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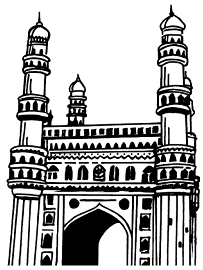

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C O N T E N T S

DIFFERENTIAL EQUATIONS

STUDY MANUAL

SOLVED MODEL PAPERS

SYLLABUS

UNIT - I

Differential Equations of first order and first degree: Introduction - Equations in which Variables are Separable - Homogeneous Differential Equations - Differential Equations Reducible to Homogeneous Form - Linear Differential Equations - Differential Equations Reducible to Linear Form - Exact differential equations - Integrating Factors - Change in variables - Total Differential Equations - Simultaneous Total Differential Equations - Equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

UNIT - II

Differential Equations first order but not of first degree: Equations Solvable for p - Equations Solvable for y - Equations Solvable for x - Equations that do not contain x (or y) - Equations Homogeneous in x and y - Equations of the First Degree in x and y - Clairaut's equation. Applications of First Order.

Differential Equations: Growth and Decay-Dynamics of Tumour Growth -Radio activity and Carbon Dating - Compound Interest - Orthogonal Trajectories.

UNIT - III

Higher order Linear Differential Equations: Solution of homogeneous linear differential equations with constant coefficients - Solution of non-homogeneous differential equations P(D)y = Q(x) with constant coefficients by means of polynomial operators when Q(x) = be^{ax}, b sin ax/b cos ax, bx^k, V e^{ax} - Method of undetermined coefficients

UNIT - IV

Method of variation of parameters - Linear differential equations with non constant coefficients - The Cauchy - Euler Equation - Legendre's Linear Equations - Miscellaneous Differential Equations. Partial Differential Equations: Formation and solution- Equations easily integrable - Linear equations of first order.

Contents

UNIT I

Differential Equations of first order and first degree: Introduction - Equations in which Variables are Separable - Homogeneous Differential Equations - Differential Equations Reducible to Homogeneous Form - Linear Differential Equations - Differential Equations Reducible to Linear Form - Exact differential equations - Integrating Factors - Change in variables - Total Differential Equations - Simultaneous Total Differential Equations - Equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

1.1 Introduction to Differential Equations of First Order and First Degree

Q1. Define

a) Differential equations

b) Ordinary differential equation

c) Partial differential equation

d) Order of a differential equation

e) Degree of differential equation

Ans :

a) Differential Equations

An equation involving derivaties or differentials of one or more dependent vaiables with respect to one or more independent variables is called differential equation.

Eg: $\frac{dy}{dx} = \sin x + \cos x$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$$\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^4 = e^t$$

b) Ordinary differential equations

A differential equation which involves derivatives with respect to single independent variable is known as an Ordinary equation.

Eg: $xy'' + y' + xy = 0$

c) Partial differential equations

A differential equation which contains two or more independent variables and partial derivatives with respect to them is called a partial differential equation.

Eg: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

d) Order of a differential equation

The order of the highest order derivative involved in a differential equation is called the order of the differential equation.

Eg: $dy = (x + \sin x)dx$ is first order equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is second order equation.

e) Degree of differential equation

The degree of a differential equation is the degree of the highest derivative which occurs in it, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned.

Eg: $\frac{\partial^2 v}{\partial t^2} = K\left(\frac{\partial^3 v}{\partial x^3}\right)^2$

The degree of differential equation is '2'.

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t$$

of degree is '1'.

➢ An ordinary differential equation of the first order and first degree is of the form

$$\frac{dy}{dx}+f(x,y)=0$$

which is written as $M\,dx+N\,dy=0$ where M and N are functions of x and y or constant.

1.2 EQUATIONS IN WHICH VARIABLES ARE SEPARABLE

Q2. Explain the working rule of variable separable.

Ans :

If the differential equation $\frac{dy}{dx}=f(x,y)$ can be expressed in the form $\frac{dy}{dx}=\frac{f_1(x)}{f_2(y)}$ where f and g are continuous functions of a single variable, It is said to be of the form variable separable.

Working Rule to find the General Solution

1. The given equation $\frac{dy}{dx}=\frac{f_1(x)}{f_2(y)}$ can be written by separating variable as

$$f_1(x)\,dx = f_2(y)\,dy \quad ...\ (1)$$

2. Integrable both sides of (1) and adding an arbitary constant of integration to any one of the two sides.

3. General Solution of (1) is $\int f_1(x)dx = \int f_2(y)dy + C$ where C is constant of Integration, is the required solution.

Q3. Solve $(1-x)dy+(1-y)dx=0$

Sol :

The given equation is $(1-x)dy+(1-y)dx=0$ which can be written as

$$\frac{dx}{1-x}+\frac{dy}{1-y}=0$$

By Integrating, $\int\frac{1}{1-x}dx+\int\frac{1}{1-y}dy=C$

$$\log(1-x)(-1)+\log(1-y)(-1)=-\log C_1$$

$$-\log(1-x)-\log(1-y)=-\log C_1$$

$$-[\log(1-x)+\log(1-y)]=-\log C_1$$

$$\log(1-x)(1-y)=\log C_1$$

$$(1-x)(1-y)=C_1$$

$\therefore$ The required solution is $(1-x)(1-y)=C$.

Q4. Solve $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$

Sol :

Given that $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$ which may be written as

$$(\sin y + y\cos y)dy = x(2\log x + 1)dx$$

By Integrating, we have

$$\int \sin y\,dy + \int y\cos y\,dy = 2\int x\log x\,dx + \int x\,dx$$

We know that $\int UV = U\int V dx - \int U^1\left[\int V dx\right]dx$

$$\Rightarrow\ -\cos y + y\int \cos y\,dy - \int \frac{d}{dy}(y)\left[\int \cos y\,dy\right]dy = 2\left[\log x\int x\,dx - \int \frac{1}{x}\left[\int x\,dx\right]dx\right] + \int x\,dx$$

$$\Rightarrow -\cos y + y\sin y - \int \sin y\,dy = 2\left[\log x\frac{x^2}{2} - \int \frac{x}{2}dx\right] + \int x\,dx$$

$$\Rightarrow\ \cancel{-\cos y} + y\sin y + \cancel{\cos y} = 2\left[\log x\frac{x^2}{2} - \frac{x^2}{4}\right] + \frac{x^2}{2} + C$$

$$y\sin y = 2\frac{x^2}{2}\log x - \frac{2x^2}{4} + \frac{x^2}{2} + C$$

$$y\sin y = x^2\log x - \cancel{\frac{x^2}{2}} + \cancel{\frac{x^2}{2}} + C$$

$y\sin y = x^2\log x + C$ which is the required solution.

Q5. Solve $(e^y + 1)\cos x\,dx + e^y\sin x\,dy = 0$

Sol :

Given that $(e^y + 1)\cos x\,dx + e^y\sin x\,dy = 0$ which can be written as

$$\frac{\cos x}{\sin x}dx + \frac{e^y}{(e^y + 1)}dy = 0$$

By Inegrating, we have

$$\int \frac{\cos x}{\sin x} dx + \int \frac{e^y}{(e^y + 1)} dy = C$$

$$\int \cot x dx + \int \frac{e^y}{(e^y + 1)} dy = C$$

$$\log \sin x + \log(e^y + 1) = \log C$$

$$\log\left[\sin x + (e^y + 1)\right] = \log C$$

$\therefore \sin x(e^y + 1) = C$ which is required solution.

Q6. Solve $\frac{dy}{dx} = (4x + y + 1)^2$

Sol :

Given that $\frac{dy}{dx} = (4x + y + 1)^2$

Let 4x + y + 1 = V ... (1)

differentiating (1) with respect to x.

Then, we get

$$4 + \frac{dy}{dx} = \frac{dV}{dx}$$

$$\frac{dy}{dx} = \frac{dV}{dx} - 4 \qquad \text{... (2)}$$

By using (1) and (2) in given equation

$$\frac{dV}{dx} - 4 = V^2$$

$$\frac{dV}{dx} = V^2 + 4$$

Now separating variable of x and V.

Then we have, $dx = \frac{dV}{V^2 + 4}$

By Integrating, $\int dx = \int \frac{1}{V^2 + 4} dV$

$$\int dx = \int \frac{dV}{V^2 + 2^2} \qquad \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$x + C = \frac{1}{2}\tan^{-1}\frac{V}{2}$$

$$2x + C = \tan^{-1}\frac{V}{2}$$

$$\tan(2x + C) = \frac{V}{2}$$

$$V = 2\tan(2x + C)$$

$\therefore 4x + y + 1 = 2\tan(2x + C)$ which is required solution.

Q7. Solve $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$.

Sol :

Given that $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

Let $x + y = V$ ... (1)

Differentiating (1) with respect to 'x'

Then we get

$$1 + \frac{dy}{dx} = \frac{dV}{dx}$$

$$\frac{dy}{dx} = \frac{dV}{dx} - 1 \quad \text{... (2)}$$

By using (1) and (2) in given equation

$$\frac{dV}{dx} - 1 = \sin V + \cos V$$

$$\frac{dV}{dx} = \sin V + \cos V + 1$$

But $\left[\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \quad \cos\theta + 1 = 2\cos^2\frac{\theta}{2}\right]$

$$\frac{dV}{dx} = 2\sin\frac{V}{2}\cos\frac{V}{2} + 2\cos^2\frac{V}{2}$$

$$= 2\cos^2\frac{V}{2}\left[\frac{\sin\frac{V}{2}}{\cos\frac{V}{2}} + 1\right]$$

$$\frac{dV}{dx} = 2\cos^2\frac{V}{2}\left[\tan\frac{V}{2} + 1\right]$$

$$\frac{dV}{2\cos^2\frac{V}{2}\left[\tan\frac{V}{2}+1\right]} = dx$$

$$\frac{\frac{1}{2}\sec^2\frac{V}{2}}{1+\tan\frac{V}{2}}dV = dx$$

By Integrating,

$$\int\frac{\frac{1}{2}\sec^2\frac{V}{2}}{1+\tan\frac{V}{2}}dV = \int dx + C$$

$$\log\left(1+\tan\frac{V}{2}\right) = x + C$$

$$\log\left(1+\tan\frac{x+y}{2}\right) = x + C$$

which is required solution.

Q8. Solve $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

Sol :

Given that $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$ which can be written as

$$y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$$

$$y - ay^2 = (a+x)\frac{dy}{dx}$$

$$y(1-ay) = (a+x)\frac{dy}{dx}$$

By variable separable

$$\frac{dy}{y(1-ay)} = \frac{dx}{(a+x)} \qquad \text{... (1)}$$

Resolving the left hand side into partial fraction, we get

consider

$$\frac{1}{y(1-ay)} = \frac{A}{y} + \frac{B}{1-ay}$$

$$1 = A(1-ay) + B(y)$$

If $y = \frac{1}{a}$

$$1 = \left(1 - a\left(\frac{1}{a}\right)\right) + B\left(\frac{1}{a}\right)$$

$$1 = B\left(\frac{1}{a}\right)$$

$$\boxed{B = a}$$

If $y = 0$

$$1 = A(1 - a(0)) + B(0)$$

$$A = 1$$

$$\therefore \quad \frac{1}{y(1-ay)} = \frac{1}{y} + \frac{a}{1-ay}$$

$$\left(\frac{1}{y} + \frac{a}{1-ay}\right)dy = \frac{dx}{a+x}$$

By Integrating, we get

$$\int\left(\frac{a}{1-ay} + \frac{1}{y}\right)dy = \int\frac{dx}{(a+x)} + C$$

$$\log(1-ay)(-1) + \log y = \log(x+a) + \log C$$

$$-\log(1-ay) + \log y = \log(x+a) + \log C$$

$$\log\left(\frac{y}{1-ay}\right) = \log\left(C(x+a)\right)$$

$$\frac{y}{1-ay} = C(x+a)$$

$y = C(x+a)(1-ay)$ which is required solution.

Q9. Solve $\frac{dy}{dx} + x^2 = x^2e^3y$

Sol :

Given that $\frac{dy}{dx} + x^2 = x^2e^3y$ which may be written as,

$$\frac{dy}{dx} = x^2 e^3 y - x^2$$

$$\frac{dy}{dx} = x^2(e^3 y - 1)$$

$$\frac{dy}{e^3 y - 1} = x^2 dx$$

$$\frac{e^{-3y}}{e^{-3y}(e^3 y - 1)} dy = x^2 dx$$

$$\frac{e^{-3y}}{1 - e^{-3y}} dy = x^2 dx$$

By Integrating,

$$\frac{1}{3}\log(1 - e^{-3y}) = \frac{x^3}{3} + C$$

$$\log(1 - e^{-3y}) = x^3 + 3C$$

$$1 - e^{-3y} = e^{x^3 + 3C_1}$$

where $3C = C_1$

$$= e^{x^3} e^{C_1}$$

$$= C_2 e^{x^3}$$

where $C_2 = e^{C_1}$

$$1 - e^{\frac{1}{3y}} = C_2 e^{x^3}$$

$$e^{3y} - 1 = C_2 e^{x^3} e^{3y}$$

$$e^{3y} - 1 = C_2 e^{x^3 + 3y}$$

which is required solution.

Q10. Solve $(x+y)^2 \frac{dy}{dx} = a^2$

Sol :

Given that $(x+y)^2 \frac{dy}{dx} = a^2$

Let $x + y = V$ ... (1)

differentiating (1) with respect to 'x'

$$1 + \frac{dy}{dx} = \frac{dV}{dx}$$

$$\frac{dy}{dx} = \frac{dV}{dx} - 1 \qquad \text{... (2)}$$

By using (1) and (2) in given equation

$$V^2\left[\frac{dV}{dx} - 1\right] = a^2$$

$$V^2 \frac{dV}{dx} - V^2 = a^2$$

$$V^2 \frac{dV}{dx} = a^2 + V^2$$

$$\frac{dV}{dx} = \frac{a^2 + V^2}{V^2}$$

By separating variable.

$$\frac{V^2}{V^2 + a^2} dV = dx$$

$$\left[1 - \frac{a^2}{V^2 + a^2}\right] dV = dx$$

By Integrating,

$$V - a^2 \frac{1}{a} \tan^{-1}\frac{V}{a} = x + C$$

$$(x + y) - a\tan^{-1}\frac{x + y}{a} = x + C$$

$$x + y - x - a\tan^{-1}\frac{x + y}{a} = C$$

$$\therefore y = a\tan^{-1}\frac{x + y}{a} = x + C$$

which is required solution.

Q11. Solve $\frac{x+y-a}{x+y-b}\frac{dy}{dx} = \frac{x+y+a}{x+y+b}$

Sol :

Given that

$$\frac{x+y-a}{x+y-b}\frac{dy}{dx} = \frac{x+y+a}{x+y+b}$$

Let $x + y = V \longrightarrow (1)$

differentiating (1) with respect to 'x'

$$1 + \frac{dy}{dx} = \frac{dV}{dx} \Rightarrow \frac{dy}{dx} = \frac{dV}{dx} - 1 \longrightarrow (2)$$

By using (1) and (2) in given equation

$$\frac{V-a}{V-b}\left[\frac{dV}{dx} - 1\right] = \frac{V+a}{V+b}$$

$$\frac{dV}{dx} - 1 = \frac{V+a}{V+b} \times \frac{V-b}{V-a}$$

$$\frac{dV}{dx} - 1 = \frac{(V+a)(V-b)}{(V+b)(V-a)}$$

$$\frac{dV}{dx} = \frac{V^2 - Vb + aV - ab}{V^2 - aV + bV - ab} + 1$$

$$= \frac{V^2 - \cancel{Vb} + \cancel{aV} - ab + V^2 - \cancel{aV} + \cancel{bV} - ab}{V^2 - aV + bV - ab}$$

$$\frac{dV}{dx} = \frac{2V^2 - 2ab}{V^2 - aV + bV - ab}$$

$$\frac{dV}{dx} = \frac{2[V^2 - ab]}{V^2 - aV + bV - ab}$$

By variable separable.

$$\left[\frac{V^2 - ab}{V^2 - aV + bV - ab}\right] dV = 2dx$$

$$\left[1 + \frac{V(b-a)}{V^2 - ab}\right] dV = 2dx$$

By Integrating,

$$\int dV + \frac{b-a}{2}\int \frac{2V}{V^2 - ab} dV = 2\int dx + C$$

$$V + \frac{1}{2}(b-a)\log(V^2 - ab) = 2x + C$$

$$x+y+\frac{1}{2}(b-a)\log\left[(x+y)^2-ab\right]=2x+C$$

$$(b-a)\log\left[(x+y)^2-ab\right]=4x-2x-2y+2C$$

$$(b-a)\log\left[(x+y)^2-ab\right]=2x-2y+2C$$

$$(b-a)\log\left[(x+y)^2-ab\right]=2(x-y+C)$$

which is reuired solutiion.

Q12. Solve $3e^x \tan y dx + (1-e^x)\sec^2 y dy = 0$

Sol :

Given that $3e^x \tan y dx + (1-e^x)\sec^2 y dy = 0$ which can be written as

$$\frac{3e^x}{1-e^x}dx+\frac{\sec^2 y}{\tan y}dy=0$$

By Integrating,

$$\int\frac{3e^x}{1-e^x}dx+\int\frac{\sec^2 y}{\tan y}dy=C$$

$$-3\log(1-e^x)+\log\tan y=\log C \qquad \left[\because \int\frac{f'(x)}{f(x)}=\log f(x)\right]$$

$$\log(1-e^x)^{-3}+\log\tan y=\log C$$

$$\log\left[(1-e^x)^{-3}\tan y\right]=\log C$$

$\tan y = C(1-e^x)^3$ which is required solution.

Q13. Solve $(xy^2+x)dx+(yx^2+y)dy=0$

Sol :

Given that $(xy^2+x)dx+(yx^2+y)dy=0$ which can be written as

$$x(y^2+1)dx+y(x^2+1)dy=0$$

divide by $(y^2+1)(x^2+1)$

$$\frac{x}{x^2+1}dx+\frac{y}{y^2+1}dy=0$$

By Integrating,

$$\int \frac{x}{x^2+1}dx + \int \frac{y}{y^2+1}dy = C$$

$$\frac{1}{2}\log(x^2+1) + \frac{1}{2}\log(y^2+1) = \frac{1}{2}\log C$$

$$\log(x^2+1) + \log(y^2+1) = \log C$$

$$\log\left[(x^2+1)(y^2+1)\right] = \log C$$

$(x^2+1)(y^2+1) = C$ which is required solution.

1.3 Homogeneous Differential Equations

Q14. Define homogeneous differential equations.

Ans :

A differential equation of first order and first degree is said to be homogeneous if it can be put in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

*** Working Rule for Solving Homogeneous Equation**

Let the given equation be homogeneous then by definition, the given equation can be put in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ ... (1)

To Solve (1) , Let, $\frac{y}{x} = v$ ie $y = vx$ ... (2)

Differentiating with respect to x (2) gives

$$\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right) \quad \text{... (3)}$$

Using (2) & (3) in (1) becomes

$$v + x\left(\frac{dv}{dx}\right) = f(v)$$

$$x\frac{dv}{dx} = f(v) - v$$

Separating the variables x and v.

We have

$$\frac{dx}{x} = \frac{dv}{f(v) - v}$$

By Integrating,

$$\log x + c = \int \frac{dv}{f(v) - v}$$

Where c is arbitary constant, after integration, replace v by $\frac{y}{x}$.

Q15. Solve $\left(x^2 - y^2\right)dx + 2xy\,dy = 0$.

Sol :

The given equation is $(x^2 - y^2)\,dx + 2xy\,dy = 0$.

Which can be written as

$$\left(x^2 - y^2\right)dx = -2xy\,dy$$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{-2xy}$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \quad \text{... (1)}$$

Which is a homogeneous differential equation

putting $y = vx$ ... (2)

differentiating $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ... (3)

Using (2) & (3) in (1)

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x(vx)}$$

$$= \frac{\cancel{x^2}\left(v^2 - 1\right)}{\cancel{x^2}.2v}$$

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

Separating the variable, we have

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$

$$x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{-v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{-(v^2 + 1)}{2v}$$

$$\frac{2v}{v^2 + 1}dv = -\frac{dx}{x}$$

By integrating

$$\int \frac{2v}{v^2 + 1}dv = -\int \frac{1}{x}dx + c$$

$$\log\left(v^2 + 1\right) = -\log x + \log c$$

$$\log\left(v^2 + 1\right) + \log x = \log c$$

$$\log\left(v^2 + 1\right)x = \log c$$

$$\left(v^2 + 1\right)x = c$$

$$\left[\left(\frac{y}{x}\right)^2 + 1\right]x = c$$

$$\left[\frac{y^2 + x^2}{x^2}\right]x = c$$

$$\frac{y^2 + x^2}{x} = C$$

$$y^2 + x^2 = xC$$

Q16. Solve $\left(x^2y - 2xy^2\right)dx = \left(x^3 - 3x^2y\right)dy$.

Sol :

The given equation is

$$\left(x^2y - 2xy^2\right)dx = \left(x^3 - 3x^2y\right)dy$$

Which can be written as

$$\frac{dy}{dx} = \frac{x^2y - 2xy^2}{x^3 - 3x^2y}$$

$$= \frac{\cancel{x}\left[xy - 2y^2\right]}{\cancel{x}\left[x^2 - 3xy\right]}$$

$$\frac{dy}{dx} = \frac{xy - 2y^2}{x^2 - 3xy} \quad \text{... (1)}$$

Which is a homogeneous equation.

Putting $y = vx$ ... (2)

By differentiating,

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{... (3)}$$

Using (3) (2) in (1)

$$v + x\frac{dv}{dx} = \frac{x[vx] - 2\left[v^2x^2\right]}{x^2 - 3x[vx]}$$

$$v + x\frac{dv}{dx} = \frac{x^2v - 2x^2v^2}{x^2 - 3x^2v}$$

$$= \frac{x^2\left[v - 2v^2\right]}{x^2[1 - 3v]}$$

$$x\frac{dv}{dx} = \frac{v - 2v^2}{1 - 3v} - v$$

$$x\frac{dv}{dx} = \frac{\cancel{v} - 2v^2 - \cancel{v} + 3v^2}{1 - 3v}$$

$$x\frac{dv}{dx} = \frac{v^2}{1 - 3v}$$

By separating the variables,

$$\frac{1 - 3v}{v^2}dv = \frac{dx}{x}$$

By Integrating,

$$\int\left(\frac{1}{v^2} - \frac{3}{v}\right)dv = \int\frac{1}{x}dx + c$$

$$\frac{v^{-2+1}}{-2+1} - 3\log v = \log x + \log c$$

$$-\frac{1}{v} - 3\log v = \log x + \log c$$

sub $v = \frac{y}{x}$

$$-\frac{x}{y} - 3\log\frac{y}{x} = \log x + \log c$$

$$\log x + 3\log\frac{y}{x} + \frac{x}{y} + \log c = 0$$

$$\log x + 3(\log y - \log x) + \frac{x}{y} + \log c = 0$$

$$\log x + 3\log y - 3\log x + \frac{x}{y} + \log c = 0$$

$$\frac{x}{y} - 2\log x + \log y^3 + \log c = 0$$

$$\frac{x}{y} - \log x^2 + \log y^3 + \log c = 0$$

$$\frac{x}{y} - \log x^2 + \log(cy^3) = 0$$

$$\frac{x}{y} = \log x^2 - \log\left(cy^3\right)$$

$$\frac{x}{y} = \log\left[\frac{x^2}{cy^3}\right]$$

$$e^{x/y} = \frac{x^2}{cy^3}$$

$$x^2 = cy^3e^{x/y}$$

Which is required solution

Q17. Solve $xdy - ydx = \sqrt{x^2 + y^2}dx$.

Sol :

The given equation is,

$$xdy - ydx = \sqrt{x^2 + y^2}dx$$

Which can be written as

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$$

$$= \frac{y + x\left(\sqrt{1 + \left(\frac{y}{x}\right)^2}\right)}{x}$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} \qquad \text{... (1)}$$

Putting $\frac{y}{x} = v$ ie, $y = vx$

so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ... (2)

from (1) & (2)

$$v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}$$

By separating the variables

$$\frac{dx}{x} = \frac{dv}{\sqrt{1 + v^2}}$$

By Integrating

$$\int \frac{dx}{x} + c = \int \frac{dv}{\sqrt{1 + v^2}}$$

$$\log x + \log c = \log\left[v + \sqrt{v^2 + 1}\right]$$

$$\log(xc) = \log\left[v + \sqrt{v^2 + 1}\right]$$

$$xc = v + \sqrt{v^2 + 1} \qquad \because v = \frac{y}{x}$$

$$xc = \frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 1}$$

$$x^2c = y + \sqrt{y^2 + x^2}$$

Which is required solution

Q18. Solve $x\frac{dy}{dx} = y[\log y - \log x + 1]$.

Sol :

The given equation is,

$$x\frac{dy}{dx} = y[\log y - \log x + 1]$$

putting $y = vx$ in above equation

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\cancel{x}\left[v + x\frac{dv}{dx}\right] = v\cancel{x}[\log vx - \log x + 1]$$

$$v + x\frac{dv}{dx} = v[\log vx - \log x + 1]$$

$$= v\left[\log\frac{v\cancel{x}}{\cancel{x}} + 1\right]$$

$$\cancel{v} + x\frac{dv}{dx} = v\log v + \cancel{v}$$

$$\frac{dv}{v\log v} = \frac{dx}{x}$$

By Integrating

$$\int \frac{dv}{v\log v} = \int \frac{dx}{x} + c$$

$$\log(\log v) = \log x + \log c$$

$$\log(\log v) = \log xc$$

$$\log v = xc$$

$$v = e^{xc}$$

$$\frac{y}{x} = e^{xc}$$

$$y = x.e^{xc}$$

Which is required solution.

Q19. Solve $x(x-y)\frac{dy}{dx} = y(x+y)$.

Sol :

Given equation is $x(x-y)\frac{dy}{dx} = y(x+y)$

Which may written as

$$\left(x^2 - xy\right)\frac{dy}{dx} = yx + y^2$$

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2 - xy} \qquad \text{... (1)}$$

which is a homogeneous equation

putting $y = vx$ & $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in(1)

We get $v + x\frac{dv}{dx} = \frac{x(vx) + v^2x^2}{x^2 - x(vx)}$

$$= \frac{vx^2 + v^2x^2}{x^2 - vx^2} \Rightarrow \frac{x^2\left[v + v^2\right]}{x^2[1-v]}$$

$$v + x\frac{dv}{dx} = \frac{v + v^2}{1-v}$$

$$x\frac{dv}{dx} = \frac{v + v^2}{1-v} - v$$

$$x\frac{dv}{dx} = \frac{v + v^2 - v + v^2}{1-v}$$

$$x\frac{dv}{dx} = \frac{2v^2}{1-v}$$

By separating variable

$$\frac{1-v}{v^2}dv = 2\frac{dx}{x}$$

By Integrating

$$\int\frac{1-v}{v^2}dv = 2\int\frac{dx}{x} + c$$

$$\int\left(\frac{1}{v^2} - \frac{1}{v}\right)dv = 2\int\frac{dx}{x} + c$$

$$-\frac{1}{v} - \log v = 2\log x + \log c$$

$$-\frac{1}{v} = 2\log x + \log v + \log c$$

$$-\frac{1}{v} = \log x^2 + \log v + \log c$$

$$-\frac{1}{\left(\frac{y}{x}\right)} = \log x^2 + \log v + \log c$$

$$-\frac{x}{y} = \log\left(x^2 v\right) + \log c$$

$$-\frac{x}{y} = \log\left(x^2 \cdot \frac{y}{x}\right) + \log c$$

$$-\frac{x}{y} = \log xy + \log c$$

$$-\frac{x}{y} - \log xy = \log c$$

$$\frac{x}{y} + \log(xy) = c$$

Q20. Solve $\frac{dy}{dx} + \frac{x^2 + 3y^2}{3x^2 + y^2} = 0$

Sol :

The given equation is $\frac{dy}{dx} + \frac{x^2 + 3y^2}{3x^2 + y^2} = 0$

putting $y = vx$ & $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in above equation

$$v + x\frac{dv}{dx} + \frac{x^2 + 3v^2x^2}{3x^2 + v^2x^2} = 0$$

$$v + x\frac{dv}{dx} + \frac{x^2\left[1+3v^2\right]}{x^2\left[3+v^2\right]} = 0$$

$$x\frac{dv}{dx} = -\left[\frac{1+3v^2}{3+v^2}\right] - v$$

$$x\frac{dv}{dx} = -\left[\frac{1+3v^2}{3+v^2} + v\right]$$

$$x\frac{dv}{dx} = -\left[\frac{1+3v^2+3v+v^3}{3+v^2}\right]$$

By separating variables

$$\frac{3+v^2}{v^3+3v^2+3v+1}dv = -\frac{dx}{x}$$

$$\int\frac{3+v^2}{v^3+3v^2+3v+1}dv = -\int\frac{dx}{x} + c \qquad \text{... (1)}$$

consider $\int\frac{v^3+3}{v^3+3v^3+3v+1}dv = \int\frac{v^2+3}{(v+1)^3}dv$ ... (2)

$$\frac{v^2+3}{(v+1)^3} = \frac{A}{v+1} + \frac{B}{(v+1)^2} + \frac{C}{(v+1)^3}$$

$$v^2+3 = A(v+1)^2 + B(v+1) + C$$

$$v^2+3 = A\left[v^2+1+2v\right] + B[v+1] + C$$

Comparing coefficients on both sides

$v^2 = Av^2 \Rightarrow A = 1$

v coeff $\Rightarrow 0 = 2A + B$

$A = 1 \Rightarrow 0 = 2(1) + B$

$B + 2 = 0 \Rightarrow B = -2$

constant $\Rightarrow 3 = A + B + C$

$3 = 1 - 2 + C$

$3 = -1 + C \Rightarrow C = 4$

by (2)

$$\int \frac{v^3+3}{(v+1)^3} = \int \frac{1}{v+1}dv - \int \frac{2}{(v+1)^2}dv + \int \frac{4}{(v+1)^3}dv$$

$$= \log(v+1) + 2\frac{1}{(v+1)} - \frac{4}{2(v+1)^2}$$

$$\int \frac{v^3+3}{(v-1)^3} = \log(v+1) + \frac{2}{v+1} - \frac{2}{(v+1)^2} \qquad \text{... (3)}$$

Substitute (3) in equation (1)

$$\log(v+1) + \frac{2}{v+1} - \frac{2}{(v+1)^2} = -\log x + \log c$$

$$\log(v+1) + \frac{2}{v+1} - \frac{2}{(v+1)^2} + \log x = \log c$$

$$\log\left(\frac{y}{x}+1\right) + \frac{2}{\frac{y}{x}+1} - \frac{2}{\left(\frac{y}{x}+1\right)^2} + \log x = \log c$$

$$\log\left(\frac{y+x}{x}\right) + \frac{2x}{y+x} - \frac{2x^2}{(y+x)^2} + \log x = \log c$$

$$\log(y+x) - \cancel{\log x} + \frac{2x}{y+x} - \frac{2x^2}{(y+x)^2} + \cancel{\log x} = \log c$$

$$\log(y+x) + \frac{2x}{y+x} - \frac{2x^2}{(y+x)^2} = c$$

Q21. Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$.

Sol :

The given equation is $x^2y\,dx - (x^3+y^3)\,dy = 0$

Which can be written as

$$x^2y\,dx = (x^3+y^3)\,dy$$

$$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3} \qquad \text{... (1)}$$

putting $y = vx$ & $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in above equation.

$$v + x\frac{dv}{dx} = \frac{x^2[vx]}{x^3 + v^3x^3} = \frac{x^3v}{x^3\left[1+v^3\right]}$$

$$v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$x\frac{dv}{dx} = \frac{v}{1+v^3} - v \Rightarrow \frac{\not{v} - \not{v} - v^4}{1+v^3}$$

$$x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$$

By separating variables

$$\frac{1+v^3}{v^4}dv = -\frac{dx}{x} + c$$

By Integrating

$$\int\frac{1+v^3}{v^4}dv = -\int\frac{dx}{x} + c$$

$$\int\left[\frac{1}{v^4} + \frac{1}{v}\right]dv = -\int\frac{dx}{x} + \log c$$

$$-\frac{1}{3v^3} + \log v + \log x = \log c$$

since $v = \frac{y}{x}$

$$-\frac{1}{3}\left(\frac{x}{y}\right)^3 + \log\left(\frac{y}{x}\right) + \log x = \log c$$

$$-\frac{1}{3}\left(\frac{x}{y}\right)^3 + \log y - \log x + \log x = \log c$$

$$-\frac{x^3}{3y^3} + \log y = \log c$$

$$-\frac{x^3}{3y^3} = \log c - \log y$$

$$-\frac{x^3}{3y^3} = \log\frac{c}{y}$$

$$e^{-x^3/3y^2} = \frac{c}{y}$$

$$y = c\,e^{-x^3/3y^2}$$

which is required solution.

Q22. Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Sol :

The given equation,

$$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x} \quad \text{... (1)}$$

Since the RHS of the given equation is function of $\frac{y}{x}$ alone,

We conclude that it must be a homogeneous equation

Take $\frac{y}{x} = v$ ... (2)

$y = vx$, By differentiating

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{... (3)}$$

By using (2) & (3) in (1)

$$v + x\frac{dv}{dx} = v + \tan v$$

$$\frac{dv}{dx} = \frac{\tan v}{x} \Rightarrow \frac{dv}{dx} = \frac{\sin v}{\cos v}.\frac{1}{x}$$

$$\frac{dx}{x} = \frac{\cos v}{\sin v}dv$$

By Integrating

$$\int\frac{dx}{x} + c = \int\frac{\cos v}{\sin v}dv$$

$$\log x + \log c = \int\frac{1}{t}dt \qquad t = \sin v$$

$$dt = \cos v\,dv$$

$$\log x + \log c = \log t$$
$$\log(xc) = \log(\sin v)$$
$$xc = \sin v$$
$$\therefore cx = \sin\left(\frac{y}{x}\right)$$

Which is required solution

1.4 Differential Equations Reducible to Homogeneous Form

Q23. Derive differential equations reducible to homogeneous form.

Ans :

Equation of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ where $\frac{a}{a'} \neq \frac{b}{b'}$ (1) can be reduced to homogeneous differential equation.

put $x = X + h$ and $y = Y + k$................(2) where X and Y are variables and h, k are constant to be so choosen.

by (2) dx = dX and dy = dY

So that $\frac{dY}{dX} = \frac{dy}{dx}$

using (2) & (3) in (1) becomes

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'}$$

$$\frac{dY}{dX} = \frac{ax+bY+(ah+bk+c)}{a'x+b'Y+(a'h+b'k+c)}\text{.........}(4)$$

In order to make (4) homogeneous, choose h & k so as to satisfy the following two equations

$$\left.\begin{aligned} ah+bk+c=0 \\ a'h+b'k+c'=0 \end{aligned}\right\}\text{.............}(5)$$

solving (5) $h = \frac{bc'-b'c}{ab'-a'b}$ & $k = \frac{ca'-c'a}{ab'-a'b}$............(6)

Given that $\frac{a}{a'} \neq \frac{b}{b'}$

$\therefore ab' - a'b \neq 0$

Hence h and k given by (6) are meaning ful i.e., h and k will exist.

Now, h and k are known,

so from (2), we get

$$X = x - h, \quad Y = y - k$$

by (4) & (5) reduces to $\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$

$$= \frac{X\left[a+b\left(\frac{Y}{X}\right)\right]}{X\left[a'+b'\left(\frac{Y}{X}\right)\right]}$$

$$= \frac{a+b\left(\frac{Y}{X}\right)}{a'+b'\left(\frac{Y}{X}\right)}$$

which is surely homogeneous equation in X and Y can be solved by putting $\frac{Y}{X} = v$. After getting solution interms of X & Y we remove X and Y using (7)

Obtain solution in terms of the original variable x and y.

Q24. Solve $(2x+y-3)dy = (x+2y-3)dx$

Sol :

The given differential equation is $(2x+y-3)dy = (x+2y-3)dx$

Which can be written as

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

putting $x = X+h, \quad y = Y+k$

$$\frac{dY}{dX} = \frac{X+h+2(Y+k)-3}{2(X+h)+(Y+k)-3}$$

$$= \frac{X+h+2Y+2k-3}{2X+2h+Y+k-3}$$

$$= \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \text{............}(1)$$

choose h and k such that

$h+2k-3=0$

$2h+k-3=0$

solving above two equations

h		k	I
2	−3	1	2
1	−3	2	1

$$\frac{h}{-6+3}=\frac{k}{-6+3}=\frac{1}{1-4}\Rightarrow\frac{h}{-3}=\frac{k}{-3}=\frac{1}{-3}$$

$\therefore$ h = 1 and k = 1

sub h and k values in (1)

$$\frac{dY}{dX}=\frac{X+2Y+(1+2(1)-3)}{2X+Y+(2(1)+1-3)}$$

$$\frac{dY}{dX}=\frac{X+2Y}{2X+Y}..........(2)$$

where

$X=x-1$ and $Y=y-1$

putting Y = VX ....(2)

then we get

$$\frac{dY}{dX}=v+x\frac{dV}{dX}...........(3)$$

Substitute corresponding values in (2)

$$V+X\frac{dV}{dX}=\frac{X+2VX}{2X+VX}$$

$$V+X\frac{dV}{dX}=\frac{X[1+2V]}{X[2+V]}$$

$$X\frac{dV}{dX}=\frac{1+2V}{2+V}-V$$

$$X\frac{dV}{dX}=\frac{1+2V-2V-V^2}{2+V}$$

$$X\frac{dV}{dX}=\frac{1-V^2}{2+V}$$

$$\frac{2+V}{1-V^2}dV=\frac{dV}{V}............(4)$$

Resolving into partial fraction

$$\frac{2+V}{1-V^2}=\frac{A}{1-V}+\frac{B}{1+V}$$

$$=A(1+V)+B(1-V)$$

$$2+V=A(1+V)+B(1-V)$$

If V = – 1

$$2-1=0+B(1-(-1))$$

$$1=2B$$

$$B=\frac{1}{2}$$

If $V=1$

$$2+1=A(1+1)+B(1-1)$$

$$3=2A$$

$$A=\frac{3}{2}$$

$$\therefore\frac{2+V}{1-V^2}=\frac{3}{2}.\frac{1}{1-V}+\frac{1}{2}.\frac{1}{1+V}$$

Now integrating (4) by substituting above

$$\int\frac{3}{2}\frac{1}{1-v}dv+\int\frac{1}{2}\frac{1}{1+v}dv=\int\frac{dx}{x}$$

$$\frac{3}{2}\log(1-V)(-1)+\frac{1}{2}\log(1+V)=\log X+\log c$$

$$\log(1+V)^{1/2}-\log(1-V)^{3/2}=\log(Xc)$$

$$\log\left[\frac{1+V}{(1-V)^3}\right]^{1/2}=\log(Xc)$$

$$\log\left[\frac{1+V}{(1-V)^3}\right]=\log(Xc)^2$$

$$\frac{1+V}{(1-V)^3}=(Xc)^2$$

Now, substitute $V=\frac{Y}{X}$

$$\frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^3}=(cX)^2$$

$$\frac{\frac{X+Y}{\frac{(X-Y)^3}{X^2}}}{} = (cX)^2$$

$$\frac{X^2(X+Y)}{(X-Y)^3} = C^2X^2$$

$$\frac{x-1+y-1}{(x-1-y+1)} = C^2$$

$$\frac{x+y-2}{(x-y)^3} = C^2 \Rightarrow (x+y-2) = C^2(x-y)^3$$

which is required solution.

Q25. Solve $(2x-y+1)dx+(2y-x-1)dy=0$

Sol :

The given equation is

$$(2x-y+1)dx+(2y-x-1)dy=0 \quad ... (1)$$

which can be written as

$$\frac{dy}{dx}+\frac{2x-y+1}{2y-x-1}=0 \quad ... (2)$$

putting $x = X+h, \quad y = Y+k$ ... (3)

$$\frac{dY}{dX}+\frac{2(X+h)-(Y+k)+1}{2(Y+k)-(X+h)-1}=0$$

$$\frac{dY}{dX}+\frac{2X-Y+(2h-k+1)}{2Y-X+(2k-h-1)}=0 \quad ... (4)$$

choose h and k so that

$$2h-k+1=0.......(i)$$

$$2k-h-1=0.......(ii)$$

$$(i)\times 2 \Rightarrow -2k+4h+2=0$$

$$2k-h-1=0$$

$$3h+1=0$$

$$h=-\frac{1}{3}$$

substitute $h=-\frac{1}{3}$

$$2\left(\frac{-1}{3}\right)-k+1=0$$

$$\frac{-2}{3}-k+1=0$$

$$-k+\frac{1}{3}=0$$

$$k=\frac{1}{3}$$

Substitute h and k in (3)

Then we have,

$$x = X-\frac{1}{3}, \quad y = Y+\frac{1}{3} \quad ... (5)$$

$$\frac{dY}{dX}+\frac{2X-Y+\left[2\left(\frac{-1}{3}\right)-\frac{1}{3}+1\right]}{2Y-X+\left[2\left(\frac{1}{3}\right)+\frac{1}{3}-1\right]}=0$$

$$\frac{dY}{dX}+\frac{2X-Y+[0]}{2Y-X+[0]}=0$$

$$\frac{dY}{dX}+\frac{2X-Y}{2Y-X}=0$$

$$\Rightarrow \frac{dY}{dX}=\frac{2X-Y}{X-2Y}$$

Putting $Y = VX$ in above equation

$$\frac{dY}{dX}=V+X\frac{dV}{dX}$$

$$V+X\frac{dV}{dX}=\frac{2X-VX}{X-2VX}$$

$$V+X\frac{dV}{dX}=\frac{X[2-V]}{X[1-2V]}$$

$$X\frac{dV}{dX}=\frac{2-V}{1-2V}-V$$

$$X\frac{dV}{dX} = \frac{2 - V - V + 2V^2}{1 - 2V}$$

$$X\frac{dV}{dX} = \frac{2 + 2V^2 - 2V}{1 - 2V}$$

$$\frac{1 - 2V}{V^2 - V + 1}dV + \frac{2}{X}dX = 0$$

By Integrating

$$\int \frac{1 - 2V}{V^2 - V + 1}dV + 2\int \frac{1}{X}dX = 0$$

$$\log\left(V^2 - V + 1\right) + 2\log X = \log C$$

$$\log\left(V^2 - V + 1\right) + \log X^2 = \log C$$

$$X^2\left(V^2 - V + 1\right) = C$$

Substitute $V = \frac{Y}{X}$

$$X^2\left[\frac{Y^2}{X^2} - \frac{Y}{X} + 1\right] = C$$

$$Y^2 - YX + X^2 = C$$

$$\left(y - \frac{1}{3}\right)^2 - \left(y - \frac{1}{3}\right)\left(x + \frac{1}{3}\right) + \left(x + \frac{1}{3}\right)^2 = C$$

$$y^2 + \frac{1}{9} - \frac{2}{3}y - \left[xy + \frac{y}{3} - \frac{x}{3} - \frac{1}{9}\right] + x^2 + \frac{1}{9} + \frac{2}{3}x = C$$

$$y^2 + \frac{1}{9} - \frac{2}{3}y - xy - \frac{y}{3} + \frac{x}{3} + \frac{1}{9} + x^2 + \frac{1}{9} + \frac{2}{3}x = C$$

$$y^2 + x^2 - y + x - xy + \frac{1}{3} = C$$

$$3x^2 + 3y^2 - 3y + 3x - 3xy + 1 - 3C = 0$$

which is required solution.

Q26. Solve $(2x+4y+3)\frac{dy}{dx} = x+2y+1$.

Sol :

The given equation is,

$$(2x+4y+3)\frac{dy}{dx} = x+2y+1$$

This can be written as

$$\frac{dy}{dx} = \frac{x+2y+1}{2x+4y+3}$$

$$\frac{dy}{dx} = \frac{(x+2y)+1}{2(x+2y)+3} \quad \ldots (1)$$

put $x+2y = v$ so that

$$1+2\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{dv}{dx}-1\right)$$

$\Rightarrow$ by (1)

$$\frac{1}{2}\frac{dv}{dx} - 1 = \frac{v+1}{2v+3}$$

$$\frac{dv}{dx} = \frac{2[v+1]}{2v+3} + 1$$

$$\frac{dv}{dx} = \frac{2v+2+2v+3}{2v+3}$$

$$\frac{dv}{dx} = \frac{4v+5}{2v+3}$$

$$\frac{2v+3}{4v+5}dv = dx$$

$$\frac{1}{2}\left[1+\frac{1}{4v+5}\right]dv = dx$$

By Integrating,

$$\int \frac{1}{2}\left[1+\frac{1}{4v+5}\right]dv = \int dx$$

$$\int \frac{1}{2}dv + \frac{1}{2}\int \frac{1}{4v+5}dv = x+c$$

$$\frac{1}{2}v + \frac{1}{2}\log(4v+5).\frac{1}{4} = x+c$$

$$\frac{1}{2}v + \frac{1}{8}\log(4v+5) = x+c$$

$$4v + \log(4V+5) = 8(x+C)$$

$$4v + \log(4v+5) = 8x+8c$$

$$\log(4v+5) = 8x-4v+8c$$

$$\log(4(x+2y)+5) = 8x-4(x+2y)+8c$$

$$\log(4(x+2y)+5) = 8x-4x-8y+8c$$

$$\log(4x+8y+5) = 4(x-2y)+8c$$

$$4x+8y+5 = e^{8c}\ e^{4(x-2y)}$$

$$4x+8y+5 = c_1 e^{4(x-2y)}$$

which is required solution.

Q27. Solve $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$

Sol:

The given equation is,

Which can written as $\frac{dy}{dx} = \frac{x-y+3}{2(x-y+3)-1}$

Here $x-y+3 = v$

$$1-\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = 1+\frac{dv}{dx}$$

$$-\frac{dv}{dx} = \frac{v}{2v-1} - 1$$

$$-\frac{dv}{dx} = \frac{v-2v+1}{2v-1}$$

$$-\frac{dv}{dx} = \frac{-v+1}{2v-1}$$

$$\not{-}\frac{dv}{dx} = \frac{\not{-}(v-1)}{2v-1}$$

$$\frac{2v-1}{v-1}dv = dx$$

$$\left(2+\frac{1}{v-1}\right)dv = dx$$

By Integrating,

$$\int\left(2+\frac{1}{v-1}\right)dv = \int dx + C$$

$$2v + \log(v-1) = x + C$$

$$2(x - y + 3) + \log(x - y + 3 - 1) = x + C$$

$$2x - 2y + 6 + \log(x-y+2) - x = C$$

$$x - 2y + \log(x-y+2) = C - 6$$

$$x - 2y + \log(x-y+2) = C_1$$

which is required solution.

Q28. Solve $(x-y-2)dx = (2x-2y-3)dy$.

Sol :

The given equation is

$$(x-y-2)dx = (2x-2y-3)dy$$

which can be written as

$$\frac{dy}{dx} = \frac{x-y-2}{2x-2y-3}$$

$$\frac{dy}{dx} = \frac{x-y-1-1}{2x-2y-2-1} = \frac{(x-y-1)-1}{2(x-y-1)-1}$$

$$1-\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = 1-\frac{dv}{dx}$$

$$1-\frac{dv}{dx} = \frac{v-1}{2v-1}$$

$$-\frac{dv}{dx} = \frac{v-1}{2v-1}-1$$

$$-\frac{dv}{dx} = \frac{v-1-2v+1}{2v-1}$$

$$-\frac{dv}{dx} = \frac{v-2v}{2v-1}$$

$$-\frac{dv}{dx} = \frac{-v}{2v-1}$$

$$\frac{dv}{dx} = \frac{v}{2v-1}$$

$$\frac{2v-1}{v}dv = dx$$

$$\left(\frac{2v}{v}-\frac{1}{v}\right)dv = dx$$

$$\left(2-\frac{1}{v}\right)dv = dx$$

By Integrating,

$$\int\left(2-\frac{1}{v}\right)dv = \int dx + C$$

$$2v - \log v = x + C$$

$$2(x-y-1) - \log(x-y-1) = x + C$$

$$2x - 2y - 2 - \log(x-y-1) - x - C = 0$$

$$x - 2y - 2 - C = \log(x-y-1)$$

$$\log(x-y-2) = x - 2y - C$$

which is requried solution.

1.5 Linear Differential Equation

Q29. Explain the working rule for solving linear equation.

Ans :

A differential equation which is of the form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x alone.

Working Rule for Solving Linear Equation

1. First put the given equation in the standard form
2. Find integrating factor (I.F)by using formula

$$I.F = e^{\int P.dx}$$

Two formulas $e^{m\log A} = A^m$

$e^{-m\log A} = \frac{1}{A^m}$ will be often used in simplifying I.F

3. Lastly, the required solution is obtained by using the result y × (I.F) = $\int$[Q × (I.F)] dx + C, where C is an arbitrary constant.

Q30. Solve $ydx - xdy + \log x dx = 0$.

Sol :

The given equation is,

$$y\,dx - x\,dy + \log x\,dx = 0$$

and which can be written as

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x}\log x$$

which is in the form of $\frac{dy}{dx} + Py = Q$

where $P = \frac{-1}{x}$ $Q = \frac{1}{x}\log x$

$$\therefore \text{I.F} = e^{\int p\,dx} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

$\therefore$ The solution is $y e^{\int p\,dx} = \int Q e^{\int p\,dx} dx + C$

$$y\frac{1}{x} = \int \frac{1}{x}\log x \ \frac{1}{x}dx + C$$

$$y\frac{1}{x} = \int \frac{\log x}{x}.\frac{1}{x}dx + C$$

$$= \int \log x . x^{-2} dx + C$$

$$= \log x\left(\frac{-1}{x}\right) - \int \frac{1}{x}\left(\frac{-1}{x}\right)dx + C$$

$$y\frac{1}{x} = \frac{-\log x}{x} - \frac{1}{x} + C$$

$$y = Cx - (1 + \log x)$$

Which is required solution.

Q31. Solve $(1+x)\frac{dy}{dx} - xy = 1 - x$

Sol :

The given equation is $(1+x)\frac{dy}{dx} - xy = 1 - x$

Divide $(1+x)$ on both side,

$$\frac{dy}{dx} - \frac{xy}{1+x} = \frac{1-x}{1+x}$$

which is in the form of $\frac{dy}{dx} + Py = Q$

where $P = \frac{-x}{1+x}$; $Q = \frac{1-x}{1+x}$

$$\therefore \text{I.F} = e^{\int p\,dx} = e^{-\int \frac{x}{1+x}dx} = e^{-\int \frac{(x+1)-1}{x+1}dx}$$

$$= e^{-\int\left(1-\frac{1}{x+1}\right)dx}$$

$$= e^{-(x - \log(x+1))}$$

$$= e^{-x}.e^{\log(x+1)}$$

$$\text{I.F} = (x+1)e^{-x}$$

$\therefore$ The solution of the given differential equation is

$$y\,e^{\int p\,dx} = \int Q e^{\int p\,dx} dx + C$$

$$y(x+1)e^{-x} = \int \frac{1-x}{1+x}.(1+x)e^{-x}\,dx + C$$

$$= \int (1-x)e^{-x}\,dx + C$$

$$= \int \left(e^{-x} - xe^{-x}\right)dx + C$$

$$= \int e^{-x}dx - \int xe^{-x}dx + C$$

$$= \frac{e^{-x}}{-1} - \left[x\int e^{-x} - \int e^{-x}dx\ dx\right] + C$$

$$= -e^{-x} - \left[x\frac{e^{-x}}{-1} - \int \frac{e^{-x}}{1}dx\right] + C$$

$$= -e^{-x} + xe^{-x} + \frac{e^{-x}}{1} + C$$

$$= -e^{-x} + xe^{-x} + e^{-x} + C$$

$$= xe^{-x} + C$$

$$\therefore y(x+1)e^{-x} = xe^{-x} + C$$

$$y(x+1) = x\,e^{-x}.e^{x} + Ce^{x}$$

$$y(x+1) = x + Ce^{x}$$

which is required solution.

Q32. Solve $\left(1-x^2\right)\frac{dy}{dx}+2xy=x\left(1-x^2\right)^{1/2}$

Sol :

The given equation is $\left(1-x^2\right)\frac{dy}{dx}+2xy=x\left(1-x^2\right)^{1/2}$

and which can be written as

$$\frac{dy}{dx}+\frac{2xy}{1-x^2}=\frac{x\left(1-x^2\right)^{1/2}}{1-x^2}$$

$$\frac{dy}{dx}+\frac{2x}{1-x^2}y=\frac{x}{\sqrt{1-x^2}}$$

Which is of the form $\frac{dy}{dx}+py=Q$

where $P=\frac{2x}{1-x^2},\ Q=\frac{x}{1-x^2}$

$$\therefore \text{I.F} = e^{\int p\,dx}=e^{-\int\frac{2x}{1-x^2}dx}=e^{-\int\frac{2x}{1-x^2}dx}$$

$$=e^{-\log\left(1-x^2\right)}$$

$$=\frac{1}{1-x^2}$$

$\therefore$ The solution of the given differential equation is

$$y\,e^{\int p\,dx}=\int Q e^{\int p\,dx}dx+C$$

$$y\frac{1}{1-x^2}=\int\frac{x}{\sqrt{1-x^2}}.\frac{1}{1-x^2}\,dx+C$$

$$\frac{y}{1-x^2}=\int\frac{x}{\left(1-x^2\right)^{3/2}}dx+C$$

Put $1-x^2=t$

$$-2x\,dx=dt$$

$$x.dx=\frac{-dt}{2}$$

$$\frac{y}{1-x^2} = -\int \frac{1}{t^{3/2}}.\frac{dt}{2} + C$$

$$\frac{y}{1-x^2} = -\frac{t^{1/2}}{-\frac{1}{2}}.\frac{1}{2} + C$$

$$\frac{y}{1-x^2} = -2t^{-1/2}.\frac{1}{2} + C$$

$$\frac{y}{1-x^2} = \left(1-x^2\right)^{-1/2}$$

$$\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + C$$

$$y = \sqrt{1-x^2} + C\left(1-x^2\right)$$

Which is required solution.

Q33. Solve $x\frac{dy}{dx} + 2y = x^2 \log x$

Sol :

The given equation is $x\frac{dy}{dx} + 2y = x^2 \log x$

which can be written as

$$\frac{dy}{dx} + \frac{2}{x}.y = \frac{x^2}{x}\log x$$

$\Rightarrow \frac{dy}{dx} + \frac{2}{x}.y = x\log x$ which is in the form of $\frac{dy}{dx} + Py = Q$

where $P = \frac{2}{x}$, $Q = x\log x$

Now, $I.F = e^{\int P\,dx} = e^{\int \frac{2}{x}dx}$

$= e^{2\log x} = x^2$

$\therefore I.F = x^2$

The solution of the given differential equation is

$$y\,I.F = \int Q.I.F\,dx + C$$

$$y.x^2 = \int x \log x . x^2 \, dx + C$$

$$= \int x^3 \log x \; dx + C$$

$$= \log x \int x^3 - \int \frac{1}{x}\left(x^3 \, dx\right) dx + C$$

$$= \log x . \frac{x^4}{4} = \int \frac{1}{x} . \frac{x^4}{4} dx + C$$

$$= \log x . \frac{x^4}{4} - \frac{1}{4} \int x^3 dx + C$$

$$y.x^2 = \log x . \frac{x^4}{4} - \frac{1}{4} . \frac{x^4}{4} + C$$

$$16yx^2 = 4x^4 \log x - x^4 + C$$

which is required solution.

Q34. Solve $(x + y + 1)\frac{dy}{dx} = 1$

Sol :

The given equation is $(x + y + 1)\frac{dy}{dx} = 1$

which can be written as

$$\frac{dy}{dx} = \frac{1}{x + y + 1}$$

$$\frac{dx}{dy} = x + y + 1$$

$$\frac{dx}{dy} - x = y + 1$$

which i in the form of $\frac{dx}{dy} + P y = Q$

where $P = -1, \; Q = y + 1$

$\therefore$ I.F $= e^{\int p dy} = e^{\int -1 dy}$

I.F $= e^{-y}$

$\therefore$ The solution of the given differential equation is

$$x.I.F = \int Q.I.F \, dy + C$$

$$x.e^{-y} = \int (y+1)e^{-y}\,dy + C$$

$$xe^{-y} = \int (ye^{-y} + e^{-y})\,dy + c$$

$$xe^{-y} = \int ye^{-y}dy + \int e^{-y}\,dy + C$$

$$xe^{-y} = y\int e^{-y} - \int 1\left(\int e^{-y}dy\right)dy + \left[\frac{e^{-y}}{-1}\right] + C$$

$$xe^{-y} = y\left[\frac{e^{-y}}{-1}\right] - \int \frac{e^{-y}}{-1}dy - e^{-y} + C$$

$$xe^{-y} = -ye^{-y} - \left[\frac{e^{-y}}{-(-1)}\right] - e^{-y} + C$$

$$xe^{-y} = -ye^{-y} - e^{-y} - e^{-y} + C$$

$$xe^{-y} = -ye^{-y} - 2e^{-y} + C$$

$$x = -y - 2 + Ce^{y}$$

which is required solution.

Q35. Solve $\sin 2x \frac{dy}{dx} - y = \tan x$.

Sol :

The given equation is $\sin 2x \frac{dy}{dx} - y = \tan x$

and which can be written as

$$\frac{dy}{dx} - \frac{y}{\sin 2x} = \frac{\tan x}{\sin 2x}$$

$$\frac{dy}{dx} - \text{cosec}\,2x.y = \frac{\tan x}{\sin 2x}$$

$$\frac{dy}{dx} - \text{cosec}\,2xy = \frac{\sin x}{\cos x}.\frac{1}{\sin 2x}$$

$$\frac{dy}{dx} - \text{cosec}\,2xy = \frac{1}{2}\sec^2 x$$

which in the form of $\frac{dy}{dx} + P\,y = Q$

where $P = -\text{cosec}\,2x,\ Q = \frac{1}{2}\sec^2 x$

$$\therefore \text{I.F} = e^{\int p\,dx} = e^{-\int \text{cosec}\,2x\,dx}$$

$$= \exp\left[\log(\tan x)^{-1/2}\right]$$

$$\text{I.F} = \frac{1}{\sqrt{\tan x}}$$

The solution of the given differential equation is

$$y.\text{I.F} = \int (Q\ \text{I.F})dx + C$$

$$y.\frac{1}{\sqrt{\tan x}} = \int \frac{1}{2}\sec^2 x.\frac{1}{\sqrt{\tan x}}dx + C$$

let $t = \tan x$

$dt = \sec^2 x\,dx$

$$y.\frac{1}{\sqrt{\tan x}} = \frac{1}{2}\int \frac{1}{\sqrt{t}}dt + C$$

$$= \frac{\frac{1}{2}t^{-\frac{1}{2}+1}}{\frac{-1}{2}+1} + C$$

$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right] + C = t^{1/2} + C$$

$$y.\frac{1}{\sqrt{\tan x}} = \sqrt{\tan x} + C$$

$$y = \tan x + C\sqrt{\tan x}$$

Which is required solution.

Q36. Solve $\left(x + 2y^3\right)\frac{dy}{dx} = y$

Sol :

The given equation is $\left(x + 2y^3\right)\frac{dy}{dx} = y$

Which can be written as

$$\frac{dy}{dx} = \frac{y}{x + 2y^3} \Rightarrow \frac{dx}{dy} = \frac{x + 2y^3}{y}$$

$$\frac{dx}{dy} = \frac{x}{y} + \frac{2y^3}{y}$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2$$

Which is of the form

$$\frac{dx}{dy} + Px + Q$$

where

$$P = \frac{-1}{y} \text{ and } Q = 2y^2$$

$$\therefore \text{I.F } e^{\int p.dy} = e^{-\int \frac{1}{y}dy} = e^{-\log y} = e^{\log y^{-1}} = \frac{1}{y}$$

$\therefore$ The solution of the given differential equation is

$$x.\text{ I.F} = \int (Q\,IF)dy + C$$

$$x.\frac{1}{y} = \int 2y^2.\frac{1}{y}dy + C$$

$$x.\frac{1}{y} = \int 2y\,dy + C$$

$$\frac{x}{y} = \frac{2y^2}{2} + C$$

$$x = y^3 + yC$$

Which is required solution.

Q37. Solve $(1+y^2)dx = (\tan^{-1}y - x)dy$.

Sol :

The given equation is,

$$(1+y^2)dx = (\tan^{-1}y - x)dy$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

which is of the form $\frac{dx}{dy} + Px = Q$

where,

$$P = \frac{1}{1+y^2} \text{ and } Q = \frac{\tan^{-1}y}{1+y^2}$$

$$\therefore \text{I.F } e^{\int p\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$\therefore$ The solution of the given differential equation is

$$x.e^{\int p\,dy} = \int Q.e^{\int p\,dy}dy + C$$

$$x\,e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}.e^{\tan^{-1}y}dy + C$$

Let $\tan^{-1}y = t$

$$\frac{1}{1+y^2}dy = dt$$

$$x\,e^t = \int t\,e^t dt + C$$

$$x\,e^t = t\int e^t - \int 1\left(\int e^t dt\right)dt + C$$

$$x\,e^t = t\int e^t - \int e^t\,dt + C$$

$$x\,e^t = t\,e^t - e^t + C$$

$$x.t = e^t(t-1)$$

$$x.e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + C$$

Which is required solution.

1.6 Differential Equation Reducible to Linear Form (or) Non Linear Differential Equation or Bernoulis Equation

Q38. Derive the Bernoulis Equation.

Ans :

An equation of the form $\frac{dy}{dx} + Py = Qy^n........(1)$ where P and Q are constant or function of x alone and n is constant except 0 and 1 is called Bernoulli's equation.

Working Rule of Bernoulli's

Firstly multiply by y^{-n} to (1)

$$y^{-n}\frac{dy}{dx} + P\,y.y^{-n} = Q.y^n.y^{-n}$$

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q.........(2)$$

Put $y^{1-n} = v$ in (2)

differentiate $(1-n)y^{(1-n)-1}\frac{dy}{dx} = \frac{dv}{dx}$

$$(1-n)y^{1-n-1}\frac{dy}{dx} = \frac{dv}{dx}$$

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}y^n$$

Substitute (2), (3) in (1)

$$\frac{1}{1-n}\frac{dv}{dx}y^n + P\,y = Q.y^n$$

$$\frac{1}{1-n}\frac{dv}{dx} + P\,y^{-n}y = Q.y^n.y^{-n}$$

$$\frac{1}{1-n}\frac{dv}{dx}P\,y^{1-n} = Q$$

$$\frac{1}{1-n}\frac{dv}{dx} + P\,v = Q$$

$\frac{dv}{dx} + P(1-n)v = Q(1-n)$ which is linear in v and x.

$$\text{Its I.F} = e^{\int P(1-n)\,dx} = e^{(1-n)\int P\,dx}$$

$$v.e^{(1-n)\int P\,dx} = \int Q.e^{(1-n)\int P\,dx}dx + C$$

$$y^{1-n}e^{(1-n)\int P\,dx} = \int Q.e^{(1-n)\int P\,dx}dx + C$$

Where C is an arbitary constant.

Q39. Solve $\left(1-x^2\right)\frac{dy}{dx} + xy = xy^2$.

Sol :

The given equation is $\left(1-x^2\right)\frac{dy}{dx} + xy = xy^2$

which can be written as $\frac{dy}{dx} + \frac{x}{1-x^2}y = \frac{x}{1-x^2}y^2$

$$y^{-2}\frac{dy}{dx} + \frac{x}{1-x^2}y.y^{-2} = \frac{x}{1-x^2}$$

$$y^{-2}\frac{dy}{dx} + \frac{x}{1-x^2}y^{-1} = \frac{x}{1-x^2}$$

Let $y^{-1} = v$

$$-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$-\frac{dv}{dx} + \frac{x}{1+x^2}v = \frac{x}{1-x^2}$$

which is a linear equation in v.

$$\text{I.F} = e^{\int P\,dx} = e^{-\int \frac{x}{1-x^2}dx} = e^{\frac{-1}{2}\log\left(1-x^2\right)}$$

$$\text{I.F} = \left(1-x^2\right)^{\frac{1}{2}}$$

$\therefore$ The required solution of the given equation is $v\,e^{\int P\,dx} = \int Q e^{\int P\,dx}dx + C$

$$v\left(1-x^2\right)^{\frac{1}{2}} = -\int \frac{x}{1-x^2}\left(1-x^2\right)^{\frac{1}{2}}dx + C$$

$$= -\int \frac{x}{\sqrt{1-x^2}}dx + C$$

Let $1-x^2 = t$

$$-2x\,dx = dt$$

$$V(1-x^2)^{1/2} = -\frac{1}{2}\int \frac{2x}{\sqrt{1-x^2}}dx + C$$

$$v\left(1-x^2\right)^{\frac{1}{2}} = \frac{-1}{2}\int \frac{dt}{\sqrt{t}} + C$$

$$v\left(1-x^2\right)^{\frac{1}{2}} = \frac{-1}{2}\left[\frac{t^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}\right] + C$$

$$v\left(1-x^2\right)^{\frac{1}{2}} = \frac{-1}{2}t^{\frac{1}{2}}.2 + C$$

$$v\left(1-x^2\right)^{\frac{1}{2}} = -\left(t^{1/2}\right) + C$$

$$v\left(1-x^2\right)^{\frac{1}{2}} = -\sqrt{\left(1-x^2\right)} + C$$

$$y^{-1}\left(1-x^2\right)^{\frac{1}{2}} = -\left(1-x^2\right)^{\frac{1}{2}} + C$$

$$y^{-1}\left(1-x^2\right)^{\frac{1}{2}} + \left(1-x^2\right)^{\frac{1}{2}} = C$$

$$(1-y)\sqrt{1-x^2} = Cy$$

Which is required solution.

