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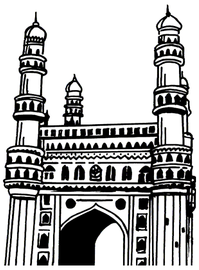

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UNIT I

Introduction to Operations Research: Nature and scope of Operations research: Origins of OR, Applications of OR in different Managerial areas, Problem solving and decision making, Quantitative and qualitative analysis, Defining a model, types of model, Process for developing an operations research model, Practices, opportunities and short comings of using an OR model.

1.1 INTRODUCTION TO OPERATIONS RESEARCH

Q1. Define Operations Research.

Ans : **(Sep. - 15)**

Introduction

The term, "Operations Research" was first coined by Mc Closky and Trefthen in 1940 in a small town, Bowdsey of United Kingdom.

The name operations research was given to this subject because it has started with the research of (military) operations. During world war - II, the military commands of UK and USA engaged several teams of scientists to discover tactical and strategic military operations. Their mission was to formulate specific proposals and to arrive to the decisions that can optimally utilize the scarce resources to acquire maximum possible level of effective results. In simple words, it was to uncover the methods that can yield greatest results with little efforts. Thus it has gained popularity and was called "an art of winning the war without actually fighting it".

Following the end of the war, the success and encouraging results of British teams have attracted industrial managers to apply these methods to solve their complex problems. The first method in this direction was simplex method (LPP) developed in 1947 by G.B. Dantzig, USA. Since then several scientists have been developing this science in the interest of making operations to yield high profits or least costs.

Definitions

(i) **According to Operations Research, Society - UK** "Operations research is the application of the methods of science to complex problems in the direction and management of large systems of men, machines, materials and money in the industry, business, government and defence. The distinctive approach is to develop a scientific model of the system by incorporating measurement of factors such as chance and risk, with which to predict and compute the outcomes of alternative decisions, strategies and controls. The purpose is to help management in determining the policy and actions scientifically".

(ii) **According to Operations Research Society, America,** "Operations Research is concerned with scientifically deciding how to best design and operate man-machine systems usually requiring the allocation of scarce resources".

(iii) **According to Thierauf and Klekamp (1975)** "Operations Research utilises the planned approach (updated scientific method) and an interdisciplinary team in order to represent complex financial relationships as mathematical models for the purpose of providing a quantitative basis for decision making and uncovering new problems for quantitative analysis".

(iv) **According to Churchman, Ackoff and Arnoff (1957)** "Operations Research is in the most general sense can be characterised as the application of scientific methods, techniques and tools, to problems involving the operations of a system so as to provide those in control of the operations with optimal solutions to the problems".

(v) **According to Dallenbach and George (1978)** "Operations Research is the systematic application of quantitative methods, techniques and tools to the analysis of problems involving operation of systems".

(vi) According to Dellenbach and George (1978), "Operations Research is essentially a collection of mathematical techniques and tools which in conjunction with a systems approach, are applied to solve practical decision problems of an economic (or) engineering nature".

(vii) According to D.W. Miller and M.K. Stan, "Operations Research is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a through-going rationality in dealing with his decision problems".

(viii) According to S.L. Cook (1977) "Operations Research has been described as a method, an approach, a set of techniques, a team activity, a com-bination of many disciplines, an extension of particular disciplines (mathematics, enginee-ring, economics), a new discipline, a vocation, even a religion. It is perhaps some of all these things".

(ix) According to P.S. Hill and G.J. Lieberman (1980), "Operations Research may be described as a scientific approach to decision making that involves the operations of organisational systems."

(x) According to RM. Morse and G.E. Kimball, "Operations Research is a scientific method of providing executive departments with a quantitative basis for decisions under their control".

(xi) According to H.M. Wagner, "Operations Research is a scientific approach to problem solving for executive management".

(xii) According to Churchman "Operations Research is the application of scientific methods, techniques and tools to problems involving the operations of systems to provide those in control of operations with optimum solution to the problems".

1.1.1 Nature and Scope of Operations Research

Q2. Explain briefly about Nature and Scope of Operations Research.

Ans : **(March - 15, Aug. - 17)**

Nature of Operations Research

The nature of operations research deals with the following,

(a) Operations Research is a Scientific Method

Operations research is of scientific nature as it solves the problems scientifically. It helps in providing quantitative basis for decisions related to the operations under their control.

(b) Operations Research Provides Optimal Solution to the Problem

Operations research determines the root cause of the problem and helps in selecting the best alternative among the. various alternatives.

(c) Helps the Executive Management

Operations research helps the executive management in solving the problems related to management scientifically by providing them analytical and objective basis for making decisions.

(d) Uses Interdisciplinary Team

Operations research uses interdisciplinary team for representing difficult functional relationships as mathematical models as this helps in providing quantitative basis for taking effective decisions and detecting new problems for quantitative analysis.

(e) Operations Research is an Experimental and Applied Science

The nature of operations research deals with experimental and applied science which predicts, understands and observes the behaviour of man-machine system and operations research workers and utilizes all these resources to solve the practical problems effectively. The problems may be related to business, government and society at large.

Scope of Operations Research

In its recent years of organized development, OR has entered successfully many different areas of research for military, government and industry. The basic problem in most of the developing countries in Asia and Africa is to remove *poverty* and *hunger* as quickly as possible. So there is a great scope for economists, statisticians, administrators, politicians and the technicians working in a team to solve this problem by an OR approach. Besides this, OR is useful in the following various important fields.

1. **In Agriculture :** With the explosion of popula-tion and consequent shortage of food, every country is facing the problem of :-
   - (i) Optimum allocation of land to various crops in accordance with the climatic conditions; and
   - (ii) Optimum distribution of water from various resources like canal for irrigation purposes.

   Thus there is a need of determining best policies under the prescribed restrictions. Hence a good amount of work can be done in this direction.
2. **In Finance :** In these modern times of economic crisis, it has become very necessary for every government to have a careful planning for the economic development of country. OR-techniques can be fruitfully applied.
   - (i) To maximize the per capita income with minimum resources;
   - (ii) To find out the profit plan for the company
   - (iii) To determine the best replacement policies, etc.
3. **In Industry :** If the industry manager decides his policies (not necessarily optimum) only on the basis of his past experience (without using OR techniques) and a day comes when he gets retirement, then a heavy loss is encountered before the Industry. This heavy loss can immediately be compensated by newly appointing a young specialist of OR techniques in business management. Thus OR is useful to the Industry Director in deciding optimum allocation of various limited resources such as men, machines, material, money, times etc., to arrive at the optimum decision.
4 **In Marketing:** With the help of OR techniques a Marketing Administrator (Manager) can decide:
   - (i) where to distribute the products for sale so that total cost of transportation etc. is minimum,
   - (ii) the minimum size per unit sale price,
   - (iii) the size of the stock to meet the future demand
   - (iv) how to select the best advertizing media with respect to time, cost, etc.
   - (v) how, when and what to purchase at the minimum can use OR techniques.
5. **In Personnel Management:** A personnel manager can use OR techniques :
   - (i) To appoint the most suitable persons on minimum salary,
   - (ii) To determine the best age of retirement for the employees
   - (iii) To find out the number of persons to be appointed on full time basis when the work load is seasonal (not continuous).
6. **In Personnel Management:** A production manager can use OR techniques:
   - (i) To find out the number and size of the items to be produced
   - (ii) In scheduling and sequencing the production run by proper allocation of machines;
   - (iii) In calculating the optimum product mix, and
   - (iv) To select, locate, and design the sites for the production plants.
7. **In L.I.C. :** OR approach is also applicable to enable the L.I.C. offices to decide:
   - (i) what should be the premium rates for various modes of policies
   - (ii) how best the profits could be distributed in the cases with profit policies? etc.

Q3. Explain the characteristics of Operations Research.

Ans :

OR plays a crucial role in various fields due to the following striking features :

(a) Distinct Direction in Decision Making

Operations Research model provides a clear and distinct direction to the managers in decision making and problem solving. A major premise of Operations Research is that decision making irrespective of the situation involved, can be considered as a general systematic process.

(b) Scientific Approach

Operations Research employs scientific reasoning to its problems. Therefore the mangers can confidently implement their decisions.

(c) Objective Orientation

Operations Research is oriented to locate the best possible or optimal solution to the problem. As the approach itself is embedded with setting of the goal or objectives, it becomes easy to use this as a measure to compare the alternative courses of action.

(d) Inter - Disciplinary Team Approach

Operations Research is inter disciplinary in nature and therefore needs a team approach. It is a blend of the aspects of various disciplines such as economics, physics, physiology, sociology, anatomy, engineering, technology, mathematics, statistics and management.

(e) Compatible to Digital Computer

The use of digital computer has become an internal part of the Operations Research approach to decision making. There are several software packages developed with the help of Operations Research approach to problems with high volume and complexity in nature.

Q4. Discuss the three main phases of operations research.

Ans :

Main Phases of Operations Research

The scientific method on which Operations Research is based has three phases:

(i) Judgement phase

(ii) Research phase and

(iii) Action phase

These three phases are briefly described in the following paragraphs:

(i) Judgement Phase : This phase includes

(a) Identification of the real-life problem (i.e., problem as faced in real life).

(b) Selection of an appropriate organizational objective.

(c) Selection of the values of various variables related to the specific objective.

(d) Application of the appropriate scale of measurement, i.e., deciding the measure of effectiveness of the objective to the organisation.

(e) Formulation of an appropriate model (i.e., structural form) of the problem with relevant essential information for the decision maker.

(ii) Research Phase

(a) In this phase, relevant data is collected for a better understanding of the problem.

(b) Based on the data, an appropriate hypothesis and model are formulated.

(c) Observation and experimentation to test the hypothesis on the basis of additional data. Additional data may become necessary to test the applicability of the hypothesis over a wide range of observations and variability.

(d) Analysis of the available information and verification of the hypothesis using pre-established measures of desirability of the method or model.

(e) Prediction of various results from the hypothesis.

(f) Generalisation of results and consideration of alternative methods for "what-if" systems is then standardized.

(iii) Action Phase : In this phase, the recommendations for the implementation of the decision so arrived are made by the individual who is in a position to implement the results. The individual must be aware of the actual problem and the environment in which the problem occurred along with various assumptions, limitations and omissions of the model formulated for the problems.

1.1.2 Origins of Operations Research

Q5. State the evolution of operations research.

Ans : **(July - 18)**

The origin and evolution of operations research can be known from the following classification,

1. Pre-world War-II Developments

The development of operations research had been started in the early 1800s, but it came into practise when Frederick W.Taylor used the methods of scientific analysis for the production techniques.

Most of the techniques of operations research such as Inventory control, Queueing theory and Statistical quality control were developed and applied before the introduction and coinage of the term “operations research”.

In 1885, F.W. Taylor carried-out his experiments on a simple shovel to determine the optimal capacity of a Shovel to carry maximum load with minimum fatigue.

Although, the jobs on machines are performed perfectly, there was a delay in their movement for further processing. But with the introduction of job scheduling techniques developed by Henry L.Gnatt, movement from one machine to other is found on a timely manner without any delays. This helps in minimizing the process time and enables the firm to plan for monthly machine loadings in advance by accurately stating the delivery dates. The total cost of inventory can be optimized by Economic Order Quantity (EOQ) model which was developed by Ford Harris in 1915 and analyzed by R.H. Wilson in 1934.

Queueing theory was very much developed in 1916 by A.K. Erlang.

In 1924, control charts were developed by Shwart in the inspection engineering department of Western Electric’s Bell Laboratory. Such control charts are used for controlling the quality and cost of raw materials, components and finished products.

Industrial revolution is responsible for bringing the rapid pace in the growth and development of operations research. Prior to the industrial revolution, most of the industries were small scale employing only few people but due to automation, when the man is replaced by machine, improvements were seen in transportation and communication and management of an organization has undergone a division wherein the entire organization was classified into separate departments such as marketing, finance, production, IT which were headed by individual managers for each department.

Each and every department of an organization has its own objectives whose accomplishment contributes to the achievement of organizational objectives. An individual department cannot contribute to the whole objectives of the organization. They just constitute a part. All the departments together achieve the established objectives of the organization.

All the departments must strive hard for the achievement of their goals and objectives.

Hence, a policy should be set up which serves the organizations interest but not the interest of individual department.

2. Developments During World War-II

During this phase, there was a great concern about the allocation of scarce military resources which were restricted only to the different military operations. For this, a team of scientists, under the guardianship of professor and a naval officer P.M.S. Blackett was appointed by the military management. Their role was to study and examine various strategic and tactical problems related to air and land defence and to formulate the optimal solution by optimally allocating the scarce resources. As, they have performed efficiently, they were able to achieve their objectives. Many such teams were spread to western allied countries.

3. Post-world War-II Developments

Fascinated by the results obtained by military teams in the world war-II, the industrial managers tried to follow the operation research techniques in their production operations to solve their problems and to achieve maximum profitability out of the available scarce resources.

Primarily, Britain and America focussed on these operations research techniques to be applied in their industries.

In U.S.A, second industrial revolution led to the growth of industrial operation research which resulted into the replacement of man by machine. After the world war II, a new revolution had been witnessed due to the easy availability of electronic computers.

4. Computer Era

As most of the operations research techniques are associated with complex calculations, longer times are required for solving the real-life problems. But, due to the development of high speed digital computers, even the complex computations can be easily done in few seconds.

5. Inclusion of Uncertainty Models

Operations research techniques are able to deal with the undeterministic situations more realistically by using probability theory and statistics than deterministic situations.

1.1.3 Applications of OR in Different Managerial Areas

Q6. State the applications of operations research.

Ans : **(May-19, Dec.-18, July -18)**

1. **In Agriculture :** With the explosion of population and consequent shortage of food, every country is facing the problem of :-
   (i) Optimum allocation of land to various crops in accordance with the climatic conditions; and
   (ii) Optimum distribution of water from various resources like canal for irrigation purposes.

   Thus there is a need of determining best policies under the prescribed restrictions. Hence a good amount of work can be done in this direction.

2. **In Finance:** In these modern times of economic crisis, it has become very necessary for every government to have a careful planning for the economic development of country. OR-techniques can be fruitfully applied.
   (i) To maximize the per capita income with minimum resources;
   (ii) To find out the profit plan for the company
   (iii) To determine the best replacement policies, etc.

3. **In Industry :** If the industry manager decides his policies (not necessarily optimum) only on the basis of his past experience (without using OR techniques) and a day comes when he gets retirement, then a heavy loss is encountered before the Industry. This heavy loss can immediately be compensated by newly appointing a young specialist of OR techniques in business management. Thus OR is useful to the Industry Director in deciding optimum allocation of various limited resources such as men, machines, material, money, times etc., to arrive at the optimum decision.

4 **In Marketing :** With the help of OR techniques a Marketing Administrator (Manager) can decide:
   (i) where to distribute the products for sale so that total cost of transportation etc. is minimum,
   (ii) the minimum size per unit sale price,
   (iii) the size of the stock to meet the future demand
   (iv) how to select the best advertizing media with respect to time, cost, etc.
   (v) how, when and what to purchase at the minimum can use OR techniques.

5. **In Personnel Management :** A personnel manager can use OR techniques :
   (i) to appoint the most suitable persons on minimum salary,
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6. **In Personnel Management :** A production manager can use OR techniques:
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   - (ii) in scheduling and sequencing the production run by proper allocation of machines;
   - (iii) in calculating the optimum product mix, and
   - (iv) to select, locate, and design the sites for the production plants.
7. **In L.I.C. :** OR approach is also applicable to enable the L.I.C. offices to decide:
   - (i) What should be the premium rates for various modes of policies
   - (ii) How best the profits could be distributed in the cases with profit policies? etc.

Q7. State any five areas for the application of OR techniques in financial management with suitable examples. How it improves the performance of the organization?

Ans : **(Feb. -18)**

Following are the areas of financial management where Operations Research (OR) techniques are applied,

1. Cash Management

A finance manager is accountable for proper supply of funds to various divisions and departments of an organization, because availability of sufficient funds is very important for smooth running of business. For this purpose, Linear programming techniques of OR are very useful in ascertaining allocation of funds to various divisions.

For example, Linear programming techniques not only used to find out the divisions/departments which are having un-used funds but they also helps to allocate such funds to needed departments.

2. Simulation Techniques

In simulation, different aspects are taken into consideration which have impacts on present and projected costs of borrowed funds from banks and tax rates and gives a favourable mix of financing for required capital amount.

For example: Simulation may help in replacing intuitive estimation, judgements or ideas of management by dependable or valid information.

3. Capital Budgeting

In capital budgeting, many investment projects can be evaluated such as launch of a new product, buying of new machineries in place of old ones etc.

For example: Techniques of operations research OR such as linear programming dynamic programming and integer programming assists in choosing best investments proposals.

4. Inventory Control

In large scale companies, inventory control techniques of OR assists management in formulating proper inventory policies and reduces the scale of investment in inventories. The Inventory control techniques assists in bringing stability available between shortage costs, ordering costs and inventory carrying costs.

For example: OR techniques help in determining the quantity of stock available and when it is required to order.

5. Queue Management

Queuing is one of the major technique of Operation Research (OR). This technique helps to find out the optimal arrival time of customers.

For example: Queuing may help the firm to find out the number at customers waiting for service and required time to deliver the service etc. If firm efficiently managed its customers then it will automatically increase the sales.

1.2 PROBLEM SOLVING AND DECISION MAKING

Q8. Define the terms:

(a) Problem Solving

(b) Decision Making

Ans :

(a) Problem Solving

Problem solving can be defined as the process of identifying a difference between the actual and the desired state of affairs and then taking action to resolve the difference. Seven steps involved in the problem solving process (for problems which are important and time and effort of analysis can be justified) are :

(i) Identify and define the problem

(ii) Determine the set of alternative solutions

(iii) Determine the criteria that will be used to evaluate the alternatives

(iv) Evaluate the alternatives

(v) Choose an alternative

(vi) Implement the selected alternative

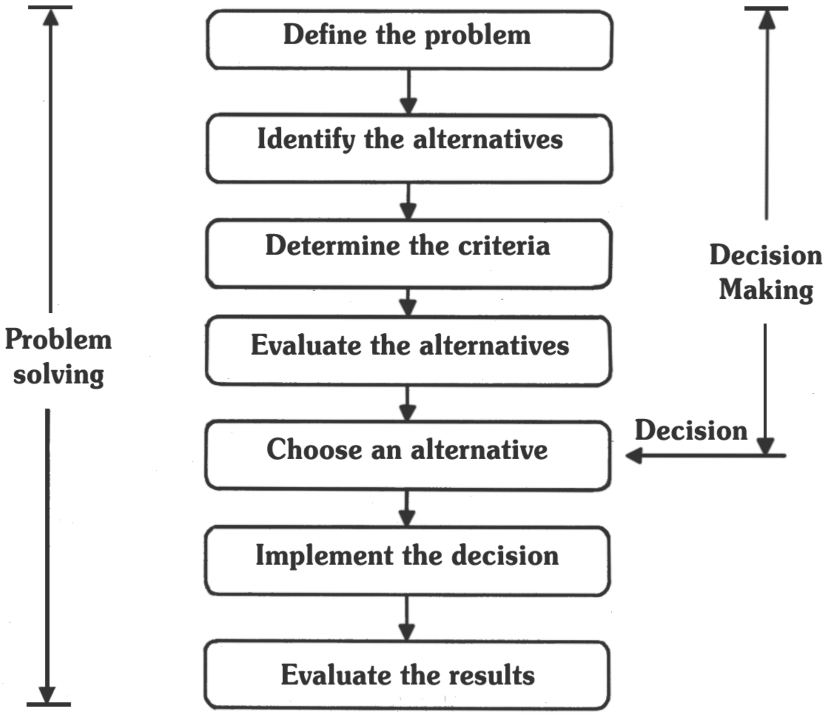
(vii) Evaluate the results to determine whether a satisfactory solution has been obtained.

(b) Decision Making

Decision making is “a process of identification and ‘election of an action from a number of alternative courses of action for resolving a problem in the organization”.

Decision making acts as the basis for planning an activity in the organization. It is one of the important managerial function. Decision making must be rational for achieving the set goals successfully. It is very important to take the decisions at every stage of the organization. The decisions which are taken by top management are called as strategic decisions and the decisions which are related to the normal day-to-day activities of organization are called as tactical or operational decisions.

Relationship



Q9. Explain different types of decision makings.

Ans :

The decisions are categorized broadly into six categories based on the different criteria. They are as follows,

1. Classification based on their impact on organization.
2. Classification based on the nature of decision and the nature of problems involved.
3. Classification based on the number of individuals involved in the process.
4. Classification based on their importance.
5. Classification based on the extent of freedom to decide.
6. Classification based on the persons involved.

1. Classification Based on their Impact on Organization

The decisions are classified into two types based on their impact on organization. They are as follows,

(a) Basic Decisions

The decisions which are important to the organization and are strategic in nature are called as basic decision. These decisions are also called as strategic decision. These decisions have a major impact on the success of the organization.

Example

Location of plant, decisions relating to distribution channels design of the organizational structure etc.

(b) Routine Decisions

The decisions which are related to the routine day-to- day activities are called as routine decisions. These decisions does not have a significant impact on the performance of the organization.

Example

The decisions related to the movement of raw materials to production process, marketing of a product at a selected place etc.

2. Classification Based on Nature of Problems Involved

The decisions are classified into two types based on the nature of the problems involved. They are as follows,

(a) Programmed Decisions

The decisions which are taken by the management based on its past experience for resolving the structured problems are called as programmed decisions. Structured problems are clear and definite. It can be anticipated and are routine in nature. For solving these problems, the management can plan the decisions before its occurrence. The programmed decisions follows the policies, procedures and rules which are fixed by the organization for a particular period of time.

Programmed decisions may limit the freedom of the (employees) managers as they are taken as per the policies. It is a time saving process as the problems are anticipated and the management can plan to resets the problems in advance.

(b) Non-programmed Decisions

The decisions which are taken by the management for resolving the complex, unanticipated and exceptional problems are called as non-programmed decisions. These problems are also called as unstructured problems.

It is very important to take non-programmed decisions at every level of management. These decisions are specific in nature for resolving the non routine problems.

Example

Decisions relating to the allocation of resources, improvement of community relations etc., are the examples of non-programmed decisions.

3. Classification Based on Number of Individuals Involved

The decisions are classified into two types based on the number of individuals involved in the decision making process. They are as follows,

(a) Individual Decisions

The decisions which are taken by managers individually or by any individual person of the organization without consulting others are called as individual decisions. These are routine in nature and do not have any major impact on the organizational success. The managers have the right to take individual decisions in certain conditions.

(b) Group Decisions

The organizations mostly opt for group decision making. The decisions in which a group of members or managers and associates consult with the other group members are called as group decisions. These decisions are taken after reviewing the advantages and limitations of each alternative proposed by all the members. All the group members collectively resolve the problem. Strategic decisions are a type of group decisions which are taken by a group of managers from each department and board of directors.

4. Classification Based on their Importance

The decisions are classified into two types based on their importance in the organization activities. They are as follows,

(a) Major Decisions

The decisions which are strategic in nature and are related to the significant aspects of organization like construction of building for production, processing of products, business expansion are called as major decisions. These decisions require huge amount of money for implementation and are very specific decisions.

(b) Minor Decisions

The decisions which are routine in nature and deals with the organizational aspect are less significant are called as minor decisions. These decisions require low cost for implementation and are not specific decisions.

5. Classification Based on the Extent of Freedom to Decide

The decisions which are classified into two types based on the extent of freedom given to the managers to decide. They are as follows,

(a) Personal Decisions

The decisions are taken by manager without consulting others in the organization are called as personal decisions. The manager has the freedom to take personal decisions.

(b) Organizational Decisions

The decisions which a manager take by considering the organizational and environmental conditions and factors that are within the boundaries of the organizational policy are called as organizational decisions. The manager either consults with the colleagues or subordinates or superiors while taking decisions or takes the decisions independently.

6. Classification Based on the Persons Involved in Taking Decisions

The decisions are classified into two types based on the persons which are involved in taking the decisions. They are as follows,

(a) Departmental Decisions

The decisions which are taken by the head of the department or chief of the department are called as departmental decisions. These decisions are mostly based on the past performance and the opinions of the people in the department.

Example

Decisions relating to the development of the department etc.

(b) Inter Departmental Decisions

The decisions which are taken by the heads of two separate departments or the chiefs of all the departments of the organization are called as interdepartmental decisions. In some cases, instead of the chief of the departments a group of senior managers of the department or general managers takes the related decisions.

7. Classification Based on the Decision-making Model

The decisions are classified into two types based on the decision-making model.

(a) Rational Decisions

The rational decision maker makes effective, consistent and profit maximizing decisions within specified limits. Rational decisions are taken by following the rational decision making model/process which includes the following six steps,

(i) Identifying and defining the problem.

(ii) Specifying the criteria.

(iii) Assigning weights to the selected criteria.

(iv) Generating alternatives.

(v) Evaluating alternatives.

(vi) Choosing the best alternative.

Some of the assumptions of rational decision-making model are,

(i) The decision maker must posses complete information.

(ii) The decision maker should have the capability to identify the alternatives without any bias.

(iii) The decision maker should have the capability to select the best alternative from the identified alternatives.

Generally, people do not follow the process of rational decision-making model because they focus more on finding reasonable solution rather than optimal solution.

(b) Bounded Rational Decisions

Bounded rational decision are also known as 'Bounded rationality'. It is a kind of perception associated with decision-making, wherein rationality of individuals is limited due to the coincised information and the time within which they have to make decisions. Bounded rationality considers rationality as optimization. Thus, in bounded rationality the decision maker is a satisfier, who seeks form satisfactory solution rather than optimal solution.

Q10. Outline the process of decision making.

Ans :

The decision-making model (or the decision making process) involves six steps. These six steps have to be carried out by a decision maker in order to arrive at a decision.

The following figure depicts the six steps in the decision-making model.

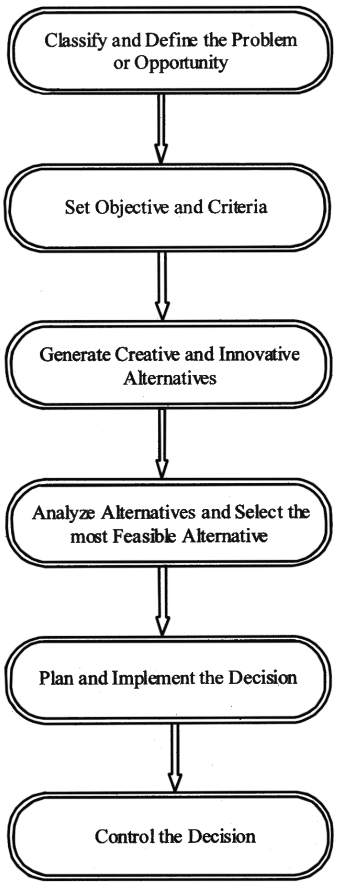


Fig.: The Decision-making Model

It is not mandatory that all the six steps have to be carried out in the above mentioned sequence. At times, the decision maker may require to go back to the previous step rather than going to the next step. This is usually done when the decision maker wants to make changes in the previous steps.

Following all the steps in the decision making model may not ensure a good decision but will enhance the chances of success in problem solving and decision making.

Step 1: Classify and Define the Problem or Opportunity

Defining a problem is the first step in the process of problem solving. Collecting information processing the information and careful consideration of the collected information are the activities to be undertaken in this step.

The problem has to be defined clearly and correctly because the solution to a problem relys upon how effectively the problem was defined in the initial stage.

A problem could be of many types, but in simple terms it can be stated as "the difference between what is and what should be". A manager should try to define the problem as clearly as possible because a well-defined problem is halfway solved. The efficiency of the decision making process and quality of decision highly depends upon how clearly the problem was defined.

Generally, defining a problem is quite difficult is majority of the cases because the actual problem may be different from the appearing problem. Thus, the situation of the problem has to be defined and described in terms of its importance, scope, symptoms, causes etc.

Step 2: Set Objectives and Criteria

In order to ensure good decision making, clear objectives must be set for the organization. These objectives assist the managers in making good decisions. Objectives give the description of what the decisions is supposed to do i.e., whether they have to resolve a problem or they have to grab an opportunity.

Step 3: Generate Creative and Innovative Alternatives

The process of implementing the new ideas is called as innovation. The innovation is a sequential process which basically deals with identifying problem generating ideas for solving the problems following the best ideas to complete and generate value from these ideas. The process of innovation always includes the generation of ideas and implementation of them.

Creativity in the organizational decision-making is the most significant feature of employees and process. It is often regarded as a mental process and is related with developing a new product or problem solving. It is not possible for a company to create a unique image and gain competitive advantage, if it cannot effectively differentiate its functioning style and nature of product offerings. A product has a broad scope and can either be tangible or intangible. Creative thinking and linear thinking are different from each other as linear thinking is convergent thinking which basically emphasizes on solution, while creative thinking is divergent or different. Hence, creative thinking is also known as 'out- of-the box thinking' or 'right-brain thinking'.

Step 4: Analyze Alternatives and Select the most Feasible Alternative

Analyzing/evaluating the alternatives and selecting the most feasible alternatives is the fourth step in the decision-making model. At the time of evaluating alternatives, each alternative has to be compared with the objectives and criteria set in step-2 of the decision-making process. Apart from this, alternatives need to be compared with one another to identify the most feasible alternative. While evaluating alternatives, the decision maker should try to estimate the possible results from each alternative.

It is essential for a decision-maker to select the best alternative from the list of alternatives. The decision maker requires to compare the results of various alternatives and select the one which yield maximum benefits. Selecting the best alternative from available alternatives is not so simple as it seems to be. Usually, it give rise to same complications. The reputation of the decision maker may be at risk. As decision maker is the one who has to face criticism against his wrong decision (so this fear stops the manager from making any decision) Sometimes, the fear of risk prevents him from taking correct decision. Thus, the capability to select the best alternative from the available alternatives differentiate the successful managers from unsuccessful or less successful managers.

Step 5: Plan and Implement the Decision

In decision-making model planning and implementing the decision (step-5) and controlling (step-6) plays an active role in execution of decision. After taking decision the manager develops an action plan with a schedule to implement it.

A 'plan' is defined as a specific action which is proposed to help the organization to achieve its objectives. In an organization, it is very essential for the managers to develop the organizational plan. In other words, "planning is deciding in the present what to do in future. It is the process whereby companies reconcile their resources with their objectives and opportunities".

After developing the action plan, that plan must be implemented. The manager must identify and remove the constraint if any in the way of implementation. The implementation of action plan needs the active support of the organization and employees at all levels of the organization.

Step 6: Control the Decision

Controlling is the last step in the decision-making process. At the time of planning, control methods are developed to determine the decision outcomes. Some check points have to be set during the implementation process to find out whether the selected alternative is capable of solving the problem or not. It the alternative chosen is failing in solving the problem then the decision maker has to go back to the previous steps to make necessary corrections in the selection stage (selection of best alternative stage).

Q11. Explain the process for developing an operations research model.

Ans : **(Sep. - 15, March - 16, Aug. - 17)**

The various phases of an operations research study are as follows:

1. Define the problem of interest and gather relevant data.
2. Formulate a mathematical model to represent the problem.

3. Develop a computer-based procedure for deriving solutions to problem from the model.
4. Test the model and refine it as needed.
5. Prepare for the ongoing application of the model as prescribed by the management.
6. Implement each of these phases will be discussed in the following sections:

Step 1: Defining the Problem and Gathering Data

Most practical problems encountered by operations research teams are usually described vaguely. Therefore, the first step is to study the relevant system and develop a well-defined statement of the problem to be considered. This includes determining.

(i) The appropriate objectives,

(ii) The constraints on what can be done,

(iii) Interrelationships between the area of study and other areas of the organisation,

(iv) Possible alternative courses of action and

(v) Time limits for making a decision.

The process of defining the problem is crucial because it greatly affects how relevant the conclusions of the study would be. A right answer can be extracted only from a right problem.

The operations research team is normally working in an advisory capacity, (i.e., advising management-decision maker).

The team performs a detailed technical analysis of the problem and then presents recommendations to the management. The report of the operations research team usually will identify a number of alternatives under different assumptions or over different range of values of some policy parameters that can be evaluated only by the management (e.g., trade-off between costs and benefits).

Management evaluates the study report and its recommendations, takes into account a variety of intangible factors and make the final decision based on its best judgement.

Ascertaining the appropriate objectives is very important aspect of problem definition. Operations research is concerned with the welfare of the entire organisation rather than of only certain of its components (i.e., units, divisions, departments).

An operations research study seeks optimal solutions for the overall organisation rather than suboptimal solutions that are best for only one component. However, many problems primarily concern only a portion of the organisation. In such cases the objectives stated should not be too general. Instead the objectives should be as specific as possible and consistent with the higher level objectives of the organisation.

Operations research teams spend a large amount of time gathering relevant data about the problem. Much data are needed both to gain an accurate understanding of the problem and to provide the needed input for the mathematical model being formulated in the next phase of the study.

Frequently much of the needed data will not be available when the study begins. Therefore it is often necessary to install a new computer-based management information system (MIS) to collect the necessary data, on an ongoing basis and in the needed form.

Step 2 : Formulating a Mathematical Model

After defining the problem, the problem is reformulated in a form that is convenient for analysis. The operations research approach is to construct a mathematical model that represents the essence of the problem. Mathematical models are expressed in terms of mathematical symbols and expressions.

For example, $E = mc^2$ is a mathematical model. Mathematical models of a business problem is the system of equations and related mathematical expressions that describe the essence of the problem. If there are 'n' related quantifiable decisions to be made, they are represented as decision variables (say x_1, x_2, ... x_n), whose respective values are to be determined. The appropriate measure of performance is then expressed as a mathematical function of these decision variables (for example $Z = ax_1 + bx_2 + cx_3 + nx_n$)- This function is referred to as "objective function".

Any restrictions on the values that can be assigned to these decision variables are expressed by means of inequalities or equations. Such mathematical expressions for the restrictions are called "constraints".

The constants (i.e., the coefficients and right-hand sides) in the constraints and the objective function are called the parameters of the model. The mathematical model represents the problem to choose the values of the decision variables so as to maximise the objective function, subject to the specified constraints.

Step 3 : Deriving Solutions from the Model

After a mathematical model is formulated for the problem under consideration, the next phase is to develop a procedure for deriving solutions to the problem from this model.

One of the standard algorithms (systematic solution procedures) of operations research is applied on a computer using one of a number of readily available software packages.

If is common in operations research to search for an optimal or best solution. But since the model is an idealised rather than the exact representation of the real problem, there cannot be any guarantee that the optimal solution for the model will prove to be the best possible solution that could be implemented for the real problem.

However, if the model is well formulated and tested, the resulting solution should tend to be a good approximation to an ideal course of action for the real problem.

Since an optimal solution may not always be feasible, managers tend to seek a solution that is "good enough" for the problem on hand, that is a satisfying solution which is much more prevalent than optimising solution in practice (satisficing is a combination of the words satisfying and optimising).

In the words of operations research leader Samuel Eiron "optimising is the science of the ultimate, satisficing is the art of the feasible". The goal of the operations research study should be to conduct the study in an optimal manner, regardless of whether this involves finding an optimal solution for the model.

Thus in addition to pursuing the science of the ultimate, the team should also consider the cost of the study and the disadvantages of delaying its completion and then attempt to maximise the net benefits of the study. Therefore, operations research teams occasionally use only heuristic procedures to find a good suboptimal solution.

Step 4 : Testing the Model

The first version of a large mathematical model inevitably contains many flaws. Therefore, before the model is used, it must be thoroughly tested to try to identify and correct as many flaws as possible. After a long succession of improved models, the current model gives reasonably valid results. The major flaws have been sufficiently eliminated and the model now can be reliably used.

This process of testing and improving a model to increase its validity is commonly referred to as model validation.

Step 5 : Preparing to Apply the Model

After the testing phase has been completed and an acceptable model has been developed, for the model to be used repeatedly, a well documented system has to be established for applying the model as prescribed by management. This system includes the model, solution procedure and operating procedures for implementation. This system usually is computer based.

A considerable number of computer programmes often need to be used and integrated. Data bases and management information systems may provide uptodate input for the model whenever it is used.

In such cases, interface programmes are needed. After a solution procedure (another programme) is applied to the model, additional computer programmes may trigger the implementation of the results automatically. In other cases, an interactive computer-based system called a decision support system is installed to help managers use data and models to support their decision making us needed.

Step 6 : Implementation

After a system is developed for applying the model, the last phase of operations research study is to implement this system as prescribed by management.

It is important that the operations research team participate in launching this phase both to ensure that model solutions are accurately translated to an operating procedure and to rectify any flaws in the solutions that are then uncovered.

The implementation phase involves several steps. These are:

(a) The operations research team gives operating management a careful explanation of the new system to be adopted and how it relates to operating realities.

(b) These two parties share the responsibility for developing the procedures required to put this system into operation.

(c) Operating management sees that a detailed indoctrination is given to the personnel involved and the new course of action is initiated.

(d) The operations research team monitors the initial experience with the course of action taken and seeks to identify any modifications that should be made in the future.

Throughout the entire period during which the new system is being used, it is important to continue to obtain feedback regarding how well the system is working and whether the assumptions of the model continue to be satisfied. When significant deviations from the original assumptions occur, the model should be examined to determine if any modifications should be made in the system.

1.2.1 Quantitative and Qualitative Analysis

Q12. What do you mean by Quantitative and Qualitative Analysis of Decision Making?

Ans : **(Feb. - 17)**

The analysis phase of the decision making process having two basic forms: qualitative and quantitative.

i) Qualitative Analysis

Qualitative analysis is primarily based on the judgement and experience of the managers and it includes the manager's intuition regarding the problem. Qualitative analysis is emphasised in situations where managers had experience with similar problems or if the problem is relatively simple. Quantitative analysis is recommended if the problem is quite complex and the manager is not having enough experience with similar problems.

ii) Quantitative Analysis

Quantitative analysis is the scientific approach to managerial decision making. This approach starts with data which are manipulated or processed into information that is valuable for making decisions. This processing and manipulating of raw data into meaningful information is the heart of quantitative analysis. Computers have greatly enhanced the use of quantitative analysis for solving managerial problems.

Techniques

Quantitative techniques (or) quantitative methods are primarily mathematical and statistical techniques used in managerial decision making. Quantitative approach to problem solving requires collection of large amount of data (quantitative in nature) and development of mathematical models. Quantitative techniques such as linear programming, transportation method, assignment method, queuing models, game theory, simulation etc., are used to support management to make decisions under complex and uncertain situations in order to achieve the objectives and goals of the organisation. Other names used synonymously or interchangeably with quantitative analysis or quantitative techniques are:

(a) Operations research,

(b) Management Science,

(c) Decision Analysis,

(d) Operations Analysis and

(e) Systems Analysis.

Limitations of Quantitative Methods

(i) Time consuming and elaborate.

(ii) Difficulty in identifying uncertainties may make coordination with the system difficult.

(iii) Treated as a supporting tool to analyse mathematical models and may not be acceptable to many decision makers.

(iv) Can be highly expensive and time and cost may not be proportional to the size and complexity of the problem.

