 0$

If $s = 0$, then $\lim S_n = 0$

$\forall \in > 0 \ \exists \ m \in N$ such that $|s_n - 0| < \in^2 \ \forall n \geq m$

$\Rightarrow \quad 0 \leq s_n - 0 < \in^2$

$\Rightarrow \quad 0 \leq \sqrt{s_n} < \in \quad \forall \ n \geq m$

**Case (ii) :**

Let $S > 0$ then $\sqrt{s} > 0$

$$\sqrt{s_n} - \sqrt{s} = \frac{(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}} = \frac{s_n - s}{\sqrt{s_n} + \sqrt{s}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{s_n} + \sqrt{s}} \leq \frac{1}{\sqrt{s}} \qquad \therefore \quad \left|\sqrt{s_n} - \sqrt{s}\right| \leq \left(\frac{1}{\sqrt{s}}\right)|s_n - s|$$

Since $\lim S_n = s$ we have $\lim \sqrt{s_n} = \sqrt{s}$ .

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2. Prove that convergent sequences are bounded. **(Unit - I, Q.No.2)**

3. If the sequence $\{s_n\}$ converges, prove that every subsequence of it converges to the same limit. **(Unit - II, Q.No.1)**

4. If $a_n = \sin\left(\frac{n\pi}{3}\right)$, then find lim sup $a_n$ and lim inf $a_n$.

*Ans :*

Given $a_n = \sin\left(\frac{n\pi}{3}\right) \ \forall \ n \in z^+$

But we know that $\forall\ n \in z^+$

$$-1 \le \sin\frac{n\pi}{3} \le 1$$

$$\Rightarrow \left|\sin\frac{n\pi}{3}\right| \le 1$$

$\therefore$ $\{s_n\}$ is bonded.

$\therefore$ lim inf $fs_n = -1$ and lim sup $s_n = 1$

---

5. Check whether the power series $\sum_{n=0}^{\infty}\left(\frac{2^n}{n!}\right)x^n$ converges for every $x \in R$. **(Out of Syllabus)**

6. If $f_n(x) = \frac{1}{n}$ sin nx, $x \in R$, then prove that $f_n \to 0$ uniformly on R. **(Out of Syllabus)**

7. Define Riemann integral $\int_a^b f(x)dx$.

*Ans :*

**Definition :**

Let $f : [a, b] \to R$ be a bounded function and $P = \{a = x_0, x_1, .... x_n = b\}$ be a partition of [a,b]

If $\int_a^b f(x)dx = \sup\ \{L(p,f) / P \in \phi[a,b]\}$ is equal to $\int_a^b f(x)dx = \inf\ \{U(p,f)/P \in \phi[a,b]\}$

then f is Riemann integrable over [a, b].

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8. Prove that every monotonic function f on [a, b] is integrable. **(Unit-IV, Q.No. 5)**

**PART - B (4 × 5 = 60 Marks)**

**[Essay Answer Type]**

**Note : Answer ALL the questions**

9. a) (i) Let $\{s_n\}$ be an increasing sequence of positive numbers. **(Unit-I, Q.No. 45)**

Define $\sigma_n = \frac{1}{n}(s_1 + s_2 + ... + s_n)$. Prove that $\{\sigma_n\}$ is also an increasing sequence.

(ii) Prove that Cauchy sequences are bounded. **(Unit - I, Q.No. 50)**

(OR)

b) (i) Let $t_1 = 1$ and $t_{n+1} = \left(1-\frac{1}{(n+1)^2}\right)t_n$ for $n \ge 1$. Prove by **(Unit - I, Q.No. 47)**

induction that $t_n = \frac{n+1}{2n}$ and hence find the lim $t_n$.

(ii) Prove that Cauchy sequences are convergent. **(Unit-I, Q.No. 51)**

10. a) (i) If $\{s_n\}$ converges to s and $\{t_n\}$ converges to t, then prove that $\{s_n t_n\}$ converges to st. **(Unit - I, Q.No. 26)**

(ii) Calculate $\lim_{n\to\infty} (n!)^{\frac{1}{n}}$.

*Sol. :*

Let $y = (n!)^{\frac{1}{n}}$

Applying logoritham on both sides

$$\log y = \frac{1}{n} \log n!$$

Applying $\underset{n\to\infty}{\text{Lt}}$ on both sides

$$\underset{n\to\infty}{\text{Lt}} \log y = \underset{n\to\infty}{\text{Lt}} \frac{\log n!}{n}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{n} \times \underset{n\to\infty}{\text{Lt}} \log n!$$

$$= \frac{1}{\infty} \times \underset{n\to\infty}{\text{Lt}} \log n!$$

$$= 0 \times \underset{n\to\infty}{\text{Lt}} \log n! = 0$$

$$\underset{n\to\infty}{\text{Lt}} \log y = 0$$

$$\log \underset{n\to\infty}{\text{Lt}} y = 0$$

$$\underset{n\to\infty}{\text{Lt}} y = e^0 = 1$$

(OR)

b) (i) Prove that a series converges if and only if it satisfies the Cauchy criterion.