Q40. Solve $x\frac{dy}{dx} + y = y^2 \log x$

Sol :

The given equation is $x\frac{dy}{dx} + y = y^2 \log x$

Which can be written as

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2 \log x}{x}$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{y}{y^2.x} = \frac{\log x}{x}$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}.\frac{1}{y} = \frac{\log x}{x}$$

Put $y^{-1} = v \Rightarrow -y^{-1-1}\frac{dy}{dx} = \frac{dv}{dx}$

$$-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$-\left(\frac{dv}{dx} + \frac{1}{x}v\right) = \frac{1}{x}\log x$$

$$\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}\log x$$

which is a linear equation is v and

$$I.F = e^{\int P\,dx} = e^{-\int \frac{1}{x}dv} = e^{-\log x}$$

$$I.F = \frac{1}{x}$$

$\therefore$ The solution of the given equation is

ie., $v\,I.F = \int (Q\ I.F)\,dx + C$

$$v.\frac{1}{x} = -\int \frac{1}{x}\log x.\frac{1}{x}dx + C$$

$$Vx^{-1} = \int x^{-2}\log x\ dx + C$$

$$y^{-1}.x^{-1} = -\left[\log x\int x^{-2} - \int \frac{1}{x}\left(\int x^{-2}dx\right)dx\right] + C$$

$$= -\left[\log x\frac{x^{-2+1}}{-2+1} - \int \frac{1}{x}.\frac{x^{-2+1}}{-2+1}.dx\right] + C$$

$$= -\left[\log x\frac{x^{-1}}{-1} - \int \frac{1}{x}.\frac{x^{-1}}{-1}dx\right] + C$$

$$= \log x\, x^{-1}\int \frac{x^{-2}}{1} + C$$

$$= \frac{\log x}{x} - \frac{x^{-2+1}}{-2+1} + C$$

$$= \frac{\log x}{x} + \frac{x^{-1}}{1} + C$$

$$y^{-1}x^{-1} = \frac{\log x}{x} + \frac{1}{x} + C$$

$$y^{-1} = \log x + 1 + Cx$$

which is required solution.

Q41. Solve $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$.

Sol :

The given equation is,

$$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$$

divide by $\cos^2 y$ on both sides

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{x\sin 2y}{\cos^2 y} = x^3$$

$$\sec^2 y\frac{dy}{dx} + \frac{2x\sin y\cos y}{\cos^2 y} = x^3$$

$$\sec^2 y\frac{dy}{dx} + \frac{2x\sin y}{\cos y} = x^3$$

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$$

Put $\tan y = v$

$$\sec^2 y\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + 2.x.v = x^3$$

Which is a linear differential equation

$$IF = e^{\int P\,dx} = e^{\int 2x.dx} = e^{x^2}$$

$\therefore$ The solution of the given equation is

$$v\,IF = \int (Q\,I.F)\,dx + C$$

$$v\,e^{x^2} = \int x^3.e^{x^2}dx + C$$

$$v\,e^{x^2} = \int x.x^2\,e^{x^2}dx + C$$

Let $x^2 = t$

$$2x\,dx = dt$$

$$ve^{x2} = \int \frac{1}{2}t.e^t dt + C$$

$$ve^{x2} = \frac{1}{2}\int te^t dt + C$$

$$v\,e^x = \frac{1}{2}\left[t\int e^t - \int 1\int e^t dt\,dt\right] + C$$

$$v\,e^{x^2} = \frac{1}{2}\left[te^t - e^t\right] + C$$

$$v\,e^{x^2} = \frac{1}{2}e^t\left[t-1\right] + C$$

$$\tan y\,e^{x^2} = \frac{1}{2}e^{x^2}\left[x^2-1\right] + C$$

Which is required solution.

Q42. Solve $y\left(2xy + e^x\right)dx - e^x dy = 0$

Sol :

The given equation is,

$$y\left(2xy + e^x\right)dx - e^x dy = 0$$

Which can be written as

$$\left(2xy^2 + e^x y\right)dx - e^x dy = 0$$

$$e^x dy = \left(2xy^2 + e^x y\right)dx$$

$$e^x\frac{dy}{dx} = 2xy^2 + e^x y$$

$$\frac{dy}{dx} = \frac{2xy^2}{e^x} + \frac{e^x y}{e^x}$$

$$\frac{dy}{dx} = 2xe^{-x}y^2 + y$$

$$\frac{1}{y^2}\frac{dy}{dx} = 2x\,e^{-x}\frac{y^2}{y^2} + \frac{y}{y^2}$$

$$y^{-2}\frac{dy}{dx} = 2x\,e^{-x} + \frac{1}{y}$$

$$y^{-2}\frac{dy}{dx} - y^{-1} = 2xe^{-x}$$

put $y^{-1} = v$

$$-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$-\frac{dv}{dx} - v = 2xe^{-x}$$

$$\frac{dv}{dx} + v = -2xe^{-x}$$

which is a linear equation in v,

$\therefore$ The solution of the given equation is

$$v\,IF = \int (Q\,I.F)\,dx + C$$

But, $IF = e^{\int P\,dx} = e^{\int dx} = e^x$

$$y^{-1}e^x = -\int 2xe^{-x}.e^x\,dx + C$$

$$y^{-1}e^x = -2\int x\,dx + C$$

$$y^{-1}e^x = -2\frac{x^2}{2} + C$$

$$\frac{1}{y}e^x = -x^2 + C$$

which is required solution.

Q43. Solve $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$

Sol :

The given equation is $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$

Rewrite the given equation

$$2xy\frac{dy}{dx} = x^2 + y^2 + 1\frac{dy}{dx}$$

$$2xy\frac{dy}{dx} - y^2 = 1 + x^2$$

Divide by 'x'

$$2y\frac{dy}{dx} - \frac{y^2}{x} = \frac{1}{x} + x$$

Put $y^2 = v \Rightarrow 2y\frac{dy}{dx} = \frac{dv}{dx}$

$$\frac{dv}{dx} - \frac{v}{x} = \frac{1}{x} + x$$

which is in the form of $\frac{dv}{dx} + Pv = Q$

where $P = \frac{-1}{x}$, $Q = \frac{1}{x} + x$

$$\therefore I.F = e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

$\therefore$ The solution of the given equation

$$v\,IF = \int Q.I.F\,dx + C$$

$$y^2.\frac{1}{x} = \int \left(\frac{1}{x} + x\right)\frac{1}{x}\,dx + C$$

$$y^2 x^{-1} = \int \left(\frac{1}{x^2} + 1\right)dx + C$$

$$y^2 x^{-1} = -x^{-1} + x + C$$

$$y^2 = -1 + x.x + Cx$$

$$y^2 = (C + x)x - 1$$

which is required solution.

Q44. Solve $x\left(\frac{dy}{dx}\right) + y\log y = xye^x$

Sol :

The given equatio is

$$x\frac{dy}{dx} + y\log y = xye^x$$

divide by 'xy'

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^x$$

put $\log y = v$

$$\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + \frac{1}{x}.v = e^x$$

which is in the form of $\frac{dv}{dx} = Pv = Q$

where $P = \frac{1}{x}, Q = e^x$

$\therefore$ I.F $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$\therefore$ The solution of the given equation is

$$v.x = \int e^x . x\, dx + C$$

$$\log y.x = \int e^x . x\, dx + C$$

$$\log y.x = x\int e^x - \int 1 \int e^x dx . dx + C$$

$$\log y.x = xe^x - e^x + C$$

$$\log y.x = e^x (x - 1) + C$$

which is required solution.

Q45. Solve $(1 + y^2) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$

Sol :

The given equation is $(1 + y^2) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$

Rewritting the given equation

$$\frac{dy}{dx} + \frac{x.e^{\tan^{-1} y}}{1 + y^2} = 0$$

$$\frac{dy}{dx} + \frac{1}{1 + y^2} x = \frac{e^{-\tan^{-1} x}}{1 + y^2}$$

which is of the form $\frac{dy}{dx} + Px = Q$

Where $P = \frac{1}{1 + y^2}$, $Q = \frac{e^{-\tan^{-1} x}}{1 + y^2}$

$\therefore$ IF $= e^{\int p\, dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

$\therefore$ The solution of the given differential equation

$$x\, I.F = \int (Q\, IF) dy + C$$

$$x.e^{\tan^{-1} y} = \int \frac{e^{-\tan^{-1} x}}{1 + y^2} . e^{\tan^{-1} x} dy + C$$

$$xe^{\tan^{-1} y} = \int \frac{1}{1 + y^2} dy + C$$

$$xe^{\tan^{-1} y} = \tan^{-1} y + C$$

Which is required solution.

1.7 Exact Differential Equations

Q46. Define exact differential Equations.

Ans :

If M and N are functions of x and y the equation $Mdx + Ndy = 0$ is called exact. When there exists a function $f(x,y)$ of x and y ∍ $d[f(x,y)] = Mdx + Ndy$. ie., $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}fy = Mdx + Ndy$.

Prove That

The Necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Proof :

Necessary Condition

Suppose that the differential equation $Mdx + Ndy = 0$ be exact

By the definition

There must exists a function $f(x,y)$ of x and y

such that $d[f(x,y)] = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy = Mdx + Ndy ----(1)$

Equating coefficients of dx and dy (1)

We have $M = \frac{\partial f}{\partial x}, N = \frac{\partial f}{\partial y} ----(2)$

and

To remove the unknown function $f(x,y)$

We differentiate partially (2) with respect to y and x respectively.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

Since $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Sufficient Condition

Let us suppose that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$,

To show that $Mdx+Ndy=0$ is an exact equation we must find a function $f(x,y)$ such that $P=\int Mdx$ then $\frac{\partial P}{\partial x}=M$ so that $\frac{\partial^2 P}{\partial x\,\partial y}=\frac{\partial M}{\partial y}$.

But $\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}=\frac{\partial^2 P}{\partial y\,\partial x}=\frac{\partial^2 P}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right)$

Integrating both sides with respect to x we get

$$N=\frac{\partial P}{\partial y}+\phi(y)$$

Where $\phi(y)$ is a function of y only.

$$Mdx+Ndy=\frac{\partial P}{\partial y}dx+\left[\frac{\partial P}{\partial y}+\phi(y)\right]dy$$

$$=dP+dF(y)$$

where $dF(y)=\phi(y)dy$

$$=d\left[P+F(y)\right]$$

Which shows that $Mdx+Ndy$ is an exact differential and this proves the sucifficient part.

1.7.1 Working Rule for Solving an Exact Differential Equation

Q47. Explain the working rule for solving an exact differential equation.

Ans :

Compare the given equation with $Mdx+Ndy=0$ and find out M and N

Then find out $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$

If $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, we conclude that the given equation is exact.

If the equation is exact

Step 1: Integrate M with respect to x treating y as a constant.

Step 2: Integrate with respect to y only those terms of N which do not contain x.

Step 3: Equate the sum of these two integrals [found in step 1 & 2] to an arbitary constant. and thus we obtain the required solution.

i.e., $Mdx+Ndy=0$ is

$$\underset{\text{(Treating y as constant)}}{\int Mdx}+\int(\text{containing x})\,dy=C$$

Q48. Solve $x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$

Sol :

The given equation is $x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$

and which can be written as

$$\left(x - \frac{y}{x^2+y^2}\right)dx + \left(y + \frac{x}{x^2+y^2}\right) = 0$$

Which is in the form of $Mdx + Ndy = 0$

Here, $M = x - \frac{y}{x^2+y^2}$, $N = y + \frac{x}{x^2+y^2}$

$$M = \frac{x^3 + xy^2 - y}{x^2+y^2},\ N = \frac{x^2y + y^3 + x}{x^2+y^2}$$

Partial differentiation with respect to 'y'

Now $\frac{\partial M}{\partial y} = \frac{\left[x^2+y^2\right]\left[2xy-1\right] - \left[x^3+xy^2-y\right]\left[2y\right]}{\left(x^2+y^2\right)^2}$

$$= \frac{2x^3y - x^2 + 2xy^3 - y^2 - 2x^3y - 2xy^3 + 2y^2}{\left(x^2+y^2\right)^2}$$

$$= \frac{-x^2+y^2}{\left(x^2+y^2\right)^2}$$

Partial differentiation with respect to 'x'

$$\frac{\partial N}{\partial x} = \frac{\left[x^2+y^2\right]\left[2xy+1\right] - \left[x^2y+y^3+x\right]\left[2x\right]}{\left(x^2+y^2\right)^2}$$

$$= \frac{2x^3y + x^2 + 2xy^3 + y^2 - 2x^3y - 2xy^3 - 2x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial N}{\partial x} = \frac{-x^2+y^2}{\left(x^2+y^2\right)^2}$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is exact equation.

Now $\int\limits_{(y\text{ as constant})} M\,dx = \int\left(x - \frac{y}{x^2+y^2}\right)dx = \int x\,dx - y\int\frac{1}{x^2+y^2}dx$

$$= \frac{x^2}{2} - y\left(\frac{1}{y}\right)\tan^{-1}\frac{x}{y}$$

$$= \frac{x^2}{2} - \tan^{-1}\frac{x}{y}$$

$$\int_{\text{Only y terms}} N\,dy = \int y + \frac{x}{x^2+y^2}\,dy = \int y\,dy = \frac{y^2}{2}$$

$\therefore$ The required solution is

$$\int M\,dx + \int N\,dy = C$$

$$\frac{x^2}{2} - \tan^{-1}\frac{x}{y} + \frac{y^2}{2} = C$$

$$x^2 - 2\tan^{-1}\frac{x}{y} + y^2 = 2C$$

Q49. Solve $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$

Sol :

The given equation is $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$

Which can be written as

$$\frac{dy}{dx} = \frac{-ax+hy+g}{hx+by+f}$$

$$(hx+by+f)dy = -(ax+hy+g)dx$$

$$(ax+hy+g)dx + (hx+by+f)dy = 0$$

which is of the form $Mdx + Ndy = 0$

here $M = ax+hy+g,\ N = hx+by+f$

Now, partial differentiation of M, N with respect to y and x respectively

$$\frac{\partial M}{\partial y} = h\ ,\ \frac{\partial N}{\partial x} = h$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is exact equation.

Now, $\int M\,dx \ + \ \int N\,dy = C$

(y constant) (terms not having x)

$$\int (ax+hy+g)dx + \int (hx+by+f)dy = C_1$$

$$\frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = C_1$$

$$ax^2 + 2hxy + 2gx + by^2 + fy = 2C_1 \qquad \because C = 2C_1$$

$$ax^2 + 2hxy + 2gx + by^2 + fy = C$$

Which is required solution.

Q50. Solve $\left(\sin x\cos y + e^{2x}\right)dx + \left(\cos x\sin y + \tan y\right)dy = 0$

Sol :

The given equation is $\left(\sin x\cos y + e^{2x}\right)dx + \left(\cos x\sin y + \tan y\right)dy = 0$

Which is in the form of $Mdx + Ndy = 0$

Here $M = \sin x\cos y + e^{2x}$, $N = \cos x\sin y + \tan y$

Now, $\frac{\partial M}{\partial y} = -\sin x\sin y$

$\frac{\partial N}{\partial x} = -\sin x\sin y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is a exact equation.

Now, $\int M\,dx \;+\; \int N\,dy = C$
(y constant) (terms not having x)

$$\int(\sin x\cos y + e^{2x})\,dx + \int(\cos x\sin y + \tan y)\,dy = C$$

$$\left[-\cos x\cos y + \frac{e^{2x}}{2}\right] + \log\sec y = C$$

$$-\cos x\cos y + \frac{1}{2}e^{2x} + \log\sec y = C$$

Which is the required solution.

Q51. Solve $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

Sol :

The given equation is $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

Which is of the form $Mdx + Ndy = 0$

Here $M = 1 + e^{x/y}$, $N = e^{x/y}\left(1 - \frac{x}{y}\right)$

Now, $\frac{\partial M}{\partial y} = e^{x/y}.x.\frac{-1}{y^2}$

$N = e^{x/y} - \frac{x}{y}e^{x/y}$

$$\frac{\partial N}{\partial x} = e^{x/y}.\cancel{\frac{1}{y}} - \frac{x}{y}e^{x/y}.\frac{1}{y} - \cancel{\frac{1}{y}}.e^{x/y}$$

$$\frac{\partial N}{\partial x} = -xe^{-x/y}.\frac{1}{y^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Now, $\int M\,dx + \int N\,dy = C$

(y constant) (terms not having x)

$$\int\left(1+e^{x/y}\right)dx + \int\left(e^{x/y} - \frac{x}{y}e^{x/y}dy\right)$$

$$x + \frac{e^{x/y}}{\frac{1}{y}} + 0 = C$$

$$x + ye^{x/y} = C$$

$\therefore$ The required solution is $x + ye^{x/y} = C$.

1.8 Integrating Factors

Q52. Write the methods for finding the Integrating Factors.

Ans :

Method I

In some case the integrating factor is found by inspection.

Using the following few exact differential we can easy to find the integrating factor.

(a) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

(b) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

(c) $d(xy) = x\,dy + y\,dx$

(d) $d\left(\frac{x^2}{y}\right) = \frac{2xy\,dx - x^2\,dy}{y^2}$

(e) $d\left(\frac{y^2}{x}\right) = \frac{2xy\,dy - y^2\,dx}{x^2}$

(f) $d\left(\frac{x^2}{y^2}\right) = \frac{2xy^2\,dx - 2x^2y\,dy}{y^4}$

(g) $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2y\,dy - 2xy^2\,dx}{x^4}$

(h) $d\left(\frac{1}{xy}\right) = -\frac{x\,dy + y\,dx}{x^2y^2}$

(i) $d\left(\log\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{xy}$

(j) $d\left(\log\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{xy}$

(k) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2 + y^2}$

(l) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$

(m) $d\left(\frac{e^x}{y}\right) = \frac{y\,e^x dx - e^x\,dy}{y^2}$ (n) $d\left(\frac{e^y}{x}\right) = \frac{x\,e^x dy - e^y\,dx}{x^2}$

(o) $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2y^2}$ (p) $d\left[\frac{1}{2}\log\left(x^2 + y^2\right)\right] = \frac{xdx + ydy}{x^2 + y^2}$

Q53. Solve $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$

Sol :

The given equation is $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$

Which can be written as

$$y\,dx + xy^2\,dx + x\,dy - x^2y\,dy = 0$$

$$(y\,dx + x\,dy) + \left(xy^2\,dx - x^2y\,dy\right) = 0$$

divide by x^2y^2

$$\frac{y\,dx + x\,dy}{x^2y^2} + \frac{xy^2}{x^2y^2}dx - \frac{x^2y}{x^2y^2}dy = 0$$

Since $d\left(\frac{1}{xy}\right) = -\frac{x\,dy + y\,dx}{x^2y^2}$

$$d\left(\frac{-1}{xy}\right) + \frac{1}{x}dx - \frac{1}{y}dy = 0$$

By integrating we get

$$-\int d\left(\frac{1}{xy}\right) + \int\frac{1}{x}dx - \int\frac{1}{y}dy = 0$$

$$-\frac{1}{xy} + \log x - \log y = C$$

$$-\frac{1}{xy} + \log\frac{x}{y} = C$$

$$\log\frac{x}{y} = C + \frac{1}{xy}$$

Which is required solution.

Q54. Solve $\left(x^3e^x - my^2\right)dx + mxy\,dy = 0$

Sol :

The given equation is,

$$\left(x^3e^x - my^2\right)dx + mxy\,dy = 0$$

Which can be written as

$$x^3e^x\,dx - my^2\,dx + mxy\,dy = 0$$

$$x^3e^x\,dx + m\left(xy\,dy - y^2\,dx\right) = 0$$

dividing by x^3

$$\frac{x^3e^x}{x^3}dx + \frac{m\left(xy\,dy - y^2\,dx\right)}{x^3} = 0$$

$$e^x\;dx + \frac{m}{2}\;\frac{x^2 2y\,dy - 2xy^2\,dx}{x.x^3} = 0$$

$$e^x\,dx + \frac{m}{2}\;\frac{x^2 2y\,dy - 2xy^2\,dx}{x^4} = 0$$

$$e^x dx + \frac{1}{2}m\,d\left(\frac{y^2}{x^2}\right) = 0$$

$$\text{since } d\left(\frac{y^2}{x^2}\right) = \frac{2x^2y\,dy - 2xy^2\,dx}{x^4}$$

By Integrating

$$\int e^x\,dx + \frac{1}{2}m\int d\left(\frac{y^2}{x^2}\right) = 0$$

$$e^x + \frac{1}{2}m\;\frac{y^2}{x^2} = C$$

$$2x^2e^x + my^2 = 2Cx^2$$

Which is required solution.

Q55. Solve $x\,dy - y\,dx = a\left(x^2 + y^2\right)dy$

Sol :

The given equation is,

$$x\,dy - y\,dx = a\left(x^2 + y^2\right)dy$$

Which can be written as

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = a\,dy$$

$$\text{since } d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

$$d\left(\tan^{-1}\frac{y}{x}\right) = a\,dy$$

By Integrating

$$\int d\left(\tan^{-1}\frac{y}{x}\right) = \int a\,dy$$

$$\tan^{-1}\frac{y}{x} - ay = C$$

Which is required solution.

Method II

If the given equation $Mdx + Ndy = 0$ is homogeneous and $\left(Mx + Ny\right) \neq 0$ then $\dfrac{1}{Mx + Ny}$ is integrating factor.

Q56. Solve $x^2y\,dx - \left(x^3 + y^3\right)dy = 0$

Sol :

The given equation is,

$$x^2y\,dx - \left(x^3 + y^3\right)dy = 0 \text{ -------(1)}$$

Clearly the equation is homogeneous differential equation.

Comparing with $Mdx + Ndy = 0$

where $M = x^2y,\; N = -\left(x^3 + y^3\right)$

$$\frac{\partial M}{\partial y} = x^2,\; \frac{\partial N}{\partial x} = -3x^2$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Which is not exact.

$$\therefore Mx + Ny \Rightarrow (x^2y)x + (-x^3 - y^3)y = \cancel{x^3y} - \cancel{x^3y} - y^4 \neq 0$$

Then I.F $\dfrac{1}{Mx+Ny} = -\dfrac{1}{y^4}$

Multiply I.F to equation (1)

$$-\frac{1}{y^4}(x^2y)dx - \left(\frac{-1}{y^4}\right)(x^3+y^3)dy = 0$$

$$-\frac{x^2y}{y^4}dx + \left(\frac{x^3}{y^4} + \frac{y^3}{y^4}\right)dy = 0$$

$$-\frac{x^2}{y^3}dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{x^2}{3y^4},\ \frac{\partial N}{\partial x} = \frac{3x^2}{y^4}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The resulting differential equation is exact.

Then $\int M\,dx \quad + \quad \int N\,dy = C$

(y constant) (terms not having x)

$$\int -\frac{x^2}{y^3}dx + \int\left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = C$$

$$\frac{-x^3}{3y^3} + \log y = C$$

$$x^3 = 3y^3(\log y - C)$$

Which is required solution.

Q57. Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Sol :

The given equation is $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0 -----(1)$

Clearly the differential equation is homogeneous

Then $M = x^2y - 2xy^2,\ N = -(x^3 - 3x^2y)$

$$Mx + Ny = \left(x^2y - 2xy^2\right)x + \left(-x^3 + 3x^2y\right)y$$

$$= x^3y - 2x^2y^2 - x^3y + 3x^2y^2$$

$$= x^2y^2 \neq 0$$

$\therefore\ Mx + Ny \neq 0$

Then the I.F. $= \dfrac{1}{Mx + Ny} = \dfrac{1}{x^2y^2}$

Multiply I.F to equation (1)

$$\frac{1}{x^2y^2}\left[x^2y - 2xy^2\right]dx - \frac{1}{x^2y^2}\left[x^3 - 3x^2y\right]dy = 0$$

$$\left[\frac{x^2y}{x^2y^2} - \frac{2xy^2}{x^2y^2}\right]dx - \left[\frac{x^3}{x^2y^2} - \frac{3x^2y}{x^2y^2}\right]dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

Which is of the term $Mdx + Ndy = 0$

$$M = \frac{1}{y} - \frac{2}{x},\ N = -\left(\frac{x}{y^2} - \frac{3}{y}\right)$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2},\ \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The differential equation is exact

Then $\int M\,dx + \int N\,dy = C$

(y constant) (terms not having x)

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int -\left(\frac{x}{y^2} - \frac{3}{y}\right)dy = C$$

$$\frac{x}{y} - 2\log x + 3\log y = C$$

$$\frac{x}{y} - \log x^2 + \log y^3 = C$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = C$$

Which is required solution.

Method III

If the equation $Mdx + Ndy = 0$ is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$, then $\frac{1}{Mx - Ny}$ is an integrating factor of $Mdx + Ndy = 0$ provided $(Mx - Ny) \neq 0$.

Q58. Solve $y(1 - xy)dx - x(1 + xy)dy = 0$

Sol :

The given equation is $y(1-xy)dx - x(1+xy)dy = 0$

Here $M = yf_1(xy) = y(1-xy)$

$$N = xf_2(xy) = -x(1+xy) = -x - x^2y$$

$$\Rightarrow Mx - Ny = [y(1-xy)]x - [-x(1+xy)]y$$

$$= xy - \cancel{x^2y^2} + xy + \cancel{x^2y^2}$$

$$= 2xy \neq 0$$

$\Rightarrow$ The IF is $\frac{1}{Mx - Ny} = \frac{1}{2xy}$

Multiplying the given equation by IF

$$\frac{1}{2xy}(y - xy^2)dx - \frac{1}{2xy}(x + x^2y)dy = 0$$

$$\left(\frac{y}{2xy} - \frac{xy^2}{2xy}\right)dx - \left(\frac{x}{2xy} + \frac{x^2y}{2xy}\right)dy = 0$$

$$\left(\frac{1}{2x} - \frac{y}{2}\right)dx - \left(\frac{1}{2y} + \frac{x}{2}\right)dy = 0$$

Which is of the form $Mdx + Ndy = 0$

$$M = \frac{1}{2x} - \frac{y}{2},\ N = -\left(\frac{1}{2y} + \frac{x}{2}\right)$$

$$\frac{\partial M}{\partial y} = -\frac{1}{2},\ \frac{\partial N}{\partial x} = -\frac{1}{2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Which is exact

Then $\int M\,dx + \int N\,dy = C$

(y constant) (terms not having x)

$$\int\left(\frac{1}{2x}-\frac{y}{2}\right)dx+\int\left(\frac{-1}{2y}-\frac{x}{2}\right)dy=C$$

$$\frac{1}{2}\log x-\frac{xy}{2}-\frac{1}{2}\log y=C$$

$$\frac{1}{2}[\log x-\log y]-\frac{xy}{2}=C$$

$$\frac{1}{2}\log\left(\frac{x}{y}\right)-\frac{xy}{2}=C$$

Which is required solution.

Q59. Solve $\left(x^4y^4+x^2y^2+xy\right)y\,dx+\left(x^4y^4-x^2y^2+xy\right)x\,dy=0$.

Sol :

The given equation is $\left(x^4y^4+x^2y^2+xy\right)y\,dx+\left(x^4y^4-x^2y^2+xy\right)x\,dy=0$ ... (1)

Here $M=\left(x^4y^4+x^2y^2+xy\right)y,\ N=\left(x^4y^4-x^2y^2+xy\right)x$

Then the IF is $=\dfrac{1}{Mx-Ny}$

$$\Rightarrow Mx-Ny=\left[\left(x^4y^4+x^2y^2+xy\right)y\right]x-\left[\left(x^4y^4-x^2y^2+xy\right)x\right]y$$

$$=x^5y^5+x^3y^3+x^2y^2-x^5y^5+x^3y^3-x^2y^2$$

$$=2x^3y^3$$

$$\Rightarrow \frac{1}{Mx-Ny}=\left[\frac{1}{2x^3y^3}\right]\neq 0$$

Multiply IF to equation (1)

$$\therefore \frac{1}{2x^3y^3}\left[x^4y^4+x^2y^2+xy\right]y\,dx+\frac{1}{2x^3y^3}\left[x^4y^4-x^2y^2+xy\right]x\,dy=0$$

$$\left[\frac{x^4y^5}{2x^3y^3}+\frac{x^2y^3}{2x^3y^3}+\frac{xy^2}{2x^3y^3}\right]dx+\left[\frac{x^5y^4}{2x^3y^3}-\frac{x^3y^2}{2x^3y^3}+\frac{x^2y}{2x^3y^3}\right]dy=0$$

$$\left[\frac{xy^2}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right]dx+\left[\frac{x^2y}{2}-\frac{1}{2y}+\frac{1}{2xy^2}\right]dy=0$$

Which is of the form $Mdx + Ndy = 0$

$$M = \frac{xy^2}{2} + \frac{1}{2x} + \frac{1}{2x^2y},\ N = \frac{x^2y}{2} - \frac{1}{2y} + \frac{1}{2xy^2}$$

$$\frac{\partial M}{\partial y} = \frac{2xy}{2}; \frac{1}{2x^2y^2} = xy - \frac{1}{2x^2y^2}$$

$$\frac{\partial N}{\partial x} = \frac{2xy}{2} - \frac{1}{2x^2y^2} = xy - \frac{1}{2x^2y^2}$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Which is exact.

Then $\int M\,dx \ + \ \int N\,dy = C$

(y constant) (terms not having x)

$$\int\left(\frac{xy^2}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \int\left(\frac{x^2y}{2} - \frac{1}{2y} + \frac{1}{2xy^2}\right)dy = C$$

$$\frac{x^2y^2}{4} + \frac{1}{2}\log x - \frac{1}{2yx} - \frac{1}{2}\log y = C$$

$$\frac{x^2y^2}{4} - \frac{1}{2xy} - \frac{1}{2}(\log y - \log x) = C$$

$$\frac{x^2y^2}{4} - \frac{1}{2xy} - \frac{1}{2}\log\left(\frac{y}{x}\right) = C$$

Which is required solution.

Q60. Solve $(xy + 2x^2y^2)y\,dx + (xy - x^2y^2)x\,dy = 0$

Sol :

The given equation is $(xy + 2x^2y^2)y\,dx + (xy - x^2y^2)x\,dy = 0$ ------(1)

Here $M = (xy + 2x^2y^2)y,\ N = (xy - x^2y^2)x$

Then the IF is $\frac{1}{Mx - Ny}$

$$Mx - Ny = \left[(xy + 2x^2y^2)y\right]x - \left[(xy - x^2y^2)x\right]y$$

$$= x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3$$

$$Mx - Ny = 3x^3y^3 \neq 0$$

$$IF = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3} \neq 0$$

Multiply IF to the equation (1)

$$\frac{1}{3x^3y^3}\left[xy^2 + 2x^2y^3\right]dx + \frac{1}{3x^3y^3}\left[x^2y - x^3y^2\right]dy = 0$$

$$\left(\frac{xy^2}{3x^3y^3} + \frac{2x^2y^3}{3x^3y^3}\right)dx + \left(\frac{x^2y}{3x^3y^3} - \frac{x^3y^2}{3x^3y^3}\right)dy = 0$$

$$\Rightarrow \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

Which is of the form $Mdx + Ndy = 0$

$$\therefore M = \frac{1}{3x^2y} + \frac{2}{3x}, \quad N = \frac{1}{3xy^2} - \frac{1}{3y}$$

$$\frac{\partial M}{\partial x} = \frac{-1}{3x^2y^2}, \frac{\partial N}{\partial y} = \frac{-1}{3x^2y^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ Which is exact.}$$

Then $\int M\,dx \quad + \quad \int N\,dy = C$

(y constant) (terms not having x)

$$\int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int\left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = C$$

$$\left[\frac{-1}{3xy} + \frac{2}{3}\log x\right] - \frac{1}{3}\log y = C$$

$$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = C$$

$$-\frac{1}{3}\frac{1}{xy} + \frac{1}{3}\left[2\log x - \log y\right] = C$$

$$\frac{1}{3}\left[2\log x - \log y\right] = \frac{1}{3}\frac{1}{xy} + C_1$$

$$2\log x - \log y = \frac{1}{xy} + 3C_1 \qquad \text{where } 3C_1 = C$$

$$2\log x - \log y = \frac{1}{xy} + C$$

Which is required solution.

Q61. Solve $\left(x^2y^2+xy+1\right)y\,dx+\left(x^2y^2-xy+1\right)x\,dy=0$.

Sol :

The given equation is $\left(x^2y^2+xy+1\right)y\,dx+\left(x^2y^2-xy+1\right)x\,dy=0$ -------(1)

Here $M=\left(x^2y^2+xy+1\right)y,\;\; N=\left(x^2y^2-xy+1\right)x$

Then the IF is $\dfrac{1}{Mx-Ny}$

$$Mx-Ny=\left[\left(x^2y^2+xy+1\right)y\right]x-\left[\left(x^2y^2-xy+1\right)x\;\right]y$$

$$=\cancel{x^3y^3}+x^2y^2+\cancel{xy}-\cancel{x^3y^3}+x^2y^2-\cancel{xy}$$

$$=2x^2y^2$$

$$\therefore \frac{1}{Mx-Ny}=\frac{1}{2x^2y^2}$$

Multiply IF to the equation in (1)

$$\frac{1}{2x^2y^2}\left[\left(x^2y^2+xy+1\right)y\right]dx+\frac{1}{2x^2y^2}\left[\left(x^2y^2-xy+1\right)x\;\right]dy=0$$

$$\left(\frac{x^2y^3}{2x^2y^2}+\frac{xy^2}{2x^2y^2}+\frac{y}{2x^2y^2}\right)dx+\left(\frac{x^3y^2}{2x^2y^2}-\frac{x^2y}{2x^2y^2}+\frac{x}{2x^2y^2}\right)dy=0$$

$$\left(\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right)dx+\left(\frac{x}{2}-\frac{1}{2y}+\frac{1}{2xy^2}\right)dy=0$$

Which is of the form $Mdx+Ndy=0$

$$M=\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y},\;\; N=\frac{x}{2}-\frac{1}{2y}+\frac{1}{2xy^2}$$

$$\frac{\partial M}{\partial x}=\frac{1}{2}-\frac{1}{2x^2y^2},\;\; \frac{\partial N}{\partial y}=\frac{1}{2}-\frac{1}{2x^2y^2}$$

$\therefore \dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ Which is exact.

Then $\int M\,dx \;+\; \int N\,dy=C$

(y constant) (terms not having x)

$$\int\left(\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right)dx+\int\left(\frac{x}{2}-\frac{1}{2y}+\frac{1}{2xy^2}\right)dy=C$$

$$\frac{xy}{2}+\frac{1}{2}\log x-\frac{1}{2xy}-\frac{1}{2}\log y=C$$

$$\frac{xy}{2}-\frac{1}{2xy}+\frac{1}{2}[\log x-\log y]=C$$

$$\frac{xy}{2}-\frac{1}{2xy}+\frac{1}{2}\log\left(\frac{x}{y}\right)=C$$

$$\frac{1}{2}\left(xy+\log\left(\frac{x}{y}\right)-\frac{1}{xy}\right)=\frac{1}{2}C$$

$$xy+\log\left(\frac{x}{y}\right)-\frac{1}{xy}=C$$

Which is required solution.

Q62. Solve $(xy\sin xy+\cos xy)y\,dx+(xy\sin xy-\cos xy)x\,dy=0$.

Sol :

The given equation is $(xy\sin xy+\cos xy)y\,dx+(xy\sin xy-\cos xy)x\,dy=0$ ... (1)

Here $M=(xy\sin xy+\cos xy)y$; $N=(xy\sin xy-\cos xy)x$

Then the Then the IF is $\frac{1}{Mx-Ny}$

i.e., $Mx-Ny=[(xy\sin xy+\cos xy)y]x-[(xy\sin xy-\cos xy)x]y$

$$=xy(xy\sin xy+\cos xy)-xy(xy\sin xy-\cos xy)$$

$$=x^2y^2\sin xy+xy\cos xy-x^2y^2\sin xy+xy\cos xy$$

$$Mx-Ny=2xy\cos xy\neq 0$$

$$\therefore\ \frac{1}{Mx-Ny}=\frac{1}{2xy\cos xy}\neq 0$$

Multiply in I.F to the equation (1)

$$\frac{1}{2xy\cos xy}(xy^2\sin xy+y\cos xy)dx+\frac{1}{2xy\cos xy}(xy^2\sin xy-x\cos xy)dy=0$$

$$\left[\frac{xy^2\sin xy}{2xy\cos xy}+\frac{y\cos xy}{2xy\cos xy}\right]dx+\left[\frac{xy^2\sin xy}{2xy\cos xy}-\frac{x\cos xy}{2xy\cos xy}\right]dy=0$$

$$\left[\frac{y}{2}\tan xy+\frac{1}{2x}\right]dx+\left[\frac{x}{2}\tan xy-\frac{1}{2y}\right]dy=0$$

Which is of the form $Mdx + Ndy = 0$

Here $M = \frac{y}{2}\tan xy + \frac{1}{2x}$; $N = \frac{x}{2}\tan xy + \frac{1}{2y}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Which is exact.

The solution of given differential equation

Then $\int M\,dx + \int N\,dy = C$

(y = constant) (term not having x)

$$\int\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \int\left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)dy = C$$

$$\frac{1}{2}(\log\sec xy + \log x) - \frac{1}{2}\log y = \frac{1}{2}\log C$$

$$\log\sec xy + \log\left(\frac{x}{y}\right) = \log C$$

$$\left(\frac{x}{y}\right)\sec xy = C$$

Which is required solution.

Q63. Solve $\left(x^3y^3 + x^2y^2 + xy + 1\right)y\,dx + \left(x^3y^3 - x^2y^2 - xy + 1\right)x\,dy = 0$.

Sol :

The given equation is $\left(x^3y^3 + x^2y^2 + xy + 1\right)y\,dx + \left(x^3y^3 - x^2y^2 - xy + 1\right)x\,dy = 0$ ------(1)

Comparing the given equation with $Mdx + Ndy = 0$

$$M = \left(x^3y^3 + x^2y^2 + xy + 1\right)y; \quad N = \left(x^3y^3 - x^2y^2 - xy + 1\right)x$$

$$\therefore Mx - Ny = xy\left(x^3y^3 + x^2y^2 + xy + 1\right) - xy\left(x^3y^3 - x^2y^2 - xy + 1\right)$$

$$= \cancel{x^4y^4} + x^3y^3 + x^2y^2 + \cancel{xy} - \cancel{x^4y^4} + x^3y^3 + x^2y^2 - \cancel{xy}$$

$$= 2xy\left(x^2y^2 + xy\right)$$

$$= 2x^2y^2\,(xy + 1) \neq 0$$

On multiplying the given equation by its IF

We have

$$\frac{x^2y^2(xy+1)+(xy+1)}{2x^2y^2(xy+1)}y\,dx+\frac{(xy+1)(x^2y^2-xy+1)-xy(xy+1)}{2x^2y^2(xy+1)}x\,dy=0$$

$$\frac{x^2y^2+1}{x^2y^2}y\,dx+\frac{(x^2y^2-xy+1)-xy}{x^2y^2}x\,dy=0$$

$$(y\,dx+x\,dy)+\frac{y\,dx+x\,dy}{x^2y^2}-\frac{2x^2y}{x^2y^2}dy=0$$

$$d(xy)+\frac{d(xy)}{(xy)^2}-\frac{2}{y}dy=0$$

$$d(xy)+\left(\frac{1}{z^2}\right)dz-\left(\frac{2}{y}\right)dy=0 \qquad \text{Putting } xy=z$$

By Integrating

$$xy-\frac{1}{z}-2\log y=C$$

$$xy-\frac{1}{z}-2\log y=C$$

Method IV:

If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function x alone say f(x), then $e^{\int f(x)dx}$ is an integrating factor of $Mdx+Ndy = 0$.

Q64. Solve $(x^2+y^2)dx-2xy\,dy=0$

Sol :

The given equation is $(x^2+y^2)dx-2xy\,dy=0$ ... (1)

Comparing (1) with $Mdx+Ndy=0$

We get $M=x^2+y^2, \quad N=-2xy$

Here $\frac{\partial M}{\partial y}=2y, \frac{\partial N}{\partial x}=-2y$

$$\therefore \frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=2y-(-2y)=4y$$

$$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right) = \frac{-1}{2xy}(4y)$$

$= \frac{-2}{x}$ Which is a function of x alone

$$\therefore IF = e^{\int f(x)dx}$$

$$= e^{-\int \frac{2}{x}dx}$$

$$= e^{-2\log x}$$

$$= e^{-\log x^2}$$

$$IF = \frac{1}{x^2}$$

Now, multiply the given equation by IF

Then we get $\frac{1}{x^2}(x^2 + y^2)dx - \frac{1}{x^2}(2xy)dy = 0$

$$\Rightarrow \left(1 + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

Which is of the form $Mdx + Ndy = 0$

Then $\int \left(1 + \frac{y^2}{x^2}\right)dx - \int \frac{2y}{x}dy = C$

y = constant term is not having x

$$\int dx + \int \frac{y^2}{x^2}dx = C$$

$$x - \frac{y^2}{x} = C$$

Which is required solution.

Q65. Solve $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}(x + xy^2)dy = 0$

Sol :

The given equation is $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}(x + xy^2)dy = 0$ ... (1)

Comparing (1) with $Mdx + Ndy = 0$

We get $M = y + \frac{1}{3}y^3 + \frac{1}{2}x^2;\ N = \frac{1}{4}\left(x + xy^2\right)$

Here $\frac{\partial M}{\partial y} = 1 + \frac{3y^2}{3},\ \frac{\partial N}{\partial x} = \frac{1}{4} + \frac{y^2}{4}$

$$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 + y^2 - \frac{1}{4} - \frac{y^2}{4}$$

$$= \frac{3}{4} + \frac{3}{4}y^2$$

$$= \frac{3}{4}\left[1 + y^2\right]$$

$$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\frac{1}{4}\left(x + xy^2\right)}\left[\frac{3}{4}\left(1 + y^2\right)\right]$$

$$= \frac{4}{x(1 + y^2)}\frac{3}{4}(1 + y^2)$$

$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3}{x}$ which is a function of x alone

$$\therefore IF = e^{\int f(x)dx}$$

$$= e^{\int \frac{3}{x}dx}$$

$$= e^{3\log x}$$

$$= e^{\log x^3}$$

$IF = x^3$

Now, multiplying (1) with IF

$$x^3\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + x^3\left[\frac{1}{4}\left(x + xy^2\right)dy\right] = 0$$

$$\left(x^3y + \frac{y^3x^3}{3} + \frac{x^5}{2}\right)dx + \left(\frac{x^4}{4} + \frac{x^4y^2}{4}\right)dy = 0$$

Which is of the form $Mdx + Ndy = 0$

$$M = x^3y + \frac{y^3x^3}{3} + \frac{x^5}{2},\ N = \frac{x^4}{4} + \frac{x^4y^2}{4}$$

which must be exact equation

So, its solution is

Then $\int M dx \quad + \quad \int N dy = C$

(y = constant) (term is not having x)

$$\int\left(x^3y+\frac{y^3x^3}{3}+\frac{x^5}{2}\right)dx+\int\left(\frac{x^4}{4}+\frac{x^4y^2}{4}\right)dy=C$$

$$\Rightarrow \frac{x^4y}{4}+\frac{x^4y^3}{12}+\frac{x^6}{12}=C$$

$$3x^4y+x^4y^3+x^6=C$$

Which is required solution.

Q66. Solve $(x^2+y^2+2x)dx+2y\ dy=0$.

Sol :

The given equation is $(x^2+y^2+2x)dx+2y\ dy=0$ ... (1)

Comparing (1) with $Mdx+Ndy=0$

Here $M=x^2+y^2+2x;\quad N=2y$

$$\frac{\partial M}{\partial y}=2y,\ \frac{\partial N}{\partial x}=0$$

$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2y}(2y-0)=1$ which is alone [Let us say f(x)]

$$\therefore IF=e^{\int f(x)dx}$$

$$e^{\int 1dx}=e^x$$

$$IF=e^x$$

Multiplying (1) with IF we get

$$e^x(x^2+y^2+2x)dx+e^x(2y)dy=0$$

Which is of the form $Mdx+Ndy=0$

which must to be exact equation,

So, its solution is

$$\int(e^xx^2+e^xy^2+e^x2x)dx+2\int e^xy\ dy=C$$

$$\int e^xx^2dx+\int e^xy^2dx+2\int e^xx\,dx=C$$

$$x^2\int e^x - \int 2x\left(\int e^x\,dx\right)dx + e^x y^2 + 2\left[e^x(x-1)\right] = C$$

$$x^2 e^x - 2\int xe^x\,dx + e^x y^2 + 2e^x(x-1) = C$$

$$x^2 e^x - 2\left[e^x(x-1)\right] + e^x y^2 + 2e^x(x-1) = C$$

$$x^2 e^x + e^x y^2 = C$$

$$e^x\left(x^2+y^2\right) = C$$

Which is required solution.

Q67. Solve $\left(x^3-2y^2\right)dx + 2xy\,dy = 0$.

Sol :

The given equation is $\left(x^3-2y^2\right)dx + 2xy\,dy = 0$ ... (1)

Comparing (1) with $Mdx + Ndy = 0$

Here $M = x^3 - 2y^2;\ \ N = 2xy$

$$\frac{\partial M}{\partial y} = -4y,\ \frac{\partial N}{\partial x} = 2y$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ which is not exact

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(-4y-2y)$$

$= \frac{-6y}{2xy} = \frac{-3}{x}$ which is a function of x alone.

$$\therefore\ IF = e^{\int f(x)dx}$$

$$= e^{-\int \frac{3}{x}dx}$$

$$= e^{-3\log x}$$

$$= e^{-\log x^3}$$

$$IF = \frac{1}{x^3}$$

Multiplying (1) with IF, we get

$$\frac{1}{x^3}\left(x^3-2y^2\right)dx + \frac{1}{x^3}(2xy)\,dy = 0$$

$$\left(1-\frac{2y^2}{x^3}\right)dx+\frac{2y}{x^2}dy=0$$

Which must to be exact equation.

So, its solution is

$$\int\left(1-\frac{2y^2}{x^3}\right)dx+\int\frac{2y}{x^2}dy=C$$

(y = constant) (term not having x)

$$x+\frac{2y^2}{2x^2}=C$$

$$x+\frac{y^2}{x^2}=C$$

Which is required solution.

Method V

If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is a function of y alone say f(y), then $e^{\int f(y)dy}$ is an integrating factor of Mdx + Ndy = 0.

Q68. Solve $\left(3x^2y^4+2xy\right)dx+\left(2x^3y^3-x^2\right)dy=0$

Sol :

The given equation is $\left(3x^2y^4+2xy\right)dx+\left(2x^3y^3-x^2\right)dy=0$ ... (1)

Comparing (1) with $Mdx+Ndy=0$

We get, $M=3x^2y^4+2xy,\ N=2x^3y^3-x^2$

$$\Rightarrow\frac{\partial M}{\partial y}=12x^2y^3+2x,\ \frac{\partial N}{\partial x}=6x^2y^3-2x$$

$$\therefore\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=6x^2y^3-2x-12x^2y^3-2x$$

$$=-6x^2y^3-4x$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{3x^2y^4+2xy}\left[-6x^2y^3-4x\right]$$

$$=\frac{1}{y\left[3x^2y^3+2x\right]}-2\left[3x^2y^3+2x\right]$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{-2}{y}$$ Which is a function of y alone.

$$IF = e^{\int f(y)\,dy}$$

$$= e^{-\int \frac{2}{y}\,dy}$$

$$= e^{-2\log y}$$

$$= e^{-\log y^2}$$

$$IF = \frac{1}{y^2}$$

Multiplying (1) with IF we get

$$\frac{1}{y^2}\left(3x^2y^4+2xy\right)dx+\frac{1}{y^2}\left(2x^3y^3-x^2\right)dy=0$$

$$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0$$

Which must to be exact equation

So its solution is

Then $\int Mdx + \int Ndy = C$

(y = constant) (term not having x)

$$\int\left(3x^2y^2+\frac{2x}{y}\right)dx+\int\left(2x^3y-\frac{x^2}{y^2}\right)dy=C$$

$$\frac{3x^3y^2}{3}+\frac{2x^2}{2y}=C$$

$$x^3y^2+\frac{x^2}{y}=C$$

$$x^3y^3+x^2=C$$

Which is required solution.

Q69. Solve $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0$.

Sol :

The given equation is $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0$ ... (1)

Comparing (1) with $Mdx+Ndy=0$

Here $M=xy^3+y,\ N=2\left(x^2y^2+x+y^4\right)$

$$\frac{\partial M}{\partial y} = 3xy^2 + 1, \ \frac{\partial N}{\partial x} = 4xy^2 + 2$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 4xy^2 + 2 - 3xy^2 - 1$$

$$= xy^2 + 1$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy^3 + y}\left(xy^2 + 1\right)$$

$$= \frac{xy^2 + 1}{y\left(xy^2 + 1\right)}$$

$$= \frac{1}{y} \text{ which is a function of y alone}$$

$$IF = e^{\int f(y)\, dy}$$

$$= e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Multiplying (1) with IF, we have

$$y\left[xy^3 + y\right]dx + y2\left[x^2y^2 + x + y^4\right]dy = 0$$

$$\left(xy^4 + y^2\right)dx + 2\left(x^2y^3 + xy + y^5\right)dy = 0$$

which must to be exact equation.

So, its solution is

$$\int\left(xy^4 + y^2\right)dx + 2\int\left(x^2y^3 + xy + y^5\right)dy = C$$

(y = constant) (term is not having x)

$$\frac{x^2}{2}y^4 + xy^2 + 2\frac{y^6}{6} = C$$

$$3x^2y^4 + 3xy^2 + y^6 = 6C$$

Which is required solution.

Q70. Solve $\left(xy^2 - x^2\right)dx + \left(3x^2y^2 + x^2y - 2x^3 + y^2\right)dy = 0$.

Sol :

The given equation is $\left(xy^2 - x^2\right)dx + \left(3x^2y^2 + x^2y - 2x^3 + y^2\right)dy = 0$

Comparing (1) with $Mdx + Ndy = 0$

Here $M = xy^2 - x^2,\ N = 3x^2y^2 + x^2y - 2x^3 + y^2$

$$\frac{\partial M}{\partial y} = 2xy \quad , \frac{\partial N}{\partial x} = 6xy^2 + 2xy - 6x^2$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 6xy^2 + 2xy - 6x^2 - 2xy$$

$$= 6xy^2 - 6x^2$$

$$= 6x\left[y^2 - x\right]$$

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy^2 - x^2}\left[6\left(xy^2 - x^2\right)\right] = 6$ which is a function of y alone

$$IF = e^{\int f(y)\,dy} = e^{\int 6\,dy} = e^{6y}$$

$$IF = e^{6y}$$

Multiplying (1) with IF, we have

$$e^{6y}\left[xy^2 - x^2\right]dx + e^{6y}\left[3x^2y^2 + x^2y - 2x^3 + y^2\right]dy = 0$$

which must to be exact equation.

So, its solution is

$$\int e^{6y}\left[xy^2 - x^2\right]dx + \int e^{6y}\left[3x^2y^2 + x^2y - 2x^3 + y^2\right]dy = C$$

$$\int\left(e^{6y}xy^2 - e^{6y}x^2\right)dx + \int e^{6y}3xy^2dy + \int e^{6y}x^2y\,dy - 2\int e^{6y}x^3dy + \int e^{6y}y^2dy = C$$

$$\frac{x^2}{2}y^2e^{6y} - \frac{x^3}{3}e^{6y} + \left[y^2\int e^{6y} - \int 2y\left(\int e^{6y}dy\right)dy\right] = C$$

$$\frac{x^2y^2}{2}e^{6y} - \frac{x^3}{3}e^{6y} + \left[y^2.\frac{e^{6y}}{6} - \int 2y.\frac{e^{6y}}{6}dy\right] = C$$

$$e^{6y}\left[\frac{x^2y^2}{2} - \frac{x^3}{3}\right] + \frac{1}{6}y^2e^{6y} - \frac{1}{3}\left[y\int e^{6y} - \int 1\left(\int e^{6y}dy\right)dy\right] = C$$

$$e^{6y}\left[\frac{x^2y^2}{2} - \frac{x^3}{3}\right] + \frac{1}{6}y^2e^{6y} - \frac{1}{3}\left[y\frac{e^{6y}}{6} - \int\frac{e^{6y}}{6}dy\right] = C$$

$$e^{6y}\left[\frac{x^2y^2}{2} - \frac{x^3}{3}\right] + \frac{1}{6}y^2e^{6y} - \frac{1}{3}\left[y\frac{e^{6y}}{6} - \frac{1}{6}\frac{e^{6y}}{6}\right] = C$$

$$e^{6y}\left[\frac{1}{2}x^2y^2 - \frac{1}{3}x^3 + \frac{1}{6}y^2 - \frac{1}{18}y + \frac{1}{108}\right] = C$$

Which is required solution.

Method VI

If the given equation Mdx + Ndy = 0is of the form $x^a y^b$ (my dx + nx dy) + $x^c y^d$ (py dx + qx dy) = 0, where a, b, c, d, m, n, p and q are constants then $x^h y^k$ is the integrating factor, where h, k are constants and can be obtained by applying the condition that after multiplying by $x^h y^k$ the given equation is exact.

Q71. Solve $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$.

Sol :

The given equation is $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$ ... (1)

Let $x^h y^k$ be the integrating factor, multiplying the given equation by this factor

$x^h y^k [y^2 + 2x^2y]dx + x^h y^k [2x^3 - xy]dy = 0$

$[x^h y^{k+2} + 2x^{h+2}y^{k+1}]dx + [2x^{h+3}y^k - x^{h+1}y^{k+1}]dy = 0$

Here $M = x^h y^{k+2} + 2x^{h+2}y^{k+1}$

$N = 2x^{h+3}y^k - x^{h+1}y^{k+1}$

Which is exact, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

ie., $\frac{\partial M}{\partial y} = (k+2)x^h y^{(k+2)-1} + 2(k+1)x^{h+2}y^{k+1-1}$

$= (k+2)x^h y^{k+1} + 2(k+1)x^{h+2}y^k$

and $\frac{\partial N}{\partial x} = (h+3)2x^{h+3-1}y^k - (h+1)x^{h+1-1}y^{k+1}$

$= 2(h+3)x^{h+2}y^k - (h+1)x^h y^{k+1}$

$\Rightarrow (k+2)x^h y^{k+1} + 2(k+1)x^{h+2}y^k = 2(h+3)x^{h+2}y^k - (h+1)x^h y^{k+1}$

Now equating the coefficients of $x^h y^{k+1}$ and $x^{h+2}y^k$

$k + 2 = -(h+1)$ $\quad$ $2(k+1) = 2(h+3)$

$k + h = -3$ $\quad$ $2k - 2h = 4$

$k + h = -3$ ____ (i) $\quad$ $k - h = 2$ ______ (ii)

by solving above equations

$2k = -1$

$k = \frac{-1}{2}$

substitute $k = \frac{-1}{2}$ in (i)

$$-\frac{1}{2}+h=-3$$

$$h=-3+\frac{1}{2}$$

$$h=\frac{-5}{2}$$

$\therefore$ $IF = x^h y^k = x^{\frac{-5}{2}} y^{\frac{-1}{2}}$

ie., $IF = x^{\frac{-5}{2}} y^{\frac{-1}{2}}$

multiplying (1) by IF, we get

$$x^{\frac{-5}{2}} y^{\frac{-1}{2}}\left[y^2+2x^2y\right]dx + x^{\frac{-5}{2}} y^{\frac{-1}{2}}\left[2x^3-xy\right]dy=0$$

$$\left[x^{\frac{-5}{2}} y^{\frac{3}{2}} + 2x^{\frac{-1}{2}} y^{\frac{1}{2}}\right]dx + \left[2x^{\frac{1}{2}} y^{\frac{-1}{2}} - x^{\frac{-3}{2}} y^{\frac{1}{2}}\right]dy=0$$

Which must be exact.

So its solution is

$$\int\left[x^{\frac{-5}{2}} y^{\frac{3}{2}} + 2x^{\frac{-1}{2}} y^{\frac{1}{2}}\right]dx + \int\left[2x^{\frac{1}{2}} y^{\frac{-1}{2}} - x^{\frac{-3}{2}} y^{\frac{1}{2}}\right]dy = C$$

y constant term is not having x

$$\frac{x^{\frac{-5}{2}+1}}{\frac{-5}{2}+1} y^{\frac{3}{2}} + \frac{2x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} y^{\frac{1}{2}} = C$$

$$\frac{-2}{3} x^{\frac{-3}{2}} y^{\frac{3}{2}} + 2.2x^{\frac{1}{2}} y^{\frac{1}{2}} = C$$

$$\frac{-2}{3} x^{\frac{-3}{2}} y^{\frac{3}{2}} + 4x^{\frac{1}{2}} y^{\frac{1}{2}} = C$$

Which is required solution.

Q72. Solve $(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$.

Sol :

The given equation is $(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$ ... (1)

We can rewrite the given equation into $2y\,dx + 3x\,dy + 6xy^2\,dx + 8x^2y\,dy = 0$

$$\left(2y+6xy^2\right)dx + \left(3x+8x^2y\right)dy = 0$$

Let $x^h y^k$ be the integrating factor, multiplying the given equation by this factor.

$$x^h y^k\left[2y+6xy^2\right]dx+x^h y^k\left[3x+8x^2y\right]dy=0$$

$$\left[2x^h y^{k+1}+6x^{h+1}y^{k+2}\right]dx+\left[3x^{h+1}y^k+8x^{h+2}y^{k+1}\right]dy=0$$

Here $M=2x^h y^{k+1}+6x^{h+1}y^{k+2}$

$N = 3x^{h+1}y^k + 8x^{h+2}y^{k+1}$

Which is exact, we must have

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y}=(k+1)2x^h y^{k+1-1}+6(k+2)x^{h+1}y^{k+2-1}$$

$$=(k+1)2x^h y^k+6(k+2)x^{h+1}y^{k+1}$$

$$\frac{\partial N}{\partial x}=3(h+1)x^{h+1-1}y^k+8(h+2)x^{h+2-1}y^{k+1}$$

$$=3(h+1)x^h y^k+8(h+2)x^{h+1}y^{k+1}$$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$(k+1)2x^h y^k+6(k+2)x^{h+1}y^{k+1}=3(h+1)x^h y^k+8(h+2)x^{h+1}y^{k+1}$$

Now, equating the coefficients of $x^h y^k$ and $x^{h+1}y^{k+1}$

$2(k+1)=3(h+1)$ and $6(k+2)=8(h+2)$

$2k-3h=1$ ____ (i) $\quad$ $6k-8h=4$ ______ (ii)

$(i)\times 3-(ii)$

$$\begin{aligned}\Rightarrow\ \ & 6k-9h=3\\ & 6k-8h=4\\ \hline & -h=-1\end{aligned}$$

sub $h=1$ in (1)

$2k-3(1)=1$

$2k=4$

$k=2$

$\therefore h=1,\ k=2 \quad IF=x^1y^2=xy^2$

Multiplying (1) by IF, we have

$$xy^2\left[2y+6xy^2\right]dx+xy^2\left[3x+8x^2y\right]dy=0$$

$$\left[2xy^3+6x^2y^4\right]dx+\left[3x^2y^2+8x^3y^3\right]dy=0$$

which must to be exact.

so its solution is

$$\left(2xy^3+6x^2y^4\right)dx+\int\left(3x^2y^2+8x^3y^3\right)dy=C$$

y constant term is not having x

$$\frac{2x^2}{2}y^3+\frac{6x^3}{3}y^4=C$$

$$x^2y^3+2x^3y^4=C$$

Which is required solution.

1.9 Change in Variables

Q73. Define change in variable.

Ans :

By suitable substitution we can reduce a given differential equation which does not directly come under any of the forms discussed so far to one of these forms. This procedure of reducing the given differential equation by substitution is called the change of dependent (or independent) variable.

Q74. Solve $x\,dx+y\,dy=\frac{a^2(x\,dx-y\,dy)}{x^2+y^2}$.

Sol :

The given equation is $x\,dx+y\,dy=\frac{a^2(x\,dx-y\,dy)}{x^2+y^2}$ ... (1)

Let $x=r\cos\theta$ ______(i)

$y=r\sin\theta$ ______(ii)

by (i) and (ii) $x^2+y^2=r^2\cos^2\theta+r^2\sin^2\theta$

$x^2+y^2=r^2$ ____(iii)

(i) and (ii) $\frac{y}{x}=\frac{r\sin\theta}{r\cos\theta}=\tan\theta$

$\therefore \frac{y}{x}=\tan\theta$

differentiating (i) and (ii), we get

$dx=-r\sin\theta\,d\theta+dr\cos\theta,\qquad dy=r\cos\theta\,d\theta+dr\sin\theta$

Substituting the corresponding in (1)

$$\Rightarrow r\cos\theta(-r\sin\theta d\theta + dr\cos\theta) + r\sin\theta(r\cos\theta\ d\theta + dr\sin\theta)$$

$$= \frac{a^2\left(r\cos\theta(r\cos\theta\ d\theta + dr\sin\theta) - r\sin\theta(-r\sin\theta\ d\theta + dr\cos\theta)\right)}{r^2}$$

$$\Rightarrow \cancel{-r^2\cos\theta\sin\theta d\theta} + r\cos^2\theta dr + \cancel{r^2\sin\theta\cos\theta\ d\theta} + r\,dr\ \sin^2\theta$$

$$= \frac{a^2\left(r^2\cos^2\theta\ d\theta + \cancel{r\ dr\cos\theta\sin\theta} + r^2\sin^2\theta\ d\theta - \cancel{r\ dr\sin\theta\cos\theta}\right)}{r^2}$$

$$\Rightarrow r\ dr\left[\cos^2\theta + \sin^2\theta\right] = \frac{a^2\left[r^2\left(\sin^2\theta + \cos^2\theta\right)d\theta\right]}{r^2}$$

$r\ dr = a^2\ d\theta$

By integrating

$\int r\ dr = \int a^2\ d\theta + C$

$\frac{r^2}{2} = a^2\theta + C$

$r^2 = 2a^2\theta + C$

But $r^2 = x^2 + y^2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore$ The required solution is

$x^2 + y^2 = 2a^2\tan^{-1}\left(\frac{y}{x}\right) + C$

Q75. Solve $\sec^2 y\left(\frac{dy}{dx}\right) + 2x\tan y = x^3$.

Sol :

The given equation is $\sec^2 y\left(\frac{dy}{dx}\right) + 2x\tan y = x^3$ ... (1)

We can rewrite the given equation into by dividing " $\sec^2 y$ "

Let $\tan y = v$ ________(i)

$\sec^2 y\frac{dy}{dx} = \frac{dv}{dx}$ ______(ii)

Let us sub (i), (ii) in (i)

$\frac{dv}{dx} + 2x.v = x^3$ which is linear differential equation.

Where $P = 2x$, $Q = x^3$

$IF = e^{\int P dx} = e^{\int 2x\,dx} = e^{x^2}$

$\therefore$ The solution of the given differential equation is

$V.IF = \int (Q.IF)\,dx + C$

$v.e^{x^2} = \int \left(x^3.e^{x^2}\right) dx + C$

Let $x^2 = t$

$2x\,dx = dt$

$v.e^{x^2} = \int x^2.x\ e^{x^2}\ dx + C$

$= \frac{1}{2}\int x^2\ e^{x^2}\ 2x\,dx + C$

$= \frac{1}{2}\int t\ e^t\ dt + C$

$= \frac{1}{2}e^t (t-1) + C$

$\tan y\, e^{x^2} = \frac{1}{2}e^{x^2}\left(x^2 - 1\right) + C$

$\tan y = \frac{1}{2}\left(x^2 - 1\right) + C e^{x^2}$

Which is required solution.

Q76. Solve $\frac{x\,dx + y\,dy}{x\,dx - y\,dy} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$.

Sol :

The given equation is $\frac{x\,dx + y\,dy}{x\,dx - y\,dy} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$ ... (1)

Let $x = r\cos\theta$, $y = r\sin\theta$

Then $r^2 = x^2 + y^2$, $\theta = \tan^{-1}\frac{y}{x}$.

$dx = r\cos\theta - r\sin\theta\, d\theta;\; dy = r\cos\theta\, d\theta + dr\sin\theta$

substituting corresponding values in (1)

$$\frac{r\cos\theta(dr\cos\theta - r\sin\theta\, d\theta) + r\sin\theta(r\cos\theta\, d\theta + dr\sin\theta)}{r\cos\theta(r\cos\theta\, d\theta + dr\sin\theta) - r\sin\theta(dr\cos\theta - r\sin\theta\, d\theta)} = \sqrt{\frac{a^2 - r^2\cos^2\theta - r^2\sin^2\theta}{r^2}}$$

$$= \frac{r\cos^2\theta dr - r^2\sin\theta\cos\theta\, d\theta + r^2\cos\theta\sin\theta d\theta + r\,dr\sin^2\theta}{r^2\cos^2\theta d\theta + r\sin\theta\cos\theta dr - r\sin\theta\cos\theta dr + r^2\sin^2\theta d\theta} = \sqrt{\frac{a^2 - r^2(\cos^2\theta + \sin^2\theta)}{r^2}}$$

$$\frac{r(\cos^2\theta + \sin^2\theta)dr}{r^2(\cos^2\theta + \sin^2\theta)d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}}$$

$$\frac{r\,dr}{r^2\,d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sqrt{a^2 - r^2}}{r}$$

$$\frac{dr}{\sqrt{a^2 - r^2}} = d\theta$$

By Integrating

$$\int \frac{dr}{\sqrt{a^2 - r^2}} = \int d\theta + C$$

$$\sin^{-1}\frac{r}{a} = d\theta$$

$$\frac{r}{a} = \sin(\theta + C)$$

$$r = a(\sin\theta + C)$$

$\therefore$ The general solution (1) is $\sqrt{x^2 - y^2} = a\sin\left[\tan^{-1}\left(\frac{y}{x}\right) + C\right]$.