(v) Selection of wrong variables or insufficient variables for constructing the model can result in erroneous decisions.

Q13. What are the differences between quantitative analysis and qualitative analysis. ?

Ans :

Sl. No.	Criteria	Qualitative Analysis	Quantitative Analysis
1.	**Purpose / Need**	Understand and interpret social interactions	Test hypothesis - Check cause and effect relationship, develops prediction about future.
2.	**Studied group**	Small, selected internationally	Large, selected random
3.	**Data types**	Words, Images, Objects	Numbers and statistics.
4.	**Data form**	Open ended responses, interviews, participants observations.	Presize measurements using structures and validated Instruments data collect.
5.	**Type of data analysis**	Patterns, features, thems, Identification.	Statistical relationship Identification.
6.	**Result**	Particular Findings, less generalisable	Generalisation findings can be applied.

1.3 DEFINING A MODEL

Q14. Define a model ?

Ans : **(Dec.-18)**

The essence of operations research lies in the construction and use of models. A model in the sense intended in operations research is just a simplified representation of something real. It is implied that a model is always, necessarily a representation that it is less than perfect.

When a real life situation is represented in some abstract form, whether physical or mathematical, bringing out relationships of its major ingredients, a model is said to be formed. The model so formed, need not describe all the aspects of this situation, but it should signify and identify important factors and their interrelationships to describe the total situation. Hence, models do not and cannot represent every aspect of reality because of the real life problems of innumerable and changing characteristics to be represented. Models are limited approximation of reality.

A model is constructed to analyse and understand the given system for the purpose of improving its performance. The reliability of the solution obtained from a model depends on the validity of the model in representing the system being studied. A model, however, allows the analyst (model builder) to examine the behaviour changes of the system without affecting the on-going process in the system.

1.3.1 Types of Model

Q15. Explain the different types of models.

Ans : **(Sep.-14, March - 15, Sep.-15)**

These models are briefly discussed below:

1. **Models Based on Function**

   (a) Descriptive

   Descriptive models are those which simply describe some aspects of a situation, based on observation, survey, questionnaire results or other variable data concerned with a situation. These models do not predict or recommend anything.

   Example : Plant layout diagrams, flow charts, organisation charts etc.

(b) Predictive

Predictive models are those which indicate "if something happens what will follow". They indicate the relationship between dependent and independent variables and permit trying out "what-if" questions. Using these models, one can predict the outcomes due to a given set of alternatives for the problem.

(c) Normative

Normative or optimisation models are those which provide the "best" or "optimal" solution to problems subject to certain limitations on the use of resources.

Example : Mathematic model formulating an objective function subject to restrictions or constraints on resource in the context of the problem. These models are also referred to as "prescriptive models" because they prescribe what the decision maker has to do.

Basis Types

1. Function
   - (a) Descriptive
   - (b) Predictive
   - (c) Normative/optimisation
2. Structure
   - (a) Physical
     - (i) Iconic
     - (ii) Analogue
   - (b) Symbolic
     - (i) Verbal
     - (ii) Mathematical
3. Time reference
   - (a) Static
   - (b) Dynamic
4. Degree of certainty
   - (a) Deterministic
   - (b) Probabilistic (stochastic)
5. Degree of quantification
   - (a) Qualitative
     - (i) Mental
     - (ii) Verbal
   - (b) Quantitative
     - (i) Heuristic
     - (ii) Analytical
     - (iii) Simulation

2. Models Based on Structure

(a) Physical models: These models provide a physical representation of the real object under study in a reduced size (scaled model). Example: Scale models of proposed aircraft under design/prototype construction.

Physical models are classified as:

(i) Iconic models: An icon is the depiction of an object as its image. An iconic model is a scaled version of the system it represents. It represents the system as it is by scaling it up or down.

Examples are blue prints of buildings, houses, photographs, drawings etc.

(ii) Analog models: represent a system like an iconic model but not as the exact replica of the actual system. They represent a system by a set of properties different from those of the original system and does not resemble it physically.

Examples: Maps, organisation charts, graphs of time series, frequency curves etc.

(b) Symbolic models: Symbolic models are those used to represent actual problems using symbols (letters, numbers) to represent variables and their relationships to describe the properties of the system. Two types of symbolic models are:

(i) Verbal models

Verbal models which describe a situation in written or spoken words or sentences. E.g.. Books, reports etc.

(ii) Mathematical models

Mathematical models which involve the use of mathematical operators (+, -, x, -r etc.) to represent relationships among various variables of the system in order to describe the properties or behaviour of the system. Example : "Economic order quantity" (EOQ) model in inventory management, cost-volume-profit model in break-even-analysis etc.

3. Models Based on Time Reference

(a) Static models

Static models represent a system at some specified time and do not account for change over time. E.g., EOQ model.

(b) Dynamic model

Dynamic model which considers time as one of the variables and allows the impact of changes due to change in time. E.g., Dynamic programming.

4. Models Based on Degree of Certainty

(a) Deterministic

Deterministic models in which all the parameters, constants and functional relationships are assumed to be known with certainty when the decision is taken. E.g., linear programming models.

(b) Probabilistic

Probabilistic (stochastic) models are those in which at least one parameter or decision variable is a random variable. In such cases, consequences or payoffs due to certain changes in the independent variable cannot be predicted with certainty, but it is possible to predict a pattern of values of both variables by their probability of distribution. E.g., models representing insurance against risk of fire, accidents, sickness of employees etc.

5. Models Based on Quantification

(a) Qualitative models

Qualitative models which are descriptive models - mental or verbal description is used to represent the situation.

(b) Quantitative Models

(i) Heuristic models

Heuristic models employ some set of rules which, though not optimal, do facilitate problem solving when applied consistently.

(ii) Analytical models

Analytical models which have a specific mathematical structure and can be solved using analytical or mathematical techniques. E.g., linear programming model for determining the optimal product mix.

(iii) Simulation models

Simulation models are those that have mathematical structure but are not solved using mathematical techniques. They use essentially a computer assisted experimentation of a mathematical model of a real-life problem. Simulation models are used to describe and evaluate the behaviour of a system under certain assumptions over a period of time.

Apart from the above models, operations research models can be either general or specialised models based on degree of generality, either two-dimensional or multidimensional based on dimensionality and either closed or open based on the degree of closure.

1.3.2 Process for Developing an Operations Research Model

Q16. Outline the general principles used in model building within the context of OR. Briefly explain the scientific method in OR.

Ans: **(May-19, Dec.-18, Sep.-14, March-16)**

Process for Developing an OR Model

Formulation of the problem is the first stage in the construction of a model. It involves the analysis of the system which is under study. The next stage in model construction is to define a measure of effectiveness. In this stage, a model is constructed wherein the effectiveness of the system is indicated as a function of the variables that defines the system.

The general form of OR model is,

$$\boxed{E = f(x_i, y_j)}$$

Where,

E = Effectiveness of the system.

x_1 = The controllable variables of the system

y_1 = The uncontrollable variables of the system.

(These variables can affect E)

In order to derive a solution from the model determination of values of control variables *x* is required. For these variables, the measure of effectiveness is optimized. Optimization involves maximization (when there are profits and production capacity etc.,) and minimization (when there are losses and cost of production).

The different steps involved in the model construction are as follows,

Step 1: Selection of Various System Components

The components of the system which leads to effectiveness of the system should be listed out in this step.

Step 2: Suitability of the Components

After preparing a list of components, it is necessary to determine whether these components are to be considered or not. To do this, the effect of alternative actions on these components are determined. The components which are independent of the changes (such as fixed costs) are temporarily ignored.

Step 3: Integration of Components

Now, the components are integrated. Example: It is convenient to combine purchase price, receiving cost of material and freight costs and this integration is called as 'raw material acquisition cost'.

The next step is to find whether the value of components on the modified list are fixed or variable. In case of variable component, it is essential to determine the factors affecting the component value

Step 4: Substituting Symbols

When the variable component in the modified list is broken down into sub-components, symbols are assigned to each sub-component.

1.3.3 Practices, Opportunities and Short Comings of using an OR Model

Q17. Explain the various models of OR in practice.

Ans :

Some of the operations research models applied in business decisions are as follows,

1. Allocation Models (Distribution Models)

These models are concerned with the allocation of available resources, so as to maximize profit or minimize loss or (cost) subject to known and or predicted restriction. Methods for solving allocation models are

(a) Linear Programming Problems (LPP)

Linear programming problem is defined as a method which is adopted to optimize an objective function which is subjected to set of constraints. It is named linear programming since objective function and set of constraints are linear equations.

To solve an LPP some basic requirements must be fulfilled. They are,

(i) Objective function availability

(ii) Set of constraints

(iii) Both objective function and constraints must be linear functions,

(b) Transportation Problems

Transportation Problem (TP) is a special case of LPP in which goods or products are transferred from sources to destination for minimizing the total cost of transportation. An Initial Basic Feasible Solution (IBFS) can be calculated for TP by using North West Comer Rule, row and column minimum methods, matrix minimum method and VAM. After obtaining IBFS, optimal solution is calculated for TP. The two methods available for obtaining optimal solution are, stepping stone method and MODI method. Unbalanced TP and degeneracy are the special cases in transportation problem.

(c) Assignment Problems

Assignment problem is a special case of transportation problem in which the objective is to assign number of tasks (jobs, origins and sources) to an equal number of facilities (machines, persons or destinations) at a minimum cost (or maximum profit). Hungarian method is an important method used for solving assignment problem. Assignment problem can be classified into two types - Maximization case and minimization case. In maximization case the objective function is maximization of profit, revenue, returns etc. In minimization case, objective function is to minimize the cost.

2. Waiting Line Models (Queueing)

This model is an attempt made to predict,

(a) How much average time will be spent by the customer in a queue?

(b) What will be an average length of the queue?

(c) What will be the utilization factor of a queue system? etc.

This model tries to minimize the sum of costs of providing service and cost of obtaining service or costs associated - with the value of time spent by the customer in a queue.

3. Game Theory (Competitive Strategy Models)

These models are used to determine the behaviour of decision-making under competition or conflict. Methods for s: king such models have not been found suitable for industrial applications because they are referred to as idealistic world neglecting many essential features of reality.

4. Inventory Models (Production)

These models are concerned with the determination of the optimal (economic) order scientific advancement or determination due to wear and tear, accidents etc., individual and group replacement policies can be used in case of equipments that fail completely and instantaneously (electric bulbs, decorative items etc).

5. Replacement Models

These models are used in situations where the decision has to be taken regarding the replacement of equipments. The time of replacement is the crucial decision as the objective is to minimize the total cost of investments in new machine and operating costs of old machine. Normally, such problems arise when equipment fail completely, deteriorate over time or become obsolete.

6. Job Sequencing Models

These models involve the selection of such a sequence of performing a series of jobs to be done on machines that optimize the efficiency measure of performance of the system.

7. Network Models

These models are applicable in large projects involving complexities and interdependencies of activities. CPM (Critical Part Method) and PERT (Project Evaluation and Review Technique) are used for planning, scheduling and controlling activities of complex project which can be characterized as network diagram (Arrow diagram).

8. Simulation Models

These models are used for solving when the number of variables and constrained relationships are very large.

9. Markovian Models

These models are applicable in such situations where the state of the system can be defined by same descriptive measure of numerical value and where the system moves from one state to another on a probability basis.

Q18. Explain the opportunities and Short comings of using an OR Model.

(OR)

Explain limitations of operation research.

Ans : **(Nov.-20, Dec.-18, July - 18)**

The use of quantitative methods is appreciated to improve managerial decision-making. However, besides certain opportunities OR approach has not been without its shortcomings. The main reasons for its failure are due to unawareness on the part of decision-makers about their own role, as well as the avoidance of behavioural/organizational issues while constructing a decision model. A few opportunities and threats of the OR Approach are listed below.

a) Opportunities of OR

The OR approach of problem solving facilitates.

1. Decision-maker has to define explicitly his/her objectives, assumptions and constraints.
2. Decision-maker has to identify variables that might influence decisions.
3. In identifying gaps in the data required to support solution to a problem.
4. The use of computer software to solve a problem.

b) Short Comings of OR

Following are the limitations or shortcomings of Operations Research (OR) model,

1. The model cannot be considered as valid until and unless experiments are conducted on it.
2. Models are just idealised representation of reality. Thus, it must not be considered as perfect in any case.
3. The problems are solved either by simplifying assumptions or by making it simpler. Therefore, solution derived have limitations.
4. All realistic problems cannot be solved by Operations Research.
5. Many a times, the decision-maker does not have clear idea about the limitations of the models which he is using.
6. At times, models may fail to represent the real world situations wherein the decisions should be taken.
7. Lack of knowledge in decision maker regarding their own roles and responsibilities.
8. Not considering organizational issues at the time of developing decision model is one of the short coming of Operations Research approach.

Short Question and Answers

1. Define Operations Research.

Ans :

The term, “Operations Research” was first coined by Mc Closky and Trefthen in 1940 in a small town, Bowdsey of United Kingdom.

The name operations research was given to this subject because it has started with the research of (military) operations. During world war - II, the military commands of UK and USA engaged several teams of scientists to discover tactical and strategic military operations. Their mission was to formulate specific proposals and to arrive to the decisions that can optimally utilize the scarce resources to acquire maximum possible level of effective results. In simple words, it was to uncover the methods that can yield greatest results with little efforts. Thus it has gained popularity and was called “an art of winning the war without actually fighting it”.

Following the end of the war, the success and encouraging results of British teams have attracted industrial managers to apply these methods to solve their complex problems. The first method in this direction was simplex method (LPP) developed in 1947 by G.B. Dantzig, USA. Since then several scientists have been developing this science in the interest of making operations to yield high profits or least costs.

Definitions

(i) **According to Operations Research, Society - UK** “Operations research is the application of the methods of science to complex problems in the direction and management of large systems of men, machines, materials and money in the industry, business, government and defence. The distinctive approach is to develop a scientific model of the system by incorporating measurement of factors such as chance and risk, with which to predict and compute the outcomes of alternative decisions, strategies and controls. The purpose is to help management in determining the policy and actions scientifically”.

(ii) **According to Operations Research Society, America,** “Operations Research is concerned with scientifically deciding how to best design and operate man-machine systems usually requiring the allocation of scarce resources”.

(iii) **According to Thierauf and Klekamp (1975)** “Operations Research utilises the planned approach (updated scientific method) and an interdisciplinary team in order to represent complex financial relationships as mathematical models for the purpose of providing a quantitative basis for decision making and uncovering new problems for quantitative analysis”.

(iv) **According to Churchman, Ackoff and Arnoff (1957)** “Operations Research is in the most general sense can be characterised as the application of scientific methods, techniques and tools, to problems involving the operations of a system so as to provide those in control of the operations with optimal solutions to the problems”.

2. Nature of Operations Research

Ans :

The nature of operations research deals with the following,

(a) Operations Research is a Scientific Method

Operations research is of scientific nature as it solves the problems scientifically. It helps in providing quantitative basis for decisions related to the operations under their control.

(b) Operations Research Provides Optimal Solution to the Problem

Operations research determines the root cause of the problem and helps in selecting the best alternative among the. various alternatives.

(c) Helps the Executive Management

Operations research helps the executive management in solving the problems related to management scientifically by providing them analytical and objective basis for making decisions.

(d) Uses Interdisciplinary Team

Operations research uses interdisciplinary team for representing difficult functional relationships as mathematical models as this helps in providing quantitative basis for taking effective decisions and detecting new problems for quantitative analysis.

3. Problem Solving

Ans :

Problem solving can be defined as the process of identifying a difference between the actual and the desired state of affairs and then taking action to resolve the difference. Seven steps involved in the problem solving process (for problems which are important and time and effort of analysis can be justified) are :

(i) Identify and define the problem

(ii) Determine the set of alternative solutions

(iii) Determine the criteria that will be used to evaluate the alternatives

(iv) Evaluate the alternatives

(v) Choose an alternative

(vi) Implement the selected alternative

(vii) Evaluate the results to determine whether a satisfactory solution has been obtained.

4. Decision Making

Ans :

Decision making is "a process of identification and 'election of an action from a number of alternative courses of action for resolving a problem in the organization".

Decision making acts as the basis for planning an activity in the organization. It is one of the important managerial function. Decision making must be rational for achieving the set goals successfully. It is very important to take the decisions at every stage of the organization. The decisions which are taken by top management are called as strategic decisions and the decisions which are related to the normal day-to-day activities of organization are called as tactical or operational decisions.

5. Explain different types of decision makings.

Ans :

The decisions are categorized broadly into six categories based on the different criteria. They are as follows,

1. Classification based on their impact on organization.
2. Classification based on the nature of decision and the nature of problems involved.
3. Classification based on the number of individuals involved in the process.
4. Classification based on their importance.
5. Classification based on the extent of freedom to decide.
6. Classification based on the persons involved.

6. What are the differences between quantitative analysis and qualitative analysis. ?

Ans :

Sl. No.	Criteria	Qualitative Analysis	Quantitative Analysis
1.	**Purpose / Need**	Understand and interpret social interactions	Test hypothesis - Check cause and effect relationship, develops prediction about future.
2.	**Studied group**	Small, selected internationally	Large, selected random
3.	**Data types**	Words, Images, Objects	Numbers and statistics.
4.	**Data form**	Open ended responses, interviews, participants observations.	Presize measurements using structures and validated Instruments data collect.
5.	**Type of data analysis**	Patterns, features, thems, Identification.	Statistical relationship Identification.
6.	**Result**	Particular Findings, less generalisable	Generalisation findings can be applied.

7. Define a model ?

Ans :

The essence of operations research lies in the construction and use of models. A model in the sense intended in operations research is just a simplified representation of something real. It is implied that a model is always, necessarily a representation that it is less than perfect.

When a real life situation is represented in some abstract form, whether physical or mathematical, bringing out relationships of its major ingredients, a model is said to be formed. The model so formed, need not describe all the aspects of this situation, but it should signify and identify important factors and their interrelationships to describe the total situation. Hence, models do not and cannot represent every aspect of reality because of the real life problems of innumerable and changing characteristics to be represented. Models are limited approximation of reality.

8. Explain limitations of operation research.

Ans :

i) The model cannot be considered as valid until and unless experiments are conducted on it.

ii) Models are just idealised representation of reality. Thus, it must not be considered as perfect in any case.

iii) The problems are solved either by simplifying assumptions or by making it simpler. Therefore, solution derived have limitations.

iv) All realistic problems cannot be solved by Operations Research.

v) Many a times, the decision-maker does not have clear idea about the limitations of the models which he is using.

vi) At times, models may fail to represent the real world situations wherein the decisions should be taken.

9. “Operations Research (OR) is a tool for decision support system”. Justify the statement.

Ans :

Operations Research (OR) is considered as a important and crucial tool of decision making in business organizations. It helps in making effective business decisions. It can be applied in every functional and managerial area of decision making like production, marketing, finance etc. As such, it is also considered as a tool for Decision Support System (DSS).

Decision Support System (DSS) is an interactive computer-based information system which helps manager and business professionals in making appropriate decisions. However, for making appropriate decisions, DSS analyses large amount of data and information in a fast and sophisticated manner.

In the process of DSS, Operations Research (OR) plays a crucial role in collecting and sorting data from different sources. With the help of OR techniques or models (L.RP, inventory models, game theory, simulation etc). DSS is able to support decision makers in finding solution to the problems which are rapidly changing and non-understandable. Addition to this, by using OR, DSS can provide following type of decisions to an business organization,

1. Consequences related to various decision alternatives based on previous experience.
2. Comparative sales figure on weekly, monthy or yearly basis.
3. Inventory turnovers and so on.

UNIT II

Linear Programming Method: Structure of LPP, Assumptions of LPP, Applications areas of LPP, Guidelines for formulation of LPP, Formulation of LPP for different areas, solving of LPP by Graphical Method: Extreme point method, simplex method, converting primal LPP to dual LPP, Limitations of LPP.

2.1 LINEAR PROGRAMMING

Q1. Define Linear Programming.

Ans :

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective functions. It is subject to a set of linear equalities and/or inequalities known as constraints. Linear programming is a mathematical technique which involves the allocation of limited resources in an optimal manner, on the basis of a given criterion of optimality.

To solve a LPP some basic requirements must be fulfilled. They are,

1. Objective function availability
2. Set of constraints
3. Both objective function and constraints must be linear functions. (or) Decision Variables.

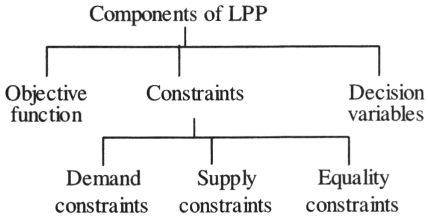
2.1.1 Structure of LPP

Q2. Elucidate the Structure of LPP.

Ans :

The major components of LPP are,

1. Objective function
2. Constraints
3. Decision variables.



Figure

1. **Objective Function**

This is the function which is formulated by using contributions (i.e., profits, cost) and this function is to be optimized. That is to maximize the profit and minimize the loss respectively.

It is formulated by considering contributions and decision variables. It should be a linear equation.

Example

Profit per doll of type A = 30/- and profit per doll of type B = 40/-. Express the objective function.

If profit is given, objective function is to be maximized and if cost is given it should be minimized.

Given that profit per doll of type A = 30/-

So, if 'x' dolls of type A are produced, total profit would be 30x.

Similarly, if dolls of type B are produced, then total profit would be 40y.

Objective function is denoted by 'Z'. It can be written as,

$$Z = 30x + 40y$$

$$\therefore \quad \text{Maximize } Z = 30x + 40y$$

This is the expression for objective function.

2. **Constraints**

An objective function is always limited by some restrictions called as constraints.

Example

An examination is conducted for 100 marks. So, objective of a student is to obtain maximum marks which is limited to only 100 that is the student can require exactly 100 or less than 100. Therefore the objective is said to be subjected to some constraints.

Constraints are further classified as,

i) Demand constraint

ii) Supply constraint

iii) Equality constraint.

i) **Demand Constraint:** Demand is analogous to requirement. It has lower limit but upper limit is not mentioned.

Example

At least 50,000/- is required to buy an excellent quality guitar.

Therefore, in demand constraint ($\geq$) inequality is used.

ii) **Supply Constraint:** Supply is related to availability. So it has upper limit.

Example

From given quantity of raw material, not more than 100 products can be produced.

Therefore, in supply constraint ($\leq$) inequality is used.

iii) **Equality Constraint:** Here the availability and requirement should be exactly equal.

Example

Customer who purchase a pair of shoe will not compromise with the feet size. He wants the exact size of shoe.

Therefore, equality constraint (=) symbol is used.

3. **Decision Variables:** The variables which are required to be determined using different techniques are said to be decision variables.

Example

The variables in objective function x, *y* are nothing but the decision variables.

a) **Slack Variables:** When a supply constraint is given i.e., '$\leq$' type of inequality. Thus, in order to change inequality to the equality a non-negative variable should be added on left side of an equation.

Example

If $3x_1 + 5x_2 \leq 15$ is an equation then (s_1) a slack variable must be added on L.H.S of inequality. So, it becomes,

$$3x_1 + 5x_2 S_1 = 15$$

Where, $x_1\ x_2, S_1$ are ≥ 0

b) **Surplus Variables:** When a requirement constraint is given i.e., '$\geq$' type of inequality. Thus in order to change inequality to equality, variable should be added to inequality on right hand side of an equation.

Example

If $3x_1 + 5x_2 \geq 15$ is in equation, then S_2 is surplus variable, it should be added on R.H.S so,

$$3x_1 + 5x_2 = 15 + S_2$$

$$\Rightarrow \quad 3x_1 + 5x_2 - S_2 = 15$$

Where, $x_1, x_2, S_2 \geq 0$.

c) **Artificial Variables:** The variable which should be added in case of minimization Linear Programming Problem (LPP) along with surplus variable is called an artificial variable.

Example

Let, Z = 40x + 120y be minimization function i.e.,

Min Z = 40x + 120y

Subject to constraints,

$x - 3y \geq 8$

$- I5x + 40y \leq - 600$

Since a constraint has $\geq$ inequality we make it into equality by deducting surplus value (S_1) and by adding an artificial variables (A_1).

Such that, $x - 3y - S_1 + A_1 = 8$

S_1 = Surplus variables

A_1 = Artificial variable.

and $-15x + 40y + S_2 = -600$

S_2 = Slack variable.

2.1.2 Assumptions of LPP

Q3. What are the Assumptions of LPP ?

(or)

Elucidate the various assumptions of LPP ?

Ans : **(Aug.-15, Aug.-16, Sep.-14)**

Following are the assumptions in linear programming problem that limit its applicability.

(a) Proportionality : A primary requirement of linear programming problem is that the objective function and every constraint function must be linear. Roughly speaking, it simply means that if 1 kg of a product costs Rs. 2, then 10 kg will cost Rs. 20. If a steel mill can produce 200 tons in 1 hour, it can produce 1000 tons in 5 hours.

Intuitively, linearity implies that the product of variables such as $x_1 x_2$, powers of variables such as x_3^2, and combination of variables such as $a_1 x_1 + a_2 \log x_2$, are not allowed.

(b) Additivity: Additivity means if it takes t_1 hours on machine G to make product A and t_2 hours to make product B, then the time on machine G devoted to produce A and B both is $t_1 + t_2$, provided the time required to change the machine from product A to B is negligible.

The additivity may not hold, in general. If we mix several liquids of different chemical composition, then the total volume of the mixture may not be the sum of the volume of individual liquids.

(c) Multiplicativity : It requires:

(i) If it takes one hour to make a single item on a given machine, it will take 10 hours to make 10 such items; and

(ii) The total profit from selling a given number of units is the unit profit times the number of units sold.

(d) Divisibility: It means that the fractional levels of variables must be permissible besides integral values.

(e) Deterministic: All the parameters in the linear programming models are assumed to be known exactly. While in actual practice, production may depend upon chance also. Such type of problems, where some of the coefficients are not known, are discussed in the extension of sensitivity analysis known as parametric programming.

2.1.3 Applications areas of LPP

Q4. State the various Applications of LPP.

Ans :

1. Production Management

Linear programming can be applied in production management for determining product mix, product smoothing, assembly line balancing, production scheduling and blending problem.

2. Marketing Management

Linear programming can be applied in marketing area by helping in analyzing the effectiveness of advertising campaign, advertising media mix and the shortest route for travelling salesman.

3. Manpower Management

LP allows the human resource to analyses personal policy combinations in terms of their appropriateness for maintaining a steady-state flow of people into the firm and out of the firm.

4. Transportation Problem

LP helps in determining the optimum transportation schedule with minimum total transportation cost of moving goods from various origins to various destinations.

5. Assignment Problem

LP models give the best assignment schedule with minimum total cost of assignment of various resources to various activities on a one-to-one basis.

6. Military Applications

LP models can be used for various military operations like wars, terrorists and defense against communal clashes etc. The problem is to determine the optimum number of weapons to be used that minimizes the cost of operations.

7. Agricultural Applications

LP can be applied in agricultural planning for allocating the limited resources such as water, labour, working capital etc., so as to maximize the net income.

8. Facilities Location

LP determines the most economic and efficient manner of locating manufacturing plants and distribution centres for physical distribution of goods.

9. **Hydel Power Generation**

In this problem, variations in storage of dams which generate power is determined so as to maximize the energy obtained from the entire system.

10. Portfolio Selection

LP gives the optimal solution to the selection of specific investments from among a wide variety of alternatives. This helps the managers of banks, insurance companies, investment services etc.

11. Financial Mix Strategy

Financial mix strategy problems can be solved using LP technique. This involves the selection of means for financing projects, production operations and various other activities.

12. Airline Routine

LP can be used to determine the most economic pattern and timing for flights with the objective of efficiently using aircrafts, crews and money.

13. Diet Problems

LP helps the hospitals in determining the most economical diet for patients to meet the nutrients requirements of patients.

14. Environmental Protection

LP may be used to analyze alternatives for handling liquid waste material in order to satisfy antipollution requirements of patients.

15. Urban Development

Of late, LP is being used to analyze public expenditure planning, school busing, drug control, garbage collection etc.

2.1.4 Guidelines for formulation of LPP

Q5. How LPP can be formulated? Discuss the mathematical formulation of LPP.

Ans :

The various steps/guidelines involved in the formulation of LPP are as follows,

(a) Identification and Selection of Variables

The first step in the formulation of LPP is identification of the variables called decision variables or design variables. After the identification of variables, it is then expressed in the form of symbolic notation along with their units of measurement. These variables may be seen as number of units to be ordered, number of units to be sold, number of units to be manufactured etc.

(b) Construction of Objective Function

The second step in the formulation of LPP is the setting of goal or objective function. It may be in two forms i.e., maximization or minimization form. If the data reveals profit in the system, then the objective function will be in maximization form. For example, output, profit, sales etc., are to be maximized. If the data is in terms of cost, then the objective function will be written in the minimization form. For example, production cost, spoilage, overheads, loss etc.

(c) Developing the Constraints Equation

The third step in the formulation of LPP is the development of the constraints equation. Constraints are the restrictions applied to the objective function. These constraints may be demand, supply and equality constraints. Demand constraints are represented by ‘ > ’

sign and is used when a minimum restriction is applied to the function.

Supply constraints are represented by ‘ $<$ ’ sign and issued when a maximum restriction is applied to the function.

Equality constraints are represented by ‘$=$’ sign and is used when an exact value should be used.

(d) Deciding the Variable Status

The fourth step in the formulation of LPP is deciding the status of decision variable. As the variables are generally numbers, they may be positive, negative or zero. For most of the cases it will be positive. For example, allocation of material for transportation cannot be negative, it should be either zero or greater than that. But, in some cases, it may be positive or negative, like in inventory. Therefore, the condition of the variable can be non-negative or unrestricted.

Mathematical Formulation of LPP

The general formulation of the LPP can be stated as follows:

In order to find the values of n decision variables $x_1, x_2, \ldots x_n$ to maximize or minimize the objective function.

$$Z = c_1x_1 + c_2 x_2 + \ldots. + c_nx_n \qquad \ldots (1)$$

and also satisfy m-constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots. + a_{1n}\ x_n(\le = \ge)b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots. + a_{2n}\ x_n(\le = \ge)b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots. + a_{in}\ x_n(\le = \ge)b_i \\ \vdots \\ am_1x_1 + a_{m2}x_2 + \ldots. + a_{mn}\ x_n(\le = \ge)b_m \end{array}\right\} \ldots (2)$$

where constraints may be in the form of inequality $\le$ or $\ge$ or even in the form an equation $(=)$ and finally satisfy the non negative restrictions.

$$x_1 \ge 0,\ x_2 \ge 0,\ \ldots.\ x_n \ge 0.$$

2.1.5 Formulation of LPP for different areas

Q6. Illustrate the formulation of LPP in financial management.

Ans :

The board of directors of a company has given approval for the construction of a new plant. The plant will require an investment of `50 lakh. The required funds will come from the sale of a proposed bond issue and by taking loans from two financial corporations. For the company, it will not be possible to sell more than `20 lakh worth of bonds at the proposed rate of 12%. Financial corporation A will give loan upto `30 lakh at an interest rate of 16% but.insists that the amount of bond debt plus the amount owned to financial corporation B be no Z more than twice the amount owned to financial corporation A. Financial corporation B will loan the same amount as that loaned by financial corporation A but it would do so at an interest rate of 18%. Formulate this problem as an L.P model to determine the amount of funds to be obtained from each source in a manner that minimizes the total annual interest charges.

Objective Function

In the given problem, the objective function is to minimizes the total annual interest charges, after fulfilling the requirement of `50 lakhs for constructing new plant.

Let,

‘x_1’ - Be the bond debt

‘x_2’ - Be the loan from A

‘x_3’ - Be the loan from B

The interest rates of x_1, x_2 and x_3 are 12, 16 and 18 percents respectively.

Thus, minimize

$$z = 0.12\ x_1 + 0.16\ x_2 + 0.18\ x_3$$

Subject to Constraints

The requirements of new plant is `50 lakhs which has to be fulfilled from these three sources.

Thus, $x_1 + x_2 + x_3 = 50{,}00{,}000$

Through bond issue, the company can obtain `20 lakhs.

Thus, $x_1 \le 20{,}00{,}000$

Financial corporation 'A' can provide loan upto `30 lakhs.

Thus, $x_2 < 30,00,000$

The condition of A, states that the bond debt and loan from B should not be more than twice the amount of A.

Thus, $X_1 + x_3 \leq 2x_2$

Financial corporation '5' can also provide loan upto `30 lakhs.

Thus, $x_3 \leq 30,00,000$

Therefore,

$\text{Min } z = 0.12\, x_1 + 0.16\, x_2 + 0.18\, x_3$

STC,

$x_1 + x_2 + x_3 = 50,00,000$

$x_1 \leq 20,00,000$

$x_2 \leq 30,00,000$

$x_3 \leq 30,00,000$

$x_1 + x_3 \leq 2x_2$

and $x_1, x_2, x_3 \geq 0$.

Q7. Explain the formulation of LPP in personnel management.

Ans :

A machine tool company conducts a job training program for machinists. Trained machinists are used as teachers in the programme at a ratio of one for every ten trainees. The training lasts for one month. From past experience it has been found that out of ten trainees hired, only six complete the programme successfully. The unsuccessful trainees are released. Trained machinists are needed for machining and company's requirements for the next three months are January: 120, February: 180 and March: 220. Further, the company requires 250 trained machinists by April. There are 150 trained machinists available at the beginning of the year. The relevant costs per month are,

Each trainee	: `1,000.00
Each trained machinists (Machining or Teaching)	: `2,000.00
Each trained machinist idle	: ` 1,500.00

Formulate as an LPP that will result in minimum cost hiring and training schedule and meet the company's need.

LP Model Formulation

Let,

$x_1, x_2 =$ Trained machinist teaching and idle in January respectively.

$x_3, x_4 =$ Trained machinist teaching and idle in February respectively.

x_5, x_6 = Trained machinist teaching and idle in March respectively.

LP Model

Minimize (total cost) Z = Cost of training programme (teachers and trainees) + Cost of idle machinists + Cost of machinists doing machine work (constant)

$= 1000\ (10x_1 + 10x_3 + 10x_5) + 2000\ (x_1 + x_3 + x_5) + 1500\ (x_2 + x_4 + x_6)$

Subject to the constraints,

(i) Total trained machinists available at the beginning of January = Number of machinists doing machining + Teaching + Idle

$150 = 120 + x_1 + x_2$

$x_1 + x_2 = 150 - 120 = 30$ ... (1)

(ii) Total trained machinists available at the beginning of February = Number of machinists in January + Joining after training programme

$150 + 6x_1 = 180 + x_3 + x_2$

$6x_1 - x_3 - x_4 = 180 - 150 = 30$ ... (2)

(iii) In January there are $10x_1$ trainees in the programme and out of those only $6x_1$ will become trained machinists.

Total trained machinists available at the beginning of March = Number of machinists in January + Joining after training programme in January and February

$150 + 6x_1 + 6x_3 = 220 + x_5 + x_6$

$6x_1 + 6x_3 - x_5 - x_6 = 220 - 150$

$6x_1 + 6x_3 - x_5 - x_6 = 70$ ... (3)

(iv) Company requires 250 trained machinists by April

$150 + 6x_1 + 6x_3 + 6x_5 = 250$

$6x_1 + 6x_3 + 6x_5 = 250 \quad 150$

$6x_1 + 6x_3 + 6x_5 = 100$ ... (4)

LP Model (Simplification)

Minimize (Cost of hiring and training schedule)

$Z = 1000\ (10\ x_1 + 10\ x_3 + 10x_5) + 2000\ (x_1 + x_3 + x_5) + 1500\ (x_2 + x_4 + x_6)$

$= 10000x_1 + 10000x_3 + 10000x_5 + 2000x_1 + 2000x_3 + 2000x_5 + 1500x_2 + 1500x_4 + 1500x_6$

$Z = (10000\ x_1 + 2000\ x_1) + (1500\ x_2) + (10000\ x_3 + 2000\ x_3) + (1500\ x_4) + (10000\ x_5 + 2000\ x_5) + (1500\ x_6)$

$Z = 12000\ x_1 + 1500\ x_2 + 12000\ x_3 + 1500\ x_4 + 12000\ x_5 + 1500\ x_6$

Subject,

$x_1 + x_2 = 30$ ... (1)

$6x_1 - x_3 - x_4 = 30$ ... (2)

$6x_1 + 6x_3 - x_5 - x_6 = 70$ ... (3)

$6x_1 + 6x_3 + 6x_5 = 100$ ... (4)

Standard form of LP Model

Minimize,

$Z = 12000\ x_1 + 1500\ x_2 + 12000\ x_3 + 1500\ x_4 + 12000\ x_5 + 1500\ x_6$

Subject to the constraints,

$x_1 + x_2 \geq 30$

$6x_1 - x_3 - x_4 \geq 30$

$6x_1 - 6x_3 - x_5 - x_6 \geq 70$

$6x_1 + 6x_3 + 6x_5 \geq 100$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

PROBLEMS

1. A manufacturer produces two types of models M_1, and M_2. Each model of the type M_1, requires 4 hrs of grinding and 2 hours of polishing; where as each model of the type M_2, requires 2 hours of grinding and 5 hours of polishing. The manufacturers have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. Profit on M_1, model is Rs. 3.00 and on model M_2 is Rs. 4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a week?

Sol :

i) **Decision Variables** Let x_1, and x_2 be the number of units of M_1, and M_2, model.

ii) **Objective function** Since the profit on both the models are given, we have to maximize the profit viz.

Max $Z = 3X_1 + 4X_2$

iii) **Constraints**

There are two constraints one for grinding and the other for polishing.

No. of hrs. available on each grinder for one week is 40 hrs. There are 2 grinders. Hence the manufacturer does not have more than 2 × 40 = 80 hrs of grinding.

M_1, requires 4 hrs of grinding and M_2 requires 2 hrs of grinding.

The grinding constraint is given by

$4x_1 + 2x_2 \leq 80.$

Since there are 3 polishers, the available time for polishing in a week is given by 3 × 60 = 180. M_1, requires 2 hrs of polishing and M_2, requires 5 hrs of polishing.

Hence we have $2x_1 + 5x_2 \leq 180$

Finally we have

$$\text{Max } Z = 3x_1 + 4x_2,$$

Subject to $4x_1 + 2x_2 \leq 80$

$2x_1 + 5x_2 \leq 180$

$x_1\, x_2 \geq 0$

---

2. A company manufactures two products A and B. These products are processed in the same machine. It takes 10 minutes to process one unit of product A and 2 minutes for each unit of product B and the machine operates for a maximum of 35 hrs in a week. Product A requires 1 kg. and B 0.5 kg. of raw material per unit the supply of which is 600 kg. per week. Market constraint on product B is known to be 800 unit every week. Product A costs Rs. 5 per unit and sold at Rs. 10. Product B costs Rs.6 per unit and can be sold in the market at a unit price of Rs. 8. Determine the number of units of A and B per week to maximize the profit.

Sol :

i) Decision Variables

Let x_1 and x_2 be the number of products A and B.

ii) Objective function

Costs of product A per unit is Rs. 5 and sold at Rs. 10 per unit.