*Ans :*

Let $s_n$ be the $n^{th}$ partial sum of $\Sigma U_n$

$S_q = u_1 + u_2 + .... + u_p + u_{p+1} + .... + u_q$

$S_q = u_1 + u_2 + .... + u_p$

$S_q - s_p = U_{p+1} + U_{p+2} + .... + U_q$

The series $\Sigma U_n$ converges $\Leftrightarrow$ the sequence $\{s_n\}$ converges

$\Leftrightarrow$ For each $\varepsilon > 0 \ \exists m \in z^+$. such that $|s_p - s_q| < \varepsilon \ \forall \ q \geq p \geq m$

$\Leftrightarrow$ for each $\varepsilon > 0 \ \exists m \in z^+$ such that

$$| U_{P+1} + U_{P+2} + .... + U_q | < \varepsilon \ \forall \ q \geq p \geq m$$

(ii) Check whether the series $\sum_{n=0}^{\infty} 2^{(-1)^n - n}$ converges.

*Ans :*

Given $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^{n-1}}$

which is an alternating series

$U_n = \frac{n}{2n-1}$ then

$U_n - U_{n+1} = \frac{n}{2n-1} - \frac{n+1}{2n+1} = \frac{1}{(2n-1)(2n+1)}$ $U_n > U_{n+1}$ $\forall n \in N$

Also $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n}{2n-1}$

$$= \lim_{n\to\infty} \frac{1}{2-\left(\frac{1}{n}\right)} = \frac{1}{2} \neq 0$$

By Leibnitz's Test

$\Sigma(-1)^{n-1} U_n$ is not convergent.

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11. a) (i) Let $f_n(x) = \frac{1+2\cos^2 nx}{\sqrt{n}}$. Prove that $\{f_n\}$ converges uniformly to 'a' on R. **(Out of Syllabus)**

(ii) If g and h are integrable on [a, b] and if $g(x) \le h(x)$ for all $x \in [a,b]$ then prove that $\int_a^b g(x)dx \le \int_a^b h(x)dx$. **(Unit - I, Q.No. 17)**

(OR)

b) Let $\{f_n\}$ be a sequence of continuous functions on [a, b] and suppose that $f_n \to f$ uniformly on [a, b]. Then prove that **(Out of Syllabus)**

$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)dx.$

12. a) (i) Prove that every continuous function f on [a, b] is integrable. **(Unit - IV, Q.No. 6)**

(ii) If f is integrable on [a, b] then prove that $|f|$ is integrable on [a,b] and $\left|\int_a^b f\right| < \int_a^b |f|$. **(Unit - IV, Q.No. 22)**

(OR)

b) (i) State and prove intermediate value theorem for integrals. **(Unit - IV, Q.No. 19)**

(ii) Prove that $\left|\int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)\,dx\right| \le \frac{16\pi^3}{3}$. **(Unit - IV, Q.No. 24)**

# FACULTY OF SCIENCE

**B.Sc. III - Semester (CBCS) Examination**

**November / December - 2018**

## MATHEMATICS

REAL ANALYSIS

Time : 3 Hours] [Max. Marks : 80

**PART - A (5 × 4 = 20 Marks)**

**(Short Answer Type)**

***Note :*** *Answer any* ***FIVE*** *of the following questions.*

1. Determine the limit of the sequence $\{s_n\}$, where $s_n = \sqrt{n^2+1-n}$

***Sol :***

Given, $S_n = \sqrt{n^2+1}-n$

$$= \sqrt{n^2+1}-n \times \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$$

$$= \frac{\sqrt{\left(n^2+1\right)^2}-n^2}{\sqrt{n^2+1}+n}$$

$$= \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$

$$S_n = \frac{1}{\sqrt{n^2+1}+n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n}$$

$$= \frac{1}{\infty}$$

$$= 0$$

$\therefore S_n = \sqrt{n^2+1}-n$ is converges to '0'