1.10 Total Differential Equation

Q77. Explain the working rule of total differential equation.

Ans :

An equation of the form $P\,dx + Q\,dy + R\,dz = 0$, where P, Q, R are functions of x, y, z is called a total differential equation in three variables.

The differential equation $P\,dx + Q\,dy + R\,dz = 0$ (P, Q, R are functions of x, y, z) to be integrable is

$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$.

*** Working Rule to solve total differential equation of the form** $P\,dx + Q\,dy + R\,dz = 0$ **--------(1)**

1. Verify the condition of integrability of (1).
2. After verifying the condition of integrability of consider one of the variable say z to be constant $dz = 0\cdot$

   Then (1) becomes $P\,dx + Q\,dy = 0$ --------(2)
3. After integrating (2), Let the solution be $U(x, y) = \phi(z)$ -------(3)

   where $\phi(z)$ is an arbitary function of z.
4. Now differentiate (3) totally w. r. to x, y, z and compare the result with (1)

   If the coefficients of dz involve functions of x and y, remove then by using (3)
5. Then solve the equation got in (4) to obtain $\phi(z)$ Using any method discussed in the previous concepts.

   Putting this value of f in (3) we get the required solution of (1)

Q78. Solve $\mathbf{(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0}$.

Sol :

The given equation is $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0$ ... (1)

Which is of the form $P\,dx + Q\,dy + R\,dz = 0$ ... (2)

here $P = y^2 + yz;\ Q = z^2 + zx;\ R = y^2 - xy$

$\therefore$ the equation of (2) is integrable, when

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$$\frac{\partial Q}{\partial z} = 2z + x;\ \frac{\partial R}{\partial y} = 2y - x;\ \frac{\partial R}{\partial x} = -y$$

$$\frac{\partial P}{\partial z} = y;\ \frac{\partial P}{\partial y} = 2y + z;\ \frac{\partial Q}{\partial x} = z$$

sub all the above values into (3)

$$(y^2 + yz)(2z + x - 2y + x) + (z^2 + zx)(-y - y) + y^2 - xy(2y + z - z) = 0$$

$\therefore$ Let z be a constant so that the given equation takes the form.

$$(y^2 + yz)dx + (z^2 + zx)dy = 0$$

$$\frac{dx}{z(z + x)} + \frac{dy}{y(y + z)} = 0$$

By integrating

$$\int \frac{1}{z(z+x)}dx + \int \frac{1}{y(y+z)}dy = 0$$

$$\log(z+x)(1) + \log y - \log(y+z) = C$$

$$\log y(z+x) - \log(y+z) = C$$

$$\frac{y(z+x)}{y+z} = \phi(z) \text{ -------(4)}$$

$$y(z+x) - \phi(z)(y+z) = 0$$

Differentiate with respect to x, y, z to obtain

$$y\,dx + [z + x - \phi(z)]dy + [y - (y+z)\phi'(z) - \phi(z)(1)]dz = 0$$

comparing the above equation with given equation

$$\frac{y^2+yz}{y} = \frac{z^2+zx}{z+x-\phi(z)} = \frac{y^2-xy}{y-(y+z)\phi'(z)-\phi(z)}$$

$$\frac{y^2+yz}{y} = \frac{z^2+zx}{z+x-\phi(z)}$$

reduces to (4) and thus given no information about $\phi(z)$

$$\therefore \frac{y^2+yz}{y} = \frac{y^2-xy}{y-(y+z)\phi^1(z)-\phi(z)}$$

simplify to get

$$y^2 - xy = y^2 - xy - (y-z)^2\phi^1(z) \qquad [\because \text{By using (4)}]$$

Which gives $\phi^1(z) = 0$ and $\phi(z) = C$

Hence the required solution is

$$y(z+x) = (y+z)C$$

Q79. Solve $(y+z)dx + (z+x)dy + (x+y)dz = 0$.

Sol :

The given equation is $(y+z)dx + (z+x)dy + (x+y)dz = 0$ ... (1)

$$y\,dx + z\,dx + z\,dy + x\,dy + x\,dz + y\,dz = 0$$

After regrouping of the form

$$(x\,dy + y\,dx) + (y\,dz + z\,dy) + (z\,dx + x\,dz) = 0$$

$$d(xy) + d(yz) + d(zx) = 0$$

By Integrating

$$\int d(xy) + \int d(yz) + \int d(zx) = C$$

$xy + yz + zx = C$ Which is the general solution of (1)

Q80. Solve $x\,dx + z\,dy + (y+2z)\,dz = 0$

Sol :

The given equation is $x\,dx + z\,dy + (y+2z)\,dz = 0$ ... (1)

$x\,dx + (y\,dz + z\,dy) + 2z\,dz = 0$

By regrouping the term 2 can be written as

$x\,dx + (y\,dz + z\,dy) + 2z\,dz = 0$

$x\,dx + d(yz) + 2z\,dz = 0$

By Integrating

$\int x\,dx + \int d(yz) + \int 2z\,dz = C$

$\frac{x^2}{2} + yz + \frac{2z^2}{2} = C$

Which is required solution of (1)

1.11 Simultaneous Total Differential Equations

The equation $\left.\begin{array}{l} P\,dx + Q\,dy + R\,dz = 0 \\ P^{I}\,dx + Q^{I}\,dy + R^{I}\,dz = 0 \end{array}\right\}$ ... (1)

Where P, Q, R and P^{I}, Q^{I}, R^{I} are any functions of 'x' , are called simultaneous total differential equations.

(a) If each of the above equation is integrable and has solution $\phi(x,y,z) = C$ and $\varphi(x,y,z) = C^{I}$ respectively then taken these equations together form the solution of (1).

(b) If one or both of the equation (1) is not integrable then we write them as

$$\frac{dx}{QR^{I} - RQ^{I}} = \frac{dy}{RP^{I} - PR^{I}} = \frac{dz}{PQ^{I} - QP^{I}}$$

1.12 Equation of the Form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

1.12.1 Method of grouping

Q81. Write the step for method of grouping method of grouping.

Ans :

Given equation are $\frac{dx}{P} = \frac{dy}{Q}, \frac{dx}{P} = \frac{dz}{R}, \frac{dy}{Q} = \frac{dz}{R}$ ---------(1) consider tge three sets of equation $\frac{dx}{P} = \frac{dy}{Q}, \frac{dx}{P} = \frac{dz}{R}, \frac{dy}{Q} = \frac{dz}{R}$ ------(2)

In set of equation (2) any one can be obtained from the remaining two equations so we can take any two equations of set (2) as equivalent to the set (1)

Case (1)

If any two equation of the set (2) are integrable, we can find their solutions by using the method of variables separable the pair of such solutions given the general solution of the given equation in the system (1)

Case (2)

If one equation only of the set (2) is integrable, we can find its general solution by the method of variables separable. This solution may be used to find the solution of another set. The pair of these solution gives the complete solution of the system(1).

Q82. Solve $\frac{dx}{z^2y} = \frac{dy}{z^2x} = \frac{dx}{y^2x}$

Sol :

The given that $\frac{dx}{z^2y} = \frac{dy}{z^2x} = \frac{dx}{y^2x}$

taking $\frac{dx}{z^2y} = \frac{dy}{z^2x}$

$$x\,dx = y\,dy$$

$$x\,dx - y\,dy = 0$$

By Integrating

$$\int x\,dx - \int y\,dy = C$$

$$\frac{x^2}{2} - \frac{y^2}{2} = C$$

$$x^2 - y^2 = C$$

Now take $\frac{dy}{z^2x} = \frac{dz}{y^2x}$

$$y^2\,dy = z^2\,dz$$

$$y^2\,dy - z^2\,dz = 0$$

By Integrating

$$\int y^2\,dy - \int z^2\,dz = C$$

$$\frac{y^3}{3} - \frac{z^3}{3} = C$$

$$y^3 - z^3 = C^I$$

$\therefore$ The complete solution is

$$x^2 - y^2 = C,\; y^3 - z^3 = C^I$$

1.12.2 Method of Multipliers

Q83. Write the step of method of multipliers.

Sol :

By a proper choice of multipliers l, m, n which are not necessary constants we write $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$ such that 1P + mQ + nR = 0. Then $ldx + mdy + ndz = 0$ can be solved giving the $\phi(x,y,z) = C$ ---(1) Again look for another set of multipliers λ,μ,γ such that $\lambda P + \mu Q + \gamma R = 0$ giving $\lambda\,dx + \mu\,dy + \gamma\,dz$. Which on integrating, Then which gives the solution as $\varphi(x,y,z) = C^I$ -----(2)

(1) & (2) are taken together form the required solution.

Q84. Solve $\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}$

Sol :

The given equation is,

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}$$

choose the multipliers x, y, z

$$= \frac{x\,dx + y\,dy + z\,dy}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dy}{x^2y^2 - x^2z^2 - y^2z^2 + y^2x^2 + z^2x^2 + z^2y^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

Thus $x\,dx + y\,dy + z\,dz = 0$

By integrating

$$\int x\,dx + \int y\,dy + \int z\,dz = C$$

$$x^2 + y^2 + z^2 = C$$

Again using the multipliers , $\frac{1}{x}, \frac{-1}{y}, \frac{-1}{z}$

$$\text{each fraction } = \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{\frac{1}{x}.x\left(y^2 - z^2\right) + \frac{1}{y}.y\left(z^2 + x^2\right) - \frac{1}{z}.z\left(x^2 + y^2\right)}$$

$$= \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{y^2 - z^2 + z^2 + x^2 - x^2 - y^2}$$

$$= \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{0}$$

Thus $\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz = 0$

By Integrating

$$\int \frac{1}{x}dx - \int \frac{1}{y}dy - \int \frac{1}{z}dz = C^{I}$$

$$\log x - \log y - \log z = C^{I}$$

$$\log x - \left(\log y + \log z\right) = C^{I}$$

$$\log x - \log yz = C^{I}$$

$$\log \frac{x}{yz} = \log C^{I}$$

$$\frac{x}{yz} = C^{I}$$

$$yz = C^{I}x$$

Hence the solution of the given equation is

$$x^2 + y^2 + z^2 = C, \quad yz = C^{I}x$$

Q85. Solve $\dfrac{dx}{mz-ny}=\dfrac{dy}{nx-lz}=\dfrac{dz}{ly-mx}$

Sol :

The given equation is $\dfrac{dx}{mz-ny}=\dfrac{dy}{nx-lz}=\dfrac{dz}{ly-mx}$

choose the multipliers l, m, n

$$=\frac{l\,dx+m\,dy+n\,dz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}$$

$$=\frac{l\,dx+m\,dy+n\,dz}{\cancel{lmz}-\cancel{lny}+\cancel{mnx}-\cancel{mlz}+\cancel{nly}-\cancel{mnx}}$$

$$=\frac{l\,dx+m\,dy+n\,dz}{0}$$

$\therefore l\,dx+m\,dy+n\,dz=0$

By Integrating

$\int l\,dx+\int m\,dy+\int n\,dz=C$

$lx+my+nz=C$

Again, choosing x, y, z as multipliers, each fraction of the given equations.

$$=\frac{x\,dx+y\,dy+z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}$$

$$=\frac{x\,dx+y\,dy+z\,dz}{mxz-nxy+nxy-lyz+lyz-zmx}$$

$$=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$\therefore\ x\,dx+y\,dy+z\,dz=0$

By Integrating

$\int x\,dx+\int y\,dy+\int z\,dz=C'$ where $2C=C'$

Hence $\int x\,dx+\int y\,dy+\int z\,dz=C'$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=C'$$

$x^2+y^2+z^2=C'$ where $2C=C'$

Hence are complete solution

Q86. Solve

Sol :

The given that

choose y, x, -1 as multipliers, each fraction of the given equation

By Integrating

Now, again choose as multipliers, each fraction of the given equation

By Integrating

Hence, , are complete solution.

Q87. Solve

Sol :

The given equation is

Taking the first two fractions.

By Integrating

again, choose as multipliers each fraction of the given equations.

By Integrating

The required general solution is

,

Q88. Solve

Sol :

Given that,

choose as multipliers, each fraction

By Integrating

again choose x, y, z as multipliers

By Integrating

Hence, the complete solution consists of

Q89. Solve

Sol :

Given that ... (1)

Taking last two fractions

By Integrating

Choosing x. y, z as multipliers, each fraction in (1)

.... (2)

Combining the third fraction in (1) with fraction (2), we get

By Integrating,

... (4)

Hence, the complete solution consist of

Multiple Choice Questions

1. $3e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$, Then the solution is [b]

   (a) $\tan y = C - e^x$ (b) $\tan y = C(1 - e^x)^3$

   (c) $\tan y = C (1- e^x)$ (d) $\tan y = C(1 - e^x)^2$

2. $(x^2 - y^2)\, dx + 2xy\, dy = 0$ [a]

   (a) $(y^2 + x^2) = Cx$ (b) $(y^2 - x^2) = Cx$

   (c) $(y^2 + x^2) = C$ (d) $(y^2 + x^2) = Cx^2$

3. x = y(log y – log x + 1) then [d]

   (a) (b)

   (c) (d)

4. Then I.F is [a]

   (a) (b)

   (c) (d)

5. I.F for [b]

   (a) x (b)

   (c) (d) y

6. [a]

   (a) (b)

   (c) (d)

7. Integrating factor for is [a]

   (a) (b)

   (c) (d) None of these

8. Bernoullise equation [b]

(a) (b)

(c) (d) None of these

9. [a]

(a) (b) –x

(c) (d) x

10. Then is [c]

(a) (b)

(c) (d) None of these

Fill in the blanks

1. ______.
2. If , Then which is ______ equation.
3. If is a function of y alone Then IF is ______.
4. A first order differential equation is called ______.
5. The linear differential equation is . Then the I.F is ______.
6. The solution for linear differential equation is ______.
7. If the differential equation if and Then ______ is an integrating factor.
8. If the differential equation is homogeneous and Then ______ is the integrating factor.
9. Then ______ and ______.
10. If the given equation is exact then the solution will be calculated as ______.

ANSWERS

1.
2. Exact
3. .
4. Linear differential equation
5.
6.
7.
8.
9. ,
10.

UNIT II	**Differential Equations first order but not of first degree:** Equations Solvable for p - Equations Solvable for y - Equations Solvable for x - Equations that do not contain x (or y) - Equations Homogeneous in x and y - Equations of the First Degree in x and y - Clairaut's equation. Applications of First Order. **Differential Equations:** Growth and Decay-Dynamics of Tumour Growth - Radio activity and Carbon Dating - Compound Interest - Orthogonal Trajectories.

2.1 Introduction to Differential Equations First Order but not of First Degree

The general first order differential equation of degree n > 1 (or not of the 1st degree) is,

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + ... + P_{n-1}\left(\frac{dy}{dx}\right) P_n = 0 \qquad ... (1) \text{ (or)}$$

$P_0P^n + P_1 P^{n-1} + P_2 P^{n-2} + ... + P_{n-1} P + P_n = 0$ where $P = \frac{dy}{dx}$ and $P_0, P_1, P_2 ..... P_n$ are functions of x and y

This can also be written as ... (2)

The above equation however cannot be solved in this general form. We will discuss here the situation where a solution of this equation exists. Let us consider two case.

Case - I

The first member of equation (1) can be resolved into rational factors of the first order.

Case - II

Here the member cannot be thus factored.

2.1.1 Equations Solvable for P

Q1. Derive equations for solvable for P.

Ans :

Let $P_0P_n + P_1 P^{n-1} + P_2 P^{n-2} + ... + P_{n-1} P + P_n = 0$ ... (1) be the given differential equation of first order and degree n > 1. Since (1) is solvable for P, it can be put in the form $[P - f_1(x, y)]\,[P - f_2(x, y)] \;......\; [P - f_n(x, y)] = 0$ ... (2) equating each factor to zero we get equations of first order and the first degree.

Let their solutions be $\phi_1(x, y, C_1) = 0, \phi_2(x, y, C_2) = 0, ..., \phi_n(x, y, C_n) = 0$ with out any loss of generality, we can write

$C_1 = C_2 = ... = C_n = C$ as they are arbitrary constants.

Therefore the solution of equation (1) can be put in the form.

$$\phi_1(x, y, C_1) = 0, \phi_2(x, y, C_2) = 0, ..., \phi_n(x, y, C_n) = 0$$

Q2. Solve (P – xy) (P – x²) (P – y²) = 0.

Sol :

Solving for P

If $P = xy$

$$\frac{dy}{dx} = xy \Rightarrow \frac{dy}{d} = x\,dx$$

By Integrating

$$\Rightarrow \int \frac{dy}{d} = \int x\,dx + C_1$$

$$\log y = \frac{x^2}{2} + C_1$$

$$\Rightarrow \log y - \frac{x^2}{2} - C_1 = 0$$

$$(P - xy) = \log y - \frac{x^2}{2} - C_1$$

If $P = x^2$

$$\frac{dy}{dx} = x^2$$

$$\Rightarrow dy = x^2 dx$$

By Integrating

$$\Rightarrow \int dy = \int x^2 dx + C_2$$

$$y = \frac{x^3}{3} + C_2$$

$$\Rightarrow y - \frac{1}{3}x^3 - C_2 = 0$$

$$(P - x^2) = \left(y - \frac{1}{3}x^3 - C_2\right)$$

If $P = y^2$

$$\frac{dy}{dx} = y^2$$

$$\Rightarrow \frac{dy}{y^2} = dx$$

By Integrating

$$\int \frac{1}{y^2} dy = \int dx + C_3$$

$$-\frac{1}{y} = x + C_3$$

$$x + \frac{1}{y} + C_3 = 0$$

$$(P - y^2) = x + \frac{1}{y} + C_3$$

Since $C_1 = C_2 = C_3 = C$

∴ The required solution is

$$\left(\log y - \frac{x^2}{2} - C\right)\left(y - \frac{1}{3}x^3 - C\right)\left(x + \frac{1}{y} + C\right) = 0.$$

Q3. Solve P² – 7P + 12 = 0.

Sol :

Given $P^2 - 7P + 12$, resolving into linear factor.

$$P^2 - 7P + 12 = 0$$

$$\Rightarrow P^2 - 3P - 4P + 12 = 0$$

$$\Rightarrow P(P-3) - 4(P-3) = 0$$

$$(P-3)(P-4) = 0$$

Its component equations are $P - 3 = 0$ and $P - 4 = 0$.

If $P - 3 = 0 \Rightarrow P = 3$

$$\Rightarrow \frac{dy}{dx} = 3$$

$$\Rightarrow dy = 3dx$$

By Integrating

$$\Rightarrow \int dy = \int 3dx + C_1$$

$$\Rightarrow y = 3x + C_1$$

$$\Rightarrow y - 3x - C = 0$$

If $P - 4 = 0 \Rightarrow P = 4$

$$\Rightarrow \frac{dy}{dx} = 4$$

$$\Rightarrow dy = 4\,dx$$

By Integrating

$$\Rightarrow \int dy = \int 4\,dx + C_2$$

$\Rightarrow y = 4x + C_2$

$\Rightarrow y - 4x - C = 0$

Hence the required solution is

$(y-3x-C)(y-4x-C) = 0$,

C being arbitrary constant.

Q4. Solve $P + 2xP^2 - y^2P^2 - 2xy^2P = 0$

Sol :

Given equation is.

$$P + 2xP^2 - y^2P^2 - 2xy^2P = 0$$

can be rewritten as

$P(P^2 + 2xP - y^2P - 2xy^2) = 0$

$P[P(P+2x) - y^2(P+2x)] = 0$

$P(P+2x) - y^2(P+2x) = 0$

Its component equations are,

$P = 0,\ P + 2x = 0,\ P - y^2 = 0$

If $P = 0$

$\Rightarrow \dfrac{dy}{dx} = 0$

$\Rightarrow dy = 0$

By Integrating

$\Rightarrow \int dy = C_1$

$\Rightarrow y - C = 0$

If $P + 2x = 0$

$\Rightarrow P = -2x$

$dy = -2x\ dx$

By Integrating

$\int dy = -2\int x\ dx + C_2$

$y + \dfrac{2x^2}{2} - C = 0$

$y + x^2 - C = 0$

If $P - y^2 = 0$

$\Rightarrow P = y^2$

$\Rightarrow \dfrac{dy}{dx} = y^2 \Rightarrow \dfrac{dy}{y^2} = dx$

By Integrating

$\int \dfrac{dy}{y^2} = \int dx + C_3$

$-\dfrac{1}{y} - x = C_3$

$-\left(x - \dfrac{1}{y}\right) = -C_3$

$x + \dfrac{1}{y} = C_3$

$\Rightarrow x + \dfrac{1}{y} - C_3 = 0$

$\Rightarrow x + \dfrac{1}{y} - C = 0$

Hence the combined solution of the given equation is

$$(y-C)(y-x^2-C)\left(x+\frac{1}{y}-C\right) = 0.$$

Q5. Solve $x^2P^2 - 2xyP + (2y^2 - x^2) = 0$

Sol :

Given equation is,

$$x^2P^2 - 2xyP + (2y^2 - x^2) = 0$$

can be rewritten as

$x^2P^2 = 2xyP - (2y^2 - x^2)$

$x^2P^2 - 2xyP + (2y^2 - x^2) = 0$

$$P = \frac{2xy \pm \left[4x^2y^2 - 4x^2(2y^2 - x^2)\right]^{\frac{1}{2}}}{2x^2}$$

$$\frac{dy}{dx} = \frac{1}{2x}\left[2xy \pm 2x\left(x^2 - y^2\right)^{\frac{1}{2}}\right]$$

$$= \left[\frac{y \pm \left(x^2 - y^2\right)^{\frac{1}{2}}}{x}\right]$$

Putting $\frac{y}{x} = v$ or $y = xy$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{xv \pm \left(x^2 - x^2v^2\right)^{\frac{1}{2}}}{x} = v \pm \left(1 - v^2\right)^{\frac{1}{2}}$$

$$\left(\frac{1}{x}\right)dx = \pm\frac{1}{\sqrt{1-v^2}}dv$$

By Integrating

$$\log x - \log C = \pm\sin^{-1} v$$

$$\log\frac{x}{C} = \pm\sin^{-1}\frac{y}{x}$$

$$x = Ce^{\pm\sin^{-1}\frac{y}{x}}$$

Q6. Solve (p+y+x) (xp+y+x) (p+2x) = 0.

Sol :

Given that

(p+y+x) (xp+y+x) (p+2x) = 0 ... (1)

Here we have,

$$p + y + x = 0,\ xp + y + x = 0,\ p + 2x = 0$$

If $p + y + x = 0 \Rightarrow \frac{dy}{dx} + y + x = 0$

Put $x + y = v \Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} = -1$$

$$\Rightarrow \frac{dv}{dx} - 1 + v = 0$$

$$\Rightarrow \frac{dv}{dx} = 1 - v$$

$$\Rightarrow \frac{dv}{1-v} = dx$$

By Integrating

$$\int\frac{dv}{1-v} = \int dx + C_1$$

$$-\log(1-v) = x + C_1$$

$$-(1-v) = e^{x+C_1}$$

$$1 - v = -e^{-x-C_1}$$

$$1 - x - y - Ce^{-x} = 0 \_\_\_\_\_ (2)$$

If $xp + y + x = 0 \Rightarrow x.\frac{dy}{dx} + y + x = 0$

$$\frac{dy}{dx} + \frac{1}{x}y + 1 = 0$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x}y = -1$$

$\frac{dy}{dx} + \frac{1}{x}y = -1$ which is a linear equation.

Then the integrating factor

$$IF = e^{\int p\,dx}$$

$$= e^{\int\frac{1}{x}dx}$$

$$= e^{\log x}$$

$$IF = x$$

Then its solution is $y.IF = \int Q(IF)dx + C_2$

$$y.x = \int(-1)(x)dx + C_2$$

$$yx = -\left[\frac{x^2}{2}\right] + C_2$$

$$2xy + x^2 - C = 0 \_\_\_\_\_\_ (3)$$

If $p + 2x = 0 \Rightarrow p = -2x$

$$\Rightarrow \frac{dy}{dx} = -2x$$

$\Rightarrow dy = -2x\,dx$

By Integrating

$\Rightarrow \int dy = -2\int x\,dx + C_3$

$\Rightarrow y = \dfrac{-2x^2}{2} + C_3$

$\Rightarrow y + x^2 - C = 0$ ______ (4)

Hence the combined solution of the given equation is

$\left(1 - x - y - Ce^{-x}\right)\left(2xy + x^2 - C\right)\left(y + x^2 + C\right) = 0$

Q7. Solve $P^2 + 2Py\cot x = y^2$.

Sol :

Given that $P^2 + 2Py\cot x = y^2$ which can be written as $P^2 + 2Py\cot x - y^2 = 0$

Solving this for p, we get

$$P = \frac{-2y\cot x \pm \sqrt{(2y\cot x)^2 - 4(1)(-y^2)}}{2}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\cot x^2 + 4y^2}}{2}$$

$$= \frac{-2y\cot x \pm 4y\sqrt{\cot^2 x + 1}}{2}$$

$$= \frac{-2y\cot x \pm 2y\sqrt{\operatorname{cosec}^2 x}}{2}$$

$$= \frac{-2y\cot x \pm 2y\operatorname{cosec} x}{2}$$

$$P = -y\cot x \pm y\operatorname{cosec} x$$

$$\Rightarrow P = -y\left(\frac{\cos x}{\sin x}\right) \pm y\left(\frac{1}{\sin x}\right)$$

$$\Rightarrow P = y\left[\frac{-\cos x \pm 1}{\sin x}\right]$$

$$\Rightarrow P = y\left[\frac{-\cos x + 1}{\sin x}\right] \quad \text{or} \quad \Rightarrow P = y\left[\frac{-\cos x - 1}{\sin x}\right]$$

$$\Rightarrow P = 2y\frac{\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} \quad \text{or} \quad P = \frac{-2y\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$\Rightarrow P = y + \tan\frac{x}{2} \quad \text{or} \quad P = -y\cot\frac{x}{y}$$

If $P = y + \tan\frac{x}{2}$

$$\Rightarrow \frac{dy}{dx} = y\tan\frac{x}{2}$$

$$\Rightarrow \frac{dy}{d} = \tan\frac{x}{2}dx$$

$$\int\frac{dy}{d} = \int\tan\frac{x}{2}dx + C_1$$

$$\log y = 2\log\left(\sec\frac{x}{2}\right) + \log C_1$$

$$\Rightarrow \log y = \log\left(\sec\frac{x}{2}\right)^2 + \log C_1$$

$$\Rightarrow \log y = \log\left(C_1\sec^2\frac{x}{2}\right)$$

$$\Rightarrow y = C_1\sec^2\frac{x}{2}$$

$$\Rightarrow \frac{y}{\sec^2\frac{x}{2}} = C_1$$

$$\Rightarrow y\cos^2\frac{x}{2} = C_1$$

$$\Rightarrow y(1+\cos x) = 2C_1 = C$$

$$\Rightarrow y(1+\cos x) - C = 0 \_\_\_\_\_ (1)$$

If $P = -y\cot\frac{x}{y}$

$$\Rightarrow \frac{dy}{dx} = -y\cot\frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx} = -\cot\frac{x}{2}dx$$

By Integrating

$$\int\frac{dy}{dx} = -\int\cot\frac{x}{2}dx + C_2$$

$$\Rightarrow \log y = -2\log\left(\sin\frac{x}{2}\right) + \log C_2$$

$$\Rightarrow \log y = \log\left(\sin\frac{x}{2}\right)^{-2} + \log C_2$$

$$\Rightarrow \log y = \log\left(C_2\sin^{-2}\frac{x}{2}\right)$$

$$\Rightarrow y = \frac{C_2}{\sin^2\frac{x}{2}}$$

$$\Rightarrow y = \sin^2\frac{x}{2} = C_2$$

$$\Rightarrow y(1-\cos x) = 2C_2 = C$$

$$\Rightarrow y(1-\cos x) - C = 0 \_\_\_\_ (2)$$

Hence the combined solution of the given equation is

$$\big(y(1+\cos x) - C\big)\big(y(1-\cos x) - C\big) = 0.$$

Q8. Solve $x^2p^2 + xyp - 6y^2 = 0$.

Sol :

Given that $x^2p^2 + xyp - 6y^2 = 0$ which can be written as,

$$x^2p^2 + 3xyp - 2xyp - 6y^2 = 0 \_\_\_\_\_ (1)$$

$$xp[xp+3y] - 2y[xp+3y] = 0$$

$$[xp+3y][xp-2y] = 0$$

$$xp - 2y = 0,\ xp + 3y = 0$$

$$xp - 2y = 0$$

$$\Rightarrow x\frac{dy}{dx} - 2y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \frac{dy}{d} = \frac{2}{x}dx$$

By Integrating,

$$\Rightarrow \int \frac{dy}{d} = 2\int \frac{1}{x}dx + C_1$$

$$\Rightarrow \log y = 2\log x + \log C_1$$

$$\Rightarrow \log y - \log x^2 = \log C_1$$

$$\Rightarrow \log\left(\frac{y}{x^2}\right) = \log C_1$$

$$\frac{y}{x^2} - C = 0 \_\_\_\_\_ (2)$$

$$xp + 3y = 0$$

$$\Rightarrow x\left(\frac{dy}{dx}\right) + 3y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3y}{x}$$

$$\Rightarrow \frac{dy}{y} = -\frac{3}{x}dx$$

By Integrating

$$\Rightarrow \int \frac{dy}{y} = -3\int \frac{1}{x}dx + C_2$$

$$\Rightarrow \log y = -3\log x + \log C_2$$

$$\Rightarrow \log y = \log^{x^{-3}} + \log C_2$$

$$\Rightarrow \log y - \log^{x^{-3}} = \log C_2$$

$$\Rightarrow \log \frac{y}{x^3} = \log C_2$$

$$\Rightarrow \frac{y}{x^3} = C_2$$

$$\Rightarrow yx^3 - C = 0 \_\_\_\_\_ (3)$$

Hence the combined solution of the given equation is

$$\left(\frac{y}{x^2} - C\right)\left(yx^3 - C\right) = 0$$

Q9. Solve $xy^2\left(p^2 + 2\right) = 2py^3 + x^3$.

Sol :

Given that,

$$xy^2\left(p^2 + 2\right) = 2py^3 + x^3 \quad \text{... (1)}$$

Which can be written into,

$$xy^2 + 2xy^2 = 2py^3 + x^3$$

adding and subtracting xy^2p

$$xy^2p^2 - x^2yp + x^2yp + 2xy^2 - 2py^3 - x^3 = 0$$

$$xy^2p^2 - x^2yp + x^2yp - x^3 + 2xy^2 - 2py^3 = 0$$

$$xyp[yp - x] + x^2[yp - x] - 2y^2[yp - x] = 0$$

$$(yp - x)\left(xyp + x^2 - 2y^2\right) = 0$$

$$(yp - x)\left(xyp + \left(x^2 - 2y^2\right)\right) = 0$$

$\therefore$ Its components equations are $yp - x = 0$; $xyp + \left(x^2 - 2y^2\right) = 0$

If $yp - x = 0$

$$\Rightarrow y\frac{dy}{dx} - x = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow y\,dy = x\,dx$$

By Integrating

$$\Rightarrow \int y\,dy = \int x\,dx$$

$$\Rightarrow y^2 - x^2 - C = 0 \_\_\_\_\_ (2)$$

If $xyp + x^2 - 2y^2 = 0$

$$\Rightarrow xy\frac{dy}{dx} + x^2 - 2y^2 = 0$$

$$\Rightarrow y\frac{dy}{dx} + x - \frac{2y^2}{x} = 0$$

$$\Rightarrow y\frac{dy}{dx} - \frac{2}{x}y^2 = -x$$

multiplying by '2'

$$\Rightarrow 2y\frac{dy}{dx} - \frac{4}{x}y^2 = -2x$$

Let $v = y^2$

$$\Rightarrow \frac{dv}{dx} = 2y\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} - \frac{4}{x}.v = -2x$$

Which is a linear equation, Then the IF is differential

$$IF = e^{-\int \frac{4}{x}dx}$$

$$= e^{-4\log x}$$

$$= e^{\log x^{-4}}$$

$$= \frac{1}{x^4}$$

Then its solution is,

$$v(IF) = \int Q(IF)dx + C_2$$

$$v\frac{1}{x^4} = -\int 2x.\frac{1}{x^4}dx + C_2$$

$$y^2.\frac{1}{x^4} = -2\int \frac{1}{x^3}dx + C_2$$

$$\frac{y^4}{x^4} = \frac{2}{2x^3} + C_2$$

$$y^2 = \frac{x^4}{x^4} + x^4C_2$$

$$y^2 = x^2 - x^4C = 0$$

$$y^2 - x^2 + x^4C = 0 \_\_\_\_\_\_ (3)$$

$\therefore$ From (2) and (3) the combined solution of (1) is

$$(y^2 - x^2 - C)(y^2 - x^2 + x^4C) = 0$$

Q10. Solve $4y^2p^2 + 2xy(3x+1)p + 3x^3 = 0$.

Sol :

Given that,

$$4y^2p^2 + 2xy(3x+1)p + 3x^3 = 0 \_\_\_\_\_\_ (1)$$

Which can be written as,

$$4y^2p^2 + 6x^2yp + 2xyp + 3x^3 = 0$$

$$2yp(2yp + 3x^2) + x(2yp + 3x^2) = 0$$

$$(2yp + x)(2yp + 3x^2) = 0$$

Its component equations are $2yp + x = 0$; $2yp + 3x^2 = 0$

$$2yp + x = 0$$

$$\Rightarrow 2y\frac{dy}{dx} + x = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{2y}$$

$$\Rightarrow 2y\,dy = -x\,dx$$

By Integrating

$$\Rightarrow 2\int y\,dy = -\int x\,dx + C_1$$

$$\Rightarrow \frac{2y^2}{2} = \frac{-x^2}{2} + C_1$$

$$\Rightarrow y^2 + \frac{x^2}{2} - C_1 = 0$$

$$\Rightarrow 2y^2 + x^2 - C = 0 \_\_\_\_ (2)$$

If $2yp + 3x^2 = 0$

$$\Rightarrow 2y\frac{dy}{dx} + 3x^2 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3x^2}{2y}$$

$$\Rightarrow 2y\,dy = -3x^2\,dx$$

$$\Rightarrow 2\int y\,dy = -3\int x^2\,dx + C_2$$

$$\Rightarrow \frac{2y^2}{2} = \frac{-3x^2}{3} + C_2$$

$$\Rightarrow y^2 + x^3 - C_2 = 0$$

$$\Rightarrow y^2 + x^3 - C = 0 \_\_\_\_\_\_ (3)$$

$\therefore$ From (2) and (3) The combined solution of given equation is,

$$\left(2y^2 + x^2 - C\right)\left(y^2 + x^3 - C\right) = 0 .$$

Q11. Solve $yp^2 + (x-y)p - x = 0$.

Sol :

Given that $yp^2 + (x-y)p - x = 0$ _______ (1)

Which can be rewritten as

$$yp^2 + xp - yp - x = 0$$

$$yp[p-1] + x[p-1] = 0$$

$$[p-1][yp-x] = 0$$

Its components equations are $p-1=0$; $yp-x=0$

If $p-1=0$

$$\Rightarrow \frac{dy}{dx} = 1$$

$$\Rightarrow dy = dx$$

By Integrating

$$\Rightarrow \int dy = \int dx + C_1$$

$$\Rightarrow y = x + C_1$$

$$\Rightarrow y - x - C = 0 \_\_\_\_\_ (2)$$

If $yp - x = 0$

$$\Rightarrow y\frac{dy}{dx} + x = 0$$

$$\Rightarrow y\frac{dy}{dx} = -x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$$\Rightarrow y\,dy = -x\,dx$$

By Integrating

$$\Rightarrow \int y\,dy = -\int x\,dx + C_2$$

$$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} - C_2 = 0$$

$$\Rightarrow y^2 + x^2 - 2C_2 = 0$$

$$\Rightarrow x^2 + y^2 - C = 0 \_\_\_\_\_\_\_ (3)$$

From (2) and (3) The combined solution of the given equation is $(y-x-C)\left(x^2+y^2-C\right) = 0$.

Q12. Solve $x^2p^3 + y\left(1+x^2y\right)p^2 + y^3p = 0$.

Sol :

Given that,

$$x^2p^3 + y\left(1+x^2y\right)p^2 + y^3p = 0 \_\_\_\_ (1)$$

Which can be simplified as

$$p\left[p^2x^2 + y\left(1+x^2y\right)p + y^3\right] = 0$$

$$p\left[p^2x^2 + yp + x^2y^2p + y^3\right] = 0$$

$$p\left[x^2p\left[p+y^2\right] + y\left[p+y^2\right]\right] = 0$$

$$p\left(p+y^2\right)\left(x^2p+y\right) = 0$$

Its components equations are,

$$p = 0, p + y^2 = 0, x^2p + y = 0$$

If $p = 0$

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\Rightarrow dy = 0$$

By Integrating

$$\Rightarrow y = C_1$$

$$\Rightarrow y - C = 0 \_\_\_\_\_\_ (2)$$

If $p + y^2 = 0$

$$\Rightarrow \frac{dy}{dx} = -y^2$$

$$\Rightarrow \frac{dy}{y^2} = -dx$$

$$\Rightarrow \int \frac{dy}{y^2} = -\int dx$$

$$\Rightarrow \frac{-1}{y} = -x - C_2$$

$$\Rightarrow \frac{-1}{y} = x + C_2 = 0$$

$$\Rightarrow -1 + xy + C_2 y = 0$$

$$\Rightarrow xy + Cy - 1 = 0 \_\_\_\_\_\_ (3)$$

If $x^2 p + y = 0$

$$x^2 \frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x^2}$$

$$\Rightarrow \frac{dy}{y} = -\frac{1}{x^2} dx$$

By Integrating

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{1}{x^2} dx + C_3$$

$$\log y = \frac{1}{x} + C_3$$

$$y = e^{\frac{1}{x}} + C$$

$$ye^{-\frac{1}{x}} + C = 0 \_\_\_\_\_ (4)$$

$\therefore$ From (2) (3) and (4) The combined solution of equation (1) is

$$(y - c)(xy + xy - 1)\left(ye^{-\frac{1}{x}} + C\right) = 0$$

2.1.2 Equation Solvable For Y

Q13. Define equation for solvable for y.

Ans :

If the differential equation f(x, y, p) = 0 is solvable for y, then $y = f(x, p)$ ____(1)

Differentiating with respect to x, gives

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right) \_\_\_\_ (2)$$

Which is an equation in two variables x and p and it will give rise to a solution $F(x, y, p) = 0$

The elimination of p between (1) and (2) gives a solution between x, y and e which is required solution.

Q14. Solve $y + px = x^4 p^2$.

Sol :

Given that $y + px = x^4 p^2$ ____(1)

Where $p = \frac{dy}{dx}$

Solving 1 for y

$$\Rightarrow y = x^4 p^2 - px \_\_\_ (2)$$

Differentiating (2) with respect to 'x' and writing p for $\frac{dy}{dx}$

$$\frac{dy}{dx} = 4x^3 p^2 + 2x^4 p.\frac{dp}{dx} - \left[p + x\frac{dp}{dx}\right]$$

$$p = 4x^3 p^2 + 2x^4 p\frac{dp}{dx} - \left[p + x\frac{dp}{dx}\right]$$

$$p + p - 4x^3 p^2 + \frac{dp}{dx}(x - 2x^4 p) = 0$$

$$2p - 4x^3 p^2 + \frac{dp}{dx}(x - 2x^4 p) = 0$$

$$2p(1 - 2x^3 p) + x\frac{dp}{dx}(1 - 2x^3 p) = 0$$

$$(1 - 2x^3 p)\left(2p + x\frac{dp}{dx}\right) = 0 \_\_\_\_\_ (3)$$

Neglecting the first factor which does not involve $\frac{dp}{dx}$. Then equation (3) reduces to,

$$2p + x\left(\frac{dp}{dx}\right) = 0$$

$$x\frac{dp}{dx} = -2p$$

$$\frac{dp}{p} = -\frac{2}{x}dx$$

By Integrating

$$\int\frac{dp}{p} = -2\int\frac{1}{x}dx + C_1$$

$$\log p = -2\log x + \log C_1$$

$$\log p + 2\log x = \log C$$

$$\log p + \log x^2 = \log C$$

$$\log px^2 = \log C$$

$$px^2 = C$$

$$p = \frac{C}{x^2}$$

Now substitute p in (1)

Then the required solution is,

$$y + \left(\frac{C}{x^2}\right)x = x^4\left(\frac{C}{x^2}\right)^2$$

$$y + \frac{C}{x} = C^2$$

$$xy + C = C^2x$$

Q15. Solve $y = 3x + \log p$.

Sol :

Given that $y = 3x + \log p$ Where ,

$$p = \frac{dy}{dx} \quad \_\_\_\_\_\_\_(1)$$

differentiating (1) with respect to 'x' and writing p for $\frac{dy}{dx}$

$$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$\Rightarrow p = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$\Rightarrow p - 3 = \frac{1}{p}\frac{dp}{dx}$$

$$\Rightarrow p(p-3) = \frac{dp}{dx}$$

$$\Rightarrow dx = \frac{dp}{p(p-3)}$$

By resolving into partial fractions

consider $\frac{1}{p(p-3)} = \frac{A}{p} + \frac{B}{p-3}$

$$1 = A(p-3) + B(p)$$

If $p = 0$

$1 = A(0-3)$

$\Rightarrow 1 = -3A$

$\Rightarrow A = \frac{-1}{3}$

If $p = 3$

$1 = B(3)$

$B = \frac{1}{3}$

$$\frac{1}{p(p-3)} = \frac{-1}{3p} + \frac{1}{3(p-3)}$$

$$\frac{1}{p(p-3)} = \frac{-1}{3}\left[\frac{-1}{p} + \frac{1}{p-3}\right]$$

$$\Rightarrow dx = \frac{1}{3}\left[\frac{1}{p-3} - \frac{1}{p}\right]dp$$

By Integrating

$$\int dx = \frac{1}{3}\left[\int\frac{1}{p-3} - \int\frac{1}{p}\right]dp$$

$$\int dx = \frac{1}{3}\left[\log(p-3) - \log p\right] + \log C_2$$

$$x = \frac{1}{3}\left[\log\left(\frac{p-3}{p}\right)\right] + \log C_2$$

$$3x = \log\left(\frac{p-3}{pC}\right)$$

$$\frac{p-3}{PC} = e^{3x}$$

$$\therefore p = \frac{3}{1-Ce^{3x}}$$

Putting p in (1) then the required solution is $y = 3x + \log\left(\frac{3}{1-Ce^{3x}}\right)$.

Q16. Solve $y = yp^2 + 2px$.

Sol :

Given that $y = yp^2 + 2px$ _____(1) Where $p = \frac{dy}{dx}$

solving (1) for y,

$$y - yp^2 = 2px$$

$$y(1-p^2) = 2px$$

$$y = \frac{2px}{1-p^2} \_\_\_\_(2)$$

differentiating (2) with respect to 'x'

$$\frac{dy}{dx} = \frac{(1-p^2)\left[2p + 2x\frac{dp}{dx}\right] - 2px\left[-2p\frac{dp}{dx}\right]}{(1-p^2)^2}$$

$$p(1-p^2)^2 = 2p(1-p^2) + 2x(1-p^2)\frac{dp}{dx} + 4p^2x\left(\frac{dp}{dx}\right)$$

$$p(1-p^2)^2 - 2p(1-p^2) - 2x(1-p^2)\frac{dp}{dx} - 4p^2x\left(\frac{dp}{dx}\right) = 0$$

$$p(1-p^2)\left[1-p^2-2\right] - 2x\frac{dp}{dx}(1-p^2+2p^2) = 0$$

$$p(p^2-1)(-p^2-1) - 2x\frac{dp}{dx}(1+p^2) = 0$$

$$-p(p^2-1)(p^2+1) - 2x\frac{dp}{dx}(1+p^2) = 0$$

$$(p^2+1)\left[p(p^2-1) - 2x\left(\frac{dp}{dx}\right)\right] = 0 \_\_\_\_\_(3)$$

Neglecting first factor which does not involves $\frac{dp}{dx}$

Then (3) becomes

$$p(p^2-1)-2x\left(\frac{dp}{dx}\right)=0$$

$$p(p+1)(p-1)-2x\left(\frac{dp}{dx}\right)=0$$

$$2x\left(\frac{dp}{dx}\right)=p(p+1)(p-1)$$

$$\frac{dp}{p(p+1)(p-1)}=\frac{dx}{2x} \quad \text{\_\_\_\_\_\_} (4)$$

By resolving into partial fractions

$$\frac{1}{p(p+1)(p-1)}=\frac{A}{P}+\frac{B}{p+1}+\frac{C}{p-1}$$

1 = A(p+1) (p–1) + B(p–1) p+C(p) (p+1)

If p = 0

$$\Rightarrow 1=A(1)(-1)$$

$$\Rightarrow -A=1$$

$$\Rightarrow A=-1$$

If p = 1

$$\Rightarrow 1=C(1)(2)$$

$$\Rightarrow 2C=1$$

$$\Rightarrow C=\frac{1}{2}$$

If p = –1

$$1=B(-1-1)(-1)$$

$$\Rightarrow 2B=1$$

$$\Rightarrow B=\frac{1}{2}$$

$$\frac{1}{p(p+1)(p-1)}=\frac{-1}{P}+\frac{1}{2(p+1)}+\frac{1}{2(p-1)}$$

$\therefore$ By 4

$$2\left[\frac{-1}{P}+\frac{1}{2(p+1)}+\frac{1}{2(p-1)}\right]dp=\frac{dx}{x}$$

$$\left[\frac{-2}{P}+\frac{1}{p+1}+\frac{1}{p-1}\right]dp=\frac{dx}{x}$$

By Integrating

$$\int\frac{-2}{P}dp+\int\frac{1}{p+1}dp+\int\frac{1}{p-1}dp=\int\frac{1}{x}dx+C$$

$$-2\log p+\log(p+1)+\log(p-1)=\log x+\log C$$

$$-\log p^2+\left[\log(p+1)+\log(p-1)\right]=\log x+\log C$$

$$\log\frac{(p+1)(p-1)}{p^2}=\log xC$$

$$\frac{(p+1)(p-1)}{p^2}=xC$$

$$\Rightarrow\frac{p^2-1}{p^2}=Cx$$

$$\Rightarrow p^2-1=Cxp^2$$

$$\Rightarrow p^2(1+Cx)=1$$

$$\Rightarrow p^2=\frac{1}{1+Cx}$$

$$\Rightarrow p^2=\frac{1}{(1+Cx)^{1/2}}$$

Sub p in (1)

$$y=y\left[\frac{1}{(1-Cx)^{1/2}}\right]^2+2\left[\frac{1}{(1-Cx)^{1/2}}\right]x$$

$$=y\frac{1}{(1-Cx)}+\frac{2x}{(1-Cx)^{1/2}}$$

$$\frac{2x}{(1-Cx)^{1/2}}=y\left[1-\frac{1}{1+Cx}\right]$$

$$=y[1+Cx-1]=\frac{2x}{(1-Cx)^{1/2}}$$

$$\frac{Cxy}{(1-Cx)^{1/2}} = 2$$

$$Cy = 2(1-Cx)^{1/2}$$

Squaring on both sides

$$C^2y^2 = 4(1+C_1)^2$$

Q17. Solve $y = 2px + \tan^{-1}(xp^2)$

Sol :

Given that $y = 2px + \tan^{-1}(xp^2)$ ______(1) Where $p = \frac{dy}{dx}$

Differentiating (1) with respect to 'x'

$$\frac{dy}{dx} = 2p + 2\frac{dp}{dx} + \frac{p^2}{1+(xp^2)^2}\left[(p^2) + x(2p)\frac{dp}{dx}\right]$$

$$p = 2p + 2x\frac{dp}{dx} + \frac{p^2}{1+x^2p^4} + \frac{2xp}{1+x^2p^4}\frac{dp}{dx}$$

$$0 = p\left(1+\frac{p}{1+x^2p^4}\right) + 2x\frac{dp}{dx}\left[1+\frac{p}{1+x^2}\right]$$

$$0 = \left(x+\frac{p}{1+x^2p^4}\right) + \left(p+2x\frac{dp}{dx}\right) \text{______}(2)$$

Neglecting first factors which does not involves $\frac{dp}{dx}$

Then (2) becomes

$$p + 2x\frac{dp}{dx} = 0$$

$$\Rightarrow 2x\frac{dp}{dx} = -p$$

$$\frac{dp}{dx} = \frac{-p}{2x}$$

$$\frac{dp}{p} = -\frac{dx}{2x}$$

By Integrating

$$\int\frac{dp}{p} = -\frac{1}{2}\int\frac{dx}{x}$$

$$\log p = \frac{-1}{2}\log x + \log C$$

$$\log p = \log x^{\frac{-1}{2}} + \log C$$

$$\log p = \log x^{-1/2}C$$

$$p = x^{\frac{-1}{2}}C$$

Substituting p value in (1)

$$y = 2\left(x^{\frac{-1}{2}}C\right) + \tan^{-1}(x)\left(x^{\frac{-1}{2}}C\right)$$

$$y = 2x^{\frac{-1}{2}}C + \tan^{-1}C$$

Which is required solution.

Q18. Solve $y = \sin p - p\cos p$

Sol :

Given that $y = \sin p - p\cos p$ ____(1)

Differentiating (1) with respect to 'x' we get

$$\frac{dy}{dx} = \cos p\frac{dp}{dx} - \left[p(-\sin p)\frac{dp}{dx} + \cos p\frac{dp}{dx}\right]$$

$$= \cos p\frac{dp}{dx} + p\sin\frac{dp}{dx} - \cos p\frac{dp}{dx}$$

$$\frac{dy}{dx} = p\sin p\frac{dp}{dx}$$

$$p = p\sin p\frac{dp}{dx}$$

$$p - p\sin p\frac{dp}{dx} = 0$$

$$p\left[1 - \sin p\frac{dp}{dx}\right] = 0$$

Neglecting 'p'

$$\sin p\frac{dp}{dx} = 1$$

$$\sin p\, dp = dx$$

By Integrating

$$\int \sin p\, dp = \int dx + C$$

$$-\cos p = x + C$$

$$\cos p = C - x$$

$$\Rightarrow p = \cos^{-1}(C - x)$$

by (1)

$$\Rightarrow p\cos p = \sin p - y$$

$$p\cos p = \sqrt{1-\cos^2 p} - y$$

$$p = \frac{\sqrt{1-\cos^2 p} - y}{\cos p}$$

$$\cos^{-1}(C - x) = \frac{\sqrt{1-\cos^2 p} - y}{\cos p}$$

$$C - x = \cos\left(\frac{\sqrt{1-\cos^2 p} - y}{\cos p}\right)$$

$$C - x = \cos\left(\frac{\sqrt{1-c^2-x^2+2Cx} - y}{\cos p}\right)$$

Which is required solution.

2.1.3 Equation solvable for x.

Q19. Define equation for solvable for x.

Ans :

- When the differential equation $F(x,y,p)$ is solvable for x, then we have $x = f(y,p)$
- Differentiating with respect to y, gives

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right)$$

- From which a relation between p and y may be obtained $F(y,p,C) = 0$
- Between this and the given equation p, may be eliminates or x and y expressed in terms of p.

Q20. Solve $x(1+p^2) = 1$.

Sol :

Given that $x(1+p^2) = 1$

$$\Rightarrow x = \frac{1}{1+p^2}$$

$$\Rightarrow x = \left(1+p^2\right)^{-1} \qquad \text{Where } p = \frac{dy}{dx}$$

Differentiating (1) with respect to y.

$$\frac{dx}{dy} = \frac{-1}{\left(1+p^2\right)^2}.2p\frac{dp}{dy} \qquad \text{where } \frac{dx}{dy} = \frac{1}{p}$$

$$\frac{1}{p} = \frac{-2p}{\left(1+p^2\right)^2}.\frac{dp}{dy}$$

$$dy = \frac{-2p^2}{\left(1+p^2\right)^2}dp$$

By Integrating

$$\int dy = \int \frac{-2p^2}{\left(1+p^2\right)^2}dp + C$$

$$\text{Let } p = \tan\theta$$

$$dp = \sec^2\theta\, d\theta$$

$$y = -2\int \frac{\tan^2\theta}{\left(1+\tan^2\theta\right)^2}\sec^2\theta\, d\theta + C$$

$$y = C - 2\int \frac{\tan^2\theta}{\left(1+\tan^2\theta\right)^2}\sec^2\theta\, d\theta$$

$$= C - 2\int \frac{\tan^2\theta}{\sec^2\theta}d\theta$$

$$= C - 2\int \frac{\tan^2\theta}{\cancel{\cos^2\theta}}.\cancel{\cos^2\theta}\, d\theta$$

$$= C - 2\int \sin^2\theta\, d\theta$$

$$= C - \int\left(1-\cos^2\theta\right) d\theta$$

$$y = C - \theta + \frac{1}{2}\sin^2\theta$$

$$y = C + \theta + \frac{1}{2}\frac{2\tan\theta}{1+\tan^2\theta}$$

$$y = C - \tan^{-1}p + \frac{p}{1+p^2} \_\_\_\_ (2)$$

From (1) and (2) together from the solution in parametric form.

Q21. Solve $y^2 \log p = xyp + p^2$

Sol :

Given that $y^2 \log p = xyp + p^2$ ____(1)

Which can be written as

$$xyp + p^2 = y^2 \log y$$

$$xyp = y^2\log y - p^2$$

$$x = \frac{y^2\log y}{yp} - \frac{p^2}{yp}$$

$$x = \frac{y\log y}{p} - \frac{p}{y}$$

$$x = \frac{1}{p}y\log y - \frac{p}{y}$$

Now, differentiating with respect to 'y'

$$\frac{dx}{dy} = (1+\log y)\frac{1}{p} - \frac{y\log y}{p^2}\frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

$$\frac{1}{p} = \frac{1}{p} + \frac{1}{p}\log y + \frac{p}{y^2} - \left[\frac{y\log y}{p^2} + \frac{1}{y}\right]\frac{dp}{dy}$$

$$0 = \frac{-1}{p} + \frac{1}{p} + \frac{1}{p}\log y + \frac{p}{y^2} - \left[\frac{y\log y}{p^2} + \frac{1}{y}\right]\frac{dp}{dy}$$

$$\frac{1}{p}\log y + \frac{p}{y^2} - \left[\frac{y\log y}{p^2} + \frac{1}{y}\right]\frac{dp}{dy} = 0$$

$$\frac{p}{y}\left[\frac{y\log y}{p^2} + \frac{1}{y}\right] - \left[\frac{y\log y}{p^2} + \frac{1}{y}\right]\frac{dp}{dy} = 0$$

$$\left[\frac{y\log y}{p^2} + \frac{1}{y}\right]\left[\frac{p}{y} - \frac{dp}{dy}\right] = 0 \_\_\_\_\_\_ (2)$$

Neglecting first factor from (2) which is does not involves $\frac{dp}{dy}$

$$\frac{p}{y} = \frac{dp}{dy} = 0$$

$$\frac{dp}{p} = \frac{dy}{y}$$

By Integrating on

$$\int \frac{dp}{p} = \int \frac{dy}{y}$$

$$\log p = \log y + C$$

$$p = Cy$$

eliminating p from (1)

$$y^2 \log y = x(Cy)y + (Cy)^2$$

$$y^2 \log y = xCy^2 + C^2y^2$$

$$\log y = xC + C^2$$

The general solution of (1) is $\log y = xC + C^2$

Q22. Solve $xp^3 = a + bp$

Sol :

Given that $xp^3 = a + bp$

Can be written as

$$x = \frac{a}{p^3} + \frac{b}{p^2} \text{ ____ } (1)$$

Differentiating (1) with respect to 'y'

$$\frac{dx}{dy} = a\left(\frac{-3}{p^4}\right)\frac{dp}{dy} + b\left(\frac{-2}{p^3}\right)\frac{dp}{dy}$$

$$\frac{1}{p} = -\frac{1}{p}\left(\frac{3a}{p^3} + \frac{2b}{p^2}\right)\frac{dp}{dy}$$

$$\frac{p}{p} = -\left(\frac{3a}{p^3} + \frac{2b}{p^2}\right)\frac{dp}{dy}$$

$$1 = -\left(\frac{3a}{p^3} + \frac{2b}{p^2}\right)\frac{dp}{dy}$$

$$dy + \left(\frac{3a}{p^3} + \frac{2b}{p^2}\right)dp = 0$$

Integrating

$$\int dy + \left(\int \frac{3a}{p^3} + \frac{2b}{p^2}\right)dp + C$$

$$y = -\frac{3a}{2p^2} + \frac{2b}{p} + C$$

Which is required solution.

Q23. Solve $x = y + a \log p$

Sol :

Given that $x = y + a \log p$ ______(1)

Differentiating with respect to 'y'

$$\frac{dx}{dy} = 1 + a\frac{1}{p}\frac{dp}{dy}$$

$$\frac{1}{p} = 1 + \frac{a}{p}\frac{dp}{dy}$$

$$\frac{1}{p} - 1 = \frac{a}{p}\frac{dp}{dy}$$

$$\frac{1-p}{p} = \frac{a}{p}\frac{dp}{dy}$$

$$\frac{\not{p}}{a}\left(\frac{1-p}{\not{p}}\right) = \frac{dp}{dy}$$

$$\frac{1-p}{p} = \frac{dp}{dy}$$

$$\Rightarrow dy = \frac{a}{1-p}dp$$

By Integrating

$$\int dy = \int \frac{a}{1-p}dp + C$$

$$\Rightarrow y = a\log(1-p)(-1) + C$$

$$y = C - a\log(1-p) \text{ ______ } (2)$$

Substitute y in (1)

$$x = C - a\log(1-p) + a\log p$$

Which is required solution.

Q24. Solve $x = y - p^2$

Sol :

Given that $x = y - p^2$ _____(1)

Differentiating (1) with respect to 'y'

$$\frac{dx}{dy} = 1 - 2p\frac{dp}{dy}$$

$$\frac{1}{p} = 1 - 2p\frac{dp}{dy}$$

$$\frac{1}{p} - 1 = -2p\frac{dp}{dy}$$

$$\Rightarrow \frac{1-p}{p} = -2p\frac{dp}{dy}$$

$$\frac{1-p}{2p^2} = -\frac{dp}{dy}$$

By separating variable

$$dy = -\left(\frac{2p^2}{1-p}\right)dp$$

$$dy = \frac{2p^2}{1-p}dp$$

By Integrating

$$\int dy = 2\int \frac{p^2}{1-p}dp$$

$$\int dy = 2\int \left(p + 1 + \frac{1}{p-1}\right)dp$$

$$y = 2\left[\frac{p^2}{2} + p + \log(p-1)\right] + C$$

$$y = \left[p^2 + 2p + 2\log(p-1)\right] + C$$

Sub y in (1)

$$x = \left[p^2 + 2p + 2\log(p-1)\right] + C - p^2$$

$$x = C + 2p + 2\log(p-1)$$

$\therefore$ x = C + 2p + 2log (p – 1),

y = C + p^2 + 2p + 2log y(p–1)

Which is required solution.

Q25. Solve $p^3 - p(y+3) + x = 0$

Sol :

Given that $p^3 - p(y+3) + x = 0$ ______(1)

Which can be written as

$$x = p(y+3) - p^3 \quad \text{\_\_\_\_\_\_}(2)$$

Differentiating (2) with respect to 'y'

$$\frac{dx}{dy} = p + (y+3)\frac{dp}{dy} - 3p^2\frac{dp}{dy}$$

$$\frac{1}{p} - p = \frac{dp}{dy}[y + 3 - 3p]$$

$$\frac{1-p^2}{p}.\frac{dy}{dp} = y + 3 - 3p^2$$

$$\frac{dy}{dp} = \frac{p}{1-p^2}\left[y + 3(1-p^2)\right]$$

$$\frac{dy}{dp} = \frac{p}{1-p^2}y + \frac{3\cancel{(1-p^2)}p}{\cancel{(1-p^2)}}$$

$$\frac{dy}{dp} - \frac{p}{1-p^2}y = 3p \quad \text{\_\_\_\_\_\_}(3)$$

Which is linear equation.

Then its Integrating factor

$$IF = e^{-\int \frac{p}{1-p^2}dp}$$

$$= e^{-\frac{1}{2}\log(1-p^2)}$$

$$IF = (1-p^2)^{\frac{1}{2}}$$

Then the solution of (3) is

$$y(IF) = \int Q(IF)dp + C$$

$$y(1-p^2)^{\frac{1}{2}} = \int 3p(1-p^2)^{\frac{1}{2}}dp + C$$

put $1 - p^2 = v$

$$\Rightarrow -2p\,dp = dv$$

$$p\,dp = \frac{-1}{2}dv$$

$$y(1-p^2)^{\frac{1}{2}} = \int 3v^{\frac{1}{2}}.\frac{1}{2}dv + C$$

$$= \frac{-3}{2}\int v^{\frac{1}{2}}dv + C$$

$$= \frac{-3}{2}\left(\frac{v^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + C$$

$$= \frac{-3}{2}.\frac{2}{3}\left(v^{\frac{3}{2}}\right) + C$$

$$y\left(1-p^2\right)^{\frac{1}{2}} = -\left(v^{\frac{3}{2}}\right) + C$$

$$y\left(1-p^2\right)^{\frac{1}{2}} = -\left(1-p^2\right)^{\frac{-1}{2}} - \left(1-p^2\right)$$

$$y = C\left(1-p^2\right)^{\frac{-1}{2}} - \left(1-p^2\right) \_\_\_\_\_\_ (4)$$

Let substitute (4) in (2)

$$x = p\left[C\left(1-p^2\right)^{\frac{1}{2}} - \left(1-p^2\right) + 3\right] - p^3$$

$$x = Cp\left(1-p^2\right)^{-\frac{1}{2}} - p + p^3 + 3p - p^3$$

$$x = Cp\left(1-p^2\right)^{\frac{1}{2}} + 2p \_\_\_\_\_\_ (5)$$

(5) and (6) together form the solution of (1) in parametric form p being treated as parameter.

2.1.4 Equation that do not Contain x (or y)

Q26. Write Equation that do not Contain x (or y).

Ans :

- If the equation has the form $f(y, p) = 0$ and is solvable for p, $\Rightarrow \frac{dy}{dx} = \phi(y)$ which is integrable.
- If it is solvable for y, then $y = F(p)$.
- When the equation is of the form $f(x, p) = 0$, it will gives $\Rightarrow \frac{dy}{dx} = \phi(x)$, which is also integrable

  But, if it is solvable for x, then $x = F(p)$.
- It may be mentioned that in equations having either of properties, and not solvable for p, on solving for x or y, the differentiation is made with respect to absent variable.
- By differentiating the equation given, we have a chance of obtaining a differential equation.
- These two relations will then be used either for the elimination of p or for the expression of x and y term of p.

2.1.5 Equations Homogeneous in x and y

Q27. Explain the procedure for equations homogeneous in x and y.

Ans :

When the equation is homogeneous in x and y it can be written as $F\left(\frac{dy}{dx}, \frac{y}{x}\right) = 0$.

It is then possible to solve it for $\frac{dy}{dx}$ and proceed as Homogeneous differential equations, or solve it for $\frac{y}{x}$, and obtain $y = x\,f(p)$ which is given in equation solvable for y.

Proceeding as solvable for y and differentiating with respect to x, we get

$$p = f(p) + x f^1(p)\frac{dp}{dx} \quad , \quad \frac{dx}{x} = \frac{f^1(p)\,dp}{p - f(p)}$$ where the variable are separated.

2.1.6 Equations of the first degree in x and y - Clairaut's Equation.

Clairaut's Equation Definition: An equation of the form $y = px + f(p)$ is known as Clairaut's equation.

Q28. State and prove General solution of Clairaut's equation.

Sol :

Statement

Show that the general solution of Clairaut's equation $y = px + f(p)$ is $y = Cx + f(C)$ Which is obtained by replacing p by C, where C is an arbitrary constant.

Proof:

Given Clairaut's equation is $y = px + f(p)$____(1)

Differentiating (1) with respect to 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f^1(p)\frac{dp}{dx}$$

$$p = p + x\left(\frac{dp}{dx}\right) + f^1(p)\frac{dp}{dx}$$

$$x\frac{dp}{dx} + f^1(p)\frac{dp}{dx} = 0$$

$$\left[x + f^1(p)\right]\frac{dp}{dx} = 0$$

Neglecting $x + f^1(p)$ which does not involve $\frac{dp}{dx}$

$$\Rightarrow \frac{dp}{dx} = 0$$

$dp = 0$

By Integrating

$p = C$, C being on arbitrary constant putting the value of p given by $p = C$ in (1)

Then we get required solution is $y = Cx + f(C)$.

Q29. Write a Working Rule for solving Clairaut's equation

Ans :

1. The given equation can be written in the form $y = xp + f(C)$_____(1)
2. Replace p, by C in (1) to obtain the general solution (1) C being an arbitart constant
3. Putting p = C in (i) we get $y = Cx + f(C)$.

Q30. Solve $p = \log(px - y)$

Sol :

The given equation is,

$$p = \log(px - y) \_\_\_\_\_\_ (1)$$

This can be written as $e^p = px - y$

$$y = px - e^p \_\_\_\_\_\_\_ (2)$$

Which is of the form $y = p(x) + f(p)$ which is called as Clairaut's form.