∴ Profit on one unit of product A.

= 10 – 5 = 5

∴ x_1, units of product A contributes a profit of Rs. 5 x_1, profit contribution from one unit of product

B = 8 – 6 = 2

∴ x_2 units of product B contribute a profit of Rs. $2x_2$

∴ The objective function is given by,

$$\text{Max } Z = 5x_1 + 2x_2$$

iii) Constraints

Time requirement constraint is given by

$10x_1 + 2x_2 \leq (35 \times 60)$

$10x_1 + 2x_2 \leq 21.00$

Raw material constraint is given by

$x_1 + 0.5\, x_2 \leq 600$

Market demand on product B is 800 units every week

$\therefore x_2 \geq 800$

The complete LPP is

$$\text{Max } Z = 5x_1 + 2x_2$$

Subject to $10x_1 + 2x_2 \leq 2100$

$x_1 + 0.5x_2 \leq 600$

$x_2 \geq 800$

$x_1, x_2 \geq 0$

3. A person requires 10, 12 and 12 units chemicals A, B and C respectively for his garden. A liquid product contains 5, 2 and 1 units of A, B and C respectively per jar. A dry product contains 1, 2 and 4 units of A, B, C per carton. If the liquid product sells for Rs. 3 per jar and the dry product sells for Rs. 2 per carton, how many of each should be purchased, in order to minimize the cost and meet the requirements?

Sol :

i) Decision variable

Let X_1 and X_2 be the number of units of liquid and dry products.

ii) Objective function

Since the cost for the products are given we have to minimize the cost

$$\text{Min } Z = 3x_1 + 2x_2$$

Constraints

As there are 3 chemicals and its requirement are given. We have three constraints for these three chemicals.

$5x_1 + x_2 > 10$

$2x_1 + 2x_2 > 12$

$x_1 + 4x_2 > 12$

Finally the complete L.P.P is

$\text{Min } Z = 3x_1 + 2x_2$

Subject to

$5x_1 + x_2 \geq 10$

$2x_1 + 2x_2 \geq 12$

$x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0.$

---

4. A company manufactures two products A and B. Each unit of B takes twice as long to produce as one unit of A and if the company were to produce only A it would have time to produce 2000 units per day. The availability of the raw material is sufficient to produce 1500 units per day of both A and B combined. Product B requiring a special ingredient only 600 units can be made per day. If A fetches a profit of Rs. 2 per unit and B a profit of Rs. 4 per unit, find the optimum product mix.

Sol :

Let x_1 and x_2 be the number of units of the products A and B respectively.

The profit after selling these two products is given by the objective function

$\text{Max } Z = 2x_1 + 4x_2$

Since the company can produce at the most 2000 units of the product in a day and type B requires twice as much time as that of type A, production restriction is given by

$x_1 + 2x_2 \leq 2000$

Since the raw material are sufficient to produce 1500 units per day both A and B combined.

We have $x_1 + x_2 \leq 1500$.

There are special ingredients for the product B we have $x_2 \leq 600$.

Also, since the company cannot produce negative quantities $x_1 \geq 0$ and $x_2 \geq 0$.

Hence the problem can be finally put in the form:

Find x_1 and X_2 such that the profits

$Z = 2x_1 + 4x_2$ is maximum

Subject to $x_1 + 2x_2 \leq 2000$

$x_1 + x_2 \leq 1500$

$x_2 \leq 600$

$x_1, x_2 \leq 0$.

---

5. A firm manufacturers 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has 2 machines and given below is the required processing time in minutes for each machine on each product.

	Product			
		A	B	C
Machines	C	4	3	5
	D	2	2	4

Machine C and D have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 A's 200 B's and 50 C's but no more than 150 A's. Setup an LP problem to maximize the profit.

Sol :

Let x_1, x_2, x_3, be the number of units of the product A, B, C respectively.

Since the profits are Rs. 3, Rs. 2 and Rs. 4 respectively, the total profit gained by the firm after selling these three product is given by $Z = 3x_1 + 2x_2 + 4x_3$. The total number of minutes required in producing these three products at machine C is given by $4x_1 + 3x_2 + 5x_3$ and at machine D is given by $2x_1 + 2x_2 + 4x_3$.

The restrictions on the machine C and D are given by 2000 minutes and 2500 minutes.

$4x_1 + 3x_2 + 5x_3 \leq 2000$

$2x_1 + 2x_2 + 4x_3 \leq 2500$

Also since the firm manufactures 100 A's, 200 B's and 50 C's but not more than 150 A's the further restriction becomes

$100 \leq x_1 \leq 150$

$200 \leq x_2 \leq 0$

$50 \leq x_3 \leq 0$

Hence the allocation problem of the firm can be finally put in the form:

Find the value of x_1, x_2, x_3 so as to maximise

$$Z = 3x_1 + 2x_2 + 4x_3$$

Subject to the constraints

$$4x_1 + 3x_2 + 5x_3 \leq 2000$$

$$2x_1 + 2x_2 + 4x_3 \leq 2500$$

$$100 \leq x_1 \leq 150,$$

$$200 \leq x_2 \geq 0,$$

$$50 \leq x_3 \geq 0.$$

2.2 SOLVING OF LPP BY GRAPHICAL METHOD

2.2.1 Extreme point method

Q8. What is Graphical Method LPP ? Explain the characteristics of Graphical Method.

Ans : **(Feb.-17)**

Graphical method is a simple method to understand and also to use. This is effectively used in LPP's which involves only 2 variables. It gives the graphical representation of the solutions.

All types of solutions are highlighted in this method very clearly. The only drawback is that more the number of constraints, more will be the straight lines which makes the graph difficult to understand.

Characteristics of Graphical Method

The following are the characteristics of graphical method of LPP,

1. Method is very simple and easy to understand.
2. Very sensitive analysis and can be illustrated very easily by drawing graphs.
3. Very easy to obtain optimal solution.
4. It consumes very less time.

Q9. Describe the steps involved in graphical solution to linear programming models.

Ans :

Simple linear programming problems of two decision variables can be easily solved by graphical method. The outlines of graphical procedure are as follows :

Step 1 : Consider each inequality-constraint as equation.

Step 2 : Plot each equation on the graph, as each one will geometrically represent a straight line.

Step 3 : Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality-constraint corresponding to that lines is '$\leq$', then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality-constraint with '$\geq$' sign, the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.

Step 4 : Choose the convenient value of z (say = 0) and plot the objective function line.

Step 5 : Pull the objective function line until the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Step 6 : Read the coordinates of the extreme point(s) selected in *step 5* and find the maximum or minimum (as the case may be) value of z.

PROBLEMS ON GRAPHICAL METHOD

6. Given the LPP as

Max 3A + 2B

Such that,

$A + B \geq 4$

$3A + 4B \leq 24$

$A \geq 2$

$A - B \leq 0$

$A, B \geq 0$

Solve the above problem graphically, mark the feasible region and explain the optimal solution.

Sol :

Step 1

Objective Function

Max 3A + 2B

Such that,

$A + B \geq 4$ ... eq(1)

$3A + 4B \leq 24$ ... eq(2)

$A \geq 2$ ... eq(3)

$A - B \leq 0$ ... eq(4)

$A, B \geq 0$

Step 2

Converting inequality constraints into equality constraints

Equation I

By solving equation (1), we get the values of P and Q points.

Case I

If A = 0, then by solving equation (1), we get,

$A + B = 4$

$= 0 + B = 4$

$B = 4$

$$\boxed{\therefore P = (0, 4)}$$

Case II

If B = 0, then by solving equation (1), we get,

$$A + B = 4$$

$$A + 0 = 4$$

$$A = 4$$

$$\boxed{\therefore Q = (4, 0)}$$

Equation II

By solving equation (2), we get the values of R and S points.

Case I

If A = 0, then by solving equation (2), we get

$$3A + 4B = 24$$

$$3(0) + 4B = 24$$

$$4B = 24$$

$$B = \frac{24}{4}$$

$$B = 6$$

$$\boxed{\therefore R = (0, 6)}$$

Case II

If B = 0, then by solving equation (2), we get,

$$3A + 4B = 24$$

$$3A + 4(0) = 24$$

$$3A = 24$$

$$A = \frac{24}{3}$$

$$A = 8$$

$$\boxed{\therefore S = (8, 0)}$$

Equation III

By solving equation (3), we get,

$$A = 2$$

$$\boxed{\therefore T = (2, 0)}$$

Equation IV

By solving equation (4), we get,

$$U \Rightarrow A - B = 0$$

$$(0)$$

$$\boxed{\therefore J = (2, 0)}$$

Step 3

Plot the points P, Q, R, S, T and U on the graph to obtain a feasible solution.

P(0, 4), Q(4, 0), R(0, 6), S(8, 0), T(2, 0), U(0, 0).

Step 4

The common feasible region is V, W and X

V = (2, 2)

W = (3, 4, 3.4)

X = (2, 4.5)

Step 5

Substitute the coordinates of feasible region in objective function.

Max 3A + 2B

$$V(2, 2) = 3(2) + 2(2)$$

$$= 6 + 4$$

$$= 10$$

$$W(3.4, 3.4) = 3(3.4) + 2(3.4)$$

$$= 10.20 + 6.80$$

$$= 17$$

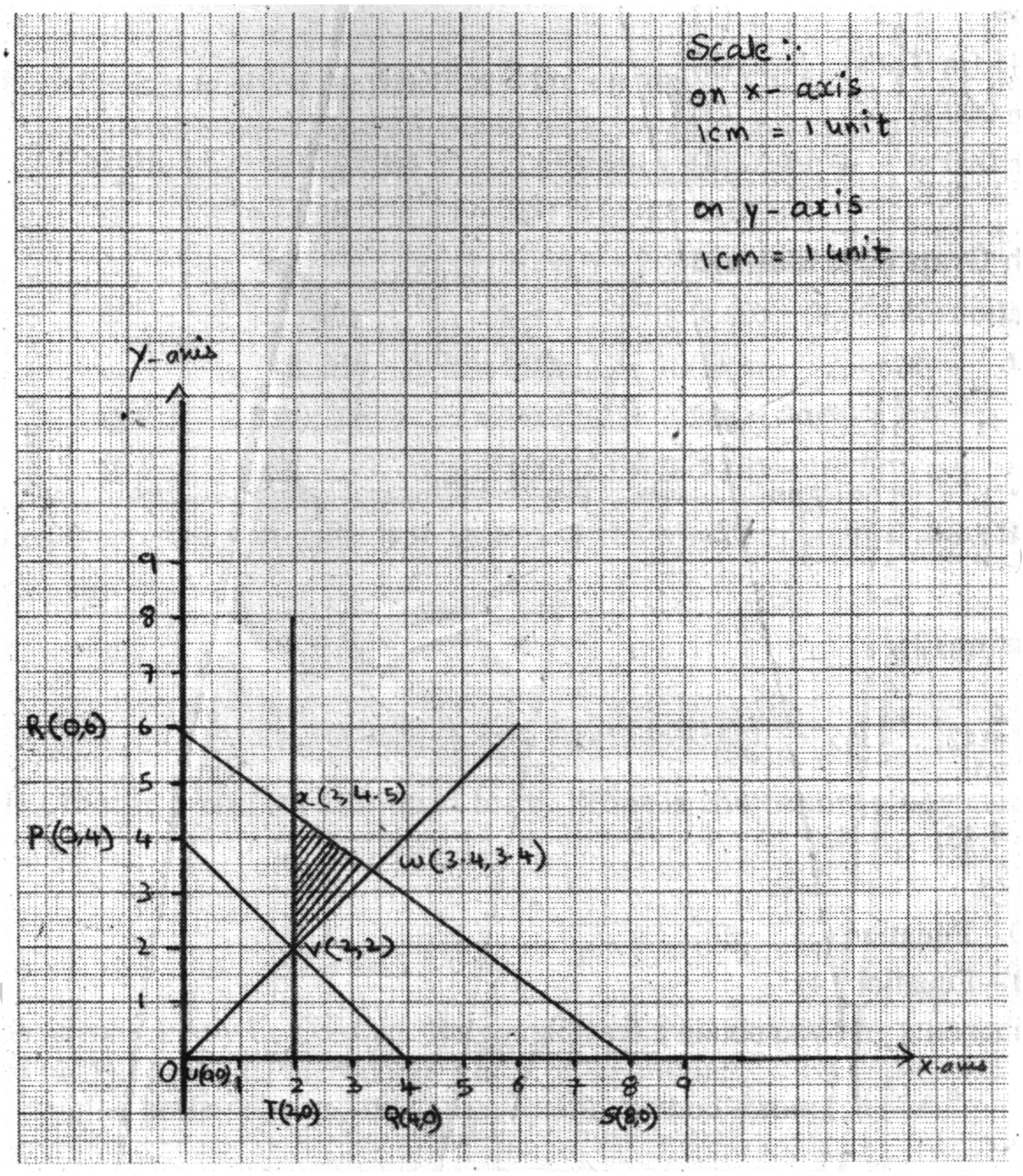
$$X(2, 4.5) = 3(2) + 2(4.5)$$

$$= 6 + 9$$

$$= 15$$

$\therefore$ Maximum value of ` is obtain at,

$$W(3.4, 3.4) = 17$$



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7. Solve the following LPP graphically

Min Z = $4x_1 + 2x_2$

S.T.C. :

$x_1 + 2x_2 \geq 2$

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1, x_2 \geq 0$

Sol :

Convert inequations into equations considering equations (1), (2) and (3)

$x_1 + 2x_2 = 2 \Rightarrow$ (1)

$3x_1 + x_2 = 3 \Rightarrow$ (2)

$4x_1 + 3x_2 = 6 \Rightarrow$ (3)

To plot $x_1 + 2x_2 = 2$

put $x_1 = 0, x_2 = 1 \Rightarrow (0,1)$

$x_2 = 0, x_1 = 2 \Rightarrow (2, 0)$

To plot $3x_1 + x_2 = 3$

put $x_1 = 0, x_2 = 3 \Rightarrow (0,3)$

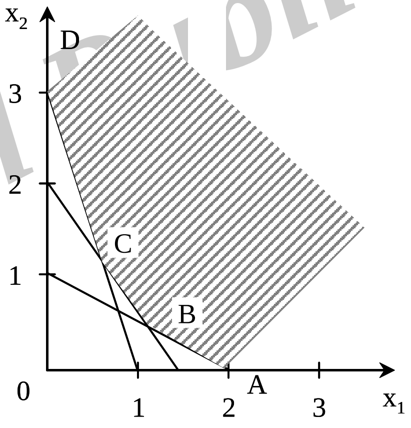
$x_2 = 0, x_1 = 1 \Rightarrow (1, 0)$

To plot $4x_1 + 3x_2 = 6$

put $x_1 = 0, x_2 = 2 \Rightarrow (0,2)$

$x_2 = 0, x_1 = 1.5 \Rightarrow (1.5, 0)$

Plotting these equations in the graphs, we get



Corner Points	Coordinates	Max Z = $4x_1 + 2x_2$	Value
A	(2, 0)	4(2) + 2(0)	8
B	(1.2, 0.4)	4(1.2) + 2(0.4)	5.6
C	(0.6, 1.2)	4(0.6) + 2(1.2)	4.8
D	(3, 0)	4(3) + 2(0)	6

The minimum value of z is 4.8 which occurs at C = (0.6, 1.2).

Hence, the solution to the above problem is $x_1 = 0.6$; $x_2 = 1.2$, min z = 4.8

8. The manufacturer of patent medicines has proposed to prepare a production plan for medicines A and B. There are sufficient ingredients available to make 20,000 bottles of medicine A and 40,000 bottles of medicine B but there are only 45,000 bottles into which either of the medicines can be filled. Further, it makes three hours to prepare enough material to fill 100 bottles of medicine A and one hour to prepare enough material to fill 1000 bottles of medicine B and there are 66 hours available for this operation. The profit is ` 8 bottle for medicine A and ` 7 per bottle for medicine B. Formulate this problem as a LPP and solve it by graphical method.

Sol :

Formulating LPP

Let us assume that the two types of medicines A and B are x_1 and x_2.

The sufficient ingredients available to make x, is 20,000.

Thus, $x_1 \leq 20,000$.

The sufficient ingredients available to make x_2 is 40,000.

Thus, $x_2 \leq 40,000$.

Total number of bottles that can be filled is 45,000.

Thus, $x_1 + x_2 \leq 45,000$.

Number of hours needed to fill x_1 is 3 and x_2 is 1.

Availability of hours is 66.

Thus, $3x_1 + x_2 \leq 66$

Profit on x_1 is 8 and on x_2 is 7.

$x_1 = 100; x_2 = 1000$

Thus, $z = 8 \times 100\, x_1 + 7 \times 1000\, x_2$

$= 800\, x_1 + 7000\, x_2$

This can be summarized as,

Objective function, max $z = 800\, x_1, + 7000\, x_2$.

STC, $3x_1 + x_2 \leq 66$ ... (1)

$x_1 + x_2 \leq 45$ (in 000's) ... (2)

$x_1 \leq 20$ (in 000's) ... (3)

$x_2 \leq 40$ (in 000's) ... (4)

$x_1, x_2, \geq 0$

Solving the LPP with Graphical Method

Considering equation (1),

$3x_1 + x_2 = 66$

Put $x_1 = 0$; $3(0) + x_2 = 66$, $x_2 = 66$

∴ (0, 66)

Put $x_2 = 0; 3x_1 + 0 = 66; x_1 = \frac{66}{3} = 22$

∴ (22,0)

Considering equation (2)

$x_1 + x_2 = 45$

Put $x_1 = 0; 0 + x_2 = 45, x_2 = 45$

∴ (0,45)

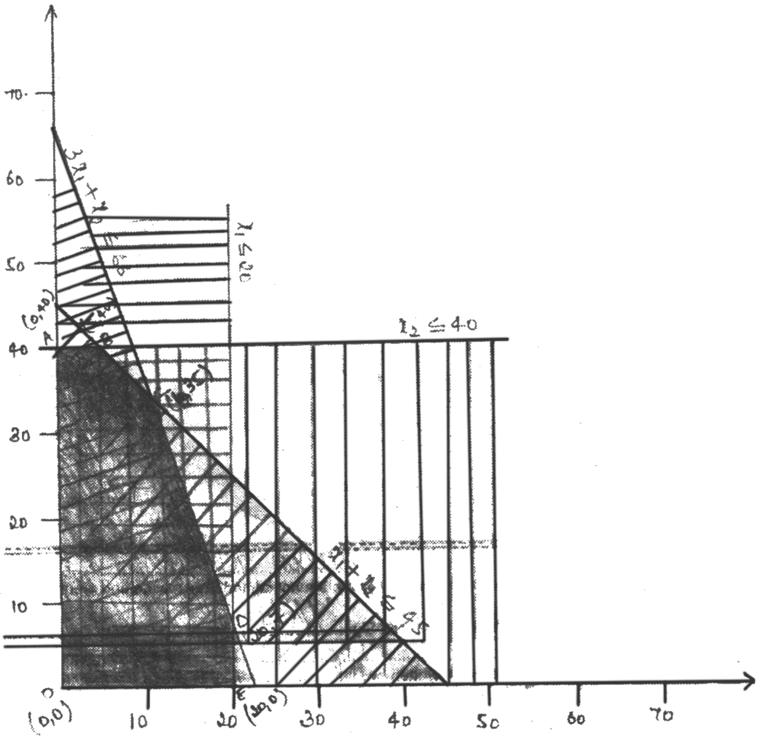
Put $x_2 = 0, 0 + x_2 = 45, x_2 = 45$

(45, 0)

$x_1 = 20$ – Equation (3)

$x_2 = 40$ – Equation (4)

Plotting points on a graph to obtain a feasible region.



The points obtained from the feasible region on graph are,

0 (0,0)

A (0, 40)

B (5,40)

C (12, 35)

D (20, 7)

E (20, 0)

Substituting the obtained points in objective function (Max $z = 800\ x_1 + 7000\ x_2$)

Points	**Coordinates**	**Objective Function**
0	(0, 0)	$z = 800\ (0) + 7000(0) = 0$
A	(0,40)	$z = 800(0) + 7000(40) = 280000$
B	(5, 40)	$z = 800(5) + 7000(40) = 284000^*$
C	(12,35)	$z = 800(12)+ 7000(35) = 254600$
D	(20, 7)	$z = 800(20) + 7000(7) = 65000$
E	(20, 0)	$z = 800(20) + 7000(0) = 16000$

Since, it is a profit function, maximum value is to be considered which is obtained at point B.

$\text{Max } z = 2,84,000$

$x_1 = 5, x_2 = 40$

9. A company produces 2 types of hats. Every hat A require twice as much labour time as the second hat B. If the company process only hat B then it can produce a total of 500 hats a day. The market limits daily sales of the hat A and hat B to 150 and 250 hats. The profits on hat A and B are Rs.8 and Rs.5 respectively. Solve graphically to get the optimal solution.

Sol :

Let X_1 and X_2 be the number of units of type A and type *B* hats respectively.

$\text{Max } Z = 8X_1 + 5X_2$

Subject to $2X_1 + X_2 \le 500$

$X_1 \ge 150$

$X_2 \ge 250$

$X_1, X_2 \ge 150$

First rewrite the inequality of the constraint into an equation and plot the lines in the graph.

$2X_1 + X_2 = 500$ passes through (0, 500) (250, 0)

$X_1 = 150$ passes through (150, 0)

$X_2 = 250$ passes through (0, 250)

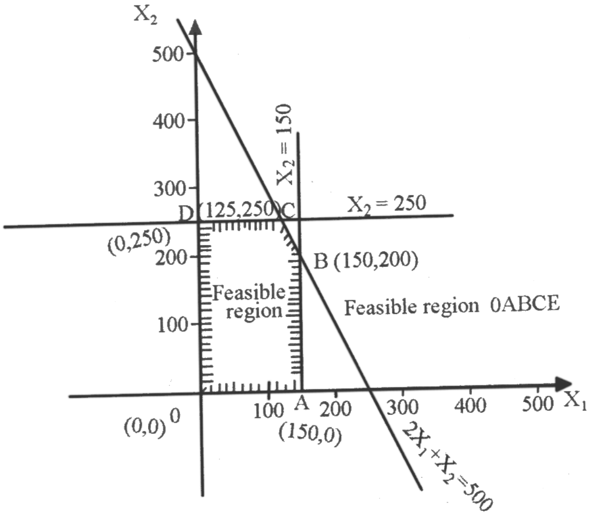
We mark the region below the lines lying in the first quadrant as the inequality of the constraints are $\le$. The feasible region is OABCD B and C are point of intersection of lines.

$2X_1 + X_2 = 500, \quad X_1 = 150$ and

$2X_1 + X_2 = 500, \quad X_2 = 250$

On solving, we get B = (150, 200)

C = (125, 250)



Corner points	Value of Z = $8X_1$ + 5X2
O(0, 0)	0
A(150, 0)	1200
B(150,200)	2200
C(125,250)	2250 (Maximum Z = 2250)
D(0, 250)	1250

The maximum value of Z is attained at C (125, 250)

$\therefore$ The optimal solution is $X_1 = 125$, $X_2 = 250$.

i.e., The company should produce 125 hats of type A and 250 hats of type B in order to get the maximum profit of Rs. 2250.

2.2.2 Special Cases In Graphical Approach

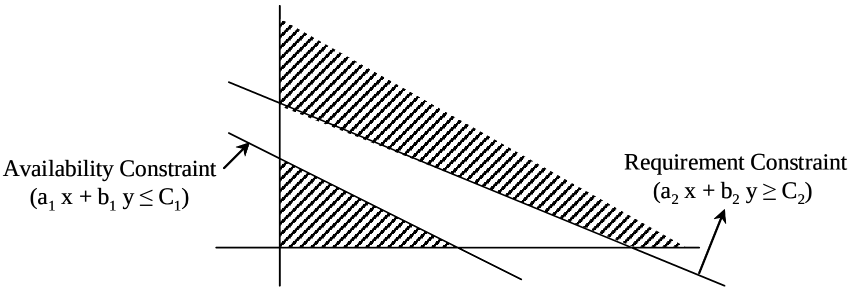
2.2.2.1 Infeasible Solutions

Q10. Define infeasible solution.

Ans : **(Aug.-17)**

When feasible region does not exist, the solution we get is infeasible.

In graphical solution it is found when one constraint is availability ($\leq$) type and the other is requirement ($\geq$) type and these two can not produce any common area (non intersecting) in the specific quadrant (such as $x_1 \geq 0$, $x_2 \geq 0$ indicates first quadrant).



PROBLEMS ON INFEASIBLE SOLUTIONS

10. Solve Graphically

Maximise $Z = 50x_1 + 60x_2$

Subject to $x_1 + x_2 \leq 12$

$2x_1 + 3x_2 \geq 60$

$x_1, x_2 \geq 0$

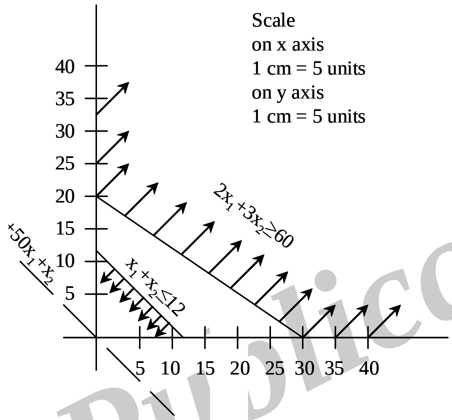
Sol :

$x_1 + x_2 = 12$

x_1	0	2
x_2	12	0

$2x_1 + 3x_2 = 60$

x_1	0	30
x_2	20	0



There is no feasible region and so the solution is infeasible.

11. Solve graphically the following LPP.

$Z = 4x_1 + 5x_2$

Subject to constraints

$x_1 + x_2 \geq 1$

$-2x_1 + x_2 \leq 1$

$4x_1 + 2x_2 \leq 1$

and $x_1, x_2 \geq 0$

Sol :

Step - 1

To identify the objective function and given constraints.

Step - 2

Considering inequality constraints as equality constraints and solve them.

$x_1 + x_2 = 1$

$-2x_1 + x_2 = 1$

$4x_1 + 2x_2 = 1$

Solving equation (1),

Put $x_1 = 0 \Rightarrow x_2 = 1$

Put $x_2 = 0 \Rightarrow x_1 = 1$

$\therefore$ P(0, 1) Q(1, 0)

Mark the points P, Q on graph and join them to obtain straight line PQ.

Solving equation (2),

Put $x_1 = 0 \Rightarrow x_2 = 1$

Put $x_2 = 0 \Rightarrow x_1 = -0.5$

$\therefore$ R(0, 1) S(–0.5, 0)

Mark points R and S on graph and join them to obtain straight line RS.

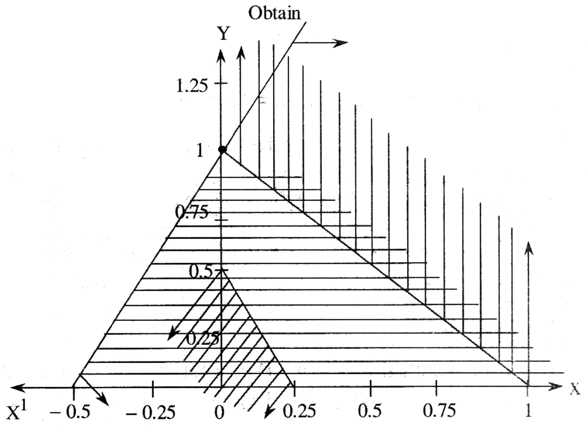
Solving equation (3),

Put $x_1 = 0 \Rightarrow x_2 = 0.5$

Put $x_2 = 0 \Rightarrow x_1 = 0.25$

$\therefore$ L(0, 0.5) M(0.25, 0)

Mark points L and M on graph and join them to obtain straight line LM.



Step 3 : Identifying Feasible Region

Since equation (1) is $\geq$ type, shading is done above the line and equations (2) and (3) are $\leq$ type and shading is done below the line.

There is no common feasible region obtained.

Step 4

Thus the solution is infeasible.

12. Minimize $Z = x_1 + 2x_2$

Subject to $x_1 - x_2 \geq 3$

$-x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$

Sol :

Convert the constraints into equation

$$x_1 - x_2 = 3$$
$$-x_1 + x_2 = 4$$
$$\Rightarrow \quad x_1 - x_2 = -4$$

To plot $x_1 - x_2 = 3$

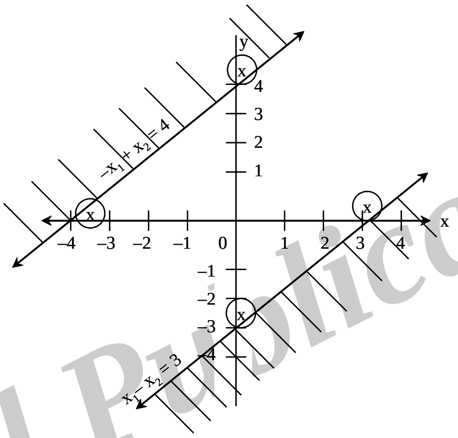
put $x_1 = 0 \Rightarrow x_2 = -3 \Rightarrow (0, -3)$

$x_2 = 0 \Rightarrow x_1 = 3 \Rightarrow (3, 0)$

To plot $x_1 - x_2 = -4$

put $x_1 = 0 \Rightarrow x_2 = 4 \Rightarrow (0, 4)$

$x_2 = 0 \Rightarrow x_1 = -4 \Rightarrow (-4, 0)$



Since there is no feasible region, the solution is infeasible.

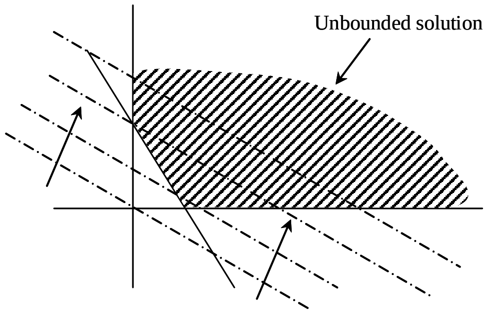
2.2.2.2 Unbounded Solution

Q11. Define Unbounded Solution.

Ans : **(Aug.-17)**

If a distinct and finite solution can not be found or the solution exists at infinity, the solution is said to be unbounded.

In graphical solution, unbounded solutions are obtained if the feasible region is unbounded (formed by requirement constraints i.e., $\geq$ type) while the objective function is maximization.



(Since it has to be taken to infinity to locate maximum value we have no finite or unbounded solution).

Problems on Unbounded Solutions

13. Solve the following LPP graphically

Maximize $Z = 3x_1 + 2z_2$

s.t.c. $x_1 - x_2 \leq 1$

$x_1 + x_2 \geq 3$; $x_1, x_2 \geq 0$

Sol :

Convert inequations into equations

$x_1 - x_2 = 1$ and $x_1 + x_2 = 3$

To plot $x_1 - x_2 = 1$

put $x_1 = 0, \; x_2 = -1 \Rightarrow (0, -1)$

$x_2 = 0, \; x_1 = 1 \Rightarrow (1, 0)$

To plot $x_1 + x_2 = 3$

put $x_1 = 0, \; x_2 = 3 \Rightarrow (0, 3)$

$x_2 = 0, \; x_1 = 3 \Rightarrow (3, 0)$

Here the shaded region is unbounded. The two vertices of the region are B = (0, 3); A = (2, 1). The values of the objective function at these vertices are Z(A) = 6 and Z(B) = 8. But there exists points in the region for which the values of the objective function is more than 8. For example, the point (5, 5) lies in the region and the function value at this point is 25 which is more than 8. Hence, the maximum value of Z occurs at the point at infinity only and thus the problem has an unbounded solution.

2.2.2.3 Problem with Inconsistent System of Constraints

Q12. Solve the following LPP

Maximize $Z = x_1 + x_2$ subject to constraints

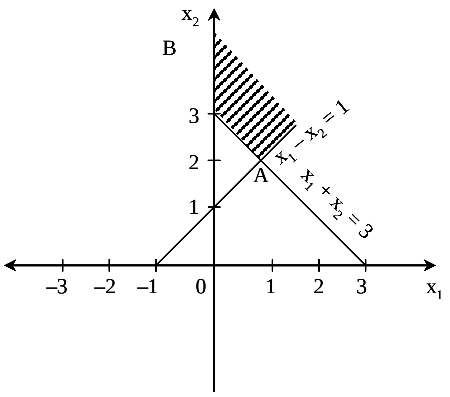
$x_1 + x_2 \leq 1$; $-3x_1 + x_2 \geq 3$;

$x_1 \geq 0, \; x_2 \geq 0.$

Ans :

Consider each inequality as equation

$x_1 + x_2 = 1$; $-3x_1 + x_2 = 3$



To plot $x_1 + x_2 = 1,$

put $x_1 = 0,\ x_2 = 1 \Rightarrow (0, 1)$

$x_2 = 0,\ x_1 = 1 \Rightarrow (1, 0)$

To plot $-3x_1 + x_2 = 3$

put $x_1 = 0,\ x_2 = 3 \Rightarrow (0, 3)$

$x_2 = 0,\ x_1 = -1 \Rightarrow (0, -1)$

The figure shows that there is no point (x_1, x_2) which satisfies both constraints simultaneously.

Hence, the problem has no solution because the constraints are inconsistent.

2.3 SIMPLEX METHOD

Q13. Define Simplex Method.

Ans :

Simplex method is an iterative procedure for solving LPP in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be than at the previous vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps (or) indicates the existence of unbounded solution.

Q14. Explain basic terminology is used in Simplex Method for solving LPP.

Ans :

Terminology of Simplex Method

The following are the terminologies used in solving LPP through simplex method,

1. Standard Form

A LPP in which all the constraints are written as equalities.

2. Slack Variable

A variable added to LHS of ≤ constraints (maximizations LPP) to convert the constraints into equality. The value of this variable is equal to the amount of unused resource.

3. Surplus Variable

A variable subtracted from the LHS of ≥ constraints (minimization LPP) to convert the constraints into equality. The value of this variable is equal to the amount over and above the required minimum level.

4. Basic Solution

For a general LP with 'n' variables and 'm' constraints, a basic solution can be found by setting (n –m) variables equal to zeros and solving the constraint equations for the value of other m variables. If a unique solution exists, it is a basic solution.

5. Basic Feasible Solution

It refers to the set of constraints corresponding to the extreme point of the feasible region.

6. Simplex Table

A table used to keep track of the calculations made at each iteration when the simplex solution method is employed.

7. Product Mix

A column in the simplex table that contains all of the variables in the solution.

8. Basic

A set of variables which are not restricted or equal to zero in the current basic solution are listed in the product mix column. These variables are called basic variables.

9. Iteration

The sequence of steps performed in moving from one basic feasible solution to another.

10. Z_j Row

The row containing the figures for gross profit or loss given up by adding one unit of a variable into the solution.

11. $C_j - Z_j$ / Net Evaluation index Row

The-row- containing the net profit or loss that will result from introducing one unit of the variable indicated in that column in the solution. Number in the index row are also shadow prices or according prices.

12. Pivot/Key Column

The column with the largest positive number in the $C_j - Z_j$ row of a maximization problem (largest negative for minimization problems). It indicates which variables will enter the solution next (i.e., entering variable/incoming variable).

13. Pivot/Key Row

The row' corresponding to the variable that will leave the basis in order to make room for the entering variable. The departing variable (outgoing variable) will correspond to the smaller positive ratio found by dividing the quantity column values by the key column values for each row.

14. Pivot/Key Element

The element of the intersection of key row and key column.

2.3.1 Maximization Case

Q15. Write the computational procedure for simplex method.

(OR)

Explain the procedure for simplex method.

Ans :

At each step it projects the improvement in the objective function over its previous step. Thus, the solution becomes optimum when no further improvement is possible on the objective function.

Simplex Algorithm

The algorithm goes as follows :

Step 1 : Formulation of LPP :

- Selection of decision variables
- Setting of objective function
- Identification of constraint set
- Writing the conditions of variables.

Step 2 : Convert constraints into equality form.

- Add slack variable if constraints is $\leq$ type.
- Subtract surplus and add an artificial variable if the constraint is $\geq$ type.
- Add an artificial variable if constraint is exact $(=)$ type.

Step 3 : Find, Initial Basic Feasible Solution (IBFS)

- If m non identical equations have n variables $(m < n)$ including all decision, slack / surplus and artificial variables, we get m number of variables basic and $(n - m)$ variables non basic (i.e., equated to zero).
- First make all decision variables (and surplus) as non basic i.e., equate to zero to identity the IBFS.
- Find solution values for basic variables.

Step 4 : Construct, Initial Simplex Tableau as given above with the following notations.

- C_B : Coefficient of basic variable in the objective functions (or contribution of basic variables)
- BV : Basic variables (form IBFS)
- SV : Solution value (from IBFS)
- C_j : Contribution of j^{th} variables or coefficient of each variable (j^{th}) in objective function.
- $Z_j - C_j$: Net contribution.

C_B	BV	C_j SV	C_j x_i, S & A	Min-Ratio	Remarks
Contribution of basic variable in objective function	Basic variables	Solution variables	y_i Key Element KE	Most min ratio of SV/key co. vlaue	← Key row
		Z_j	Sum of products of C_B and y_i		
		$Z_j - C_j$	Most negative value		

Key column

Step 5 : Find 'out going' and 'incoming' variables.

- Find Z_n by summation of products of C_B and y_i for each column
- Computed $Z_j - C_j$ value for each column
- To find key column use most negative value of $Z_j - C_j$
- Variable in key column is 'in coming variable' or 'entering' variable.
- The variable of key row is 'out going' or 'existing' variable.
- Find the minimum ratio of solution value to corresponding key column value to identify key row.
- The cross section of key column and key row is key element with which the next iteration is carried out.

Step 6 : Re-write next tableau as per given set of rules.

- Replace the existing variable from the basis with the entering variable along with its coefficient (or contribution).
- You have to make key element as unity (i.e., 1) and other element in the key column as zeros.
- To make key element as unity, divide the whole key row by the key element. This is supposed as the new row in the place of key row in the next iteration table.
- To find other rows of next iteration table, use this new row. By appropriate adding or subtracting entire new row in the old rows, make other elements of the key column as zeros.

Step 7 : Check whether all the values of $Z_j - C_j$ are positive. If all are positive, the optimal solution is reached. Write the solution values and find Z_{opt}. (i.e., Z_{max} or Z_{min} as the case may be).

If $Z_j - C_j$ values are still negative, again choose most negative among these and go to step 5 and repeat the iteration till all the values of $Z_j - C_j$ become positive.