2. Let $t_1 = 1$ and $t_{n+1} = \frac{t_n^2 + 2}{2t_n}$ for $n \geq 1$. Find the $\lim t_n$.

*Sol :*

Given that,

$$t_1 = 1, t_{n+1} = \frac{tn^2 + 2}{2t_n}$$

Let us assume that $\{t_n\}$ converges to t i.e., $\lim t_n = t$

$$\lim t_{n+1} = \lim \left(\frac{t_n{}^2 + 2}{2t_n}\right) = \frac{\lim t_n^2 + 2}{2 \lim t_n}$$

$$\lim t_{n+1} = \frac{tn^2 + 2}{2t}$$

To find the limit , $t_{n+1} = \frac{tn^2 + 2}{2t_n}$ for $n \geq 1$

If n =1, $t_2 = \frac{t_1^2 + 2}{2t_1} = \frac{1+2}{2(1)} = \frac{3}{2} = 1.5$

If n = 2, $t_3 = \frac{t_2^2 + 2}{2t_2} = \frac{\left(\frac{3}{2}\right)^2 + 2}{2\left(\frac{3}{2}\right)}$

$$= \frac{9+8}{12} = \frac{17}{12} = 1.416.....$$

If n = 3, $t_4 = \frac{t^2{}_3 + 2}{2t_3}$

$$= \frac{\left(\frac{17}{12}\right)^2 + 2}{2\left(\frac{17}{12}\right)}$$

$$= \frac{289 + 288}{144} \times \frac{6}{17}$$

$$= \frac{577}{408} = 1.4142156$$

$\therefore$ The given sequence, is converges to $\cong$ 1.414

i.e., $t = \sqrt{2}$

3. If $a_n = \sin\left(\frac{n\pi}{3}\right)$ then find lim sup $a_n$ and lim inf $a_n$.

*Sol :*

$a_n = \sin\left(\frac{n\pi}{3}\right)$ n = 1, 2, 3......

$a_1 = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$a_3 = \sin\left(\frac{3\pi}{3}\right) = 0$

$a_4 = \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$a_5 = \sin\left(\frac{5\pi}{3}\right) = \frac{-\sqrt{3}}{2}$

$a_6 = \sin\left(\frac{6\pi}{3}\right) = 0$ ........

$\therefore$ The set $\left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$ is a subsequential limit

hence the lim sup $a_n = \frac{\sqrt{3}}{2}$

lim inf $a_n = \frac{-\sqrt{3}}{2}$

4. Show that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$. **(Unit-I, Q.No. 68(c))**

5. For n = 0, 1, 2, 3, ..., let $a_n = \left(\frac{4+2(-1)^n}{5}\right)^n$. Find lim sup $(a_n)^{\frac{1}{n}}$ lim inf $(a_n)^{\frac{1}{n}}$ **(Out of Syllabus)**

6. Let $f_n(x) = \frac{1+2\cos^2 nx}{\sqrt{n}}$. Prove that $\{f_n\}$ converges uniformly to 0 on R. **(Out of Syllabus)**

7. Prove that every contiuous function f on [a, b] is integrable.

*Sol :*

Given that,n f is continuous on [a, b]

for each $\varepsilon < 0 \exists$ a partition P on [a, b] $\ni \left|f(x_r) - f(x_{r-1})\right| < \frac{\varepsilon}{b-a}$ & $I_r \in [x_{r-1}, x_r]$

$I_r = [x_{r-1}, x_r]$

sup of f= $M_r$ = $f(M_r)$

Inf of f = $m_r$ = $f(m_r)$

Consider U(P, f) – L (P,f) = $\sum_{r=1}^{n} M_r \delta_r - \sum_{r=1}^{n} m_r \delta_r$

$$= \sum_{r=1}^{n} (M_r - m_r)\delta_r$$

$$= \sum_{r=1}^{n} \left(f(M_r) - f(m_r)\right)\delta_r$$

$$= \sum_{r=1}^{n} \left(f(x_r) - f(x_{r+1})\right)\delta r$$

$$< \frac{\varepsilon}{b-a} \sum_{r=1}^{n} \delta r$$

$$< \frac{\varepsilon}{b-a} \sum_{r=1}^{n} (x_r - x_{r-1})$$

$$< \frac{\varepsilon}{b-a}\left[(x_1 - x_0) + (x_2 - x_1) + \ldots.. + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1})\right]$$

$$\frac{\varepsilon}{b-a}\left[\cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + \ldots.. + \cancel{x_{n-1}} - \cancel{x_{n-2}} + x_n - \cancel{x_{n-1}}\right]$$

$$< \frac{\varepsilon}{b-a}(x_n - x_0)$$

$$< \frac{\varepsilon}{b-a}(b - a)$$

$\therefore$ U (P,f) – L (P, f) < $\varepsilon$

$\therefore$ f is Reimann integrable on [a, b]