Differentiating (2) with respect to 'x'

We get

$$\frac{dy}{dx} = p(1) + x\frac{dp}{dx} - e^p\frac{dp}{dx}$$

$$p = p + (x - e^p)\frac{dp}{dx}$$

$$0 = (x - e^p)\frac{dp}{dx}$$

Neglecting $x - e^p$, which does not involves $\frac{dp}{dx} = 0$

$$\Rightarrow \frac{dp}{dx} = 0$$

$$\Rightarrow dp = 0$$

By Integrating $p = C$, C being an arbitrary constant.

Putting the value of p in (1)

Then $C = \log C(x - y)$

Which is required solution.

Q31. Solve $(y - px)(p - 1) = p$

Sol :

Given that $(y - px)(p - 1) = p \_\_\_\_\_ (1)$

which can be written as $y - px = \frac{p}{p-1}$

$$y = px + \frac{p}{p-1} \_\_\_\_\_ (2)$$

which is of the form $y = p(x) + f(p)$.

Differentiating (2) with respect to 'x'

$$\frac{dy}{dx} = p(1) + x\frac{dp}{dx} + \left[\frac{(p-1)\frac{dp}{dx} - \frac{dp}{dx}p}{(p-1)^2}\right]$$

$$p = p + x\frac{dp}{dx} + \left[\frac{p\frac{dp}{dx} - \frac{dp}{dx} - \frac{dp}{dx}p}{(p-1)^2}\right]$$

$$p = p + x\frac{dp}{dx} + \left[\frac{-\frac{dp}{dx}}{(p-1)^2}\right]$$

$$0 = \frac{dp}{dx}\left[x - \frac{1}{(p-1)^2}\right]$$

$\therefore$ Neglecting $x - \frac{1}{(p-1)^2}$, which does not involves $\frac{dp}{dx} = 0$.

$$\frac{dp}{dx} = 0 \Rightarrow dp = 0$$

By Integrating $p = C$

C being an arbitrary constant.

putting the value of p in (1)

$$(y - Cx)(C - 1) = C$$

Which is required solution.

Q32. Solve $\sin px \cos y = \cos px \sin y + p$

Sol :

Given that

$$\sin px \cos y = \cos px \sin y + p \_\_\_\_\_\_ (1)$$

$$\sin px \cos p - \cos px \sin y = p$$

$$\sin(px - y) = p$$

$$y = px \sin^{-1} p \_\_\_\_\_\_ (2)$$

which is in Clairaut's form

Differentiating (2) with respect to 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}$$

$$p = p + \frac{dp}{dx}\left[x + \frac{2p}{\sqrt{1-p^2}}\right]$$

$$p - p = \frac{dp}{dx}\left[x + \frac{2p}{\sqrt{1-p^2}}\right]$$

$$\frac{dp}{dx}\left[x + \frac{2p}{\sqrt{1-p^2}}\right] = 0$$

Neglecting $\left[x + \frac{2p}{\sqrt{1-p^2}}\right]$, Which does not involves $\frac{dp}{dx}$

$$\Rightarrow \frac{dp}{dx} = 0$$

$$\Rightarrow dp = 0$$

By Integrating

$$p = C$$

$$\sin Cx \cos y = \cos Cx \sin y + C$$

(or)

By (2) we can say $y = Cx - \sin^{-1} C$

Which is required general solution.

2.1.7 Equations Reducible to Clairaut's form

Q33. Write a short notes on reducible to clairaut's form.

Ans :

To solve $y^2 = \left(\frac{py}{x}\right)x^2 + f\left(\frac{py}{x}\right)$, put $x^2 = u$ and $y^2 = v$

$x^2 = u$ and $y^2 = v$

$2x\,dx = du$ and $2y\,dy = dv$

$$\frac{2y}{2x}\frac{dy}{dx} = \frac{dv}{du} \quad \Rightarrow \frac{y}{x}.p = p$$

where $p = \frac{dv}{du}$

Hence the given equation becomes $v = pu + f(p)$.

Which is in Clairaut's form and so its solution is $v = Cu + f(C)$ (or) $y^2 = Cx^2 + f(C)$, C being an arbitrary constant.

Q34. Solve $axyp^2 + (x^2 - ay^2 - b) - xy = 0$ by putting $u = x^2$, $v = y^2$.

(OR)

Use the transformation $x^2 = u$ and $y^2 = v$ to solve the equation $axyp^2 + (x^2 - ay^2 - b) - xy = 0$.

Sol :

Given that,

$$axyp^2 + (x^2 - ay^2 - b) - xy = 0 \quad \_\_\_\_\_\_ (1)$$

Putting $x^2 = u$ and $y^2 = v$

By Differentiating,

$2x\,dx = du$ and $2y\,dy = dv$

We get $\frac{2y\,dy}{2x\,dx} = \frac{dv}{du}$

$\frac{y}{x}p = P$ where $P = \frac{dv}{du}$

$$p = \frac{xP}{y}$$

replacing P by $\frac{xP}{y}$ in (1)

$$axy\left(\frac{xP}{y}\right)^2 + (x^2 - ay^2 - b)\left(\frac{xP}{y}\right) - xy = 0$$

$$axy\frac{x^2y^2}{y^2} + (x^2 - ay^2 - b)\left(\frac{xP}{y}\right) - xy = 0$$

$$ax^3P^2 + (x^2 - ay^2 - b)xp - xy^2 = 0$$

$$x\left[ax^2P^2 + (x^2 - ay^2 - b)P - y^2\right] = 0$$

$$ax^2P^2 + (x^2 - ay^2 - b)P - y^2 = 0$$

as $x^2 = u$ and $y^2 = v$

$$auP^2 + (u - av - b)P - v = 0$$

$$uP(aP+1) - bP = v(1 + aP)$$

$$v = \frac{uP(aP+1) - bP}{1 + aP}$$

$$v = \frac{uP(1+aP)}{1+aP} - \frac{bP}{1+aP}$$

$$v = uP - \frac{bP}{1+aP}$$

Which is in clairaut's form so replacing P by arbitrary constant C,

$$v = uC - \frac{bC}{1+aC}$$

$$y^2 = Cx^2 - \frac{bC}{1+aC}$$

Which is required solution.

Q35. Reduce $xyp^2 - (x^2 + y^2 + 1)p + xy = 0$

Sol :

Given

$$xyp^2 - (x^2 + y^2 + 1)p + xy = 0 \quad \_\_\_\_\_\_(1)$$

at $x^2 = u$ and $y^2 = v$

By Differentiating,

$2x\,dx = du$ and $2y\,dy = dv$

We get $\frac{2y\,dy}{2x\,dx} = \frac{dv}{du}$

$$\frac{y}{x}p = P \quad \text{where } P = \frac{dv}{du}$$

$$p = \frac{xP}{y}$$

replacing P by $\frac{xP}{y}$ in (1)

$$xy\left(\frac{x}{y}P\right)^2 - (x^2 + y^2 + 1)\left(\frac{x}{y}P\right) + xy = 0$$

$$xy\,\frac{x^2}{y^2}P^2 - (x^2 + y^2 + 1)\frac{x}{y}P + xy = 0$$

$$x^3\frac{P^2}{y} - (x^2 + y^2 + 1)xp + xy^2 = 0$$

$$x\left[x^2P^2 - (x^2 + y^2 + 1)P + y^2\right] = 0$$

$$x^2P^2 - (x^2 + y^2 + 1)P + y^2 = 0$$

as $x^2 = u$ and $y^2 = v$

$$uP^2 - (u + v + 1)P + v = 0$$

$$uP^2 - uP - vP - P + v = 0$$

$$uP(P-1) - v(P-1) = 0$$

$$(p-1)(uP - v) - P = 0$$

$$(p-1)(uP - v) = -P$$

$$uP - v = \frac{P}{P-1}$$

$$-v = \frac{P}{P-1} - uP$$

$$v = uP + \frac{P}{P-1} \quad \_\_\_\_\_ (2)$$

Which is in clairaut's form.

Differentiating with respect to u.

$$\frac{dv}{du} = (1)P + u\frac{dp}{du} + \left[\frac{(P-1)\frac{dp}{du} - P\left(\frac{dp}{du}\right)}{(P-1)^2}\right]$$

$$P = P + u\frac{dp}{du} + \left[\frac{P\frac{dp}{du} - \frac{dp}{du} - P\left(\frac{dp}{du}\right)}{(P-1)^2}\right]$$

$$0 = u\frac{dp}{du} + \frac{1}{(P-1)^2}\frac{dp}{du}$$

$$\left[u - \frac{1}{(P-1)^2}\right]\frac{dp}{du} = 0$$

To get the singular solution, we consider,

$$u - \frac{1}{(P-1)^2} = 0$$

$$\frac{1}{(P-1)^2} = u$$

$$-\frac{1}{u} = (P-1)^2$$

$$\Rightarrow \frac{1}{\sqrt{u}} = P - 1$$

$$\Rightarrow P = \frac{1}{\sqrt{u}} + 1$$

Substitute p in (2)

$$v = -u\left[\frac{1}{\sqrt{u}} + 1\right] + \frac{\frac{1}{\sqrt{u}} + 1}{\frac{1}{\sqrt{u}} + 1 - 1}$$

$$-\sqrt{u} - u + \frac{\frac{1+\sqrt{u}}{\sqrt{u}}}{\frac{1}{\sqrt{u}}}$$

$$v = \sqrt{u} + u + 1 + \sqrt{u}$$

$$v = 2\sqrt{u} + u + 1$$

$$y^2 = 2\sqrt{(x^2)} + x^2 + 1$$

$$y^2 = 2x + x^2 + 1$$

$$y^2 = (x+1)^2$$

which is required solution.

Q36. Solve $xy(y - px) = x + py$.

Sol :

Given that $xy(y - px) = x + py$ _______(1)

Which can be written as $xy^2 - x^2yp = x + py$

$$xy^2 = px^2y + x + py$$

$$y^2 = \frac{px^2y}{x} + \frac{x}{x} + \frac{py}{x}$$

$$y^2 = px^2\frac{y}{x} + 1 + p\frac{y}{x} \text{ ______ } (2)$$

Putting $x^2 = u$ and $y^2 = v$

By Differentiating,

$2x\,dx = du$ and $2y\,dy = dv$

We get $\frac{2y\,dy}{2x\,dx} = \frac{dv}{du}$

$\frac{y}{x}p = P$ where $P = \frac{dv}{du}$

Using (3) and (4) in (2)

$$v = uP + 1 + P$$

$$v = uP + (1 + P)$$

which is in Clairaut's form.

so replacing p by arbitrary constant C,

$v = uC + (1 + C)$ as $x^2 = u$ and $y^2 = v$

$$y^2 = x^2C(1 + C)$$

which is required general solution.

Q37. Solve $(px - y)(py + x) = h^2p$.

Sol :

Given that

$(px - y)(py + x) = h^2p$ _______(1)

Which can be written as

$$p^2xy + px2 - py^2 - xy = h^2p$$

$$(pxy - y^2)x\left\{1 + \left(\frac{py}{x}\right)\right\} = h^2p$$

$$pxy - y^2 = \frac{h^2\left(p\frac{y}{x}\right)}{1+\left(p\frac{y}{x}\right)}$$

$$y^2 = \left(\frac{py}{x}\right)x^2 - \frac{h^2\left(p\frac{y}{x}\right)}{1+\left(p\frac{y}{x}\right)} \_\_\_\_\_ (2)$$

Putting $x^2 = u$ and $y^2 = v$ _______(3)

By Differentiating,

$$2x\,dx = du \quad \text{and} \quad 2y\,dy = dv$$

We get $\frac{2y\,dy}{2x\,dx} = \frac{dv}{du} \quad = \frac{y}{x}p = P$ _____(4)

Using (3) and (4) in (2)]

$$v = Pu^2 - \frac{h^2P}{1+P}$$ which is Clairaut's form.

So, Replacing P by C

$$v = Cu^2 - \frac{h^2C}{1+C}$$

$$y^2 = x^2C - \frac{h^2C}{1+C}$$ which is required solution.

Q38. Solve $y = 2px + y^2p^3$.

Sol :

Given that $y = 2px + y^2p^3$ _____(1)

By multiplying the equation by 'y'

$$y^2 = 2px + y^3p^3$$

$$y^2 = x(2py) + \frac{1}{8}(2yp)^3 \_\_\_\_\_ (2)$$

Put $y^2 = v$

$$2y\frac{dy}{dx} = \frac{dv}{dx}$$

$2yp = P$ where $P = \frac{dv}{dx}$

Then the (2) becomes

$v = xP + \frac{1}{8}P^3$ _____(3) which is in clairaut's form.

So replacing P by arbitrary constant C in (3)

Then the required solution is

$$v = xC + \frac{1}{8}C^3$$

As $v = y^2$

$$\Rightarrow y^2 = xC + \frac{1}{8}C^3$$

Q39. Solve $x^2y^2 + yp(2x+y) + y^2 = 0$, by reducing it to Clairaut's form by using the substitution $y = u$ and $xy = v$.

Sol :

Given equation is $x^2y^2 + yp(2x+y) + y^2 = 0$ ________(1)

Given $y = u$ and $xy = v$ _______(2)

Differentiating (2)

$dy = du$ and $x\,dy + y\,dx = dv$

$$\frac{x\,dy + y\,dx}{dy} = \frac{dv}{du}$$

$x + y\frac{dx}{dy} = \frac{dv}{du}$ where $P = \frac{dy}{dx}$

$$x + \frac{y}{p} = P$$

putting $P = \frac{y}{P-x}$ in (1)

$$\Rightarrow y = P(P-x)$$

$$\frac{x^2y^2}{(P-x)^2} + y.\frac{y}{P-x}(2x+y) + \left[P(P-x)\right]^2 = 0$$

$$\frac{x^2y^2}{(P-x)^2} + y.\frac{y}{P-x}(2x+y) + \left[P(P-x)\right]^2 = 0$$

$$xP^2 + P^2(P-x)(2x+y) + P(P-x)^2 = 0$$

$$P^2\left[x+(P-x)(2x+y)+(P-x)^2\right]=0$$

$$x+(P-x)(2x+y)+(P-x)^2=0$$

$$\cancel{x^2}+\cancel{2Px}+Py-\cancel{2x^2}-xy+P^2+\cancel{x^2}-\cancel{2Px}=0$$

$$Py-xy+P^2=0$$

as $xy=v$ and $y=u$

$$Pu-v+P^2=0$$

$$v=uP+P^2 \_\_\_\_\_\_(3)$$

(3) is in Clairaut's form,

So, by replacing P by C,

Then its general solution is

$$v=uC+C^2$$

$xy=yC+C^2$, C being an arbitrary constant

Q40. Solve $(x^2+y^2)(1+P^2)-2(x+y)(1+P)(x+Py)+(x+yP)^2=0$.

Sol :

Given that $(x^2+y^2)(1+P^2)-2(x+y)(1+P)(x+Py)+(x+yP)^2=0$ _____(1)

Let $x^2+y^2=u$ and $x+y=u$ ______(2)

Differentiating (2)

$2x\,dx+2y\,dy=du$ $\qquad dx+dy=du$

$$\frac{du}{dv}=\frac{2(x\,dx+y\,dy)}{dx+dy}$$

$$\frac{du}{dv}=\frac{2\left(x+y\left(\frac{dy}{dx}\right)\right)}{\left(1+\frac{dy}{dx}\right)}$$

$$\frac{du}{dv}=\frac{2(x+yP)}{1+P}$$

$$\frac{1}{2P}=\frac{x+yP}{1+P}$$

$$P=\frac{1+P}{2(x+yP)}$$

and rewriting the given equation.

$$\left(x^2+y^2\right)-2(x+y)\frac{(x+yP)}{1+P}+\left(\frac{x+yP}{1+P}\right)^2=0$$

$$u-v\frac{1}{P}+\left(\frac{1}{2P}\right)^2=0$$

$$u=\frac{v}{P}-\frac{1}{4P^2}$$

$$u=\frac{v}{C}-\frac{1}{4C^2}$$

$$x^2+y^2=\frac{x+y}{C}-\frac{1}{(2C)^2}$$

Which required solution.

Q41. Solvable for x for $y=2px+p^2y$

Sol :

Given that $y=2px+p^2y$ ______(1)

Solving for x, We get

$$-2px=-y+p^2y$$

$$-2x=-\left(\frac{y}{p}-yp\right) \text{\_\_\_\_\_(2)}$$

Differentiating (2) with respect to 'y' we get

$$2\frac{dy}{dx}=-\frac{y}{p^2}\frac{dp}{dy}+\frac{1}{p}-\left(p+y\frac{dp}{dy}\right)$$

$$2\frac{1}{p}=-p-y\frac{dp}{dy}-\frac{y}{p^2}\frac{dp}{dy}+\frac{1}{p} \quad \text{as } \frac{dx}{dy}=\frac{1}{p}$$

$$\frac{2}{p}=p+\frac{1}{p}-y\left(\frac{dp}{dy}\right)\left[1+\frac{1}{p^2}\right]$$

$$\frac{2}{p}-\frac{1}{p}+p=-y\left(\frac{dp}{dy}\right)\left[1+\frac{1}{p^2}\right]$$

$$\frac{1}{p}+p=-y\left(\frac{dp}{dy}\right)\left[1+\frac{1}{p^2}\right]$$

$$p\left(1+\frac{1}{p^2}\right)=-y\left(\frac{dp}{dy}\right)\left[1+\frac{1}{p^2}\right]$$

$$p\left(1+\frac{1}{p^2}\right)+y\left(\frac{dp}{dy}\right)\left[1+\frac{1}{p^2}\right]=0$$

$$\left(1+\frac{1}{p^2}\right)\left[p+y\frac{dp}{dy}\right]=0$$

Neglecting the first factors which does not contains $\frac{dp}{dy}$

$$\Rightarrow p+y\frac{dp}{dy}=0$$

$$\Rightarrow -p=y\frac{dp}{dy}$$

$$\Rightarrow \frac{dy}{dp}=-\frac{y}{p}$$

$$\Rightarrow \frac{dy}{y}+\frac{dp}{p}=0$$

By Integrating

$$\int\frac{dy}{y}+\int\frac{dp}{p}=C$$

$$\log y+\log p=\log C$$

$$p=C$$

$$p=\frac{C}{y}$$

Substitute P in (1)

$$y=2\left(\frac{C}{y}\right)x+\left(\frac{C}{y}\right)^2y$$

$$y=2\frac{C}{y}x+\frac{C^2}{y^2}y$$

$$y^2=2Cx+C^2$$

Which is required solution.

Q42. Find the general and singular solution of the differential equation $y = px + \sqrt{a^2p^2 + b^2}$

Sol :

Given differential equation is $y = px + \sqrt{a^2p^2 + b^2}$ _______(1)

Differentiating (1) with respect to 'x' we have

$$\frac{dy}{dx} = p(1) + x\frac{dp}{dx} + \frac{1}{\not{2}\sqrt{a^2p^2+b^2}}\not{2}pa^2\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + \frac{a^2P}{\sqrt{a^2p^2+b^2}}\frac{dp}{dx} \quad \text{as } \frac{dy}{dx} = p$$

$$p - p = x\frac{dp}{dx} + a^2p\left(a^2p^2+b^2\right)^{\frac{-1}{2}}\frac{dp}{dx}$$

$$\left[x + a^2p\left(a^2p+b^2\right)^{\frac{-1}{2}}\right]\frac{dp}{dx} = 0 \text{\_\_\_\_\_\_\_}(2)$$

The component equations $\left[x + a^2p\left(a^2p+b^2\right)\right]^{\frac{-1}{2}} = 0$ and $\frac{dp}{dx} = 0$

Neglecting the first factor i.e $x + a^2p\left(a^2p+b^2\right)^{\frac{-1}{2}} = 0$

$$\frac{dp}{dx} = 0$$

$$dp = 0$$

By Integrating

$$p = C$$

Substitute $p = C$ in (1)

$$y = Cx + \sqrt{a^2p^2+b^2}$$

and also from (2)

$$x + a^2p\left(a^2p+b^2\right)^{\frac{-1}{2}} = 0$$

$$p = \frac{bx}{a\sqrt{a^2-x^2}}$$

Using this value of p and equation (2)

The singular solution is obtained as

$$y^2a^2\left(a^2-x^2\right) = b^2\left(x^2+a^2\right)$$

Q43. Solve $y = 2px + yp^2$

Sol :

Given that $y = 2px + yp^2$ ______(1)

multiplying both sides by 'y' we get

$$y^2 = 2pxy + y^2p^2$$

$$y^2 = x(2py) + \frac{1}{4}(2py)^2 \text{ ______(2)}$$

Put $y^2 = v$

$$\Rightarrow 2y\frac{dy}{dx} = \frac{dv}{dx}$$

$$2y\,p = P \qquad P = \frac{dv}{dx}$$

Then by substituting corresponding values in (2)

Then we get

$$v = xP + \frac{1}{4}(P)^2$$

$$v = xP + \frac{1}{4}P^2$$ which is clairaut's form

so replacing p by an arbitrary constant 'C'

Then the required solution is

$$y^2 = xC + \frac{1}{4}C^2$$

Q44. Solve $y = 2px + y^{n-1}p^n$

Sol :

Given that $y = 2px + y^{n-1}p^n$ _____(1)

Multiplying both sides by 'y' we get

$$y^2 = 2pxy + y^{n-1}.y\,p^n$$

$$y^2 = 2pxy + y^np^n$$

$$y^2 = x(2py) + \frac{1}{2^n}(2yp)^n \text{ _______}(2)$$

Put $y^2 = v$

$$\Rightarrow 2y\frac{dy}{dx} = \frac{dv}{dx}$$

$2y\,p = P \qquad P = \dfrac{dv}{dx}$

Then by substituting corresponding values in (2)

$$v = xp + \frac{1}{2^n}p^n \text{ which is Clairaut's form}$$

Replacing p by C

$$v = xC + \frac{1}{2^n}C^n$$

$\therefore$ The required solution is $y^2 = xC + \dfrac{1}{2^n}C^n$

Q45. Solve $p^3(x+2y)+3p^2(x+y)+(y+2x)p=0$

Sol :

Given that $p^3(x+2y)+3p^2(x+y)+(y+2x)p=0$ ______(1)

$$p\left[p^2(x+2y)+3p(x+y)+(y+2x)\right]=0$$

Then its component equations are

$$p=0, \qquad p^2(x+2y)+3p(x+y)+(y+2x)=0$$

Consider $p^2(x+2y)+3p(x+y)+(y+2x)=0$

$$p = \frac{-3(x+y)\pm\sqrt{9(x+y)^2-4(x+2y)(y+2x)}}{2(x+2y)}$$

$$= \frac{-3(x+y)\pm\sqrt{9(x^2+y^2+2xy)-4(xy+2x^2+2y^2+4xy)}}{2(x+2y)}$$

$$= \frac{-3(x+y)\pm\sqrt{9x^2+9y^2+18xy-4xy-8x^2-8y^2-16xy}}{2(x+2y)}$$

$$= \frac{-3(x+y)\pm\sqrt{x^2+y^2-2xy}}{2(x+2y)}$$

$$= \frac{-3(x+y)\pm\sqrt{(x-y)^2}}{2(x+2y)}$$

$$p = \frac{-3(x+y)\pm(x-y)}{2(x+2y)}$$

$\therefore\ p = \dfrac{-3(x+y)+(x-y)}{2(x+2y)}$ (or) $p = \dfrac{-3(x+y)-(x-y)}{2(x+2y)}$

$p = \dfrac{-3x-3y+x-y}{2(x+2y)}$ (or) $p = \dfrac{-3x-3y-x+y}{2(x+2y)}$

$p = \dfrac{-2x-4y}{2(x+2y)}$ (or) $p = \dfrac{-4x-2y}{2(x+2y)}$

$p = \dfrac{-2(x+2y)}{2(x+2y)}$ (or) $p = \dfrac{-2(2x+y)}{2(x+2y)}$

$p = -1$ (or) $p = \dfrac{-(2x+y)}{(x+2y)}$

Consider $p = 0$

$\Rightarrow \dfrac{dy}{dx} = 0$

$dy = 0$

By Integration

$y = C \quad \Rightarrow y - C = 0$

If $p = -1$

$\Rightarrow \dfrac{dy}{dx} = -1$

$\Rightarrow dy = -dx$

By Integrating

$y = -x + C$

$\Rightarrow y + x - C = 0$

If $p = \dfrac{-(2x-y)}{2+2y}$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{2x+y}{x+2y} = 0$ ______ (2)

Which is a homogeneous differential equation.

Put $y = vx$

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

By (1) $\Rightarrow\ v + x\dfrac{dv}{dx} + \dfrac{2x+vx}{x+2vx} = 0$

$$v + x\frac{dv}{dx} + \frac{x(2+v)}{x(1+2v)} = 0$$

$$v + x\frac{dv}{dx} + \frac{2+v}{1+2v} = 0$$

$$x\frac{dv}{dx} + \left(v + \frac{2+v}{1+2v}\right) = 0$$

$$x\frac{dv}{dx} + \frac{v + 2v^2 + 2 + v}{1+2v} = 0$$

$$x\frac{dv}{dx} + \left(\frac{2v^2 + 2v + 2}{1+2v}\right) = 0$$

By Variable separable

$$\left(\frac{1+2v}{2v^2+2v+2}\right)dv + \frac{1}{x}dx = 0$$

By Integrating

$$\frac{1}{2}\int\frac{1+2v}{v^2+v+1}dv + \int\frac{1}{x}dx = 0$$

$$\frac{1}{2}\log\left(v^2+v+1\right) + \log x = \log C$$

$$\log\sqrt{v^2+v+1} + \log x = \log C$$

$$\log x\sqrt{v^2+v+1} = \log C$$

$$x\sqrt{v^2+v+1} = \log C$$

$$x\sqrt{\frac{y^2}{x^2} + \frac{y}{x} + 1} = C$$

$$x\sqrt{\frac{y^2+xy+x^2}{x^2}} = C$$

$$\sqrt{y^2+xy+x^2} = C$$

$$\therefore \sqrt{y^2+xy+x^2-C} - C = 0$$

$\therefore$ The required complete solution is

$$(y-C)(y+x-C)\left(\sqrt{y^2+xy+x^2} - C\right) = 0$$

Q46. Solve $x^2p^2 - 2xyp + \left(2y^2 - x^2\right) = 0$.

Sol :

Given that $x^2p^2 - 2xyp + \left(2y^2 - x^2\right) = 0$

$$p = \frac{2xy \pm \sqrt{4x^2y^2 - 4\left(x^2\right)\left(2y^2 - x^2\right)}}{2x^2}$$

$$= \frac{2xy \pm \sqrt{4x^2y^2 - 8x^2y^2 + 4x^2}}{2x^2}$$

$$= \frac{2xy \pm \sqrt{-4x^2y^2 - 4x^2}}{2x^2}$$

$$= \frac{2xy \pm \sqrt{4x^2\left(x^2 - y^2\right)}}{2x^2}$$

$$= \frac{2xy \pm 2x\sqrt{x^2-y^2}}{2x^2}$$

$$= \frac{2x\left[y \pm \sqrt{x^2-y^2}\right]}{2x^2}$$

$$= \frac{y \pm \sqrt{x^2-y^2}}{x}$$

$$\therefore P = \frac{y + \sqrt{x^2-y^2}}{x} \quad \text{(or)}$$

$$P = \frac{y - \sqrt{x^2-y^2}}{x}$$

If $P = \dfrac{y + \sqrt{x^2-y^2}}{x}$

Which is a homogeneous differential equation.

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2-y^2}}{x}$$

$$v+x\frac{dv}{dx}=\frac{vx+\sqrt{x^2-v^2x^2}}{x}$$

$$=\frac{vx+x\sqrt{1-v^2}}{x}$$

$$v+x\frac{dv}{dx}=\frac{x\left(v-\sqrt{1-v^2}\right)}{x}$$

$$\not{v}+x\frac{dv}{dx}=\not{v}+\sqrt{1-v^2}$$

By variable separable

$$\frac{dv}{\sqrt{1-v^2}}=\frac{dx}{x}$$

By Integrating

$$\int\frac{dv}{\sqrt{1-v^2}}=\int\frac{dx}{x}+C$$

$$\sin^{-1}v=\log x+C$$

$$\sin^{-1}\left(\frac{y}{x}\right)-\log x-C=0\_\_\_\_\_(1)$$

If $P=\frac{y-\sqrt{x^2-y^2}}{x}$ which is a homogeneous differential equation.

Put $y=vx$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\frac{dy}{dx}=\frac{y-\sqrt{x^2-y^2}}{x}$$

$$v+x\frac{dv}{dx}=\frac{vx+\sqrt{x^2-v^2x^2}}{x}$$

$$=\frac{vx-x\sqrt{1-v^2}}{x}$$

$$v+x\frac{dv}{dx}=\frac{x\left(v-\sqrt{1-v^2}\right)}{x}$$

$$\not{v}+x\frac{dv}{dx}=\not{v}-\sqrt{1-v^2}$$

By variable separable

$$\frac{dv}{\sqrt{1-v^2}}=-\frac{dx}{x}$$

By Integrating

$$\int\frac{dv}{\sqrt{1-v^2}}=-\int\frac{dx}{x}+C$$

$$\sin^{-1}v=-\log x+C$$

$$\sin^{-1}\left(\frac{y}{x}\right)+\log x-C=0\_\_\_\_\_(2)$$

By using (1) and (2) the required complete solution is

$$\left(\sin^{-1}\left(\frac{y}{x}\right)-\log x-C\right)\left(\sin^{-1}\left(\frac{y}{x}\right)+\log x-C\right)=0$$

Q47. Solve $x^2+p^2x=yp$.

Sol :

Given that $x^2+p^2x=yp$ ... (1)

Which can be written as,

$$y=\frac{x^2+p^2x}{p}\_\_\_\_\_(2)$$

$$p=\frac{p\left[2x+p^2(1)+2px\frac{dp}{dx}\right]-\left(x^2+p^2x\right)\frac{dp}{dx}}{p^2}$$

$$p^3=p\left[2x+p^2+2px\frac{dp}{dx}-x^2\frac{dp}{dx}-p^2x\frac{dp}{dx}\right]$$

$$p^3=2px+p^3+2p^2x\frac{dp}{dx}-x^2\frac{dp}{dx}-p^2x\frac{dp}{dx}$$

$$p^3-p^3=2px+2p^2x\frac{dp}{dx}-x^2\frac{dp}{dx}-p^2x\frac{dp}{dx}$$

$$0=2px+p^2x\frac{dp}{dx}-x^2\frac{dp}{dx}$$

$$2px+x\left(p^2-x\right)\frac{dp}{dx}=0$$

$$\frac{2px}{p^2-x}=-x\frac{dp}{dx}$$

$$\frac{dp}{dx}=\frac{-2p}{p^2-x}$$

$$\frac{dp}{dx}=\frac{2p}{x-p^2}$$

$$\Rightarrow \frac{dx}{dp}=\frac{x-p^2}{2p}$$

$$\Rightarrow \frac{dx}{dp}-\frac{x-p^2}{2p}=0$$

$$\Rightarrow \frac{dy}{dx}-\frac{x}{2p}=\frac{-p^2}{2p}$$

$$\Rightarrow \frac{dx}{dp}-\frac{x}{2p}=-\frac{p}{2}$$

Which is line as differential equation in p

Then the IF is $e^{-\int \frac{1}{2p}dp}$

$$=e^{-\frac{1}{2}\log p}$$

$$=e^{\log p^{-\frac{1}{2}}}$$

$$=e^{\log \sqrt{p}}$$

$$IF=\frac{1}{\sqrt{p}}$$

$\therefore$ The required solution for linear differential equations.

$$x.IF=\int Q\times(IF)+C$$

$$x\frac{1}{\sqrt{p}}=\int \frac{-p}{2}.\frac{1}{\sqrt{p}}dp+C$$

$$x\frac{1}{\sqrt{p}}=\int -\frac{\sqrt{p}}{2}dp+C$$

$$\frac{x}{\sqrt{p}}=-\frac{1}{2}\int \sqrt{p}\ dp+C$$

$$\frac{x}{\sqrt{p}}=\frac{-1}{2}\frac{p^{3/2}}{\frac{3}{2}}+C$$

$$\frac{x}{\sqrt{p}}=-\frac{1}{2}.\frac{3}{2}p^{3/2}+C$$

$$x=\frac{-1}{3}\sqrt{p}.\ p\sqrt{p}+\sqrt{pC}$$

$$x=-\frac{p^2}{3}+\sqrt{pC}$$

$$x=-\frac{p^2}{3}+C\sqrt{p}\ \_\_\_\_\_(3)$$

Substitute (3) in (1)

$$y=\frac{\left[\frac{-p^2}{3}+C\sqrt{p}\right]^2+p^2\left[\frac{-p^2}{3}+C\sqrt{p}\right]}{p}$$

$$y=\frac{\left(\frac{-p^2}{3}+C\sqrt{p}\right)^3}{p}+p\left(\frac{-p^2}{3}+C\sqrt{p}\right)\_\_\_\_\_(4)$$

Equation (3) and (4) constitute the required solution.

Q48. Solve $xp^2-2yp+ax=0$.

Sol :

Given that $xp^2-2yp+ax=0$ ______(1)

Which can be written as

$$2yp=xp^2+ax$$

$$y=\frac{xp^2+ax}{2p}\ \_\_\_\_\_\_(2)$$

differentiating (2) with respect to 'x'

$$\frac{dy}{dx}=\frac{2p\left[1.p^2+2px\frac{dp}{dx}+a\right]-\left(xp^2+ax\right)2\frac{dp}{dx}}{4p^2}$$

$$p=\frac{\frac{2}{4}\left[p^3+2p^2x\frac{dp}{dx}+ap-xp^2\frac{dp}{dx}-ax\frac{dp}{dx}\right]}{p^2}$$

$$p^3=\frac{1}{2}\left[2xp^2\frac{dp}{dx}+p^3+ap-xp^2\frac{dp}{dx}-ax\frac{dp}{dx}\right]$$

$$2p^3-p^3=2xp^2\frac{dp}{dx}-xp^2\frac{dp}{dx}+ap-ax\frac{dp}{dx}$$

$$xp^2\frac{dp}{dx}-p^3+ap-ax\frac{dp}{dx}=0$$

$$x\left(p^2-a\right)\frac{dp}{dx}-p\left(p^2-a\right)=0$$

$$\left(p^2-a\right)\left(x\frac{dp}{dx}-p\right)=0$$

neglecting first factor i.e., p^2-a

consider

$$x\frac{dp}{dx}-p=0$$

$$\frac{dp}{dx}=\frac{p}{x}$$

By variable separable

$$\frac{dp}{p}=\frac{dx}{x}$$

By Integrating

$$\Rightarrow \int\frac{dp}{p}=\int\frac{dx}{x}+C$$

$$\log p=\log x+\log C$$

$$p=xC$$

Let substitute $p=xC$ in (2)

$$y=\frac{x(xC)^2+ax}{2(xC)}$$

$$=\frac{x\left(x^2C^2\right)+ax}{2xC}$$

$$=\frac{x^3C^2+ax}{2xC}$$

$$\Rightarrow\frac{x\left(x^2C^2+a\right)}{2xC}$$

$$y=\frac{x^2C^2+a}{2C}$$

$$y=\frac{x^2C}{2}+\frac{a}{2C}$$

$$2y=x^2C+\frac{a}{C}$$

Which is required solution.

Q49. Solve $y=2p+3p^2$

Sol :

Given that $y=2p+3p^2$ ___(1)

Differentiating (2) with respect to 'x'

$$\frac{dy}{dx}=2\frac{dp}{dx}+6p\frac{dp}{dx}$$

$$p=\frac{dp}{dx}(6p+2)$$

By variable separable.

$$dx=\frac{dp}{p}(6p+2)$$

$$\int dx=\int\frac{1}{p}(6p+2)dp+C$$

$$=\int\frac{6p}{p}+\frac{2}{p}dp+C$$

$$=\int 6\,dp+2\int\frac{1}{p}dp+C$$

$$x=6p+2\log p+C$$

$$x=\log p^2+6p+C$$

Which is required solution.

Q50. Solve $(x-a)p^2+(x-y)p-y=0$.

Sol :

Given that,

$(x-a)p^2+(x-y)p-y=0$ ________(1)

$$xp^2-ap^2+xp-yp-y=0$$

$$xp(p+1)-y(p+1)-ap^2=0$$

$$xp(p+1)-y(p+1)=ap^2$$

$$(p+1)(xp-y)=ap^2$$

$$xp-y=\frac{ap^2}{p+1}$$

$$y=px-\frac{ap^2}{p+1}$$

Which is in Clairaut's form

So, By replace p by C

Then its general solution is

$$y=Cx-\frac{aC^2}{C+1}.$$

2.2 Applications of First Order Different Equations

2.2.1 Growth and Decay

Q51. Define growth and decay.

Ans :

The rate at which the substance changes is proportional to the quantity od substance present at ant time.

$$\frac{dx}{dt} = kx$$

By variable separable

$$\frac{dx}{x}\ k\,dt$$

Where x = Quantity of substance present at any time

t = time

k = Proportionality constant

Integration

$$\log x = kt + C$$

$$x = e^{kt+C}$$

$$x = e^{kt}.e^{C}$$

$$x = Ce^{kt}$$

Q52. A culture initially has N_0 number of bacteria. At $t = 1hr$, the number of bacteria is measured to be $\left(\frac{3}{2}\right)N_0$. If the rate of growth is proportional to the number of bacteria present, determine the time necessary for the number of bacteria to triple.

Sol :

N_0 is number of bacteria

At $t = 1hr$, the number of bacteria is $\left(\frac{3}{2}\right)N_0$.

The rate of growth is proportional to the number of bacteria present $\frac{dN}{dt} = kN$ subject to $N(0) = N_0$.

where k is proportional constant

t = time.

By variable separable

$$\frac{dN}{N} = k\,dt$$

By Integrating

$$\int \frac{dN}{N} = \int k\,dt + C$$

$$\log N = k\int dt + C$$

$$N = e^{kt+C}$$

$$N = e^{kt}.e^{C} \quad \text{Where } e^{C} = C$$

$$N = Ce^{kt}$$

At t = 0

$$N = Ce^{k(0)} = Ce^{0} = C(1) = C$$

so, $N(t) = N_0e^{kt}$

At t = 1

$$\text{We have } \left(\frac{3}{2}\right)N_0 = N_0e^{k(1)}$$

$$\frac{3}{2}N_0 = N_0e^{k}$$

$$\frac{3}{2} = e^{k}$$

$$k = \log\left(\frac{3}{2}\right) = 0.4055$$

$$\therefore N(t) = N_0e^{0.4055t}$$

We have to find the number of bacteria to triple

$$3N_0 = N_0e^{0.4055t}$$

$$3 = e^{0.4055t}$$

$$0.4055t = \log 3$$

$$t = \frac{\log 3}{0.4055} \simeq 2.71\text{hr}$$

Q53. Bacteria in certain culture increase at a rate proportional to the number present. If the number N increases from 1000 to 2000 in 1 hour. How many are present at the end of 1.5 hours?

Sol :

The differential equation is $\frac{dN}{dt} = KN$

Variable separable

$$\frac{dN}{N} = K\,dt$$

By Integration

$$\int \frac{dN}{N} = \int K\,dt + C$$

$$\log N = Kt + C$$

$$\Rightarrow N = e^{kt}e^{C}$$

Where $e^{C} = N_0$

$$N = N_0e^{kt}$$

Since $N = 1000$ when $t = 0$

$$N_0 = 1000$$

Since $N = 2000$ and $t = 1$

$$2000 = 1000e^{k(t)}$$

$$2 = e^{k}$$

Thus $N(t) = 1000e^{k(t)} = 1000(e_2)^{t}$

i.e., $t = 1.5$

$$N(1.5) = 1000(2)^{1.5}$$

$$N(1.5) \simeq 2828.43$$

$\therefore \simeq 2828.43$ are present at the end of 105 hours.

Q54. In a culture of yeast, the amount A of acting yeast, the amount A of active yeast grows at a rate proportional to the amount present. If the original amount A_0 doubles in 2 hours, how long does it take for the original amount of triple.

Sol :

The differential equation is $\frac{dA}{dt} = KA$

By variable separable $\frac{dA}{A} = K\,dt$

By Integrating

$$\log A = kt + C$$

$$A = e^{kt+C}$$

$$A = e^{kt}e^{C}$$

$$A = A_0e^{kt}$$

The amount A of active yeast grows at a rate proportional to the amount present A_0

$A = A_0e^{kt}$ ____(1)

The original amount A_0 doubles in 2 hours

$A = 2A_0$ ______(2) when $t = 2$

Equating (1) and (2)

$2A_0 = A_0e^{kt}$

$2 = e^{kt}$

$A = A_0e^{kt} A = A_0\left(e^{2t}\right)^{\frac{t}{2}}$

$A = A_0 = A_0(2)^{\frac{t}{2}}$

$A = 3A_0$

$3A_0 = A_0(2)^{\frac{t}{2}}$

$3 = 2^{\frac{t}{2}}$

$\log 3 = \log 2^{\frac{t}{2}}$

$\log 3 = \frac{t}{2}\log 2$

$2\log 3 = t\log 2$

$\frac{2\log 3}{\log 2} = t$

$\frac{2(0.4771)}{0.30103} = t$

$\Rightarrow \frac{0.9542}{0.30103}$

$t \simeq 3.1698$ hr

Q55. Bacteria in certain culture increase at a rate proportional to the number present. If the number doubles in one hour. How long does it takes for the number to triple?

Sol :

Let y denote the number present at time t,

Then the function denoted by $y = f(t)$

Satisfies the differential equation

$\frac{dy}{dx} = Ky$ ______(1)

and the condition $t = 0$, $y = y_0$

If the number doubles in one hour

$t = 1$; $y = 2y_0$

By (1) $\frac{dy}{dt} - Ky = 0$

Which is a linear differential equation

$\therefore$ By Integrating factor

$IF = e^{\int -k\,dt}$

$= e^{-kt}$

$\therefore$ It's solution is

$y\,IF = \int Q.IF\,dt + C$

$ye^{-kt} = \int 0.e^{-kt}\,dt + C$

$y.e^{-kt} = C$

$y = Ce^{kt}$ ____(2)

as we have condition $t = 0$, $y = y_0$

$y = Ce^{k(0)}$

$y = C \quad \Rightarrow \quad y_0 = C$ ____(3)

If $t = 1 \Rightarrow y = 2y_0$ _____(4) is now used to find the constant of proportionality k

By using (3) and (4) in (2)

$2y_0 = y_0e^{kt}$

as $t = 1 \Rightarrow 2\cancel{y_0} = \cancel{y_0}e^{k(1)}$

$2 = e^k$

$k = \log 2$

Hence $y = y_0e^{\log 2.t}$ which is a condition for number double in 1hour

Now we find the number to triple.

i.e., $y = 3y_0$

$3\cancel{y_0} = \cancel{y_0}e^{t\log 2}$

$3 = e^{t\log 2}$

$\log 3 = t\log 2$

Hence the number will be triple

$$t = \frac{\log 3}{\log 2}$$

$$= \frac{0.47712}{0.301030} = 1.5850 \text{ hr}$$

$$t = 1.5850 \text{ hr}.$$

2.2.2 Dynamics and tumour Growth

Q56. Derive dynamics and tumour growth.

Ans :

It has been observed experimentally that free - living dividing cells, such as Bacteria cells grow at a rate proportional to the volume of dividing cells at that moment.

Let $v(t)$ denote the volume of dividing cells at time t.

Then $\frac{dv}{dt} = kv$ ___ (1) for some positive constant k.

By variable separable

$$\frac{dv}{v} = k\,dt \text{ and}$$

By Integrating

$$\int \frac{dv}{v} = \int k\,dt + C$$

$$\log v = kt + C$$

$$v = e^{kt+C}$$

$$\Rightarrow v = e^{kt}.e^{C}$$

$$v = v_0 e^{k(t-t_0)}$$

Where v_0 is the volume of dividing cells at time to (initial time).

Thus, free living dividing cell grow exponentially with time, whereas solid tumours do not grow expotentially with time.

As the toumor becomes larger, the doubling time of the total tomour volume continuously increases.

$$v(t) = v_0 \exp\left[\frac{k}{a}\left(1 - e^{-at}\right)\right]$$

k and a are positive constants

$$\frac{dv}{dt} = v_0 k e^{-at} \exp\left[\frac{k}{a}\left(1 - e^{-at}\right)\right]$$

$$\frac{dv}{dt} = ke^{-at}v \;\_\_\_\_ (2)$$

$$\frac{dv}{dt} = \left(k e^{-at}\right)v$$

$$\frac{dv}{dt} = k\left(e^{-at}v\right)$$

with these arrangements of (2) two theories have been evolved for the dynamics of tumour growth.

2.2.3 Radioactivity and Carbon Dating

Q57. State and explain half life of a radio active substance.

Sol :

Half Life

Half life is defined as the time taken by the radio active substance to disintegrate by half of its initial amount. It is used to measure the stability of radioactive materials.

Examples

(i) Half life of radium (Ra – 226) $\approx$ 1700 years (disintegrated to Radon (Rn – 222)).

(ii) Half life of uranium isotope (U – 238) $\approx$ 4.5 billion years (disintegrated to lead (Pb – 206)).

(iii) Half life of iodine – 131 $\approx$ 8.1 days.

(iv) Half life of carbon – 14 $\approx$ 5568 years.

Q58. It is found that 22 percent of the original radiocarbon in a wooden archaeological specimen has decomposed, use the half-life T = 5568 yrs of ^{14}C to compute the number of years since the specimen was a part of living free. [This should yield a good estimate of the time elasped since the SP]

Sol :

Let C be the amount of ^{14}C (Carbon-14) present at time t.

C_0 the amount present at $t = 0$

If k denotes the decay constant ^{14}C

(half life 5568 years), then $\frac{dC}{dt} = kC$

$$\frac{dC}{C} = k\,dt$$

$$\int \frac{dC}{C} = \int k\,dt + C$$

$$\log C = kt + C$$

$$C = e^{kt+C}$$

$$C = e^{kt}.e^{C}$$

$C = C_0 e^{kt}$ Where $C_0 = e^{C}$

$C = C_0 e^{kt}$ give $0.78C_0 = C_0 e^{kt}$

$$0.78 = e^{kt}$$

$$\log 0.78 = kt$$

$$t = \frac{\log 0.78}{7}$$

From $T = -\log\frac{2}{k} = 5568$

We obtain $k = -\log\frac{2}{5568}$

$$t = \frac{5568\log 0.78}{-\log 2} = 1996\text{yrs}$$

Its arbitrary time t

$$C = C_0 \exp\left(\frac{-1092}{5568}\right)t$$

$$C = C_0 (2)^{\frac{-t}{5568}}$$

Q59. It is found that 0.5 percent of radium disappear in 12 years

(a) What percentage will disappear in 100 years?

(b) What is the half life of radium?

Sol :

Let A be the quantity of radium in grammer, present after t years.

Then $\frac{dA}{dt}$ represents the rate of dis-integration of radium

According to the radius activity decay

We have $\frac{dA}{dt} \alpha A$ or $\frac{dA}{dt} = aA$

Since A is positive and is decreasing then $\frac{dA}{dt} < 0$.

We see that the constant of proportionality a must be negative.

$$a = -k$$

$$\frac{dA}{dt} = -kA$$

Let A_0 be the amount in grammes of radium present initially.

Then $0.005\,A_0\,g$ disappears in 12 years.

$A = A_0$ at to and $A = 0.995\,A_0$ at $t = 12$

Since $A = A_0$ at $t = 0$ and $C = A_0$

Hence $A = A_0 e^{-kt}$

Also at $t = 12$ and $A = 0.995\,A_0$

Then $0.995\,A_0 = A_0 e^{-kt}$

$$0.995\,A_0 = A_0 e^{-12k}$$

$$e^{-12k} = 0.995$$

$$e^{-k} = (0.995)^{\frac{t}{12}}$$

$$k = -0.000418$$

$$A = A_0e^{-kt} = A_0\left(e^{-k}\right)^t = A_0\left(0.995\right)^{\frac{t}{12}}$$

$$A = A_0\left(0.995\right)^{\frac{t}{12}} \text{\_\_\_\_\_\_\_\_\_\_} (1)$$

$$A = A_0e^{-0.000418t} \text{\_\_\_\_\_\_\_\_} (2)$$

(a) When $t = 1000$

by (1) $\Rightarrow A = A_0\left(0.995\right)^{\frac{1000}{12}}$

$$A = A_0\left(0.6584\right)$$

So that 34.2 percent will disappear in 1000 years.

(b) The half life of a radioactive substance is defined as the time it takes for 50 percent of the substance to disappear.

We have $A = \frac{1}{2}A_0$ and using (2)

$$e^{-0.000418t} = \frac{1}{2} \Rightarrow t = 16721770 \text{ years}.$$

Q60. A fossilized bone is found to contain $\frac{1}{1000}$ the original amount of ^{14}C. Determine the age of the fossil.

Sol :

We have the first differential equation is $A(t) = A_0e^{kt}$

The original amount of ^{14}C is $t = 5568$ years

$A(t) = \frac{A_0}{2}$ from which we can find the value of k as

$$A(t) = \frac{A_0}{2}e^{k(5568)}$$

$$k = \frac{-\log 2}{5568} = -0.0001244$$

$$\therefore A(t) = A_0e^{-0.0001244t}$$

Where $A(t) = \frac{1}{1000}$

$$t = \frac{\log 2}{0.0001244} = 55489.32 \text{ years}$$

Q61. A breeder reactor connects the relatively stable uranium 238 into the isotope plutonium 239. After 15 year it is found that 0.043 percent of the initial amount A_0 of the plutonium has disintegrated. Find the half-life of this isotope. If the rate of disintegration is proportional to the remaining amount.

Sol :

Let A(t) denote the amount of the plutonium remaining at any time.

Then the solution of the initial value problem

$$\frac{dA}{dt} = kA, \quad A(0) = A_0$$

is $A(t) = A_0e^{kt} \Rightarrow A(t) = A_0e^{k(15)}$

If 0.043 percent of the atoms of A_0 have disintegrated then 99.957 percent of the substance remains.

To find k, we solve $0.99957A_0 = A_0e^{k(15)}$

$$0.99957 = e^{k15}$$

$$\log(0.99957) = k15$$

$$k = \frac{\log(0.99957)}{15}$$

$$k = -0.000028$$

Hence $A(t) = A_0e^{-0.000028t}$

Now, the half-life is the corresponding value of time for which $A(t) = \frac{A_0}{2}$

Solving for t,

$$\frac{A_0}{2} = A_0e^{-0.000028t}$$

$$t = \frac{\log 2}{0.00002867}$$

$$t = 24176.741 \text{ years}$$

2.2.4 Compound Interest

Q62. Define compound Interest with example.

Ans :

Interest is defined as a charge for the borrowed money. It a principal of p rupees, invested at an interest rate r per annum grows to $p(1+r)$ rupees in 1 years.

$p(1+r)^2$ rupees for 2 years

$p(1+r)^t$ rupees for t years. r is called the rate of interest per annum compounded annually.

If the interest rate per annum is r and interest is compounded twice year.

If p rupees intrested at interest rate r per annum with interest compounded k time per year, then the amount a of the original investment at the end of t year is

$$a = p\left(1+\frac{r}{k}\right)^{kt} = f(t) \_\_\_\_ (1)$$

The quantity $\frac{r}{k}$ is the interest rate applied at each compounding, and e^{kt} is the total number of compounding in t years.

For example,

1 rupee invested at 10 percent per annum compounded annually.

$p = 1, \; t = 1, \; r = 10$

$$a = p\left(1+\frac{r}{100}\right)^t = 1\left(1+\frac{10}{100}\right)^1$$

$$= 1(1+0.1)$$

$$= 1.01 \text{ rupees in 1 year}$$

For fixed values of p, r and t, the value of a in equation (1) increases as k increases.

As $k \longrightarrow \infty$ the value of a does not increase without limit.

$$\lim_{k\to\infty} a = \lim_{k\to+\infty} p\left(1+\frac{r}{k}\right)^{kt}$$

$$= p \lim_{k\to+\infty}\left[\left(1+\frac{r}{k}\right)^{k/r}\right]^{rt} = p\left[\lim_{k\to+\infty}\left(1+\frac{r}{k}\right)^{k/r}\right]^{rt}$$

$$= pe^{rt} \text{ where } e \simeq 2.71828$$

In order to have a continuous model for continuous compounding defines A as $A = pe^{rt} = F(t)$ equation fir differentiation $\frac{dA}{dt} = pre^{rt}$

$$\frac{dA}{dt} = rA$$

Q63. If Rs. 10,000 is invested at 6 percent per annum. find what amount has accumulated after 6 years, if interset is compounded

(a) Annually

(b) Quarterly and

(c) Continuously.

Sol :

(a) p = 10,000 is invested at r = 6% per annum. The amount has accumulate after 6 years of compound interest is for annually

$$p\left(1+\frac{r}{100}\right)^6 \Rightarrow 10000\left(1+\frac{6}{100}\right)^6$$

$$=10000(1+0.06)^6$$

$$=10000(1.06)^6$$

$$=10000(1.41852)$$

Rs.14,185.19

(b) Rs. 10000 is invested 6% per annum the amount accumulated after 6 years if interest is compounded for quarterly (k = 4)

$$a=p\left(1+\frac{r}{k}\right)^{rt}=10000\left(1+\frac{0.06}{4}\right)^{4(6)}$$

$$r=6\% \Rightarrow \frac{6}{100}=0.06$$

$$A=10000(1+0.015)^{24}$$

$$=10000(1.4295)$$

$$A=Rs.14295.03$$

(c) Rs. 10000 is invested 6% per annum the amount accumulated after 6 years if interest is compounded for continuously

$$A=Pe^{rt}$$

$$P=10000, \quad r=6\%,$$

$$\Rightarrow \frac{6}{100}=0.06$$

$$t=6$$

$$A=10000\left(e^{(0.06)6}\right)$$

$$=10000e^{0.36}$$

$$A=Rs.14333.29$$

Q64. How long does it takes for a given amount of money to double at 6 percent per annum compounded,

(a) Annually and

(b) Continuously

Sol :

(a) We know that the compound interest is

$$p\left(1+\frac{r}{100}\right)^{rt}$$

But, the amount of money to double at 6% per annum.

$$2p\left(1+\frac{r}{100}\right)^t=p\left(1+\frac{6}{100}\right)^t$$

$$2p=p(1.06)^t$$

$$\log 2=\log(1.06)^t$$

$$\log 2=t\log(1.06)$$

$$t=\frac{\log 2}{\log(1.06)}$$

$$t=\frac{0.30103}{0.02531}$$

$$t=11.89375 \text{ yr}$$

(b) The given amount of money to double at 6% per annum compound for continuously

$$2p=pe^{rt} \quad r=6\% \Rightarrow \frac{6}{100}=0.06$$

$$2p=pe^{(0.06)t}$$

$$2=e^{0.06t}$$

$$\log 2=0.06t$$

$$t=\frac{\log 2}{0.06}=\frac{0.62347}{0.06}=11.5523 \text{ yrs}$$

Q65. Find the time required for money to double when invested at 7 percent annum compounded continuously.

Sol :

The continuous compounding is $A = pe^{rt}$

The time required for money to double when invested at 7%per annum componded continuously.

$$2p = pe^{(0.07)t} \quad r = 7\% \Rightarrow \frac{7}{100} = 0.07$$

$$2p = pe^{(0.07)t}$$

$$2 = e^{0.07t}$$

$$\log 2 = (0.07)t$$

$$t = \frac{\log 2}{0.07} = \frac{0.69313}{0.07}$$

$$t = 9.9021 \text{ yrs}$$

2.2.5 Orthogonal Trajectories

Q66. Define orthogonal trajectories.

Ans :

A curve which cuts every member of a given family of curves in accordance with some give law is called trajectory of the given family of curves.

If a curve cuts every member of given family of curves at right angles, it is called orthogonal trajectory.

To find the orthogonal trajectories of a given family of curves we first find the differential equation.

$\frac{dy}{dx} = f(x,y)$ which describes the family, The differential equation of the second and orthogonal family is then $\frac{dy}{dx} = \frac{-1}{f(x,y)}$.

Q67. Find an equation of the family orthogonal to the family $y = Cx$.

Sol :

Given family is $y = Cx$

eliminating C from $y = Cx \Rightarrow \frac{y}{x} = C$ ___ (i)

$$\frac{dy}{dx} = C \text{ ___ (ii)}$$

(i) and (ii) $\frac{dy}{dx} = \frac{y}{x}$ differential equation of the family.

Hence a differential equation of the required family is

$$\frac{dy}{dx} = -\frac{1}{\frac{y}{x}}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

By variable separable

$$y\,dy = -x\,dx$$

By Integrating

$$\int y\,dy + \int x\,dx = C$$

$$\frac{y^2}{2} + \frac{x^2}{2} = C$$

$$y^2 + x^2 = 2C$$

$$x^2 + y^2 = k^2 \quad \text{where } k^2 = 2C$$

Q68. Find the orthogonal trajectories of $x^2 + y^2 = Cx$.

Sol :

Given that $x^2 + y^2 = Cx$

differentiating with respect to 'x'

we get $2x + 2y\frac{dy}{dx} = C$

$$\frac{dy}{dx} = \frac{C - x}{2y} \text{ ___ (1)}$$

But $x^2 + y^2 - Cx = 0$

We can solve C interms of x, y

$$C = \frac{x^2 + y^2}{x}$$

putting C in (1)

$$\frac{dy}{dx}=\frac{\frac{x^2+y^2}{x}-2x}{2y}$$

$$=\frac{x^2+y^2-2x^2}{2xy}$$

$$=\frac{-x^2+y^2}{2xy}$$

$$\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$$

Thus the of the orthogonal trajectories satisfy

$$\frac{dy}{dx}=\frac{-1}{\frac{y^2-x^2}{2xy}}=\frac{-2xy}{y^2-x^2}$$

$$=\frac{-2xy}{-(x^2-y^2)}$$

$$\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$$

$$=x^2\frac{\left[2\frac{y}{x}\right]}{x^2\left[1-\left(\frac{y}{x}\right)^2\right]}$$

$$\frac{dy}{dx}=\frac{2(y/x)}{1-(y/x)^2}$$

By change of variable

$$y=vx\Rightarrow v=\frac{y}{x}$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

The above differential equation can be transformed into

$$v+x\frac{dv}{dx}=\frac{2v}{1-v^2}$$

$$v+x\frac{dv}{dx}=\frac{2v}{1-v^2}$$

$$x\frac{dv}{dx}=\frac{2v}{1-v^2}-v$$

$$x\frac{dv}{dx}=\frac{2v-v+v^3}{1-v^2}$$

$$x\frac{dv}{dx}=\frac{-v+v^3}{1-v^2}$$

By variable separable and integration

$$\int\frac{1-v^2}{v+v^3}dv=\int\frac{dx}{x}+C$$

$$\int\frac{1-v^2}{v(1+v^2)}dv=\int\frac{dx}{x}+k$$

Solve $\frac{1-v^2}{1+v^2}$ by partial fractions

$$\frac{1-v^2}{1+v^2}=\frac{A}{v}+\frac{B+Dv}{1-v^2}=\frac{A+Av^2+Bv+Cv^2}{v(1+v^2)}$$

$$1-v^2=A(1-v^2)+Bv$$

If $v=0$

$$1=A(1-0)+B(0)$$

$A=1$, $B=0$, $D=-2$

The original problem reduces to

$$\int\frac{dv}{v}-\int\frac{2v}{1+v^2}dv=\int\frac{dx}{x}+k$$

$$\Rightarrow \log v-\log(1+v^2)=\log x+k$$

$$\log\frac{v}{1+v^2}=\log x+k$$

$$\frac{v}{1+v^2}=kx$$

$$\frac{\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2}=kx$$

$$\frac{xy}{x^2+y^2}=kx$$

$$\frac{y}{x^2+y^2}=k$$

$x^2+y^2=kx$ which is required solution.

Q69. Find the family orthogonal to the family $y = Ce^{-x}$ of exponential curves. Determine the number of each family through (0, 4).

Sol :

Given family of curve is $y = Ce^{-x}$

differentiating with respect to 'x'

$$\frac{dy}{dx} = -Ce^{-x}$$

We obtain $y^1 = -y$ a differential equation of the family of exponential curves solving

$$\frac{dy}{dx} = \frac{1}{y} \Rightarrow y\,dy = dx$$

By Integrating

$$\int y\,dy = \int dx + K$$

$$\frac{y^2}{2} = x + K$$

$y^2 = 2(x + K)$ a one parameter family of parabolas.

The parabolas are orthogonal to the exponents curves.

Putting $x = 0$ and $y = 4$

and $y = Ce^{-x}$ and $y^2 = 2(x + K)$

$u = Ce$

$C = u$ and $16 = 2(0 + k)$

$2K = 16$

$K = 8$

The required family member through (0, 4) are $-u$ and $\frac{1}{u}$ respectively.

Q70. Find the orthogonal trajectories of the family of rectangular hyperbolas $y = \frac{C_1}{x}$

Sol :

Given that the family of rectangular hyperbola is $y = \frac{C_1}{x}$ _____(1)

By eliminating $C_1 \Rightarrow C_1 = \frac{y}{x}$

differentiating (1) with respect to 'x'

$$\frac{dy}{dx} = C_1 - \frac{-1}{x^2}$$

$$\frac{dy}{dx} = \frac{-C_1}{x^2} \text{ \_\_\_\_}(2)$$

By (1) $C_1 = xy$

Sub C_1 in (2)

$$\frac{dy}{dx} = \frac{-xy}{x^2} = \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

The differential equation of the given orthogonal family is

$$\frac{dy}{dx} = \frac{-1}{\frac{-y}{x}}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Solving by variable separable

$$y\,dy = x\,dx$$

By Integrating

$$\int y\,dy = \int x\,dx + C_2$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_2$$

$$y^2 - x^2 = 2C_2$$

$$-(x^2 - y^2) = 2C_2$$

$$x^2 - y^2 = C_2$$

Which required equation of the family of orthogonal trajectories.

Q71. Find the orthogonal trajectories of family $y = x + Ce^{-x}$ and determine that particular member of each family that passes through (0, 3)

Sol :

The given equation is $y = x + Ce^{-x}$ ____(1)

differentiating (1) with respect to 'x'

$$\frac{dy}{dx} = 1 - Ce^{-x} \text{ ____} (2)$$

By (1) eliminating C

$$y - x = Ce^{-x}$$

$$\frac{y-x}{e^{-x}} = C$$

Sub C value in (2)

$$\frac{dy}{dx} = 1 - \frac{(y-x)}{\cancel{e^{-x}}}\left(\cancel{e^{-x}}\right)$$

$$= 1 - y + x$$

$$\therefore \frac{dy}{dx} = 1 + x - y$$

Thus, the differential equation for the family of orthogonal trajectories.

$$\frac{dy}{dx} = \frac{-1}{\frac{1}{1+x-y}}$$

$$\frac{dx}{dy} = -(1 + x - y)$$

$$\frac{dx}{dy} = -1 - x + y$$

$$\frac{dx}{dy} + x = y - 1$$

Which is linear differential equation

Then the IF is $e^{\int 1 dy} = e^y$

$$IF = e^y$$

So, it's solution

$$xIF = \int Q(IF)dy + C$$

$$xe^y = \int (y-1)e^y \, dy + C$$

$$xe^y = \int \left(ye^y - e^y\right)dy + C$$

$$xe^y = y\int e^y - \int 1\left(\int e^y dy\right)dy - \int e^y dy + C$$

$$xe^y = ye^y - e^y - e^y + C$$

$$xe^y = ye^y - 2e^y + C$$

$$xe^y = (y-2)e^y + C$$

$$xe^y - e^y(y-2) = C$$

$\therefore$ The required curve passing through (0, 3) found to be $y = x + 3e^{-x}$, $x - y + 2 + e^{3-y} = 0$.

Q72. Find an equation of the orthogonal trajectories of the family of circles having a polar equation r = f(θ) = 2a cosθ.

Sol :

Given polar equations is $r = f(\theta) = 2a\cos\theta$

$$f^1(\theta) = -2a\sin\theta$$

$$\tan\psi = \frac{f(\theta)}{f^1(\theta)} = \frac{2a\cos\theta}{-2a\sin\theta}$$

$$= \frac{\cos\theta}{-\sin\theta} = -\cot\theta = g(\theta)$$

$$\tan\psi = \frac{-1}{g(\theta)}$$

$$\tan\psi = \frac{-1}{(-\cot\theta)}$$

$$\tan\psi = \tan\theta$$

$$r\left(\frac{d\theta}{dr}\right) = \frac{1}{g(\theta)}$$

$$r\frac{d\theta}{dr} = \tan\theta$$

is differential equation of the required family

Separating variables $\frac{d\theta}{\tan\theta} = \frac{dr}{r}$

$$\Rightarrow \frac{dr}{r} = \frac{\cos\theta}{\sin\theta} d\theta$$

By Integrating

$$\int \frac{dr}{r} = \int \frac{\cos\theta}{\sin\theta} d\theta$$

$$\log r = \log(\sin\theta) + \log C$$

$$\log r = \log(\sin\theta) C$$

$$r = C \sin\theta$$

The required family of circle.

Q73. Find the orthogonal trajectory of $r = C_1(1 - \sin\theta)$.