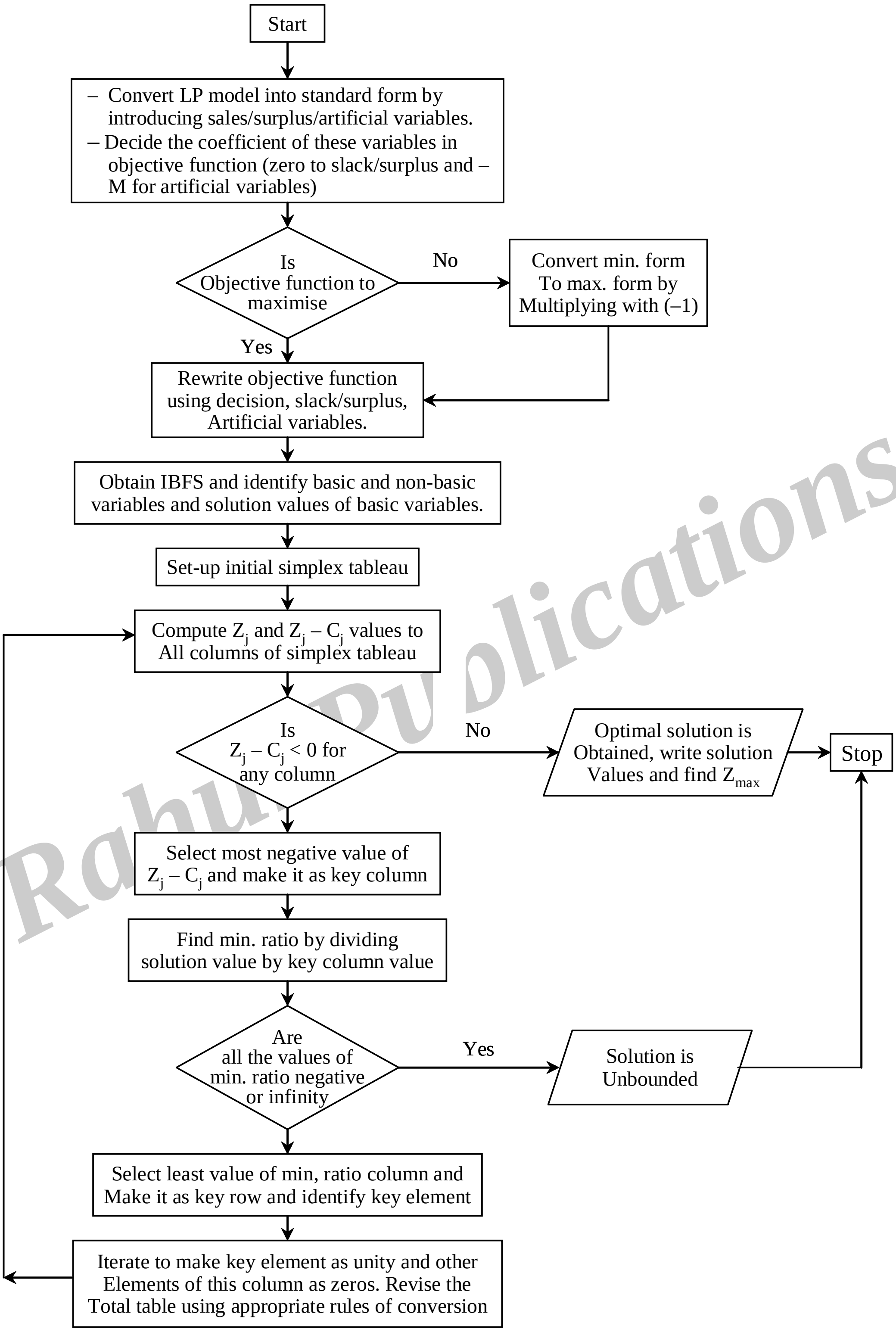


Figure : Flow Chart or Simplex Method

PROBLEMS

14. Use simplex method to solve the LPP.

$$\text{Max } Z = 3X_1 + 2X_2$$

Subject to $X_1 + X_2 \le 4$

$X_1 - X_2 \le 2$

$X_1, X_2 \ge 0$

Sol :

By introducing the slack variables S_1, S_2, convert the problem in standard form.

$$\text{Max } Z = 3X_1 + 2X_2 + 0S_1 + 0S_2$$

Subject to $X_1 + X_2 + S_1 = 4$

$X_1 - X_2 + S_2 = 2$

$X_1, X_2, S_1, S_2 \ge 0$

$$\begin{bmatrix} X_1 & X_2 & S_1 & S_2 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

An initial basic feasible solution is given by

$X_B = B^{-1}$ b, where

$B = I_2, X_B = (S_1, S_2)$.

i.e., $(S_1\ S_2) = I_2(4, 2) = (4, 2)$.

Initial Simplex Table

$$Z_j = C_B\, a_j$$

$$Z_1 - c_1 = C_B\, a_1 - c_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3 = -3$$

$$Z_2 - c_2 = C_B\, a_2 - c_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1 - 1) - 2 = -2$$

$$Z_3 - c_3 = C_B\, a_3 - c_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1 - 0) - 0 = -0$$

$$Z_4 - c_4 = C_B\, a_4 - c_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (0 - 1) - 0 = -0$$

		C_j	3	2	0	0	
C_B	BV	X_B	X_1	X_2	S_1	S_2	$\text{Min}\frac{X_B}{X_1}$
0	S_1	4	1	1	1	0	4/1 = 4
←0	S_2	2	①	–1	0	1	2/1 = 2
	Z_j	0	0	0	0	0	
	$Z_j – C_j$		–3↑	–2	0	0	

Since there are some $Z_j – C_j < 0$, the current basic feasible solution is not optimum.

Since Z_1 – C, = – 3 is the most negative, the corresponding non basic variable X, enters the basis.

The column corresponding to this X, is called the key column.

$$\text{To find the ratio} = \text{Min}\left\{\frac{X_{Bi}}{X_{ir}}, X_{ir} > 0\right\}$$

$$= \text{Min}\left\{\frac{4}{1}, \frac{2}{1}\right\} = 2 \text{ which corresponds to } S_2.$$

∴ The leaving variable is the basic variable S_2. This row is called the key row. Convert the leading element X_{21} to units and all other elements in its column n i.e. (X_1) to zero by using the formula:

$$\text{New element} = \text{Old element} - \left[\frac{\text{Product of elements in key row and key column}}{\text{key element}}\right]$$

To apply this formula, first we find the ratio, namely

$$\frac{\text{the element to be zero}}{\text{key element}} = \frac{1}{1} = 1$$

Apply this ratio, for the number of elements that are converted in the key row. Multiply this ratio by key row element as shown below.

1 × 2 = 2

1 × 1 = 1

1 × – 1 = –1

1 × 0 = 0

1 × 1= 1

Now subtract this element from the old element. The elements to be converted into zero, is called the old element row. Finally we have

$4 - 1 \times 2 = 2$

$1 - 1 \times 1 = 0$

$1 - 1 \times -1 = 2$

$1 - 1 \times 0 = 1$

$0 - 1 \times 1 = -1$

∴ The improved basic feasible solution is given in the following simplex table.

First Iteration

		C_j	3	2	0	0	
C_B	BV	X_B	X_1	X_2	S_1	S_2	$\text{Min}\frac{X_B}{X_2}$
← 0	S_1	2	0	②	1	– 1	2/1 = 1
3	X_1	2	1	– 1	0	1	–
	Z_j	6	3	– 3	0	0	
	$Z_j - C_j$		0	–5↑	0	0	

Since $Z_2 - C_2$ is most negative, X_2 enters the basis.

To find $\text{Min}\left(\frac{X_B}{X_{i2}}, X_{i2} > 0\right)$

$\text{Min}\left(\frac{2}{2}\right) = 1.$

This gives the out going variables. convert the leading element into one. This is done by dividing all the elements in the key row by 2. The remaining element by zero using the formula as shown below –½ is the common ratio. Put this ratio 5 times and multiply each ratio by key row element.

$-\frac{1}{2} \times 2$

$-\frac{1}{2} \times 0$

$-\frac{1}{2} \times 2$

$-1/2 \times 2$

$-1/2 \times -1$

Subtract this from the old element. All the row elements which are converted into zero, are called the old element.

$$2-\left(-\frac{1}{2}\times 2\right) = 3$$

$1 - (-1/2 \times 0) = 1$

$-1 - (-1/2 \times 2) = 0$

$0 - (-1/2 \times 1) = 1/2$

$1 - (-1/2 \times -1) = 1/2$

Second Iteration

		C_j	3	2	0	0
C_B	BV	X_B	X_1	X_2	S_1	S_2
2	X_2	1	0	1	½	– ½
3	X_1	3	1	0	½	½
	Z_j	11	3	2	5/2	½
	$Z_j - C_j$		0	0	5/2	½

Since all $Z_j - C_j \geq 0$, the solution is optimum, The optimal solution is Max Z = 11, $X_1 = 3$, and $X_2 = 1$.

---

15. Solve the LPP

Max $Z = 3X_1 + 2X_2$

Subject to $4X_1 + 3X_2 \leq 12$

$4X_1 + X_2 \leq 8$

$4X_1 - X_2 \leq 8$

$X_1, X_2 \geq 0$

Sol :

Convert the inequality of the constraint into an equation by adding slack variables S_1, S_2, S_3,....

Max $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2 + 0S_3$

Subject to $4X_1 + 3X_1 + S_1 = 12$

$4X_1 + X_2 + S_2 = 8$

$4X_1 - X_2 + S_3 = 8$

$X_1, X_2, S_1, S_2, S_3 \geq 0$

$$\begin{bmatrix} X_1 & X_2 & S_1 & S_2 & S_3 \\ 4 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 \\ 4 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 8 \end{bmatrix}$$

Initial Table

		C_j	3	2	0	0	0	
C_B	BV	X_B	X_1	X_2	S_1	S_2	S_3	Min $\frac{X_B}{X_1}$
0	S_1	12	4	3	1	0	0	12/4 = 3
0	S_2	8	4	1	0	1	0	8/4 = 2
← 0	S_3	8	④	–1	0	0	1	8/4 = 2
	Z_j	0	0	0	0	0	0	
	$Z_j – C_j$		–3↑	–2	0	0	0	

$\because$ $Z_1 - C_1$ is most negative, X_1 enters the basis. And the min $\left(\frac{X_B}{x_{ij}}, x_{ij} > 0\right)$ = min (3, 2, 2) = 2 gives S_3 as the leaving variable.

Convert the leading element into, by dividing key row element by 4 and the remaining elements into 0.

First Iteration

		C_j	3	2	0	0	0	
C_B	BV	X_B	X_1	X_2	S_1	S_2	S_3	Min $\frac{X_B}{X_2}$
0	S_1	4	0	4	1	0	– 1	$\frac{4}{4}$ = 1
←0	S_2	0	0	②	0	1	– 1	0/2 = 1
3	X_1	2	1	–1/4	0	0	$\frac{1}{4}$	–
	Z_j	⑥	3	– 3/4	0	0	3/4	
	$Z_j – C_j$		0	–11/4↑	0	0	3/4	

$$8 - \frac{4}{4} \times 8 = 0 \qquad 12 - \frac{4}{4} \times 8 = 4$$

$$4 - \frac{4}{4} \times 4 = 0 \qquad 4 - \frac{4}{4} \times 4 = 0$$

$$1 - \frac{4}{4} \times -1 = 2 \qquad 3 - \frac{4}{4} \times -1 = 4$$

$$0 - \frac{4}{4} \times 0 = 0 \qquad 1 - \frac{4}{4} \times 0 = 1$$

$$1 - \frac{4}{4} \times 0 = 1 \qquad 0 - \frac{4}{4} \times 0 = 0$$

$$0 - \frac{4}{4} \times 1 = -1 \qquad 10 - \frac{4}{4} \times 1 = -1$$

Since $Z_2 - C_2 = -3/4$ is the most negative x_2 enters the basis. To find the outgoing variable, find min $\left({}^{x_B}\!/_{x_{i2}}, x_{i2} > 0 \right)$ min $\left(\frac{4}{4}, \frac{0}{2}, -1 \right) = 0$.

First Iteration

Therefore S_2 leaves the basis. Convert the leading element into 1 by dividing the key row elements by 2 and remaining element in that column as zero using the formula.

$$\text{New element} = \text{old element} - \left[\frac{\text{Pr oduct of elements in key row and key column}}{\text{key element}} \right]$$

Initial Table

		C_j	3	2	0	0	0	
C_B	BV	X_B	X_1	X_2	S_1	S_2	S_3	Min $\frac{X_B}{S_3}$
←0	S_1	4	0	0	1	– 2	①	4/1 = 1
2	X_2	0	0	1	0	1/2	–1/2	–
3	X_1	2	1	0	0	1/8	1/8	2/1/8 = 16
	Z_j	6	3	2	0	11/8	–5/8	
	$Z_j - C_j$		0	0	0	11/8	–5/8 ↑	

Second Iteration

Since $Z_5 - C_5 = -5/8$ is mot negative S_3 enters the basis and

$$\text{Min}\left(\frac{X_B}{S_{13}}, S_{i3}\right) = \text{Min}\left(\frac{4}{1}, -1\frac{2}{1/18}\right) = 4.$$

Therefore, S_1 leaves the basis, Convert the leading element into one and remaining elements as zero.

Third iteration

		C_j	3	2	0	0	0
C_B	BV	X_B	X_1	X_2	S_1	S_2	S_3
0	S_3	4	0	0	1	– 2	1
2	X_2	2	0	1	1/2	–1/2	0
3	X_1	3/2	1	0	–1/8	3/8	0
	Z_j	17/2	3	2	5/8	1/8	0
	$Z_j - C_j$		0	0	5/8	1/8	0

Since all $Z_j - C_j \geq 0$, the solutionis optimum and it is given by $X_1 = 3/2$, $X_2 = 2$ and. Max $Z = 17/2$.

2.3.2 Artificial Variables Techniques.

Q16. Define Artificial Variables Techniques.

Ans :

LPP in which constraints may also have $\geq$ and = signs after ensuring that all $b_i \geq 0$ are considered in this section. In such cases basis matrix cannot be obtained as an identify matrix in the starting simplex table, therefore we introduce a new type of variable called the artificial variable. These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. To solve such LPP there are two methods.

(i) The Big M Method (or) Method of Penalties

(ii) The Two-phase Simplex Method.

2.3.2.1 Big M Method

Q17. Explain the various steps involved in Big M Method.

Ans :

The following steps are involved in solving an LPP using the Big M method.

Step 1

Express the problem in the standard form.

Step 2

Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type $\geq$ or $=$. However, addition of these artificial variable causes violation of the corresponding constraints. Therefore, we would like to get rid of these variables and would not allow them to appear in the final solution. This is achieved by assigning a very large penalty (-M for maximization and M for minimization) in the objective function.

Step 3

Solve the modified LPP by simplex method, until any one of the three cases may arise.

1. If no artificial variable appears in the basis and the optimality conditions are satisfied, then the current solution is an optimal basic feasible solution.
2. If at least one artificial variable in the basis at zero level and the optimality condition is satisfied then the current solution is an optimal basic feasible solution (though degenerated solution).
3. If at least one artificial variable appears in the basis at positive level and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function, since it contains a very large penalty M and is called pseudo optimal solution.

Note

While applying simplex method, whenever an artificial variable happens to leave the basis, we drop that artificial variable and omit all the entries corresponding to its column from the simplex table.

PROBLEMS

16. Use penalty method to

Maximize $Z = 3x_1 + 2x_2$

Subject to the constraints

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Sol :

By introducing slack variable $S_1 \geq 0$, surplus variable $S_2 \geq 0$ and artificial variable $A_1 \geq 0$, the given LPP can be reformulated as:

Maximize $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1$

Subject to $2x_1 + x_2 + S_1 = 2$

$3x_1 + 4x_2 - S_2 + A_1 = 12$

The starting feasible solution is $S_1 = 2$, $A_1 = 12$.

Initial Table

		C_j	3	2	0	0	– M	
C_B	BV	x_B	x_1	x_2	S_1	S_2	A_1	Min x_B/x_2
← 0	S_1	2	2	①	1	0	0	2/1 = 2
– M	A_1	12	3	4	0	–1	1	12/4 = 3
	Z_j	–12M	–3M	–4M	0	M	–M	
	$Z_j - C_j$	–	– 3M –3	– 4M – 2	0	M	0	
				↑				

Since some of $Z_j - C_j \leq 0$ the current feasible solution is not optimum. Choose the most negative $Z_j - C_j = -4M - 2$.

$\therefore$ x_2 variable enters the basis, and the basic variable S_1 leaves the basis.

First iteration

		C_j	3	2	0	0	– M
C_B	BV	X_B	X_1	X_2	S_1	S_2	A_1
2	x_2	2	2	1	1	0	0
– M	A_1	4	– 5	0	– 4	– 1	1
	Z_j	4 – 4M	4 + 5M	2	2 + 4M	M	–M
	$Z_j - C_j$		5M + 1	0	4M + 2	M	0

Since all $Z_j - C_j \geq 0$ and an artificial variable appears in the basis at positive level, the given LPP does not possess any feasible solution. But the LPP possesses a pseudo optimal solution.

17. Solve the LPP

Minimize $Z = 4x_1 + x_2$

Subject to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0.$

Sol :

Since the objective function is minimization, we covert it into maximization using.

Min Z = – Max (z)

Maximize $Z = -4x_1 - x_2$

Subject to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0.$

Convert the given LPP into standard form by adding artificial variables A_1, A_2, surplus variable S_1 and slack variable S_2 to get the initial basic feasible solution.

Maximize $Z = -4x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject $3x_1 + x_2 + A_1 = 3$

$4x_1 + 3x_2 - S_1 + A_2 = 6$

$x_1 + 2x_2 + S_2 = 3$

The starting feasible solution is $A_1 = 3$, $A_2 = 6$, $S_2 = 3$.

Initial Solution

		C_j	– 4	– 1	– M	0	–M	0	
C_B	BV	X_B	X_1	X_2	A_1	S_2	A_2	S_2	Min $\frac{X_B}{S_2}$
– M	A_1	3	3	1	1	0	0	0	3/3 = 1
– M	A_2	6	4	3	0	– 1	1	0	6/4 = 1.5
← 0	S_2	3	1	②	0	0	0	1	3/1 = 3
	Z_j	– 9M	– 7M	– 4M	– M	M	– M	0	
	$Z_j – C_j$		–7M + 4	–4 M+1↑	0	M	0	0	

Since some of the $Z_j - C_j \leq 0$, the current feasible solutio is not optimum. As $Z_1 - C_1$ is most negative, x_1 enters the basis and the basic variable A_2 leaves the basis.

First Iteration

		C_j	– 4	– 1	– M	0	– M	0	
C_B	BV	X_B	X_1	X_2	A_1	S_1	A_2	S_2	Min $\frac{X_B}{X_1}$
– M	A_1	3/2	5/2	0	1	0	0	– 1/2	3/5
← – M	A_2	3/2	$\boxed{5/2}$	0	0	– 1	1	– 3/2	3/5
– 1	x_2	3/2	1/2	1	0	0	0	1/2	3
	Z_j	– 3M – 3/2	– 5M – 1/2	– 1	– M	+ M	– M	2M – 1/2	
	$Z_j – C_j$		–5M + 7/2 ↑	0	0	M	0	2M – 1/2	

Since $Z_i - C_j$ is negative, the current feasible solution is not optimum. Therefore, x_1 variable enters the basis and the artificial variable A_2 leaves the basis.

Second Iteration

		C_j	– 4	– 1	– M	0	0	
C_B	BV	X_B	X_1	X_2	A_1	S_1	S_2	$\text{Min}\,\frac{X_B}{X_1}$
← – M	A_1	0	0	0	1	①	1	0
– 4	x_1	3/5	1	0	0	$-\frac{2}{5}$	$-\frac{3}{5}$	–
– 1	x_2	6/5	0	1	0	$-\frac{1}{5}$	$\frac{4}{5}$	–
	Z_j	$-\frac{18}{5}$	– 4	– 1	– M	$-M+\frac{9}{5}$	$-M+\frac{8}{5}$	
	$Z_j - C_j$		0	0	0	$-M+\frac{9}{5}$ ↑	$-M+\frac{8}{5}$	

Since $Z_4 - C_4$ is most negative, S1 enters the basis and the artificial variable A_1 leaves the basis.

Third Iteration

		C_j	– 4	– 1	0	0	
C_B	BV	X_B	X_1	X_2	S_1	S_2	$\text{Min}\,\frac{X_B}{S_2}$
← 0	S_1	0	0	0	1	①	0
– 4	x_1	3/5	1	0	0	$-\frac{1}{5}$	–
– 1	x_2	6/5	0	1	0	1	6/5
	Z_j	$-\frac{18}{5}$	– 4	– 1	0	–1/5	
	$Z_j - C_j$		0	0	0	– 1/5↑	

Since $Z_4 - C_4$ is most negative, S_2 enters the basis and S_1 leaves the basis,

Fourth Iteration

		C_j	– 4	– 1	0	0
C_B	BV	X_B	X_1	X_2	S_1	S_2
0	S_2	0	0	0	1	1
– 4	x1	3/5	1	0	1/5	0
– 1	x_2	6/5	0	1	– 1	1
	Z_j	– 18/5	– 4	– 1	1/5	0
	$Z_j - C_j$		0	0	1/5	0

Since all $Z_j - C_j \geq 0$ the solution is optimum and is given by $x_1 = 3/5$. $x_2 = 6/5$, and Max $Z = -18/5$.

$\therefore$ Min $Z = -$ Max $(-Z) = 18/5$.

---

18. Solve by Big M method.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_2 + x_4 = 10$

Sol :

Since the constraints are equations, introduce artificial variables $A_1, A_2 \geq 0$. The reformulated problem is given as follows.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

Subject to $x_1 + 2x_2 + 3x_3 + A_1 = 15$

$2x_1 + x_2 + 5x_3 + A_2 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

Initial solution is given by $A_1 = 15$, $A_2 = 20$ and $x_4 = 10$.

Initial table

		C_j	1	2	3	– 1	– M	– M	
C_B	BV	X_B	X_1	X_2	X_3	x_4	A_1	A_2	Min $\frac{X_B}{X_3}$
– M	A_1	15	1	2	3	0	1	0	15/3 = 5
← – M	A_2	20	2	1	⑤	0	0	1	20/5 = 4
– 1	x_4	10	1	2	1	1	0	0	10/1 = 10
	Z_j	– 35 M – 10	– 3M – 1	– 3M – 2	– 8 M – 1	– 1	– M	– M	
	$Z_j - C_j$		– 3M – 2	– 3M – 4	– 8M – 4	0	0	0	
					↑				

Since $Z_3 - C_3$ is most negative, x_3 enters the basis and the basic variable A_2 leaves the basis.

First Iteration

		C_j	1	2	3	– 1	– M	
C_B	BV	X_B	X_1	X_2	X_3	x_4	A_1	$\text{Min}\frac{X_B}{X_2}$
←– M	A_1	3	–1/5	$\boxed{7/5}$	0	0	1	$\frac{3}{7/5} = \frac{15}{7}$
3	x_3	4	2/5	1/5	1	0	0	$\frac{4}{1/5} = \frac{20}{1}$
– 1	x_4	6	3/5	9/5	0	1	0	$\frac{6}{9/5} = \frac{30}{9}$
	Z_j	– 3M + 6	$\frac{1}{5}M + \frac{3}{5}$	$-\frac{7}{5}M - \frac{6}{5}$	3	–1	– M	
	$Z_j - C_j$		$\frac{1}{5}M - \frac{2}{5}$	$-\frac{7}{5}M - \frac{16}{5}$	0	0	0	
				↑				

Since $Z_2 - C_2$ is most negative x_2 enters the basis and the basic variable A_1 leaves the basis.

Second Iteration

		C_j	**1**	**2**	**3**	**– 1**	
C_B	BV	X_B	X_1	X_2	X_3	x_4	$\text{Min}\frac{X_B}{X_1}$
2	x_2	15/7	$-\frac{1}{7}$	1	0	0	–
3	x_3	25/7	3/7	0	1	0	25/3
← – 1	x_4	15/7	$\boxed{6/7}$	0	0	1	15/6
	Z_j	$\frac{90}{7}$	$\frac{1}{7}$	2	3	–1	
	$Z_j - C_j$		– 6/7	0	0	0	
			↑				

Since $Z_1 - C_1 = -6/7$ in negative, the current feasible solution is not optimum. Therefore, x_1 enters the basis and the basic variable x_4 leaves the basis.

Third Iteration

		C_j	1	2	3	– 1
C_B	BV	X_B	X_1	X_2	X_3	x_4
2	x_2	15/6	0	1	0	1/6
3	x_3	15/6	0	0	1	3/6
1	x_1	15/6	1	0	0	7/6
	Z_j	15	1	2	3	3
	$Z_j – C_j$		0	0	0	4

Since all $Z_j – C_j \geq 0$ the solution is optimum and is given by $x_1 = x_2 = x_3 = 15/6 = 5/2$, and Max $Z = 15$.

2.3.2.2 Two Phase Method

Q18. What is Two Phase Method ? Explain the steps involved in Two Phase Method to solve a LPP.

Ans :

The two-phase simplex method is another method to solve a given LPP involving some artificial variable. The solution is obtained in two phases.

Phase I

In this phase, we construct an auxiliary LPP leading to a final simplex lable containing a basic feasible solution to the original problem.

Step 1

Assign a cost-1 to each artificial variable and a cost 0 to all other variables and get a new objective function $Z^* = – A_1 – A_2 – A_3$ ... where A_1 are artificial variable.

Step 2

Write down the auxiliary LPP in which the new objective function is to be maximized subject to the given set of constraints.

Step 3

Solve the auxiliary LPP by simplex method until either of the following three cases arise:

(i) Max $Z^* < 0$ and at least one artificial variable appears in the optimum basis at positive level.

(ii) Max $Z^* = 0$ and at least one artificial variable appears in the optimum basis at zero level.

(iii) Max $Z^* = 0$ and no artificial variable appears in the optimum basis.

In case (i), given LPP does not possess any feasible solution, where-as in case (ii) and (iii) we go to phase II.

Phase II

Use the optimum basic feasible solution of phase I as a starting solution for the original LPP. Assign the actual costs to the variable in the objective function and a zero cost to every artificial variable in the basis at zero level. Delete the artificial variable column from the table which is eliminated from the basis in phase I. Apply simplex method to the modified simplex table obtained at the end of phase I till an optimum basic feasible is obtained or till there is an indication of unbounded solution.

PROBLEMS

19. Using two phase simplex method to solve.

Maximize Z= $5x_1 - 4x_2 + 3x_3$

Subject to $2x_1 + x_2 - 6x_3 = 20$

$6x_1 + 5x_2 + 10x_3$

$8x_1 - 3x_2 + 6x_3 \leq 50$

$x_1, x_2, x_3 \geq 0$

Sol :

Introducing slack variables $S_1, S_2 \geq 0$ and an artificial variable $A_1 \geq 0$ in the constraints of the given LPP, the problem is reformulated in the standard form. Initial basic feasible solution is given by $A_1 = 20$, $S_1 = 76$ and $S_2 = 50$.

Assigning a cost -1 to the artificial variable A_1 and cost 0 to other variables, the objective function of the auxiliary LPP is

Maximize $Z^* = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - 1A_1$

Subject to $2x_1 + x_2 - 6x_3 + A_1 = 20$

$6x_1 + 5x_2 + 10x_3 + S_1 = 76$

$8x_1 - 3x_2 + 6x_3 + S_2 = 50$

$x_1, x_2, x_3, S_1, S_2, A_1 \geq 0$

Initial table

		C_j	0	0	0	–1	0	0	
C_B	BV	X_B	X_1	X_2	X_3	A_1	S_1	S_2	Min $\frac{X_B}{X_1}$
– 1	A_1	20	2	1	–6	1	0	0	20/2 = 10
0	S_1	76	6	5	10	0	1	0	76/6 = 12.66
←0	S_2	50	⑧	– 3	6	0	0	1	50/8 = 6.25
	Z_j	– 20	– 2	– 1	6	– 1	0	0	
	$Z_j - C_j$		– 2↑	– 1	6	0	0	0	

		C_j	0	0	0	–1	0	0	
C_B	BV	X_B	X_1	X_2	X_3	A_1	S_1	S_2	Min $\frac{X_B}{X_2}$
←– 1	A_1	15/2	0	$\boxed{7/4}$	– 15/2	1	0	–1/4	30/7
0	S_1	77/2	0	29/4	11/2	0	1	–3/4	154/29
0	x_1	25/4	1	–3/8	3/4	0	0	1/8	–
	Z_j	– 15/2	0	– 7/4	15/2	– 1	0	1/4	
	$Z_j – C_j$		0	– 7/4↑	15/2	0	0	1/4	
0	x_2	30/7	0	1	–30/7	4/7	0	–1/7	
0	S_1	52/7	0	$\boxed{1}$	256/7	–29/7	1	2/7	
0	x_1	55/7	1	0	–6/7	3/4	0	1/14	
	Z_j	0	0	0	0	0	0	0	
	$Z_j – C_j$		0	0	0	0	0	0	

Since all $Z_j = C_j \geq 0$, an optimum solution to the auxiliary LPP has been obtained. Also Max Z* = 0 with no artificial variable in the basis. We go to phase II.

Phase II

Consider the final simplex table of phase I. Consider the actual cost associated with the original variables. Delete the artificial variable A_1 column from the table as it is eliminated in phase I.

		C_j	5	–4	3	0	0
C_B	BV	X_B	X_1	X_2	X_3	S_1	S_2
– 4	x_2	30/7	0	1	–30/7	0	–1/7
0	S_1	52/7	0	0	256/7	1	2/7
5	x_1	55/7	1	0	–6/7	0	1/14
	Z_1	155/7	5	–4	90/7	0	13/14
	$Z_j – C_j$	0	0	0	69/7	0	13/14

Since all $Z_j – C_j \geq 0$ an optimum basic feasible solution has been reached. Hence, an optimum feasible solution to the given LPP is $x_1 = 55/7$, $x_2 = 30/7$, $x_3 = 0$ and Max Z = 155/7.

20. Solve using, two-phase simplex method

Solve the following LPP

Min $Z = 10X + 15Y$

S.T.C. ...,

$Y \geq 3$

$X - Y \geq 0$

$Y \leq 12;$

$X + Y \leq 30$

$X \leq 20$ **and** $X, Y \geq 0$

Sol :

Rewritting the objective function and constraint set in standard form by introducing slack, surplus and artificial variables, we get,

Maxmize $Z = 10X + 15Y$

Subject to $Y - S_1 + A_1 = 3$

$X - Y - S_2 + A_2 = 0$

$Y + S_3 = 12$

$X + Y + S_4 = 30$

$X + S_5 = 20$

$X, Y \geq 0, S_1, S_2, S_3, S_4, S_5 \geq 0$ and $A_1, A_2 \geq 0$

(where X, Y are decision variables, S_1 and S_2 are surplus variables, S_3, S_4 and S_5 are variables and A_1 and A_2 are artificial variables).

Phase - I

For Phase - I, we consider the objective function as

Max. $Z_1 = -A_1 - A_2$

$Y - S_1 + A_1 = 3;\ X - Y - S_2 + A_2 = 0;\ Y + S_3 = 12,$

$X + Y + S_4 = 30;\ X + S_5 = 20$ and

$X, Y \geq 0,\ S_j \geq 0,\ A_1, A_2 \geq 0\ (j = 1, 2, 3, 4, 5)$

IBFS

Basic variables : $A_1 = 3,\ A_2 = 0,\ S_3 = 12,\ S_4 = 30,\ S_5 = 20$

Non basic variables : $X = Y = S_1 = S_2 = 0$

Iteration Tableau I

Entering Variable

Delete column for next Iteration

C_B	BV	C_j SV	0 X	0 Y	0 S_1	0 S_2	0 S_3	0 S_4	0 S_5	–1 A_1	–1 A_2	Min ratio
–1	A_1	3	0	$\boxed{1}$ KE	–1	0	0	0	0	1	0	3/1 ← KR
–1	A_2	0	1	–1	0	–1	0	0	0	0	1	– ve (Ignore)
0	S_3	12	0	1	0	0	1	0	0	0	0	12/1
0	S_4	30	1	1	0	0	0	1	0	0	0	30/1
0	S_5	20	1	0	0	0	0	0	1	0	0	∞ (Ignore)
Leaving variable		Z_j	–1	–1	1	1	0	0	0	–1	–1	
		$Z_j – C_j$	–1	–1	0	0	0	0	0	0	0	

Key column

Note :

We an choose X or Y as key column here, but better to choose Y and if X is taken, it may lead to confusions.

Iteration Tableau II

Entering Variable

Delete column

C_B	BV	C_j SV	0 X	0 Y	0 S_1	0 S_2	0 S_3	0 S_4	0 S_5	–1 A_2	Min ratio	Remarks
0	Y	3	0	1	–1	0	0	0	0	0	∞	$R_1^N \rightarrow R_1^0$
–1	A_2	3	$\boxed{1}$	0	–1	–1	0	0	0	1	3/1 ←	$R_2^N \rightarrow R_2^0 + R_1^N$
0	S_3	9	0 KE	0	–1	0	1	0	0	0	∞	$R_3^N \rightarrow R_3^0 - R_1^N$
0	S_4	27	1	0	–1	0	0	1	0	0	27/1	$R_4^N \rightarrow R_4^0 - R_1^N$
0	S_5	20	1	0	0	0	0	0	1	0	20/1	$R_5^N \rightarrow R_5^0$
Leaving variable		Z_j	–1	0	1	1	0	0	0	–1		
		$Z_j – C_j$	–1	0	1	1	0	0	0	0		

Key column

Note :

In the above iteration, we can observe that there is no need of changing R_1 and R_5 since the desired 1 and 0 are already available.

Iteration Tableau III

C_B	BV	C_j	0	0	0	0	0	0	0	Min ratio	Remarks
		SV	X	Y	S_1	S_2	S_3	S_4	S_5		
0	Y	3	0	1	–1	0	0	0	0		$R_1^N \to R_1^0$
0	X	3	1	0	–1	–1	0	0	0		$R_2^N \to R_2^0$
0	S_3	9	0	0	–1	0	1	0	0		$R_3^N \to R_3^0$
0	S_4	24	0	0	0	1	0	1	0		$R_4^N \to R_4^0 - R_2^N$
0	S_5	17	0	0	1	1	0	0	1		$R_5^N \to R_5^0 - R_2^N$
		Z_j	0	0	0	0	0	0	0		
		$Z_j - C_j$	0	0	0	0	0	0	0	This is now called auxiliary simplex tableau	

Since $Z_j - C_j = 0$ for all the variable phase - I computation is complete at this state. Both artificial variables have been replaced from the basis. Therefore we proceed to phase - II now. Now for phase - II, we take above auxiliary simplex tableau with all numericals 'as is' except for the values of C_j. These are taken from the new objective function of second place.

Phase - II :

Objective Function : Max. $Z2 = 10X + 15Y$

Entering Variable

C_B	BV	C_j	0	0	0	0	0	0	0	Min ratio	Remarks
		SV	X	Y	S_1	S_2	S_3	S_4	S_5		
0	Y	3	0	1	–1	0	0	0	0	– ve	
0	X	3	1	0	–1	–1	0	0	0	– ve	
0	S_3	9	0	0	–1	0	1	0	0	– ve	
0	S_4	27	0	0	0	1	0	1	0	∞	
0	S_5	17	0	0	1	1	0	0	1	20/1 ←	
		Z_j	10	15	–25	–10	0	0	0		
		$Z_j - C_j$	0	0	–25	–10	0	0	0		

Leaving variable

Key column

C_B	BV	C_j	10	15	0	0	0	0	0	Min ratio	Remarks
		SV	X	Y	S_1	S_2	S_3	S_4	S_5		
15	Y	20	0	1	0	1	0	0	1		$R_1^N \to R_1^0 + R_5^N$
10	X	20	1	0	0	0	0	0	1		$R_2^N \to R_2^0 + R_5^N$
0	S_3	26	0	0	0	1	1	0	0		$R_3^N \to R_3^0 + R_5^N$
0	S_4	24	0	0	0	1	0	1	0		
0	S_5	17	0	0	1	1	0	0	1		$R_5^N \to R_5^0$
		Z_j	10	15	0	15	0	0	25		Zmax = 10×20 + 5×20 = 500
		$Z_j - C_j$	0	0	0	15	0	0	25		

Since all the values of $Z_j - C_j$ are positive ($\geq$ 0), we arrived, at optimal solution.

The optimal solution is $X = 20; \; Y = 20$

$$Z_{max} = 10 \times 20 + 15 \times 20 = 200 + 300 = 500.$$

2.3.3 Special Cases in Simplex Method

2.3.3.1 Unbounded Solution

Q19. Define Unbounded Solution.

Ans :

In some cases if the value of a variable is increased indefinitely, the constraints are not violated. This indicates that the feasible region is unbounded at least in one direction. Therefore, the objective function value can be increased indefinitely. This means that problem has been poorly formulated or conceived.

In simplex method, this can be noticed if $Z_j – Z_j$ value is negative to a variable (entering) which is notified as key column and the ratio of solution value to key column value is either negative or infinity (both are to be ignored) to all the variables. This indicates that no variable is ready to leave the basis, though a variable is ready to enter. We cannot proceed further and the solution is unbounded or not finite.

PROBLEMS

21. Maximise $Z = 4x_1 + 3x_2$

Subject to $x_1 \leq 5$

$x_1 – x_2 \leq 8$

$x_1, x_2 \geq 0$

Sol :

Let us introduce slack variables to express inequalities;

Maximise $Z = 4x_1 + 3x_2 + 0.S_1 + 0.S_2$

Subject to $x_1 + S_1 = 5$

$x_1 – x_2 + S_2 = 8$

$x_1, x_2 \geq 0,\ S_1, S_2 \geq 0$

IBFS $x_1 = 0,\quad x_2 = 0$ (Non basic)

$s_1 = 5,\quad s_2 = 8$ (basic)

With usual steps it is solved through the following iterations.

Iteration Tableau 1 :

Entering Variable

C_B	BV	C_j SV	4 x_1	3 x_2	0 S_1	0 S_2	Min ratio	Remarks
0	S_1	5	1	0	1	0	$\frac{5}{1}=5$	Key Row
0	S_2	8	1	–1	0	1	$\frac{8}{1}=8$	
Leaving variable		Z_j	0	0	0	0		
		Z_j - C_j	–4	–3	0	0		

Key Row

Iteration Tableau 2 :

$R_1^N \to R_1^0; \quad R_2^N \to R_2^N - R_1^0$

Entering Variable

C_B	BV	C_j	4	3	0	0	Min ratio	Remarks
		SV	x_1	x_2	S_1	S_2		
4	x_1	5	1	0	1	0	$\frac{5}{0} = \infty$	to be ignored
0	S_2	3	0	–1	–1	1	$\frac{3}{-1} = -\text{ve}$	to be ignored
Leaving variable		Z_j	0	0	0	0	No variable is ready to leave the basis	
		$Z_j - C_j$	0	–3	4	0	Solution is UNBOUNDED	

Key Row

From the above tableau - II, it is clear that x_2 is entering variable into the basis (key column variable whose $Z_j - C_j$ is negative) but no variable is read to leave since the ratio of solution value to key column value is infinity for R_1 and negative for R_2 both of which are to be ignored. Thus we cannot proceed further because key row (leaving variable) cannot be found. Thus, this problem yields *no finite solution* or in other words, as *unbounded solution*.