8. Show that $\left|\int_{-2\pi}^{2\pi} \sin^8(e^x)dx\right| \le \frac{16\pi^3}{3}$. **(Unit-IV, Q.No. 24)**

**PART - B (4 × 15 = 20 Marks)**
**(Essay Answer Type)**

***Note :*** *Answer* ***ALL*** *the following questions.*

9. (a) (i) If ($S_n$) converges to s, ($t_n$) converges to t, then prove that ($s_n$ $t_n$) converges to s t. **(Unit-I, Q.No. 26)**

(ii) If ($S_n$) converges to s and $s_n \neq 0$ for all n, and if $s \neq 0$, then show that $\left(\frac{1}{s_n}\right)$ converges to $\frac{1}{s}$. **(Unit-I, Q.No. 27)**

**(OR)**

(b) (i) Prove that $\lim_{n\to\infty} a_n = 0$ of $|a_n| < 1$

(ii) Prove that $\lim_{n\to\infty} n^{1/n} = 1$ **(Unit-I, Q.No. 7)**

10. (a) (i) If the sequence ($s_n$) converges, then prove that every subsequence converges to the same limit. **(Unit-I, Q.No. 49)**

(ii) State and prove Bolzano - Weierstrass theorem. **(Unit-I, Q.No. 55)**

**(OR)**

(b) If ($s_n$) converges to a positive real number s and ($t_n$) is any sequence then prove that lim sup $s_n$ $t_n$ = s lim sup $t_n$ **(Unit-I, Q.No. 60)**

11. (a) Let ($f_n$) be a sequence of functions defined and uniformly Cauchy on a set $S \subseteq R$. Then prove that there exists a function f on S such that $f_n \to f$ uniformly on S. **(Out of Syllabus)**

**(OR)**

(b) Derive an explicit formula for $\sum_{n=1}^{\infty} n^2x^n$ for |x| < | and hence evaluate $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$. **(Out of Syllabus)**

12. (a) Let f be a bounded function on [a, b]. If P and Q are partitions of [a, b] and $P \subseteq R$, then prove that $L(f, P) \leq L(f, Q) \leq U(f,Q) \leq U(f,P)$ **(Unit-IV, Q.No. 45)**

**(OR)**

(b) Prove that a bounded function f on [ a, b] in Riemann integrable on [a, b] $\Leftrightarrow$ it is Darboux integrable, in which case the values of the integrals agree.

*Sol :*

Suppose first that f is Darboux integrable on [a,b]

Let $\varepsilon > 0$, and Let $\delta > 0$ be choosen

We know that $\left| s - \int_a^b f \right| < \varepsilon$

for everuy Riemann sum S = $\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$

associated with a partition P having mesh (P) < $\delta$

Clearly, we have $L(f,P) \le S \le U(f, P)$

$U(f,P) < L(f,P) + \varepsilon \le L(f) + \varepsilon = \int_a^b f + \varepsilon$ and $L(f,P) > U(f,P) - \varepsilon \ge U(f) - \varepsilon = \int_a^b f - \varepsilon$

Hence f is Riemann Integrable

$$R\int_a^b f = \int_a^b f$$

Now suppose that f is Riemann Integrable and consider $\varepsilon > 0$. Let $\delta > 0$ and r be as given

$P = \{a = t_0 < t_1 < .....< t_n = b\}$ with mesh (P) < $\delta$

for each k = 1, 2 .....n select $x_n$ in $[t_{k-1}, t_k]$

so that

$f(x_k) < m(f[t_{k-1}, t_k]) + \varepsilon$

The Riemann sums for this choice of $x_k$'s satisfies

$S \le L(f, P) + \varepsilon (b - a)$ as well as $|s - r| < \varepsilon$

It followes that $L(f) \ge L(f,P) \ge S - \varepsilon (b - a) > r - \varepsilon - \varepsilon(b - a)$

Since $\varepsilon$ is arbitary

We have $L(f) \ge r$

Similarly $U(f) \le r$

Since $L(f) \le U(f)$

as we see that $L(f) = U(f) = r$

This showes that f is integrable and $\int_a^b f = r = R\int_a^b f$

# FACULTY OF SCIENCE

**B.Sc. III - Semester (CBCS) Examination**

**JUNE / JULY - 2019**

## MATHEMATICS

## REAL ANALYSIS

Time : 3 Hours] [Max. Marks : 80

**PART - A (5 × 4 = 20 Marks)**

**(Short Answer Type)**

***Note :*** *Answer any* ***FIVE*** *of the following questions.*

1. Compute $\lim_{n\to\infty} (\sqrt{4n^2+n}) - 2n.$ **(Out of Syllabus)**

2. Computer $\lim_{n\to\infty}\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+....\frac{1}{3^n}\right).$ **(Unit-I, Q.No. 35)**