Sol :

Given that $r = C_1(1 - \sin\theta)$ ____ (1)

Differentiating (1) with respect to 'θ'

$$\frac{dr}{d\theta} = C_1 \cos\theta \text{ ____ } (2)$$

By (1) $C_1 \frac{r}{1 - \sin\theta}$ ____ (3)

Substitute (3) in (2)

$$\frac{dr}{d\theta} = \frac{-r\cos\theta}{1 - \sin\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1 - \sin\theta}$$

$$\Rightarrow r\frac{d\theta}{dr} = \frac{\cos\theta}{1 - \sin\theta}$$

$$r\frac{d\theta}{dr} = \tan\psi$$

$$\frac{d\theta}{\frac{\cos\theta}{1 - \sin\theta}} = \frac{dr}{r}$$

$$\frac{1 - \sin\theta}{\cos\theta} d\theta = \frac{dr}{r}$$

By Integrating

$$\int \frac{1 - \sin\theta}{\cos\theta} d\theta = \int \frac{dr}{r}$$

$$\int (\sec\theta - \tan\theta) d\theta = \int \frac{dr}{r}$$

$$\int \sec\theta \, d\theta - \int \tan\theta \, d\theta = \frac{dr}{r}$$

$$\log r = \log(\sec\theta + \tan\theta) - (-\log\cos x) + \log C_2$$

$$\log r = \log(\sec\theta + \tan\theta) + \log\cos x + \log C_2$$

$$\log r = \log(C_2(1 + \sin\theta))$$

$$r = C_2(1 + \sin\theta)$$

Multiple Choice Questions

1. Solution for $p^2 - 5p + 6 = 0$ [a]

(a) $(y - 3x - C)(y - 2x - C) = 0$ (b) $(y - 3x + C)(y + 2x + C) = 0$

(c) $(y + 3x + C)(y + 2x - C) = 0$ (d) $(y + 3x - C)(y + 2x - C) = 0$

2. Solution for $p^2 + 2py\cot x = y^2$ [c]

(a) $y(1 - \cos x) - C = 0$

(b) $[y(1 + \cos x) + C][y(1 - \cos x) - C] = 0$

(c) $[y(1 - \cos x) - C][y(1 + \cos x) - C] = 0$

(d) $[y(1 + \cos x) + C][y(1 + \cos x) - C] = 0$

3. Solution for $x^2p^2 + xyp - 6y^2 = 0$ [d]

(a) $(y - Cx^2)(yx^3 + C) = 0$ (b) $(y + Cx^2)(yx^3 + C) = 0$

(c) $(y + Cx^2)(yx^3 - C) = 0$ (d) $(y - Cx^2)(yx^3 - C) = 0$

4. Bacteria in certain culture increase at a rate proportional to the number present, if the number doubles in one hour then how long does it takes for the number to triple [a]

(a) $t = 1.5850\,hr$ (b) $t = 2.5850\,hr$

(c) $t = 0.0580\,hr$ (d) $t = 0.5850\,hr$

5. Compound Interest [a]

(a) $A = p\left(1 + \frac{r}{100}\right)^t$ (b) $A = p\left(1 - \frac{r}{100}\right)^t$

(c) $A = \frac{ptr}{100}$ (d) None

6. Which is the half life of radium if it is found that 0.5% of radium disappear in 12 years [b]

(a) 16721.70 years (b) 1672.1770 years

(c) 16.72170 years (d) None

7. If Rs 10,000 is invested at 6 percent per annum accumlated after 6 years if interest is compounded for quarterly [c]

(a) Rs. 1433.029 (b) Rs 1433.29

(c) Rs 14333.29 (d) Rs 143.29

8. How long does it takes for a given amount of money to double at 6 percent per annum compounded [b]

(a) 10 years (b) 11.89375 years

(c) 11.00937 years (d) 12 years

9. The orthogonal trajectories of $x^2 + y^2 = Cx$ [c]

(a) $x^2 - y^2 = k$ (b) $x^2 - y^2 = kx$

(c) $x^2 + y^2 = kx$ (d) None

10. The orthogonal trajectories of the family of rectangular hyperbola is [d]

(a) $x^2 + y^2 = C$ (b) $x - y = C_1$

(c) $x = \frac{C_1}{y}$ (d) $y = \frac{C_1}{x}$

Fill in the blanks

1. When the differential equation $F(x, y, p) = 0$ is solvable for x, then we have ________ .

2. If the differential equation $f(x, y, p) = 0$ is ________ then $y = f(x, p)$.

3. If the equation has the form $f(y, p) = 0$ and is solvable for p, it will then give $\frac{dy}{dx} = \phi(y)$ which is ________ .

4. The equati $F(x, y, p) = 0$ is of the first order degree in x and y then ________.

5. $y = x f_1(p) + f_2(p)$ is known as ________ .

6. Solution for $(y - px)(p - 1) = p$ is ________ .

7. The rate at which the substance changes is proportional to the quality of substance present at any time is ________ .

8. The solution for $\frac{dx}{dt} = Kx$ is ________ .

9. ________ percentage will disappears in 100 years if it is found that 0.5% of radium disappear in 12 years.

10. If Rs. 10,000 is invested at 6%/annum. Then the C.I is ________ .

ANSWERS

1. $x = f(y, p)$
2. solvable for y,
3. integrable
4. $y = x f_1(p) + f_2(p)$
5. Lagrange's equation.
6. $(y - Cx)(C - 1) = C$
7. $\frac{dx}{dt} = Kx$
8. $x = Ce^{kt}$
9. 34.2
10. Rs. 14,185.19.

UNIT III	**Higher order Linear Differential Equations:** Solution of homogeneous linear differential equations with constant coefficients - Solution of non-homogeneous differential equations P(D)y = Q(x) with constant coefficients by means of polynomial operators when Q(x) = be^{ax}, b sin ax/b cos ax, bx^k, V e^{ax} - Method of undetermined coefficients

3.1 Solution of Homogeneous Linear Differential Equation of Order 'n' with Constant Coefficients

Q1. Define non homogeneous linear differential equation and homogeneous linear differential equation.

Ans :

A linear differential equation with constant coefficients is that in which the dependent variable and its differential coefficients are only in the first degree and are not multiplied together and the coefficients are all constant.

The general form of the equation is

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = Q(x) \quad ... (1)$$

where

$a_0, a_1, ... a_n$ and Q(x) are continuous real functions on a common interval and $a_n(x) \neq 0$

[This can also written as by using symbols $a_n D^n y + a_{n-1} D^{n-1} y + a_{n-2} D^{n-2} y + ... + a_0 y = Q(x)$]

The equation (1) is called the non homogeneous equation

If Q(x) is identically zero. Then (1) will be becomes

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = 0$$

which called a homogeneous linear equation of order n.

Q2. Write a short note on auxiliary equation and complimentary function.

Ans :

The differential equation is,

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = 0 \quad ... (2)$$

When

$a_0, a_1, ... a_n$ are all constant & $a_n \neq 0$

Suppose that the solution of equation (2) is

$y = e^{mx}$

Sine $\frac{dy}{dx} = me^{mx} \cdot \frac{d^2y}{dx^2} = m^2 e^{mx} ..... \frac{d^ny}{dx^n} = m^n e^{mx}$ by equation (2)

$a_n me^{mx} + a_{n-1}m^{n-1} e^{mx} + ... + a_1 me^{mx} + a_0 e^{mx} = 0$ ... (3)

as $e^{mx} \neq 0 \;\; \forall \; m \in x$

Divide (3) be e^{mx}. Then we get

$a_0 m^n + a_{n-1} m^{n-1} + ... + a_1 m + a_0 = 0$

Which is an algebraic equation in m of degree n and which is also called "Auxillary equation" or "characteristic equation".

By fundamental theorem of algebra it has at least one and not more than n distinct roots.

We denote by roots $m_1, m_2 ... m_n$ where 'm' is need not all be distinct.

Then each function

$y_1 = e^{m_1x}, \; y_2 = e^{m_2x}$ ....

$y_n = e^{m_nx}$ is a solution of equation (2).

By Auxillary equation, we can obtained the equation by replacing y' with m, y'' with m^2 and so on and $y^{(n)}$ with m^n.

By solving the auxillary equation the following three case may occur.

Case (1) :

If the n roots $m_1, m_2 ... m_n$ of A.E are distinct and real. Then the solution of A.E. is

$y_1 = e^{m_1x}, \; y_2 = e^{m_2x} \; ... \; y_n = e^{m_nx}$

But these n solutions are different and

Linearly independent and the general solution of equation (2) is

$y_c = y = c_1 e^{m_1x} + c_2 e^{m_2x} + ... + c_n e^{m_nx}$

Where

y_c is known as the complementary function

Case (2)

If the characteristic equation (3) has a root m = a which repeat 'n' times.

Then the general solution of equation (3) is

$y = (c_1 + c_2 x + c_3 x^2 + ... + c_n x^{n-1})e^{ax}$

- One real root m_1 then the nature of roots of A.E is $y = c_1e^{m_1x}$
- If one pair of complex roots $\alpha \pm i\beta$

  Then the nature of rots of A.E is

  $e^{ax} (c_1 \cos\beta x + c_2 \sin\beta x)$
- If two pairs of complex and equal roots

  $\alpha \pm i\beta, \; \alpha \pm i\beta$ Then the nature of root of

  A.E is $e^{ax} [(c_1 + c_2 x) \cos\beta x + (c_3 + c_4 x) \sin\beta x]$

Q3. Solve $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

Sol :

The given equation is

$$\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$$

Then the A.E is

$m^3 + 6m^2 + 11m + 6 = 0$

If m = –1

$\Rightarrow (-1)^3 + 6(-1)^2 + 11(-1) + 6$

$-1 + 6 - 11 + 6 = 0$

m = –1				
	1	6	11	6
	0	–1	–5	–6
	1	5	6	0

$\therefore \quad m^2 + 5m + 6 = 0$

$m^2 + 2m + 3m + 6 = 0$

$(m + 2)(m + 3) = 0$

$\therefore \quad m = -2, -3$

$\therefore$ The real roots are –1, –2, –3 which are distinct. Then the general solution is

$$y_c = y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$$

Q4. Solve $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2y = 0$.

Sol :

The given equation is

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2y = 0$$

The A.E is $m^2 - 2am + a^2 = 0$

$(m - a)^2 = 0$

$(m - a)(m - a) = 0$

$\Rightarrow m = a, \alpha$

which are some roots.

Then the general solution is

$$y_c = y = (c_1 + c_2x)e^{ax}$$

Q5. Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$.

Sol :

The given equation $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Then the A.E is $m^3 - 3m + 2 = 0$

If $m = -1 \Rightarrow (-1)^3 - 3(-1) + 2$

$-1 + 3 + 2 \neq 0$

If $m = 1 \Rightarrow (1)^3 - 3(1) + 2$

$= 1 - 3 + 2 = 3 - 3 = 0$

m = 1				
	1	0	–3	2
	0	1	1	–2
	1	1	–2	0

$m^2 + m - 2 = 0$

$\Rightarrow m^2 + 2m - m - 2 = 0$

$m(m + 2) - 1(m + 2) = 0$

$\therefore \quad m = 1, -2$

$\therefore$ The root are 1, 1, –2

$\therefore$ The solution is $y = (c_1 + c_2x)e^x + c_3e^{-2x}$

Q6. Solve $16\frac{d^2y}{dx^2} + 24\frac{dy}{dx} + 9y = 0$.

Sol :

The given equation is $16\frac{d^2y}{dx^2} + 24\frac{dy}{dx} + 9y = 0$

Then the A.E is $16m^2 + 24m + 9 = 0$

$(4m + 3)^2 = 0$

$(4m + 3)(4m + 3) = 0$

$m = \frac{-3}{4}, \frac{-3}{4}$

$\therefore$ The solution is $y = (c_1 + c_2x)\, e^{\frac{-3}{4}x}$

Q7. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

Sol :

The given equation is $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Then the A.E is

$m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m - 2) - 1(m - 2) = 0$

$(m - 1)(m - 2) = 0$

$m = 1, 2$, which are two distinct roots,

Then the solution is

$y = c_1e^{x} + c_2e^{2x}$

Q8. Solve $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = 0.$

Sol :

The given equation is

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 0$$

Then the A.E. is

$m^2 - m^2 - 6m = 0$

$m(m^2 - m - 6) = 0$

$m = 0, \; m^2 - m - 6 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2^a} = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1 + 24}}{2} = \frac{1 \pm \sqrt{25}}{2}$$

$$m = \frac{1 \pm 5}{2}$$

$m = 3, -2$

$\therefore$ The roots are real and distinct

Then $y_c = c_1e^{0x} + c_2e^{3x} + c_3e^{-2x}$

$y_c = c_1 + c_2e^{3x} + c_3e^{-2x}$

Q9. Solve $\dfrac{d^2y}{dx^2} + 4y = 0.$

Sol :

The given equation is $\dfrac{d^2y}{dx^2} + 4y = 0$

Then the A.E is $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

$\therefore$ The roots are imaginary roots

Then the solution is

$y_c = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$

$y = c_1 \cos 2x + c_2 \sin 2x$

Q10. Solve $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$

Sol :

The given equation is $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$

The A.E is $m^4 + 8m^2 + 16 = 0$

$(m^2)^2 + m^2 + 16 = 0$

$(m^2 + 4)^2 = 0$

$(m^2 + 4)(m^2 + 4) - 0$

$m = \pm 2i, \; m = \pm 2i$

These roots, imaginary roots

$y = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$

Q11. Solve $\dfrac{d^3y}{dx^3} + y = 0$

Sol :

The equation is $\dfrac{d^3y}{dx^3} + y = 0$

The A.E. is $m^3 + 1 = 0$

$(m + 1)(m^2 - m + 1) = 0$

$m = -1, \; m^2 - m + 1 = 0$

$$m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1 - 4}}{2}$$

$$m = \frac{1 \pm \sqrt{3}\, i}{2}$$

$\therefore$ The roots are real and imaginary

The solution is

$$y = c_1e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$$

Q12 Solve $\left(\frac{dy}{dx} - y\right)^2 \left(\frac{d^2y}{dx^2} + y\right)^2 = 0$

Sol :

The equation is $\left(\frac{dy}{dx} - y\right)^2 \left(\frac{d^2y}{dx^2} + y\right)^2 = 0$

Then the A.E is

$(m - 1)^2 (m^2 + 1)^2 = 0$

$(m - 1)(m - 1)(m^2 + 1)(m^2 + 1) = 0$

$m = 1, 1$ & $(m^2 + 1)(m^2 + 1) = 0$

$m = \pm i, \pm i$

∴ The roots are $1, 1, \pm i, \pm i$

The solution is

$y = (c_1 + c_2 x) e^x + (c_3 + c_4 x) \cos x + (c_5 + c_6 x) \sin x$

Q13. Solve $\frac{d^2y}{dx^2} + (a + b)\frac{dy}{dx} + aby = 0$

Sol :

The given equation is

$$\frac{d^2y}{dx^2} + (a + b)\frac{dy}{dx} + aby = 0$$

Then the A.E. is $m^2 + (a + b)m = ab = 0$

$$m = \frac{-(a+b) \pm \sqrt{(a+b)^2 - 4(1)(ab)}}{2(1)}$$

$$= \frac{-(a+b) \pm \sqrt{(a+b)^2 - 4ab}}{2}$$

$$= \frac{-(a+b) \pm \sqrt{(a-b)^2}}{2}$$

$$m = \frac{-(a+b)+(a-b)}{2}, \frac{-(a+b)-(a-b)}{2}$$

$$m = \frac{-a-b+a-b}{2}, \frac{-a-b-a+b}{2}$$

$$m = \frac{-2b}{2}, \frac{-2a}{2}$$

∴ $m = -a, -b$ which are the distinct roots

The solution is

$y = c_1 e^{-ax} + c_2 e^{-bx}$

3.2 Solution of Non-Homogeneous Linear Differential Equations with Constant Coefficients by Means of Polynomial Operators

Q14. Write a short note on solution of homogeneous equation to solve particular integral.

Ans :

The general solution of a non homogeneous linear differential equation

$$a_n \frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0 y = Q(x)$$

where $a_n \neq 0$, $Q(x) \neq 0$

& $a_0, a_1, ... a_n$ are constant is

$y = y_c + y_p$

Here y_c is known as complementary function y_p is known as Particular Integral (P.I)

Q(x) consists of such terms as $b, x^k, e^{ax}, \sin ax, \cos ax$ and a finite number of combination of such terms. Where a & b are constants & k is positive integer.

3.2.1 When $Q(x) = bx^m$ and m being a Positive Integer

Q15. Write working rule for evaluating $Q(x) = bx^m$ where m being positive integer.

Ans :

Short method to find P.I.

Working rule for evaluating

$(D - a_0)\, y = bx^m$

Step I :

Bringout the lowest degree term from f(D) so that the remaining factor in the denominator is of the form $[1 + \phi(D)]^n$ or $[1 - \phi(D)]^n$, n being a positive integer.

Step II :

We take $[1 + \phi(D)]^n$ or $[1 - \phi(D)]^n$ in numerator so that it takes the form

$[1 + \phi(D)]^{-n}$ or $[1 - \phi(D)]^{-n}$

Step III :

We expand $[1 + \phi(D)]^n$ by the binomial theorem.

* In particular some of binomial expansion should be remembered.

1. $(1-x)^{-1} = 1 + x + x^2 + x^3 + ...$
2. $(1+x)^{-1} = 1 - x + x^2 + x^3 + ...$
3. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...$
4. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ...$

Q16. Solve $(D^2 - 4)\, y = x^2$.

Sol :

The given equation is $(D^2 - 4)\, y = x^2$

The A.E is $m^2 - 4 = 0$

$$m^2 = 4 \Rightarrow m = \pm 2$$

Thus which roots are two distinct real roots

Thus,

$$y_c = c_1e^{-2x} + c_2e^{2x}$$

Now, $\quad P.I = y_p = \dfrac{1}{f(D)}x^2$

$$= \frac{1}{D^2-4}x^2$$

$$= \frac{1}{4\left[1-\frac{D^2}{4}\right]}x^2$$

$$= \frac{-1}{4}\left[1-\frac{D^2}{4}\right]^{-1}x^2$$

$[\because [1-x]^{-1} = 1 + x + x^2 + ...]$

$$= \frac{-1}{4}\left[1+\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+...\right]x^2$$

$$= \frac{-1}{4}\left[x^2+\frac{D^2}{4}(x^2)+\left(\frac{D^2}{4}\right)^2x^2+...\right]$$

$$= \frac{-1}{4}\left[x^2+\frac{2}{4}+0...\right]$$

$$y_p = \frac{-1}{4}\left[x^2+\frac{1}{2}\right]$$

$\therefore$ The general solution is

$$y = y_c + y_p$$

$$= c_1e^{-2x} + c_2e^{2x} - \frac{1}{4}\left[x^2+\frac{1}{2}\right]$$

Q17. Solve $(D^3 - 2D + 4)\, y = x^4 + 3x^2 - 5x + 2$.

Sol :

The given equation is

$(D^3 - 2D + 4)\, y = x^4 + 3x^2 - 5x + 2$

Then the A.E is

$m^3 - 2m + 4 = 0$

$(m + 2)(m^2 - 2m + 2) = 0$

$(m + 2) = 0 \Rightarrow m = -2$

$m^2 - 2m + 2 = 0$

$$\frac{-(-2)\pm\sqrt{4-4(1)(2)}}{2(1)} = \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm 2i}{2} = 1 \pm i$$

$\therefore$ The roots are $m = -2,\ 1 \pm i$

$\therefore$ $y_c = c_1e^{-2x} + e^{2x}(c_2\cos x + c_3\sin x)$

Now

To find y_p

$$P.I = y_p = \frac{1}{D^3-2D+4}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4\left[1-\frac{2D}{4}+\frac{D^3}{4}\right]}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1-\frac{D}{2}+\frac{D^3}{4}\right]^{-1}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1-\frac{1}{2}\left(D-\frac{D^3}{2}\right)\right](x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1+\frac{1}{2}\left(D-\frac{D^3}{2}\right)+\frac{1}{4}\left(D-\frac{D^3}{2}\right)^2+...\right](x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[x^4 + 3x^2 - 5x + 2 + \frac{1}{2}\left[D(x^4 + 3x^2 - 5x + 2) - \frac{D^3}{2}(x^4 + 3x^2 - 5x + 2)\right]\right.$$

$$\left. + \frac{1}{4}\left[D^2(x^4 + 3x^2 - 5x + 2) + \frac{D^6}{4}(x^4 + 3x^2 - 5x + 2) - D^4(x^4 + 3x^2 - 5x + 2)\right]\right]$$

$$= \frac{1}{4}\left[x^4 + 3x^2 - 5x + 2 + \frac{1}{2}\left[(4x^3 + 6x - 5) - \frac{1}{2}(24x)\right] + \frac{1}{4}[(12x^2 + 6)] + 0 - (24)\right]$$

$$= \frac{1}{4}\left[x^4 + 3x^2 - 5x + 2 + 2x^3 + 3x - \frac{5}{2} - 6x + 3x^2 + \frac{3}{2} - 6\right]$$

$$= \frac{1}{4}[x^4 - 2x^3 + 6x^2 - 8x - 5]$$

$\therefore$ $y = y_c + y_p$

$$= c_1e^{-2x} + e^x(c_1 \cos x + c_3 \sin x) + \frac{1}{4}[x^4 + 2x^3 + 6x^2 - 8x - 5]$$

Q18. Solve $(D^2 + 2D + 1)y = 2x + x^2$.

Sol :

The given equation $(D^2 + 2D + 1)\ y = 2x + x^2$

The A.E is $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$m = -1, -1$ are the roots

$\therefore$ $y_c = (c_1 + c_2 x)e^{-x}$

Now

$$P.I = y_p = \frac{1}{f(D)}(2x + x^2)$$

$$= \frac{1}{(D+1)^2}(2x + x^2)$$

$= (D + 1)^{-2}(2x + x^2)$

$= (1 - 2D + 3D^2 + ...)(2x + x^2)$

$= 2x + x^2 - 2D(2x + x^2) + 3D^2(2x + x^2)$

$= 2x + x^2 - 2[2 + 2x] + 3[2]$

$= 2x + x^2 - 4 - 4x + 6$

$y_p = x^2 - 2x + 2$

$\therefore$ The general solution is

$y = y_c + y_p$

$= (c_1 + c_2 x)e^{-x} + x^2 - 2x + 2$

Q19. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x$.

Sol :

The given equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x$$

i.e., $(D^2 + D)\,y = x^2 + 2x$

$\therefore$ The A.E is $m^2 + m = 0$

$m(m + 1) = 0$

$m = 0, \; m = -1$

$\therefore$ The roots are $m = 0, -1$ which are real and distinct

$y_c = c_1e^{0x} + c_2e^{-x}$

$y_c = c_1 + c_2\,e^{-x}$

Now,

P.I $\quad y_p = \frac{1}{f(D)}\,x^2 + 2x$

$= \frac{1}{D^2 + D}\,x^2 + 2x$

$= \frac{1}{D(1+D)}\,x^2 + 2x$

$= \frac{1}{D}\,(1 + D)^{-1}\,x^2 + 2x$

$= \frac{1}{D}\,[1 - D + D^2 - ...]\,x^2 + 2x$

$= \frac{1}{D}\,[x^2 + 2x - D\,(x^2 + 2x) + D^2\,(x^2 + 2x)]$

$= \frac{1}{D}\,[x^2 + 2x - 2x - 2 + 2]$

$= \frac{1}{D}\,[x^2]$

$y_p = \frac{x^3}{3}$

$\therefore$ The general solution is

$$y = y_c + y_p \Rightarrow c_1 + c_2\,e^{-x} + \frac{x^3}{3}$$

Q20. Solve $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 2x^3$.

Sol :

The given equation is $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 2x^3$

i.e., $D^3 - D^2 = 2x^3$

The A.E is

$m^3 - m^2 = 0$

$m^2\,(m - 1) = 0$

$m^2 = 0, \; m - 1 = 0$

$m = 0, \; m = 1$

$\therefore \quad y_c = c_1 + c_2\,e^x$

Now,

P.I $= y_p = \frac{1}{D^3 - D^2}\,2x^3$

$= -\frac{1}{D^2[1 - D]}\,2x^3$

$= \frac{-1}{D^2}\,[1 - D]^{-1}\,2x^3$

$= \frac{-1}{D^2}\,[1 + D + D^2 + D^3]\,2x^3$

$= \frac{-2}{D^2}\,[x^3 + D(x^3) + D^2(x^3) + D^3(x^3)]$

$= \frac{-2}{D^2}\,[x^3 + 3x^2 + 6x + 6]$

$= \frac{-2}{D}\left[\frac{x^4}{4} + \frac{3x^3}{3} + \frac{6x^2}{2} + 6x\right]$

$= -2\left[\frac{x^5}{20} + \frac{x^4}{4} + \frac{3x^3}{3} + \frac{6x^2}{2}\right]$

$= -2\left[\frac{x^5}{20} + \frac{x^4}{4} + x^3 + 3x^2\right]$

$y_p = \frac{-x^5}{10} - \frac{x^4}{2} - 2x^3 - 6x^2$

$\therefore$ The required solution is

$$y = y_c + y_p \Rightarrow c_1 + c_2e^x - \frac{x^5}{10} - \frac{x^4}{2} - 2x^3 - 6x^2$$

Q21. Solve $y'' + y' + y = x^2$

Sol :

The given equation $y'' + y' + y = x^2$

The A.E is $(m^2 + m + 1) = 0$

$$m = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2} = \frac{-1 \pm \sqrt{1-3}}{2}$$

$$= \frac{-1 \pm \sqrt{-2}}{2}$$

$$= \frac{-1 \pm \sqrt{2}\, i}{2}$$

$$= \frac{-1}{2} \pm \frac{i}{\sqrt{2}}$$

$$\therefore \quad y_c = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{1}{\sqrt{2}}x + c_2 \sin\frac{1}{\sqrt{2}}x\right)$$

Now,

$$\text{P.I} = y_p = \frac{1}{D^2 + D + 1} x^2$$

$$= \frac{1}{[1 + (D + D^2)]} x^2$$

$$= [1 + (D + D^2)]^{-1} x^2$$

$$= [1 - (D + D^2) + (D + D^2)] x^2$$

$$y_p = x^2 - 2x - 2$$

$\therefore$ The required solution is

$$y = y_c + y_p$$

$$y = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{1}{\sqrt{2}}x + c_2 \sin\frac{1}{\sqrt{2}}x\right) + x^2 - 2x - 2$$

Q22. Solve $y'' + 3y' + 2y = 4$.

Sol :

The given equation $y'' + 3y' + 2y = 4$

The A.E is

$$m^2 + 3m + 2 = 0$$

$$m^2 + 2m + m + 2 = 0$$

$$(m + 2)(m + 1) = 0$$

$$\therefore \quad m = -1, -2$$

$$\therefore \quad y_c = c_1 e^{-x} + c_2 e^{-2x}$$

Now,

$$\text{P.I} = y_p = \frac{1}{D^2 + 3D + 2} 4.x^0$$

$$= 4 \frac{1}{2\left[1 + \frac{3D}{2} + \frac{D^2}{2}\right]} x^0$$

$$= 2\left[1 + \left(\frac{3D}{2} + \frac{2D}{2}\right)\right]^{-1} x^0$$

$$= 2\left[1 + \left(\frac{3D}{2} + \frac{D^2}{2}\right) + \left(\frac{3D}{2} + \frac{D^2}{2}\right)^2\right] x^0$$

$$= 2[x^0 + 0]$$

$$= 2(1)$$

$$y_p = 2$$

$\therefore$ The required solution is

$$y = y_c + y_p$$

$$y = c_1 e^{-x} + c_2 e^{-x} + 2$$

Q23. Solve $(D^3 - D^2 - 6D)\, y = x^2 + 1$.

Sol :

The given equation $(D^3 - D^2 - 6D)\, y = x^2 + 1$

The A.E is

$$m^3 - m^2 - 6m = 0$$

$$m(m^2 - m - 6) = 0$$

$$m = 0, \; m^2 - m - 6 = 0$$

$$m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2}$$

$$= \frac{1 \pm \sqrt{1 + 24}}{2}$$

$$= \frac{1 \pm 5}{2} = \frac{6}{2}, \frac{-4}{2}$$

$$m = 3, -2$$

$\therefore$ The roots are real and distinct

Then

$$y_c = c_1 e^{0x} + c_2 e^{-2x} + c_3 e^{3x}$$

$$y_c = c_1 + c_2 e^{-2x} + c_3 e^{3x}$$

Now,

$$\text{P.I} = y_p = \frac{1}{D^3 - D^2 - 6D} x^2 + 1$$

$$= \frac{1}{6D\left[1+\frac{D^2}{6D}-\frac{D^3}{6D}\right]}\,(x^2+1)$$

$$= \frac{-1}{6D}\left[1+\left(\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}(x^2+1)$$

$$= \frac{-1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2\right](x^2+1)$$

$$= \frac{-1}{6D}\left[x^2+x-\frac{D}{6}(x^2+1)+\frac{D^2}{6}(x^2+1)+\frac{D^2}{36}x^2\right]$$

$$= \frac{-1}{6D}\left[x^2+1-\frac{2x}{6}+\frac{2}{6}+\frac{2}{36}\right]$$

$$= \frac{-1}{6D}\left[x^2-\frac{x}{3}+\frac{4}{3}+\frac{1}{18}\right]$$

$$y_p = \frac{-1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

∴ The required solution is

$$y = y_c + y_p$$

$$= c_1 + c_2e^{-2x} + c_3e^{3x} - \frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

Q24. Solve $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$.

Sol :

The given equation

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$$

i.e., $(D^3 + 3D^2 + 2D)\,y = x^2$

The A.E. is

$m^3 + 3m^2 + 2m = 0$

$m(m^2 + 3m + 2) = 0$

$m = 0,\ m^2 + 3m + 2 = 0$

$m^2 + 2m + m + 2 = 0$

$(m + 2)(m + 1) = 0$

$m = -2, -1, 0$

∴ The roots are real and distinct

$$y_c = c_1 + c_2e^{-x} + c_3e^{-2x}$$

Now,

$$\text{P.I} = y_p = \frac{1}{D^3+2D^2+2D}\,x^2$$

$$= \frac{1}{D(D+1)(D+2)}$$

$$= \frac{1}{2D}\left[(1+D)^{-1}\left(1+\frac{D}{2}\right)^{-1}\right]x^2$$

$$= \frac{1}{2D}\left[(1-D+D^2+\ldots)\left(1-\frac{D}{2}+\frac{D^2}{4}+\ldots\right)\right]x^2$$

$$= \frac{1}{2D}\left[1-\frac{D}{2}+\frac{D^2}{4}-D+\frac{D^2}{2}-\frac{D^3}{4}+D^2-\frac{D^3}{2}+\frac{D^4}{4}\right]x^2$$

$$= \frac{1}{2D}\left[x^2-\frac{1}{2}(2x)+\frac{1}{4}(2)-2x+\frac{2}{2}-0+2-0+0\right]$$

$$= \frac{1}{2D}\left[x^2-x+\frac{1}{2}-2x+1+2\right]$$

$$= \frac{1}{2D}\left(x^2-3x+\frac{7}{2}\right)$$

$$= \frac{1}{2}\left[\frac{x^3}{3}-\frac{3x^2}{2}+\frac{7x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{4x^3-18x^2+42x}{12}\right]$$

$$y_p = \frac{1}{12}\,[2x^3 - 9x^2 + 21x]$$

∴ The required solution is

$$y = y_c + y_p$$

$$y = c_1 + c_2e^{-x} + c_3e^{-2x} + \frac{1}{12}[2x^3-9x^2+21x]$$

3.2.2 (i) When $Q(x) = be^{ax}$ & $f(a) \neq 0$

In this case $f(D)\,y = Q(x)$ becomes $f(D)\,y = be^{ax}$

Then the particular integral is

$$y_p = \frac{1}{f(D)} be^{ax} = \frac{be^{ax}}{f(a)} \quad , \; f(a) \neq 0$$

(ii) When $Q(x) = be^{ax}$ & $f(a) = 0$

In this case $(D - a)$ is factor of $f(D)$

Suppose that $(D - a)^n$ is a factor of $f(D)$

We can write $f(D) = (D - a)^n . f(D)$; $f(D) \neq 0$

$$y_p = \frac{1}{f(D)} be^{ax} = \frac{1}{(D-a)^n f(D)} be^{ax} = \frac{bx^n e^{ax}}{n! f(a)}$$

Q25. Solve $(D^2 - 2D + 5)\, y = e^{-x}$.

Sol :

The given equation is

$(D^2 - 2D + 5)\, y = e^{-x}$

One the A.E is $m^2 - 2m + 5 = 0$

$$m = \frac{-(-2) \pm \sqrt{4 - 4(1)(5)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$

$= 1 \pm 2i$

Thus C.F $= y_c = e^x (c_1 \cos 2x + c_2 \sin 2x)$

Now,

$$P.I = y_p = \frac{1}{f(D)} e^{-x}$$

$$= \frac{1}{D^2 - 2D + 5} e^{-x} = \frac{1}{(-1)^2 - 2(-1) + 5} e^{-x}$$

$$= \frac{1}{1 + 2 + 5} e^{-x}$$

$$y_p = \frac{1}{8} e^{-x}$$

$\therefore$ The required solution is

$y = y_c + y_p$

$$= e^x (c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{8} e^{-x}$$

Q26. Solve $y'' + 3y' + 2y = 12e^x$.

Sol :

The given equation is

$y'' + 3y' + 2y = 12e^x$

The A.E is $m^2 + 3m + 2 = 0$

$m^2 + 2m + m + 2 = 0$

$(m + 2)(m + 1) = 0$

$m = -1, -2$ which are real and distinct roots

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

Now,

$$P.I = y_p = \frac{1}{f(D)} 12e^x$$

$$= 12 \frac{1}{D^2 + 3D + 2} e^x$$

$$= 12 \frac{1}{(1)^2 + 3(1) + 2} e^x$$

$$= 12 \frac{1}{6} e^x$$

$y_p = 2e^x$

$\therefore$ The required solution

$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 2e^x$

Q27. Solve $y'' + y = 3e^{-2x}$.

Sol :

The given equation is $y'' + y = 3e^{-2x}$

The A.E is $(m^2 + 1) = 0$

$m^2 = -1$

$m = \pm i$

Which is a complex root

Thus $y_c = e^{0.x} (c_1 \cos x + c_2 \sin x)$

$y_c = c_1 \cos x + c_2 \sin x$

Now,

$$P.I = y_p = \frac{1}{f(D)} 3e^{-2x}$$

$$= 3 \frac{1}{D^2 + 1} e^{-2x}$$

$$= 3 \frac{1}{(-2)^2 + 1} e^{-2x}$$

$$= 3 \frac{1}{4 + 1} e^{-2x}$$

$$y_p = \frac{3}{5} e^{-2x}$$

$\therefore$ The required solution is

$y = y_c + y_p$

$$y = c_1 \cos x + c_2 \sin x + \frac{3}{5} e^{-2x}.$$

Q28. Solve $y'' + y' + 2y = 3e^{-2x}$.

Sol :

The given $y'' + y' + 2y = 3e^{-2x}$.

The A.E is $m^2 + m - 2 = 0$

$$m = \frac{-1 \pm \sqrt{1-4(1)(-2)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1+8}}{2}$$

$$= \frac{-1 \pm \sqrt{9}}{2}$$

$$m = \frac{-1 \pm 3}{2}$$

$$m = \frac{-1+3}{2}, \frac{-1-3}{2}$$

$m = 1, -2$ which are real and distinct.

$\therefore$ $y_c = c_1e^x + c_2e^{-2x}$

Now,

$$P.I = y_p = \frac{1}{f(D)} 3e^{-2x}$$

$$= 3\frac{1}{D^2+D-2} e^{-2x}$$

$$= 3\frac{1}{(-2)^2+2-2} e^{-2x}$$

$$= 3\frac{1}{4} e^{-2x}$$

$$y_p = \frac{3}{4} e^{-2x}$$

$\therefore$ The required solution is

$y = y_c + y_p$

$$y = c_1 e^x + c_2 e^{-2x} + \frac{3}{4} e^{-2x}.$$

Q29. Solve $y'' - y = 2e^x$.

Sol :

The given equation is $y'' - y = 2e^x$

The A.E. is $m^2 - 1 = 0$

$m^2 = 1$

$m = 1, 1$

$\therefore$ $y_c = (c_1 + c_2 x) e^x$

Now,

$$P.I = y_p = \frac{1}{f(D)} 2e^x$$

$$= \frac{1}{D^2-1} 2e^x$$

$$= 2\frac{1}{D^2-1} e^x$$

$$2\frac{1}{(D+1)(D-1)} e^x$$

$$= \frac{2}{D+1}\frac{x}{1!} e^x$$

$$= \frac{2}{(1+1)} x e^x$$

$= x e^x$

$\therefore$ The required solution is

$y = y_c + y_p = (c_1 + c_2 x) e^x + xe^x$

Q30. Solve $y'' - 2y' + y = 7e^x$.

Sol :

The given equation is $y'' - 2y' + y = 7e^x$

Then the A.E. is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

Thus the roots are same real roots

$y_c = (c_1 + c_2 x) e^x$

Now

$$P.I = y_p = \frac{1}{f(D)} 7e^x = 7\frac{1}{D^2-2D+1}\bigg|_{D=1} e^x$$

$$= 7\frac{1}{(D-1)^2-2(D-1)+1} e^x$$

$$= 7\frac{1}{D^2+1-2D-2(D-1)+1}$$

$$= 7\frac{1}{D^2+1-2D-2D+2+1} e^x$$

$$= 7\frac{1}{D^2-4D+4} e^x$$

$$= \frac{7}{(D-1)^2} e^x$$

$$= 7 \frac{x^2}{2!} e^x$$

$$y_p = \frac{7x^2}{2} e^x$$

$\therefore$ The required solution is

$y = y_c + y_p$

$$y = (c_1 + c_2 x) e^x + \frac{7}{2} x^2 e^x.$$

Q31. Solve $(D^3 - D^2 - 4D + 4) y = e^{3x}$.

Sol :

The given equation is

$(D^3 - D^2 - 4D + 4) y = e^{3x}$

and the A.E. is $m^2 - m^2 - 4m + 4 = 0$

$(m - 1)(m^2 - 4) = 0$

$m = 1, 2, -2$

Thus C.F $= c_1e^x + c_2e^{2x} + c_3e^{-2x}$

Now

$$P.I = y_p = \frac{1}{D^3 - D^2 - 4D + 4} e^{3x}$$

$$= \frac{1}{(3)^3 - (3)^2 - 4(3) + 4} e^{3x} = \frac{1}{10} e^{3x}$$

$\therefore$ The equation solution is

$y = y_c + y_p$

$$= c_1e^x + c_2e^{2x} + c_3e^{-2x} + \frac{1}{10} e^{3x}$$

Q32. Solve $(D^3 + 3D^2 + 3D + 1) y = e^{-x}$.

Sol :

The given equation is

$(D^3 + 3D^2 + 3D + 1) y = e^{-x}$

and the A.E is $m^3 + 3m^2 + 3m + 1 = 0$

$(m + 1)^3 = 0$

$m = 1, 1, 1$

$\therefore$ The roots are same and real

Thus $y_c = (c_1 + c_2 x + c_3 x^2) e^x$

Now

$$P.I = y_p = \frac{1}{(D+1)^3} e^{-x}$$

$$= \frac{x^3}{3!} e^{-x}$$

$$y_p = \frac{x^3}{6} e^{-x}$$

The required solution is

$y = y_c + y_p$

$$= (c_1 + c_2 x + c_3 x^2) e^x + \frac{x^3}{6} e^{-x}$$

Q33. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 12e^x$.

Sol :

The given equation is

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 12e^x$$

i.e., $(D^2 + 3D + 2) y = 12e^x$

and the A.E is

$m^2 + 3m + 2 = 0$

$m^2 + 2m + m + 2 = 0$

$(m + 2)(m + 1) = 0$

$m = -2, -1$ are the real and distinct roots

Thus $y_c = c_1e^{-x} + c_2e^{-2x}$

Now,

$$P.I = y_p = 12 \frac{1}{f(D)} e^x$$

$$= 12 \frac{1}{D^2 + 3D + 2} e^x$$

(Put D = 1)

$$= 12 \frac{1}{1 + 3(1) + 2} e^x$$

$$= \frac{12}{6} e^x - 2e^x$$

$\therefore$ The required solution is

$y = y_c + y_p$

$y = c_1e^{-x} + c_2e^{-2x} + 2e^x$

3.2.3 When Q(x) = bsinax (or) bcosax

Q34. Derive particular integral when Q(x) = b sinax or bcosax.

Ans :

When f(D)y = Q(x) where Q(x) = bsinax or bcos ax.

Then P.I is $\frac{1}{f(D)}$bsinax or $\frac{1}{f(D)}$bcosax

Case(i)

In this case put $D^2 = -a^2$

Then $y_p = \frac{1}{f(D)}$ bsinax = $\frac{1}{f(-a^2)}$bsinax;

$f(-a^2) \neq 0$

Similarly

$y_p = \frac{1}{f(D)}$bcosax = $\frac{1}{f(-a^2)}$bcosax;

$f(-a^2) \neq 0$

Case(ii)

When $f(-a^2) = 0$

Then $\frac{1}{f(D)}$bcos ax or bcos ax becomes,

we know that

$e^{i\theta} = \cos\theta + i\sin\theta$

let $\theta = ax \Rightarrow e^{iax} = \cos ax + i\sin ax$

$\frac{1}{D^2+a^2}$cosax = Real part of $\frac{1}{D^2+a^2}e^{iax}$

$= \frac{1}{D^2+a^2}e^{iax}$

$= \frac{1}{(D-ai)(D+ai)}e^{iax}$

$= \frac{1}{ai+ai}\frac{x}{1!}e^{iax}$

$= \frac{x}{2ai}\ e^{iax}$

$= \frac{-xi}{2a}(\cos ax + i\sin ax)$

$= \frac{x}{2a}(-i\cos ax + \sin ax)$

Equating the real part, we have

$$\boxed{\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax}$$

Simillarly

$\frac{1}{D^2+a^2}$sinax = Im $\frac{1}{D^2+a^2}e^{iax}$

$\frac{1}{D^2+a^2}e^{iax} = \frac{1}{(D+ai)(D-ai)}e^{iax}$

$= \frac{1}{2ai}\ \frac{x}{1!}e^{iax}$

$= \frac{x}{2ai}e^{iax}$

$= \frac{-xi}{2a}(\cos ax + i\sin ax)$

$\frac{1}{D^2+a^2}e^{iax} = \frac{x}{2a}(-i\cos ax + \sin ax)$

Comparing imaginary part we have

$$\boxed{\frac{1}{D^2+a^2}\sin ax = \frac{-x}{2a}\cos ax}$$

Q35. Solve $(D^2 - 3D + 2)\,y = 3\sin 2x$

Sol :

The given equation is

$(D^2 - 3D + 2)\,y = 3\sin 2x$

Thus the A.E is

$m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-1)(m-2) = 0$

m = 1, 2 which are real and distinct

Thus $y_c = c_1e^x + c_2e^{2x}$

Now,

P.I $= y_p = \frac{1}{f(D)}\ 3\sin 2x$

$= 3\ \frac{1}{D^2-3D+2}\ \sin 2x$

(Put $D^2 = -2^2$)

$$= 3\,\frac{1}{(-2)^2 - 3D + 2}\,\sin 2x$$

$$= 3\,\frac{1}{-4 - 3D + 2}\,\sin 2x$$

$$= 3\,\frac{1}{-2 - 3D}\,\sin 2x$$

Multiply and divide by $3D - 2$

$$= -3\,\frac{1}{(3D+2)} \times \frac{3D-2}{3D^2 - 2}\,\sin 2x$$

$$= \frac{-3(3D-2)}{9D^2 - 4}\,\sin 2x$$

put $D^2 = -2^2$

$$= \frac{-3(3D-2)}{9(-2^2) - 4}\,\sin 2x$$

$$= \frac{-3(3D-2)}{-40}\,\sin 2x$$

$$= \frac{3}{40}\,[3D(\sin 2x) - 2\sin 2x]$$

$$= \frac{3}{40}\,[6\cos 2x - 2\sin 2x]$$

$$= \frac{6}{40}\,[3\cos 2x - \sin 2x]$$

$$= \frac{3}{20}\,[3\cos 2x - \sin 2x]$$

$\therefore$ The required solution is

$y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{3}{20}(3\cos 2x - \sin 2x)$$

Q36. Solve $(D^3 + 1)\,y = \cos 2x$.

Sol :

The given equation is

$(D^3 + 1)\,y = \cos 2x$

and the A.E is

$m^3 + 1 = 9$

$(m + 1)(m^2 - m + 1) = 0$

$m + 1 = 0,\ m^2 - m + 1 = 0$

$$m = -1 \qquad m = \frac{-(-1) \pm \sqrt{1 - 4(1)(1)}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\,i}{2}$$

$$m = -1 \qquad m = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\therefore \quad \text{Thus, } y_c = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

Now,

$$\text{P.I} = y_p = \frac{1}{f(D)}\cos 2x$$

$$= \frac{1}{D^3 + 1}\cos 2x$$

$$= \frac{1}{D.D^2 + 1}\cos 2x$$

Put $D^2 = -2^2$

$$= \frac{1}{D(-2^2) + 1}\cos 2x$$

$$= \frac{1}{-4D + 1}\cos 2x$$

Multiply and divide by $1 - 4D$

$$= \frac{1+4D}{1-4D} \cdot \frac{1}{1-4D}\cos 2x$$

$$= \frac{1+4D}{1-16D^2}\cos 2x$$

$$= \frac{1+4D}{1-16(-2^2)}\cos 2x$$

$$= \frac{1+4D}{1+64}\cos 2x$$

$$= \frac{1}{65}(1 + 4D)\cos 2x$$

$$= \frac{1}{65}[\cos 2x - 4\sin 2x \,.\, 2]$$

$= \frac{1}{65}$ [cos2x – 8 sin2x]

$\therefore$ The required solution is

$y = y_c + y_p$

$= c_1e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$

$+ \frac{1}{65}$ (cos2x – 8 sin2x)

Q37. Solve $(D^3 + D^2 – D – 1)$ y = cos2x.

Sol :

The given equation is

$(D^3 + D^2 - D - 1)\, y = \cos 2x$

and the A.E is $m^3 + m^2 - m - 1 = 0$

If $m = 1 \Rightarrow (-1)^3 + (-1)^2 - (-1) - 1$

$1 + 1 - 1 - 1 = 0$

$$m = \begin{array}{c|cccc} & 1 & 1 & -1 & -1 \\ & 0 & 1 & 2 & 1 \\ \hline & 1 & 2 & 1 & \lfloor 0 \end{array}$$

$m^2 + 2m + 1 = 0$

$(m + 1) = 0$

$m = -1, 1$

$\therefore$ The roots are m = 1, –1, –1

Thus, $y_c = c_1e^x (c_2 + c_3 x)\, e^{-x}$

$P.I = y_p = \frac{1}{D^3 + D^2 - 1} \cos 2x$

$= \frac{1}{D.D^2 + D^2 - 1} \cos 2x$

Put $D^2 = -2^2$

$= \frac{1}{D(-2^2) + (-2^2) - D - 1} \cos 2x$

$= \frac{1}{-4D - 4 - D - 1} \cos 2x$

$= \frac{1}{-5D - 5} \cos 2x$

Multiply and divide by D – 1

$= \frac{-1}{5} \frac{1}{(D+1)} \frac{D-1}{D-1} \cos 2x$

$= \frac{-1}{5} \frac{D-1}{D^2-1} \cos 2x$

Put $D^2 = -2^2$

$= \frac{-1}{5} \frac{D-1}{-2^2-1} \cos 2x$

$= \frac{-1}{5} \frac{D-1}{-5} \cos 2x$

$y_p = \frac{1}{25}$ (–2 sinx2x – cos2x)

The required solution is

$y = y_c + y_p$

$= c_1e^x (c_2 + c_3 x)\, e^{-x} + \frac{1}{25}$ (–2 sin2x – cos2x)

Q38. Solve $(D^4 – 1)$ y = sinx

Sol :

The given equation is

$(D^4 - 1)\, y = \sin x$

and the A.E is $= m^4 - 1 = 0$

$= (m^2)^2 - (1^2) = 0$

$= (m^2 - 1)(m^2 - 1) = 0$

$m = \pm 1, \pm i$

Thus, $y_c = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x$

Now,

$P.I = y_p = \frac{1}{D^4 - 1} \sin x$

$= \frac{1}{(D^2-1)(D^2+1)} \sin x$

Put $D^2 = -1^2$

$= \frac{1}{(-1^2)-1} \frac{1}{D^2+1} \sin x$

$= \frac{-1}{2} \frac{1}{D^2+1} \sin x$

$y_p = \frac{-1}{2}\left[\frac{-x}{2(1)}\cos x\right]$

$\therefore$ The required solution is

$y = y_c + y_p$

$= c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x + \frac{x}{4}\cos x$

Q39. Solve $(D^2 + 4)y = \cos 2x$.

Sol :

The given equation is

$(D^2 + 4)y = \cos 2x$

& the A.E. is $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

Thus, $y_c = c_1\cos 2x + c_2\sin 2x$

Now

$$P.I = y_p = \frac{1}{D^2+4}\cos 2x$$

$$= \frac{x}{2(2)}\sin 2x$$

$$y_p = \frac{x}{4}\sin 2x$$

$\therefore$ The required solution is

$$y = c_1\cos 2x + c_2\sin 2x + \frac{x}{4}\sin 2x$$

Q40. Solve $\frac{d^2y}{dx^2} - y = \sin x$.

Sol :

The given equation is $\frac{d^2y}{dx^2} - y = \sin x$.

and the A.E is $m^2 - 1 = 0$

$(m-1)(m+1) = 0$

$m = 1, -1$

$\therefore$ $y_c = c_1e^x + c_2e^{-x}$

Now

$$P.I = y_p = \frac{1}{D^2-1}\sin x$$

Put $D^2 = -1^2$

$$= \frac{1}{-1^2-1}\sin x = \frac{-1}{2}\sin x$$

$\therefore$ The required solution is

$y = y_c + y_p$

$= c_1e^x + c_2e^{-x} - \frac{1}{2}\sin x.$

Q41. Solve $\frac{d^2y}{dx^2} - y = \cos x$.

Sol :

The given equation is $\frac{d^2y}{dx^2} - y = \cos x$

and the A.E is $m^2 - 1 = 0$

$m = \pm 1$

$\therefore$ $y_c = c_1e^x + c_2e^{-x}$

Now

$$P.I = y_p = \frac{1}{D^2-1}\cos x$$

Put $D^2 = -1$

$$= \frac{1}{-1^2-1}\cos x$$

$$= \frac{-1}{2}\cos x$$

$\therefore$ $y = y_c + y_p$

$$y = c_1e^x + c_2e^{-x} - \frac{1}{2}\cos x$$

which is required solution.

Q42. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3\sin x$.

Sol :

The given equation is $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3\sin x$.

and the A.E is $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-4(1)(2)}}{2}$$

$$= \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$$

$m = 1 \pm i$

$\therefore \quad y_c = e^x (c_1\cos x + c_2 \sin x)$

Now,

$$P.I = y_p = \frac{1}{D^2 + 2D + 2} 3\sin x$$

Put $D^2 = -1^2$

$$= 3 \frac{1}{(-1^2) - 2D + 2} \sin x$$

$$= \frac{3}{-1 - 2D + 2} \sin x$$

$$= \frac{3}{2D + 1} \sin x$$

Multiply and divide by 2D – 1

$$= \frac{3}{2D+1} \cdot \frac{2D-1}{2D-1} \sin x$$

$$= \frac{3}{4D^2 - 1} (2D - 1) \sin x$$

Put $D^2 = -1^2$

$$= \frac{3}{4(-1^2) - 1} (2D - 1) \sin x$$

$$y_p = \frac{3}{-5} (2\cos x - \sin x)$$

$\therefore$ The required slution is

$$y = e^x (c_1\cos x + c_2 \sin x) - \frac{3}{5} (2\cos x - \sin x)$$

Q43. Solve $y'' + 3y' + 2y = 8 + 6e^x + 2\sin x$.

Sol :

The given equation is $y'' + 3y' + 2y = 8 + 6e^x + 2\sin x$

and the A.E is $m^2 + 3m + 2 = 0$

$(m + 1)(m + 2) = 0$

$m = -1, -2$

$\therefore \quad y_c = c_1e^{-x} + c_2e^{-2x}$

Now,

$$P.I = y_p = \frac{1}{D^2 + 3D + 2} 8 + 6e^x + 2\sin x$$

$$= \frac{1}{D^2+3D+2} 8x^0 + \frac{1}{D^2 + 3D + 2} e^x + 2 \frac{1}{D^2 + 3D + 2} \sin x$$

Put $D^2 = -1$ Put $D^2 = -1^2$

$$= \frac{1}{2\left[1+\frac{3D}{2}+\frac{D^2}{2}\right]} x^0 + 6 \frac{1}{(1)^2+3(1)+2} e^x + 2 \frac{1}{-1^2+3D+2} \sin x$$

$$= \frac{1}{2}\left[1+\left(\frac{3D}{2}+\frac{D^2}{2}\right)\right]^{-1} x^0 + \frac{6}{6} e^x + \frac{1}{3D+1} \sin x$$

$$= \frac{1}{2}[1] + e^x + \frac{3D-1}{9D^2-1} \sin x$$

Put $D^2 = -1^2$

$$= \frac{1}{2} + e^x + \frac{3D-1}{-10} \sin x$$

$$y_p = \frac{1}{2} + e^x - \frac{3}{10}\cos x + \frac{1}{10}\sin x$$

$\therefore$ The required solution is

$$y = c_1e^{-x} + c_2e^{-2x} + \frac{1}{2} + e^x - \frac{3}{10}\cos x + \frac{1}{10}\sin x$$

Q44. Solve $y'' + 3y' + 2y = 2(e^{-2x} + x^2)$

Sol :

The given equation is $y'' + 3y' + 2y = 2(e^{-2x} + x^2)$

and the A.E. is $m^2 + 3m + 2 = 0$

$(m + 2)(m + 1) = 0$

$m = -2, -1$

$\therefore$ The roots ae $m = -1, -2$

Thus, $y_c = c_1e^{-2x} + c_2e^{-2x}$

$$\text{P.I} = y_p = \frac{1}{D^2+3D+2} 2e^{-2x} + 2x^2$$

$$= 2 \frac{1}{D^2+3D+2} e^{-2x} + 2 \frac{1}{D^2+3D+2} x^2$$

$$= 2 \frac{x^2}{2!} e^{-2x} + \frac{1}{1+\left(\frac{3D}{2}+\frac{D^2}{2}\right)} x^2$$

$$= 2 \frac{x^2e^{-2x}}{2!} + \left[1+\left(\frac{3D}{2}+\frac{D^2}{2}\right)\right]^{-1} x^2$$

$$= x^2 e^{-2x} + \left[1-\left(\frac{3D}{2}+\frac{D^2}{2}\right)+\left(\frac{3D}{2}+\frac{D^2}{2}\right)^2\right] x^2$$

$$= x^2 e^{-2x} + x^2 - \frac{3(2x)}{2} - \frac{2}{2} + \frac{9(2)}{4}$$

$$= x^2 e^{-2x} + x^2 - 3x - 1 + \frac{9}{2}$$

$$y_p = x^2 - 3x + \frac{7}{2} + x^2 e^{-2x}$$

$\therefore$ The required solution is

$$y = c_1 e^{-2x} + c_2 e^{-2x} + x^2 - 3x + \frac{7}{2} + x^2 e^{-2x}$$

Q45. Solve $(D^2 + 6D + 9)\, y = 2e^{-3x}$.

Sol :

The given equation is $(D^2 + 6D + 9)\, y = 2e^{-3x}$

and the A.E is $m^2 + 6m + 9 = 0$

$m^2 + 3m + 3m + 9 = 0$

$(m + 3)(m + 3) = 0$

$\therefore$ The roots are $m = -3, -3$

Thus $y_c = (c_1 + c_2 x)\, e^{-3x}$

Now,

$$\text{P.I} = y_p = \frac{1}{D^2 + 6D + 9} 2e^{-3x}$$

$$= \frac{2e^{-3x}}{(D-3)^2 + 6(D-3) + 9}$$

$$= 2e^{-3x} \frac{1}{D^2 + 9 - 6D + 6D - 18 + 9}$$

$$= 2e^{-3x} \frac{1}{D^2} (1)$$

$$= 2e^{-3x} \left(\frac{x^2}{2}\right)$$

$$y_p = e^{-3x} x^2$$

$\therefore$ The required solution is

$$y = (c_1 + c_2 x)e^{-3x} + x^2 e^{-3x}.$$

Q46. Solve $(D^2 + 4D + 4)\, y = e^{2x} - e^{-2x}$.

Sol :

The given equation is $(D^2 + 4D + 4)\, y = e^{2x} - e^{-2x}$

and the A.E. is $m^2 + 4m + 4 = 0$

$(m + 1)^2 = 0$

$m = -2, -2$

Thus,

$y_c = (c_1 + c_2 x)\, e^{-2x}$

Now

$$P.I = y_p = \frac{1}{D^2+4D+4}\, e^{2x} - \frac{1}{D^2+4D+4}\, e^{-2x}$$

$$= \frac{1}{2^2+4(2)+4}\, e^{2x} - \frac{1}{(D-2)^2+4(D-2)+4}\ (1)$$

$$= \frac{1}{16}\, e^{2x} - e^{-2x}\, \frac{1}{D^2+4-4D+4D-8+4}$$

$$= \frac{1}{16}\, e^{2x} - e^{-2x}\, \frac{1}{D^2}\ (1)$$

$$y_p = \frac{1}{16}\, e^{2x} - e^{-2x}\, \frac{x^2}{2}$$

$\therefore$ The required solution is

$$y = (c_1 + c_2 x)\, e^{-2x} + \frac{1}{16}\, e^{2x} - e^{-2x}\, \frac{x^2}{2}$$

Q47. Solve $(D^2 - 4D + 4)\, y = 8(x^2 + e^{2x} + \sin 2x)$.

Sol :

The given equation is $(D^2 - 4D + 4)\, y = 8(x^2 + e^{2x} + \sin 2x)$.

The A.E is $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

Thus,

$y_c = (c_1 + c_2 x)\, e^{2x}$

Now

$$P.I = y_p = \frac{1}{D^2-4D+4}\, 8(x^2 + 8e^{2x} + 8\sin 2x)$$

$$y_p = \frac{1}{(D-2)^2}\, 8x^2 + 8\, \frac{1}{(D-2)^2}\, e^{2x} + 8\, \frac{1}{(D-2)^2}\, \sin 2x$$

$$= 8\, \frac{1}{4\left[1-\frac{D}{2}\right]^2}\, x^2 + 8\, \frac{1}{(D+2-2)^2}\, e^{2x} + 8\, \frac{1}{D^2+4-4D}\, \sin 2x$$

$$= 2\left[1-\frac{D}{2}\right]^{-2} x^2 + \frac{8}{D^2} e^{2x} + 8\frac{1}{-4+4-4D} \sin 2x$$

$$= 2\left[1+2\left(\frac{D}{2}\right)+3\left(\frac{D}{2}\right)^2\right]x^2+8e^{2x}\frac{1}{D}(1)-\frac{8}{40}\sin 2x$$

$$= 2\left[x^2+2x+\frac{3}{4}(2)\right]+8e^{2x}x^2.\frac{x^2}{2}+\frac{2\cos 2x}{2}$$

$y_p = 2x^2 + 4x + 3 + 4e^{2x}x^2 + \cos 2x$

$\therefore$ The required solution is

$y = (c_1 + c_2 x) e^{2x} + 2x^2 + 4x + 3 + 4e^{2x} x^2 + \cos 2x.$

3.2.4 When $Q(x) = e^{ax} v$ where v is function of x

Q48. Working Rule for $Q(x) = e^{ax} v$ is a function of x.

Ans :

To find particular integral when Q(x) is contains $e^{ax} v$ where v is a function of x [x may be any polynomial or sinax, cosax]

follows the below steps

Step (i) Write in $\frac{1}{f(D)} e^{ax}v$

Step (ii) Substitute D = a in the given f(D)

$\Rightarrow e^{ax}\frac{1}{f(a)}v$

Step (iii) Check the function of x and as per the rules continue the derivation to get P.I.

49. Solve $(D^2 + 4D - 12) y = (x - 1) e^{2x}$.

Sol :

The given equation is

$(D^2 + 4D - 12) y = (x - 1) e^{2x}$

and the A.E is $m^2 + 4m - 12 = 0$

$m^2 + 6m - 2m - 12 = 0$

$m(m + 6) - 2(m + 6) = 0$

$m = -6, 2$

$\therefore$ $y_c = c_1 e^{2x} + c_2 e^{-6x}$

Now,

$$P.I = y_p = \frac{1}{D^2+4D-12}(x-1)e^{2x}$$

$$= e^{2x}\frac{1}{(D+2)^2+4(D+2)-12}(x-1)$$

$$= e^{2x}\frac{1}{D^2+4+4D+4D+8-12}(x-1)$$

$$= e^{2x} \frac{1}{D^2 + 8D} (x - 1)$$

$$= e^{2x} \frac{1}{8D\left(1 + \frac{D}{8}\right)} (x - 1)$$

$$= e^{2x} \frac{1}{8D} \left[1 + \frac{D}{8}\right]^{-1} (x - 1)$$

$$= \frac{e^{2x}}{8D} \left(1 - \frac{D}{8} + \frac{D^2}{64} + ...\right) (x - 1)$$

$$= \frac{e^{2x}}{8D} \left(x - 1 - \frac{1}{8}\right)$$

$$= \frac{e^{2x}}{8D} \left(x - \frac{9}{8}\right) \Rightarrow \frac{e^{2x}}{8} \left[\frac{x^2}{2} - \frac{9}{8}x\right]$$

$$= \frac{e^{2x}}{8} \left[\frac{4x^2 - 9x}{8}\right]$$

$$y_p = \frac{1}{64} e^{2x} [4x^2 - 9x]$$

$\therefore$ The required solution is

$$c_1 e^{2x} + c_2 e^{-6x} + \frac{1}{64} e^{2x} (4x^2 - 9x)$$

Q50. Solve **$(D^2 - 2D + 1) y = e^x x^2$.**

Sol :

The given equation is $(D^2 - 2D + 1) y = e^x x^2$

The A.E. is $m^2 - 2m + 1 = 0$

$(m - 1)^2 = 0$

$m = 1, 1$

Thus $y_c = (c_1 + c_2 x) e^x$

Now,

$$\text{P.I} = y_p = \frac{1}{D^2 - 2D + 1} e^x x^2$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x^2$$

$$= e^x \frac{1}{D^2 + 1 + 2D - 2D - 2 + 1} x^2$$

$$= e^x \frac{1}{D^2}(x^2)$$

$$y_p = e^x \frac{x^4}{12}$$

$\therefore$ The required solution is

$$y = (c_1 + c_2 x)e^x + \frac{1}{12} e^x e^4$$

Q51. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 3x^2 e^x$.

Sol :

The given equation is $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 3x^2 e^x$

and the A.E. is $m^2 + m + 1 - 0$

$$m = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

$$\therefore m = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

Thus $y_c = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$

Now,

$$\text{P.I} = y_p = \frac{1}{D^2 + D + 1} 3x^2 e^x$$

$$= 3e^x \frac{1}{(D+1)^2 + (D+1) + 1} x^2$$

$$= 3e^x \frac{1}{D^2 + 1 + 2D + D + 1 + 1} x^2$$

$$= 3e^x \frac{1}{D^2 + 3D + 3} x^2$$

$$= 3e^x \frac{1}{3\left[1 + D + \frac{D^2}{3}\right]} x^2$$

$$= e^x \left[1 + \left(D + \frac{D^2}{3}\right)\right]^{-1} x^2$$

$$= e^x \left[1 - \left(D + \frac{D^2}{3}\right) + \left(D + \frac{D^2}{3}\right)^2\right] x^2$$

$$= e^x \left[x^2 - 2x - \frac{2}{3} - 2\right]$$

$$= e^x \left[x^2 - 2x - \frac{8}{3}\right]$$

$$y_p = \frac{1}{3} e^x (3x^2 - 6x - 8)$$

$\therefore$ The required solution is

$$y = e^{-x/2} \left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{3} e^x (3x^2 - 6x - 8)$$

Q52 Solve $(D^2 - 2D + 5)\, y = e^{2x} \sin x$.

Sol :

The given equation is $(D^2 - 2D + 5)\ y = e^{2x} \sin x$

and the A.E. is $m^2 - 2m + 5 = 0$

$$m = \frac{2 \pm \sqrt{4 - 4(1)(8)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$

$$m = 1 \pm 2i$$

$\therefore$ $y_c = e^x (c_1 \cos 2x + c_2 \sin 2x)$

Now,

$$P.I = y_p = \frac{1}{D^2 - 2D + 5} e^{2x} \sin x$$

put $D^2 = D + 2$

$$= e^{2x} \frac{1}{(D+2)^2 - 2(D+2) + 5} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 4 + 4D - 2D - 4 + 5} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 2D + 5} \sin x$$

put $D^2 = -1^2$

$$= e^{2x} \frac{1}{-1^2 + 2D + 5} \sin x$$

$$= e^{2x} \frac{1}{2D + 4} \sin x$$

$$= e^{2x} \frac{1}{2D + 4} \times \frac{2D - 4}{2D - 4} \sin x$$

$$= e^{2x} \frac{2D - 4}{4D^2 - 16} \sin x$$

$$= \frac{e^{2x}}{-4 - 16} (2D - 4) \sin x$$

$$= \frac{e^{2x}}{-20} (2 \cos x - 4 \sin x)$$

$$y_p = \frac{e^{2x}}{-10} (\cos x - 2\sin x)$$

∴ The required solution is

$$y = e^x (c_1 \cos 2x + c_2 \sin 2x) - \frac{e^{2x}}{10} (\cos x - 2\sin x)$$

Q53. Solve $4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} = x^2 e^x$.