2.3.3.2 Multiple Optimal Solution

Q20. Define Multiple Optimal Solution.

Ans :

When the objective function is parallel to one of the constraints, the multiple optimal solutions may exist. As we have seen from graphical solutions, that the optimal solution exists at the extreme point on the feasible region, the multiple optimal solutions will be noticed on at least two points of the binding constraint parallel to that of objective function. Thus in simplex method also at least two solution can be found.

This alternate optima is identified in simplex method by using the following principle.

After reaching the optimality, if at least one of the non-basic (decision) variables possess a zero value in $Z_j - C_j$, the multiple optimal solutions exist.

---

22. Maximise $Z = 3x_1 + 6x_2$

Subject to $x_1 + x_2 \le 5$

$x_1 + 2x_2 \le 6$

$x_1, x_2 \ge 0$

Sol :

Converting inequalities into equations, we get

Maximise $Z = 3x_1 + 6x_2 + 0.S_1 + 0.S_2$

Subject to $x_1 + x_2 + S_1 = 5$

$x_1 + 2x_2 + S_2 = 6$

$$x_1, x_2, S_1, S_2 \geq 0$$

IBFS $S_1 = 5,\ S_2 = 6$ (Basic)

$x_1 = 0,\ x_2 = 0$ (Non basic)

Iteration Tableau I

Entering Variable

C_B	BV	C_j SV	3 x_1	6 x_2	0 S_1	0 S_2	Min ratio	Remarks
0	S_1	5	1	1	1	0	5	
0	S_2	6	1	$\boxed{2}$ Key element	0	1	3	Key row
Leaving variable		Zj	0	0	0	0		
		$Z_j - C_j$	–3	–6	0	0		

Key Row

Iteration Tableau II

Entering Variable

C_B	BV	C_j SV	3 x_1	6 x_2	0 S_1	0 S_2	Min ratio	Remarks
0	S_1	2	$\boxed{\frac{1}{2}}$ Key element	0	1	$-\frac{1}{2}$	5 ← Key row	$R_1^N \rightarrow R_1^0 - R_2^N$
6	x_2	3	$\frac{1}{2}$	1	0	$\frac{1}{2}$	3	$R_2^N \rightarrow \frac{1}{2} R_2^0$
Leaving variable		Zj	3	6	0	3	Since $Z_j - C_j \geq 0\ \forall$ all variables,	
		$Z_j - C_j$	0	0	0	3	Optimal solution is $Z_{max} = 18$	

Key Row

From the above Iteration Tableau - II the optimality is already reached since, $Z_j - C_j \geq 0$ i.e., positive for all the variables. Therefore the solution is

$$x_1 = 0,\ x_2 = 3,$$

and $Z_{max} = 3 \times 0 + 6 \times 3 = 18$

However, the value of $Z_j - C_j = 0$ for the non basic decision variable x_1 in tableau - II. This indicates alternative optimal solution. Therefore choosing first column as key column, if we further iterate, we get the following tableau - III.

Iteration Tableau III

C_B	BV	C_j SV	3 x_1	6 x_2	0 S_1	0 S_2	Min ratio	Remarks
3	x_1	4	1	0	2	–1	5	$R_1^N \rightarrow 2R_1^0$
6	x_2	1	0	1	–1	1	3	$R_2^N \rightarrow R_2^0 - \frac{1}{2}R_1^N$
		Zj	3	6	0	3	$Z_j - C_j \geq$ for all variable and hence	
		Z_j - C_j	0	0	0	3	Optimal solution is $Z_{max} = 18$	

This tableau - III yields another solution as,

$$x_1 = 4, \; x_2 = 1,$$

$$\text{and} \quad Z_{max} = 3 \times 4 + 6 \times 1 = 18$$

We can observe that $Z_j - C_j$ value will be always zero for basic variables in any simplex tableau [observe for S_1 and S_2 in Tableau - I, for S_1, and x_2 in Tableau - II and for x_1 and x_2 in Tableau - III they have $Z_j - C_j$ values zero]. However, in Tableau - II, apart from basic variables S_1 and x_2, the other variable x_1 is also showing $Z_j - C_j = 0$. This indicates that x_1 is also worth coming into basis, as it is behaving similar to any basic variables. Thus it provides can alternate optimal solution.

However, if all the basic variables are not replaced by decision variables it does not mean any multiple solution unless this condition (i.e., $Z_j - C_j = 0$ is obeyed any non-basic decision variable.

2.3.3.3 Infeasible Solution

Q21. Define Infeasible Solution.

Ans :

There may not exist any solution to certain LPP. This in LPP jargon is said to be infeasible solution. In this type of solution, there exists no feasible region. We do not get any infeasible solution with all constraints as ‘less than or equal to type’.

23. Maximise $Z = 2x_1 + 3x_2$

Subject to $x_1 \leq 5$

$$x_1 - x_2 \geq 10$$

$$x_1 \geq 0, \; x_2 \geq 0$$

Sol :

Converting the problem into canonical form,

Maximise $Z = 2x_1 + 3x_2$

Subject to $x_1 \leq 5$

$$-x_1 + x_2 \leq -10$$

$$x_1 \geq 0, \; x_2 \geq 0$$

Now introduce slack variables, to get equations

$$x_1 + S_1 = 5$$

$$-x_1 + x_2 + S_2 = -10$$

$$x_1, x_2, S_1 \text{ and } S_2 \geq 0$$

IBFS : $\left.\begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}\right\}$ Non basic

$\left.\begin{array}{l} S_1 = 5 \\ S_2 = -10 \end{array}\right\}$ Basic

From the condition $S_2 \geq 0$, S_2 can not take value –10, therefore no IBFS can exist with such less than or equal to type constraints.

So also, this method of converting '$\geq$' type constraint to '$\leq$' type with a negative sign can not yield any result and not suitable in such cases.

2.4 CONVERTING PRIMAL LPP TO DUAL LPP

Q22. Define duality principle. What is primal dual relationship. Write the rules for converting primal to dual.

Ans :

Duality Principle

For every linear programming problem there is a unique linear programming problem associated with it, involving the same data and closely related optimal solutions. The original (given) problem is called "the primal problem" while the other is called its "dual". But, in general, the two problems are said to be duals to each other.

The importance of the duality concept is due to two main reasons. Firstly, if the primal contains a large number of constraints and a smaller number of variables, the work of a computations can be considerably reduced by converting into the dual problem and then solving it. Secondly, the interpretation of the dual variables from the cost or economic point of view proves extremely useful in making future decisions in the activities being programmed.

Primal-Dual Relationship

	If Primal	Then Dual
1.	Objective : Maximize	Objective : Minimize
2.	Variable x_j	Constraint, j
3.	Constraint, i	Variable y_i
4.	Constraint i: '=' type	Constraint j: '=' type
5.	Variable x_j: unrestricted	Variable y_i: unrestricted
6.	$\leq$ type constraints	$\geq$ type constraints
7.	$\geq$ 0 variables	$\geq$ inequality constraints
8.	$\leq$ inequality constraints	$\geq$ 0 variables

Rules for Converting Primal to Dual

Following rules/steps are followed for converting primal to dual,

Step 1

Check the following condition,

For Max Z, all constraints are " < " type.

For Min Z, all constraints are " > " type.

If the condition is not satisfied convert the constraints sign by multiplying by '–1 '.

If any of the constraint is equal to ('=') type, ignore at this step. This will be considered later.

Step 2

Decide on the decision variables of the dual LPP. The number of decision variables should be equal to the number of constraints of the primal LPP.

Step 3

(a) If the objective function is maximization convert it to minimization and vice versa.

(b) The contributions of variables to the objective function of the dual will be the RHS (Right Hand Side) values of the constraints of the primal.

Step 4

(a) The values of the coefficients of variables in the constraints for the dual LPP is obtained from that of primal LPP but transposed.

Example

Primal Dual

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 6 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 6 \\ 2 & 5 & 7 \end{bmatrix}$$

(b) The RHS values of dual is taken from the contributions in objective function of primal LPP.

(c) The equation type of constraints is decided as follows,

Primal Objective	Primal variable	Dual constraints
Max Z	(i) $\geq$ (ii) unrestricted	$\geq$ =
Min Z	(i) ≥ 0 (ii) unrestricted	$\leq$ =

Step 5

The status of the variables are decided as follows,

Primal Objective	Primal variable	Dual constraints
Max Z	(i) $\leq$	≥ 0
	(ii) =	unrestricted
Min Z	(i) ≥ 0	≥ 0
	(ii) =	unrestricted

Step 6: Testing of Optimality

In case of maximization, $C_j - Z_j$ values are either negative or zero then the given solution is said to be an optimal one.

Obtaining Primal Solution from Dual Solution

The steps of obtaining an optimum solution to one of the problems from an optimum solution to the dual may be given as follows,

(a) First locate the starting solution primal variables. These variables are corresponding to the primal basic variables in the optimum solution.

(b) Read the net evaluations or index row corresponding to the above located starting variables in the optimum dual simplex table. The optimal solution to the primal is given by the optimal dual index row directly. If the dual is a maximization problem, then the positive of the $C_j - Z_j$ values will give the optimal values.

Primal (Max)	Dual (Min)
Basic variable values	$(C_j - Z_j)$ under starting solution columns.
$(C_j + Z_j)$ under starting	Basic variable values solution columns.

PROBLEMS ON DUAL FROM PRIMAL METHOD

Case I : All constraints are "$\leq$" type.

24. Write the dual of the following LPP.

Maximize $Z = 3x_1 + 5x_2$

s.t.c. $2x_1 + 6x_2 \leq 50$

$3x_1 + 2x_2 \leq 35,\ 5x_1 - 3x_2 \leq 10,\ x_2 \leq 20$ where $x_1 \geq 0,\ x_2 \geq 0$

Sol :

Let y_1, y_2, y_3 and y_4 be the corresponding dual variables, then the dual problem is given by

Min $Z^* = 50y_1 + 35y_2 + 10y_3 + 20y_4$

s.t.c.

$2y_1 + 3y_2 + 5y_3 \geq 3$; $6y_1 + 2y_2 - 3y_3 + y_4 \geq 5$

where y_1, y_2, y_3, y_4 all ≥ 0.

25. Write the dual of the following LPP.

Max $Z = x_1 - x_2 + 3x_3$

s.t.c. $x_1 + x_2 + x_3 \leq 10,$

$2x_1 + - 0.x_2 - x_3 \leq 2,\ 2x_1 - 2x_2 - 3x_3 \leq 6$ **where** $x_1, x_2, x_3 \geq 0.$

Sol :

Let y_1, y_2 and y_3 be the corresponding dual variables, then the dual problem is given by

Min $Z^* = 10y_1 + 2y_2 + 6y_3$

s.t.c. $y_1 + 2y_2 + 2y_3 \geq 1$

$y_1 + 0.y_2 - 2y_3 \geq -1,\ y_1 - y_2 - 3y_3 \geq 3$

where y_1, y_2, y_3 all ≥ 0

Case II : One or more constraints are "$\geq$" type.

26. Write the dual of the following LPP.

Max $Z = 3x_1 + 17x_2 + 9x_3$

s.t.c. $x_1 - x_2 + x_3 \geq 3,$

$-3x_1 + 2x_2 \leq 1 \quad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$

Sol :

We convert the $\geq$ constraint into $\leq$ constraint by multiplying both sides by –1.

i.e., $-x_1 + x_2 - x_3 \leq -3$

Let y_1 and y_2 be the corresponding dual variables, then the dual problem is given by

Min. $Z^* = -3y_1 + y_2$

s.t.c. $-y_1 - 3y_2 \geq 3, \quad y_1 \geq 17,$

$-y_1 + 2y_2 \geq 9$ where $y_1 \geq 0, \quad y_2 \geq 0$

---

27. Write the dual of the following LPP.

Max $Z = 2x_1 + 5x_2 + 3x_3$

s.t.c. $2x_1 + 4x_2 - x_3 \leq 8$

$2x_1 + 2x_2 + 3x_3 \geq -7, \quad x_1 + 3x_2 - 5x_3 \geq -2$

$4x_1 + x_2 + 3x_3 \leq 4$ **where** $x_1, x_2, x_3 \geq 0$

Sol :

We first convert "$\geq$" constraint into "$\leq$" constraints by multiplying both sides by –1 i.e., $-2x_1 + 2x_2 - 3x_3 \leq 7, \quad -x_1 - 3x_2 + 5x_3 \leq 2$

Let y_1, y_2, y_3 and y_4 be the corresponding dual variables, then the dual problem is given by

Min. $Z^* = 8y_1 + 7y_2 + 2y_3 + 4y_4$

s.t.c. $2y_1 - 2y_2 - y_3 + 4y_4 \geq 2$

$4y_1 - 2y_2 - 3y_3 + y_4 \geq 5, \ -y_1 - 3y_2 + 5y_3 + 3y_4 \geq 3$ where $y_1, y_2, y_3, y_4 \geq 0$

Case III : All constraints are "$\geq$" type.

28. Write the dual of the following LPP.

Max. $Z = 3x_1 + 4x_2$

s.t.c. $x_1 + x_2 \geq 4$

$-x_1 + 3x_2 \geq 4$ **where** $x_1 \geq 0, \quad x_2 \geq 0$

Sol :

We first convert all "$\geq$" constraints into "$\leq$" constraint by multiplying both sides by –1.

i.e., $-x_1 - x_2 \leq -4, \quad x_1 - 3x_2 \leq 4$

Let y_1 and y_2 be the corresponding dual variables then the dual problem is given by

Min $Z^* = 4y_1 + 4y_2$

s.t.c. $-y_1 + y_2 \geq 3$

$-y_1 - 3y_2 \geq 4$ where $y_1 \geq 0, \quad y_2 \geq 0$.

2.5 LIMITATIONS OF LPP

Q23. What are the advantages and limitations of LPP ?

Ans : **(Mar.-15, Aug.-17)**

Advantages

(i) Provides an optimal solution(s).

(ii) Fast determination of the solution if a computer is used.

(iii) Finds solutions to a wide variety of problems which can be formulated with L.P.

(iv) Finds solutions to problems with a very large or infinite number of possible solutions.

(v) Provides a natural sensitivity analysis.

Limitations

The major limitations are:

(i) Certainty

It is assumed that all the data in the LP are known with certainty.

(ii) Linear Objective Function

This means that per unit cost, price and profit are assumed to be unaffected by changes in production methods or quantities produced or sold.

(iii) Linear Constraints

The linearity assumptions in LP are reflected in additivity, independence and proportionality.

(a) Additivity

It is assumed that the total utilization of each resource is determined by adding together that portion of the resource required for the production of each of the various products or activities.

(b) Independence

Complete independence of coefficients is assumed both among activities and among resources.

(c) Proportionality

It is assumed that the objective function and constraints must be linear as a requirement of proportionality. This means that the amount of resources used and the resulting value of the objective function will be proportional to the value of the decision variables.

(iv) Non-negativity

Negative activity levels or negative production are not permissible.

(v) Divisibility

Variables are classified as continuous or discrete. In LP it is assumed that the unknown variables x_1, x_2 .... are continuous, i.e., they can take any fractional values (divisibility assumption).

Exercises Problems

1. Three articles A, B and C have weight, volume and cost as given below : The total weight cannot exceed 2000 units and total volume cannot exceed 2500 units. Find the number of articles to be selected from each type such that the total cost is maximum.

	Weight	**Volume**	**Cost (Rs.)**
A	4	9	5
B	8	7	6
C	2	4	3

Sol :

Maximize $Z = 5x_1 + 6x_2 + 3x_3$

S.T.C.

$4x_1 + 8x_2 + 2x_3 \leq 2000$

$9x_1 + 7x_2 + 4x_3 \leq 2500$

$x_1, x_2, x_3 \geq 0$

2. An electronic company produces three types of parts for washing machine. It purchases unfinished articles from a foundary and does the jobs of drilling, shaping and polishing. The selling prices of the parts A, B and C are Rs. 8, Rs. 10 and Rs. 14 respectively. Their purchase costs are Rs. 5, Rs. 6 and Rs. 10 respectively. The company has only one machine for each job. Machine costs per hour are Rs. 20 for drilling, Rs. 30 for shaping and Rs. 30 for shaping and Rs. 30 for polishing. The capacities of each machine (parts per hour) are given below :

Machine	**Capacity per hour**		
	A	**B**	**C**
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

It is required to determine how many parts of each type should be produced per hour in order to maximize the profit.

Sol :

Maximize $P = (0.25)\, x_1 + (1.00)\, x_2 + (0.95)\, x_3$

S.T.C.

$8x_1 + 5x_2 + 8x_3 \leq 200$

$4x_1 + 5x_2 + 5x_3 \leq 100$

$3x_1 + 4x_2 + 3x_3 \leq 120$

$x_1, x_2, x_3 \geq 0$

3. A furniture manufacturing company plans to make chairs and tables. The requirements are given below :

Requirements			
Item	**Timber (in board ft)**	**Man-Hours**	**Profits (`)**
Chair	5	10	45
Table	20	15	80

Totally 400 board feet of timber and 450 man-hours are available. Determine the number of chairs and tables to be manufactured in order to maximize the profit.

Sol :

Maximize $P = 45x_1 + 80x_2$

S.T.C.

$5x_1 + 20x_2 \leq 400$

$10x_1 + 15x_2 \leq 450$

$x_1, x_2 \geq 0$

---

4. Maximize $Z = 3x_1 + 4x_2$

S.T.C.

$2x_1 + x_2 \leq 40$

$2x_1 + 5x_2 \leq 180$

$x_1, x_2 \geq 0$

Sol :

$x_1 = 5/2, \; x_2 = 35, \; Z^* = 147.5$

---

5. Maximize $Z = 3x_1 + 4x_2$

S.T.C.

$x_1 + x_2 \leq 450$

$2x_1 + x_2 \leq 600$

$x_1, x_2 \geq 0$

Sol :

$x_1 = 0, \; x_2 = 450, \; Z^* = 18000$

6. Maximize $Z = 3x_1 + 2x_2$

S.T.C.

$-2x_1 + x_2 \leq 1$

$x_1 \leq 2$

$x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$

Sol :

$x_1 = 2, \; x_2 = 1, \; Z^* = 8$

7. Minimize $Z = 5x_1 - 6x_2 - 7x_3$

S.T.C.

$x_1 + 5x_2 - 3x_3 \geq 15$

$5x_1 - 6x_2 + 10x_3 \leq 20$

$x_1 + x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 0$

Sol :

$x_1 = 0, \; x_2 = 15/4, \; x_3 = 5/4, \; Z^* = -125/4$

8. Minimize $Z = 4x_1 + 3x_2$

S.T.C.

$2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

Sol :

$x_1 = 4, \; x_2 = 2, \; Z^* = 22$

9. Minimize $Z = 2x_1 + 3x_2$

S.T.C.

$x_1 + x_2 \geq 5$

$x_1 + 2x_2 \geq 6$

$x_1, x_2 \geq 0$

Sol :

$x_1 = 4, \; x_2 = 1, \; Z^* = 11$

Short Question and Answers

1. Define Linear Programming.

Ans :

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective functions. It is subject to a set of linear equalities and/or inequalities known as constraints. Linear programming is a mathematical technique which involves the allocation of limited resources in an optimal manner, on the basis of a given criterion of optimality.

To solve a LPP some basic requirements must be fulfilled. They are,

1. Objective function availability
2. Set of constraints
3. Both objective function and constraints must be linear functions. (or) Decision Variables.

2. What are the Assumptions of LPP ?

Ans :

(a) Proportionality

A primary requirement of linear programming problem is that the objective function and every constraint function must be linear. Roughly speaking, it simply means that if 1 kg of a product costs Rs. 2, then 10 kg will cost Rs. 20. If a steel mill can produce 200 tons in 1 hour, it can produce 1000 tons in 5 hours.

Intuitively, linearity implies that the product of variables such as $x_1 x_2$, powers of variables such as x_3^2, and combination of variables such as $a_1 x_1 + a_2 \log x_2$, are not allowed.

(b) Additivity

Additivity means if it takes t_1 hours on machine G to make product A and t_2 hours to make product B, then the time on machine G devoted to produce A and B both is $t_1 + t_2$, provided the time required to change the machine from product A to B is negligible.

The additivity may not hold, in general. If we mix several liquids of different chemical composition, then the total volume of the mixture may not be the sum of the volume of individual liquids.

(c) Multiplicativity

It requires:

(i) If it takes one hour to make a single item on a given machine, it will take 10 hours to make 10 such items; and

(ii) The total profit from selling a given number of units is the unit profit times the number of units sold.

(d) Divisibility

It means that the fractional levels of variables must be permissible besides integral values.

3. Mathematical Formulation of LPP.

Ans :

The general formulation of the LPP can be stated as follows:

In order to find the values of n decision variables $x_1, x_2, \ldots x_n$ to maximize or minimize the objective function.

$$Z = c_1x_1 + c_2 x_2 + ... + c_nx_n \qquad ... (1)$$

and also satisfy m-constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}\ x_n (\le = \ge) b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}\ x_n (\le = \ge) b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}\ x_n (\le = \ge) b_i \\ \vdots \\ am_1x_1 + a_{m2}x_2 + ... + a_{mn}\ x_n (\le = \ge) b_m \end{array}\right\} \quad ... (2)$$

where constraints may be in the form of inequality $\leq$ or $\geq$ or even in the form an equation (=) and finally satisfy the non negative restrictions.

$x_1 \geq 0,\ x_2 \geq 0,\ ....\ x_n \geq 0.$

4. What is Graphical Method LPP ?

Ans :

Graphical method is a simple method to understand and also to use. This is effectively used in LPP's which involves only 2 variables. It gives the graphical representation of the solutions.

All types of solutions are highlighted in this method very clearly. The only drawback is that more the number of constraints, more will be the straight lines which makes the graph difficult to understand.

Characteristics of Graphical Method

The following are the characteristics of graphical method of LPP,

1. Method is very simple and easy to understand.
2. Very sensitive analysis and can be illustrated very easily by drawing graphs.
3. Very easy to obtain optimal solution.
4. It consumes very less time.

5. Describe the steps involved in graphical solution to linear programming models.

Ans :

Simple linear programming problems of two decision variables can be easily solved by graphical method. The outlines of graphical procedure are as follows :

Step 1 : Consider each inequality-constraint as equation.

Step 2 : Plot each equation on the graph, as each one will geometrically represent a straight line.

Step 3 : Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality-constraint corresponding to that lines is '$\leq$', then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality-constraint with '$\geq$' sign, the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.

Step 4 : Choose the convenient value of z (say = 0) and plot the objective function line.

Step 5 : Pull the objective function line until the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Step 6 : Read the coordinates of the extreme point(s) selected in *step 5* and find the maximum or minimum (as the case may be) value of z.

6. Define Simplex Method.

Ans :

Simplex method is an iterative procedure for solving LPP in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be than at the previous vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps (or) indicates the existence of unbounded solution.

7. Define Artificial Variables Techniques.

Ans :

LPP in which constraints may also have $\geq$ and = signs after ensuring that all $b_i \geq 0$ are considered in this section. In such cases basis matrix cannot be obtained as an identify matrix in the starting simplex table, therefore we introduce a new type of variable called the artificial variable. These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. To solve such LPP there are two methods.

(i) The Big M Method (or) Method of Penalties

(ii) The Two-phase Simplex Method.

8. Define Unbounded Solution.

Ans :

In some cases if the value of a variable is increased indefinitely, the constraints are not violated. This indicates that the feasible region is unbounded at least in one direction. Therefore, the objective function value can be increased indefinitely. This means that problem has been poorly formulated or conceived.

In simplex method, this can be noticed if $Z_j - Z_j$ value is negative to a variable (entering) which is notified as key column and the ratio of solution value to key column value is either negative or infinity (both are to be ignored) to all the variables. This indicates that no variable is ready to leave the basis, though a variable is ready to enter. We cannot proceed further and the solution is unbounded or not finite.

9. Define Multiple Optimal Solution.

Ans :

When the objective function is parallel to one of the constraints, the multiple optimal solutions may exist. As we have seen from graphical solutions, that the optimal solution exists at the extreme point on the feasible region, the multiple optimal solutions will be noticed on at least two points of the binding constraint parallel to that of objective function. Thus in simplex method also at least two solution can be found.

This alternate optima is identified in simplex method by using the following principle.

After reaching the optimality, if at least one of the non-basic (decision) variables possess a zero value in $Z_j - C_j$, the multiple optimal solutions exist.

10. Define Infeasible Solution.

Ans :

There may not exist any solution to certain LPP. This in LPP jargon is said to be infeasible solution. In this type of solution, there exists no feasible region. We do not get any infeasible solution with all constraints as ‘less than or equal to type’.

UNIT III

Assignment Model: Algorithm for solving assignment model, Hungarians Method for solving assignment problem, variations of assignment problem: Multiple Optimal Solutions, Maximization case in assignment problem, unbalanced assignment problem, travelling salesman problem, simplex method for solving assignment problem.

Transportation Problem: Mathematical Model of transportation problem, Methods for finding Initial feasible solution: Northwest corner Method, Least Cost Method, Vogels approximation Method, Test of optimality by Modi Method, unbalanced Supply and demand , Degeneracy and its resolution.

3.1 ASSIGNMENT PROBLEM

Q1. What is an Assignment Problem ? What are the characteristics of the Assignment Problem.

Ans : **(Nov.-20, May-19)**

Assignment problems are special type of linear programming problems where assignees are being assigned to perform tasks. The assignment problem arises in a variety of decision-making situations. In many business situations, management finds it necessary to assign personnel to jobs, jobs to machines, machines to job locations within a plant, or sales persons to territories within the distribution area of the business, or contracts to bidders etc.

In each of these cases, the management would like to make the most effective or cost-efficient assignment of a set of workers (or objects) to a set of jobs (or assignment). The criteria used to measure the effectiveness of a particular set of assignments may be total cost, total profit or total time to perform a set of operations.

A distinguishing feature of the assignment problem is that one worker (or job) is assigned to one and only one task (or machine). Specifically, we look for the set of assignments that will optimize a stated objective such as minimise cost, minimise time or maximise profit.

Assignment problems are similar to transportation problems and can be solved by any one of the transportation algorithms. But two distinct characteristics of assignment problems make the application of the transportation algorithm inconvenient in solving them. These characteristics are:

(i) The number of rows are equal to the number of columns and

(ii) In the optimal solution, there can be only one assignment in a given row and column.

When a transportation problem is having these two features, it is treated as an assignment problem. Thus, an assignment problem is a special type of transportation problems or a linear programming problem.

Assignment Model : Assignment models are characterised as follows :

(i) m workers are to be assigned to m jobs.

(ii) A unit cost (or profit) is associated with worker 'i' performing job 'j'.

Objective : Minimise the total cost (or maximise the total profit) of assigning workers to jobs so that each worker is assigned one job and each job is performed.

In formulating an assignment model, which set is considered as objects and which set as assignments is irrelevant. For example, assigning four machines to four jobs is equivalent to assigning four jobs to four machines, as long as each job is completed and each machine is assigned.

The assignment model is therefore a square matrix (since the objects and assignments are equal in numbers) for example, 'm' workers assigned to 'm' jobs or 'n' jobs assigned to 'n' machines. As in a linear programming problem, the goal or objective function is to minimise the total cost of jobs performed by the machines or workers as the case may be.

Characteristics of the Assignment Problem

The characteristics are as follows:

(i) The objects under consideration, such as service teams, jobs, employees or projects are finite in number.

(ii) The objects have to be assigned on a one-to-one basis to other objects.

(iii) The result of each assignment can be expressed in terms of pay-offs such as costs or profits.

(iv) The aim is to assign all objects (if possible) in such a way that the total cost is minimised (or the total profit is maximised)

Q2. State the assumptions of an Assignment Problem.

Ans :

Assumptions Made in Assignment Models

(i) The number of assignees (workers or machines) and the number of jobs or tasks are the same (say the number is 'n').

(ii) Each assignee is to be assigned to exactly one job or task.

(iii) Each job or task is to be performed by exactly one assignee.

(iv) There is a cost c_i associated with assignee 'i' (i = 1, 2....,n) performing task 'j' (j = 1, 2, n).

(v) The objective is to determine how all 'n' assignments should be made to minimise the total cost.

Any problem satisfying all these assumptions can be solved very efficiently by algorithms designed specifically for assignment problems.

The first three assumptions are fairly restrictive. Many applications do not quite satisfy these assumptions. When these assumptions are not satisfied, it is possible to reformulate the problem to make it fit.

For example, dummy assignees or dummy jobs or tasks can be used for this purpose.

Many solution algorithms have been proposed to assignment problems. They are:

(i) Total enumeration of all possibilities

(ii) Linear programming

(iii) A transportation approach

(iv) Dynamic programming

(v) A binary branch and bound approach and

(vi) An efficient approach developed specifically for the assignment problem known as the Hungarian method or algorithm, which is generally used.

Q3. State the mathematical formulation of the Assignment Problem.

Ans : **(May-19, Feb.-17)**

Mathematically the assignment problem can be stated as

$$\text{Minimise} \quad z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \quad i = 1,2,...n$$

$$j = 1,2,...n$$

Subject to the restrictions

$$x_{ij} = \begin{cases} 1 \text{ if the } i^{th} \text{ person is assigned } j^{th} \text{ job} \\ 0 \text{ if not} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ (one job is done by the } i^{th} \text{ person)}$$

and $\sum_{j=1}^{n} x_{ij} = 1$ (only one person should be assigned the j^{th} job)

Where x_{ij} denotes that the j^{th} job is be assigned to the i^{th} person.

3.2 ALGORITHM FOR SOLVING ASSIGNMENT MODEL

3.2.1 Hungarians Method for Solving Assignment Problem

Q4. Discuss the steps involved in the Hungarians Method used to find optimal solution to an Assignment Problem.

Ans : **(Feb.-17, Aug.-16, Sep.-15, March-15)**

Steps Involved in the Hungarian Method (in Solving Minimisation Problems)

Step 1

Check whether the numbers of rows are equal to the number of column. If the number of rows equals the number of columns, the problem is a balanced one and Hungarian method can be used. If not, then the assignment problem is unbalanced and application of Hungarian method to an unbalanced problem yields an incorrect solution. Hence any assignment problem should be balanced by the additions of one or more dummy points (i.e., rows and columns). For dummy rows and columns, the value at the point of intersection of row and column is of zero value (i.e., zero cost in a cost matrix)

Step 2

Find the minimum element (or cost) in each row of the (m × n) cost matrix. Construct a new matrix by subtracting from each cost the minimum cost in its row. For this new matrix, find the minimum cost in each column. Construct a new matrix (called the reduced cost matrix) by subtracting from each cost, the minimum cost in its column.

This step may also be stated as below:

(a) **Row Subtraction : Subtract the minimum** element (say cost) of each row from all elements in that row. (Note : If there is zero in each row, there is no need for row subtraction).

(b) **Column Subtraction :** Subtract the minimum element of each column (of the new matrix obtained after row subtraction) from all elements of that column (Note : if there is zero in each column, there is no need for column subtraction).

Step 3

Draw the minimum number of lines (horizontal, vertical or both) that are needed to cover all the zeros in the reduced cost matrix. If 'm' lines are required then an optimal solution is available among the covered zeros in the matrix. If fewer than 'm' lines are needed, then proceed to Step 4.

[**Note :** To draw the minimum number of lines the following procedure may be followed:

(a) Select a row containing exactly one uncovered zero and draw a vertical line through the column containing this zero and repeat the process till no such row is left.

(b) Select a column containing exactly one uncovered zero and draw a horizontal line through the row containing the zero and repeat the process till no such column is left].

Step 4 :

Find the smallest non-zero element (call its value K) in the reduced cost matrix that is uncovered by the lines drawn in Step 3. Now subtract K from each uncovered element of the reduced cost matrix and add K to each element that is covered by two lines.

Step 5 :

Repeat steps 3 and 4 till minimum number of lines covering all zeros is equal to the size of the matrix (i.e., 'm' lines in a 'm × n' matrix)

Step 6 :

Assignment : Use the matrix obtained in Step 5 (without horizontal or vertical lines) select a row containing exactly one unmarked zero and surround it by a □ and draw a vertical line through the column containing this zero. Repeat the process till no such row is left, then select a column containing exactly one covered zero and surround it by a □ and draw a horizontal line through the row containing this zero and repeat this process till no such column is left.

[**Note :** if there are more than one unmarked zero in any row or column, it indicates that an alternative solution exists. In this case, select any one arbitrarily and pass two lines horizontally and vertically.]

Step 7 :

Add up the value attributable to the allocation which shall be the minimum value.

Step 8 :

Alternate solution : If there are more than one covered zero in any row or column, select the other one (i.e., other than the one selected in step 6) and pass two lines horizontally and vertically. Add up the value attributable to the allocation, which shall be the minimum value.

PROBLEMS

1. Solve the following assignment problem by Hungarian assignment method.

Time (in minutes)			
Worker	**Job 1**	**Job 2**	**Job 3**
A	**4**	**2**	**7**
B	**8**	**5**	**3**
C	**4**	**5**	**6**

Sol : **(Feb. - 17)**

Given,

Time (in minutes)			
Worker	**Job 1**	**Job 2**	**Job 3**
A	**4**	**2**	**7**
B	**8**	**5**	**3**
C	**4**	**5**	**6**

Row Reduced Matrix (Row Reduction)

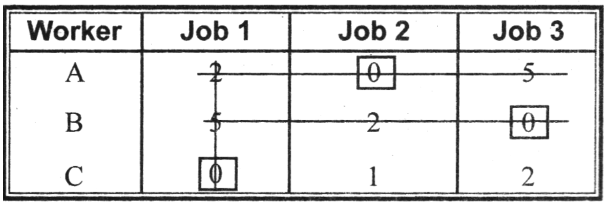
Select the minimum number from each row and subtract it from each element of row.

Worker	Job 1	Job 2	Job 3
A	2	0	5
B	5	2	0
C	0	1	2

Column Reduced Matrix (Column Reduction)

No need to do column reductions as each column is having a 'zero'.

Now, cover maximum zeros from the reduced matrix.



Worker	Job 1	Job 2	Job 3
A	2	$\boxed{0}$	5
B	5	2	$\boxed{0}$
C	$\boxed{0}$	1	2

Since, the number of assignments is equal to the order of matrix, the current solution is said to be optional.

Allocations are,

Worker A → Job 2 = 2

Worker B → Job 3 = 3

Worker C → Job 1 = 4

Minimum Cost 9

---

2. A company has 4 machines available for assignment to 4 tasks. Any machine can be assigned to any task, and each task requires processing by one machine. The time required to set up each machine for the processing of each task is given in the table below.

What kind of assignment will allow the company to minimize the total setup time needed for the processing of all four tasks?

	TIME (Hours)			
	Task 1	**Task 2**	**Task 3**	**Task 4**
Machine 1	**13**	**4**	**7**	**6**
Machine 2	**1**	**11**	**5**	**4**
Machine 3	**6**	**7**	**2**	**8**
Machine 4	**1**	**3**	**5**	**9**

Sol : **(Aug./Sept. - 16)**

	TIME (Hours)			
	Task 1	**Task 2**	**Task 3**	**Task 4**
Machine 1	13	4	7	6
Machine 2	1	11	5	4
Machine 3	6	7	2	8
Machine 4	1	3	5	9

Step 1

Row Reduced Matrix (Row Reduction)

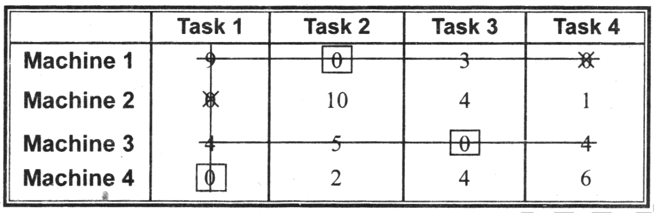
Select the minimum number from each row and subtract it from each element of row.

	Task 1	Task 2	Task 3	Task 4
Machine 1	9	0	3	2
Machine 2	0	10	4	3
Machine 3	4	5	0	6
Machine 4	0	2	4	8

Step 2

Column Reduced Matrix (Column Reduction)

Select the minimum number from each column and subtract it from each element of column.



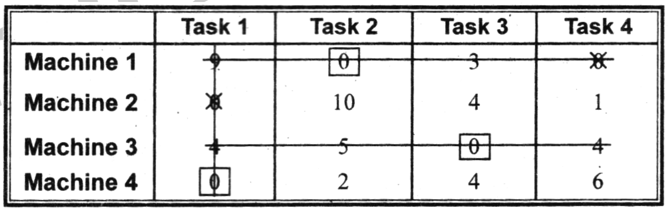
	Task 1	Task 2	Task 3	Task 4
Machine 1	9	$\boxed{0}$	3	$\cancel{0}$
Machine 2	$\cancel{0}$	10	4	1
Machine 3	4	5	$\boxed{0}$	4
Machine 4	$\boxed{0}$	2	4	6

Step 3

Assignment

As number of assignment ≠ order of matrix. Thus applying Hungarian rule. The minimum cost uncovered element is '1'. So add '1' where the lines are intersecting and subtract '1' from all uncovered elements.

New Cost Matrix



	Task 1	Task 2	Task 3	Task 4
Machine 1	9	$\boxed{0}$	3	$\cancel{0}$
Machine 2	$\cancel{0}$	10	4	1
Machine 3	4	5	$\boxed{0}$	4
Machine 4	$\boxed{0}$	2	4	6

Step 4: Optimality Test

Number of assignments = 4

Number of row/columns = 4

Since, the number of assignments = Number of row/ columns

Therefore, the current solution is optimal.

Optimal Solution

Machine 1 → Task 2 = 4

Machine 2 → Task 4 = 4

Machine 3 → Task 3 = 2

Machine 4 → Task 1 = 1

Min. time = 11

∴ The minimum total setup time is 11 hours.

3. Solve the minimal assignment problem for the cost matrix given below :

	1	2	3	4
A	12	13	14	15
B	14	15	16	17
C	17	18	19	18
D	13	15	18	14

Sol :

Step 1 : Row Subtraction :

Subtract the smallest element in the row from each element in that row. The resulting cost matrix is shown:

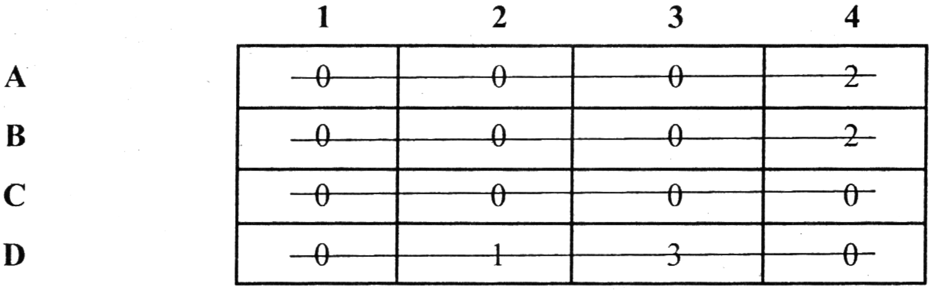
	1	2	3	4
A	0	1	2	3
B	0	1	2	3
C	0	1	2	1
D	0	2	5	1

Step 2 : Column subtraction :

In the cost matrix obtained in Step 1 : subtract the smallest element in the column from each element in that column.