3. Does the series $\sum_{n=4}^{\infty}\frac{1}{n(\log n)(\log\log n)}$ converge ? Justify your answer. **(Unit-I, Q.No. 68(c))**

4. Find the set of subsequential limits of the sequence $(s_n)$, where **(Out of Syllabus)**

   $s_n = \cos\left(\frac{n\pi}{3}\right).$

5. For $n = 0, 1, 2, ....,$ let $a_n = \left(\frac{4+2(-1)^n}{5}\right)^n \cdot$ Find lim sup $\left|\frac{a_{n+1}}{a_n}\right|$ and **(Out of Syllabus)**

   lim inf $\left|\frac{a_{n+1}}{a_n}\right|.$

6. Let $f_n(x) = \frac{nx}{1+nx}$, $x \in (0,\infty)$. Find $f(x) = \lim_{n\to\infty} f_n(x).$ **(Out of Syllabus)**

7. If $f$ and $g$ are integrable on $[a, b]$ then prove that $\int_a^b f \le \int_a^b g$ **(Unit-IV, Q.No. 21)**

   wherever for all $x$ in $[a, b]$.

8. State and prove intermediate value theorem for integrals. **(Unit-IV, Q.No. 39)**

**PART - B (4 × 15 = 60 Marks)**

**(Essay Answer Type)**

***Note :*** *Answer* **ALL** *the questions.*

9. a) i) Prove that all bounded monotone sequences converge. **(Unit-I, Q.No. 40)**

   ii) Let $(s_n)$ be an increasing sequence of positive numbers and **(Unit-I, Q.No. 45)**

   define $\sigma_n = \frac{1}{n}(s_1+s_2+...s_n)$. Prove that $(\sigma_n)$ is also an increasing sequence.

(OR)

b) i) Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$. Prove that $s_n \geq \frac{1}{2}$ for all n, by using induction. **(Unit-I, Q.No. 48)**

ii) Let $t_1 = 1$ and $t_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right)t_n$ for $n > 1$, prove that **(Unit-I, Q.No. 47)**

$t_n = \frac{n+1}{2n}$.

10. a) i) Prove that every sequence $(s_n)$ has a monotonic subsequence. **(Unit-I, Q.No. 54)**

ii) Prove that every bounded sequence has a convergent sequence. **(Unit-I, Q.No. 55)**

(OR)

b) i) Let $(s_n)$ be a sequence of non-zero real numbers. Prove that **(Unit-I, Q.No. 61)**

$$\lim\inf\left|\frac{s_{n+1}}{s_n}\right| \leq \lim\inf|s_n|^{\frac{1}{n}} \leq \lim\sup|a_n|^{\frac{1}{n}} \leq \lim\sup\left|\frac{s_{n+1}}{s_n}\right|.$$

11. a) Let $(f_n)$ be a sequence of continuous functions on [a, b], and suppose that $f_n \to f$ uniformly on [a, b]. Then prove that **(Out of Syllabus)**

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

(OR)

b) For $|x| < 1$, derive an explicit formula for $\sum_{n=1}^{\infty} n^2 x^n$ and hence evaluate $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$. **(Out of Syllabus)**

12. a) Prove that a bounded function f on [a, b] is integrable if and only if for each $\in > 0$. there exists a partition P of [a, b] such that $U(f,P) - L(f,P) < \in$. **(Unit-IV, Q.No. 28)**

(OR)

b) Let f be a function defined on [a, b]. If $a < c < b$ and f is integrable on [a, c] and on [c, b], then prove that **(Unit-IV, Q.No. 43)**

i) f is integrable on [a, b] and

ii) $\int_a^b f = \int_a^c f + \int_c^b f.$

# FACULTY OF SCIENCE

**B.Sc. III - Semester (CBCS) Examination**

**MODEL PAPER - I**

## REAL ANALYSIS

## (MATHEMATICS)

Time : 3 Hours] [Max. Marks : 80

**PART - A (8 × 4 = 32 Marks)**

**(Short Answer Type)**

***Note :*** *Answer any **Eight** of the following questions.*

1. Every convergent sequence is bounded. **(Unit-I, Q.No. 2)**

2. Every Convergent Sequence is a Cauchy Sequence. **(Unit-I, Q.No. 49)**

3. Does series converge? Justify your answer. **(Unit-I, Q.No. 68(a))**

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$$

4. Prove that $x = \cos(x)$ for some x in $\left(0, \frac{\pi}{2}\right)$. **(Unit-II, Q.No. 15)**

5. Let $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$ Prove that f is continuous at 0. **(Unit-II, Q.No. 3)**

6. Is the function $f(x) = x^2$ Uniformly continuous on [–7, 7]? **(Unit-II, Q.No. 23)**

7. Show that $\sin x \leq x$ for all $x \geq 0$. **(Unit-III, Q.No. 17)**

8. Let $f(x) = x \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. **(Unit-III, Q.No. 8)**

   (a) Observe that f is continuous at $x = 0$

   (b) Is f differentiable at $x = 0$ ? Justify your answer.