Sol :

The given equation is $(4D^2 - 5D)\, y = x^2 e^x$

and the A.E is $4m^2 - 5m = 0$

$$m^2 - \frac{5}{4} m = 0$$

$$m\left(m - \frac{5}{4}\right) m = 0$$

$$m = 0, \; m = \frac{5}{4}$$

∴ $y_c = c_1 + c_2\, e^{5/4 x}$

Now

$$\text{P.I} = y_p = \frac{1}{4D^2 - 5D} x^2 e^x$$

$$= e^x \frac{1}{4(D+1)^2 - 5(D+1)} x^2$$

$$= e^x \frac{1}{4(D^2+1+2D)-5D-5} x^2$$

$$= e^x \frac{1}{4D^2+4+8D-5D-5} x^2$$

$$= e^x \frac{1}{4D^2+3D-1} x^2$$

$$= \frac{e^x}{-1}\left[\frac{1}{1-3D-4D^2}\right] x^2$$

$= e^x [1 - (3D + 4D^2)]^{-1} x^2$

$= e^x [1 + (3D + 4D^2) + (3D + 4D^2)^2] x^2$

$= e^x [x^2 + 3(2x) + 4(2) + 9(2)]$

$= e^x [x^2 + 6x + 8 + 18]$

$y_p = e^x [x^2 + 6x + 26]$

$\therefore$ The required solution is

$y = c_1 + c_2 e^{5/4 x} + x^2 [x^2 + 6x + 26]$

Q54. Solve $(D^2 + 1) y = xe^{2x}$.

Sol :

The given solution is $(D^2 + 1) y = xe^{2x}$

The A.E. is $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

$\therefore$ $y_c = c_1 \cos x + c_2 \sin x$

Now

$$\text{P.I} = y_p = \frac{1}{D^2+1} xe^{2x}$$

$$= e^{2x} \frac{1}{(D+1)^2+1} x$$

$$= e^{2x} \frac{1}{D^2+4+4D+1} x$$

$$= e^{2x} \frac{1}{D^2+4D+5} x$$

$$= \frac{e^{2x}}{5}\left[1+\left(\frac{4D}{5}+\frac{D^2}{5}\right)\right]^{-1} x$$

$$= \frac{e^x}{5}\left[1-\left(\frac{4D}{5}+\frac{D^2}{5}\right)+\left(\frac{4D}{5}+\frac{D^2}{5}\right)^2\right] x$$

$$= \frac{e^x}{5}\left[x - \frac{4}{5}(1)\right]$$

$$y_p = \frac{1}{25} e^x [5x - 4]$$

$\therefore$ The required solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{25} e^x [5x - 4]$$

Q55. Solve $(D^2 + 1) y = e^{-x} + \cos x + x^3 + e^x \cos x$.

Sol :

The given equation is

$$(D^2 + 1) y = e^{-x} + \cos x + x^3 + e^x \cos x.$$

and the A,E is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$\therefore$ $y_c = c_1 \cos x + c_2 \sin x$

Now

$$P.I = y_p = \frac{1}{D^2+1} e^{-x} + \cos x + x^3 + e^x \cos x$$

$$= \frac{1}{D^2+1} e^{-x} + \frac{1}{D^2+1} \cos x + \frac{1}{D^2+1} x^3 + e^x \frac{1}{(D^2+1)^2+1} \cos x$$

$$= \frac{1}{(-1)^2+1} e^{-x} + \frac{x}{2} \sin x + \frac{1}{D[1+D^2]} x^3 + e^x \frac{1}{D^2+1+2D+1} \cos x$$

$$= \frac{e^{-x}}{2} + \frac{x}{2} \sin x + [1 + D^2]^{-1} x^3 + e^x \frac{1}{D^2+2D+2} \cos x$$

$$= \frac{e^{-x}}{2} + \frac{x}{2} \sin x + [1 - (D^2) + (D^2)^2] x^3 + e^x \frac{1}{-1^2+2D+2} \cos x$$

$$= \frac{e^{-x}}{2} + \frac{x}{2} \sin x + [x^3 - 6x] + e^x \frac{1}{2D+2} \cos x$$

$$= \frac{e^{-x}}{2} + \frac{x}{2} \sin x + [x^3 - 6x] + e^x \frac{2D-1}{4D^2-1} \cos x$$

$$= \frac{e^{-x}}{2} + \frac{x}{2} \sin x + [x^3 - 6x] + e^x \frac{2D-1}{4(-1)^2-1} \cos x$$

$$= \frac{e^{-x}}{2} + \frac{x}{2} \sin x + [x^3 - 6x] - \frac{e^x}{5} [2(-\sin x) - 2\cos x)$$

$$y_p = \frac{e^{-x}}{2} + \frac{x}{2} \sin x + [x^3 - 6x] + \frac{e^x}{5} [2\sin x + \cos x)$$

$\therefore$ The required solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{e^{-x}}{2} + \frac{x}{2}\sin x + (x^3 - 6x) + \frac{e^x}{5}[2\sin x + \cos x]$$

Q56. Solve $(D^2 + 1)\, y = \cos x + xe^{2x} + e^x \sin x$.

Sol :

The given equation is

$(D^2 + 1)\, y = \cos x + xe^{2x} + e^x \sin x$

and the A.E is $m^2 + 1 = 0$

$m^2 = -1 \Rightarrow m = \pm i$

$\therefore \quad y_c = c_1 \cos x + c_2 \sin x$

Now

$$\text{P.I} = y_p = \frac{1}{D^2+1}\cos x + xe^{2x} + e^x \sin x$$

$$= \frac{1}{D^2+1}\cos x + \frac{1}{D^2+1}xe^{2x} + \frac{1}{D^2+1}e^x \sin x$$

put $D = -1$ put $D = D + 2$ put $D = D + 1$

$$= \frac{x}{2}\sin x + e^{2x}\frac{1}{(D+2)^2+1}x + e^x\frac{1}{(D+1)^2+1}\sin x$$

$$= \frac{x}{2}\sin x + e^{2x}\frac{1}{D^2+4+4D+1}x + e^x\frac{1}{D^2+1+2D+1}\sin x$$

$$= \frac{x}{2}\sin x + e^{2x}\frac{1}{D^2+4D+5}x + e^x\frac{1}{D^2+2D+2}\sin x$$

$$= \frac{x}{2}\sin x + \frac{e^{2x}}{5}\left[1+\left(\frac{4D}{5}+\frac{D^2}{5}\right)\right]^{-1}x + e^x\frac{1}{-1^2+2D+2}\sin x$$

$$= \frac{x}{2}\sin x + \frac{e^{2x}}{5}\left[1-\left(\frac{4D}{5}+\frac{D^2}{5}\right)\right]x + e^x\frac{1}{2D+1}\sin x$$

$$= \frac{x}{2}\sin x + \frac{e^{2x}}{5}\left[x-\frac{4}{5}(1)\right] + e^x\frac{2D-1}{4D^2-1}\sin x$$

$D^2 = -1^2)$

$$= \frac{x}{2}\sin x + \frac{e^{2x}}{5}\left(x-\frac{4}{5}\right) + e^x\frac{2D-1}{-5}\sin x$$

$$y_p = \frac{x}{2}\sin x + \frac{e^{2x}}{5}\left(x-\frac{4}{5}\right) - \frac{e^x}{5}[2\cos x - \sin x]$$

$\therefore$ The required solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{x}{2}\sin x + \frac{e^{2x}}{5}\left(x-\frac{4}{5}\right) - \frac{e^x}{5}[2\cos x - \sin x]$$

3.2.5 When $Q(x) = x\nu$ where ν is any function of x

To find y_p, where Q(x) is consists of xv where v is any function of x
Then y_p is

Here $$y_p = \frac{1}{f(D)}(x\nu) = x\frac{1}{f(D)}\nu - \frac{f'(D)}{[f(D)]^2}\nu$$

Q57. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\sin x.$

Sol :

The given equation is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\sin x$

and the A.E. is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$\therefore \quad y_c = (c_1 + c_2 x)e^x$

Now,

$$P.I = y_p = \frac{1}{D^2 - 2D + 1} x\sin x$$

$$= x\frac{1}{D^2 - 2D + 1}\sin x - \frac{2D-2}{(D^2 - 2D + 1)^2}\sin x$$

$$= x\frac{1}{-1^2 - 2D + 1}\sin x - \frac{2D-2}{(-1^2 - 2D + 1)^2}\sin x$$

$$= x\frac{1}{-2D}\sin x - \frac{2D-2}{(-2D)^2}\sin x$$

$$= \frac{-x}{2}\frac{1}{D}\sin x - \frac{2D-2}{4D^2}\sin x$$

$$= \frac{x}{2}\cos x - \frac{2}{4D^2}[\cos x - \sin x]$$

$$= \frac{x}{2}\cos x - \frac{1}{2}[-\cos x + \sin x]$$

$$y_p = \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$$

$\therefore$ The required solution is

$$y = (c_1 + c_2 x)e^x + \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$$

Q58. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x\cos x$.

Sol :

The given equation is $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x\cos x$

and the A.E. is $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$m = -1, -1$

$\therefore \quad y_c = (c_1 + c_2 x)\, e^{-x}$

Now,

$$P.I = y_p = \frac{1}{D^2 + 2D + 1}\, x\cos x$$

$$= x\frac{1}{D^2 + 2D + 1}\cos x - \frac{2D - 2}{(D^2 + 2D + 1)^2}\cos x$$

$$= x\frac{1}{-1^2 + 2D + 1}\cos x - \frac{2D + 2}{(-1^2 + 2D + 1)^2}\cos x$$

$$= x\frac{1}{2D}\cos x - \frac{2D + 2}{4D^2}\cos x$$

$$= \frac{x}{2}\sin x - \frac{2}{4D^2}[-\sin x + \cos x]$$

$$y_p = \frac{x}{2}\sin x - \frac{1}{2}[\sin x + \cos x]$$

$\therefore$ The required solution is

$$y = (c_1 + c_2 x)\, e^{-x} + \frac{x}{2}\sin x - \frac{1}{2}[\sin x + \cos x]$$

Q59. Solve $(D^2 - 2D + 1)\, y = xe^x \sin x$.

Sol :

The given equation is $(D^2 - 2D + 1)\, y = xe^x \sin x$

and the A.E. is $m^2 - 2m + 1 = 0$

$(m - 1)^2 = 0$

$m = 1, 1$

$\therefore \quad y_c = (c_1 + c_2 x)\, e^x$

Now,

$$P.I = y_p = \frac{1}{D^2 - 2D + 1}\, xe^x \sin x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} \text{ x sinx}$$

$$= e^x \frac{1}{D^2 + 1 + 2D - 2D - 2 + 1} \text{ x sinx}$$

$$= e^x \left[\frac{1}{D^2}\right] \text{ x sinx}$$

$y_p = e^x$ (–x sinx – 2cosx)

$\therefore$ The required solution is

$y = (c_1 + c_2 x) e^x + e^x$ (–x sinx – 2cosx)

Q60. Solve $(D^2 + 1)$ y = x^2 sin2x.

Sol :

The given equation is $(D^2 + 1)$ y = x^2 sin2x.

and the A.E. is $m^2 + 1 = 0$

$m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

Now,

$$\text{P.I} = y_p = \frac{1}{D^2 + 1} x^2 \sin 2x$$

$$= \text{Im} \frac{1}{D^2 + 1} x^2 e^{2ix}$$

$$= \text{Im } e^{2ix} \frac{1}{(D + 2i)^2 + 1} x^2$$

$$= \text{Im of } e^{2ix} \frac{1}{D^2 - 4 + 4iD + 1} x^2$$

$$= \text{Im } e^{2ix} \frac{1}{-3\left[1 - \frac{4}{3} iD - \frac{1}{3} D^2\right]} x^2$$

$$= \text{Im } \frac{e^{2ix}}{-3} \left[1 - \left(\frac{4iD + D^2}{9}\right)\right]^{-1} x^2$$

$$= \text{Im } \frac{e^{2ix}}{-3} \left[1 + \frac{4iD}{3} - \frac{13}{9} D^2\right] x^2$$

$$= \text{Im } \frac{e^{2ix}}{-3} \left[x^2 + \frac{4i2x}{3} - \frac{13}{9}(2)\right]$$

$$= \text{Im}\ \frac{-1}{3}\ (\cos 2x + \sin 2x)\left[\left(x^2 - \frac{26}{9}\right) + i\frac{8}{3}x\right]$$

$$= \frac{-1}{3}\ \frac{8}{3}\ x\cos 2x - \frac{1}{3}\left(x^2 - \frac{26}{9}\right)\sin 2x$$

Q61. Solve $(D^2 - 4D + 4)\,y = x^2 + e^x + \sin 2x$.

Sol :

The given equation is $(D^2 - 4D + 4)\,y = x^2 + e^x + \sin 2x$

and the A.E is $m^2 - 4m + 4 = 0$

$(m - 2)^2 = 0$

$m = 2, 2$

$\therefore \quad y_c = (c_1 + c_2 x)\,e^{2x}$

Now,

$$\text{P.I} = y_p = \frac{1}{D^2 - 4D + 4}\,x^2 + e^x + \sin 2x$$

$$= \frac{1}{D^2 - 4D + 4}\,x^2 + \frac{1}{D^2 - 4D + 4}\,e^x + \frac{1}{D^2 - 4D + 4}\,\sin 2x$$

$D = 1 \qquad D^2 = -2^2$

$$= \frac{1}{4\left[1 - \frac{D}{2}\right]^2}\,x^2 + \frac{1}{1 - 4(1) + 4}\,e^x + \frac{1}{-2^2 - 4D + 4}\,\sin 2x$$

$$= \frac{1}{4}\left[1 - \frac{D}{2}\right]^{-2} x^2 + e^x - \frac{1}{4D}\,\sin 2x$$

$$= \frac{1}{4}\left[1 + D + \frac{3}{4}D^2\right] x^2 + e^x - \frac{1}{4}\left(\frac{-\cos 2x}{2}\right)$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{4}(2)\right] + e^x + \frac{1}{8}\cos 2x$$

$$y_p = \frac{1}{4}\left[x^2 + 2x + \frac{3}{4}\right] + e^x + \frac{1}{8}\cos 2x$$

$\therefore$ The required solution is

$$y = (c_1 + c_2 x)\,e^{2x} + \frac{1}{4}\left[x^2 + 2x + \frac{3}{4}\right] + e^x + \frac{1}{8}\cos 2x$$

Q62. Solve $(D^5 - D)\,y = 12e^x + 8\sin x - 2x$.

Sol :

The given equation is $(D^5 - D)\,y = 12e^x + 8\sin x - 2x$

and the A.E is $m^5 - m = 0$

$m(m^4 - 1) = 0$

$m = 0, \ (m^2)^2 - (1^2)^2 = 0$

$$(m^2 - 1)(m^2 + 1) = 0$$

$$m = 0, \; m = \pm 1 \quad \pm i$$

$\therefore \quad y_c = c_1 + c_2 e^x + c_3 e^{-x} + c^4 \cos x + c^5 \sin x$

Now,

$$P.I = y_p = \frac{1}{D^5 - D} 12e^x + 8\sin x - 2x$$

$$= 12 \frac{1}{(D-1)D(D+1)(D^2+1)} e^x + 8 \frac{1}{(D-1)D(D+1)(D^2+1)} \sin x - 2 \frac{1}{(D-1)D(D+1)(D^2+1)} x$$

P.I. to corresponding $12e^x$

$$\Rightarrow \quad 12 \frac{1}{(D-1)D(D+1)(D^2+1)} e^x = 12 \frac{1}{(D-1)1(1+1)(1+1)} e^x$$

$$= 12 \frac{1}{4(D-1)} e^x$$

$$= 3 \frac{x}{1!} e^x$$

P.I. corresponding 8sinx

$$\Rightarrow \quad 8 \frac{1}{(D-1)D(D+1)(D^2+1)} 8\sin x = 8 \frac{1}{(D^2-1)D(D^2+1)} 8\sin x$$

$$= 8 \frac{1}{(D^2+1)D(-1^2-1)} \sin x$$

$$= \frac{8}{-2} \frac{1}{(D^2+1)D} \sin x$$

$$= -4 \frac{1}{D^2+1} \left[\frac{1}{D} \sin x\right]$$

$$= 4 \frac{1}{D^2+1} \cos x$$

$$= 4 \frac{x}{2} \sin x = 2x \sin x$$

P.I. corresponding to (–2x)

$$\Rightarrow \quad -2 \frac{1}{D(D^2-1)(D^2+1)} x = 2 \frac{1}{D(1-D^2)(1+D^2)} x = 2\frac{1}{D} (1 - D^2)^{-1} (1 + D^2)^{-1} x$$

$$= 2 \frac{1}{D} (1 + D^2)(1 - D^2) x = 2 \frac{1}{D} [1 + D^2 - D^2 + \ldots] x$$

$$= 2\,\frac{1}{D}\,x$$

$$= 2\,\frac{x^2}{2} = x$$

$\therefore y_p = 3x\,e^x + 2x\sin x + x$

$\therefore$ The required solution is

$y = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x + 3x\,e^x + 2x\sin x + x$

Q63. Solve $(D^2 + 1)\,y = x^2 \sin 2x$.

Sol :

The given equation is $(D^2 + 1)\,y = x^2 \sin 2x$

and the A.E is $m^2 + 1 = 0$

$m = \pm\, i$

$y_c = c_1 \cos x + c_2 \sin x$

Now,

$$\text{P.I} = y_p = \frac{1}{D^2+1}\,x^2 \sin 2x$$

$$= \text{Im}\,\frac{1}{D^2+1}\,x^2 e^{2i}x$$

$$= \text{Im}\,e^{2ix}\,\frac{1}{(D+2i)^2+1}\,x^2$$

$$= \text{Im}\,e^{2ix}\,\frac{1}{-3\left(1-\frac{4}{3}iD-\frac{1}{3}D^2\right)}\,x^2$$

$$= \text{Im}\,\frac{e^{2ix}}{-3}\left[1-\left(\frac{4iD+D^2}{3}\right)\right]^{-1} x^2$$

$$= \text{Im}\,\frac{e^{2ix}}{-3}\left(1+\frac{4iD}{3}-\frac{13}{7}D^2+\ldots\right) x^2$$

$$= \text{Im}\,\frac{e^{2ix}}{-3}\left(x^2+\frac{4i}{3}(2x)-\frac{13}{9}(2)\right)$$

$$= \text{Im}\,\frac{1}{-3}\,(\cos 2x + i\sin 2x)\left[\left(x^2-\frac{26}{9}\right)+i\frac{8}{3}x\right]$$

$$y_p = \frac{-1}{3}\cdot\frac{8}{3}\cos 2x - \frac{1}{3}\left(x^2-\frac{26}{9}\right)\sin 2x$$

∴ The required solution is

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{27}[24x \cos 2x + (9x^2 - 26)\sin 2x]$$

Q64. Solve $y'' - y' - 2y = e^x$

Sol :

The given equation is $y'' - y' - 2y = e^x$

and the A.E. is $m^2 - m - 2 = 0$

$m^2 - 2m + m - 2 = 0$

$m(m - 2) + 1(m - 2)$

$(m + 1)(m - 2) = 0$

$m = -1, 2$

∴ $y_c = c_1 e^{-x} + c_2 e^{2x}$

Now,

$$P.I = y_p = \frac{1}{D^2 - D - 2} e^x$$

$D = 1$

$$= \frac{1}{1^2 - 1 - 2} e^x$$

$$y_p = \frac{-1}{2} e^x$$

∴ The required solution is

$$y = c_1 e^{-x} + c_2 e^{2x} - \frac{1}{2} e^x$$

Q65. Solve $y'' + 3y' + 2y = 12e^x$

Sol :

The given equation is $y'' + 3y' + 2y = 12e^x$

and the A.E. is $m^2 + 3m + 2 = 0$

$m^2 + 2m + m + 2 = 0$

$(m + 2)(m + 1) = 0$

$m = -2, 1$

∴ $y_c = c_1 e^{-2x} + c_2 e^x$

Now,

$$P.I = y_p = \frac{1}{D^2 + 3D + 2} 12e^x$$

$$= 12 \frac{1}{D^2 + 3D + 2} e^x$$

$D = 1$

$$= 12 \frac{1}{1 + 3(1) + 2} e^x$$

$$= \frac{12}{6} e^x = 2e^x$$

$\therefore \quad y_p = 2e^x$

$\therefore$ The required solution is

$$y = c_1 e^{-2x} + c_2 e^x + 2e^x$$

Q66. Solve $(D^2 - 2D - 8)\, y = 9xe^x + 10e^{-x}$

Sol :

The given equation is $(D^2 - 2D - 8)\, y = 9xe^x + 10e^{-x}$

and the A.E is $m^2 - 2m - 8 = 0$

$$m^2 - 4m + 2m - 8 = 0$$

$$(m - 4)(m + 2)$$

$$m = 4, -2$$

$$\therefore \quad y_c = c_1 e^{-2x} + c_2 e^{4x}$$

Now,

$$P.I = y_p = \frac{1}{D^2 - 2D - 8} 9xe^x + 10e^{-x}$$

$$= 9e^x \frac{1}{D^2 - 2(D) - 8} x + 10 \frac{1}{D^2 - 2D - 8} e^{-x}$$

put $D = D + 1$ put $D = -1$

$$= 9e^x \frac{1}{(D+1)^2 - 2(D+1) - 8} x + 10 \frac{1}{(-1)^2 - 2(-1) - 8} e^{-x}$$

$$= 9e^x \frac{1}{D^2 + 1 + 2D - 2D - 2 - 8} x + 10 \frac{1}{1 + 2 - 8} e^{-x}$$

$$= 9e^x \frac{1}{D^2 - 9} x + 10 \frac{1}{-5} e^{-x}$$

$$= \frac{-9e^x}{9} \left[1 - \frac{D^2}{9}\right]^{-1} x - 2e^{-x}$$

$$= \frac{-9}{9} e^x [x] - 2e^{-x}$$

$$y_p = -xe^x - 2e^{-x}$$

$\therefore$ The required solution is

$$y = c_1 e^{-2x} + c_2 e^{4x} - xe^x - 2e^{-x}$$

Q67. Solve $(D^4 + D^2 + 16)\, y = 16x^2 + 256$

Sol :

The given equation is $(D^4 + D^2 + 16)\, y = 16x^2 + 256$

and the A.E is $m^4 + m^2 + 16 = 0$

$$(D^2 + 4)^2 - (D\sqrt{7})^2 = 0$$

$$(D^2 + D\sqrt{7} + 4)(D^2 - D\sqrt{7} + 4) = 0$$

$$D^2 + D\sqrt{7} + 4 = 0, \quad D^2 - D\sqrt{7} + 4 = 0$$

$$\therefore \quad D = \frac{-\sqrt{7} \pm \sqrt{7-16}}{2}, \frac{\sqrt{7} \pm \sqrt{7-16}}{2}$$

$$= \frac{-\sqrt{7}}{2} \pm \frac{3i}{2}, \frac{\sqrt{7}}{2} \pm \frac{3i}{2}$$

$$C.F = e^{-x\sqrt{7}/2}\left(c_1 \cos\frac{3x}{2} + c_2 \sin\frac{3x}{2}\right) + e^{x\sqrt{7}/2}\left(c_3 \cos\frac{3x}{2} + c_4 \sin\frac{3}{2}x\right)$$

and

$$P.I = \frac{1}{D^4 + D^2 + 16}(16x^2 + 256)$$

$$= \frac{1}{16\left[1 + \left(\frac{D^2}{16} + \frac{D^4}{16}\right)\right]}(16x^2 + 256)$$

$$= \frac{1}{16} \frac{1}{\left[1 + \left(\frac{D^2}{16} + \frac{D^4}{16}\right)\right]^{-1}}(16x^2 + 256)$$

$$= \frac{1}{16}\left[1 - \left(\frac{D^2}{16} + \frac{D^4}{16}\right)\right](16x^2 + 256)$$

$$= \frac{1}{16}\left[(16x^2 + 256) - \frac{16}{16}(2)\right]$$

$$= \frac{1}{16}[16x^2 + 256 - 2]$$

$$y_p = \frac{1}{16}[16x^2 + 254]$$

$\therefore$ The required solution is

$$y = e^{-x\sqrt{7}/2}\left(c_1 \cos\frac{3x}{2} + c_2 \sin\frac{3x}{2}\right) + e^{x\sqrt{7}/2}\left(c_3 \cos\frac{3x}{2} + c_4 \sin\frac{3}{2}x\right) + \frac{1}{16}[16x^2 + 254]$$

3.3 METHOD OF UNDERTERMINED COEFFICIENTS

Q68. Define method of undermine coefficients

Ans :

a

Method of undertermined coefficients for solving linear differential equation, with constant coefficients f(D) y = X.

We know that the general solution of the differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = Q(x) \qquad ... (1)$$

where $a_n \neq 0$, and $Q(x) \neq 0$ is $y = y_c + y_p$

where y_c is complementary function, is the general solution of the related homogeneous equation.

Here we shall now give yet another method of finding y_p of equation (1). This method is known as the method of undetermined coefficients. Here Q(x) can only contain terms such as b, x^k, e^{ax} sin ax, cos ax and a finite number of combination of such term.

Case (i)

No. term of Q(x) in equation (1) is the same as a term y_c,

In this case y_p will be a linear combination of the term in Q(x) and all its linearly independent derivatives.

Case (ii)

When Q(x) in equation (1) contains a term wifh is x^k times a term f(x) of y_c, where k is zero or a positive integer. Here, the particular integral y_p, of equation (1) will be a linear combination of x^{k+1} f(x) and all its linearly independent derivatives (ignoring the constant coefficients) If Q(x) contains terms which correspond to case I. then the proper terms required by this case must be included in y_p .

Case (iii)

If (i) the A. E of equation (1) has an r multiple root.

(ii) Q(x) contains a term which (neglecting the constant coefficients) is x^k f(x), f(x) is a term in y_c and is obtained from the r multiple root, then y_1, will be linear combination of x^{k+r} f(x) and all its linealy independent derivatives

If Q(x) contains terms that belongs to case (I) and (II) then the proper terms, which these cases demand, must also be included in y_p.

Q69. Solve $(D^2 + 4D + 4)y = 4x^2 + 6e^x$. by method of undermined coefficients.

Sol :

The given equation is $(D^2 + 4D + 4)\, y = 4x^2 + 6\, e^x$ ...(1)

and the A. F is $m^2 + 4m + 4 = 0$

$(m + 2)^2 = 0$

$m = -2, -2$

$\therefore y_c = (c_1 + c_2 x)\, e^{-2x}$

Since $Q(x) = 4x^2 + 6e^x$ has n0 term common with y_c, y_p will be linear combination of Q(x) and all its linearly independent derivatives

[Which are neglecting the constant coeffients x^2, x, 1, e^x)

$y_p = Ax^2 + Bx + C + De^x$ ...(2)

Where A, B, C, D are to be determined

diffentiating (2)

$y_p^1 = 2Ax + B + De^x$ ...(3)

again differentiating (3)

$y_p^{11} = 2A + De^x$ ...(4)

equation (2) be solution of equation (1)

We make use of (2) (3) (4) in (1)

$2A + De^x + 4(2A\,x + B + De^x) + 4\,(Ax^2 + Bx + C + De^x) = 4x^2 + 6e^x$

$2A + De^x + 8Ax + 4B + 4De^x + 4Ax^2 + 4\,B\,x + 4C + 4De^x = 4x^2 + 6e^x$

Simplifying & equating the coefficients of like term in two members of the above equation, we get

$\rightarrow$ Coefficient of x^2

$4\,A = 4$

$A = 1$

$\rightarrow$ Coefficient of x

$8\,A + 4\,B = 0$

$A = 1 \Rightarrow 8\,(1) + 4B = 0$

$4B = -8$

$B = -2$

$\rightarrow$ Coefficient of 1

$2A + 4B + 4C = 0$

$2(1) + 4(-2) + 4C = 0$

$4C = 8 - 2$

$C = \frac{6}{4}$

$C = \frac{3}{2}$

$\rightarrow$ Coefficient of e^x

$D + 4D + 4D = 6$

$9D = 6$

$D = \frac{6}{9} = \frac{2}{3}$

$D = \frac{2}{3}$

$\therefore$ Substituting these values in (2)

$$y_p = x^2 - 2x + \frac{3}{2} + \frac{2}{3}\ e^x$$

$\therefore$ The general solution of (1) is $y = (c_1 + c_2x)e^{-x} + x^2 - 2x + \frac{2}{3}e^x + \frac{3}{2}$

Q70. Solve $(D^2 - 3D + 2)\,y = 2x^2 + 3e^{2x}$

Sol :

The given equation is $(D^2 - 3D + 2)\,y = 2x^2 + 3e^{2x}$ ...(1)

and the A.E is $m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$(m - 2)(m - 1) = 0$

$m = 1,2$

$\therefore y_c = c_1 e^x + c_2 e^{2x}$

Since $Q(x) = 2x^2 + 3e^{2x}$ has no term common with y_c,y_p will be linear combination of Q(x)

$y_p = Ax^2 + Bx + C + D\,x\,e^{2x}$ ...(2)

Where A, B, C & D are to be determined differentiating (2)

$y_p^1 = 2Ax + B + D\,x\,2e^{2x} + D\,e^{2x}$

again differentiating (3)

$y_p^{11} = 2A + D\,4\,e^{2x}\,x + D\,2\,e^{2x} + De^{2x}$

$y_p^{11} = 2A + D\,xe^{2x} + 4De^{2x}$

We make use of (2), (3) & (4) in (1)

$\Rightarrow 2A + 4D\,xe^{2x} + 4\,De^x - 3\,(2A\,x + B + 2x\,De^{2x} + De^{2x}) + 2\,(Ax^2 + Bx + C + D\,xe^{2x})$

$= 2x^2 + 3e^{2x}$

$\Rightarrow 2A + 4D\,xe^{2x} + 4De^{2x} - 6\,Ax - 3B - 6x\,De^{2x} - 3De^{2x} + 2Ax^2 + 2Bx + 2C + 2Dxe^{2x}$

$= 2x^2 + 3e^{2x}$

Comparing Coefficients

Coefficients of x^2

$2A = 2$

$A = 1$

Coefficient of x

$-6A + 2B = 0$

$-6\,(1) + 2B = 0$

$2B = 6$

$B = 3$

Coefficients of 1

$2A - 3B + 2C = 0$

$2(1) - 3(3) + 2C = 0$

$2C = 9 - 2$

$2C = 7$

$C = \frac{7}{2}$

Coefficient of e^{2x}

$4D - 3D = 3$

$D = 3$

$\therefore$ Putting above values in (2)

$$y_p = x^2 + 3x + \frac{7}{2} + 3\,x\,e^{2x}$$

$\therefore$ The complete solution is

$$y = c_1e^x + c_2e^{2x} + x^2 + 3x + 3xe^{2x} + \frac{7}{2}$$

Q71. Solve $(D^2 + 2D + 5)\,y = 12e^x - 34 \sin 2x$.

Sol :

The given equation is $(D^2 + 2D + 5)y = 12e^x - 34 \sin 2x$ ...(1)

and the A.E is $m^2 + 2m + 5 = 0$

$$\frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

$y_c = e^{-x} (c_1 \cos 2x + c_2 \sin 2x)$

Also, $y_p = Ae^x + B \sin 2x + c \cos 2x$ ....(2)

differentiating (2)

$y_p^1 = Ae^x + B \cos 2x\,(2) + c\,(-\sin 2x)\,(2)$

$y_p^1 = Ae^x + 2B \cos 2x - 2\,C \sin 2x$ ...(3)

again differentiating (3)

$y_p^{11} = Ae^x + 2B\,(-\sin 2x)\,2 - 2C \cos 2x\,(2)$

$y_p^{11} = Ae^x - 4B \sin 2x - 4C \cos 2x$ ...(4)

By using (2), (3), (4) in (1)

$Ae^x - 4\,B \sin 2x - 4C \cos 2x + 2\,(Ae^x + 2B \cos 2x - 2\,(C \sin 2x) + 5\,(Ae^x + B \sin 2x + C \cos 2x)$

$= 12\,e^x - 34 \sin 2x$

$Ae^x - 4B\sin 2x - 4C\cos 2x + 2Ae^x + 4B\cos 2x - 4C\sin 2x + 5Ae^x + 5B\sin 2x + 5C\cos 2x$
$= 12e^x - 34\sin 2x$

Comparing coefficients

Coefficients of xe^x

$A + 2A + 5A = 12$

$8A = 12$

$A = \frac{12}{8} = \frac{3}{2}$

Coefficients of sin 2x

$-4B - 4C + 5B = -34$

$B - 4C = -34$

Coefficients of cos 2x

$-4C + 4B + 5C = 0$

$C + 4B = 0$

$$\begin{array}{r} 4C + 16B = 0 \\ -\ 4C + B = -34 \\ \hline 17B = -34 \end{array}$$

$B = \frac{-34}{17} = -2$

$C + 4B = 0 \Rightarrow C + 4(-2) = 0$

$C = 8$

Sub all above values in (2)

$y_p = \frac{3}{2}e^x - 2\sin 2x + 8\cos 2x$

$\therefore$ The complete solution is

$$y = e^{-x}(c_1\cos 2x + c_2\sin 2x) + \frac{3}{2}e^x - 2\sin 2x + 8\cos 2x$$

Q72. Solve $(D^2 - 3D + 2)y = xe^{2x} + \sin x$

Sol :

The given equation is $(D^2 - 3D + 2)y = xe^{2x} + \sin x$ ...(1)

and the A.E is $m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 1, 2$

$y_c = c_1e^x + c_2e^{2x}$

Now P.I $= y_p = \frac{1}{D^2 - 3D + 2} xe^{2x} + \sin x$

$$= \frac{1}{D^2 - 3D + 2}\, x\, e^{2x} + \frac{1}{D^2 - 3D + 2} \sin x$$

$$= e^{2x} \frac{1}{(D+2)^2 - 3(D+2) + 2}\, x + \frac{1}{-1^2 - 3D + 2} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 4 + 4D - 3D - 6 + 2}\, x + \frac{1}{-3D + 1} \sin x$$

$$= e^{2x} \frac{1}{D^2 - D}\, x - \frac{1}{3D-1}\frac{3D+1}{3D+1} \sin x$$

$$= -\, e^{2x} \frac{1}{D[1-D]}\, x - \frac{3D+1}{9D^2 - 1} \sin x$$

$$= \frac{-\, e^{2x}}{D}[1-D]^{-1}\, x - \frac{3D+1}{-9-1} \sin x$$

$$= \frac{-e^{2x}}{D}[1 + D + D^2]\, x + \frac{1}{10}(3 \cos x + \sin x)$$

$$= \frac{-e^{x}}{D}[x + 1] + \frac{1}{10}(3 \cos x + \sin x)$$

$$y_p = -\, e^{2x}\left[\frac{x^2}{2} + x\right] + \frac{1}{10}(3 \cos x + \sin x)$$

$\therefore$ The required solution is

$$y = c_1 e^x + c_2 e^{2x} - e^{2x}\left[\frac{x^2}{2} + x\right] + \frac{1}{10}(3 \cos x + \sin x)$$

Q73. Solve $(D^2 + 4D + 4)y = 3xe^{-2x}$

Sol :

The given equation is $(D^2 + 4D + 4) = 3xe^{-2x}$

and A.E is $m^2 + 4m + 4 = 0$

$(m + 2)^2 = 0$

$m = -2, -2$

$y_c = (c_1 + c_2)e^{-2x}$

$y_c = c_1 e^{-2x} + c_2\, xe^{-2x}$

Note that

the A.E has a multiple root $m = -2$ and Q(x) contains the term xe^{-2x}

Which is x times the term e^{-2x} in y_c

Hence $r = 2$ and $k = 1$

y_p must be a linear combination of $x^3 e^{-2x}$ and all its linearly independent derivatives.

We can reglect e^{-2x} & xe^{-2x} ad they are already in y_c

$y_p = Ax^3 e^{-2x} + B x^2 e^{-2}$

After calculating y_p^1 , y_p^{11} and sub in y_p

$6 A xe^{-2x} + 2B e^{-2x} + 2 B e^{-2x} = 3xe^{-2x}$

Which gives $A = \frac{-1}{2}, B = 0$

$y_p = \frac{1}{2} x^3 e^{-2x}$

$\therefore$ The required solution is

$y = c_1 e^{-2x} + c_2 xe^{-2x} + \frac{1}{2} x^3 e^{-2x}$

Q74. Solve $(D^2 + D) y = x^2 + 2x$

Sol :

The given equation is $(D^2 + D) y = x^2 + 2x$

and the A.E is $m^2 + m = 0$

$m(m^2 + 1) = 0$

$m = 0, \pm -1$

$y_c = c_1 + c_2 e^{-x}$

Now

$P.I = y_p = \frac{1}{D^2 + D} x^2 + 2x$

$= \frac{1}{D[1 + D]} x^2 + 2x$

$= \frac{1}{D} [1 + D]^{-1} x^2 + 2x$

$= \frac{1}{D} [1 - D + D^2] x^2 + 2x$

$= \frac{1}{D} [x^2 - 2x - 2x - 2 + 2]$

$= \frac{1}{D} [x^2]$

$y_p = \frac{x^3}{3}$

$\therefore$ The required solution is $y = c_1 + c_2 e^{-x} + \frac{x^3}{3}$

Q75. Solve $(D^2 - 3D)\, y = 2e^{2x} \sin x$

Sol :

The given equation is $(D^2 - 3D)\, y = 2e^{2x} \sin x$

and the A.E $= m^2 - 3m = 0$

$$m(m-3) = 0$$

$$m = 0, 3$$

$\therefore y_c = c_1 + c_2 e^{3x}$

Now

$$P.I = y_p = \frac{1}{D^2 - 3D} 2e^{2x} \sin x$$

$$= 2e^{2x} \frac{1}{(D+2)^2 - 3(D+2)} \sin x$$

$$= 2e^{2x} \frac{1}{D^2 + 4 + 4D - 3D - 6} \sin x$$

$$= 2e^{2x} \frac{1}{D^2 + D - 2} \sin x$$

put $D^2 = -1^2$

$$= 2e^{2x} \frac{1}{-1^2 + D - 2} \sin x$$

$$= 2e^{2x} \frac{1}{D - 3} \sin x$$

$$= 2e^{2x} \frac{D+3}{D^2 - 9} \sin x$$

$$= 2e^{2x} \frac{1}{-10} (d + 3) \sin x$$

$$= \frac{-2e^{2x}}{10} (\cos x + 3 \sin x)$$

$$y_p = \frac{-1}{5} e^{2x} (\cos x + 3 \sin x)$$

$\therefore$ The required solution is $y = c_1 + c_2 e^{3x} - \frac{1}{5}(\cos x + 3 \sin x)$

Q76. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2$

Sol :

The given equation is $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2$

and the A. E is $m^2 + m + 1 = 0$

$$\frac{-1 \pm \sqrt{1-4(1)(1)}}{2} = \frac{-1 \pm \sqrt{-3}i}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$m = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\therefore y_c = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

Now $\quad P.I = y_p = \frac{1}{D^2 + D + 1}x^2$

$$= \frac{1}{\left[1+\left(D+D^2\right)\right]}x^2$$

$$= [1+(D + D)^2]^{-1} x^2$$

$$= [1 - (D + D^2) + (D + D^2)^2]x^2$$

$$= [x^2 - 2x - 2 + 2]$$

$$y_p = x^2 - 2x$$

$\therefore$ The required solution is $y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$

Q77. Solve $(D^4 - 2D^2 + 1)\, y = x - \sin x$

Sol :

Given equation is $(D^4 - 2D^2 + 1)\, y = x - \sin x$ ...(1)

and the A. E is $m^4 - 2m^2 + 1 = 0$

$(m^2 - 1)^2 = 0 \Rightarrow m = \pm 1, \pm 1$

$\therefore y_c = (c_1 + c_2x)\, e^x + (c_3 + c_4x)\, e^{-x}$

Since

No term of y_c is same as that of Q(x), we write y_p as Linear combination of Q(x) and its Linear Independent derivatives.

$\therefore y_p = A \sin x + B \cos x + C x + D$ ...(2)

differentiating (2)

$y_p^1 = A \cos x - B \sin x + C$ ...(3)

again differentiating (3)

$y_p^{11} = -A \sin x - B \cos x$

Similarly $y_p^{111} = -A \cos x + B \sin x$

$y_p^{iv} = A \sin x + B \cos x$

Substitute the values of y_p^1, y_p^{11}, y_p^{111} ,y_p^{iv} in the equation (1)

$A \sin x + B \cos x - 2(-A \sin x - B \cos x) + A \sin x + B \cos x + Cx + D = x - \sin x$

$A \sin x + B \cos x + 2A \sin x + 2B \cos x + A \sin x + B \cos x + C x + D = x - \sin x$

Coefficient of sinx

$$A + 2A + A = -1$$
$$4A = -1$$
$$A = \frac{-1}{4}$$

Coefficient of cosx

$$B + 2B + B = 0$$
$$B = 0$$

Coefficient of x

$$C = 1$$

and $D = 0$

Sub all above values in (2)

$$y_p = \frac{-1}{4} \sin x + 0. \cos x + 1.x + 0$$

$$y_p = \frac{-1}{4} \sin x + x$$

$\therefore$ The complete solution is $y = (c_1 + c_2 x) e^x + (c_3 + c_4 x) e^{-x} - \frac{1}{4} \sin x + x$

Q78. Solve $y^{11} + 3y^1 + 2y = xe^x$ by the method of undetermined coefficients.

Sol :

The given equation is $y^{11} + 3y^1 + 2y = xe^x$

and the A.E is $m^2 + 3m + 2 = 0$

$$(m + 2)(m + 1) = 0$$
$$m = = -1, -2$$

$\therefore y_c\ c_1 e^{-x} + c_2\, e^{-2x}$

Since

No term of y_c is same as that of Q(x) we write y_p as linear combination of Q(x) and its linear. Independent derivative,

$\therefore y_p = A\,xe^x + B\,e^x$ ...(2)

differentiating (2)

$y_p^1 = A\,xe^x + A\,e^x + B\,e^x$ ...(3)

$y_p^{11} = A\,xe^x + A\,e^x + A\,e^x + Be^x$

$y_p^{11} = A\,xe^x + 2\,A\,e^x + B\,e^x$ ...(4)

By using (2), (3), (4) in equation (1)

$Axe^x + 2Ae^x + Be^x + 3(Axe^x + Ae^x + Be^x) + 2(Axe^x + Be^x) = xe^x$

$Axe^x + 2Ae^x + Be^x + 3Axe^x + 3Ae^x + 3Be^x + 2Axe^x + 2Be^x = xe^x$

Comparing coefficients

Coefficient of xe^x

$A + 3 + 2A = 1$

$6A = 1 \Rightarrow A = \frac{1}{6}$

Coefficient of e^x

$2A + B + 3A + 3B + 2B = 0$

$5A + 6B = 0$

$5\left(\frac{1}{6}\right) + 6B = 0$

$6B = \frac{-5}{6}$

$B = \frac{-5}{36}$

Sub A & B in equation (2)

$\therefore y_p = \frac{1}{6}xe^x - \frac{5}{36}e^x$

$\therefore$ The complete solution is $y = c_1e^{-x} + c_2e^{-2x} + \frac{1}{6}xe^x - \frac{5}{36}e^x$

Q79. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin x$ by using undetermined coefficients.

Sol :

The given equation is $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin x$ ...(1)

and the A. E is $m^2 + 3m + 2 = 0$

$(m + 2)(m + 1) = 0$

$m = -1, -2$

$\therefore y_c = c_1e^{-x} + c_2e^{-2x}$

Since $Q(x) = \sin x$ has no term common with y_c, y_p will be linear combination of Q(x)

$y_p = A\sin x + B\cos x$ ...(2)

Differentiating (2)

$y_p^1 = A\cos x - B\sin x$ ...(3)

again differentiating (3)

$y_p^{11} = -A\sin x - B\cos x$ ...(4)

By Using (2), (3), (4) in (1) then, we get

$-A\sin x - B\cos x - 3(A\cos x - B\sin x) + 2(A\sin x + B\cos x) = \sin x$

$-A\sin x - B\cos x - 3A\cos x + 3B\sin x + 2A\sin x + 2B\cos x = \sin x$

Comparing the coefficients

The coefficient of sinx

$-A + 3B + 2A = 1$

$A + 3B = 1$ ...(i)

The coefficient of cosx

$-B - 3A + 2B = 0$

$-3A + B = 0$ ...(ii)

Solving (i), (ii)

$$\begin{array}{r} -9A + \cancel{3B} = 0 \\ -\ \ A \pm \cancel{3B} = 1 \\ \hline -10A \quad = -1 \end{array}$$

$$\Rightarrow A = \frac{1}{10}$$

Sub $A = \frac{1}{10}$ in (i) $\Rightarrow \frac{1}{10} + 3B = 1$

$$3B = 1 \frac{-1}{10}$$

$$B = \frac{-9}{10} \times \frac{1}{3}$$

$$B = \frac{-3}{10}$$

Now substitute $A = \frac{1}{10}$, $B = \frac{-3}{10}$ in (2)

$\therefore y_p = \frac{1}{10}$ sinx $\frac{-3}{10}$ cosx

$\therefore$ The complete solution is $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{10}\sin x - \frac{3}{10}\cos x$

Q80. Solve $y^{11} + y^1 + y = x^2$ by using undermined coefficients.

Sol :

The given equation is $y^{11} + y^1 + y = x^2$

and the A.E is $m^2 + m + 1$

$$m = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\therefore y_c = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

Since

Q(x) = x^2 has no term common with y_c, y_p will be linear combination of Q(x)

$y_p = Ax^2 + Bx + C$ ....(2)

differentiating (2)

$y_p = 2Ax + B$ ....(3)

again differentiating (3)

$y_p^{11} = 2A$ ....(4)

By using (2), (3) & (4) in (1)

$2A + 2Ax + B + Ax^2 + Bx + C = x^2$

Coefficient of x^2

$A = 1$

Coefficient of x

$2A + B = 0$

$2(1) + B = 0 \Rightarrow B = -2$

Coefficient of 1

$2A + B + C = 0$

$2(1) + (-2) + C = 0$

$C = 0$

Sub A, B, C in (2)

$y_p = 1.x^2 - 2x + 0$

$y_p = x^2 - 2x$

$\therefore$ The required solution is $y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + x^2 - 2x$

Q81. Solve $(D^2 + D)\, y = x^2 + 2x$ By using undetermined coefficients

Sol :

The given equation is $(D^2 + D)\, y = x^2 + 2x$ . ...(1)

and the A. E is $m^2 + m = 0$

$m = 0, -1$

$\therefore y_c = c_1 + c_2 e^{-x}$

The term of Q(x) are x^2 & x

above the 1st term of $y_c = c_1 = c_1.1 = c_1$ f(x)

we have x^2(term of Q(x)) = $x^2 .1 = x^2$, f(x)

$k = 2$

Similarly x (term of Q(x) = $x^1.1 = x^1 f(x)$

$\Rightarrow k = 1$

$\therefore y_p$ is writter as Linear combination of x^{2+1} & x^{1+1}

y_p is written as Linear combination of x^3, x^2 & its Linear Independent derivative

$\therefore y_p = Ax^3 + Bx^2 + Cx$ ...(2)

Differentiating (2)

$y_p^1 = 3Ax^2 + 2Bx + C$ ...(3)

again differentiating (3)

$y_p^{11} = 6Ax + 2B$ ...(4)

Using (2), (3) & (4) in (1)

$6Ax + 2B + 3A x^2 + 2Bx + C = x^{2 + 2x}$

Coefficient of x^2

$3A = 1$

$A = \frac{1}{3}$

Coefficient of x

$2B + 6A = 2$

$\Rightarrow 2B + 6\left(\frac{1}{3}\right) = 2$

$2B + 2 = 2$

$B = 0$

Coefficient of 1

$2B + C = 0$

$2(0) + C = 0$

$C = 0$

Sub A, B, C in (2)

$\therefore y_p = \frac{1}{3} x^3 + D.x^2 + 0.x$

$y_p = \frac{1}{3} x^3$

$\therefore$ The required solution is $y = c_1 + c_2 e^{-x} + \frac{1}{3}x^3$

Q82. Solve $y^v + 2y^{111} + y^1 = 2x + \sin x + \cos x$.

Sol :

The given equation is $(D^5 + 2D^3 + D)y = 2x + \sin x + \cos x$

and the A. E is $m^5 + 2m^3 + m = 0$

$m(m^4 + 2m^2 + 1) = 0$

$m(m^2 + 1)^2 = 0$

$m = 0,\ m = \pm i, \pm i$

$\therefore y_c = c_1 + (c_2 + c_3x)\cos x + (c_4 + c_5 x)\sin x$

Now, $P.I = y_p = \frac{1}{D^5 + 2D^3 + D} 2x + \sin x + \cos x$

$= \frac{1}{D^5 + 2D^3 + D}2x + \frac{1}{D^5 + 2D^3 + D}\sin x + \frac{1}{D^5 + 2D^3 + D}\cos x$

(a) (b) (c)

(a) Consider $\dfrac{1}{D^5+2D^3+D}\ 2x = 2\dfrac{1}{D\left[D^4+2D^2+1\right]}x$

$= \dfrac{2}{D}\dfrac{1}{1+\left(D^4+2D^2\right)}x = \dfrac{2}{D}\left[1+\left(D^4+2D^2\right)\right]^{-1}x$

$= \dfrac{2}{D}\left[1-\left(D^4+2D^2\right)\right]x$

$= \dfrac{2}{D}[x] = 2.\dfrac{x^2}{2}$

$= x^2$

Consider (b) and (c)

Consider $\dfrac{1}{D^5+2D^3+D}$ (sinx + cosx) = $\dfrac{1}{D\left(D^2+1\right)^2}$ (sinx + cosx)

$= \dfrac{1}{\left(D^2+1\right)^2}\ \dfrac{1}{D}$ sinx + cosx

$= \dfrac{1}{\left(D^2+1\right)^2}$ (–cosx + –sinx)

$= -\dfrac{1}{\left(D^2+1\right)^2}\cos x + \dfrac{1}{\left(D^2+1\right)^2}\sin x$ ...(4)

(b_1) (b_2)

Consider (b_1) $\dfrac{1}{\left(D^2+1\right)^2}\cos x = \dfrac{1}{D^2+1}\left[\dfrac{1}{D^2+1}\cos x\right]$

$= \dfrac{1}{D^2+1}\ \dfrac{x}{2}\ \sin x$

$\dfrac{1}{\left(D^2+1\right)^2}\cos x = \dfrac{1}{2}\dfrac{1}{D^2+1}x\sin x$ ....(B)

Now,

Consider $\dfrac{1}{D^2+1}$ x sinx

$\Rightarrow$ Im $\dfrac{1}{D^2+1}\ x\ e^{ix}$

$=$ Im $e^{ix}\ \dfrac{1}{(D+i)^2+1}x$

$$= \text{Im } e^{ix} \frac{1}{D^2 + 2Di - 1 + 1} x$$

$$= \text{Im } e^{ix} \frac{1}{2Di + D^2} x$$

$$= \text{Im } e^{ix} \frac{1}{2Di\left[1 + \frac{D^2}{2Di}\right]} x$$

$$= \text{Im } e^{ix} \text{ x } \frac{1}{2Di}\left[1 + \frac{D}{2i}\right]_x^{-1}$$

$$\text{Im } e^{ix} \frac{1}{2Di}\left[1 - \frac{D}{2i}\right] x$$

$$= \text{Im } e^{ix} \frac{1}{2Di}\left[x - \frac{1}{2i}\right]$$

$$= \text{Im } e^{ix} \frac{1}{D} \frac{1}{2i}\left[\frac{2ix - 1}{2i}\right]$$

$$= \text{Im } \frac{-e^{ix}}{4} \frac{1}{D}[2ix - 1]$$

$$= \text{Im } \frac{-e^{ix}}{4}\left[2i\frac{x^2}{2} - x\right]$$

$$= \text{Im } \frac{-1}{4} (\cos x + i \sin x)(ix^2 - x)$$

$$\frac{1}{D^2 + 1} x \sin x = \frac{-1}{4} (x^2 \cos x - x \sin x)$$

By (B)

$$\frac{1}{\left(D^2 + 1\right)^2} \cos x = \frac{1}{2}\left[\frac{-1}{4}\left(x^2 \cos x - x \sin x\right)\right]$$

$$= \frac{-1}{8} (x^2 \cos x - x \sin x) \text{ ....}(B_1)$$

Consider $(b_2) \Rightarrow \frac{1}{\left(D^2 + 1\right)^2} \sin x$

$$\frac{1}{\left(D^2 + 1\right)^2} \sin = \frac{1}{D^2 + 1}\left[\frac{1}{D^2 + 1} \sin x\right]$$

$$= \frac{1}{D^2+1}\left(\frac{-x}{2}\cos x\right)$$

$$= \frac{-1}{2}\frac{1}{D^2+1} x \cos x \quad ....(c)$$

Now consider $\frac{1}{D^2+1} x \cos x$

$\Rightarrow$ R.P $\frac{1}{D^2+1} xe^{ix}$

$= \text{R.P} \frac{1}{-4}(\cos x + i \sin x)(ix^2 - x)$

$= \frac{-1}{4}(-x \cos x - x^2 \sin x)$

$= \frac{1}{4}(x \cos x + x^2 \sin x)$

$\therefore \frac{1}{(D^2+1)^2} \sin x = \frac{-1}{2}\cdot\frac{1}{4}(x \cos x + x^2 \sin x)$

$\frac{1}{(D^2+1)^2} \sin x = \frac{-1}{8}(x \cos x + x^2 \sin x) \quad ....(c)^1$

Sub (B^1) & (C^1) in (A)

$$\therefore \frac{1}{D^5+2D^3+D}(\sin x + \cos x) = \frac{-1}{8}(x^2 \cos x - x \sin x) - \frac{1}{8}(x \cos x + x^2 \sin x)$$

$$\therefore y_p = x^2 - \frac{1}{8}(x^2 \cos x - x \sin x) - \frac{1}{8}(x \cos x + x^2 \sin x)$$

$\therefore$ The required solution is

$$y = c_1 + (c_2 + c_3x)\cos x + (c_4 + c_5x)\sin x + x^2 - \frac{1}{8}(x^2 \cos x - x \sin x) - \frac{1}{8}(x\cos x + x^2 \sin x)$$

Q83. Solve $(D^2 + 1)(D^2 + 4)y = \cos\frac{x}{2}\cos\frac{3x}{2}$

Sol :

The given equation is $(D^2 + 1)(D^2 + 4)y = \cos\frac{x}{2} \cos\frac{3x}{2}$

and the A.E is $(m^2 + 1)(m^2 + 4) = 0$

$m^2 + 1 = 0 \quad m^2 + 4 = 0$

$m = \pm i \; ; \; m = \pm 2i$

$\therefore$ $y_c = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$

Now P.I = y_p

$$y_p = \frac{1}{(D^2+1)(D^2+4)} \cos\frac{x}{2} \cos\frac{3x}{2}$$

$$= \frac{1}{2} \frac{1}{(D^2+1)(D^2+4)} 2\cos\frac{x}{2} \cos\frac{3x}{2}$$

$$= \frac{1}{2} \frac{1}{(D^2+1)(D^2+4)} \cos\left(\frac{x}{2}+\frac{3x}{2}\right) + \cos\left(\frac{x}{2}-\frac{3x}{2}\right)$$

$$= \frac{1}{2} \frac{1}{(D^2+1)(D^2+4)} [\cos 2x + \cos x]$$

$$= \frac{1}{2}\left[\frac{1}{(D^2+1)(D^2+4)} \cos 2x + \frac{1}{(D^2+1)(D^2+4)} \text{cox}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(-4+1)D^2+4} \cos 2x + \frac{1}{(-1+4)(D^2+1)} \cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-3(D^2+4)} \cos 2x + \frac{1}{3}\frac{1}{(D^2+1)} \cos x\right]$$

$$= \frac{-1}{6}\left[\frac{1}{D^2+4} \cos 2x + \frac{1}{D^2+1} \cos x\right]$$

$$= \frac{-1}{6}\left[\frac{-x}{2(2)} \sin 2x + \frac{x}{2} \sin x\right]$$

$$y_p = \frac{1}{6}\left[\frac{x}{4} \sin 2x + \frac{x}{2} \sin x\right]$$

$\therefore$ The required solution is

$$y = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x + \frac{1}{6}\left[\frac{x}{4} \sin 2x + \frac{x}{2} \sin x\right]$$

Q84 Solve $y^{11} - 3y^1 + 2y = 32x^3$ by using method of undeterned coefficient.

Sol :

The given equation is $y^{11} - 3y^1 + 2y = 32x^3$ ...(1)

$(m-2)(m-1) = 0$

$m = 1,2$

$\therefore$ $y_c = c_1 e^x + c_2 e^{2x}$

Since Q(x) has no term common with y_c , y_p will be linear combination of Q(x)

$y_p = Ax^3 + Bx^2 + Cx + D$ ...(2)

differentiating (2)

$y_p^1 = 3Ax^2 + 2Bx + C$ ....(3)

again differentiate (3)

$y_p^{11} = 6Ax + 2B$ ....(4)

By using (2), (3) & (4) in (1)

$6Ax + 2B - 3(3Ax^2 + 2Bx + e) + 2(Ax^3 + Bx^2 + Cx + D) = 32x^2$

Coefficient of x^3

$2A = 32$

$A = \frac{32}{2} = 16$

Coefficient of x^2

$-9A + 2B = 0$

$-9(16) + 2B = 0$

$-144 + 23 = 0$

$2B = 144$

$B = \frac{144}{4} = 72$

Coefficient of x

$6A - 6B + 2C = 0$

$6(16) - 6(72) + 2C = 0$

$96 - 432 + 2C = 0$

$2C = 432 - 96$

$2C = 336$

$C = \frac{336}{2} = 168$

$\Rightarrow$ Coefficient of 1

$2B - 3C + 2D = 0$

$2(72) - 3(168) + 2D = 0$

$144 - 504 + 2D = 0$

$-360 + 2D = 0$

$D = \frac{360}{2}$

$D = 180$

$\therefore y_p = 16x^3 + 72x^2 + 168x + 180$

$\therefore$ The required solution is

$y = c_1e^x + c_2e^{2x} + 16x^3 + 72x^2 + 168x + 180$

Multiple Choice Questions

1. If m = 2,2 then the complimentary function is [b]

   (a) $y_c = (C_1 + C_2)e^{2x}$ (b) $y_c = (C_1 + C_2x)e^{2x}$

   (c) $y_c = (C_1 + 2C_1x)e^{x}$ (d) $y_c = C_1e^{2x} + C_2e^{2x}$

2. A homogeneous linear differential equation of n is ________ [a]

   (a) $(D^2 - 3D)y = 0$ (b) $(D^2 - 3D)y = 3$

   (c) $((D^3)^{2/3} - D)y = 0$ (d) None

3. Solution for $(D^3 - 3D - 2)y = 0$ [b]

   (a) $y = (C_1 + C_2x)e^{x} + C_3e^{2x}$ (b) $y = (C_1 + C_2x)e^{-x} + C_3e^{2x}$

   (c) $y = (C_1 + C_2x)e^{-x} + C_3e^{-2x}$ (d) $y = (C_1 + C_2x)e^{x} + C_3e^{2x}$

4. Solution for $y^{II} + 3y^{I} + 2y = 4$ [c]

   (a) 4 (b) 3

   (c) 2 (d) None

5. Solution for $y^{II} - 2y^{I} - 3y = 2e^{4x}$ [a]

   (a) $y = C_1e^{3x} + C_2e^{-x} + \frac{2}{5}e^{4x}$ (b) $y = C_1e^{3x} + C_2e^{x} + \frac{2}{5}e^{4x}$

   (c) $y = C_1e^{-3x} + C_2e^{x} + \frac{2}{5}e^{4x}$ (d) $y = C_1e^{-3x} + C_2e^{-x} + \frac{2}{5}e^{4x}$

6. Solution for $(D^2 + 2D + 1)y = 2x + x^2$ [b]

   (a) $x^2 + 2x + 2$ (b) $x^2 - 2x + 2$

   (c) $x^2 - 2x - 2$ (d) None

7. y_p for $(D^4 - 1)y = \sin x$ [c]

   (a) $\frac{1}{4}x\sin x$ (b) $-\frac{1}{4}x\cos x$

   (c) $\frac{1}{4}x\cos x$ (d) $\frac{1}{2}x\cos x$

8. y_p for $\left(D^2 - 2D + 1\right)y = xe^x \sin x$ [a]

(a) $e^x\left(-x\sin x - 2\cos x\right)$ (b) $e^x\left(x\sin x + 2\cos x\right)$

(c) $e^x\left(-x\sin x + 2\cos x\right)$ (d) $e^{-x}\left(-x\sin x + 2\cos x\right)$

9. y_c for $y^{II} + 3y^{I} + 2y = 4$ [a]

(a) $C_1e^{-2x} + C_2e^{-x}$ (b) $C_1e^{2x} + C_2e^{x}$

(c) $C_1e^{-2x} + C_2e^{x}$ (d) None

10. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\sin x$ [a]

(a) $y = \left(C_1 + C_2x\right)e^x + \frac{1}{2}\left(x\cos x + \cos x - \sin x\right)$

(b) $y = \left(C_1 - C_2x\right)e^x + \frac{1}{2}\left(x\cos x - \sin x\right)$

(c) $y = \left(C_1 + C_2x\right)e^{-x} + \frac{1}{2}\left(x\cos x + \cos x + \sin x\right)$

(d) $y = \left(C_1 + C_2x\right)e^{-x} + \frac{1}{2}\left(x\cos x + \cos x - \sin x\right)$

Fill in the blanks

1. If all the roots are distinct and real then y_c is ________ .
2. A linear differential equation of order n is ________.
3. Solution for $\left(D^3 - D^2\right)y = 2x^3$ is ________ .
4. $[1-D]^{-1}x^2$ is ________ .
5. Binomial expansion for $[1+X]^{-1}$ is ________ .
6. y_p for $\left(D^2 - 2D + 1\right)y = e^x x^2$ is ________ .
7. y_p for $\frac{d^2y}{dx^2} - y = \cos x$ is ________.
8. Solution for $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 3x^2e^x$ is ________ .
9. y_p for $y'' + y' - 2y = 3e^{-2x}$ is ________ .
10. y_p for $\frac{dy}{dx} - 3y = x^3 + 3x - 5$ is ________ .

Answers

1. $\left(C_1 + C_2x + C_3x^2 .... C_nx^n\right)e^{mx}$
2. $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = Q(x)$
3. $-2\left(\frac{x^5}{20} + \frac{x^4}{4} + x^3 + 3x^2\right)$
4. $x^2 + 2x + 2$
5. $1 - x + x^2 - x^3 + ...$
6. $\frac{e^x x^4}{12}$
7. $\frac{-1}{2}\cos x$
8. $\frac{1}{3}e^x\left(3x^2 - 6x + 4\right)$
9. $-xe^{-x}$
10. $\frac{-1}{27}\left(9x^3 + 9x^2 + 33x - 34\right)$

UNIT IV

Method of variation of parameters - Linear differential equations with non constant coefficients - The Cauchy - Euler Equation - Legendre's Linear Equations - Miscellaneous Differential Equations. Partial Differential Equations: Formation and solution- Equations easily integrable - Linear equations of first order.

4.1 Method of Variation of Parameter

Q1. Write Working Rule of method of variation of parameter

Ans:

- Consider the second order Linear differentiate equation given by $a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = Q(x)$
- The related homogeneous equation is obtained by putting $Q(x) = 0$

  Which gives $a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$
- Let the complimentary function $y_c = c_1y_1 + c_2y_2$ where c_1, c_2 are constants and y_1 and y_2 are Independent solutions
- Now assume y_p as $uy_1 + vy_2$ where u and v are function of x which are given by the formula

  $$U = -\int \frac{Q(x)y_2}{a_2\left(y_1y_2^1 - y_2y_1^1\right)},$$

  $$V = \int \frac{Q(x)y_1}{a_2\left(y_1y_2^1 - y_2y_1^1\right)}$$
- Substituting the values of u, v are obtain $y_p = uy_1 + vy_2$
- the general solution is given by $y = y_c + y_p$

Q2. Solve $y^{11} + 3y^1 + 2y = 12\,e^x$ by using the method of variation of parameter.