	1	2	3	4
A	0	1	2	3
B	0	1	2	3
C	0	1	2	1
D	0	2	5	1

The reduced cost matrix is as shown below :



	1	2	3	4
A	~~0~~	~~0~~	~~0~~	~~2~~
B	~~0~~	~~0~~	~~0~~	~~2~~
C	~~0~~	~~0~~	~~0~~	~~0~~
D	~~0~~	~~1~~	~~3~~	~~0~~

Since no signal zero exists in any row column we have the following alterative columns

(i)

	1	2	3	4
A	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	2
B	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
C	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$
D	$\cancel{0}$	1	3	$\boxed{0}$

(ii)

	1	2	3	4
A	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
B	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
C	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
D	$\boxed{0}$	1	3	0

(iii)

	1	2	3	4
A	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
B	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
C	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
D	$\boxed{0}$	1	3	$\cancel{0}$

The possible optimal solutions are

		Cost (Rs)
(i)	A → 1 →	12
	B → 2 →	15
	C → 3 →	19
	D → 4 →	14
	Total	**60**

		Cost (Rs)
(ii)	A → 2 →	13
	B → 3 →	16
	C → 4 →	18
	D → 1 →	13
	Total	**60**

		Cost (Rs)
(iii)	A → 3 →	14
	B → 2 →	15
	C → 4 →	18
	D → 1 →	13
	Total	**60**

3.3 VARIATIONS OF ASSIGNMENT PROBLEM

3.3.1 Multiple Optimal Solutions

Q5. When does Multiple Optimal Solutions are said to be operated in assignment problem.

Ans : **(May-19)**

In an Assignment Problem (AP) the facilities are assigned to jobs on a one-to-one basis using Hungarian method. While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off certain number of zeros.

This indicates that there are more than the required number of independent zero elements. In such cases, there will be multiple optimal solution with the same total cost of assignment. This type of situation is helpful in management decision making as the manager has flexibility in assignments.

3.3.2 Maximization Case in Assignment Problem

Q6. Explain briefly about Maximization Case in Assignment Problem.

Ans : **(May-19)**

If instead of cost matrix, a profit (or revenue) matrix is given, then assignments are made in such a way that total profit is maximized. The profit maximization assignment problems are solved by converting them into a cost minimization problem in either of the following two ways:

i) Put a negative sign before each of the elements in the profit matrix in order to convert the profit values into cost values.

ii) Locate the largest element in the profit matrix and then subtract all the elements of the matrix from the largest element including itself.

The transformed assignment problem can be solved by using usual Hungarian method.

PROBLEMS

4. A company has four territories open and four salesmen available for assignment. The territories are not equally rich in their sales potential: It is estimated that a typical salesman operating in each territory would bring in the following sales annually,

Territory	I	II	III	IV
Annual sales (`)	60,000	50,000	40,000	30,000

Four salesmen also considered to differ in their ability: It is estimated that, working under the same conditions, their yearly sales would be proportionality as follows,

Salesman	A	B	C	D
Proportion	7	5	5	4

If the criterion is maximum expected total sales, then intuitive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest, and so on. Verify this answer by assignment technique.

Sol : **(March - 15)**

The maximum sales matrix is obtained by adding the sale proportion i.e., $\Rightarrow 7 + 5 + 5 + 4 = 21$ and considering the sales of ` 10,000 as one unit.

Now, divide the individual sales in each territory by 21 to obtain the required annual sales by each salesman.

Example

$$\frac{7}{21}\times 6 = \frac{7}{21}5, \frac{7}{21}\times 4, \frac{7}{21}\times 3$$

$$= \frac{42}{21}, \frac{35}{2}, \frac{28}{21}, \frac{21}{21}$$

∴ The maximum sales matrix is,

Sales in 10,000 Rupees

$$\begin{array}{cc|cccc} & & \mathbf{6} & \mathbf{5} & \mathbf{4} & \mathbf{3} \\ \textbf{Sale Proportion} & & \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{IV} \\ \mathbf{7} & A & 42 & 35 & 28 & 21 \\ \mathbf{5} & B & 30 & 25 & 20 & 15 \\ \mathbf{5} & C & 30 & 25 & 20 & 15 \\ \mathbf{4} & D & 24 & 20 & 16 & 12 \end{array}$$

As it is a maximization case, therefore first converting the matrix into minimization matrix, subtract from the highest element (i.e., 42) among all the elements of the given matrix.

$$\begin{array}{c|cccc} & \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{IV} \\ \mathbf{A} & 0 & 7 & 14 & 21 \\ \mathbf{B} & 12 & 17 & 22 & 27 \\ \mathbf{C} & 12 & 17 & 22 & 27 \\ \mathbf{D} & 18 & 22 & 26 & 30 \end{array}$$

(i) Row reduction

Deduct the least element of each row in that row.

∴ Table changes to,

$$\begin{array}{c|cccc} & I & II & III & IV \\ A & 0 & 7 & 14 & 21 \\ B & 0 & 5 & 10 & 15 \\ C & 0 & 5 & 10 & 15 \\ D & 0 & 4 & 8 & 12 \end{array}$$

(ii) Column Reduction

Deduction the least element of each in that column.

∴ Table changes to,

$$\begin{array}{c|cccc} & I & II & III & IV \\ A & \boxed{0} & 3 & 6 & 9 \\ B & \xcancel{0} & 1 & 2 & 3 \\ C & \xcancel{0} & 1 & 2 & 3 \\ D & \xcancel{0} & \boxed{0} & \xcancel{0} & \xcancel{0} \end{array}$$

iii) Applying Hungarian Rule

Draw lines covering more number of rows first and then draw lines covering the zeros remained. Then,

$$\begin{array}{c|cccc|} & I & II & III & IV \\ A & 0 & 3 & 6 & 9 \\ B & 0 & 1 & 2 & 3 \\ C & 0 & 1 & 2 & 3 \\ D & 0 & 0 & 0 & 0 \end{array}$$

iv) Modified Matrix

Deduct the least element from uncovered elements of a table(1) after drawing the lines and add it at intersection of lines.

$$\begin{array}{c|cccc|} & I & II & III & IV \\ A & \boxed{0} & 2 & 5 & 8 \\ B & \cancel{0} & \boxed{0} & 1 & 2 \\ C & \cancel{0} & \cancel{0} & 1 & 2 \\ D & 1 & \cancel{0} & \boxed{0} & \cancel{0} \end{array}$$

v) Apply Hungarian Rule

Draw lines again covering zeros.

∴ Table changes to,

$$\begin{array}{c|cccc|} & I & II & III & IV \\ A & 0 & 2 & 5 & 8 \\ B & 0 & 0 & 1 & 2 \\ C & 0 & 0 & 1 & 2 \\ D & 1 & 0 & 0 & 0 \end{array}$$

Number of lines drawn $\neq$ Number of rows

∴ Solution is not obtained.

vi) Again deduct the least element from uncovered elements and it at intersection of lines.

∴ Table changes to,

$$\begin{array}{c|cccc|} & I & II & III & IV \\ A & \boxed{0} & 2 & 4 & 7 \\ B & \cancel{0} & 0 & \boxed{0} & 1 \\ C & \cancel{0} & \boxed{0} & \cancel{0} & 1 \\ D & 2 & 1 & \cancel{0} & \boxed{0} \end{array}$$

∴ Each row and column had one zero. Therefore, optimal solution is obtained.

vii) Allocation is,

$A \to I$, $B \to III$, $C \to II$, $D \to IV$ = 42

= 20 + 25 + 12 = 99

∴ maximum annual sales are

$$= \text{`}\,\frac{1}{21}(99) \times 10{,}000$$

= ` 47,142.86

3.3.3 Unbalanced Assignment Problem

Q7. Define Unbalanced Assignment Problem.

Ans : **(May-19)**

The Hungarian method for solving an assignment problem requires that the number of columns and rows in the assignment matrix should be equal. However, when the given cost matrix is not a square matrix, the assignment problem is called an *unbalanced problem.* In such cases before applying Hungarian method, dummy row(s) or column(s) are added in the matrix (with zeros as the cost elements) in order to make it a square matrix.

PROBLEMS

5. Solve the following assignment problem of minimizing total time for doing all the jobs:

Operator \ Job	I	II	III	IV	V
1	6	2	5	2	6
2	2	5	8	7	7
3	7	8	6	9	8
4	6	2	3	4	5
5	9	3	8	9	7
6	4	7	4	6	8

Sol : **(Aug. - 17)**

The given matrix has 6 rows and 5 columns, thus it is an unbalanced A.P. Converting the unbalanced A.P into balanced A.P by adding one column with zero profit of assignment.

Job / Operator	I	II	III	IV	V	D₁
1	6	2	5	2	6	0
2	2	5	8	7	7	0
3	7	8	6	9	8	0
4	6	2	3	4	5	0
5	9	3	8	9	7	0
6	4	7	4	6	8	0

Row Reduced Matrix

Row iteration is not necessary as each row has a zero.

Column Reduced Matrix

Job / Operator	I	II	III	IV	V	D_1
1	4	$\boxed{0}$	2	$\cancel{0}$	1	$\cancel{0}$
2	$\boxed{0}$	3	5	5	2	$\cancel{0}$
3	5	6	3	7	3	$\boxed{0}$
4	4	$\cancel{0}$	$\boxed{0}$	2	0	$\cancel{0}$
5	7	1	5	7	2	$\cancel{0}$
6	2	5	1	4	3	$\cancel{0}$

As number of assignments $\neq$ order of matrix, applying Hungarian rule. After drawing minimum number of lines, the least uncovered element is ‘1’. So add ‘ 1’ where lines meet and deduct ‘ 1 ’ from uncovered elements, we get,

Job / Operator	I	II	III	IV	V	D_1
1	4	0	2	$\boxed{0}$	1	1
2	$\boxed{0}$	3	5	5	2	1
3	4	5	2	6	2	$\boxed{0}$
4	4	$\cancel{0}$	$\cancel{0}$	2	$\boxed{0}$	1
5	6	$\boxed{0}$	4	6	1	$\cancel{0}$
6	1	4	$\boxed{0}$	3	2	$\cancel{0}$

The allocations are,

1 → IV	2
2 → I	2
3 → D_1	0
4 → V	5
5 → II	3
6 → III	4
Total	16 hrs/min

3.4 SIMPLEX METHOD FOR SOLVING ASSIGNMENT PROBLEM

Q8. Explain briefly about the simplex method for solving Assignment Problem.

Ans :

The basic purpose behind the classical AP is to allocate jobs to the competent workers based on their skills and competencies. However, while allocating the jobs, the manager must focus on its principal goal of minimizing the cost associated with the assignment of workers to jobs. Consider ‘n’ workers and ‘n’ jobs whose assignment matrix can be formed as follows,

$$\begin{array}{r c} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & \ldots\ldots & n \end{array} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ \text{wor ker s(a)}4 \\ \vdots \\ n \end{array} & \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & \ldots\ldots & c_{1n} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & \ldots\ldots & c_{2n} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & \ldots\ldots & c_{3n} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & \ldots\ldots & c_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & c_{n4} & c_{n5} & \ldots\ldots & c_{nn} \end{bmatrix} \end{array}$$

The element C_{ab} represents the unit cost of assigning a^{th} worker with b^{th} job. Wherein a and b = 1, 2, 3, 4 .... n.

Assignment model is established based on an assumption that both the number of workers and the number of jobs are equal i.e., each job is assigned to each worker.

Formulation

$$\text{Let, } x_{ij} = \begin{cases} 1, \text{ if } b^{th} \text{ job is assigned to } a^{th} \text{ machine} \\ 0, \text{ if } b^{th} \text{ job is not assigned to } a^{th} \text{ machine} \end{cases}$$

The assignment model is then given by the following LPP.

$$\text{Minimize Z} = \sum_{a=1}^{n}\sum_{b=a}^{n} C_{ab}x_{ab}$$

Subject to the constraints,

$$\sum_{a=1}^{n} x_{ab} = 1, \ b = 1, 2, \ldots, n$$

$\sum_{b=1}^{n} x_{ab} = 1$, a = 1, 2,..., and

x_{ab} = 0 or 1

As, it is associated with the minimization of cost, its objective function of LP model becomes :

Minimiza $z = \sum_{a=1}^{n}\sum_{b=1}^{n} C_{ab}x_{ab}$ ... (1)

Proof

The optimal solution remains independent of either the addition or the deduction of constant from any row or column of the cost matrix (C_{ab}). This can be proved by adding constants M_a and N_b to row 'a' and column 'b'. After incorporating these changes, the cost element C_{ab} is changed to C_{ab}"

$C_{ab}" = C_{ab} + M_a + N_b$ ... (2)

Even, the objective function is changed to,

Minimize $z = \sum_{a=1}^{n}\sum_{b=1}^{n} C_{ab}"x_{ab}$ ... (3)

Substitute equation (2) in equation (3) we get,

$$\sum_a\sum_b C''_{ab}x_{ab} = \sum_a\sum_b = (C_{ab} + M_a + N_b)x_{ab}$$

$$\sum_a\sum_b C''_{ab}x_{ab} = \sum_a\sum_b C_{ab}x_{ab} + \sum_a M_a\left(\sum_a x_{ab}\right) + \sum_a N_b\left(\sum_a x_{ab}\right) \quad \text{... (4)}$$

From the mathematical model, it has been seen that,

x_{ab} = 0 or 1 ... (5)

Case: 1

If x_{ab} = 0 then equation (4) becomes,

$$\sum_a\sum_b C"_{ab}\,x_{ab} = \sum_a\sum_b C_{ab}x_{ab} \sum_a M_a(0) + \sum_b N_b\,(0)$$

$\boxed{\sum_a\sum_b C''_{ab}x_{ab} = \sum_a\sum_b C_{ab}x_{ab}}$ (Feasible solution)

Proved

Case: 2

If x_{ab} = 1 then equation (4) becomes,

$$\sum_a\sum_b C''_{ab}x_{ab} = \sum_a\sum_b C_{ab}x_{ab} + \sum_b M_a\,(1) + \sum_b N_b\,(1)$$

$$\boxed{\sum_a\sum_b C''_{ab}x_{ab} = \sum_a\sum_b C_{ab}x_{ab} + \text{constant}}$$

Even though, the new objective function varies from the original function in terms of the value of the constant, the optimum values of x_{ab} are found to be same in both the cases.

3.5 TRAVELLING SALESMAN PROBLEM

Q9. What is meant by Travelling Salesman Problem ?

Ans :

A travelling salesman has to visit a certain number of cities from his headquarters. The distance (or cost or time) of journey between every pair of cities denoted by a. (i.e., distance from city 'i' to city 'j') is assumed to be known.

The problem is:

Salesman starting from his home city (headquarters) visits each city only once and returns to his home city in the shortest possible total distance (or at the least total cost or least total time). The problem of the travelling salesman may be viewed as a typical assignment problem with two additional constraints.

(i) No assignment should be made along the diagonal line of the "From" - "To" matrix.

(ii) The salesman should not be required to visit a city twice until he has visited all the cities once.

Given 'n' cities and distance c_{ij} , the salesman starts from city 1, then any permutation of city 2, 3, n to represent the number of possible ways of his travel. So there are (n - 1)! possible ways for his travel to cover all 'n' cities. The cost matrix is as shown below.

To City :	**1**	**2**	**3**	**....**	**n**
1	∞	C_{12}	C_{13}	....	C_{1n}
2	C_{21}	∞	C_{23}	....	C_{2n}
3	C_{31}	C_{32}	∞	....	C_{3n}
From city					
n	C_{n1}	C_{n2}	C_{n3}	....	∞

PROBLEMS

6. Solve the following travelling salesman problem.

Sol :

To City

		W	X	Y	Z
	W	∞	36	6	30
From	X	31	∞	40	30
City	Y	72	22	∞	50
	Z	30	30	26	∞

Sol :

Step 1 : Row subtraction

	W	X	Y	Z
W	∞	30	0	24
X	1	∞	10	0
Y	50	0	∞	28
Z	4	4	0	∞

Step 2 : Column substraction

	W	X	Y	Z
W	∞	30	$\boxed{0}$	24
X	$\boxed{0}$	∞	10	$\cancel{0}$
Y	49	$\boxed{0}$	∞	28
Z	3	4	$\cancel{0}$	∞

Since only three assignments can be made, which is less than the order of the cost matrix the current assignment is not optimal.

Step 3

Subtract the smallest number from the uncovered numbers and add to cells at the intersection of lines. The resulting reduced cost matrix is as shown below:

In this case, the smallest uncovered number is 3.

	W	X	Y	Z
W	∞	27	0	21
X	0	∞	13	0
Y	46	0	∞	25
Z	0	4	0	∞

Step 4

Assignment is done as shown below :

	W	X	Y	Z
W	∞	27	$\boxed{0}$	21
X	$\cancel{0}$	∞	13	$\boxed{0}$
Y	46	$\boxed{0}$	∞	25
Z	$\boxed{0}$	4	$\cancel{0}$	∞

The optimal solution (route) is as given below :

$W \longrightarrow Y$ i.e., $W \longrightarrow Y$

$X \longrightarrow Z$ $Y \longrightarrow X$

$Y \longrightarrow X$ $X \longrightarrow Z$

$Z \longrightarrow W$ $Z \longrightarrow W$

---

7. Solve the following Travelling Salesman problem.

	Territory			
Salesmen	**P**	**Q**	**R**	**S**
A	42	35	28	21
B	30	25	20	15
C	30	25	20	15
D	24	20	16	12

Unit entries represent sales expenses in Rs. 1000.

Sol :

Since the number of rows equal the number of columns, the problem is a balanced one.

Step 1

Row Substraction : In cash row, subtract the smallest in the row. The reduced matrix is shown below.

Territory Salesmen	P	Q	R	S
A	21	14	7	0
B	15	10	5	0
C	15	10	5	0
D	12	8	4	0

Step 2

Column Subtraction : In each column subtract the smallest number from other numbers.

The resulting cost matrix is shown below:

Territory Salesmen	P	Q	R	S
A	9	6	3	0
B	3	2	1	0
C	3	2	1	0
D	0	0	0	0

Step 3

Draw the minimum number of lines to cover all zeros in the matrix. Only two lines can be drawn to cover all zeros. Since the number of lines are less than the size of the matrix (4 x 4), optimum assignment cannot be achieved.

Step 4

Subtract the smallest uncovered number (i.e., 1) from all other uncovered numbers and add it to the number at the intersection of the lines. The resulting matrix is shown below.

Territory / Salesmen	P	Q	R	S
A	8	5	2	0
B	2	1	0	0
C	2	1	0	0
D	0	0	0	1

Step 5

In the matrix obtained in Step 4 draw minimum number of lines to cover all zeros. We have four lines which equals the size of matrix (i.e., 4x4) and hence optimum assignment is possible.

Step 6

Assignment

Territory / Salesmen	P	Q	R	S
A	8	5	2	$\boxed{0}$
B	2	1	$\boxed{0}$	$\cancel{0}$
C	2	$\boxed{1}$	$\cancel{0}$	$\cancel{0}$
D	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	1

Salesmen	Territory	Expenses
A	S	21
B	R	20
C	Q	25
D	P	24
		90

8. A company has from sales representatives who are assigned to four different sales territories. The monthly sales increase estimated for each sales representative for different sales territories (in lakhs rupees) are shown in the following table.

Sales territories

Sales representative	**I**	**II**	**III**	**IV**
A	**200**	**150**	**170**	**220**
B	**160**	**120**	**150**	**140**
C	**190**	**195**	**190**	**200**
D	**180**	**175**	**160**	**190**

Suggest an optimal assignment and the total maximum sales increase per month. If for certain reasons sales representative 'B' cannot be assigned to sales territory III, will the optimal assignment schedule be different ? If so find that schedule and the effect on total sales.

Sol :

Since the given problem is a maximization problem, we have to convert it into a minimization problem (by subtracting all the elements in the matrix from the highest element).

Step 1 : Convert the maximization problem into a minimization problem by constructing the opportunity loss matrix as shown below :

Opportunity Loss Matrix

Territory **Salesmen**	**I**	**II**	**III**	**IV**
A	220 – 200 = 20	220 – 150 = 70	220 – 170 = 50	220 – 220 = 0
B	220 – 160 = 60	220 – 120 = 100	220 – 150 = 70	220 – 140 = 80
C	220 – 190 = 30	220 – 195 = 25	220 – 190 = 30	220 – 200 = 20
D	220 – 180 = 40	220 – 175 = 45	220 – 160 = 60	220 – 190 = 30

Step 2 : Row Subtraction

Territory **Sales reps.**	**I**	**II**	**III**	**IV**	**Row minimum**
A	20	70	50	0	0
B	0	40	10	20	60
C	10	5	10	0	20
D	10	15	30	0	30

Step 3 : Column Subtraction

Territory / Sales reps.	I	II	III	IV
A	20	65	40	~~0~~
B	~~0~~	~~35~~	~~0~~	~~20~~
C	~~10~~	~~0~~	~~0~~	~~0~~
D	10	10	20	~~0~~
Column minimum	0	5	10	~~0~~

Step 4 : Draw the minimum number of lines to cover all zeros. Since the number of lines (i.e., 3) is less than the size of the matrix (4 × 4) optimum assignment is not possible.

Step 5 : The matrix requires further reduction for this the smallest uncovered number is subtracted from all other uncovered number and added to the number at the intersection of the lines. The reduced matrix is shown below. (Smallest uncovered number is 10).

Territory / Sales reps.	I	II	III	IV
A	~~10~~	~~55~~	~~30~~	~~0~~
B	~~0~~	~~35~~	~~0~~	~~30~~
C	~~10~~	~~0~~	~~0~~	~~10~~
D	~~0~~	~~0~~	~~10~~	~~0~~

Step 6 : Draw lines to cover all the zeros in the matrix obtained in step 5. The minimum optimum of lines is four and hence the optimum assignment is possible.

Territory / Sales reps.	I	II	III	IV
A	10	55	30	$\boxed{0}$ 4th
B	1st $\boxed{0}$	35	$\cancel{0}$	30
C	10	$\cancel{0}$	$\boxed{0}$ 2nd	10
D	$\cancel{0}$	$\boxed{0}$ 3rd	10	$\cancel{0}$

Step 7 : Assignment

Sales Reps.	Territory	Sales increase/month (Rs. lakhs)
A	IV	220
B	I	160
C	III	190
D	II	175
		Total Rs. 745 Lakhs / months

If sales representative B cannot be assigned to territory III, there will not be any effect on optimal assignment schedule because in optimal schedule B is assigned to territory I and territory III is assigned to C.

---

9. A salesmen must travel from city to city to maintain his accounts. This week he has to leave his home base and visit each other city and return home. The table shows the distances (in kilometers) between the various cities. The home city is city A. Use the assignment method to determine the tour that will minimize the total distances of visiting all cities and returning home.

		To City				
		A	**B**	**C**	**D**	**E**
From city	**A**	**–**	**375**	**600**	**150**	**190**
	B	**375**	**–**	**300**	**350**	**175**
	C	**600**	**300**	**–**	**350**	**500**
	D	**160**	**350**	**350**	**–**	**300**
	E	**190**	**175**	**500**	**300**	**–**

Suggest an optimal assignment and the total maximum sales increase per month. If for certain reasons sales representative 'B' cannot be assigned to sales territory III, will the optimal assignment schedule be different ? If so find that schedule and the effect on total sales.

Sol : **(March - 16)**

The given cost matrix of travelling salesman problem is,

		To City				
		A	**B**	**C**	**D**	**E**
From city	A	–	375	600	150	190
	B	375	–	300	350	175
	C	600	300	–	350	500
	D	160	350	350	–	300
	E	190	175	500	300	–

i) Row Reduction

Subtract the smallest cost element of each row from all the elements of the corresponding row, we get,

City	A	B	C	D	E
A	–	225	450	0	40
B	200	–	125	175	0
C	300	0	–	50	200
D	0	190	190	–	140
E	15	0	325	125	–

ii) Column Reduction

Subtract the smallest cost element of each column from all the elements of corresponding column, we get,

City	A	B	C	D	E
A	–	225	325	0	40
B	200	–	0	175	0
C	300	0	–	50	200
D	0	190	65	–	140
E	15	0	200	125	–

iii) Assignment

Now make the assignments in rows and columns having single zero and cross off remaining zero and cross off remaining zero elements in the assigned row / column.

City	A	B	C	D	E
A	–	225	325	$\boxed{0}$	40
B	200	–	$\boxed{0}$	175	$\cancel{0}$
C	300	$\boxed{0}$	–	50	200
D	$\boxed{0}$	190	65	–	140
E	15	$\cancel{0}$	200	125	–

Since, the number of lines or assignments ≠ Number of rows/columns. Therefore, the current solution is not optimal .

Now, subtract the smallest uncovered cost element (i.e., 15) from all the uncovered element and add the same to the intersection elements.

City	A	B	C	D	E
A	–	240	325	$\boxed{0}$	40
B	200	–	$\boxed{0}$	175	$\cancel{0}$
C	285	$\boxed{0}$	–	35	185
D	$\boxed{0}$	205	65	–	140
E	$\cancel{0}$	$\cancel{0}$	85	110	–

Since, No. of lines/assignments ≠ No. of rows or columns. Therefore, continue the procedure until each row and column contains exactly one encircled zero.

City	A	B	C	D	E
A	–	275	325	$\boxed{0}$	40
B	235	–	$\boxed{0}$	175	$\cancel{0}$
C	285	$\boxed{0}$	–	$\cancel{0}$	150
D	$\boxed{0}$	205	30	–	105
E	$\cancel{0}$	$\cancel{0}$	150	75	–

City	A	B	C	D	E
A	–	305	325	$\boxed{0}$	40
B	235	–	$\cancel{0}$	175	$\boxed{0}$
C	285	$\boxed{0}$	–	$\cancel{0}$	120
D	$\cancel{0}$	205	$\boxed{0}$	–	75
E	$\boxed{0}$	$\cancel{0}$	120	45	–

∴ The optimum assignment schedule is,

A - D	150
B - E	175
C - B	300
D - C	350
E - A	190
	1,165 km

3.6 TRANSPORTATION PROBLEM

Q10. Define Transportation Problem.

Ans :

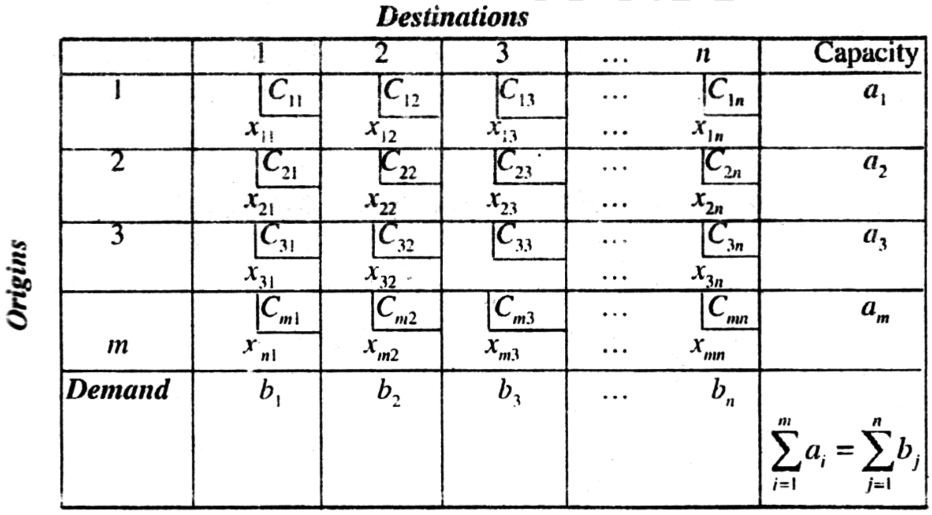
Transportation problem is another case of application to linear programming problems, where some physical distribution (transportation) of resources is to be made from one place to another to meet certain set of requirements with in the given availability. The places from where the resources are to be transferred are referred to as sources or origins. These sources or origins will have the availability or capacity or supply of resources. The other side of this transportation i.e., to where the resources are transported are called sinks or destinations such as market enters, godowns etc. These will have certain requirements or demand.

3.6.1 Mathematical Model of Transportation Problem

Q11. Explain the mathematical formulation of transportation.

Ans : **(May-19)**

Consider a transportation problem with m origins (rows) and *n* destinations (columns). Let C_{ij} be the cost of transporting one unit of the product from the ith origin to jth destination. a_i the quantity of commodity available at origin i, b_j the quantity of commodity needed at destination j. x_{ij} is the quantity transported from ith origin to jth destination. The above transportation problem can be stated in the above tabular form.



Destinations

Origins

	1	2	3	... n	Capacity
1	C_{11} x_{11}	C_{12} x_{12}	C_{13} x_{13}	... C_{1n} ... x_{1n}	a_1
2	C_{21} x_{21}	C_{22} x_{22}	C_{23} x_{23}	... C_{2n} ... x_{2n}	a_2
3	C_{31} x_{31}	C_{32} x_{32}	C_{33}	... C_{3n} ... x_{3n}	a_3
m	C_{m1} x_{n1}	C_{m2} x_{m2}	C_{m3} x_{m3}	... C_{mn} ... x_{mn}	a_m
Demand	b_1	b_2	b_3	... b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

The Linear programming model representing the transportation problem is given by

$$\text{Minimize } Z=\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1, 2, ... n$$

(Row Sum)

$$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1,2, ...n$$

(Column Sum)

$x_{ij} \geq 0$ for all i and j

The given transportation problem is said to be balanced if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} bj$$

i.e., if the total supply is equal to the total demand.

Q12. Explain the basic terminology are used in transportation problem.

Ans :

i) Feasible Solution

Any set of non negative allocations (x_{ij} > Q) which satisfies the row and column sum is called a feasible solution.

ii) Basic Feasible Solution

A feasible solution is called a basic feasible solution if the number of non negative allocations is equal to *m* + *n* – 1 where m is the number of rows, *n* the number of columns in a transportation table.

iii) Non-degenerate Basic Feasible Solution

Any feasible solution to a transportation problem containing in origins and n destina-tions is said to be non-degenerate, if it contains m + n – 1 occupied cells and each allocation is in independent positions.

The allocations are said to be in independent positions, if it is impossible to form a closed path.

Closed path means by allowing horizontal and vertical lines and all the corner cells are occupied.

The allocations in the following tables are not in independent positions.

	*	*
	*	*

*		*
*		*

	*	*	
	*		
	*	*	

The allocations in the following tables are in independent positions.

	*	
*	*	*
*		

*	*		
	*		*
		*	*

iv) Degenerate Basic Feasible Solution

If a basic feasible solution contains less than m + n – 1 non negative allocations, it is said to be degenerate.

Q13. State the assumptions of transportation problem.

Ans :

Transportation model is based on following assumptions,

1. It is assumed that quantity available at sources is equal to the quantity required at destinations.
2. Items are easily transported from source to destination.
3. Transportation cost per unit from source to destination is well known.
4. The main aim is to reduce total transportation cost for the whole organization.
5. There is direct relationship between transportation cost of specific route and number of units shipped for that specific route.

3.7 METHODS FOR FINDING INITIAL FEASIBLE SOLUTION (NWCR)

Q14. What are the different methods of Finding Initial Feasible Solution ?

Ans :

Optimal solution is a feasible solution (not necessarily basic) which minimizes the total cost.

The solution of a transportation problem can be obtained in two stages, namely initial solution and optimum solution.

Initial solution can be obtained by using any one of the three methods viz,

(i) North West Corner Rule (NWCR)

(ii) Least Cost Method (or) Matrix Minima Method (LCM)

(iii) Vogel's Approximation Method (VAM).

3.7.1 Northwest Corner Method

Q15. Define Northwest Corner Method ?

Ans :

Step 1

Starting with the cell at the upper left corner (North west) of the transportation matrix we allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied ie. $X_{11} = \min (a_1 b_1)$.

Step 2

If $b_l > a_1$ we move down vertically to the second row and make the second allocation of magnitude $x_{22} = \min(a_2\ b_l - x_{11})$ in the cell (2,1).

If $b_1 < a_1$ move right horizontally to the second column and make the second allocation of magnitude $X_{12} = \min (a_1\ X_{11} - b_1)$ in the cell (1, 2).

If $b_1 = a_1$ there is a tie for the second allocation. We make the second allocations of magnitude

$x_{12} = \min (a_1 - a_1, b_1) = 0$ in cell (1,2)

or $x_{21} = \min (a_2, b_1 - b_1) = 0$ in the cell (2,1)

Step 3

Repeat steps 1 and 2 moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

PROBLEMS

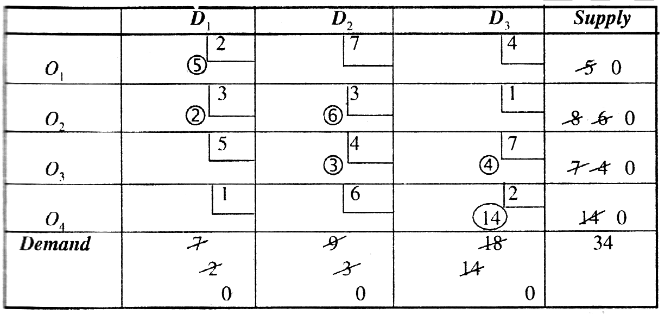
10. Obtain the initial basic feasible solution of a transportation problem whose cost and rim requirement table is given below.

Origin /Destination	D_1	D_2	D_3	Supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	34

Sol :

Since Σai = 34 = Σbj there exists a feasible solution to the transportation problem. We obtain initial feasible solution as follows.

The first allocation is made in the cell (1, 1) the magnitude being x_{11} = min (5, 7) = 5. The second allocation is made in the cell (2, 1) and the magnitude of the allocation is given by x_{21} = min (8, 7–5) = 2.



	D_1	D_2	D_3	*Supply*
O_1	2 ⑤	7	4	~~5~~ 0
O_2	3 ②	3 ⑥	1	~~8~~ ~~6~~ 0
O_3	5	4 ③	7 ④	~~7~~ ~~4~~ 0
O_4	1	6	2 ⑭	~~14~~ 0
Demand	~~7~~ ~~2~~ 0	~~9~~ ~~3~~ 0	~~18~~ ~~14~~ 0	34

The third allocation is made in the cell (2, 2) the magnitude x_{22} = min (8 –2, 9) = 6.

The magnitude of fourth allocation is made in the cell (3, 2) given by min (7, 9 – 6) = 3.

The fifth allocation is made in the cell (3, 3) with magnitude x_{33} = min (7 – 3, 14) = 4.

The final allocation is made in the cell (4, 3) with magnitude x_{43} = min (14, 18 – 4) = 14.

Hence we get the initial basic feasible solution to the given T.P. and is given by

$X_{11} = 5; X_{21} = 2; X_{22} = 6; X_{32} = 3; X_{33} = 4; X_{43} = 14$

Total cost = 2 × 5 + 3 × 2 + 3 × 6 + 3 × 4 + 4 × 7 + 2 × 14

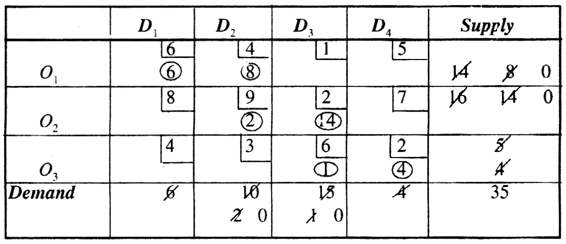
= 10 + 6 + 18 + 12 + 28 + 28 = Rs 102.

11. Determine an initial basic feasible solution to the following transportation problem using N, W, C, R.

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Required	6	10	15	4	35

Sol :

The problem is a balanced TP as the total supply is equal to the total demand. Using the steps involved R we We find the initial basic feasible solution as given in the following table.



	D_1	D_2	D_3	D_4	*Supply*
O_1	6 ⑥	4 ⑧	1	5	~~14~~ ~~8~~ 0
O_2	8	9 ②	2 ⑭	7	~~16~~ ~~14~~ 0
O_3	4	3	6 ①	2 ④	~~5~~ ~~4~~
Demand	~~6~~	~~10~~ ~~2~~ 0	~~15~~ ~~1~~ 0	~~4~~	35

Solution is given by

$X_{11} = 6$; $X_{12} = 8$; $X_{22} = 2$ $X_{23} = 14$;

$X_{33} = 1$; $X_{34} = 4$

Total cost $= 6 \times 6 + 4 \times 8 + 9 \times 2$

$+ 2 \times 14 + 6 \times 1 + 2 \times 4$

= Rs. 128.

3.7.2 Least Cost Method

Q16. What is Least Cost Method ? Explain the steps to get an initial basic feasible solution by Least Cost Method.

Ans :

Least cost method is also known as lowest cost entry method or matrix minima method. To achieve the objective of minimum transportation cost, this method consider those routes (or cells) with least unit transportation cost to transport the goods. The steps of LCM are as follows,

Step 1

Determine the smallest cost in the cost matrix of the transportation table. Let it be C_{ij}. Allocate x_{ij} = min (a_i , b_j) in the cell (i, j)

Step 2

If $X_{ij} = a_i$ cross off the i^{th} row of the transportation table and decrease b_j by a_i. Then go to step 3.

If $x_{ij} = b_j$ cross off the j^{th} column of the transportation table and decrease a_i by b_j. Go to step 3.

If $x_{ij} = a_i = b_j$ cross off either the i^{th} row or the j^{th} column but not both.

Step 3

Repeat steps 1 and 2 for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

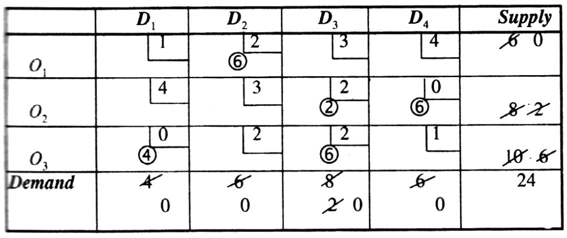
PROBLEMS

12. Obtain an Initial basic feasible solution to the following transportation problem using matrix minima method.

	D$_1$	**D$_2$**	**D$_3$**	**D$_4$**	**Supply**
O$_1$	**1**	**2**	**3**	**4**	**6**
O$_2$	**4**	**3**	**2**	**0**	**8**
O$_3$	**0**	**2**	**2**	**1**	**10**
Demand	**4**	**6**	**8**	**6**	**24**

Sol :

Since $\Sigma a_i = \Sigma b_j = 24$, there exists a feasible solution to the TP using tre steps in the least cost method, the first allocation is made in the cell (3, 1) the magnitude being $x_{31} = 4$. Which satisfies the demand at the destination $D_]$ and we delete this column from the table as it is exhausted.



	D_1	D_2	D_3	D_4	*Supply*
O_1	1	2 ⑥	3	4	~~6~~ 0
O_2	4	3	2 ②	0 ⑥	~~8~~ ~~2~~
O_3	0 ④	2	2 ⑥	1	~~10~~ ~~6~~
Demand	~~4~~ 0	~~6~~ 0	~~8~~ ~~2~~ 0	~~6~~ 0	24

The second allocation is made in the cell (2, 4). with magnitude x_{24}= min (6, 8) = 6. Since it satisfies the demand at the destination D_4.