9. Find the limit for $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$. **(Unit-III, Q.No. 30)**

10. If $f \in R[a\ b]$ and m, M are the inf. and sup. of f in [a b] then **(Unit-IV, Q.No. 19)**

    $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$ and $\int_a^b f(x)\,dx = \mu(b-a)$

    where $\mu \in [m, M]$.

11. If, $f, g \in R[a\ b]$ and $f(x) \geq g(x)\ \forall\ x \in [a, b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ **(Unit-IV, Q.No. 17)**

12. Prove that every continuous function defined on [a, b] is integrable. **(Unit-IV, Q.No. 6)**

**PART - B (4 × 12 = 48 Marks)**
**(Essay Answer Type)**

***Note :*** *Answer* **ALL** *the questions.*

13. a) All bounded monotone sequence converge. **(Unit-I, Q.No. 40)**

(i) Every monotonically increasing sequence which is bounded above is convergent.

(ii) Every monotonically decreasing sequence which is bounded below is convergent.

OR

b) (i) Let s denote the set of subsequential limit of sequence $\{s_n\}$. Suppose $\{t_n\}$ is a sequence in $S \cap R$ and that $t = \lim t_n$ then $t \in s$. **(Unit-I, Q.No. 57)**

(ii) If the sequence $\{s_n\}$ converges to $\ell$ prove that it is subsequence also converges to $\ell$. **(Unit-I, Q.No. 53)**

14. a) Verify f is continuous on set $S \subseteq$ dom(f) if an only if for each $x_0 \in s$ and $\varepsilon > 0$ there is $\delta > 0$ so that $x \in$ dom(f) and $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$ for the function $f(x) = \frac{1}{x^2}$ on $(0, \infty)$. **(Unit-II, Q.No. 20)**

OR

b) i) If f and g are real valued functions at $x_0$ then, **(Unit-II, Q.No. 5)**

(1) f + g is continuous at $x_0$

(2) fg is continuous at $x_0$

(3) f/g is continuous at $x_0$ if $g(x_0) \neq 0$.

ii) A real valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function on [a, b]. **(Unit-II, Q.No. 27)**

15. a) Let $a, b \in R$. let $f(x) = e^{ax} \cos(bx)$ and $g(x) = e^{ax} \sin(bx)$ **(Unit-III, Q.No. 20)**

(i) Compute f'(x) and g'(x)

(ii) Use (i) to compute f'' and f'''

OR

b) Let f be continuous function on [a, b] that is differentiable at (a, b). Then there exist [at least one] $c \in [a, b]$ such that **(Unit-III, Q.No. 10)**

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

16. a) If f, g $\in$ R [a b], then f + g $\in$ R [a b] and $\int_a^b (f + g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. **(Unit-IV, Q.No. 13)**

OR

b) Prove that every monotonic function on [a, b] is integrable. **(Unit-IV, Q.No. 5)**