Sol :

Given that $y^{11} + 3y^1 + 2y = 12\,e^x$ ... (1)

comparing the given differential equation with the standard equation

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = Q(x)$$

here $a_2 = 1$, $a_1 = 3$, $a_0 = 2$ and $Q(x) = 12e^x$

To find y_c

The A.E is $m^2 + 3m + 2 = 0$

$m^2 + 2m + m + 2 = 0$

$(m + 2)(m + 1) = 0$

$m = -1, -2$

$\therefore y_c = c_1 e^{-x} + c_2 e^{-2x}$ ... (2)

compare (2) with $y_c = c_1 y_1 + c_2 y_2$

here $y_1 = e^{-x}$, $y_2 = e^{-2x}$

Now, $y_1^1 = -e^{-x}$, $y_2^1 = -2e^{-2x}$

$y_p = uy_1 + vy_2$

here $u = -\int \frac{Q(x)y_2}{a_2\left(y_1y_2^1 - y_2y_1^1\right)}dx$

& $v = \int \frac{Q(x)y_1}{a_2\left(y_1y_2^1 - y_2y_1^1\right)}dx$

$\because\ y_1y_2^1 - y_2y_1^1 = e^{-x}(-2e^{-2x}) - (e^{-2x})(-e^{-x})$

$= -2e^{-3x} + e^{-3x}$

$y_1y_2^1 - y_2y_1^1 = -e^{-3x}$

$\therefore U = -\int \frac{12e^x\left(e^{-2}\right)}{-e^{-3x}}dx$

$\therefore V = \int \frac{12e^x e^{-x}}{-e^{-3x}}dx$

$= \int \frac{12e^{-x}}{e^{-3x}} dx$ $\quad = -\int 12e^{x}e^{-x}e^{3x} dx$

$= \int 12e^{2x} dx$ $\quad = -\int 12e^{3x} dx$

$= 12\int e^{2x} dx$ $\quad = -12\left[\frac{e^{3x}}{3}\right]$

$= 12\left[\frac{e^{2x}}{2}\right]$ $\quad V = -4e^{3x}$

$u = 6e^{2x}$

$\therefore y_p = uy_1 + vy_2$

$= 6e^{2x}(e^{-x}) + (-4e^{3x})(e^{-2x})$

$= 6e^{(2x-x)} - 4e^{(3x-2x)}$

$= 6e^{x} - 4e^{x}$

$y_p = 2e^{x}$

$\therefore$ The complete solution is $y = c_1 e^{-x} + c_2 e^{-2x} + 2e^{x}$

Q3. Solve $y^{11} + 2y^{1} + y = x^2 e^{-x}$ by using method of variation of parameter

Sol :

The given equation is $y^{11} + 2y^{1} + y = x^2 e^{-x}$ ... (1)

Compare (1) with $a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 = Q(x)$

here $a_2 = 1, a_2 = 2, a_0 = 1$; $Q(x) = x^2 e^{-x}$

To find y_c

The A.E is $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, -1$

$\therefore y_c = (c_1 + c_2 x)e^{-x}$

$\Rightarrow y_c = c_1 e^{-x} + c_2 xe^{-x}$ ... (2)

Compare (2) with $y_c = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^{-x}$ & $y_2 = xe^{-x}$

$\Rightarrow y_1^1 = -e^{-x}$; $y_2^1 = x[-e^{-x}] + e^{-x}$

$y_2^1 = -xe^{-x} + e^{-x}$

Now find y_p

i.e., $y_p = uy_1 + vy_2$

$\because u = -\int \frac{Q(x)y_2}{a_2\left(y_1y_2^1 - y_2y_1^1\right)} dx$;

let $y_1 y_2^1 - y_2 y_1^1 = e^{-x}(-xe^{-x} + e^{-x}) - xe^{-x}(-e^{-x})$

$= -xe^{-x-x} + e^{-x}e^{-x} + xe^{-x}e^{-x}$

$= \cancel{-xe^{-2x}} + \cancel{xe^{-2x}} + e^{-2x}$

$$y_1 y_2^1 - y_2 y_1^1 = e^{-2x}$$

$$\therefore u = -\int \frac{x^2e^{-x}\left(xe^{-x}\right)}{e^{-2x}}dx$$

$$= -\int \frac{x^3e^{-x}e^{-x}}{e^{-2x}}dx$$

$$= -\int x^3e^{-2x+2x}dx$$

$$= -\int x^3dx$$

$$u = -\left[\frac{x^4}{4}\right]$$

Consider

$$v = \int \frac{Q(x)\, y_1}{a_2(y_1y_2^1 - y_2y_1^1)}dx$$

$$v = \int \frac{x^2e^{-x}\left(e^{-x}\right)}{e^{-2x}}dx$$

$$= \int x^2e^{-2x+2x}dx$$

$$= \int x^2dx$$

$$v = \frac{x^3}{3}$$

$$\therefore y_p = \frac{-x^4}{4}y_1 + \frac{x^3}{3}y_2$$

$$= \frac{-x^4}{4}\left(e^{-x}\right) + \frac{x^3}{3}\left(xe^{-x}\right)$$

$$= -\frac{x^4e^{-x}}{4} + \frac{x^4e^{-x}}{3}$$

$$= \frac{-3x^4e^{-x} + 4x^4e^{-x}}{12}$$

$$y_p = \frac{x^4e^{-x}}{12}$$

$\therefore$ The complete solution is

$$y = c_1e^{-x} + c_2\, xe^{-x} + \frac{x^4e^{-x}}{12}$$

Q4. Solve $y^{11} + y = 4x \sin x$.

Sol :

The given equation is $y^{11} + y = 4x \sin x$ ...(1)

Compare (1) with $a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 = Q(x)$

here $a_2 = 1, a_1 = 0, a_0 = 1$ & $Q(x) = 4x \sin x$

To find y_c:

The A.E is $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

$\therefore y_c = c_1 \cos x + c_2 \sin x$ .... (2)

Compare (2) with $y = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos x,\ y_2 = \sin x$

$\Rightarrow y_1^1 = -\sin x,\ y_2^1 = \cos x$

To find y_p

$y_p = Uy_1 + Vy_2$

$$\therefore U = -\int \frac{Q(x)y_2}{a_2\left(y_1y_2^1 - y_2y_1^1\right)}dx \quad \&$$

$$V = \int \frac{Q(x)y_1}{a_2\left(y_1y_2^1 - y_2y_1^1\right)}dx$$

$\because (y_1 y_2^1 - y_2 y_1^1) = \cos x(\cos x) - \sin x(-\sin x)$

$= \cos^2 x + \sin^2 x$

$y_1 y_2^1 - y_2 y_1^1 = 1$

$$U = -\int \frac{4x \sin x(\sin x)}{1}dx$$

$$= -\int 4x \sin^2 x\, dx$$

$$= -4\int x\frac{(1-\cos 2x)}{2}dx$$

$$= -2\int x\, dx + 2\int x \cos 2x\ dx$$

$$= -\frac{2x^2}{2} + 2\left[x\frac{\sin 2x}{2} - \int 1.\frac{\sin 2x}{2}dx\right]$$

$$= -x^2 + \frac{2}{2}\left[x \sin 2x + \frac{\cos 2x}{2}\right]$$

$$U = -x^2 + x\sin 2x + \frac{\cos 2x}{2}$$

$$\because V = \int \frac{Q(x)\, y_1}{a_2(y_1\, y_2^1 - y_2\, y_1^1)} dx$$

$$V = \int \frac{4x\sin x}{1}.\cos x\ dx$$

$$= 4\int x\sin x\cos x.dx$$

$[\because 2\sin x\cos x = \sin 2x.\ \sin x\cos x = \frac{1}{2}(\sin 2x)$

$$= 4\int x.\frac{1}{2}\sin 2x\ dx$$

$$= 2\int x\sin 2x\ dx$$

$$= 2\left[x\left(\frac{-\cos 2x}{2}\right) - \int 1.\int \frac{\sin 2x}{2} dx.dx\right]$$

$$= 2\left[\frac{-x\cos 2x}{2} - \frac{1}{2}\left(\frac{-\cos 2x}{2}\right)dx\right]$$

$$= \frac{2}{2}\left[-x\cos 2x + \frac{\sin 2x}{2}\right]$$

$$V = -x\cos 2x + \frac{\sin 2x}{2}$$

$$\therefore y_p = (-x^2 + x\sin 2x + \frac{\cos 2x}{2})\cos x + (-x\cos 2x + \frac{\sin 2x}{2})\sin x$$

$$= -x^2\cos x + x\sin 2x\ \cos x + \frac{\cos 2x\cos x}{2} - x\sin x\cos 2x + \frac{\sin 2x\sin x}{2}$$

$$= -x^2\cos x + \frac{1}{2}[\cos 2x\cos x + \sin 2x\sin x] + x(\sin 2x\cos x - \cos 2x\sin x)$$

$$= -x^2\cos x + \frac{1}{2}(\cos(2x - x)) + x\sin(2x - x)$$

$$y_p = -x^2\cos x + \frac{1}{2}\cos x + x\sin x$$

$\therefore$ The complete solution is

$$y = c_1\cos x + c_2\sin x - x^2\cos x + \frac{1}{2}\cos x + x\sin x.$$

Q5 Solve $y^{11} + 2y^{1} + y = e^{-x} \log x$

Sol :

The given equation is $y^{11} + 2y^{1} + y = e^{-x} \log x$ ... (1)

Compare (1) with $a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 = Q(x)$

here $a_2 = 1, a_1 = 2, a_0 = 1, Q(x) = e^{-x} \log x$

To find y_c :

The A.E is $4m^2 + 2m + 1 = 0$

$(m+1)^2 = 0 \Rightarrow m = -1, -1$

$\therefore y_c = c_1 e^{-x} + c_2 \; xe^{-x}$ ... (2)

Compare (2) with $y_c = c_1 y_1 + c_2 y_2$

here $y_1 = e^{-x}$, $y_2 = xe^{-x}$

$\therefore$ $y_1^{1} = -e^{-x}$, $y_2^{1} = -x e^{-x} + e^{-x}$

To find y_p

$$u = -\int \frac{Q(x) y_2}{a_2 (y_1 y_2^{1} - y_2 y_1^{1})} dx \quad \& \quad v = \int \frac{Q(x) y_1}{a_2 (y_1 y_2^{1} - y_2 y_1^{1})} dx$$

Since $y_1 y_2^{1} - y_2 y_1^{1}$

$= e^{-x}(-xe^{-x} + e^{-x}) - xe^{-x}(-e^{-x})$

$= -xe^{-x}e^{-x} + e^{-x}e^{-x} + xe^{-x}e^{-x}$

$= -xe^{-2x} + e^{-2x} + xe^{-2x}$

$y_1 y_2^{1} - y_2 y_1^{1} = e^{-2x}$

$$\therefore u = -\int \frac{e^{-x} \log x . xe^{-x}}{e^{-2x}} dx$$

$$= -\int e^{-x} \log x . xe^{-x} . e^{2x} dx$$

$$= -\int x \log x . dx$$

$$= -\left[\log x \int x - \int \frac{1}{x}\left(\int x dx\right) dx\right]$$

$$= -\left[\log x . \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2} dx\right]$$

$$= -\left[\log x . \frac{x^2}{2} - \frac{1}{2}\int x \; dx\right]$$

$$= -\left[\log x . \frac{x^2}{2} - \frac{1}{2} . \frac{x^2}{2}\right]$$

$$u = \frac{-x^2 \log x}{2} + \frac{x^2}{4}$$

$$v = \int \frac{e^{-x} \log x . e^{-x}}{e^{-2x}} dx = \int \log \ x e^{-2x} \ e^{2x} dx$$

$$= \int (\log x) dx$$

Using by parts

1st function f(x) = logx & 2^{nd} function g(x) = 1

We know that $\int f(x) g(x) dx = f(x) \int g(x) dx - \int \left(f^1(x) \int g(x) dx\right) dx$

$$\therefore \int (\log x).1dx = \log x \int 1dx - \int \left(\frac{d}{dx}(\log x)\int 1dx\right)dx$$

$$= \log x. \ x - \int \left(\frac{1}{x}.\int 1 \ dx\right) dx$$

$$= x \log x - \int \frac{1}{x}.x \ dx$$

$$= x \log x - \int dx$$

$$v = x \log x - x$$

$$\therefore y_p = \left(\frac{-x^2 \log x}{2} + \frac{x^2}{4}\right) y_1 + (x \log x - x) y_2$$

$$= \left(\frac{-x^2 \log x}{2} + \frac{x^2}{4}\right) e^{-x} + (x \log x - x) \ x \ e^{-x}$$

$$= \frac{-x^2 \log x}{2} e^{-x} + \frac{x^2}{4} e^{-x} + x^2 \log x e^{-x} - x^2 e^{-x}$$

$$y_p = \frac{x^2}{2} \log x \ e^{-x} - \frac{3}{4} x^2 e^{-x}$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + c_2 x e^{-x} + \frac{x^2}{2} \log x \ e^{-x} - \frac{3}{4} x^2 \ e^{-x}$$

Q6 Solve $y^{11} - 2y^1 + y = e^x \log x$.

Sol :

The given equation is $y^{11} - 2y^1 + y = e^x \log x$ ... (1)

compare (1) with $a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 = Q(x)$

here $a_2 = 1$, $a_1 = -2$, $a_0 = 1$, $Q(x) = e^x \log x$

To find y_c :

The A. E is $m^2 - 2m + 1 = 0$

$(m - 2)^2 = 0$

$m = 1,1$

$\therefore y_c = (c_1 + c_2 x)e^x$ ... (2)

compare (2) with $y_c = c_1y_1 + c_2 y_2$

$y_1 = e^x, y_2 = xe^x$

$y_1^1 = e^x, \; y_2^1 = xe^x + e^x$

To find y_p:

$$u = -\int \frac{Q(x)y_2}{a_2\left(y_1y_2^{\,1} - y_2y_1^{\,1}\right)}dx, \; v = \int \frac{Q(x)y_1}{a_2\left(y_1y_2^{\,1} - y_2y_1^{\,1}\right)}dx$$

Since $y_1 y_2^{\,1} - y_2 y_1^{\,1} = e^x (xe^x + e^x) - x e^x (e^x)$

$= x e^{2x} + e^{2x} - xe^{2x}$

$y_1 y_2^{\,1} - y_2y_1^{\,1} = e^{2x}$

$$\therefore u = -\int \frac{e^x \log x.\, xe^x}{e^{2x}}dx$$

$$= -\int \frac{x \log x \;\; \cancel{e^{2x}}}{\cancel{e^{2x}}}dx$$

$$= -\int x \log x \, dx$$

Using by parts

1st function $f(x) = \log x$, 2nd function $g(x) = x$

We know that

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left[\frac{d}{dx}f(x)\int g(x)dx\right]dx$$

$$\Rightarrow -\left[\log x\int xdx - \int\left(\frac{1}{x}\int x \, dx\right)dx\right.$$

$$= -\left[\log x.\frac{x^2}{2} - \int\left(\frac{1}{x}.\frac{x^2}{2}\right)dx\right.$$

$$= -\left[\log x.\frac{x^2}{2} - \frac{1}{2}\int x \, dx\right.$$

$$= -\left[\log x.\frac{x^2}{2} - \frac{1}{2}\frac{x^2}{2}\right]$$

$$U = \frac{-x^2 \log x}{2} + \frac{x^2}{4}$$

$$V = \int \frac{e^x \log x.e^x}{e^{2x}}dx$$

$$= \int \frac{\log x.e^{2x}}{e^{2x}} dx$$

$$= \int \log x \; dx$$

Using by parts

1st function f(x) = logx, 2nd function g(x) = 1

$$\Rightarrow \int f(x)\, g(x) dx = \left[f(x) \int g(x) dx - \int \left(f'(x) \int g(x)\, dx \right) \right] dx$$

$$= \log x \int 1 dx - \int \left(\frac{1}{x} . \int dx \right) dx$$

$$= \log x.x - \int \left(\frac{1}{x} .x \right) dx$$

$$= \log x.x - \int dx$$

$$v = x \log x - x$$

$$\therefore y_p = \left(\frac{-x^2 \log x}{2} + \frac{x^2}{4} \right) y_1 + (x \log x - x) y_2$$

$$= \left(\frac{-x^2 \log x}{2} + \frac{x^2}{4} \right) e^x + (x \log x - x) xe^x$$

$$= \frac{-x^2 \log x}{2} e^x + \frac{x^2}{4} e^x + x^2 \log x \; e^x - x^2 \; e^x$$

$$y_p = \frac{1}{2} \log x \; e^x - \frac{3}{4} x^2 \; e^x$$

$\therefore$ The complete solution is

$$y = c_1 \, e^x + c_2 \, x \, e^x + \frac{1}{2} \log x \; e^x - \frac{3}{4} x^2 \; e^x$$

Q7. Solve $(D^2 - 3D + 2)\, y = \sin e^{-x}$

Sol :

The given equation is $(D^2 - 3D + 2)\, y = \sin e^{-x}$

$$\Rightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin e^{-x} \quad .... (1)$$

Compare (1) with $a_1 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 = Q(x)$

Where $a_2 = 1$, $a_1 = -3$, $a_0 = 2$, $Q(x) = \sin e^{-x}$

To find y_c

The A. E is $m^2 - 3m + 2 = 0$

$m - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-1)\ (m-2) = 6$

$m = 1, 2$

$\therefore\ y_c = c_1 e^x + c_2 e^{2x}$ ....(2)

Compare (2) with $y_c = c_1 y_1 + c_2 y_2$

here $y_1 = e^x$, $y_2 = e^{2x}$

$y_1^1 = e^x$, $y_2^1 = 2e^{2x}$

To find y_p

$$u = -\int \frac{Q(x)y_2}{a_2\left(y_1y_2^{\ 1} - y_2y_1^{\ 1}\right)}dx \quad ; \quad v = \int \frac{Q(x)y_1}{a_2\left(y_1y_2^{\ 1} - y_2y_1^{\ 1}\right)}dx$$

Here $y_1y_2^1 - y_2\, y_1^1 = e^x\,(2e^{2x}) - e^{2x}\,(e^x)$

$= 2e^{2x}.e^x - e^{2x}e^x$

$= 2e^{3x} - e^{3x}$

$y_1y_2^1 - y_2\, y_1^1 = e^{3x}$

$$\therefore\ U = -\int \frac{\sin e^{-x}.e^{2x}}{e^{3x}}dx$$

$$= -\int \sin e^{-x}.e^{2x}e^{-3x}dx$$

$$= -\int \left(\sin e^{-x}\right)e^{-x}dx$$

$$= \int \left(\sin e^{-x}\right)\left(-e^{-x}\right)dx$$

Let $e^{-x} = t$

$-e^{-x}dx = dt$

$$= \int \sin t.dt$$

$= -\cos t$

$U = -\cos e^{-x}$

$$\therefore\ V = \int \frac{\sin e^{-x}.e^x}{e^{3x}}dx = \int \sin e^{-x}e^x.e^{-3x}dx$$

$$= \int \sin e^{-x}.e^{-2x}dx$$

$$= \int e^{-x}\sin e^{-x}\left(e^{-x}\right)dx$$

$$= -\int e^{-x}\sin e^{-x}\left(-e^{-x}\right)dx$$

Let $t = e^{-x}$

$dt = -e^{-x}\,dx$

$$= -\int t\sin t\ dt$$

$= -\left[t\int \sin t dt - \int (1)\int (\sin t\ dt)dt\right]$

$= -\left[-t\cos t - \int (-\cos t)\, dt\right]$

$= -\ [-t\cos t + \sin t]$

$= -\sin t + t\cos t \qquad [\because e^{-x} = t]$

$= -\sin e^{-x} + e^{-x}\cos e^{-x}$

$\therefore V = -\sin e^{-x} + e^{-x}\cos e^{-x}$

$\therefore y_p = (-\cos e^{-x})\, y_1 + (-\sin e^{-x} + e^{-x}\cos e^{-x})\, y_2$

$= (-\cos e^{-x})\, e^{x} + (-\sin e^{-x} + e^{-x}\cos e^{-x})\, e^{2x}$

$= -e^{x}\cos e^{-x} - e^{2x}\sin e^{-x} + e^{x}\cos e^{-x}$

$y_p = -e^{2x}\sin e^{-x}$

$\therefore$ The complete solution is

$y = c_1 e^{x} + c_2 e^{2x} - e^{2x}\sin e^{-x}$

4.2 Linear Differential Equation with non constant coefficients

Q8. Derive Reduction of order of method.

Ans :

Consider the general linear differential equation.

$$f_n(x)\frac{d^n y}{dx^n} + f_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + f_1(x)\frac{dy}{dx} + f_0(x)y = Q(x) \qquad ...\ (I)$$

and its related homogeneous equation

$$f_n(x)\frac{d^x y}{dx^n} + f_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + f_1(x)\frac{dy}{dx} + f_0(x)y = 0 \qquad ...\ (II)$$

where $f_n(x)$, $f_{n-1}(x)$ ......$f_1(x)$, $f_0(x)$ and Q(x) are continuous function of x and $f_n(x) \neq 0$

Let us consider

$$f_2(x)\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_0(x)\,y = Q(x) \qquad ...\ (1)$$

and its related homogenous equation

$$f_2(x)\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_0(x)\,y = 0 \qquad ...\ (2)$$

Let $y_1 \neq 0$ be a independent solution of (2)

$\Rightarrow y_1$ satisfies equation (2)

$f_2(x)\, y^{11} + f_1(x)\, y_1^{1} + y_1 = 0 \qquad ...\ (3)$

Let the second independent solution be defined as

$y_2(x) = y_1(x)\int u(x)dx \qquad ...\ (4)$ (where u(x) is function x to be determined)

Now differentiate $y_2 = y_1 \int u dx$

$$y_2^1 = y_1 u + \int u dx . y_1^1$$

Again differentiate

$$y_2^{11} = y_1 u^1 + u y_1^1 + \int u\, dx\, y_1^{11} + y_1^1 u$$

$$y_2^{11} = y_1 u^1 + 2u\, y_1^1 + \int u\, dx\, y_1^{11}$$

Since y_2 is a solution of equation (2)

we have $f_2(x)\, y_2^{11} + f_1(x) y_2^1 + f_0(x)\, y_2 = 0$ ... (5)

Now substitute the values of y_2, y_2^1 , y_2^{11}

we get in equation (5)

$$f_2(x)\left[y_1 u^1 + 2u y_1^1 + \int u dx . y_1^{11}\right] + f_1(x)\left[y_1 u + \int u dx . y_1^1\right] + f_0(x)\left[y_1(x)\int u(x) dx\right] = 0$$

$$\Rightarrow f_2(x)\, y_1 u^1 + 2f_2(x)\, u y_1^1 + \int u\, dx . y_1^{11}\, f_2(x) + f_1(x) y_1 u + f_1(x)\int u\, dx .\, y_1^1 + f_0(x) . y_1(x) \int u(x) dx = 0$$

$$\Rightarrow \int u(x)\left[f_2(x) y_1^{11} + f_1(x) y_1^1 + f_0(x) y_1\right] + f_2(x) y_1 u^1 + u\left[2f_2(x) y_1^1 + f_1(x) y_1\right]$$

By using (2) we get

$$\int u dx[0] + f_2(x) y_1 u^1 + u\left[2f_2(x) y_1^1 + f_1(x) y_1\right] = 0$$

Multiple the above equation by, $\dfrac{dx}{u y_1 f_2(x)}$

$$f_2(x) y_1 u^1 . \frac{dx}{u y_1 f_2(x)} + u\left(2f_2(x) y_1^1 + f_1(x) y_1\right)\frac{dx}{u\, y_1 f_2(x)} = 0$$

$$\frac{u^1}{u} dx + 2 f_2(x) y_1^1 \frac{dx}{y_1 f_2(x)} + \frac{f_1(x) y_1}{y_1 f_2(x)} dx = 0$$

$$\frac{u^1}{u} dx + 2\frac{y_1^1}{y_1} dx + \frac{f_1(x)}{f_2(x)} dx = 0$$

Replacing u^1 by $\dfrac{du}{dx}$, y_1^1 by $\dfrac{dy_1}{dx}$

$$\frac{du}{dx} \cdot \frac{dx}{u} + \frac{2}{y_1} \cdot \frac{dy_1}{dx} \cdot dx + \frac{f_1(x)}{f_2(x)} dx = 0$$

$$\frac{du}{u} + \frac{2}{y_1} dy_1 + \frac{f_1(x)}{f_2(x)} dx = 0$$

By variation separable and Integrating Both sides

$$\int \frac{du}{u} + 2\int \frac{1}{y_1} dy_1 + \int \frac{f_1(x)}{f_2(x)} dx = 0$$

$$\log u + 2 \log y_1 = -\int \frac{f_1(x)}{f_2(x)} dx$$

$$\log u + \log y_1^2 = -\int \frac{f_1(x)}{f_2(x)} dx$$

$$\log (uy_1^2) = -\int \frac{f_1(x)}{f_2(x)} dx$$

$$uy_1^2 = e^{-\int \frac{f_1(x)}{f_2(x)} dx}$$

$$u = \frac{e^{-\int \frac{f_1(x)}{f_2(x)}}}{y_1^2} dx$$

Substituting this values of u in equation (4) we get required solutions

Q9. Solve $x^2 y^{11} - xy^1 + y = 0$ given $y_1 = x$ as a solution . By using reduction of order of method.

Sol :

Given that $x^2 y^{11} - xy^1 + y = 0$ ... (1)

Comparing (1) with $f_2(x) \frac{d^2y}{dx^2} + f_1(x) \frac{dy}{dx} + f_0(x)\, y = 0$

here $f_2(x) = x^2$, $f_1(x) = -x$

and also $y_1 = x$

$$u = \frac{\exp\left[-\int \frac{f_1(x)}{f_2(x)}\right]}{y_1^2} dx$$

$$= \frac{\exp\left[-\int \frac{-x}{-x^2}\right]}{(x)^2} dx$$

$$= \frac{\exp\left[\frac{1}{x}\right]}{x^2} dx$$

$$= \frac{\exp[\log x]}{x^2} = \frac{e^{\log x}}{x^2} \Rightarrow \frac{x}{x^2} = x^{-1}$$

$\therefore u = x^{-1}$

Since $y_2 = y_1 \int u(x) dx$

$$= x\int x^{-1} dx$$

$$= x\int \frac{1}{x} dx$$

$y_2 = x \log x$

$\therefore$ The required solution is $y = c_1 y_1 + c_2 y_2$

$\Rightarrow c_1 x + c_2 x \log x$

Q10. Given that $y_1 = x$ is a solution of $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$, $x \neq 0$ find the general solution of $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x$.

Sol :

Given that $y_1(x) = x$

Thus $y_2(x) = y_1(x) \int u(x) dx$

$y_2(x) = x\int u(x) dx$ ... (1)

differentiating (1) it twice

$y_2^1(x) = xu + \int u(x) dx$ ... (2)

$y_2^{11}(x) = xu^1 + u + u(x)$

$y_2^{11}(x) = xu^1 + 2u$ ... (3)

and also given that

$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x$ ... (4)

Since y_2 is a solution of equation (4)

we have $x^2 y_2^{11} + xy_2^1 - y_2 = x$ ... (5)

Now substitute the values (1), (2), (3) in (5)

$$x^2\left[xu^1 + 2u\right] + x\left[xu + \int u(x) dx\right] - x\int u(x) dx = x$$

$x^3u^1 + 2x^2u + x^2 u +$

$$x\int u(x) dx - x\int u(x) dx = x$$

$x^3 u^1 + u[2x^2 + x^2] = x$

divide by ' x^3 '

$$u^1 + \frac{u}{x^3}[3x^2] = \frac{x}{x^3}$$

$$u^1 + u\frac{3}{x} = x^{-2}$$

Which is a linear equation and which is of the form $\frac{du}{dx} + PU = Q$

Then the Integrating factor is

$$\text{i.e. I.F} = e^{\int p\,dx} = e^{\int \frac{3}{x}\,dx} = e^{3\log x}$$
$$= e^{\log x^3}$$
$$= x^3$$

$\therefore$ The solution of the given differential equation is

$$U.\,I.F = \int Q(x)(I.F)dx + c$$

$$U.x^3 = \int x^{-2}.x^3 dx + c$$

$$Ux^3 = \int x dx + c$$

$$Ux^3 = \frac{x^2}{2} + c$$

$$U = \frac{x^2}{2x^3} + \frac{c}{x^3}$$

$$\therefore \quad U = \frac{x^{-1}}{2} + cx^{-3}$$

Now substitute u value in equation (1)

$$y_2 = x\int\left(\frac{x^{-1}}{2} + cx^{-3}\right)dx$$

$$= x\left[\int \frac{1}{2x}dx + c\int\frac{1}{x^3}dx\right]$$

$$= x\left[\frac{1}{2}\log x - c\frac{1}{2x^2}\right]$$

$$= \frac{x}{2}\log x - c\frac{x^{-1}}{2}$$

$\therefore$ The required solution is $y = c_1 y_1 + c_2 y_2$

$$y = c_1 x + c_2\frac{x}{2}\log x - c\frac{x^{-1}}{2}$$

Q11. Solve $y^{11} - \frac{2}{x}y^1 + \frac{2}{x^2}y = 0,\ y_1 = x$ by using reduction of order method

Sol :

Given that $y^{11} - \frac{2}{x}y^1 + \frac{2}{x^2}y = 0$,

$y_1 = x$ ... (1)

Compare (1) with $f_2(x)\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_0(x)y = 0$

here $f_2(x) = 1$, $f_1(x) = \frac{-2}{x}$

and, also $y_1 = x$

$$\therefore u = \frac{\exp\left[-\int\frac{f_1(x)}{f_2(x)}\right]dx}{{y_1}^2}$$

$$u = \frac{\exp -\int\frac{\frac{-2}{x}}{1}dx}{x^2}$$

$$= \frac{\exp\left[\int\frac{2}{x}\right]dx}{x^2}$$

$$= \frac{\exp\left[\int 2\log x\right]}{x^2}$$

$$= \frac{e^{\log x^2}}{x^2} = \frac{x^2}{x^2} = 1$$

$\therefore u = 1$

Since $y_2 = y_1\int u(x)dx$

$$= x\int 1dx$$
$$= x.x$$
$$y_2 = x^2$$

$\therefore$ The required solution is

$y = c_1 y_1 + c_2 y_2$

$y = c_1 x + c_2 x^2$

Q12. Solve $x^2 y^{11} + xy^1 - y = x^2 e^{-x}$, $y_1 = x$ By using reduction of order method.

Sol :

Given that $x^2 y^{11} + xy^1 - y = x^2 e^{-x}$ ... (1)

$y_1 = x$

Compare (1) with $f_2(x) y^{11} + f_1(x) y^1 + f_0(x) y = Q(x)$

$f_2(x) = x^2$, $f_1(x) = x$, $f_0(x) = -1$

We know that $y_1 = x$

Then $y_2(x) = y_1(x) \int u(x)dx$

$\Rightarrow y_2(x) = x\int u(x)dx$ ... (2)

differentiate (2) it twice

$y_2^1(x) = x\,u + \int u(x)dx$ ... (3)

$y_2^{11}(x) = xu^1 + u + u$

$y_2^{11}(x) = xu^1 + 2u$ ... (4)

Since y_2 is a solution of (1)

we have $x^2 y_2^{11} + xy_2^1 - y_2 = x^2e^{-x}$

Now substitute (2), (3) & (4) in above equation

$$x^2\left[xu^1 + 2u\right] + x\left[xu + \int u(x)\,dx\right] - x\int u(x)dx = x^2e^{-x}$$

$$x^3u^1 + 2x^2u + x^2u + x\int u(x)dx - x\int u(x)dx = x^2e^{-x}$$

$x^3 u^1 + 3x^2 u = x^2 e^{-x}$

divide by x^3

$$u^1 + 3\frac{x^2}{x^3}u = \frac{x^2e^{-x}}{x^3}$$

$$u^1 + \frac{3}{x}u = \frac{x^2e^{-x}}{x^3}$$

$$u^1 + \frac{3}{x}u = x^{-1}e^{-x}$$

Which is a linear equation and which is of the form $\frac{du}{dx} + Pu = Q(x)$

Then the I.F. $= e^{\int Pdx} = e^{\int \frac{3}{x}dx} = e^{\log x^3}$

$IF = x^3$

$\therefore$ The solution of the given differential equation is

$$U\,(IF) = \int Q(x)\,(IF)\,dx$$

$$Ux^3 = \int x^{-1}e^{-x}.x^3 dx$$

$$Ux^3 = \int x^2 e^{-x} dx$$

$$= x^2\int e^{-x} - \int 2x\left[\int e^{-x}dx\right]dx$$

$$= \frac{x^2e^{-x}}{-1} - \int 2x\frac{e^{-x}}{-1}dx$$

$$= -x^2e^{-x} + 2\left[x\int e^{-x} - \int 1\int e^{-x}dx.dx\right]$$

$$= -x^2e^{-x} + 2\left[\frac{xe^{-x}}{-1} - \int\frac{e^{-x}}{-1}dx\right]$$

$$= -x^2e^{-1} + 2\left[-xe^{-x} + \left[\frac{e^{-x}}{-1}\right]\right]$$

$$Ux^3 = -x^2\,e^{-x} - 2x\,e^{-x} - 2e^{-x}$$

$$U = \frac{-x^2}{x^3}\,e^{-x} - \frac{2x}{x^3}\,e^{-x} - \frac{2}{x^3}\,e^{-x}$$

$$U = -x^{-1}\,e^{-x} - 2x^{-2}\,e^{-x} - 2x^{-3}\,e^{-x}$$

Now substitute u in (2)

$$y_2(x) = x\int\left(-x^{-1}e^{-x} - 2x^{-2}e^{-x} - 2x^{-3}e^{-x}\right)dx$$

Q13. Solve $y^{11} - \frac{2}{x}y^1 + \frac{2}{x^3}y = x\log x$, $y_1 = x$ by using reduction of order method.

Sol :

The given equation $y^{11} - \frac{2}{x}y^1 + \frac{2}{x^3}y = x\log x$, ... (1)

$y_1 = x$

We know that $y_1 = x$

Then $y_2(x) = y_1(x)\int u(x)dx$

$y_2(x) = x\int u(x)dx$ ... (2)

Differentiate twice

$y_2^1(x) = xu + \int u(x)dx$ ... (3)

$y_2^{11}(x) = xu^1 + u + u$

$y_2^{11}(x) = xu^1 + 2u$ ....(4)

Since y_2 is a solution of (1)

we have $y_2^{11} - \frac{2}{x} y_2^{1} + \frac{2}{x^2} y_2 = x \log x$

substitute (2), (3), (4) in above equation

$$xu^1 + 2u - \frac{2}{x}\left(xu + \int u(x)dx\right) + \frac{2}{x^2}\left(x\int u(x)dx\right) = x\log x$$

$$xu^1 + 2u - \frac{2}{x}xu - \frac{2}{x}\int u(x)dx + \frac{2}{x^2}.x\int u(x)dx = x\log x$$

$$xu^1 + 2u - \frac{2}{x}xu = x\log x$$

$xu^1 = x \log x$

$u = \log x$

$\frac{du}{dx} = \log x$

$du = \log x \, dx$

By integrating

$\int du = \int \log x dx$

$u = x. \log x - x$

Substitute u in (2)

$$y_2(x) = x\int (x\log x - x)dx$$

$$= x\left[\int x \log x \, dx - \int x \, dx\right]$$

$$= x\left[\log x\int x \, dx - \int \frac{1}{x}\int x \, dx - \int x \, dx\right]$$

$$= x\left[\log x\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2} - \frac{x^2}{2}\right]$$

$$= x\left[\log x.\frac{x^2}{2} - \frac{1}{2}\int x - \frac{x^2}{2}\right]$$

$$= x\left[\frac{x^2}{2}\log x - \frac{1}{2}\frac{x^2}{2} - \frac{x^2}{2}\right]$$

$$= \frac{x^3}{2}\log x - \frac{x^3}{4} - \frac{x^3}{2}$$

$$= \frac{x^3}{2}\log x - \frac{x^3 - 2x^3}{4}$$

$$y_2(x) = \frac{x^3}{2}\log x - \frac{3x^3}{4}$$

$\therefore$ The required solution is $y = c_1 y_1 + c_2 y_2$

$$= c_1(x) + c_2\left(\frac{x^3}{2}\log x - \frac{3}{4}x^3\right)$$

$$y = c_1 x + c_2\frac{x^3}{2}\log x - c_2\frac{3}{4}x^3$$

Q14. Solve $(2x^2+1)y^{11} - 4xy^1 + 4y = 0$, $y_1 = x$.

Sol :

Given that $(2x^2+1)y^{11} - 4xy^1 + 4y = 0$ ....(1)

compare (1) with $f_2(x)y^{11} + f_1(x)y^1 + f_0(x)y = 0$

her $f_2(x) = 2x^2+1$, $f_1(x) = -4x$ and $y_1 = x$

$$\therefore \quad u = \frac{\exp\left[-\int \frac{f_1(x)}{f_2(x)}\right]dx}{y_1^2}$$

$$= \frac{\exp\left[-\int \frac{-4x}{2x^2+1}\right]dx}{x^2}$$

$$= \frac{\exp\left[\int \frac{4x}{2x^2+1}\right]dx}{x^2} \quad \left[\because \int \frac{f'(x)}{f(x)} = \log f(x)\right]$$

$$= \frac{\exp\left[\log\left(2x^2+1\right)\right]dx}{x^2}$$

$$= \frac{e^{\log\left(2x^2+1\right)}}{x^2}$$

$$U = \frac{2x^2+1}{x^2}$$

$$U = 2 + \frac{1}{x^2}$$

Substitute U in $y_2(x) = y_1(x)\int u\,dx$

$$= x\int\left(2 + \frac{1}{x^2}\right)dx$$

$$= x\left[2\int dx + \int \frac{1}{x^2}dx\right]$$

$$= x\left[2x - \frac{1}{x}\right]$$

$$= 2x^2 - \frac{x}{x}$$

$$y_2 = 2x^2 - 1$$

$\therefore$ The required solution is

$y = c_1 y_1 + c_2 y_2$

$y = c_1 x + c_2(2x^2 - 1)$

Q15. Solve $y^{11} - \frac{2}{x}y^1 + \frac{2}{x^2}y = 0$, $y_1 = x$

Sol :

The given equation $y^{11} - \frac{2}{x}y^1 + \frac{2}{x^2}y = 0$

Compare (1) with $f_2(x)y^{11} + f_1(x)y^1 + f_0(x)y = 0$

here $f_2(x) = 1$, $f_1(x) = \frac{-2}{x}$

and $y_1 = x$

$$\therefore \quad U = \frac{\exp\left[-\int \frac{f_1(x)}{f_2(x)}\right]dx}{y_1^2}$$

$$= \frac{\exp\left[-\int \frac{\frac{-2}{x}}{1}\right]}{x^2}dx$$

$$= \frac{\exp\left[2\int \frac{1}{x}dx\right]}{x^2}$$

$$= \frac{\exp[2\log x]}{x^2} = \frac{e^{\log x^2}}{x^2} = \frac{x^2}{x^2}$$

$\therefore \quad U = 1$

Substitute U in $y_2(x) = y_1 \int u(x)\, dx$

$$= x\int 1\, dx$$

$$= x.x$$

$$y_2(x) = x^2$$

$\therefore$ The required solution is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 x + c_2 x^2$$

4.3 The Cauchy - Euler Equation

Q16. Define Cauchy Euler Equation.

Sol :

An equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_1 x \frac{dy}{dx} + a_0 y = Q(x)$$

where $a_0, a_1, a_2 \ldots\ldots a_{n-1}, a_n$ are constant is called a Cauchy Euler equation of order 'n'. Let us consider second order Cauchy. Euler equation as

$$a_2 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = Q(x) \quad \ldots (1)$$

Take $x = e^t \Rightarrow \log x = t$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Thus, $x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

hence from (1) we obtain

$$a_2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a_1\frac{dy}{dt} + a_0 y = Q(e^t)$$

$$A_2\frac{d^2y}{dt^2} + A_1\frac{dy}{dt} + A_0 y = R(t) \ldots\ldots\ldots.(2)$$

where $A_2 = a, A_1 = a_1 - a_2, A_0 = a_0$

and $R(t) = Q(e^t)$

Equation (2) is a linear differential equation with constant coefficients

Q17. Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$

Sol :

Given that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$ is a Cauchy Euler's equation.

Put $x = e^t \Rightarrow t = \log x$ and $D_1 = \frac{d}{dt}$

The given equation is

$$x^2 D^2 y + x\, Dy - 4y = x^2$$

Now, $xDy = D_1 y$ and $x = e^t$

$$x^2 D^2 y = D_1(D_1 - 1)y$$

Then the given differential equation can be written as

$$D_1(D_1 - 1)y + D_1 y - 4y = (e^t)^2$$

$$(D_1^2 - D_1 + D_1 - 4)y = e^{2t}$$

$$(D_1^2 - 4)y = e^{2t}$$

Now, the A. E is $m^2 - 4 = 0$

$$m^2 - 2^2 = 0$$

$$m = \pm 2$$

$\therefore y_c = c_1 e^{2x} + c_2 e^{-2x}$

Now, the particular Integral is

$$P.I = y_p = \frac{1}{D_1^2 - 4} e^{2t}$$

$$= e^{2t}\frac{1}{(D_1+2)^2 - 4} (1)$$

$$= e^{2t}\frac{1}{D_1^2 + 4D_1 + 4 - 4} (1)$$

$$= e^{2t}\frac{1}{D_1^2+4D_1}(1)$$

$$= e^{2t}\frac{1}{4D_1\left[1+\frac{D_1}{4}\right]}(1)$$

$$= e^{2t}\frac{1}{4D_1}\left[1+\frac{D_1}{4}\right]^{-1}(1)$$

$$= e^{2t}\frac{1}{4D_1}[1]$$

$$= \frac{e^{2t}}{4}$$

$$= \frac{e^{2t}}{4}\log x$$

$$y_p = \frac{1}{4}x^2 \log x$$

$\therefore$ The required solution is

$$y = y_c + y_p$$

$$= C_1e^{2x} + C_2 e^{-2x} + \frac{1}{4}x^2 \log x.$$

Q18. Solve $x^2 D^2y - x Dy - 3y = x^2 \log x$

Sol :

Given that $x^2 D^2 y - x Dy - 3y = x^2 \log x$ ....(1)

Putting $x = e^t \Rightarrow t = \log x$ and $D_1 = \frac{d}{dt}$

and $xDy = D_1 y$

$x^2 D^2 y = D_1 (D_1 - 1) y$

Then the equation (1) becomes

$D_1 (D_1-1)y - D_1y - 3y = (e^t)^2\, t$

$(D_1^2 - D_1 - D_1 - 3)\, y = e^{2t}.t$

$(D_1^2 - 2D_1 - 3)y = te^{2t}$

Then the A.E is $m^2 - 2m - 3 = 0$

$m^2 - 3m + m - 3 = 0$

$m (m - 3) + 1(m - 3) = 0$

$(m - 3)(m + 1) = 0$

$m = -1, 3$

$\therefore y_c = c_1e^{-x} + c_2 e^{3x}$

Now,

$$P.I = y_p = \frac{1}{D_1^2 - 2D_1 - 3}te^{2t}$$

$$(D_1 = D+2)$$

$$= e^{2t}\frac{1}{(D+2)^2 - 2(D+2) - 3}t$$

$$= e^{2t}\frac{1}{D^2+4+4D-2D-4-3}t$$

$$= e^{2t}\frac{1}{D^2+2D-3}t$$

$$= e^{2t}\frac{1}{-3\left[1-\left(\frac{D^2+2D}{3}\right)\right]}t$$

$$= \frac{e^{2t}}{-3}\left[1-\left(\frac{D^2}{3}+\frac{2}{3}D\right)\right]^{-1}t$$

$$= \frac{e^{2t}}{-3}\left[1+\left(\frac{D^2}{3}+\frac{2}{3}D\right)\right]t$$

$$= \frac{e^{2t}}{-3}\left[t+\frac{2}{3}(1)\right]$$

$$= \frac{e^{2t}}{-3}\left(t+\frac{2}{3}\right)$$

$$y_p = \frac{x^2}{-3}\left(\log x+\frac{2}{3}\right)$$

$\therefore$ The required solution is $y = y_c + y_p$

$$y = c_1 e^{-x} + c_2 e^{3x} - \frac{x^2}{3}(\log x + \frac{2}{3})$$

Q19. Solve $x^3 D^3 y + 3x^2 D^2 y + x Dy + y = x + \log x$

Sol :

The given equation is

$x^3 D^3 y + 3x^2 D^2 y + x D y + y = x + \log x$ ....(1)

Putting $x = e^t \Rightarrow t = \log x$ and $D_1 = \frac{d}{df}$

and $x^3D^3 y = D_1 (D_1 - 1) (D_1 - 2) y$

$x^2 D^2 y = D_1 (D_1 - 1) y$

$x D y = D_1 y$

Then the equation (1) can be written as

$[D_1(D_1-1) (D_2 - 2) + 3 [D_1 (D_1 -1) + D_1 +1] y = e^t + t$

$[(D_1^2 - D_1) (D_1 - 2) + 3D_1^2 - 3D_1 + D_1 +1] y = e^t + t$

$$\left[D_1^3 - 2D_1^1 - D_1^2 + 2D_1 + 3D_1^2 - 3D_1 + D_1 + 1\right] y = e^t + t$$

$$\left[D_1^3 - \cancel{3D_1^2} + \cancel{3D_1} + \cancel{3D_1^2} - \cancel{3D_1} + 1\right] y = e^t + t$$

$(D_1^3 + 1) y = e^t + t$

$\therefore$ Then the A. E is $m^3 + 1 = 0$

$(m + 1) (m^2 - m(1) + 1^2) = 0$

$(m+1) (m^2 - m + 1) = 0$

$m + 1 = 0, m^2 - m + 1 = 0$

$$m = -1, m = \frac{-(-1) \pm \sqrt{1 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2}$$

$$m = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$\therefore$ The roots are $m = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$$\therefore y_c = c_1 e^{-t} + e^{t/2}\left(c_2 \cos\frac{\sqrt{3}}{2} t + c_3 \sin\frac{\sqrt{3}}{2} t\right)$$

$$= c_1 e^{-\log x} + x^{1/2}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right)$$

$$y_c = c_1\frac{1}{x} + \sqrt{x}\left(c_2\cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right)$$

Now,

$$\text{P.I. } y_p = \frac{1}{D_1^3+1}\, e^t + t$$

$$= \frac{1}{D_1^3+1}e^t + \frac{1}{D_1^3+1}t$$

$$= e^t\,\frac{1}{1+1} + \frac{1}{\left(1+D_1^3\right)}\,t$$

$$= \frac{e^t}{2} + \left[1+D_1^3\right]^{-1} t$$

$$= \frac{e^t}{2} + [1-(D_1^3) + (D_1^3)^2]\, t$$

$$= \frac{e^t}{2} + t$$

$$y_p = \frac{x}{2} + \log x$$

$\therefore$ The required solution is $y = c_1\frac{1}{x} + \sqrt{x}\left(c_2\cos\left(\frac{\sqrt{3}}{2}\log x\right)\right) + c_3\left(\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right) + \frac{x}{2} + \log x$

Q20. Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x$

Sol :

The given equation is $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x$ .... (1)

Putting $x = e^t$, $t = \log x$

and $x^2 D^2 y = D_1(D_1 - 1)$

$x D y = D_1 y$, $\quad D_1 = \frac{d}{dt}$

Then equation (1) becomes

$D_1(D_1 - 1)y - D_1 y + y = 2t$

$(D_1^2 - D_1 - D_1 + 1)y = 2t$

$\left(D_1^2 - 2D_1 + 1\right)y = 2t$

then the Auxiliary equation is

$m^2 - 2m + 1 = 0$

$(m - 1)^2 = 0$

$m = 1, 1$

$\therefore y_c = (c_1 + c_2 x) e^x$

Now

$$\text{I. P} = y_p = \frac{1}{D_1^2 - 2D_1 + 1} 2t$$

$$= 2\frac{1}{(D_1 - 1)^2} t$$

$= 2[(1 - D_1)]^{-2} t$

$= 2[1 - 2D_1 + 3D_1^2] t$

$= 2[t - 2(1)]$

$= 2t - 4$

$y_p = 2 \log x - 4$

$\therefore$ The required solution is $y = (c_1 + c_2 x) e^x + 2 \log x - 4$

Q21. Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\log x$

Sol :

Given that $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x \log x$ ...(1)

Putting $x = e^t$ and $t = \log x$

$x^2 D^2 y = D_1 (D_1 - 1) y$

$x D y = D_1 y$

Then the equation (1) can be written as

$D_1 (D_1 - 1) y - D_1 y + 2y = e^t .t$

$(D_1^2 - D_1 - D_1 + 2) y = t e^t$

$(D_1^2 - 2D_1 + 2) y = t e^t$

Then the A. E is $m^2 - 2m + 2 = 0$

$$m = \frac{-(-2) \pm \sqrt{4 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2} \Rightarrow 1 \pm i$$

i.e. $m = 1 \pm i$

$\therefore y_c = e^t (c_1 \cos t + c_2 \sin t)$

$y_c = x(c_1 \cos (\log x) + c_2 \sin (\log x)$

Now

$$\text{P. I} = y_p = \frac{1}{D_1^2 - 2D_1 + 2} t e^t$$

$$= e^t \frac{1}{(D_1 + 1)^2 - 2(D_1 + 1) + 2} t$$

$$= e^t \frac{1}{D_1^2 + 1 + \cancel{2D_1} - \cancel{2D_1} - \cancel{2} + \cancel{2}} t$$

$$= e^t \frac{1}{D_1^2 + 1} t$$

$= e^t [1 + D_1^2]^{-1} t$

$= e^t [1 - D_1^2 + (D_1^2)^2] t$

$= e^t [t]$

$= t e^t$

$y_p = \log x. x$

$\therefore$ The required solution is $y = y_c + y_p$

$y = x(c_1 \cos(\log x) + c_2 \sin(\log x) + x \log x$

Q22. Solve $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$

Sol :

The given equation is

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x \quad ....(1)$$

Putting $x = e^t$ and $t = \log x$, $D_1 = \frac{d}{dt}$

and $x^2 D^2 y = D_1 (D_1 - 1) y$

$x D y = D_1 y$

then the equation (1) becomes

$D_1 (D_1 - 1) y + 4 D_1 y + 2 y = e^x$

$(D_1^2 - D_1 + 4 D_1 + 2) y = e^x$

$(D_1^2 + 3D_1 + 2) y = e^x$

Then the A. E is

$m^2 + 3m + 2 = 0$

$m^2 + 2m + m + 2 = 0$

$(m + 1)(m + 2) = 0$

$m = -1, -2$

$\therefore y_c = c_1 e^{-t} + c_2 e^{-2t}$

$= c_1 (e^t)^{-1} + c_2 (e^t)^{-2} = c_1 x^{-1} + c_2 x^{-2}$

Now

$$P.I = y_p = \frac{1}{D_1^2 + 3D_1 + 2} e^x$$

$$= \frac{1}{(D_1+1)(D_1+2)} e^x$$

Consider $\frac{1}{(D_1+1)(D_1+2)} = \frac{A}{D_1+1} + \frac{B}{D_1+2}$

$= A(D_1 + 2) + B(D_1 + 1)$

If $D_1 = -2$

$1 = B(-2 + 1)$

$1 = -B$

$= \boxed{B = -1}$

$\Rightarrow$ If $D_1 = -1$

$1 = A(-1 + 2)$

$1 = A$

$$\Rightarrow \frac{1}{(D_1+1)(D_2+2)} = \frac{1}{D_1+1} - \frac{1}{D_1+2}$$

$$\frac{1}{(D_1+1)(D_1+2)} e^x = \left(\frac{1}{D_1+1} - \frac{1}{D_1+2}\right) e^x$$

$$= \frac{1}{D_1+1} e^x - \frac{1}{D_1+2} e^x$$

$$= x^{-1} \int x^{1-1} e^x\, dx - x^{-2} \int x^{2-1} e^x\, dx$$

$$= x^{-1} \int e^x\, dx - x^{-2} \int x e^x\, dx$$

$$= x^{-1} e^x - x^{-2} \left[x \int e^x - \int 1\left(\int e^x dx\right) dx\right]$$

$$= x^{-1} e^x - x^{-2} \left[x\, e^x - \int e^x dx\right]$$

$= x^{-1} e^x - x^{-2} [x\, e^x - e^x]$

$= x^{-1} e^x - x^{-1} e^x + x^{-2} e^x$

$y_p = x^{-2} e^x$

$\therefore$ The required solution is

$y = y_c + y_p$

$= c_1 x^{-1} + c_2 x^{-2} + x^{-2} e^x$

➢ Key Points

➢ Definition of $\left\{\frac{1}{f(D_1)}\right\}$ X, where $D_1 = \frac{d}{dt}$, $x = e^t$ and X is a function of x.

The function $\left[\frac{1}{f(D_1)}\right]$ X is defined to be that function which when operated upon by $f(D_1)$ gives X

To find the values of $\frac{1}{D_1 - a}$ X

where $D_1 = XD \equiv x \frac{d}{dx}$

Let $\frac{1}{D_1 - a} X = U$ or $(D_1 - a)\, \nu = X$

$$x \frac{du}{dx} = au + x$$

$$\frac{du}{dx} - \frac{a}{a} u = \frac{X}{x}$$

Which is linear differential equation in variables u and x.

Its integrating factor $= e^{\int\left(-a/x\right) dx} = a^{-a \log x}$

$= x^{-a}$

and solution is

$$Ux^{-a} = \int \left(X/x\right) x^{-a}\, dx$$

$$U = x^a \int x^{-a-1} \times dx$$

Thus $\frac{1}{D_1 - a} X = x^a \int x^{-a-1} \times dx$

Replacing a by –a in the above result

$$\frac{1}{D_1 + a} X = x^{-a} \int x^{-a-1} \times dx$$

➢ An alternative method of getting P.I of homogeneous equation $f(D_1)\, y = X$

where $x = e^t$, $D_1 = \frac{d}{dt}$ and X is any function of X

$P.I = \frac{1}{f(D_1)} X$ can be obtained in either of the following two ways

(i) The operators $\frac{1}{f(D_1)}$ may be resolved into factors then

$$P.I = \frac{1}{f(D_1)} X = \frac{1}{(D_1 - a_1)} \frac{1}{(D_1 - a_2)} \ldots\ldots \frac{1}{(D_1 - a_n)} X$$

(ii) The operator $\frac{1}{f(D_1)}$ may be resolved into partial fractions

$$P.I = \frac{1}{f(D_1)} \times = \left[\frac{A_1}{D_1 - a_1} + \frac{A_2}{D_1 - a_2} + \ldots + \frac{A_n}{D_1 - a_n}\right] X$$

$$= A_1 x_1^{-a_1} \int x^{-a_1 - 1} \times dx + A_2 x^{-a_2} \int x^{-a_2 - 1} \times dx + \ldots + A_n x^{-a_n} \int x^{-a_n - 1} \times dx$$

Q23. Solve $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$

Sol :

The given equation is

$$x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$$

Which can be written as

$$(x^4 D^3 + 2x^3 D^2 - x^2 D + x)\, y = 1$$

$$x\,(x^3 D^3 + 2x^2 D^2 - x D + 1)\, y = 1$$

$$(x^3 D^3 + 2x^2 D^2 - x D + 1)\, y = x^{-1}$$

Let $x = e^t \Rightarrow t = \log x$ and let $D_1 = \frac{d}{dt}$

$$x^3 D^3 y = D_1(D_1 - 1)(D_1 - 2)\, y$$

$$x^2 D^2 y = D_1(D_1 - 1)\, y$$

$$xDy = D_1 y$$

Then (4) becomes

$$[D_1(D_1-1)(D_1-2) + 2(D_1(D_1-1) - D_1 + 1]\,y = e^{-t}$$

$$[(D_1^2 - D_1)(D_1-2) + 2D_1^2 - 2D_1 - D_1 + 1]\,y = e^{-t}$$

$$[D_1^3 - 2D_1^2 - D_1^2 + 2D_1 + 2D_1^2 - 2D_1 - D_1 + 1]\,y = e^{-t}$$

$$[D_1^3 - 3D_1^2 + 2D_1 + 2D_1^2 - 3D_1 + 1]\,y = e^{-t}$$

$$[D_1^3 - D_1^2 - D_1 + 1]\,y = e^{-t}$$

Then the A.E is

$$m^3 - m^2 - m + 1 = 0$$

$$m = -1 \Rightarrow (-1)^3 - (-1)^2 - (-1) + 1$$

$$= -1 - 1 + 1 + 1$$

$$= 0$$

$m = -1$

1	–1	–1	1
0	–1	2	–1
1	–2	1	0

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$\therefore$ The roots are $m = -1, 1, 1$

$\therefore$ $y_c = (c_1 + c_2 t)\,e^t + c_3 e^{-t}$

$y_c = (c_1 + c_2 \log x)\,x + c_3 x^{-1}$

Now

$$\text{P.I} = y_p = \frac{1}{(D_1+1)(D_1-1)^2}\,e^{-t}$$

$$= \frac{1}{(D_1+1)(-1-1)^2}\,e^{-t}$$

$$= \frac{1}{4(D_1+1)}\,e^{-t}\,.\,1$$

$$= \frac{1}{4}\,e^{-t}\,\frac{1}{D_1-1+1}\,.\,1$$

$$= \frac{1}{4}\,e^{-t}\,\frac{1}{D_1}\,(1)$$

$$= \frac{1}{4}\,e^{-t}\,(t)$$

$$y_p = \frac{1}{4}\,x^{-1}\log x$$

$\therefore$ The required solution is

$$y = (c_1 + c_2 \log x)\, x + c_3 x^{-1} + \frac{1}{4} x^{-1} \log x$$

Q24. Solve $x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 20\, y - (2 + 1)^2$

Sol :

The given equation is

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 20\, y = (x + 1)^2 \quad \text{... (1)}$$

Let $x = e^t \Rightarrow t = \log x$

rewriting (1) $\Rightarrow x^2 D^2y + 2x\, Dy - 20\, y = x^2 + 1 + 2x$

Let $x^2 D^2 y = D_1(D_1 - 1)\, y$

$xy = D_1 y$

$\Rightarrow D_1(D_1 - 1)\, y + 2D_1 y - 20\, y = (e^t)^2 + 1 + 2e^t$

$[D_1(D_1 - 1) + 2D_1 - 20]\, y = e^{2t} + 1 + 2e^t$

$[D_1^2 - D_1 + 2D_1 - 20]\ y = e^{2t} + 1 + 2e^t$

$[D_1^2 + D_1 - 20]\ y = e^{2t} + 1 + 2e^t$

Then the A.E is $m^2 + m - 20 = 0$

$m^2 + 5m - 4m - 20 = 0$

$m(m + 5) - 4(m + 5) = 0$

$(m + 5)(m - 4) = 0$

$m = -5, 4$

$\therefore\ y_c = c_1 e^{-5t} + c_2 e^{4t}$

$y_c = c_1 x^{-5} + c_2 x^4$

Now

$$\text{P.I} = y_p = \frac{1}{D_1^2 + D_1 - 20}\, e^{2t} + 1 + 2e^t$$

$$= \frac{1}{D_1^2 + D_1 - 20}\, e^{2t} + \frac{1}{D_1^2 + D_1 - 20}\, e^0 + 2\frac{1}{D_1^2 + D_1 - 20}\, e^t$$

$$= e^{2t} \frac{1}{2^2 + 2 - 20} + \frac{1}{0 + 0 - 20}\, e^t + 2\frac{1}{1 + 1 - 20}\, e^t$$

$$= \frac{1}{-14}\, e^{2t} - \frac{1}{20} + \frac{2}{-18}\, e^t$$

$$= \frac{-1}{14}\, e^{2t} - \frac{1}{20} - \frac{1}{9}\, e^t$$

$$= \frac{-1}{14}\, x^2 - \frac{1}{20} - \frac{1}{9}\, x$$

$$y_p = \frac{-x^2}{14} - \frac{x}{9} - \frac{1}{20}$$

$\therefore$ The required solution is

$$y = c_1x^{-5} + c_2x^4 - \frac{x^2}{14} - \frac{x}{9} - \frac{1}{20}$$

4.4 Legendre's Linear Equations

Q25. Derive Legendre's Linear Equations.

Ans :

An equation of the form

$$K_n (ax + 0)^n \frac{d^ny}{dx^n} + k_{n-1} (ax + b)^{n-1} \frac{d^{n-1}y}{d_x^{\,n-1}} + ... + k_0 y = Q(x)$$

Where $k_0, k_1, ... k_n$ are constants and Q(x) is a function of x, is called legendre's linear equation.

Such equations can be reduced to linear equation with constant coefficients by the substitution.

$$ax + b = e^t, \quad t = \log (ax + b)$$

$$dt = \frac{1}{ax+b} a . dx$$

Then if $D = \frac{d}{dt}$

$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{a}{(ax+b)} . \frac{dy}{dt}$$

$$(ax + b) \frac{dy}{dx} = a \frac{dy}{dt}$$

$$(ax + b) \frac{dy}{dx} = a Dy$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \frac{a}{ax+b} \frac{dy}{dt}$$

$$= \frac{a^2}{(ax+b)^2} \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$\Rightarrow (ax + b)^2 \frac{d^2y}{dx^2} = a^2 \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$(ax + b)^2 \frac{d^2y}{dx^2} = a^2 D(D - 1) y$$

Similarly

$$(ax + b)^3 \frac{d^3y}{dx^3} = a^3 D(D - 1) (D - 2)$$

and so on.