It is deleted from the table. From the reduced table the third allocation is made in the cell (3, 3) with magnitude x_{33} = min (8, 6) = 6. The next allocation is made in the cell (2, 3) with magnitude x_{23} of min (2, 2) = 2.

Finally the allocation is made in the cell (1, 2) with magnitude x_{12} = min (6,6) = 6. Now all the rim requirements have been satisfied and hence, initial feasible solution is obtained.

The solution is given by

$x_{12} = 6, x_{23} = 2, X_{24} = 6, X_{31} = 4, X_{33} = 6$

Since the total number of occupied cell = 5 < m + n – 1.

We get a degenerate solution.

Total cost = $6 \times 2 + 2 \times 2 + 6 \times 0 + 4 \times 0 + 6 \times 2$

$= 12 + 4 + 12 =$ Rs 28.

---

13. Determine an Initial basic feasible solution for the following TP, using least cost method.

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	35

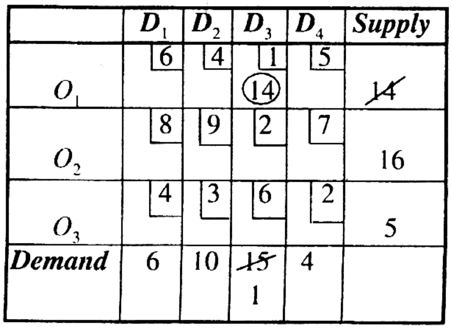
Sol :

Since $\Sigma a_i = \Sigma b_j$, there exists a basic feasible solution. Using the steps in least cost method we make the first allocation to the cell (1,3) with magnitude X_{l3} = min (14, 15) = 14. (as it is the cell having the least cost).

This allocation exhaust the first row supply. Hence, the first row is deleted. From the reduced table, the next allocation is made in the next least cost cell (2, 3) which is chosen arbitrarily with magnitude x_{23} = Min (1, 16) = 1. Which exhausts the 3rd column destination.

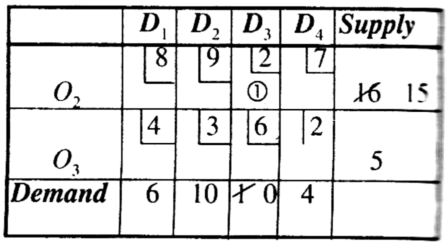
From the reduced table the next least cost cell is (3,4) which allocation > made with magnitude min (4, 5) = 4. Which exhausts the destination D_4 requirement. Delete this fourth column from the table. The next allocation is made in the cell (3,2) with magnitude X_{32} = Min (1, 10) = 1 Which exhausts the 3rd origin capacity. Hence, the 3rd row is exhausted. From the reduced table the next allocation is given to the cell (2,1) with magnitude X_{21} = min (6, 15) = 6. Which exhaust the first column require-ment. Hence it is deleted from the table.

Finally the allocation is made to the cell (2, 2) with magnitude X_{22} = Min (9, 9) = 9 which satisfies the rim requirement. These allocation are shown in the transportation table as follows.



	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1 ⑭	5	~~14~~
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	~~15~~ 1	4	

(I allocation)



	D_1	D_2	D_3	D_4	Supply
O_2	8	9	2 ①	7	~~16~~ 15
O_3	4	3	6	2	5
Demand	6	10	~~1~~ 0	4	

(II allocation)

	D_1	D_2	D_4	Supply
O_2	8	9	7	15
O_3	4	3	2 ④	~~5~~ 1
Demand	6	10	~~4~~ 0	

(III allocation)

	D_1	D_2	Supply
O_2	8	9	15
O_3	4	3 ①	~~1~~ 0
Demand	6	~~10~~ 9	

(IV allocation)

	D_1	D_2	Supply
O_2	8 ⑥	9 ⑨	15
Demand	6	9	

(V, VI allocation)

The following table gives the initial basic feasible solution.

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1 ⑭	5	14
O_2	8 ⑥	9 ⑨	2 ①	7	16
O_3	4	3 ①	6	2 ④	5
Demand	6	10	15	4	

Solution is given by

$X_{13} = 14; X_{21} = 6; X_{22} = 9; X_{23} = 1; X_{32} = 1; X_{34} = 4$

Transportation cost

$= 14 \times 1 + 6 \times 8 + 9 \times 9 + 1 \times 2 + 3 \times 1 + 4 \times 2$

= Rs 156.

3.7.3 Vogels Approximation Method

Q17. What is Vogels Approximation Method (VAM)? Explain the steps to get an initial basic feasible solution by Vogels Approxi-mation Method

Ans :

VAM or Vogel's Approximation Method is also known as penalty or regret method. It is basically a heuristic method. Allocation is done on the basis of opportunity cost (penalty) that would have incurred if allocation in certain cells with minimum costs were missed.

The steps involved in this method for finding the initial solution are as follows.

Step 1

Find the penalty cost, namely the difference between the smallest and next smallest costs in each row and column.

Step 2

Among the penalties as found in step (1) choose the maximum penalty. If this maximum penalty is more than one (i.e if there is a tie) choose any one arbitrarily.

Step 3

In the selected row or column as by step (2) find out the cell having the least cost. Allocate to this cell as much as possible depending on the capacity and requirements.

Step 4

Delete the row or column which is fully exhausted. Again compute the column and row penalties for the reduced transportation table and then go to step (2) Repeat the procedure until all the rim requirements are satisfied.

Note If the column is exhausted, then there is a change in row penalty and vice versa.

PROBLEMS

14. Find the initial basic feasible solution for the following transportation problem by VAM.

	Destination					
Origin		D_1	D_2	D_3	D_4	*Supply*
	O_1	11	13	17	14	250
	O_2	16	18	14	10	300
	O_3	21	24	13	10	400
	Demand	200	225	275	250	950

Sol :

Since $\Sigma a_i = \Sigma b_j = 950$ the problem is balanced and there exists a feasible solution to the problem.

First we find the row & column penalty P_1 as the difference between the least and next least cost. The maximum penalty is 5. Choose the first column arbitrarily. In this column choose the cell having the least cost name (1, 1). Allocate to this cell with minimum magnitude (i.e.(250,200) = 200.) This exhausts the first column. Delete this column. Since column is deleted, then there is a change in row penalty P_{II} and column penalty P_{II} remains the same. Continuing in this manner we get the remaining allocations as given in the table below.

I allocation

	D_1	D_2	D_3	D_4	*Supply*	P_I
O_1	11 (200)	13	17	14	50 ~~250~~	2
O_2	16	18	14	10	300	4
O_3	21	24	13	10	400	3
Demand	~~200~~ 0	225	275	250		
P_I	5↑	5	3	0		

II allocation

	D_2	D_3	D_4	*Supply*	P_{II}
O_1	13 (50)	17	14	~~50~~0	3
O_2	18	14	10	300	4
O_3	24	13	10	400	3
Demand	~~225~~ 175	275	250		
P_{II}	5↑	3	0		

III allocation

	D_2	D_3	D_4	*Supply*	P_{III}
O_2	$\boxed{18}$ 175	$\boxed{14}$	$\boxed{10}$	~~300~~ 125	4
O_3	$\boxed{24}$	$\boxed{13}$	$\boxed{10}$	400	3
Demand	~~175~~ 0	275	250		
P_{III}	6↑	1	0		

IV allocation

	D_3	D_4	*Supply*	P_{IV}
O_2	$\boxed{14}$	$\boxed{10}$ 125	~~125~~ 0	4 ←
O_3	$\boxed{13}$	$\boxed{10}$	400	3
Demand	275	~~250~~ 125		
P_{IV}	1	0		

V allocation

	D_3	D_4	*Supply*	P_V
O_3	$\boxed{13}$ 275	$\boxed{10}$	~~400~~ 125	3
Demand	~~275~~ 0	125		
P_V	13↑	10		

VI allocation

	D_4	*Supply*	P_{VI}
O_3	10 125	~~125~~ 0	10 ←
Demand	~~125~~ 0		
P_{VI}	10		

Finally we arrive at the initial basic feasible solution which is shown in the following table.

	D_1	D_2	D_3	D_4	*Supply*
O_1	$\boxed{11}$ 200	$\boxed{13}$ 50	$\boxed{17}$	$\boxed{14}$	250
O_2	$\boxed{16}$	$\boxed{18}$ 175	$\boxed{14}$	$\boxed{10}$ 125	300
O_3	$\boxed{21}$	$\boxed{24}$	$\boxed{13}$ 275	$\boxed{10}$ 125	400
Demand	200	225	275	250	

There are 6 positive independent allocations which equals to $m + n - 1 = 3 + 4 - 1$. This ensures that the solution is a non- degenerate basic feasible solution.

$\therefore$ The transportation cost

$= 11 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125 + 13 \times 275 + 10 \times 125$

$=$ Rs 12,075.

15. Find the initial solution to the following TP using VAM.

		Destination				
		D_1	D_2	D_3	D_4	Supply
Factory	F_1	3	3	4	1	100
	F_2	4	2	4	2	125
	F_3	1	5	3	2	75
	Demand	120	80	75	25	300

Sol :

Since $\Sigma a_i = \Sigma b_j$ the problem is a balance TP $\therefore$ there exists a feasible solution.

	D_1	D_2	D_3	D_4	Supply	P_I	P_{II}	P_{III}	P_{IV}	P_V	P_{VI}
F_1	3 ㊺	3	4 ㉚	1 ㉕	100	2	2 ←	0	1 ←	4	4
F_2	4	2 (80)	4 ㊺	2	125	0	0	2 ←	0	4 ←	–
F_3	1 (75)	5	3	2	75	1	–	– ←	–	–	–
Demand	120	80	75	25							
P_I	2↑	1	1	1							
P_{II}	1	1	0	1							
P_{III}	1	1	0	–							
P_{IV}	1	–	0	–							
P_V	–	–	0	–							
P_{VI}	–	–	4↑	–							

Finally we have the initial basic feasible solution as given in the following table.

	D_1	D_2	D_3	D_4	Supply
F_1	3 ㊺	3	4 ㉚	1 ㉕	100
F_2	4	2 (80)	4 ㊺	2	125
F_3	1 (75)	5	3	2	75
Demand	120	80	75	25	

There are 6 independent non-negative allocations equal to m + n – 1= 3 + 4 – 1= 6. This ensures that the solution is non - degenerate basic feasible .

$\therefore$ The transportation cost

$= 3 \times 45 + 4 \times 30 + 1 \times 25 + 2 \times 80 + 4 \times 45 + 1 \times 75$

$= 135 + 120 + 25 + 160 + 180 + 75 =$ Rs. 695

3.8 TEST OF OPTIMALITY BY MODI METHOD

Q18. Explain the Modi Method for obtaining optimal solution.

Ans :

Once the initial basic feasible solution has been computed, the next step in the problem is to determine whether the solution obtained is optimum or not.

Optimality test can be conducted to any initial basic feasible solution of a TP provided such allocations has exactly m + n – 1 non-negative allocations. Where *m* is the number of origins and *n* is the number of destinations. Also these allocations must be in independent positions.

To perform this optimality test, we shall discuss the modified distribution method (MODI). The various steps involved in MODI method for performing optimality test are given below.

MODI Method

Step 1

Find the initial basic feasible solution of a TP by using any one of the three methods.

Step 2

Find out a set of numbers u_i and v_j for each row and column satisfying $u_i + v_j = c_{ij}$ for each occupied cell. To start with we assign a number '0' to any row of column having maximum number of allocations. If this maximum number of allocations is more than one, choose any one arbitrarily.

Step 3

For each empty (unoccupied) cell, we find the sum u_i and v_j written in the bottom left comer of that cell.

Step 4

Find out for each empty cell the net evaluation value $\Delta_{ij} = c_{ij} - (u_i + v_j)$ and which is written at the bottom right corner of that cell. This step gives the optimality conclusion,

(i) If all $\Delta_{ij} > 0$ (ie. all the net evaluation value) the solution is optimum and a unique solution exists.

(ii) If $\Delta_{ij} \geq 0$ then the solution is optimum, but an alternate solution exists.

(iii) If atleast one $\Delta_{ij} < 0,$ the solution is not optimum. In this case we go to the next step, to improve the total transportation cost.

Step 5

Select the empty cell having the most negative value of Δ_{ij}. From this cell we draw a closed path by drawing horizontal and vertical lines with the corner cells occupied. Assign sign + and – alternately and find the minimum allocation from the cell having negative sign. This allocation should be added to the allocation having positive sign and subtracted from the allocation having negative sign.

Step 6

The above step yield a better solution by making one (or more) occupied cell as empty and one empty cell as occupied. For this new set of basic feasible allocations repeat from the step(2) till an optimum basic feasible solution is obtained.

PROBLEMS

16. Solve the following transportation problem.

		Destination				
		P	*Q*	*R*	*S*	*Supply*
Source	*A*	21	16	25	13	11
	B	17	18	14	23	13
	C	32	17	18	41	19
	Demand	6	10	12	15	43

Sol :

Origin\Dest	*P*	*Q*	*R*	*S*	*Supply*	P_I	P_{II}	P_{III}	P_{IV}	P_V	P_{VI}
A	21	16	25	13 ⑪	11	3	–	–	–	–	–
B	17 ⑥	18	14 ③	23 ④	13	4	4	4	4 ←	–	–
C	32	17 ⑩	18 ⑨	48	19	1	1	1	1	1	17
Demand	6	10	12	15	43						
P_I	4	1	4	10↑							
P_{II}	15	1	4	18↑							
P_{III}	15↑	1	4	–							
P_{IV}	–	1	4	–							
P_V	–	17	18↑	–							
P_{VI}	–	17↑	–	–							

We first find the initial basic feasible solution by using VAM. Since $\Sigma a_i = \Sigma b_j$ the given TP is a balanced one. Therefore, there exists a feasible solution.

Finally we have the initial basic feasible solution as given in the following table.

		Destination			
		P	*Q*	*R*	*S*
Source	*A*	21	16	23	13 ⑪
	B	17 ⑥	18	14 ③	23 ④
	C	32	17 ⑩	18 ⑨	41

From this table we see that the number of non-negative independent: allocation is,

$$6 = m + n - 1$$

$$= 3 + 4 - 1.$$

Hence, the solution is non-degenerate basic feasible.

The initial transportation cost

$$= 11 \times 13 + 3 \times 14 + 4 \times 23 + 6 \times 17 + 17 \times 10 + 18 \times 9$$

$$= \text{Rs } 711.$$

To Find the Optimal Solution

We apply Modi method in order to determine the optimum solution. We determine a set of numbers a and u_i and v_j for each row and column, with $u_i + v_j = c_{ij}$ for each occupied cell. To start with we give $u_2 = 0$ as the 2^{nd} row has the maximum number of allocation.

$c_{21} = u_2 + v_1 = 17 = 0 + v_1 = 17 \Rightarrow v_1 = 17$

$c_{23} = u_2 + v_3 = 14 = 0 + v_3 = 14 \Rightarrow v_3 = 14$

$c_{24} = u_2 + v_4 = 23 = 0 + v_4 = 23 \Rightarrow v_4 = 23$

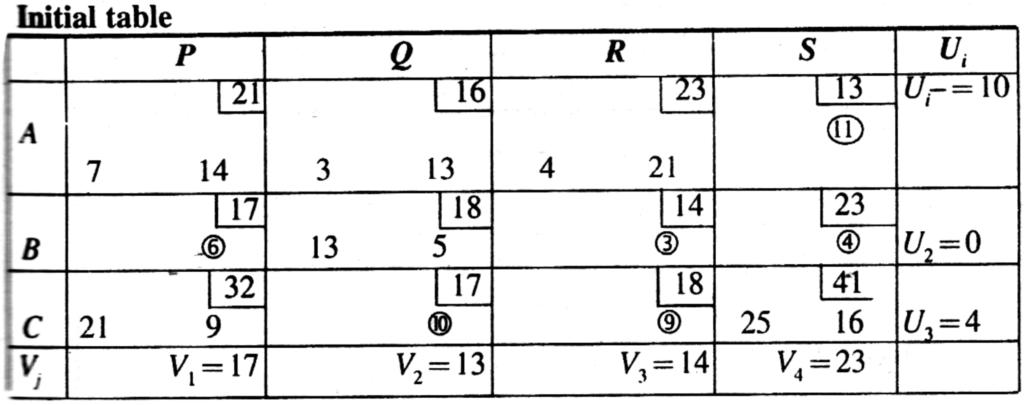
$c_{14} = u_1 + v_4 = 13 = u_1 + 23 = 13 \Rightarrow u_1 = 0$

$c_{33} = u_3 + v_3 = 18 = u_3 + 14 = 18 \Rightarrow u_3 = 4$

Now we find the sum u_i and v_j for each empty cell and enter at the bottom left corner of that cell.

Next we find the net evaluations $\Delta_{ji} = C_{ij} - \left(u_i + v_j\right)$ for each unoccupied cell and enter at the bottom right corner of that cell.

Initial table



	P	*Q*	*R*	*S*	U_i
A	21 7 14	16 3 13	23 4 21	13 ⑪	$U_i = 10$
B	17 ⑥	18 13 5	14 ③	23 ④	$U_2 = 0$
C	32 21 9	17 ⑩	18 ⑨	41 25 16	$U_3 = 4$
V_j	$V_1 = 17$	$V_2 = 13$	$V_3 = 14$	$V_4 = 23$	

Since all $\Delta_{ij} > 0$ the solution is optimal and unique. The optimum solution is given by

$$x_{14} = 11, x_{21} = 6, x_{23} = 3, x_{24} = 4, x_{32} = 10, x_{33} = 9$$

The min. transportation cost

$$= 11 \times 13 + 17 \times 6 + 3 \times 14 + 4 \times 23 + 10 \times 17 + 9 \times 18$$

$$= \text{Rs } 711$$

17. A cement company has three factories which manufacture cement which is then transported to four distribution centres. The quantity of monthly production of each factory, the demand of each distribution centre and the associated transportation cost per quintal are given below :

Factory	Distribution Centres				Monthly Production
	W	X	Y	Z	(quintals)
A	10	8	5	4	7,000
B	7	9	15	8	8,000
C	6	10	14	8	10,000
Monthly demand (in quintals)	6,000	6,000	8,000	5,000	25,000

(a) Suggest the optimum transportation schedule.

(b) Is there any other transportation schedule which is equally attractive ? If so, write that.

(c) If the company wants that atleast 5,000 quintals of cement are transported from factory C to distribution centre Y, will the transportation schedule be any different ? If so, what will be the new optimum schedule and the effect on cost?

Sol : **(Aug. - 17)**

(a) The initial feasible schedule using VAM method is as follows,

	Distribution Centres				Monthly Production (quintals)	Penalties					
Factory	W	X	Y	Z		RP$_1$	RP$_2$	RP$_3$	RP$_4$	RP$_5$	
A	10	8	$\boxed{7000}$ 5	4	~~7000~~	1	-	-	-	-	-
B	7	$\boxed{6000}$ 9	$\boxed{1000}$ 15	$\boxed{1000}$ 8	~~1000~~ ~~7000~~ 8000	1	1	1	1	1	8
C	$\boxed{6000}$ 6	10	14	$\boxed{4000}$ 8	4000 ~~10,000~~	2	2 ←	2 ←	-	-	-
Monthly demand (in quintals)	~~6000~~	~~6000~~	1000 ~~8000~~	1000 ~~5000~~							
CP$_1$	1	1	$\boxed{9}$ ↑	4							
CP$_2$	1	1	1	0							
CP$_3$	-	1	1	0							
CP$_4$	-	9	15 ↑	8							
CP$_5$	-	9 ↑	-	8							
CP$_6$	-	-	-	8							

Number of allocation = 6

M + n –1 = 4 + 3 – 1

= 7 –1= 6

∴ Total transportation cost

= (5 × 7000) + (9 × 6000) + (15 ×1000) + (8 ×1000) + (8 × 1000) + (6 × 6000)+ (8 × 4000)

= 35000 + 54000 + 15000 + 8000 + 36000 + 32000

= 1,80,000

Optimality Check through Modi Method Using Occupied Cells

$C_{ij} = u_i + v_j$; Assume $u_1 = 0$

$C_{13} = u_1 + v_3 = 5$; $0 + v_3 = 5$; $v_3 = 5$

$C_{23} = u_2 + v_3 = 15$; $u_2 + 5 = 15$; $u_2 = 10$

$C_{22} = u_2 + v_2 = 9$; $10 + u_2 = 9$; $v_2 = -1$

$C_{24} = u_2 + v_4 = 8$; $10 + v_4 = 8$; $v_4 = -2$

$C_{34} = u_3 + v_4 = 8$; $u_3 - 2 = 8$; $u_3 = 10$

$C_{31} = u_3 + v_1 = 6$; $10 + v_1 = 6$; $v_1 = -4$

Determining opportunity cost for unoccupied cells using the equations as,

$$\Delta_{ij} = C_{ij} - \left(u_j + v_j\right)$$

$\Delta_{11} = 10 - (0 - 4) = 10 + 4 = 14$

$\Delta_{12} = 8 - (0 - 1) = 8 - 1 = 9$

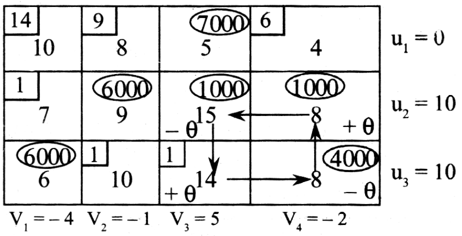
$\Delta_{14} = 4 - (0 - 2) = 4 + 2 = 6$

$\Delta_{21} = 7 - (10 - 4) = 7 - 6 = 1$

$\Delta_{32} = 10 - (10 - 1) = 10 - 9 = 1$

$\Delta_{33} = 14 - (10 + 5) = 14 - 15 = -1$

Since, we have a negative value in cell (3,3), the solution is not optimum. Construct a loop from cell (3,3) with ‘+’sign.



14 10	9 8	7000 5	6 4	$u_1 = 0$
1 7	6000 9	1000 15 $-\theta$	1000 8 $+\theta$	$u_2 = 10$
6000 6	1 10	1 14 $+\theta$	4000 8 $-\theta$	$u_3 = 10$
$V_1 = -4$	$V_2 = -1$	$V_3 = 5$	$V_4 = -2$	

θ = Min (allocations with – θ assignments)

θ = Min (4,000, 1000)

θ = 1000

The transportation table with modified allocations are as follows, The minimum negative allocation is 1000. Add this to cells with ' + ' sign and subtract from cells with '–' sign as follows,

(2,3) : 1000 – 1000 = 0

(2,4) : 1000 + 1000 = 2000

(3,3) : 0 + 1000 = 1000

(3,4) : 4000 – 1000 = 3000

10	8	7000 5	4	$u_1 = 0$
7	6000 9	15	2000 8	$u_2 = 9$
6000 6	10	1000 14	3000 8	$u_3 = 9$
$v_1 = -3$	$v_2 = 0$	$v_3 = 5$	$v_4 = -1$	

Occupied Cells

Assume $u_1 = 0$

$u_1 + v_3 = 5; 0 + v_3 = 5; v_3 = 5$

$u_3 + v_3 = 14; u_3 + 5 = 14; u_3 = 9$

$u_3 + v_4 = 8; 9 + v_4 = 8; v_4 = -1$

$u_2 + v_2 = 9; 9 + v_2 = 9; v_2 = 0$

$u_3 + v_1 = 6; 9 + v_1 = 6; v_1 = -3$

Unoccupied Cells

$\Delta_{ij} = C_{ij} - (u_i + v_j)$

$\Delta_{11} = 10 - (0 - 3) = 10 + 3 = 13$

$\Delta_{12} = 8 - (0 + 0) = 8$

$\Delta_{14} = 4 - (0 - 1) = 4 + 1 = 5$

$\Delta_{21} = 7 - (9 - 3) = 7 - 6 = 1$

$\Delta_{23} = 15 - (9 + 5) = 15 - 14 = 1$

$\Delta_{32} = 10 - (9 - 0) = 10 - 9 = 1$

Since there are no negative values in unoccupied cells, the solution is optimal,

Total transportation cost,

$= (5 \times 7000) + (9 \times 6000) + (8 \times 2000) + (6 \times 6000) + (14 \times 1000) + (8 \times 3000)$

= ` 1,79,000/-

(b) The above solution is unique there is no other transportation schedule which is equality attractive.

(c) 5000 quintals of cement are to be transported from c to y. The balance supply from factory c will be 5, 000 quintals and balance demand at y will be 3000 quintals as shown below.

Factory	**Distribution Centres**				**Monthly Production (Quintals)**
	W	**X**	**Y**	**Z**	
A	10	8	$\boxed{3000}$ 5	$\boxed{4000}$ 4	~~4000~~ ~~7000~~
B	$\boxed{1000}$ 7	$\boxed{6000}$ 9	15	$\boxed{1000}$ 8	~~1000~~ ~~2000~~ ~~8000~~
C	$\boxed{5000}$ 6	10	14	8	~~5000~~
	~~6000~~ ~~1000~~	~~6000~~	~~3000~~	~~5000~~ 1000	20000

Penalties (RP)

RP$_1$	**RP$_2$**	**RP$_3$**	**RP$_4$**	**RP$_5$**	**RP$_6$**
1	←4	–	–	–	–
1	1	1	1	1	←7
2	2	←2	–	–	–

Penalty (CP)				
CP$_1$	1	1	9↑	4
CP$_2$	1	1	–	4
CP$_3$	1	1	–	0
CP$_4$	7	1	–	8
CP$_5$	7	9↑	–	8↑
CP$_6$	7↑	–	–	–

Check for Optimality Using MODI Method

$= m + n - 1$

$= 3 + 4 - 1$

$= 7 - 1$

$= 6$

Determining Opportunity Cost for Occupied Cells Using $C_{ij} = u_i + v_j$, Assume $u_1 = 0$

$C_{13} = u_1 + v_3 = 0 + 5 \Rightarrow v_3 = 5$

$C_{14} = u_1 + v_4 = 0 + 4 \Rightarrow v_4 = 4$

$C_{24} = u_2 + v_4 = u_2 + 4 \Rightarrow u_2 = 4$

$C_{21} = u_2 + v_1 = 4 + v_1 = 7 \Rightarrow v_1 = 3(7-4)$

$C_{22} = u_2 + v_2 = 4 + v_2 = 9 \Rightarrow v_2 = 5(9-4)$

$C_{31} = u_3 + v_1 = u_3 + 3 \Rightarrow u_3 = 3$

For Unoccupied Cells Using the Equations as $\Delta_{ij} = C_{ij} -(u_i + v_j)$

$\Delta_{11} = C_{11} - (u_1 + v_1) \Rightarrow 10 - (0 + 3) = 7$

$\Delta_{12} = C_{12} - (u_1 + v_2) \Rightarrow 8 - (0 + 5) = 3$

$\Delta_{23} = C_{23} - (u_2 - v_3) \Rightarrow 15 - (4 + 5) = 6$

$\Delta_{32} = C_{32} - (u_3 + v_2) \Rightarrow 10 - (3 + 5) = 2$

$\Delta_{33} = C_{33} - (u_3 + v_3) \Rightarrow 14 - (3 + 5) = 6$

$\Delta_{34} = C_{34} - (u_3 - v_4) \Rightarrow 8 - (3 + 4) = 1$

10	8	$\boxed{3000}$ 5	$\boxed{4000}$ 4	$u_1 = 0$
$\boxed{1000}$ 7	$\boxed{6000}$ 9	15	$\boxed{1000}$ 8	$u_2 = 4$
$\boxed{5000}$ 6	10	$\boxed{5000}$ 14	8	$u_3 = 3$
$v_1 = 3$	$v_2 = 5$	$v_3 = 5$	$v_4 = 4$	

Since there are no negative values in unoccupied cells, the solution is optimal,

Total transportation cost,

= (5 × 3000) + (4 × 4000) + (7 ×1000) + (9 ×6000) + (8 × 1000) + (6 × 5000) + (14 × 5000) (Route C – Y)

= 15,000 + 16,000 + 7,000 + 54,000 +8,000 + 30,000 + 70,000

= 2,00,000.

3.9 Unbalanced Supply and Demand

Q19. What do you mean by Unbalanced Supply and Demand ? How can modify unbalanced TP into balanced TP.

Ans :

Unbalanced Transportation Problem

An unbalanced transportation problem refers to a situation where in the sum of supplies of all the sources does not match with the sum of the demands of all the destinations. Mathematically, it can be represented as,

$$\sum_{i=1}^{m} a_i \neq \sum_{i=1}^{n} b_j$$

When this rim condition is violated, i.e., the total supply is not equal to the total demand, the transportation problem is said to be an unbalanced one. There are two types of unbalanced TP are,

(a) Total supply exceeds total demand

(b) Total demand exceeds total supply.

Modifying Unbalanced TP to Balanced TP

(a) When total supply exceeds total demand, an additional column (a dummy demand centre) should be added to the transportation table to absorb the excess supply. The unit transportation cost for the cells in this column is set equal to zero because these products are neither made nor transported. By this way, the unbalanced TP becomes a balanced TP.

(b) When total demand exceeds total supply an additional row (a dummy supply centre) should be added to the transportation table to absorb the excess demand. The unit transportation cost of the cells in this row is set equal to zero and the unbalanced TP is modified to a balanced TP.

PROBLEMS

18. Solve the transportation problem when the unit transportation costs, demands and supplies are as given below.

	Destination					
		D_1	D_2	D_3	D_4	Supply
Origins	O_1	6	1	9	3	70
	O_2	11	5	2	8	55
	O_3	10	12	4	7	70
	Demand	85	35	50	45	

Sol :

Since the total demand $\Sigma b_j = 215$ is greater than the total supply $\Sigma a_i = 195$ the problem is an unbalanced TP.

We convert this into a balanced TP by introducing a dummy origin O_4 with cost zero and giving supply equal to 215 –195 = 20 units. Hence, we have the converted problem as follows

	Destination					
		D_1	D_2	D_3	D_4	Supply
Origins	O_1	6	1	9	3	70
	O_2	11	5	2	8	55
	O_3	10	12	4	7	70
	O_4	0	0	0	0	20
	Demand	85	35	50	45	215

As this problem is balanced there exists a feasible solution to this problem. Using VAM we get the following initial solution.

	D_1	D_2	D_3	D_4	Supply	P_I	P_{II}	P_{III}	P_{IV}	P_V	P_{VI}	P_{V}
O_1	6 (65)	1 ⑤	9	3	70	2	2	2	–	–	–	–
O_2	11	5 ㉚	2 ㉕	8	55	3	3	3	3	6 ←	–	–
O_3	10	12	4 ㉕	7 ㊺	70	3	3	3	3	3	3	4 ←
O_4	0 ⑳	0	0	0	20	0	–	–	–	–	–	–
Demand	85	35	50	45								
P_I	6↑	1	2	3								
P_{II}	4↑	4	2	3								
P_{III}	–	4↑	2	4								
P_{IV}	–	7↑	2	1								
P_V	–	–	2	1								
P_{VI}	–	–	4	7↑								
P_{VII}	–	–	4	–								

	D_1	D_2	D_3	D_4
O_1	6 65	1 5	9	3
O_2	11	5 30	2 25	8
O_3	10	12	4 25	7 45
O_4	0 20	0	0	0

There are 7 independent non- negative allocations equals to m + n – 1.

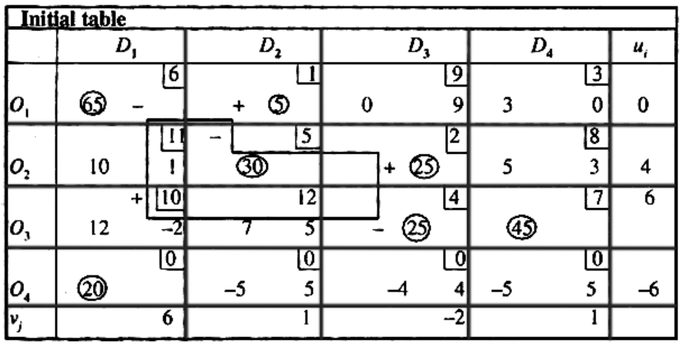
Hence, the solution is a non - degenerate one. The total transportation cost.

$= 6 \times 65 + 5 \times 1 + 5 \times 30 + 2 \times 25 + 4 \times 25 + 7 \times 45 + 20 \times 0$

$=$ Rs. 1010

To find the optimal solution

We apply the steps in MODI method to the above table.

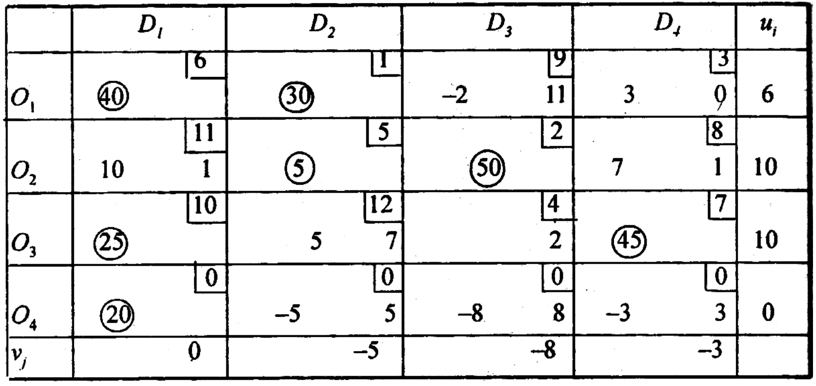


Initial table

	D_1	D_2	D_3	D_4	u_i
O_1	6 65 –	1 + 5	9 0 9	3 3 0	0
O_2	11 10 1	5 – 30	2 + 25	8 5 3	4
O_3	+ 10 12 –2	12 7 5	4 – 25	7 45	6
O_4	0 20	0 –5 5	0 –4 4	0 –5 5	–6
v_j	6	1	–2	1	

Since all $\Delta_{ij} \geq 0$ the solution is not optimum. We introduce the cell (3,1) as this cell has the most negative value of Δ_{ij}. We modify the solution by adding and subtracting the min allocation given by min (65, 30, 25). While doing this the occupied cell (3,3) becomes empty.

I Iteration table



	D_1	D_2	D_3	D_4	u_i
O_1	6 40	1 30	9 –2 11	3 3 0	6
O_2	11 10 1	5 5	2 50	8 7 1	10
O_3	10 25	12 5 7	4 2	7 45	10
O_4	0 20	0 –5 5	0 –8 8	0 –3 3	0
v_j	0	–5	–8	–3	

As the number of independent allocations are equal to m + n –1 we check the optimality.

Since all $\Delta_{ij} \geq 0$ the solution is optimal and an alternate solution exists as $\Delta_{14} = 0$ Therefore, the optimum allocation is given by X_{11} = 40, X_{12} = 30, X_{22} =5, X_{23} = 50, X_{31} = 25, X_{34} = 45, X_{41} = 20.

The optimum transportation cost is = 6 × 40 + I × 30 + 5 × 5 + 2 × 50 + 10 × 25 + 7 × 45 + 0 × 20 = Rs 960.

3.10 DEGENERACY AND ITS RESOLUTION

Q20. What is degeneracy in transportation problem ? How it is resolved ?

Ans : **(May-19)**

Degeneracy in TP

Degeneracy in TP is said to occur when the number of allocated cells is less than m + n – 1.

Where,

m = Number of rows

n = Number of columns

If degeneracy is occurred in a TP, then it is not possible to draw a closed loop for every occupied cell while solving the problem by MODI method. This degeneracy may occur during:

(a) Initial stage

(b) Testing of optimal solution.

Resolving Degeneracy TP

In order to resolve degeneracy, an artificial quantity ε with a 'o' cost is added to one or more of the unoccupied cells such that total allocated cells will be equal to m + n – 1. While placing ε in unoccupied cells, it should be considered that it has been placed in the cell containing the lowest transportation cost. After placing ε in the unoccupied cell, the problem can be solved as usual by MODI or stepping stone method. The ε will remain in the problem until degeneracy is removed or a final solution is obtained.

PROBLEMS ON DEGENERACY

19. A company has three plants at locations A, B and C which supply to warehouses located at D, E, F, G and H. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in `) are given below,

		To				
		D	E	F	G	H
From	A	5	8	6	6	3
	B	4	7	7	6	5
	C	8	4	6	6	4

Determine an optimum distribution for the company in order to minimize the total transportation cost.

OR

Solve the following transportation problem for optimal solution,

	P	Q	R	S	T	Supply
A	5	8	6	6	3	8
B	4	7	7	6	5	5
C	8	4	6	6	4	9
Demand	4	4	5	4	8	22 25

Sol :

		← Stores →					
		D	E	F	G	H	Production capacity
Factories	A	5	8	6	6	3	800
	B	4	7	7	6	5	500
	C	8	4	6	6	4	900
Requirement		400	400	500	400	800	2200 2500

As total production capacity (2200 units) ≠ total requirement (2500 units).

Hence, the given transportation problem is unbalanced. To convert into a balanced transportation problem, add a dummy factory located at *D* with production capacity 3 units and unit cost of supplying from various factories is equal to zero.

This balanced transportation problem is then solved using VAM to obtain IBFS.

Vogel's Approximation Method

	D	E	F	G	H		Penalty
A	5	8	$\boxed{5}$ 6	6	$\boxed{3}$ 3	$\cancel{8}$ $\cancel{3}$	2 2 2 3 ←3 –
B	$\boxed{4}$ 4	7	7	$\boxed{1}$ 6	5	$\cancel{5}$ $\cancel{1}$	1 1 1 1 1 1 ←1
C	8	$\boxed{4}$ 4	6	6	$\boxed{5}$ 4	$\cancel{9}$ $\cancel{5}$	0 0 0 0 2 ←2
D (Dummy)	0	0	0	$\boxed{3}$ 0	0	$\cancel{3}$	0 – – – – –
	$\cancel{4}$	$\cancel{4}$	$\cancel{5}$	$\cancel{4}$ $\cancel{1}$	$\cancel{8}$ $\cancel{5}$		
	4	4	6	6↑	3		
	4	4	6↑	0	1		
Penalty	4↑	4	–	0	1		
	–	4↑	–	0	1		
	–	–	–	0	1		
	–	–	–	0	1		

Total transportation cost = $\Sigma C_{ij} .x_{ij}$

$=(6 \times 5) + (3 \times 3) + (4 \times 4) + (6 \times 1) + (4 \times 4) + (4 \times 5) + (0 \times 3)$

= ` 97

Optimal Solution using MODI Method

Number of occupied cells = 7

$m + n - 1 = 4 + 5 - 1 = 9 - 1 = 8$

Since the number of occupied cells < m + n – 1 there is degeneracy. To resolve degeneracy, '$\in$' is added to unoccupied cells with least cost [i.e., cell (AS 1)].

Note

$\in$ is very small value but not equal to zero and

$a + \in = a - \in = a$

Compute u_i, v_j $(u_i + v_j)$ and Δ_{ij} values and check for optimality.