# FACULTY OF SCIENCE

## B.Sc. III - Semester (CBCS) Examination

### MODEL PAPER - II

## REAL ANALYSIS

## (MATHEMATICS)

Time : 3 Hours] [Max. Marks : 80

### PART - A (8 × 4 = 32 Marks)

### (Short Answer Type)

***Note :** Answer any **Eight** of the following questions.*

1. If a series $\Sigma a_n$ converges them $\lim a_n = 0$. **(Unit-I, Q.No. 71)**
2. Let $\{s_n\}$ be sequence in R prove that the $\lim s_n = 0$ iff $\lim |s_n| = 0$. **(Unit-I, Q.No. 23)**
3. Calculate, $\lim\limits_{n\to\infty}\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+....+\frac{1}{3^n}\right)$. **(Unit-I, Q.No. 35)**
4. Let f and g be continuous function, on [a, b] such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$ prove that $f(x_0) = g(x_0)$ for at lest one $x_0$ in [a, b]. **(Unit-II, Q.No. 17)**
5. Show that the function f defined by $f(x) = x^3$ is uniformly continous in [–2, 2]. **(Unit-II, Q.No. 28)**
6. Prove that $f : R \to R$ given by $f(x) = x^2$ is a continous function on R but not Uniformly continous on R. **(Unit-II, Q.No. 26)**
7. Find limit $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^x$. **(Unit-III, Q.No. 35)**
8. If f is differentiable at a and g is differentiable at f(a), then the composite function gof is differentiable at a and (gof)'(a) = g'(f(a)).f'(a). **(Unit-III, Q.No. 4)**
9. Show that $ex \leq e^x$ for all $x \in R$. **(Unit-III, Q.No. 16)**
10. Show that f(x) = 3x + 1 is integrable on [1, 2] and $\int_1^2 (3x + 1)\,dx = \frac{11}{2}$. **(Unit-IV, Q.No. 29)**
11. If $f \in R[a\ b]$ then $|f| \in R[a\ b]$ **(Unit-IV, Q.No. 12)**
12. Prove that $\left|\int_{-2\pi}^{2\pi} x^2\sin^8(e^x)dx\right| \leq \frac{16\pi^3}{3}$ **(Unit-IV, Q.No. 24)**

### PART - B (4 × 12 = 48 Marks)

### (Essay Answer Type)

***Note :** Answer **ALL** the questions.*

13. a) (i) Every bounded sequence has convergent subsequence. **(Unit-I, Q.No. 55)**

(ii) Let an = 3 + 2(–1)n for $n \in N$. **(Unit-I, Q.No. 58)**

a) List the first eight terms of the sequence $(a_n)$.

b) Give a subsequence that is constant {takes a single values specify the selection function $\sigma$.

OR

b) (i) If $\{s_n\}$ is converges to s, and $\{t_n\}$ is converges to 't'. Then $\{s_n + t_n\}$ converges to s + t that is $\lim \{s_n + t_n\} = \lim s_n + \lim t_n$. **(Unit-I, Q.No. 25)**

(ii) Let $(S_n)$ be an increasing sequence of positive number and define $\sigma_n = \frac{1}{n}(S_1 + S_2 + .... + S_n)$ prove $(\sigma_n)$ is an increasing sequence. **(Unit-I, Q.No. 45)**

14. a) Let f be a real valued function whose domain is a subset of R. Then f is continuous at $x_0$ in dom(f) if and only if for each $e > 0 \exists d > 0 \ni x \in \text{dom}(f)$ and $|x - x_0| < d \Rightarrow |f(x) - f(x_0)| < \varepsilon$. **(Unit-II, Q.No. 1)**

OR

b) (i) If f is continuous on a closed interval [a, b] then f is uniformly continous on [a, b]. **(Unit-II, Q.No. 24)**

(ii) Find the limit $\lim_{x \to b} \frac{\sqrt{x} - \sqrt{b}}{x - b}$, b > 0. **(Unit-II, Q.No. 43)**

15. a) f : [a, b] $\rightarrow$ R is such (i) f is continuous on [a, b] (ii) f is derivable on (a, b) and (iii) f(a) = f(b). The there exists $c \in (a,b)$ such that f'(c) = 0. **(Unit-III, Q.No. 9)**

OR

b) Let $f(x) = x \sin\frac{1}{x}$ for $x \neq 0$ and f(0) = 0. **(Unit-III, Q.No. 8)**

(a) Observe that f is continuous at x = 0

(b) Is f differentiable at x = 0 ? Justify your answer.

16. a) If U and V are continuous function on [a, b] that are differentiable on (a, b) and if U' and V' are integrable on [a, b] then $\int_a^b U(x)\, V'(x)\, dx + \int_a^b U'(x)\, V(x) dx = U(b)\, V(b) - U(a)V(a)$ **(Unit-IV, Q.No. 41)**

OR

b) If $f \in R$ [a b] and m, M are the inf. and sup. of f in [a b] then **(Unit-IV, Q.No. 19)**

$m(b - a) \leq \int_a^b f(x)\, dx \leq M(b - a)$ and $\int_a^b f(x)\, dx = \mu(b - a)$

where $\mu \in [m, M]$.