Q26. Solve $(1 + x)^2 \frac{d^2y}{dx^2} + (1 + x) \frac{dy}{dx} + y = 2 \sin \log y (1 + x)$

Sol :

The given equation is

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y$$

$$= 2 \sin (\log (1 + x) \text{ ....(1)}$$

Put $1 + x = e^t \Rightarrow t = \log (1 + x)$

and Also $(1 + x) D = (1)^2 D_1 = D_1$

$(1 + x^2) D^2 = (1)^2 D_1 (D_1 - 1) = D_1 (D_1 - 1)$

Then equation (1) becomes

$(D_1 (D_1 -1) + D_1 + 1) y = 2 \sin t$

$(D_1^{\,2} - D_1 + D_1 + 1) y = 2 \sin t$

$(D_1^{\,2} + 1) y = 2\sin t$

Which is linear equation with constant coefficient

Then the A. E is $m^2 + 1 = 0 \Rightarrow m^2 = -1$

$m = \pm i$

$\therefore y_c = c_1 \cos t + c_2 \sin t$

$y_c = c_1 \cos [\log (1+x)] + c_2 \sin [\log (1+x)]$

Now,

$$P.I = y_p = \frac{1}{D_1^{\,2}+1} 2 \sin t$$

$= 2\ \frac{-t}{2}\ \cos t$

$= -t \cos t$

$y_p = -[\log(1 + x)] \cos(\log(1+x))$

$\therefore$ The required solution is

$y = c_1 \cos \log(1 + x) + c_2 \sin \log(1+x) - \log(1 + x) \cos(\log(1 + x))$

Q27. Solve $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x.$

Sol :

The given equation is $(2x + 3)^2 \frac{d^2y}{dx^2} - 2(2x + 3)\frac{dy}{dx} - 12y = 6x$

Then the equation in the operator form is $[(2x + 3)^2 D^2 - 2(2x + 3) D - 12]\, y = 6x$ ......(1)

Let $2x + 3 = e^t \Rightarrow t = \log(2x + 3)$ & $\frac{d}{dt} = D_1$

and $2x = e^t - 3$

$x = \frac{e^t - 3}{2}$

Also, $(2x + 3) D = 2D_1$

$(2x + 3)^2 D^2 = 2^2 D_1 (D_1 - 1)$

Then the (1) can be written as

$[2^2 D_1 (D_1 - 1) - 2(2D_1) - 12]\, y = {}^3\not{6}\left(\frac{e^t - 3}{\not{2}}\right)$

$[4D_1^2 - 4D_1 - 4D_1 - 12]y = 3e^t - 9$

$(4D_1^2 - 8D_1 - 12)y = 3e^t - 9$

$(D_1^2 - 2D_1\ 3)\, y = \frac{1}{4}(3e^t - 9)$

Then A.E is $m^2 - 2m - 3 = 0$

$m^2 - 3m + m - 3 = 0$

$(m - 3)(m+1) = 0$

$m = -1, 3$

$\therefore y_c = c_1 e^{-t} + c_2 e^{3t}$

$\therefore y_c = c_1 (2x + 3)^{-1} + c_2 (2x + 3)^3$

Now,

$P.I = y_p = \frac{1}{D_1^2 - 2D_1 - 3}\ \frac{1}{4}\left(3e^t - 9\right)$

$$= \frac{1}{D_1^2 - 2D_1 - 3}\ \frac{3}{4}e^t - \frac{9}{4}\frac{1}{D_1^2 - 2D_1 - 3}e^0$$

$$= \frac{3}{4}\frac{1}{1-2(1)-3}e^t - \frac{9}{4}\ \frac{1}{-3}$$

$$= \frac{3}{4}\frac{1}{(-4)}e^t + \frac{3}{4}$$

$$= -\frac{3}{16}e^t + \frac{3}{4}$$

$$= \frac{3}{4}\left[-\frac{1}{4}e^t + 1\right]$$

$$y_p = \frac{3}{4}\left[-\frac{1}{4}(2x+3)+1\right]$$

$\therefore$ The required solution is

$$y = c_1\,(2x+3)^{-1} + c_2\,(2x+3)^3 + \frac{3}{4}\left[\frac{-1}{4}(2x+1)+1\right]$$

Q28. Solve $(x+3)^2 \dfrac{d^2y}{dx^2} - 4(x+3)\dfrac{dy}{dx} + 6y = \log(x+3)$

Sol :

The given equation is $(x+3)^2 \dfrac{d^2y}{dx^2} - 4(x+3)\dfrac{dy}{dx} + 6y = \log(x+3)$

Then the equation in the operator form is

$((x+3)^2 D^2 - 4(x+3) D + 6)\, y = \log(x+3)$

Let $= x + 3 = e^t \Rightarrow t = \log(x+3)$

and also $(x+3)^2 D^2 = (1)^2 D_1 (D_1 - 1)$

$(x+3)^2 D^2 = D_1 (D_1 - 1)$

$(x+3) D = D_1$

Then the equation (1) can be written as

$[D_1(D_1 - 1) - 4D_1 + 6]\, y = t$

$[D_1^2 - D_1 - 4D_1 + 6]\, y = t$

$[D_1^2 - 5D_1 + 6]\, y = t$

Then the A. E is $m^2 - 5m + 6 = 0$

$m^2 - 2m + 3m + 6 = 0$

$(m-2)(m-3) = 0$

$m = 2, 3$

$\therefore y_c = c_1 e^{2t} + c_2 e^{3t}$

$y_c = c_1 (x + 3)^2 + c_2 (x+3)^3$

Then, the P.I.

$$y_p = \frac{1}{D_1^2 - 5D_1 + 6}(t)$$

$$= \frac{1}{6\left[1-\left(\frac{5D_1}{6}-\frac{D_1^2}{6}\right)\right]} t$$

$$= \frac{1}{6}\left[1-\left(\frac{5D_1}{6}-\frac{D_1^2}{6}\right)\right]^{-1} t$$

$$= \frac{1}{6}\left[1+\left(\frac{5D_1}{6}-\frac{D_1^2}{6}\right)\right] t$$

$$= \frac{1}{6}\left[t+\frac{5}{6}(1)\right]$$

$$y_p = \frac{1}{6}\log(x+3) + \frac{5}{36}$$

$\therefore$ The required solution is

$$y = c_1 (x+3)^2 + c_2 (x+3)^3 + \frac{1}{6}\log(x+3) + \frac{5}{36}$$

4.5 Miscellaneous Differential Equations

4.5.1 Equation of the form $\frac{d^2y}{dx^2} = f(x)$

Integrating with respect to x, we get

$$\frac{dy}{dx} = \int f(x)dx + c = F(x) \text{ (say)}$$

again Integrating , we have $y = \int F(x)dx + c_1$

In general, the solution of a differential equation of the form $\frac{d^ny}{dx^n} = f(x)$ is obtained by integrating it n times successively

Q29. Solve $\frac{d^2y}{dx^2} = xe^x$

Sol :

Given that $\frac{d^2y}{dx^2} = x e^x$

By Integrating the given equation, we get

$$\int \frac{d^2y}{dx^2} = \int xe^x dx$$

$$\frac{dy}{dx} = x\int e^x - \int 1\left(\int e^x dx\right)dx + c_1$$

$$= x e^x - e^x + c_1$$

$$\frac{dy}{dx} = e^x (x-1) + c_1$$

Integrate again, $\int \frac{dy}{dx} = \int e^x(x-1) + c_1$

$$y = \int e^x x dx - \int e^x dx + \int c_1$$

$$y = x\int e^x - \int 1\left(\int e^x dx\right)dx - e^x + c_1 x + c_2$$

$$= x e^x - e^x - e^x + c_1 x + c_2$$

$$= x e^x - 2 e^x + c_1 x + c_2$$

$$y = (x-2)e^x + c_1 x + c_2$$

Which is required solution

Q30. Solve $\frac{d^3y}{dx^3} = x + \log x$

Sol :

Given that $\frac{d^3y}{dx^3} = x + \log x$

By Integrating,

$$\int \frac{d^3y}{dx^3} = \int [x + \log x]dx$$

$$\frac{d^2y}{dx^2} = \int x\,dx + \int \log x.dx$$

$$\frac{d^2y}{dx^2} = \frac{x^2}{2} + \log x \int 1 - \int \frac{1}{x}\left(\int 1 dx\right)dx$$

$$\frac{d^2y}{dx^2} = \frac{x^2}{2} + \log x.\, x - \int \frac{1}{x}.\, x\, dx$$

$$\frac{d^2y}{dx^2} = \frac{x^2}{2} + x \log x - \int 1\, dx$$

$$\frac{d^2y}{dx^2} = \frac{x^2}{2} + x \log x - x + c_1$$

Again Integrating

$$\int \frac{d^2y}{dx^2} = \int \frac{x^2}{2} + \int x \log x - \int x$$

$$\frac{dy}{dx} = \frac{x^3}{6} + \log x \int x - \int \frac{1}{x}\left(\int x\, dx\right)dx + c_1$$

$$= \frac{x^3}{6} + \log x.\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}dx + c_1x + c_2$$

$$= \frac{x^3}{6} + \frac{x^2}{2} \log x - \int \frac{x}{2}dx + c_1x + c_2$$

$$= \frac{x^3}{6} + \frac{x^2}{2} \log x - \frac{1}{2}\frac{x^2}{2} + c_1x + c_2$$

$$\frac{dy}{dx} = \frac{x^3}{6} + \frac{x^2}{2}\log x - \frac{x^2}{4} + c_1x + c_2$$

again, Integrating,

$$\int \frac{dy}{dx} = \int \frac{x^3}{6}dx + \int \frac{x^2}{2}\log x dx - \int \frac{x^2}{4}dx + c_1x^2 + c_2x + c_3$$

$$y = \frac{x^4}{24} + \frac{1}{2}\left[\int x^2 \log x\, dx\right] - \frac{x^3}{12} + c_1x^2 + c_2x + c_3$$

$$= \frac{x^4}{24} + \frac{1}{2}\left[\log x \int x^2 - \int \frac{1}{x}\left(\int x^2 dx\right)dx\right] - \frac{x^3}{12} + c_1x^2 + c_2x + c_3$$

$$= \frac{x^4}{24} + \frac{1}{2}\left[\log x.\frac{x^3}{3} - \int \frac{1}{x}.\frac{x^3}{3}dx\right] - \frac{x^3}{12} + c_1x^2 + c_2x + c_3$$

$$= \frac{x^4}{24} + \frac{1}{2}\left[\log x \frac{x^3}{3} - \frac{1}{3}\int x^2 - dx\right] - \frac{x^3}{12} + c_1x^2 + c_2x + c_3$$

$$= \frac{x^4}{24} + \frac{x^3}{6}\log x - \frac{1}{3}.\frac{x^3}{3} - \frac{x^3}{12} + c_1x^2 + c_2x + c_3$$

$$= \frac{x^4}{24} + \frac{x^3}{6}\log x - \frac{x^3}{9} - \frac{x^3}{12} + c_1x^2 + c_2x + c_3$$

$$y = \frac{x^4}{24} + \frac{x^3}{6}\log x - \frac{4x^3 - 3x^2}{36} + c_1x^2 + c_2x + c_3$$

$$\therefore y = \frac{x^4}{24} + \frac{x^3}{6}\log x - \frac{9x^3}{36} + c_1x^2 + c_2x + c_3$$

Which is required solutions

4.5.2 Equation of the form $\frac{d^2y}{dx^2} = f(y)$

Multiplying both sides by $2\left(\frac{dy}{dx}\right)$.

We have $2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) = 2\frac{dy}{dx}\ f(y)$

Integrating with respect to x, we get

$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\,dy + c = F(y)$$

$$\frac{dy}{dx} = \sqrt{F(g)}$$

Now, separating the variables and integrating we obtain,

$\int \frac{dy}{\sqrt{F(y)}} = x + c_1$ which gives the required solution.

Q31. Solve $\frac{d^2y}{dx^2} = 2(y^3 + y)$ under the condition $y = 0$, $\frac{dy}{dx} = 1$, when $x = 0$

Sol :

Given that $\frac{d^2y}{dx^2} = 2(y^3 + y)$

Which is of the form $\frac{d^2y}{dx^2} = f(y)$

Multiplying both sides by $2\left(\frac{dy}{dx}\right)$

Then we get

$$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 2(y^3 + y)\left[2\left(\frac{dy}{dx}\right)\right]$$

$$2\frac{dy}{dx}.\frac{d^2y}{dx^2} = 4\left(y^2 + y\right)\frac{dy}{dx}$$

$$\frac{dy}{dx}\frac{d^2y}{dx^2} = 2\left(y^2 + y\right)\frac{dy}{dx}$$

By Integrating, with respect to x

$$\int\frac{dy}{dx}\frac{d^2y}{dx^2} = \int 2\left(y^3 + y\right)\frac{dy}{dx}$$

$$\left(\frac{dy}{dx}\right)^2 = 2\frac{y^4}{4} + \frac{2y^2}{2} + c$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{y^4}{2} + y^2 + c$$

$$\left(\frac{dy}{dx}\right)^2 = y^4 + 2y^2 + c$$

As $\frac{dy}{dx} = 1$ for $y = 0$

$1 = 0 + 2(0) + c$

$c = 1$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = y^4 + 2y^2 + 1$$

$$\left(\frac{dy}{dx}\right)^2 = (y^2 + 1)^2$$

$$\frac{dy}{dx} = y^2 + 1$$

Now, Separate the variables

Then, we get $\frac{dy}{y^2 + 1} = dx$

By Integrating $\int\frac{dy}{y^2 + 1} = \int dx + c_1$

$\tan^{-1} y = x + c_1$

As $y = 0$ for $x = 0$

$\tan^{-1}(0) = 0 + c_1$

$c_1 = 0$

$\therefore \tan^{-1} y = x$

$y = \tan x$

Which is required solution

Q32. Solve $\frac{d^2y}{dx^2} = x^2 \sin x$

Sol :

Given that $\frac{d^2y}{dx^2} = x^2 \sin x$

By Integrating given equation,

$$\int \frac{d^2y}{dx^2} = \int x^2 \sin x + c_1$$

$$\frac{dy}{dx} = x^2 \int \sin x - \int 2x \left(\int \sin x \, dx \right) dx + c_1$$

$$= -x^2 \cos x - 2\int x(-\cos x)dx + c_1$$

$$= -x^2 \cos x + 2\int x \cos x \, dx + c_1$$

$$= -x^2 \cos x + 2\left[x\int \cos x - \int 1\left(\int \cos x \right) dx \, dx \right] + c_1$$

$$= -x^2 \cos x + 2\left[x \sin x - \int \sin x dx \right] + c_1$$

$$= -x^2 \cos x + 2[x \sin x + \cos x] + c_1$$

$$\frac{dy}{dx} = -x^2 \cos x + 2x \sin x + 2\cos x + c_1$$

Again , Integrating

$$\int \frac{dy}{dx} = -\int x^2 \cos x dx + 2\int x \sin x \, dx + 2\int \cos x dx$$

$$y = -\left[x^2 \int \cos x - \int 2x\left(\int \cos x \right) dx \, dx \right] + 2\left[x\int \sin x - \int 1\left(\int \sin x \right) dx dx \right] + 2\sin x + c_1 x + c_2$$

$$y = -\left[x^2 \sin x - 2\int x \sin x dx \right] + 2\left[-x\cos x + \int \cos x dx \right] + 2\sin x + c_1 x + c_2$$

$$= -x^2 \sin x + 2\left[x\int \sin x - \int 1\left(\int \sin x \, dx \right) dx \right] - 2x\cos x + 2\sin x + 2\sin x + c_1 x + c_2$$

$$= -x^2 \sin x + 2\left[-x\cos x + \int \cos x \, dx \right] - 2x\cos x - 2\sin x. + 2\sin x + c_1 x + c_2$$

$= -x^2 \sin x - 2x \cos x + 2 \sin x - 2x \cos x$

$+ 2 \sin x + 2 \sin x + c_1 x + c_2$

$y = -x^2 \sin x - 4x \cos x + 6 \sin x + c_1 x + c_2$

Which is required solution

4.6 Pastial Differential Equation

Q33. Define Partial differential equation .

Ans :

A differential equation which contains two or more independent variables and partial derivatives with respect to them is called a partial differential equation

Let x and y represents the independent variables and z the dependent variable so that z = f(x,y) . We will use some rotations.

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y}$$

$$= s, \quad \frac{\partial z}{\partial y^2} = t$$

4.6.1 Formation and solution of Partial Differential equations

Q34. Solve By elininating the constant, obtain the partial differential equation from the relation. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Sol :

Given equation is $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ .... (1)

differentiate the equation (1) partially with respect to x,

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2}$$

$$\frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x}$$

But we know that $\frac{\partial z}{\partial x} = P$

$$\frac{1}{a^2} = \frac{1}{x} P$$

Now, differentiate the equation (1) partially with respect to 'y'

$$2\frac{\partial z}{\partial y} = 0 + \frac{2y}{b^2}$$

$$2\frac{\partial z}{\partial y} = \frac{2y}{b^2} \Rightarrow \frac{\partial z}{\partial y} = \frac{y}{b^2}$$

$$\frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y}$$

But we know that $\frac{\partial z}{\partial y} = q$

$$\therefore \frac{1}{b^2} = \frac{1}{y} q$$

Substituting $\frac{1}{a^2}$ & $\frac{1}{b^2}$ in equation of (1)

$$2z = \left[\frac{1}{x} p\right] x^2 + \left[\frac{1}{y} q\right] y^2$$

$2z = xp + yq$

Which is required partial differential equation

Q35. Form a partial differential equation by eliminating the constant h and k from $(x-h)^2 + (y-k)^2 + z^2 = c^2$.

Sol :

The given equation is $(x-h)^2 + (y-k)^2 + z^2 = c^2$ .... (1)

differentiating the equation (1) partially with respect to x

$$2(x-h)(1) + 2z\frac{\partial z}{\partial x} = 0$$

$2[(x-h) + zp] = 0$ But $\frac{\partial z}{\partial x} = p$

$(x-h) + z.p = 0$

$x - h = -zp$

differentiating the eqution (1) partially with respect to 'y'

$$2(y-k) + 2z\frac{\partial z}{\partial y} = 0$$

$2[y - k + zq] = 0$

$y - k + zq = 0$

$y - k = -zq$

By substituting x – b & y – k in (1)

$(-zp)^2 + (-zq)^2 + z^2 = c^2$

$z^2 p^2 + z^2 q^2 + z^2 = c^2$

$z^2 (p^2 + q^2 + 1) = c^2$

Which is requird partial differential equation

Q36. By eliminating the arbitray functions. Obtain the partial differential equations form

(a) $z = f(x^2 + y^2)$

(b) $z = f(x + ct) + g(x - ct)$

Sol :

Given that $z = f(x^2 + y^2)$ ... (1)

Differentiate partically with respect to 'x'

$$\frac{\partial z}{\partial x} = f'(x^2 + y^2)\, 2x$$

$$p = 2f'(x^2 + y^2)\, x \quad \text{... (2)}$$

Differentiate partially with respect to 'y'

$$\frac{\partial z}{\partial y} = f'(x^2 + y^2)\, 2y$$

$$q = 2f'(x^2 + y^2)\, y \quad \text{... (3)}$$

$$\frac{p}{q} = \frac{2f'(x^2+y^2)\,x}{2f'(x^2+y^2)\,y} \Rightarrow \frac{p}{q} = \frac{x}{y}$$

$$yp = xq \Rightarrow yp - xq = 0$$

Which is required differential equation.

(b) $z = f(x + ct) + g(x - ct)$ ... (1)

Differentiate (1) partially with respect to x

$$\frac{\partial z}{\partial x} = f'(x + ct)(1) + g'(x - ct)$$

$$\frac{d^2 z}{dx^2} = f''(x + ct) + g''(x - ct) \quad \text{... (2)}$$

Differentiate (1) partially with respect to 't'

$$\frac{dz}{dt} = f'(x + ct)(c) + g'(x - ct)(-c)$$

$$= cf'(x + ct) - cg'(x - ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x + ct) + c^2 g''(x - ct)$$

$$= c^2 [f''(x + ct) + g''(x - ct)] \quad \text{by (2)}$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

∴ The required partial differential equation is

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Q37. By eliminating the arbitrary function $z = e^{ny} \phi(x - y)$ obtain, a partial differential equation.

Sol :

The given equation is $z = e^{ny} \phi(x - y)$... (1)

Differentiating with respect to 'y'

$$\frac{\partial z}{\partial y} = ne^{ny} \phi(x - y) + e^{ny} \phi'(x - y)(-1) \text{... (1)}$$

$$\frac{\partial z}{\partial y} = ne^{ny} \phi(x - y) - e^{ny} \phi'(x - y) \text{... (2)}$$

Differentiate with respect to 'x'

$$\frac{\partial z}{\partial y} = e^{ny} \phi'(x - y)(1) = e^{ny} \phi'(x - y) \text{... (3)}$$

By (2)

$\Rightarrow q = nz - p$

$p + q - nz = 0$ as the required solution.

Q38. By eliminating the arbitrary constants $z = (x^2 + a)(y^2 + b)$ form the partial differential equation.

Sol :

The given equation is

$$z = (x^2 + a)(y^2 + b) \quad \text{... (1)}$$

Rewrite, the given equation. then we obtain

$$z = x^2y^2 + x^2b + y^2a + ab$$

Differentiate partially with respect to 'x'

$$\frac{\partial z}{\partial x} = 2xy^2 + 2xb$$

$$p = 2x[y^2 + b] \quad \text{... (2)}$$

Differentiate partially with respect to 'y'

$$\frac{\partial z}{\partial y} = 2yx^2 + 2ya$$

$$q = 2x[x^2 + a] \quad \text{... (3)}$$

$$\text{By (2)} \Rightarrow y^2 + b = \frac{p}{2x} \quad \text{... (i)}$$

$$\text{By (3)} \Rightarrow x^2 + a = \frac{q}{2y} \quad \text{... (ii)}$$

Multiply (i) and (ii)

$$\left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right) = z$$

$$pq = 4xyz$$

4.7 Equations Easily Integrable

Some of the partial differential equation can be solved by direct integration. The usual constant of integration consists of an arbitrary function of the variable considered constant during the integration.

Q39. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0$

Sol :

The given partial differential equation is $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0$ .... (1)

Now, keeping y fixed and integrate with respect to 'x' the equation (1)

$$\frac{\partial^2 z}{\partial x \partial y} + 18\frac{x^2}{2}y^2 - \frac{1}{2}\cos(2x - y) = f(y)$$

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x - y) = f(y)$$

Again Integrate

$$\frac{\partial z}{\partial y} + 9\frac{x^3}{3}y^2 - \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) = x\,f(y) + g(y)$$

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin 2x = x\,f(y) + g(y)$$

Now, keep x as fixed, and Integrate with respect to 'y'

$$z + x^3y^3 - \frac{1}{4}\cos(x - y) = x\int f(y)dy + \int g(y)dy + h(x)$$

Denoting $\int f(y)dy = u(y),\ \int g(y)dy = v(y)$

We can write required solution as

$z = \frac{1}{4}\cos(2x - y) - x^3 y^3 + x\,u(y) + v(y) + h(x)$

where u,v and h are arbitrary functions.

Q40. Solve $y\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = 4xy$

Sol :

The given equation is $y\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = 4xy$ .... (1)

equation (1) can be written as

$$y\frac{\partial}{\partial y}.\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} = 4xy$$

We know that $\frac{\partial z}{\partial x} = p$

$y\frac{\partial}{\partial y}p + p = 4xy \Rightarrow y\frac{\partial p}{\partial y} + p = 4xy$

which is in the form of linear differential equation of the first order if x is taken as a constant.

i.e. $\frac{\partial p}{\partial y} + \frac{1}{y}p = 4x$

$\therefore$ The Integrating factor $e^{\int \frac{1}{y}dy} = e^{\log y} = y$

i.e. IF = y

$\therefore$ The solution is $P\,(IF) = \int Q(x)(IF)dy + u(x)$

$$py = \int 4x.y\,dy + u(x)$$

$$py = 4x\frac{y^2}{2} + u(x)$$

$$py = 2xy^2 + u(x)$$

$$\Rightarrow y\frac{\partial z}{\partial x} = 2xy^2 + u(x)$$

Integrating it (keeping y contant)

$$\int y\frac{\partial z}{\partial x} = \int 2xy^2 dx + \int u(x)dx + v(x)$$

$$yz = 2\frac{x^2}{2}y^2 + u_1(x) + v(x)$$

$$yz = x^2 y^2 + u_1(x) + v(x)$$

$$yz = x^2 y^2 + u_1(x) + v(x)$$

$$z = x^2 y + \frac{1}{y} u_1(x) + v(x)$$

Where $u_1(x) = \int u(x)dx$ & v(x) are arbitrary functions,

Q41. Find the differential equation of all planes which are at a constant distance r form the origin.

Sol :

The equation of the family of all the plane which are at a constant distance r from the origin is given by ax + by + cz = r

where $a^2 + b^2 + c^2 = 1$ ....(A)

differentiate partially w.r. to 'x' we get

$$a+b\frac{\partial z}{\partial x}=0$$

$$a + bp = 0 \text{ .... (1)}$$

Again differentiate partially with respect to 'y'

Then we get

$$b + c\frac{\partial z}{\partial y} = 0$$

$$b + cq = 0 \text{ ....(2)}$$

by (1) $\Rightarrow$ a = – cp

by (2) $\Rightarrow$ b = – cq

Substituting a, b in (A)

$$(-cp)^2 + (-cq)^2 + c^2 = 1$$

$$c^2 p^2 + c^2 q^2 + c^2 = 1$$

$$c^2 (p^2 + q^2 + 1) = 1$$

$$c^2 = \frac{1}{p^2+q^2+1} \Rightarrow c = \frac{1}{\sqrt{p^2+q^2+1}}$$

Substituting the value of 'c'

then $a = -\frac{1}{\sqrt{p^2+q^2+1}}\,p,\ b = \frac{-q}{\sqrt{p^2+q^2+1}}$

Now substituting a, b, & c values in the equation of plane ax + by + cz = r

$$\frac{-px}{\sqrt{p^2+q^2+1}} - \frac{qy}{\sqrt{p^2+q^2+1}} + \frac{z}{\sqrt{p^2+q^2+1}} = r - px - qy + z = r\sqrt{p^2+q^2+1}$$

$$z = px + qy + r\sqrt{p^2+q^2+1}$$

which is required solution

Q42. Form the differential equation by eliminating the arbitrary function F form $F(x^2 + y^2, z - xy) = 0$

Sol :

Given that $F(x^2 + y^2, z - xy) = 0$ ... (1)

Rewrite the equation (1) then we have

$z - xy = \phi(x^2 + y^2)$

differentiate partially with respect to 'x'

$$\frac{\partial z}{\partial x} - y = \phi^1(x^2 + y^2)(2x)$$

$$p - y = \phi^1(x^2 + y^2)(2x) \quad ... (2)$$

Again differentiate partially with respect to 'y'

$$\frac{\partial z}{\partial y} - x = \phi^1(x^2 + y^2)(2y)$$

$$q - x = \phi^1(x^2 + y^2)\, 2y \quad ... (3)$$

Now, $\frac{(2)}{(3)} \Rightarrow \frac{p-y}{q-x} = \frac{\phi^1(x^2+y^2)2x}{\phi^1(x^2+y^2)2y}$

$$\frac{p-y}{q-x} = \frac{2x}{2y}$$

$y(p - y) = x(q - x)$

$py - y^2 - xq + x^2$

$py - qx = y^2 - x^2$

which is required solution

Q43. Form the differential equation by eliminating the arbitrary function from

$$\mathbf{z = y^2 + 2F\left(\frac{1}{x} + \log y\right)}$$

Sol :

The given equation is

$$z = y^2 + 2F\left(\frac{1}{x} + \log y\right) \quad ... (1)$$

Differentiate partially with respect to 'x' we get

$$\frac{\partial z}{\partial x} = 2F^1\left(\frac{1}{x} + \log y\right)\left(\frac{-1}{x^2}\right)$$

$$p = 2F^1\left(\frac{1}{x} + \log y\right)\left(\frac{-1}{x^2}\right) \quad ... (2)$$

Differentiate partially with respect to 'y' we get

$$\frac{\partial z}{\partial y} = 2y + 2F^1\left(\frac{1}{x} + \log y\right)\frac{1}{y}$$

$$q = 2y + 2F^1\left(\frac{1}{x} + \log y\right)\frac{1}{y} \quad ... (3)$$

by (2) $\Rightarrow -Px^2 = 2F^1\left(\frac{1}{x} + \log y\right)$ ... (4)

Substitute in (3)

$$q = 2y + (-px^2)\frac{1}{y}$$

$$\Rightarrow q = 2y - \frac{px^2}{y}$$

$qy = 2y^2 - px^2$

$px^2 + qy = 2y^2$

Which is required solution

Q44. Form the differential equation by eliminating the arbitrary function from

$$\mathbf{z = xy + y\sqrt{(x^2 - a^2)} + b}$$

Sol :

Given that $z = xy + y\sqrt{(x^2 - a^2)} + b$ ... (1)

Differentiate partially with respect to 'x'

Then we get

$$\frac{\partial z}{\partial x} = y + y\frac{1}{2\sqrt{x^2 - a^2}}\, 2x$$

$$p = y + y\frac{x}{\sqrt{x^2 - a^2}}$$

$$p = y\left[1 + \frac{x}{\sqrt{x^2 - a^2}}\right]$$

$$p = y\left[\frac{\sqrt{x^2 - a^2} + x}{\sqrt{x^2 - a^2}}\right] \quad ... (2)$$

Differentiate partially with respect to 'y'

$$\frac{\partial z}{\partial y} = x + (1)\sqrt{x^2 - a^2}$$

$$q = x + \sqrt{x^2 - a^2}$$

$$\sqrt{x^2 - a^2} = q - x \text{ ....(3)}$$

Substitute (3) in (2) then we get

$$p = y\left[\frac{q - \cancel{x} + \cancel{x}}{q - x}\right]$$

$$p = y\left[\frac{q}{q - x}\right]$$

$\Rightarrow$ $p(q - x) = yq$

$\Rightarrow$ $pq - px = yq$

$\Rightarrow$ $px + qy = pq$

Which is required solution

4.8 Linear Equations of The first order

Q45. Write Working rule of Lagrange's, Linear partial differential equation.

Ans :

A linear partial differential equation of the first order is of the form $Pp + Qq = R$

Where P, Q, R are functions of x, y, z. This equation is known as Lagrange's Linear partial differential equation and the solution is $\phi(u,v) = 0$ or $u = f(v)$

To obtain the solution of $Pp + Qq = R$

We have following Rule

1. Write the given equation in form of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
2. Solve these Simultaneous equation by the method of grouping .

   Then giving $u = a$ & $v = b$ as its solutions
3. Write the solution as $\phi(u,v) = 0$ or $u = f(v)$

In the same way, we can obtain the solution of the linear partial differential equation involving more than two variables.

$$p_1\frac{\partial z}{\partial x_1} + p_2\frac{\partial z}{\partial x_2} + \text{.....} + p_n\frac{\partial z}{\partial x_n} = R \qquad \text{... (1)}$$

First find the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \text{.......} = \frac{dxn}{P_n} = \frac{dz}{R} \qquad \text{... (2)}$$

and obtain an n independent solution of above equation

Let these solution be $u_1 = c_1, u_2 = c_2 .....u_n = c_n$

Then $\phi(u_1, u_2, ...u_n) = 0$ is the solution of (1)

Where ϕ is any arbitrary function

Equation (2) is called subsidiary equation

Q46. Solve (y + z) p + (x + z) q = x + y

Sol :

The given equation is $(y + z) p + (x + z) q = x + y$

Which is of the form $Pp + Qq = R$

Where $P = y + z, Q = x + z$ & $R = x + y$

Then $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\Rightarrow \frac{dx}{y+z} = \frac{dy}{x+z} = \frac{dz}{x+y} \quad ... (1)$$

Which can be written as By choosing 1, 1, 1 as multipliers, each fraction of (1)

$$\Rightarrow \frac{dx+dy+dz}{(y+z)+(x+z)+(x+y)} = \frac{d(x+y+z)}{2(x+y+z)} \quad ....(2)$$

Choosing 1, –1, 0 as multipliess, each fraction of (1)

$$= \frac{dx-dy+0}{(y+z)-(x+z)+0} = \frac{dx-dy}{y+\cancel{z}-x-\cancel{z}}$$

$$= \frac{dx-dy}{y-x} \Rightarrow \frac{d(x-y)}{-(x-y)} \quad .... (3)$$

Again, choosing 0, 1, –1 as multipliers, each fraction of (1)

$$= \frac{0+dy-dz}{0+z+x-(x+y)} = \frac{dy-dz}{z+x-x-y}$$

$$= \frac{d(y-z)}{z-y}$$

$$= \frac{d(y-z)}{-(y-z)} \quad .... (4)$$

From (2), (3) & (4)

$$\Rightarrow \frac{d(x+y+z)}{2(x+y+z)} = \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} \quad ....(5)$$

Now taking last two fractions of (5)

$$\frac{d(x-y)}{-(x-y)} = \frac{-d(y-z)}{y-z}$$

$$\Rightarrow \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

By Integrating

$$\int \frac{1}{x-y} d(x-y) = \int \frac{1}{y-z}(y-z) + c_1$$

$\log (x - y) = \log (y - z) + \log c_1$

c_1 being an arbitrary constant

$\log (x - y) - \log (y - z) = \log c_1$

$$\log\left(\frac{x-y}{y-z}\right) = \log c_1$$

$$\frac{x-y}{y-z} = c_1 \text{ .... (6)}$$

Now, taking the first and second fraction of (5)

$$\frac{d(x+y+z)}{2(x+y+z)} = \frac{-d(x-y)}{(x-y)}$$

By Integrating,

$$\int \frac{1}{2(x+y+z)} d(x+y+z) = -\int \frac{1}{x-y} d(x-y) + c_2$$

$$\Rightarrow \frac{1}{2} \log (x + y + z) = -\log(x - y) + \log c_2$$

$$\frac{1}{2} \log (x + y + z) + \log (x - y) = \log c_2$$

$\log (x + y + z) + 2 \log (x - y) = \log c_2$

$\log (x + y + z) + \log (x - y)^2 = \log c_2$

$\log (x + y + z) (x - y)^2 = \log c_2$

$\therefore (x + y + z) (x - y)^2 = c_2$ ....(7)

from (6) and (7) the required general solution is

$$\phi\left[(x+y+z)(x-y)^2, \frac{x-y}{y-z}\right] = 0.$$ ϕ being an arbitrary function

Q47. Solve (y – z) p + (x – y) q = z – x.

Sol :

Given that (y – z) p + (x – y) q = z – x ....(1)

Which is of the form Pp + Qq = R

Where P = y –z, Q = x – y, R = z – x

Then $\frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x}$ .... (2)

Choosing 1, 1, 1 as multipliers, each fraction of (2)

$$\frac{dx+dy+dz}{y-z+x-y+z-x} = \frac{dx+dy+dz}{0}$$

$\therefore$ dx + dy + dz = 0

By Integrating

$$\int dx + \int dy + \int dz = c_1$$

x + y + z = c_1 ....(3)

Choosing x, z, y as multipliers, each fraction of (2)

$$\frac{xdx+zdy+ydz}{x(y-z)+z(x-y)+y(z-x)} = \frac{xdx+zdt+ydz}{xy-xz+zx-\cancel{zy}+yz-\cancel{yx}}$$

$\Rightarrow$ $\frac{xdx+zdy+ydz}{0}$

$\Rightarrow$ x d x + z dy + y d z = 0

Since d (yz) = zdy + ydz

x d x + d(yz) = 0

By Integrating, we get $\int x\, dx + \int d(yz) = c_2$

$$\frac{x^2}{2} + yz = c_2 \text{ ....(4)}$$

From (3) & (4) the required solution is

$$\phi\left(\frac{x^2}{2}+yz,\ x+y+z\right) = 0$$

Q48. Solve $(x^2 - y^2 - z^2)$ p + 2xyq = 2xz.

Sol :

The given equation is $(x^2 - y^2 - z^2)$ p + 2xy q = 2xz ...... (1)

which is of the form Pp + Qq = R here p = $x^2 - y^2 - z^2$,

Q = 2xy & R = 2xz

The Lagrange's auxillary equation of the given equation

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \text{ ....(2)}$$

Taking Last two fraction of (2)

$$\frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{dz}{z}$$

By Integrating, $\int \frac{dy}{y} = \int \frac{dz}{z} + c_1$

$\log y = \log z + \log c_1$
$\log y - \log z = \log c_1$

$\log \frac{y}{z} = \log c_1$

$\frac{y}{z} = c_1$ ....(3)

Choosing x, y, z as multipliers, each fraction of (2)

$$\frac{xdx + ydy + zdz}{x\left(x^2 - y^2 - z^2\right) + y(2xy) + z(2xz)}$$

$$= \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow$ x dx + y dy + z dz = 0

By Integrating $x^2 + y^2 + z^2 = c^2$ ....(4)

from (3) & (4) the required solution is

$$\phi\left(\frac{y}{z}, x^2 + y^2 + z^2\right)$$

Q49. Solve (my – ny) p + (nx – lz) q = ly – mx.

Sol :

The given equation is (mz – ny) p + (nx – lz) q = ly – mx

which is of the form Pp + Qq = R

Here P = mz – ny, Q = nx – lz, R = ly – mx

The lagrange's auxillury equation for the given equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \quad \text{........(1)}$$

Choosing x, y, z as multipliess each fraction of (1)

$$\frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow$ x dx + y dy + zdz = 0

By Integrating $\int xdx + \int ydy + \int zdz = c$

$x^2 + y^2 + z^2 = c_1$ ....(2)

Again choosing *l*, m, n as multipliers, each fraction of (1)

$$\frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + ndz}{0}$$

$\Rightarrow$ $l\,dx + m\,dy + n\,dz = 0$

By Integrating, $\int l\,dx + \int m\,dy + \int n\,dz = c$

$$lx + my + nz = c_2 \text{ ....(3)}$$

$\therefore$ From (2) and (3) the required solution is

$$\phi\left(x^2 + y^2 + z^2,\ lx + my + nz\right) = 0$$

Q50. Solve $(w + y + z)\frac{\partial w}{\partial x} + (w + x + z)\frac{\partial w}{\partial y} + (w + x + y)\frac{\partial w}{\partial z} = x + y + z$.

Sol :

Given that $(w + y + z)\frac{\partial w}{\partial x} + (w + x + z)\frac{\partial w}{\partial y} + (w + x + y)\frac{\partial w}{\partial z} = x + y + z$ ....(1)

Here the auxiliary equation of the given equation are

$$\frac{dx}{w + y + z} = \frac{dy}{w + x + z} = \frac{dz}{w + x + y} = \frac{dw}{x + y + z} \text{ ....(1)}$$

Taking first and last fraction of (1)

$$\frac{dx}{w + y + z} = \frac{dw}{x + y + z}$$

$$\frac{dw - dx}{x + \not{y} + \not{z} - w - \not{y} - \not{z}} = \frac{dw - dx}{x - w} = \frac{dw - dx}{-(w - x)}$$

Taking second and last fraction of (1)

$$\frac{dy}{w + x + z} = \frac{dw}{x + y + z} \Rightarrow \frac{dw - dy}{x + y + z - w - x - z} \Rightarrow \frac{dw - dy}{y - w} \Rightarrow \frac{dw - dy}{(w - y)}$$

Taking 3rd and last fraction of (1)

$$\frac{dw}{x + y + z} = \frac{dz}{w + x + y} \Rightarrow \frac{dw - dz}{x + y + z - w - x - y} \Rightarrow \frac{dw - dz}{z - w} \Rightarrow \frac{dw - dz}{-(z - w)}$$

Choosing 1,1,1,1 as multiplies, each fraction of (1)

$$\frac{dx + dy + dz + dw}{w + y + z + w + x + z + w + x + y + x + y + z} = \frac{dx + dy + dz + dw}{3(x + y + z + w)}$$

$\therefore$ The each fraction of (1) is

$$\frac{dw - dx}{-(w - x)} = \frac{dw - dx}{-(w - y)} = \frac{dw - dz}{-(z - w)} = \frac{dx + dy + dz + dw}{3(w + x + y + z)} \text{ .... (2)}$$

Taking first and fourth fraction of (2)

$$\frac{dw+dx+dy+dz}{3(w+x+y+z)} = \frac{dw-dx}{-(w-x)}$$

$$\frac{dw+dx+dy+dz}{3(w+x+y+z)} + \frac{dw-dx}{w-x} = 0$$

By Integrating, $\int \frac{1}{3(w+x+y+z)} d(w+x+y+z) + \int \frac{1}{w-x} d(w-x) = c_1$

$\frac{1}{3}$ log (w + x + y + z) + log (w – z) = log c_1

log $(w = x + y + z)^{1/3}$ + log (w – z) = log c_1

log $(w + x + y + z)^{1/3}$ + log (w – z) = log c_1

$(w + x + y + z)^{1/3} (w - z) = c_1$ ....(3)

Simillarly $(w + x + y + z)^{1/3} (w - y) = c_2$ ....(4)

$(w + x + y + z)^{1/3} (x - z) = c_3$ ....(5)

from (3), (4) and (5) the required general solution is

$\phi [(w + x + y + z)^{1/3} (w - z), (w + x + y + z)^{1/3} (w - y), (w + x + y + z)^{1/3} (x - z)] = 0$

Q51. Solve xp + yq = z

Sol :

Given that xp + yq = z ....(1)

Which is of the form Pp + Qq = R

The subsidary equation is $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ ....(2)

Consider first two fraction of (2)

$\frac{dx}{x} = \frac{dy}{y}$ By Integrating

$\int \frac{1}{x} dx = \int \frac{1}{y} dy + c_1 \Rightarrow$ log x – log y + log c_1

log x – log y = log c_1

log $\frac{x}{y}$ = log $c_1 \Rightarrow \frac{x}{y} = c_1$

Let choosing last two fractions

$\frac{dy}{y} = \frac{dz}{z} \Rightarrow$ By Integrating

$\int \frac{1}{y} dy = \int \frac{1}{z} dz \Rightarrow$ log y = logz + log c_2

$\Rightarrow$ logy y – log z = log c_2

$$\log \frac{y}{z} = \log c_2$$

$$\frac{y}{z} = c_2$$

$\therefore$ The required general solution is $\phi\left(\frac{x}{y} \cdot \frac{y}{z}\right) = 0$

Q52. Solve $(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy$

Sol :

Given that $(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy$ .... (1)

Which is of the form Pp + Qq = R where $P = x^2 - yz$, $Q = y^2 - zx$, $R = z^2 - xy$.

The lagrange's auxillary equation for the given equation are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \quad \text{......... (2)}$$

Taking first two tractions of (2)

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dx - dy}{x^2 - y^2 + z(x - y)} = \frac{d(x - y)}{x^2 - y^2 + z(x - y)} \quad \text{.......... (3)}$$

Taking Last two fractions of (2)

$$\frac{dy - dz}{y^2 - zx - z^2 + xy} = \frac{dy - dz}{y^2 - z^2 + x(y - z)} = \frac{d(y - z)}{y^2 - z^2 + x(y - z)} \quad \text{..........(4)}$$

by (3) and (4)

$$\frac{d(x - y)}{x^2 - y^2 + z(x - y)} = \frac{d(y - z)}{y^2 - z^2 + x(y - z)}$$

$$\frac{d(x - y)}{(x - y)(x + y) + z(x - y)} = \frac{d(y - z)}{(y - z)(y + z) + x(y - z)}$$

$$\frac{d(x - y)}{(x - y)\,(x + y + z)} = \frac{d(y - z)}{(y - z)\,(y + z + z)}$$

By Integrating

$$\int \frac{d(x - y)}{x - y} = \int \frac{d(y - z)}{y - z} + c_1$$

$\Rightarrow$ $\log(x - y) = \log(y - z) + \log c_1$

$\log(x - y) - \log(y - z) = \log c_1$

$$\log \frac{(x - y)}{(y - z)} = \log c_1$$

$$\frac{x-y}{y-z} = c_1$$

Since the system (2) is invariant under the cyclic interchange of variables $x \to y \to z \to x$.

There exists two integrals $\frac{x-y}{y-z} = c_1$; $\frac{y-z}{z-x} = c_2$

$\therefore$ Required solution is $\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$

Q53. Solve x (z² – y²) p + (x² –z²) q = z (y² – x²).

Sol :

Given that $x (z^2 - y^2) p + (x^2 - z^2) q = z (y^2 - x^2)$

Here the subsidiary equations are

$$\frac{dx}{x\left(z^2-y^2\right)} = \frac{dy}{y\left(x^2-z^2\right)} = \frac{dz}{z\left(y^2-x^2\right)} \quad \text{...............(1)}$$

Choosing x, y, z as multipliers, each fraction of (1)

$$\frac{xdx+ydy+zdz}{\cancel{x^2z^2} - \cancel{x^2y^2} + \cancel{y^2x^2} - \cancel{y^2z^2} + \cancel{z^2y^2} - \cancel{z^2x^2}} = \frac{xdx+ydy+zdz}{0}$$

$\Rightarrow$ x dx + y dy + z dz = 0

By Integrating

$$\int x\, dx + \int y\, dy + \int z\, dz = c_1$$

$x^2 + y^2 + z^2 = c_1$ ....(2)

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction of (1)

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{x}{a}\left(z^2-y^2\right) + \frac{y}{y}\left(x^2-z^2\right) + \frac{z}{z}\left(y^2-x^2\right)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{z^2} - \cancel{y^2} + \cancel{x^2} - \cancel{z^2} + \cancel{y^2} - \cancel{x^2}}$$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dzx}{0}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

By Integrating,

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = c_2$$

$\log x + \log y + \log z = \log c_2$

$\log xyz = \log c_2$

$xyz = c_2$ ....(3)

$\therefore$ From (2) and (3) the required solution is $f(x^2 + y^2 + z^2, xyz) = 0$

Q54. Solve $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$

Sol :

Given that $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$

Here the subsidiary equation are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{x^2(x-y)} \quad \text{............(1)}$$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction of (1)]

$$\Rightarrow \frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{x^2(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}.x^2(y-z) + \frac{1}{y}.y^2(z-x) + \frac{1}{z}z^2(x-y)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x(y-z) + y(z-x) + z(x-y)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{yx} + \cancel{zx} - \cancel{zy}}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{2}dz}{0}$$

$$= \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

By Integrating

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = c_1$$

$\log x + \log y + \log z = \log c_1 \Rightarrow \log(x\,y\,z) = \log c_1$

$\Rightarrow xyz = c_1$ ......(2)

Choosing $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers, each fraction of (1)

$$\Rightarrow \frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{x^2(x-y)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\frac{1}{x^2}.x^2(y-z) + \frac{1}{y^2}.y^2(z-x) + \frac{1}{z^2}z^2(x-y)}$$

$$= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y - z + z - x + x - y}$$

$$= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\Rightarrow \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

By Integrating $\int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = c_2$

$$\Rightarrow \frac{-1}{x} - \frac{1}{y} - \frac{1}{z} = \frac{-1}{c_2}$$

$\Rightarrow$ $-(x^{-1} + y^{-1} + z^{-1}) = -c_2$

$x^{-1} + y^{-1} + z^{-1} = c_2$ ....(3)

$\therefore$ From (2) and (3) the required solution is $\phi(xyz, x^{-1} + y^{-1} + z^{-1}) = 0$

Q55. Form the differential equations by eliminating the arbitrary function from $z = \frac{1}{r}[\psi_1(r - at) + \psi_2(r + at)]$

Sol :

Given that $z = \frac{1}{r}\left[\psi_1(r - at) + \psi_2(r + at)\right]$ ....(1)

Differentiate partially with respect to 't'

$$\frac{\partial z}{\partial t} = \frac{1}{r}\left[\psi_1^1(r - at)(-a) + \psi_2^1(r + at)(a)\right] \text{ ....(2)}$$

Again differentiate partially with respect to 't'

$$\frac{\partial^2 z}{\partial t^2} = \frac{1}{r}\left[\psi_1^{11}(r - at)(-a)(-a) + \psi_2^{11}(r + at)(a)(a)\right]$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{a^2}{r}\left[\psi_1^{11}(r - at) + \psi_2^{11}(r + at)\right] \text{...........(3)}$$

Differentiate partially with respect to 'r'

$$\frac{\partial z}{\partial r} = \frac{1}{r}\left[\psi_1^{11}(r - at) + \Psi_2^1(r + at)\right] - \frac{1}{r^2}\left[\psi(r - at) + \psi_2(r + at)\right]$$

$$\frac{\partial z}{\partial r} = \frac{1}{r}\left(\psi_1^1(r - at) + \psi_2^1(r - at)\right) - \frac{z}{r} \text{ ....(4)}$$

Again differentiate partially with respect to r

$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{r}\left[\psi^{11}(r-at)+\psi_2^{11}(r+at)\right]-\frac{1}{r^2}\left[\psi^{1}(r-at)+\psi_2^{1}(r+at)\right]$$

$$-\frac{1}{r^2}\left[\psi_1^{1}(r-at)+\psi_2^{1}(r+at)+\frac{2}{r^3}[\psi_1(r-at)+\psi_2(r+at)\right]$$

$$= \frac{1}{r}\left[\psi^{11}(r-at)+\psi_2^{11}(r+at)\right]-\frac{2}{r^2}\left[\psi_1^{1}(r-at)+\psi_2^{1}(r+at)\right] + \frac{2}{r^3}\left[\psi_1(r-at)+\psi_2(r+at)\right] \quad \ldots (5)$$

Using (1), (2), (3) in (5) we get

$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2} - \frac{2}{r}\left[\frac{\partial z}{\partial r}+\frac{z}{r}\right] + \frac{2}{r^2}z$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2} - \frac{2\partial z}{r\partial r} - \frac{2z}{r^2} + \frac{2z}{r^2}$$

$$\frac{\partial^2 z}{\partial r^2} + \frac{2}{r}\frac{\partial z}{\partial r} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2}$$ which is required solution.

Q56. Find the differential equations of all spheres whose centre lies on the z - axis.

Sol :

Equation of spheres whose centre lies on z-axis is given by $x^2 + y^2 + (z-c)^2 = d^2$ ....(1)

This represents a surface of revolution with z-axis

First differentiate (1) partially with respect to 'x'

We get, $2x + 2(z-c)\frac{\partial z}{\partial x} = 0$ ....(2)

Differentiate (1) partially with respect to 'y' we get

$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$ ....(3)

By (2) $\Rightarrow 2(z-c)\frac{\partial z}{\partial x} = -2x$

$(z-c)\frac{\partial z}{\partial x} = -x \Rightarrow (z-c)\,p = -x$ ....(i)

By (3) $\Rightarrow 2(z-c)\frac{\partial z}{\partial y} = -2y$

$(z-c)\frac{\partial z}{\partial y} = -y \Rightarrow (z-c)\,q = -y$ ....(ii)

By (i) & (ii) eliminating $(z-c)$

$$\frac{(z-c)}{(z-c)}\frac{p}{q} = \frac{-x}{-y} \Rightarrow -yp = -xq$$

$$q\,x - yp = 0$$

Which is required partial differential equation.

Q57. Solve the differential equation $\dfrac{\partial^2 u}{\partial x^2 \partial y} = \cos(2x + 3y)$

Sol :

Given $\dfrac{\partial^2 u}{\partial x^2 \partial y} = \cos(2x + 3y)$ ....(1)

On Integrating (1) with respect to 'x' and keeping y constant we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\sin(2x+3y)}{2} + f(y) \text{ ....(2)}$$

Again Integrating (2) with respect to x, keeping y constant, we get

$$\frac{\partial u}{\partial y} = \frac{-\cos(2x+3y)}{4} + x\,f(y) + g(y) \text{ ....(3)}$$

Now, on Integrating (3) with respect to y keeping x constant, we get

$$z = \frac{-\sin(2x+3y)}{4(3)} + x\int f(y)\,dy + \int g(y)\,dy + \gamma(x)$$

$$z = \frac{-\sin(2x+3y)}{12} + x\,\alpha(y) + \beta(y) + \gamma(x)$$

Which is required solution

Q58. Solve the differential equation

$$\frac{\partial^2 z}{\partial y^2} = z \text{ when } y = 0,\ z = e^x \text{ and } \frac{\partial z}{\partial y} = e^{-x}$$

Sol :

In the given equation $\dfrac{\partial^2 z}{\partial y^2} = z$, ...(1)

If we treat z as pure function of y only,

We could slove it like an ordinary differentiate equation with auxiliary equation as

$$D^2 = 1 \Rightarrow D = \pm 1$$

y_c: $z = c_1 e^y + c_2 e^{-y}$ ....(2)

Here z is a function of both x and y

Since we are dealing in partial differential equations

Thus, in $z = \phi(x)\, c_1 e^y + \psi(x)\, c_2 e^{-y}$

Where c_1 and c_2 are arbitrary constant

Sine $y = 0,\ z = e^x$

$\Rightarrow\ e^x = \phi(x)\, e^0 + \psi(x)\, e^0$

$e^x = \phi(x) + \psi(x)$ ....(3)

Again for $y = 0$, $\dfrac{\partial z}{\partial y} = e^{-x}$, i.e. from equation (3), we get

$e^{-x} = [\phi(x)e^y - \psi(x)\, e^{-y}]_{y=0}$

$e^{-x} = \phi(x)\, e^o - \psi(x)\dfrac{1}{e^0}$

$e^{-x} = \phi(x) - \psi(x)$ ....(4)

Now solving equation (4) and (5) for $\phi(x)$ & $\psi(x)$

(3) + (4) $\Rightarrow e^x + e^{-x} = \phi(x) + \cancel{\psi(x)} + \phi(x) - \cancel{\psi}(x)$

$e^x + e^{-x} = 2\phi(x)$

$$\phi(x) = \frac{e^x + e^{-x}}{2} = \cosh x \text{ .... (i)}$$

(3) – (4) $\Rightarrow e^x - e^{-x} = \cancel{\phi(x)} + \psi(x) - \cancel{\phi(x)} + \psi(x)$

$e^x - e^{-x} = 2\psi(x)$

$$\psi(x) = \frac{e^x - e^{-x}}{2} = \sinh x \text{ ....(ii)}$$

$\therefore$ By (i) & (ii) in (3)

$z = \cosh x\, e^y + \sinh x\, e^{-y}$

Which is required solution.

Q59. Solve $\dfrac{\partial^2 z}{\partial x^2} + z = 0$ when $x = 0,\ z = e^y$ and $\dfrac{\partial z}{\partial x} = 1$

Sol :

If z were a function of x alone, the solution would have been

$D^2 + 1 = 0$

$D^2 = -1$

$D = \pm\, i$

$y_c = c_1 \cos x + c_2 \sin x$

Where c_1 and c_2 are arbitrary constants

Since here z is a function of both x and y

$\therefore$ c_1 and c_2 be choosen arbitrary function of y

$\Rightarrow\ z = f_1(y) \sin x + f_2(y) \cos x$ ....(2)

Partial differentiating with respect to 'x'

$$\frac{\partial z}{\partial x} = f_1(y)\cos x - f_2(y)\sin x \text{ ....(3)}$$

When $x = 0$, $z = e^y \Rightarrow e^y = f_1(y)\sin(0) + f_2(y)\cos(0)$

$$e^y = f_2(y) \text{ ....(4)}$$

also, $x = 0$, $\frac{\partial z}{\partial x} = 1 \Rightarrow 1 = f_1(y)\cos(0) - f_2(y)\sin(0)$

$$1 = f_1(y) \text{ ....(5)}$$

By (4) and (5) the required solution is $z = \sin x + e^y \cos x$

Q60. Solve (yz) p + (zx) q = xy

Sol :

Given that $(yz)\,p + (zx)\,q = xy$

Hence the subsidary equation is $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$

Choosing first two fractions

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow \frac{dx}{y} = \frac{dy}{x} \Rightarrow x\,dx = y\,dy$$

By Integrating $\int x\,dx = \int y\,dy$

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{c_1}{2} \Rightarrow x^2 - y^2 = c_1 \text{ ....(1)}$$

Choosing second and third fraction

$$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow \frac{dy}{z} = \frac{dz}{y}$$

$$\Rightarrow y\,dy = z\,dz$$

By Integrating

$$\Rightarrow \int y\,dy = \int z\,dz + c_2$$

$$\Rightarrow y^2 - z^2 = c_2$$

$\therefore$ The required solution is $\phi(x^2 - y^2,\ y^2 - z^2) = 0$

Multiple Choice Questions

1. $\frac{d^2y}{dx^2} = xe^x$ [a]

(a) $y = (x-2)e^x + C_1x + C_2$ (b) $y = (x+2)e^x + C_1x + C_2$

(c) $y = (x+2)e^x + C_1x^2 + C_2x + C_3$ (d) $y = (x-2)e^x - C_1x + C_2$

2. $(ax+b)^2 \frac{d^2y}{dx^2} =$ [b]

(a) $D(D-1)y$ (b) $a^2.D(D-1)y$

(c) $a^2.D(D-1)(D-2)y$ (d) None

3. $y^{II} - \frac{2}{x}y^I + \frac{2}{x^2}y = 0,\ y_1 = x$ is [b]

(a) $y = (C_1 + C_2x)e^x$ (b) $y = C_1x + C_2x^2$

(c) $y = C_1x + C_2$ (d) $y = (C_1 + C_2x)e^{2x}$

4. $(2x^2+1)y^{II} - 4xy^I + 4y = 0$, $y_1 = x$ [c]

(a) $y = C_1x + C_2(2x^2+1)$ (b) $y = C_1x - C_2(2x^2-1)$

(c) $y = C_1x + C_2(2x^2-1)$ (d) None

5. $x^2y^{II} + xy^I - y = x^2e^{-x},\ y_1 = x$ [b]

(a) $y = C_1x + C_2x^2 + x^2\log x - \frac{3}{4}x^3$ (b) $y = C_1x + C_2x^{-1} + e^{-x}(1 + x^{-1})$

(c) $y = C_1x + C_2x^{-1}$ (d) None

6. $\frac{d^2y}{dx^2} = x^2 \sin x$ [a]

(a) $y = -x^2\sin x - 4x\cos x + 6\sin x + C_1x + C_2$ (b) $y = x^2\sin x + 4x\cos x + 6\sin x + C_1x + C_2$

(c) $y = -x^2\sin x + 4x\cos x - 6\sin x + C_1$ (d) None

7. $\left(D^2 + D\right)y = x^2 + 2x$ [a]

(a) $y = C_1 + C_2e^{-x} + \frac{x^3}{3}$ (b) $y = \left(C_1 + C_2x\right)e^{-x} + \frac{x^3}{3}$

(c) $y = C_1e^{-x} + C_2e^{-2x} + \frac{x^3}{3}$ (d) $y = \left(C_1 + C_2x\right)e^{x} + \frac{x^3}{3}$

8. y_c for $y^{II} + 2y^{I} + y = x^2e^{-x}$ is [b]

(a) $y_c = \left(C_1 + C_2x\right)e^{x}$ (b) $y_c = \left(C_1 + C_2x\right)e^{-x}$

(c) $y_c = C_1e^{x} + C_2e^{-x}$ (d) None

9. y_p for $y^{II} + y = 4x\sin x$ [b]

(a) $x^2\cos x + x\sin x$ (b) $-x^2\cos x + x\sin x$

(c) $-x^2\cos x - \sin x$ (d) $-x^2\cos x - x\sin x$

10. $\left(D^2 - 2D - 8\right)y = 9xe^{x} + 10e^{-x}$ is [a]

(a) $C_1e^{4x} + C_2e^{-2x} - xe^{x} - 2e^{-x}$ (b) $C_1e^{4x} + C_2e^{-2x} + xe^{x} + 2e^{-x}$

(c) $C_1e^{4x} + C_2e^{-2x} + xe^{x} - 2e^{-x}$ (d) None

Fill in the blanks

1. Equation of second order linear differential equation with constant coefficients is ________.
2. Homogeneous equation of second order linear differential equation is ________ .
3. Solution for $\left(D^2+4D+4\right)y=3xe^{-x}$ is ________ .
4. Equation of Legender's equation is ________ .
5. Equation of the form $\frac{d^2y}{dx^2}=f\left(y\right)$ is ________ .
6. $\left(D^4-2D^2+1\right)y=x-\sin x$ then ________ .
7. $\left(D^2-2D-8\right)y=9xe^{x}+10e^{-x}$ then y_p is ________.
8. $\left(D^2-3D\right)y=2e^{2x}\sin x$ then y_p is ________ .
9. $y^{II}+3y^{I}+2y=\sin x$ then the solution is ________ .
10. $y^{II}+3y^{I}+2y=12e^{x}$ then ________

Answers

1. $a_2\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_0y=Q\left(x\right)$
2. $a_2\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_0y=0$
3. $y=C_1e^{-2x}+C_2xe^{-2x}+\frac{1}{2}x^3e^{-2x}$
4. $k_n\left(ax+b\right)^n\frac{d^ny}{dx^n}+k_{n-1}\left(ax+b\right)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+.........+k_0y=Q\left(x\right)$
5. $\int\frac{dy}{\sqrt{F\left(y\right)}}=x+C_1$
6. $x-\frac{\sin x}{4}$
7. $-xe^{-x}-2e^{-x}$
8. $-\frac{e^{2x}}{5}\left(3\sin x+\cos x\right)$
9. $y=C_1e^{-2x}+C_2e^{-x}+2e^{x}$
10. $y_p=-2e^{x}$

FACULTY OF SCIENCE

B.Sc. II - Semester (CBCS) Examination

DIFFERENTIAL EQUATIONS

MODEL PAPER - I

Time : 3 Hours] [Max. Marks : 80

PART – A (8 × 4 = 32 Marks)

[Short Answer Type]

Note : Answer any Eight of the following questions

ANSWERS

1. Solve $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$ **(Unit - I, Q.No. 51)**

2. Solve $\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{nxy}$ **(Unit - I, Q.No. 87)**

3. Solvable for x for $y=2px+p^2y$ **(Unit - II, Q.No. 41)**

4. Solve $(p+y+x)\,(xp+y+x)\,(p+2x) = 0$. **(Unit - II, Q.No. 6)**

5. Find an equation of the orthogonal tra-jectories of the family of circles having a polar equation $r = f(\theta) = 2a\cos\theta$. **(Unit - II, Q.No. 72)**

6. Solve $\frac{d^2y}{dx^2} - y = \cos x$. **(Unit - III, Q.No. 41)**

7. Solve $y'' - 2y' + y = 7e^x$. **(Unit - III, Q.No. 30)**

8. Solve $(D^2 + D)\,y = x^2 + 2x$ **(Unit - III, Q.No. 74)**

9. Solve $(y - z)\,p + (x - y)\,q = z - x$. **(Unit - IV, Q.No. 47)**

10. By eliminating the arbitray functions. Obtain the partial differential equations form

    (a) $z = f(x^2 + y^2)$ (b) $z = f(x + ct) + g(x - ct)$ **(Unit - IV, Q.No. 36)**

11. Solve $\left(1+y^2\right)+\left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx}=0$ **(Unit - I, Q.No. 45)**

12. Solve $(D^2 - 3D + 2)\,y = \sin e^{-x}$ **(Unit - IV, Q.No. 7)**

PART – B (4 × 12 = 48 Marks)
[Essay Answer Type]

Note : Answer any Four of the following questions

13. (a) Define exact differential Equations. **(Unit - I, Q.No. 46)**

(OR)

(b) Solve $\left(xy^2 - x^2\right)dx + \left(3x^2y^2 + x^2y - 2x^3 + y^2\right)dy = 0$ **(Unit - I, Q.No. 70)**

14. (a) Solve $P^2 + 2Py\cot x = y^2$ **(Unit - II, Q.No. 7)**

(OR)

(b) Solve $p^3(x + 2y) + 3p^2(x + y) + (y + 2x)p = 0$ **(Unit - II, Q.No. 45)**

15. (a) Solve $(D^2 - 2D + 5)\,y = e^{2x}\sin x$ **(Unit - III, Q.No. 52)**

(OR)

(b) Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin x$ by using undetermined coefficients. **(Unit - III, Q.No. 79)**

16. (a) Solve $(2x+3)^2\frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$ **(Unit - IV, Q.No. 27)**

(OR)

(b) Solve $(my - ny)\,p + (nx - lz)\,q = ly - mx$ **(Unit - IV, Q.No. 49)**

FACULTY OF SCIENCE
B.Sc. II - Semester (CBCS) Examination
DIFFERENTIAL EQUATIONS
MODEL PAPER - II

Time : 3 Hours] [Max. Marks : 80

PART – A (8 × 4 = 32 Marks)
[Short Answer Type]

Note : Answer any Eight of the following questions

ANSWERS

1. Solve $(y+z)dx+(z+x)dy+(x+y)dz=0$ **(Unit - I, Q.No. 79)**

2. Solve $y(1-xy)dx-x(1+xy)dy=0$ **(Unit - I, Q.No. 58)**

3. Define growth and decay. **(Unit - II, Q.No. 51)**

4. Solve $p^3(x+2y)+3p^2(x+y)+(y+2x)p=0$ **(Unit - II, Q.No. 45)**

5. Bacteria in certain culture increase at a rate proportional to the number present. If the number N increases from 1000 to 2000 in 1 hour. How many are present at the end of 1.5 hours? **(Unit - II, Q.No. 53)**

6. Solve $(D^3 + 1)\, y = \cos 2x$. **(Unit - III, Q.No. 36)**

7. Solve $(D^3 - D^2 - 6D)\, y = x^2 + 1$. **(Unit - III, Q.No. 23)**

8. Solve $(D^2 - 2D + 1)\, y = e^x x^2$. **(Unit - III, Q.No. 50)**

9. Derive Legendre's Linear Equations. **(Unit - IV, Q.No. 25)**

10. Solve $(y + z)\, p + (x + z)\, q = x + y$ **(Unit - IV, Q.No. 46)**

11. Solve $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$ **(Unit - I, Q.No. 8)**

12. Solve $x^2 (y - z)\, p + y^2 (z - x)\, q = z^2 (x - y)$ **(Unit - IV, Q.No. 54)**

PART – B (4 × 12 = 48 Marks)
[Essay Answer Type]

Note : Answer any Four of the following questions

13. (a) Solve $\frac{x+y-a}{x+y-b}\frac{dy}{dx}=\frac{x+y+a}{x+y+b}$ **(Unit - I, Q.No. 11)**

(OR)

(b) Solve $x\,dx+y\,dy+\frac{x\,dy-y\,dx}{x^2+y^2}=0$ **(Unit - I, Q.No. 48)**

14. (a) Solve $x^2p^2+xyp-6y^2=0$ **(Unit - II, Q.No. 8)**

(OR)

(b) Solve $\left(x^2+y^2\right)\left(1+P^2\right)-2\left(x+y\right)\left(1+P\right)\left(x+Py\right)+\left(x+yP\right)^2=0$ **(Unit - II, Q.No.40)**

15. (a) (i) Solve $(D^2-4D+4)\,y = x^2 + e^x + \sin 2x$ **(Unit - III, Q.No. 61)**

(ii) Solve $(D^4-1)\,y = \sin x$ **(Unit - III, Q.No.38)**

(OR)

(b) Solve $(D^2+4D+4)y = 4x^2 + 6e^x$. by method of undermined coefficients. **(Unit - III, Q.No.69)**

16. (a) Solve $x^2D^2y - x\,Dy - 3y = x^2\log x$ **(Unit - IV, Q.No. 18)**

(OR)

(b) Write Working Rule of method of variation of parameter **(Unit - IV, Q.No. 1)**

FACULTY OF SCIENCE

B.Sc. II - Semester (CBCS) Examination

DIFFERENTIAL EQUATIONS

MODEL PAPER - III

Time : 3 Hours] [Max. Marks : 80

PART – A (8 × 4 = 32 Marks)

[Short Answer Type]

Note : Answer any Eight of the following questions

ANSWERS

1. Solve $\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$ **(Unit - I, Q.No. 89)**

2. Solve $\left(x^2 - y^2\right)dx + 2xy\ dy = 0$ **(Unit - I, Q.No. 15)**

3. Solve $(x-a)p^2 + (x-y)p - y = 0$ **(Unit - II, Q.No. 50)**

4. Solve $y = 2px + \tan^{-1}\left(xp^2\right)$ **(Unit - II, Q.No. 17)**

5. Solve $(px - y)(py + x) = h^2 p$ **(Unit - II, Q.No. 37)**

6. Solve $(D^2 + 1)\ y = x^2 \sin 2x$ **(Unit - III, Q.No. 60)**

7. Solve $(D^2 + 4D - 12)\ y = (x - 1)\ e^{2x}$ **(Unit - III, Q.No. 49)**

8. Solve $(D^2 - 2D + 1)\ y = xe^x \sin x$ **(Unit - III, Q.No. 59)**

9. Solve $y^{11} - 2y^1 + y = e^x \log x$. **(Unit - IV, Q.No. 6)**

10. Solve $\frac{d^2y}{dx^2} = 2(y^3 + y)$ under the condition $y = 0$, $\frac{dy}{dx} = 1$, when $x = 0$ **(Unit - IV, Q.No.31)**

11. Solve $x^2 y^{11} - xy^1 + y = 0$ given $y_1 = x$ as a solution . By using reduction of order of method. **(Unit - I, Q.No. 9)**

12. Derive Legendre's Linear Equations **(Unit - IV, Q.No. 25)**

PART – B (4 × 12 = 48 Marks)
[Essay Answer Type]

Note : Answer any Four of the following questions

13. (a) Solve $\left(y^2 + yz\right)dx + \left(z^2 + zx\right)dy + \left(y^2 - xy\right)dz = 0$ **(Unit - I, Q.No. 78)**

(OR)

(b) Solve $\left(1 - x^2\right)\frac{dy}{dx} + 2xy = x\left(1 - x^2\right)^{1/2}$ **(Unit - I, Q.No. 32)**

14. (a) Solve $y + px = x^4 p^2$ **(Unit - II, Q.No.14)**

(OR)

(b) If Rs. 10,000 is invested at 6 percent per annum. find what amount has accumulated after 6 years, if interset is compounded

(a) Annually (b) Quarterly and (c) Continuously. **(Unit - II, Q.No. 63)**

15. (a) Solve $(D^2 + 4D + 4)y = 3xe^{-2x}$ **(Unit - III, Q.No. 73)**

(OR)

(b) Solve $(D^2 - 4D + 4)\, y = 8(x^2 + e^{2x} + \sin 2x)$. **(Unit - III, Q.No. 47)**

16. (a) Solve $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$ **(Unit - IV, Q.No.52)**

(OR)

(b) Solve $(x + 3)^2 \frac{d^2y}{dx^2} - 4\,(x + 3)\,\frac{dy}{dx} + 6y = \log\,(x + 3)$ **(Unit - IV, Q.No. 28)**