# FACULTY OF SCIENCE

**B.Sc. III - Semester (CBCS) Examination**

**MODEL PAPER - III**

## REAL ANALYSIS
## (MATHEMATICS)

Time : 3 Hours] [Max. Marks : 80

**PART - A (8 × 4 = 32 Marks)**
**(Short Answer Type)**

***Note :*** *Answer any* ***Eight*** *of the following questions.*

1. State and prove Sandwich Theorem or Squeeze Theorem. **(Unit-I, Q.No. 5)**
2. Every convergent sequence is bounded. **(Unit-I, Q.No. 2)**
3. If $\{s_n\}$ converges to s, if $s_n \neq 0\ \forall n$ and if $s \neq 0$, then $\left\{\frac{1}{s_n}\right\}$ converges to $\frac{1}{s}$. **(Unit-I, Q.No. 27)**
4. If f is uniformly continous on an aggregate s and $\{s_n\}$ is a Cauchy sequence in s, then prove that $\{f(s_n)\}$ is also Cauchy sequence. **(Unit-II, Q.No. 29)**
5. Let $f(x) = 2x^2 + 1$ for $x \in R$, Prove f is continuous on R, by. **(Unit-II, Q.No. 2)**
   (a) Using the definition
   (b) Using the $\varepsilon - \delta$ property
6. Final $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$. **(Unit-II, Q.No. 40)**
7. Find the limit for $\lim_{x \to 0} \frac{\tan x - x}{x^3}$. **(Unit-III, Q.No. 31)**
8. If f is differentiable at a point 'a'. Then 'f' is continuous at a. **(Unit-III, Q.No. 1)**
9. Expassion of $e^x$. **(Unit-III, Q.No. 40)**
10. If $f \in R\,[a\ b]$ and $K \in R$, then $K f \in [a\ b]$ and $\int_a^b (K f)(x)\,dx = K \int_a^b f(x)\,dx$. **(Unit-IV, Q.No. 11)**
11. Given that f is a bounded function on [a, b] their exist sequence $(U_n)$ and $(L_n)$ upper and lower darboux. **(Unit-IV, Q.No. 8)**
12. If $f : [ab] \to R$ is a bounded function then $\int_a^b f(x)\,dx \leq \int_a^{\bar{b}} f(x)\,dx$. **(Unit-IV, Q.No. 1)**

**PART - B (4 × 12 = 48 Marks)**
**(Essay Answer Type)**

***Note :*** *Answer* **ALL** *the questions.*

13. a) (i) Every sequence $\{s_n\}$ has a monotonic subsequence. **(Unit-I, Q.No. 54)**

(ii) Let s denote the set of subsequential limit of sequence $\{s_n\}$. Suppose $\{t_n\}$ is a sequence in $S \cap R$ and that $t = \lim t_n$ then $t \in s$. **(Unit-I, Q.No. 57)**

OR

b) (i) If $\{s_n\}$ is converges to s and $\{t_n\}$ is converges to t, then $\{s_n\ t_n\}$ converges to st i.e., lim $(s_n\ t_n)$ = (lim $s_n$) (lim $t_n$). **(Unit-I, Q.No. 26)**

(ii) Prove that $a^n = 0$ for $|a| < |$ **(Unit-I, Q.No. 7)**

(a) $\lim_{n \to \infty} n^{1/n} = 1$

(b) $\lim_{n \to \infty} a^{1/n} = 1$ for a > 0

14. a) Show $f(x) = \frac{1}{x^2}$ is uniformly continous on $[0, \infty)$. **(Unit-II, Q.No. 32)**

OR

b) Let $f_1$ and $f_2$ be function for which the limits $L_1 = \lim_{x \to a^s} f_1(x)$ **(Unit-II, Q.No. 45)**

and $L_2 = \lim_{x \to a^s} f_2(x)$ exist and are finite. Then

(i) $\lim_{x \to a^s} (f_1 + f_2)(x)$ exists and equals $L_1 + L_2$

(ii) $\lim_{x \to a^s} (f_1\ f_2)(x)$ exits and equals $L_1\ L_2$

(iii) $\lim_{x \to a^s} (f_1 / f_2)(x)$ exits and equals $L_1/L_2$ provides $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in s$

15. a) Discuss the differentiability of $f(x) = |x - a|$ in R. **(Unit-III, Q.No. 6)**

OR

b) Let f and of be functions that are differentiable at the points each of the functions cf [c a constant], f+g, fg and f/g is also differentiable at a, except f/g if g(a) = 0 since f/g is not defined at a in this case. **(Unit-III, Q.No. 2)**

The formulas are

1. $(cf)'(a) = c\,f'(a)$
2. $(f + g)'(a) = f'(a) + g'(a)$
3. $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$
4. $(f/g)'(a) = [g(a)f'(a) - f(a)\,g'(a)]/g^2(a)$ if $g(a) \neq 0$.

16. a) Prove that every continuous function defined on [a, b] is integrable. **(Unit-IV, Q.No. 6)**

OR

b) If $f \in R$ [a b] and m, M are the infimum and Supremum of f on [a b], then $m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$. **(Unit-IV, Q.No. 3)**