	D	E	F	G	H	Supply
A	(ε) 5	8	⑤ 6	6	⑤ 3	8
B	④ 4	7	7	① 6	5	5
C	3	④ 4	6	6	⑤ 4	9
D	0	0	0	③ 0	0	4
Demand	4	4	5	4	8	25 / 25

Determining u_i and v_j values for occupied cells using,

$C_{ij} = u_i + v_j$

$u_1 + v_1 = 5$

$u_1 + v_3 = 6$

$u_1 + v_5 = 3$

$u_2 + v_1 = 4$

$u_2 + v_4 = 6$

$u_3 + v_2 = 4$

$u_3 + v_5 = 4$

$u_4 + v_4 = 0$

There are 9 shadow values, but 8 equations available. Assume $u_i = 0$ and solve remaining values.

$u_1 = 0$	$v_1 = 5$
$u_2 = -1$	$v_2 = 3$
$u_3 = 1$	$v_3 = 6$
$u_4 = -7$	$v_4 = 7, v_5 = 3$

Consider unoccupied cells and find their evaluations.

$\Delta_{ij} = C_{ij} - (u_i + v_j)$

or

$\Delta_{ij} = C_{ij} - u_j - v_j$

$C_{12} - u_1 - v_2 = 8 - 0 - 3 = +5$

$C_{14} - u_1 - v_4 = 6 - 0 - 7 = -1$

$C_{22} - u_2 - v_2 = 7 + 1 - 3 = +5$

$C_{23} - u_2 - v_3 = 7 + 1 - 6 = +2$

$C_{25} - u_2 - v_5 = 5 + 1 - 3 = +3$

$C_{31} - u_3 - v_1 = 8 - 1 - 5 = +2$

$C_{33} - u_3 - v_3 = 6 - 1 - 6 = -1$

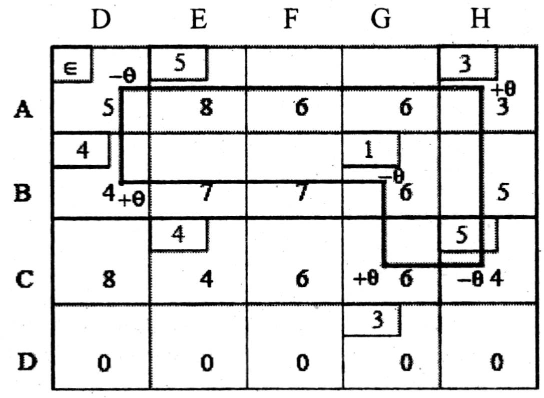
$C_{34} - u_3 - v_4 = 6 - 1 - 7 = -2$

$C_{41} - u_4 - v_1 = 0 + 7 - 5 = +2$

$C_{42} - u_4 - v_2 = 0 + 7 - 3 = +4$

$C_{43} - u_4 - v_3 = 0 + 7 - 6 = +1$

$C_{45} - u_4 - v_5 = 0 + 7 - 3 = +4$

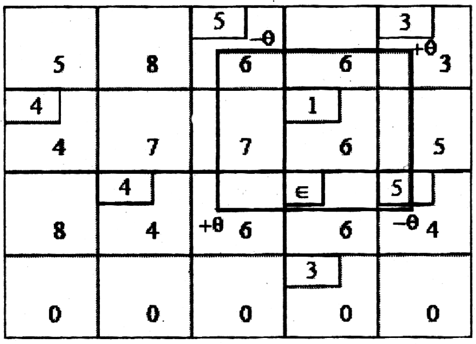


	D	E	F	G	H
A	∈ −θ 5	5 8	6	6	3 +θ 3
B	4 4 +θ	7	7	1 −θ 6	5
C	8	4 4	6	+θ 6	5 −θ 4
D	0	0	0	3 0	0

Consider the most negative all $C_{34} (\Delta_{ij} = -2)$ and use loop method to find optimum solution.

$[(\theta - \in, 500 - \theta, 100 - \theta)]$ select minimum positive value $\theta = \in$

Modified allocation table,



5	8	5 −θ 6	6	3 +θ 3
4 4	7	7	1 6	5
8	4 4	+θ 6	∈ 6	5 −θ 4
0	0	0	3 0	0

Optimality Test for Occupied Cells

$C_{ij} = u_i + v_j$ for allocated cells

Assume $u_3 = 0$

$u_i + v_3 = 6$ $\therefore$ $u_1 = -1$ $v_1 = 4$

$u_1 + v_3 = 3$ $u_2 = 0$ $v_2 = 4$

$u_2 + v_1 = 4$ $u_3 = 0$ $v_3 = 7$

$u_2 + v_4 = 6$ $u_4 = -6$ $v_4 = 6$

$u_3 + v_2 = 4$ $v_5 = 4$

$u_3 + v_4 = 6$

$u_3 + v_5 = 4$

$u_4 + v_4 = 0$

For Unoccupied Cells

$\Delta_{11} = 5 + 1 - 4 = 2$

$\Delta_{12} = 8 + 1 - 4 = 5$

$\Delta_{14} = 6 + 1 - 6 = 1$

$\Delta_{22} = 7 + 0 - 4 = 3$

$\Delta_{23} = 7 + 0 - 7 = 0$

$\Delta_{25} = 5 + 0 - 4 = 1$

$\Delta_{31} = 8 + 0 - 4 = 4$

$\Delta_{33} = 6 + 0 - 7 = -1$

$\Delta_{41} = 0 + 6 - 4 = 2$

$\Delta_{45} = 0 + 6 - 4 = 2$

Consider – ve cells & from loop from cell (3,3)

[500 – θ, 500 – θ] minimum

Modified allocation table,

5	8	$\in$ 6	6	8 3
4 4	7	7	1 6	5
8	4 4	5 6	$\in$ 6	4
0	0	0	3 0	0

As there occurs degeneracy at this stage, include one more '$\in$' at (3,1) cell to resolve it.

Check optimality for Occupied Cells

$u_1 + v_3 = 6 \quad \therefore \quad u_1 = 0 \quad v_1 = 4$

$u_1 + v_5 = 3 \quad u_2 = 0 \quad v_2 = 4$

$u_2 + v_1 = 4 \quad u_3 = 0 \quad v_3 = 6$

$u_2 + v_4 = 6 \quad u_4 = -6 \quad v_4 = 6$

$u_3 + v_2 = 4 \quad v_5 = 3$

$u_3 + v_3 = 6$

$u_3 + v_4 = 6$

$u_4 + v_4 = 0$

For Unoccupied Cells

$\Delta_{11} = 5 - 0 - 4 = 1$

$\Delta_{12} = 8 - 0 - 4 = 4$

$\Delta_{14} = 6 - 0 - 6 = 0$

$\Delta_{22} = 7 - 0 - 4 = 3$

$\Delta_{23} = 7 - 0 - 6 = 1$

$\Delta_{25} = 5 - 0 - 3 = 2$

$\Delta_{31} = 8 - 0 - 4 = 4$

$\Delta_{33} = 4 - 0 - 3 = 1$

$\Delta_{41} = 0 \ \ 6 - 4 = 2$

$\Delta_{45} = 0 + 6 - 3 = 3$

Since , all $\Delta_{ij} \geq 0$ the solution obtained here is an optimal one.

$= [(3 \times 8) + (4 \times 4) + (6 \times 1) + (4 \times 4)$
$+ (6 \times 3)] \times 10$

$= [24 + 16 + 6 + 16 + 12 + 18] \times 10$

$\therefore$ Minimum total transportation cost = 9,20

Cell	Allocation	Cost
$A \rightarrow F$	$6 \times \in$	$\in$
$A \rightarrow H$	3×8	24
$B \rightarrow D$	4×4	16
$B \rightarrow G$	6×1	6
$C \rightarrow E$	4×4	16
$C \rightarrow F$	6×5	30
$C \rightarrow G$	$6 \times \in$	$\in$
$D \rightarrow G$	0×3	0

Q21. What are the differences and similarities between Assignment and Transportation Problem.

Ans : **(Dec.-18)**

Transportation problem is sometimes referred to as Allocation Problem. Both these problems are the cases of linear programming problems with some distinctions. These are distinguished as follows :

Assignment Problem		Transportation Problem
1.	This problem is used to assign the jobs to machines or machines to mean etc.	This problem is used in transporting material from origins (like plant) to destinations (such as godown/market) etc.
2.	No availability (supply) and requirement (demand) are needed to solve the problem.	Supply & demand are needed.
3.	If number of rows is equal to no. of column the AP is said to be balanced otherwise unbalanced.	If supply is equal to demand the TP is said be balanced otherwise unbalanced.
4.	There will not be any degeneracy in this problem.	A possibility of degeneracy either at initial stage or subsequent stages may occur.
5.	This problem is solved by converting into opportunity costs (Hungarian method).	This can be solved without converting the given costs (North west corner of Vogel's approximation or (least cost). In VAM we use penalities in LCM we use least cell cost in NWC we use position of top-left in matrix.
6.	Travelling salesman problem and crew assignment are its extensions.	Trans shipment problem is its extension.

Similarities		
1.	Standard AP is to minimise the cost of time.	Standard TP is to minimise the cost.
2.	Max. problem is converted to min. problem by subtracting from highest among all or by multiplying all by (–1).	Max. problem is converted to min. problem by subtracting from highest among all or by multiplying all by (–1).
3.	Restricted assignment uses 'M', an unaffordable cost.	Restricted TP uses 'M', an unaffordable cost.

Exercise Problems

1. Solve the following transportation problem using VAM

	1	2	3	4	5	
F_1	4	2	3	2	6	8
F_2	5	4	5	2	1	12
F_3	6	5	4	7	3	14
	4	4	6	8	8	

[**Ans :** Using VAM, x_{12} = 4, x_{14} = 4, x_{24} = 4, x_{25} = 8, x_{31} = 4, x_{35} = 6, TTC = 80]

2. Determine an initial basic feasible solution using (i) Vogel's method, (ii) Row minima method, by considering the following transportation problem :

		Destination				
		1	2	3	4	Supply
Source	1	21	16	15	13	11
	2	17	18	14	23	13
	3	32	27	18	41	19
	Demand	6	10	12	15	43

[**Ans :** x_{14} = 11, x_{21} = 6, x_{22} = 3, x_{24} = 4, x_{32} = 7, x_{33} = 12, cost = Rs. 686]

3. Determine an initial basic feasible solution to the following T.P. using : (a) North-west corner rule, (b) Vogel's method.

		Destination					
		A_1	B_1	C_1	D_1	E_1	Supply
Origin	A	2	11	10	3	7	4
	B	1	4	7	2	1	8
	C	3	9	4	8	12	9
	Demand	3	3	4	5	6	21

[**Ans :** *North west corner ruel :* x_{11} = 3, x_{12} = 1, x_{22} = 2, x_{23} = 4, x_{24} = 2, x_{34} = 3,
x_{35} = 6, cost = Rs. 153
Vogel's method : x_{14} = 4, x_{22} = 2, x_{35} = 6, x_{34} = 3, x_{32} = 1, x_{33} = 4,
x_{34} = 1, cost = Rs. 68].

4. Use north west corner rule to determine an initial basic feasible solution to the following T.P. when does it have a unique solution ?

		To			
		A	B	C	Supply
From	a	2	7	4	5
	b	3	3	1	8
	c	5	4	7	7
	d	1	6	2	14
	Demand	7	9	18	34

[**Ans :** x_{11} = 5, x_{21} = 2, x_{22} = 6, x_{32} = 3, x_{33} = 4, x_{43} = 14, cost Rs. 102; Yes]

5. Solve the AP

	J_1	J_2	J_3	J_4	J_5
A	20	15	25	25	29
B	13	19	30	13	19
C	20	17	14	12	15
D	14	20	20	16	24
E	14	16	19	11	22

[Ans : A → J_2, B → J_5, C → J_3, D → J_1, E → J_4**]**

Total cost = 73

6.

	W_1	W_2	W_3	W_4	W_5
J_1	10	15	13	15	16
J_2	13	9	18	13	10
J_3	10	9	12	12	12
J_4	15	11	9	9	12
J_5	11	9	10	14	12

[Ans : J_1 → W_1, J_2 → W_5, J_3 → W_2,

J_4 → W_4, J_5 → W_3

Total cost = 48]

7. The profit per day of 4 salesmen in 4 districts are given below. Find an optimal assignment of the districts of the salesmen so as to maximize the total profit.

Districts

	1	2	3	4
A	14	8	12	9
B	12	9	13	13
C	13	13	11	10
D	11	10	12	13

[Ans : A → 1, B → 3, C → 2,

D → 4

Total profit = 53]

8. There are 5 jobs and 4 machines. The expected profits on each job on each machine is given below. Determine an optimal assignment of the machines to the jobs so that the total profit is maximum.

Job

Machines	1	2	3	4	5
I	62	78	50	101	82
II	71	84	61	73	59
III	87	92	111	71	81
IV	48	64	87	77	80

Hint : Introduce a dummy machine V with zero profit for each job. Then convert into the minimization type.

[Ans : I → 4, II → 2, III → 3, IV → 5

Job 1 remains unassigned.

Total profit = 376]

9. Solve the AP and find the minimum cost

	W_1	W_2	W_3	W_4
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22

[Ans : A → W_1, B → W_3, C → W_2

Total cost = 50]

Short Question and Answers

1. What is an Assignment Problem

Ans :

Assignment problems are special type of linear programming problems where assignees are being assigned to perform tasks. The assignment problem arises in a variety of decision-making situations. In many business situations, management finds it necessary to assign personnel to jobs, jobs to machines, machines to job locations within a plant, or sales persons to territories within the distribution area of the business, or contracts to bidders etc.

In each of these cases, the management would like to make the most effective or cost-efficient assignment of a set of workers (or objects) to a set of jobs (or assignment). The criteria used to measure the effectiveness of a particular set of assignments may be total cost, total profit or total time to perform a set of operations.

A distinguishing feature of the assignment problem is that one worker (or job) is assigned to one and only one task (or machine). Specifically, we look for the set of assignments that will optimize a stated objective such as minimise cost, minimise time or maximise profit.

Assignment problems are similar to transportation problems and can be solved by any one of the transportation algorithms. But two distinct characteristics of assignment problems make the application of the transportation algorithm inconvenient in solving them. These characteristics are:

(i) The number of rows are equal to the number of columns and

(ii) In the optimal solution, there can be only one assignment in a given row and column.

When a transportation problem is having these two features, it is treated as an assignment problem. Thus, an assignment problem is a special type of transportation problems or a linear programming problem.

2. Assumptions of an Assignment Problem.

Ans :

Assumptions

(i) The number of assignees (workers or machines) and the number of jobs or tasks are the same (say the number is 'n').

(ii) Each assignee is to be assigned to exactly one job or task.

(iii) Each job or task is to be performed by exactly one assignee.

(iv) There is a cost c_{ij} associated with assignee 'i' (i = 1, 2....,n) performing task 'j' (j = 1, 2, n).

(v) The objective is to determine how all 'n' assignments should be made to minimise the total cost.

Any problem satisfying all these assumptions can be solved very efficiently by algorithms designed specifically for assignment problems.

3. When does Multiple Optimal Solutions are said to be operated in assignment problem.

Ans :

In an Assignment Problem (AP) the facilities are assigned to jobs on a one-to-one basis using Hungarian method. While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off certain number of zeros.

This indicates that there are more than the required number of independent zero elements. In such cases, there will be multiple optimal solution with the same total cost of assignment. This type of situation is helpful in management decision making as the manager has flexibility in assignments.

4. Maximization Case in Assignment Problem.

Ans :

If instead of cost matrix, a profit (or revenue) matrix is given, then assignments are made in such a way that total profit is maximized. The profit maximization assignment problems are solved by converting them into a cost minimization problem in either of the following two ways:

i) Put a negative sign before each of the elements in the profit matrix in order to convert the profit values into cost values.

ii) Locate the largest element in the profit matrix and then subtract all the elements of the matrix from the largest element including itself.

The transformed assignment problem can be solved by using usual Hungarian method.

5. Travelling Salesman Problem.

Ans :

A travelling salesman has to visit a certain number of cities from his headquarters. The distance (or cost or time) of journey between every pair of cities denoted by a. (i.e., distance from city 'i' to city 'j') is assumed to be known.

The problem is:

Salesman starting from his home city (headquarters) visits each city only once and returns to his home city in the shortest possible total distance (or at the least total cost or least total time). The problem of the travelling salesman may be viewed as a typical assignment problem with two additional constraints.

(i) No assignment should be made along the diagonal line of the "From" - "To" matrix.

(ii) The salesman should not be required to visit a city twice until he has visited all the cities once.

6. Define Transportation Problem.

Ans :

Transportation problem is another case of application to linear programming problems, where some physical distribution (transportation) of resources is to be made from one place to another to meet certain set of requirements with in the given availability. The places from where the resources are to be transferred are referred to as sources or origins. These sources or origins will have the availability or capacity or supply of resources. The other side of this transportation i.e., to where the resources are transported are called sinks or destinations such as market enters, godowns etc. These will have certain requirements or demand.

7. What are the different methods of Finding Initial Feasible Solution ?

Ans :

Optimal solution is a feasible solution (not necessarily basic) which minimizes the total cost.

The solution of a transportation problem can be obtained in two stages, namely initial solution and optimum solution.

Initial solution can be obtained by using any one of the three methods viz,

(i) North west corner rule (NWCR)

(ii) Least cost method or Matrix minima method

(iii) Vogel's approximation method (VAM).

8. What is Vogels Approximation Method (VAM)?

Ans :

VAM or Vogel's Approximation Method is also known as penalty or regret method. It is basically a heuristic method. Allocation is done on the basis of opportunity cost (penalty) that would have incurred if allocation in certain cells with minimum costs were missed.

The steps involved in this method for finding the initial solution are as follows.

Step 1

Find the penalty cost, namely the difference between the smallest and next smallest costs in each row and column.

Step 2

Among the penalties as found in step (1) choose the maximum penalty. If this maximum penalty is more than one (i.e if there is a tie) choose any one arbitrarily.

Step 3

In the selected row or column as by step (2) find out the cell having the least cost. Allocate to this cell as much as possible depending on the capacity and requirements.

Step 4

Delete the row or column which is fully exhausted. Again compute the column and row penalties for the reduced transportation table and then go to step (2) Repeat the procedure until all the rim requirements are satisfied.

9. What do you mean by Unbalanced Supply and Demand

Ans :

An unbalanced transportation problem refers to a situation where in the sum of supplies of all the sources does not match with the sum of the demands of all the destinations. Mathematically, it can be represented as,

$$\sum_{i=1}^{m} a_i \neq \sum_{i=1}^{n} b_j$$

When this rim condition is violated, i.e., the total supply is not equal to the total demand, the transportation problem is said to be an unbalanced one. There are two types of unbalanced TP are,

(a) Total supply exceeds total demand

(b) Total demand exceeds total supply.

10. What is degeneracy in transportation problem?

Ans :

Degeneracy in TP is said to occur when the number of allocated cells is less than m + n – 1.

Where,

m = Number of rows

n = Number of columns

If degeneracy is occurred in a TP, then it is not possible to draw a closed loop for every occupied cell while solving the problem by MODI method. This degeneracy may occur during:

(a) Initial stage

(b) Testing of optimal solution.

UNIT IV

Decision Theory: Introduction, ingredients of decision problems. Decision making – under uncertainty, cost of uncertainty, under risk, under perfect information, decision tree, construction of decision tree.

Network Analysis – Network Diagram, PERT, CPM, Critical Path determination, Project Completion Time, Project Crashing.

4.1 DECISION THEORY - INTRODUCTION

4.1.1 Ingredients of Decision Problems

Q1. Define decision theory. Explain the components of decision theory.

Ans :

The success or failure that an individual or organization experiences, depends to a large extent, on the ability of making acceptable decisions on time. To arrive at such a decision, a decision-maker needs to enumerate feasible and viable courses of action (alternatives or strategies), the projection of consequences associated with each course of action, and a measure of effectiveness (or an objective) to identify the best course of action.

Decision theory is both descriptive and prescriptive business modeling approach to classify the degree of knowledge and compare expected outcomes due to several courses of action. The degree of knowledge is divided into four categories: complete knowledge (i.e. certainty), ignorance, risk and uncertainty as shown in Fig. below.

Ignorance	Uncertainty	Risk	Certainty

——— Increasing knowledge ———→

A decision problem or a decision-making situation includes two components, viz., the decision or several possible acts and the actual events that may occur in the future known as states of nature. While making the decisions the manager is uncertain which states of nature will occur in the future and also he has no control over them. For example, suppose a distribution company is considering buying a computer which will be helpful to process an increased number of orders, thereby increase its business. If the economic conditions remain good, the firm will realise a large increase in profit, if the economy is having a downward trend, the company will incur loss. In this decision situation, the possible decisions are (a) to purchase the computer or (b) not to purchase the computer. The states of nature are: (a) good economic conditions and (b) bad economic conditions. The states of nature that occurs will determine the outcome of the decision and the decision maker has no control over the states of nature that will occur in the future.

i) The Decision Problem :

A decision problem consists of several possible acts and several possible states of nature. For example, the possible acts may be

1. To decide the number units of an item to be ordered from the vendor.
2. To make or buy an item.
3. To invest or not to invest in a new business venture.
4. To add or not to add a new product line.
5. To diversify or not to diversify the business.
6. To change the price or not to change the price of a product.

The possible states of nature or events for the above acts may be,

1. The demand for the product may be 1, 2, 3, ... 50 etc.
2. If we make the product, the cost may be Rs. 10 per unit and if we buy, the cost may be Rs. 12 per unit.

3. If we invest on a new venture, the return on investment may be 15 per cent, 20 per cent or 30 per cent.
4. If we add a new product line, the sales may increase by Rs. 1 crore or 2 crores.
5. If we diversify the business, our profitability may go up by 50%, 75% or 100%.
6. If we reduce the price by Re. 1 per unit, the sales may go up by 20 per cent or 30 per cent.

ii) Decision Models

To facilitate the types of decisions mentioned above, so that the best decisions result, we have to use decision models. Decision models can be classified based on the degree of certainty into two types of models viz., (a) deterministic models which involve a single or unique pay-off for each strategy and (b) probabilistic model, in which there can be two or more pay-offs for each strategy and a probability value is associated with each pay-off.

We can visualise decision problems under three categories.

(a) Decision makingunder certainty

(b) Decision makingunder risk

(c) Decision making under uncertainty.

Q2. Explain the steps involved in decision making.

Ans :

Decision analysis is a technique of getting a clear insight about the decision-making process. The steps involved in decision-making are as follows,

Step 1

The first step involves the identification and definition of an existing problem.

Step 2

Objectives should be clearly stated along with the decision criteria.

Step 3

The third step centers around the identification and evaluation of all the possible alternatives.

Step 4

Formulation of an appropriate mathematical decision theory models.

Step 5

Applying selected model for solving the existing problem so as select the best alternative from a range of alternatives.

Step 6

Sensitivity analysis is conducted to provide optimal solution to existing problem.

Step 7

This step helps in the communication and the implementation of various decisions.

Step 8

The last step of decision-making involves follow-up and feedback of results of decision.

Q3. Explain the basic ingredients of decision.

Ans :

(i) The decision maker

The decision maker is charged with responsibility of making the decision. That is, he has to select one from a set of possible courses of action.

(ii) The acts

The acts are the alternative courses of action of strategies that are available to the decision maker. The decision involves a selection among two or more alternative courses of action. The problem is to choose the best of these alternatives to achieve an objective.

(iii) Event

Events are the occurrences which affect the achievement of the objectives. They are also called states of nature or outcomes. The events constitute a mutually exclusive and exhaustive set of outcomes, which describe the possible behaviour of the environment in which the decision is made. The decision maker has no control over which event will take place and can only attach a subjective probability of occurence of each.

(iv) Pay off table

A pay off table represents the economics of a problem, i.e. revenue and costs associated with any action with a particular outcome. It is an ordered statement of profit or costs resulting under the given situation. The pay off can be interpreted as the outcome in quantitative form if the decision maker adopts a particular strategy under a particular state of nature.

(v) Opportunity loss table

An opportunity loss is the loss incurred because of failure to take the best possible action. Opportunity losses are calculated separately for each state of nature that might occur Given the occurence of a specific state of nature we can determine the best possible act. For a given state of nature, the opportunity loss of an act is the difference between the payoff of that act, and the pay off for best act that could have been selected.

4.2 DECISION MAKING ENVIRONMENT

Q4. What are the different decision making environments?

Ans : **(May-19, Dec.-17)**

Decision theory or decision analysis focuses on identifying the best course of action (or alternative strategy) from the number of available courses of action (or alternative strategies). Depending upon the kind of information available regarding the occurrences of different outcomes/ events, decision making environment can be categorized into four types,

1. Decision-making Under Uncertainty

Decision-making under uncertainty illustrates a situation where in more than two outcomes/ events may occur from one decision point and decision maker has no knowledge regarding the probabilities to be assigned to occurrence of each event or state of nature. The lack of information regarding the probabilities of occurrences of events make the decision making process very complicated. Under uncertainty situations, decision maker cannot compute expected pay offs for each course of action due to lack of probabilities. Launching of new product into the market, setting up of new plant etc., can be taken as examples of uncertainty situation. The selection of one best course of action relies upon the nature of decision-maker and rules of the organisation.

The following choices are available before the decision maker in situ-ations of uncertainty.

(a) Maximax Criterion

(b) Minimax Criterion

(c) Maximin Criterion

(d) Laplace Criterion (Criterion of equally likelihood)

(e) Hurwicz Alpha Criterion (Criterion of Realism)

2. Decision-making Under Risk

Often business decisions are taken under risk conditions. Like uncertainty, two or more outcomes/events may occur from one single decision but decision maker possesses required knowledge regarding what probabilities can be assigned to each state of nature. The information regarding the probabilities of each state of nature can be acquired either from historical records or from personal judgements of the decision maker. As probabilities of each state of nature are known under conditions of risk, the course of action (or alternative strategy) with highest expected value is chosen as a best course of action.

The criterias used for selecting the best course of action under conditions of risk are,

(a) Expected Monetary Value (EMV)

(b) Expected Opportunity Loss (EOL)

3. Decision-making Under Certainty

Decision making under certainty pertains to a situation wherein the decision maker knows with certainty the outcome of each course of action. Each decision has only one state of nature and decision maker chooses one best pay off among the available alternative strategies. The state of nature is arraigned with the probability equal to ' 1' as only one state of nature occurs from each course of action (alternative strategy). Eventhough stage of nature is only one, courses of action may be many. We rarely find managerial decision problems with complete information regarding future outcomes.

The methods used for decision making under certainty situations are,

(a) Linear programming method

(b) Integer programming method

(c) Transportation and assignment techniques

(d) Activity analysis

(e) Input output analysis

(f) EOQ (Economic Order Quantity) method

(g) Break-Even Analysis (BEA) etc.

4. Decision-making Under Conflict

Decision-making under conflict pertains to a situation where the probability of occurrences of state of nature are neither certain nor uncertain. This situation can be called as "decision-making under conflict" or "decision-making under partial uncertainty."

'Game theory' is followed for decision making under conflict situations.

4.2.1 Decision Making under Uncertainty, Cost of uncertainty

Q5. Describe some methods which are useful for decision making under uncertainity

Ans : **(Feb.-16, Aug.-15)**

When the decision maker faces multiple states of nature but he has no "means to arrive at probability values to the likelihood of occurence of these states ctf nature, the problem is a decision problem under uncer-tainty. Such situations arise when a new product is introduced in the market or a new plant is set up. In business, there are many problems of this 'nature'. Here the choice of decision largely depends on the person-ality of the decision maker.

The following choices are available before the decision maker in situ-ations of uncertainty.

(a) Maximax Criterion

(b) Minimax Criterion

(c) Maximin Criterion

(d) Laplace Criterion (Criterion of equally likelihood)

(e) Hurwicz Alpha Criterion (Criterion of Realism)

(a) The maximax decision criterion (criterion of optimism)

The term 'maximax' is an abbreviation of the phrase maximum of the maximums, and an adventurous and aggressive decision maker may choose to take the action that would result in the maximum payoff possible. Suppose for each action there are three possible payoffs corresponding to three states of nature as given in the following decision matrix:

States of Nature	Decisions		
	A_1	A_2	A_3
S_1	220	180	100
S_2	160	190	180
S_3	140	170	200

Maximum under each decision are (220, 190, 200). The maximum of these three maximums is 220. Consequently according to the maximax criteria the decision to be adopted is A_1.

(b) The minimax decision criterion

Minimax is just the opposite of maximax. Application of the minimax criteria requires a table of losses instead of gains. The losses are the costs to be incurred or the damages to be suffered for each of the alternative actions and states of nature. The minimax rule minimizes the maximum possible loss for each course of action. The term 'minimax' is an abbreviation of the phrase minimum of the maximum. Under each of the various actions there is a maximum loss and the action that is associated with the minimum of the various maximum losses is the action to be taken according to the minimax criterion. Suppose the loss table is

States of Nature	Decisions		
	A_1	A_2	A_3
S_1	0	4	10
S_2	3	0	6
S_3	18	14	0

It shows that the maximum losses incurred by the various decisions

A_1	A_2	A_3
18	14	10

and the minimum among three maximums is 10 which is under action A_3. Thus according to minimax criterion, the decision-maker should take action A_3.

(c) The maximin decision criterion (Criterion of pessimism)

The maximin criterion of decision making stands for choice between alternative courses of action assuming pessimistic view of nature. Taking each act in turn, we note the worst possible results in terms of payoff and select the act which maximises the minimum pay off. Suppose the payoff table is

States of Nature	Decisions		
	A_1	A_2	A_3
S_1	–80	–60	–20
S_2	–30	–10	–2
S_3	30	15	7
S_4	75	80	25

Minimum under each decision are respectively

–80 –60 –20

The action A_3 is to be taken according to this criterion because it's the maximum among minimums.

(d) Laplace criterion

As the decision maker has no information about the probability of occurrence of various events, the decision maker makes a simple assumption that each probability is equally likely. The expected pay off is worked out on the basis of these probabilities. The act having maximum expected pay off is selected.

(e) Harwicz alpha criterion

This method is a combination of maximum criterion and maximax criterion. In this method, the decision maker's degree of optimism is represented by α ,the coefficient of optimism, α varies between 0 and 1. When $\alpha = 0$, there is total pessimism and when $\alpha = 1$, there is total optimism.

we find D_1, D_2, D_3 etc. connected with all strategies where $D_i = \alpha M_i + (1 - \alpha) m_i$ where M_i is the maximum payoff of 'i' the strategy and m_i is the minimum payoff of 'i'th strategy. The strategy with highest of D_1, D_2 is chosen. The decision maker will specify the value of a depending upon his level of optimism.

PROBLEMS

1. A food products' company is contemplating the introduction of a revolutionary new product with new packaging or replacing the existing product at much higher price (S_1). It may even make a moderate change in the composition of the existing product, with a new packaging at a small increase in price (S_2), or may mall a small change in the composition of the existing product, backing it with the word "New' and a negligible increase in price (S_3). The three possible states of nature or events are: (i) high increase in sales (N_1), (ii) no change in sales (N_2) and (iii) decrease in sales (N_3). The marketing department of the company worked out the payoffs in terms of yearly net profits for each of the strategies of three events (expected sales). This is represented in the following table :

Strategies	**States of Nature**		
	N_1	**N_2**	**N_3**
S_1	7,00,000	3,00,000	1,50,000
S_2	5,00,000	4,50,000	0
S_3	3,00,000	3,00,000	3,00,000

Which strategy should the concerned executive choose on the basis of

a) Maximin criterion b) Maximax criterion

c) Minimax regret criterion d) Laplace criterion

Sol :

The payoff matrix is rewritten as follows :

a) Maximin Criterion

States of Nature	**Strategies**		
	S_1	**S_2**	**S_3**
N_1	7,00,000	5,00,000	3,00,000
N_2	3,00,000	4,50,000	3,00,000
N_3	1,50,000	0	3,00,000
Column (minimum)	1,50,000	0	**3,00,000** ← Maximin Payoff

The maximum of column minima is 3,00,000. Hence, the company should adopt strategy S_3.

b) Maximax Criterion

States of Nature	Strategies		
	S_1	S_2	S_3
N_1	7,00,000	5,00,000	3,00,000
N_2	3,00,000	4,50,000	3,00,000
N_3	1,50,000	0	3,00,000
Column (maximun)	**7,00,000**	5,00,000	3,00,000

↑ Maximax Payoff

The maximum of column maxima is 7,00,000. Hence, the company should adopt strategy S_1.

c) Minimax Regret Criterion Opportunity loss table is shown below :

States of Nature	Strategies		
	S_1	S_2	S_3
N_1	7,00,000 – 7,00,000 = 0	7,00,000 – 5,00,000 = 2,00,000	7,00,000 – 3,00,000 = 4,00,000
N_2	4,50,000 – 3,00,000 = 1,50,000	4,50,000 – 4,50,000 = 0	4,50,000 – 3,00,000 = 1,50,000
N_3	3,00,000 – 1,50,000 = 1,50,000	3,00,000 – 0 = 3,00,000	3,00,000 – 3,00,000 = 0
Column (maximum)	**1,50,000**	3,00,000	4,00,000

↑ Minimax Regret

Hence the company should adopt minimum opportunity loss strategy, S_1.

d) Laplace Criterion. Asumming that each state of nature a probability 1/3 of occurrence. Thus,

Strategy	Expected Return (Rs.)
S_1	(7,00,000 + 3,00,000 + 1,50,000)/3 = 3,83,333.33 ← Largest Payoff
S_2	(5,00,000 + 4,50,000 + 0)/3 = 3,16,666.66
S_3	(3,00,000 + 3,00,000 + 3,00,000)/3 = 3,00,000

Since the largest expected return is from strategy S_1 the executive must select strategy.

---

2. From the following information Calculate Laplace criterion.

Events	Act		
	A_1	A_2	A_3
E_1	20	12	25
E_2	25	15	30
E_3	30	20	22

Sol :

We associate equal probability for each event say 1/3. Expected pay off are

$$A_1 \rightarrow \frac{(20+25+30)}{3} = \frac{75}{3} = 25$$

$$A_2 \rightarrow \frac{(12+15+20)}{3} = \frac{47}{3} = 15.67$$

$$A_3 \rightarrow \frac{(25+30+22)}{3} = \frac{77}{3} = 25.67$$

Since A_3 has maximum expected payoff, A_3 is the optimal Act.

---

3. From the following information Calculate Horwitz Alpha criterion the level of optimism is 0.6.

Events	**Act**		
	A_1	**A_2**	**A_3**
E_1	20	12	25
E_2	25	15	30
E_3	30	20	22

Sol :

Let $\alpha = 0.6$

for A_1 max. pay off = 30

Min. payoff = 20

$D_1 = \alpha M_i + (1 - \alpha) m_i$

$\therefore \quad D_1 = (0.6 \times 30) + (1 - 0.6) 20 = 26$

Similarly $D_2 - (0.6 \times 30) + (1 - 0.6) 2 = 16.8$

$D_3 - (0.6 \times 30) + (1 - 0.6) 22 = 26.8$

Since D_3 is max, select the act A_3.

4.2.2 Decision Making under Risk

Q6. Write a note on decision making under risk. Explain briefly about various criteria involved in the process of decision-making under risk.

Ans : **(Feb.-17, Feb.-16)**

In this situation the decision maker has to face several states of nature. But he has some knowledge or experience which will enable him to assign probability to the occurence of each state of nature. The objective is to optimise the expected profit, (or) minimise the opportuiniy loss.

For decision problems under risk, the most popular methods used are EMV (Expected monetary value) criterion, EOL (Expected Opportunity Loss.

(a) Expected Monetary Value (EMV)

When the probabilities ran be assigned to the various states of nature, it is possible to calculate the statistical expectation of gain for each course of action.

The conditional value of each event in the pay off table is multiplied by its probability and the product is summed up. The resulting number is the EMV for the act. The decision maker then selects from the available alternative actions, the action that leads to the maximum expected gain (that is the action with highest EMV). Consider the following example. Let the states of nature be S_1 and S_2 and the alternative strategies be A_1 and A_2. Let the pay off table be as shown below.

	A_1	A_2
S_1	30	20
S_2	35	30

Let the probabilities for the states of nature S_1 and S_2 be respectively 6 and 4.

Then,

EMV for A, $=(30 \times 0.6) + (35 \times 0.4) = 18 + 14 = 32$

EMV for A_2 $= (20 \times 0.6) + (30 \times 0.4) = 12 + 12 = 24$

∴ EMV for A_1 is greater.

∴ The decision maker will choose the strategy A_1

(b) Expected Opportunity Loss (EOL)

The difference between the greater pay off and the actual pay off is known as opportunity loss. Under this criterion the strategy which has minimum Expected Opportunity Loss (EOL) is chosen. The calculation of EOL is similar to that of EMV.

Consider the following Example: Given below is an opportunity loss table. A_t and A_2 are the strategies and S_1 and S_2 are the states of nature.

	A_1	A_2
S_1	0	10
S_2	2	–5

Let the probabilities for two states be 0.6 and 0.4

EOL for A, $= (0 \times 0.6) + (2 \times 0.4) = 0.8$

EOL for A_2 $= (10 \times 0.6) + (–5 \times 0.4) = 6 – 2 = 4$

EOL for A is least. Therefore the strategy A_1 may be chosen.

PROBLEMS

4. A conditional pay off matrix is given below. By using this matrix calculate EMV for each course of action and determine the optimum EMV.

States of Nature	Probabilities	Conditional Pay off Matrix Course of Action				
		A_1	A_2	A_3	A_4	A_5
N_1	0.03	0	–30	–70	–110	–150
N_2	0.15	0	4	–25	–65	–110
N_3	0.20	0	4	9	–20	–60
N_4	0.50	0	4	9	14	–15
N_5	0.12	0	4	9	14	10

Sol :

States of Nature	Probabilities (1)	Conditional Pay off Matrix courses of action					Expected pay off matrix courses of action				
		A_1 (2)	A_2 (3)	A_3 (4)	A_4 (5)	A_5 (6)	A_1 (1)×(2)	A_2 (1)×(3)	A_3 (1)×(4)	A_4 (1)×(5)	A_5 (1)×(6)
N_1	0.03	0	– 30	– 70	– 110	– 150	0	– 0.9	– 2.1	– 3.3	– 4.5
N_2	0.15	0	4	– 25	– 65	– 110	0	0.6	– 3.75	– 9.75	– 16.5
N_3	0.20	0	4	9	– 20	– 60	0	0.8	1.8	–4	– 12
N_4	0.50	0	4	9	14	– 15	0	2	4.5	7	– 7.5
N_5	0.12	0	4	9	14	10	0	0.48	1.08	1.68	1.2.
Expected Monetary Value (EMV) →							**0**	**2.98**	**1.53**	**– 8.37**	**– 39.3**

The optimum EMV is 2.98, which corresponds to course of action A_2.

5. A conditional profit matrix is given below,

States of Nature	Probabilities	Course of Action		Condition Profit	
		A_1	A_2	A_3	A_4
N_1	0.15	200	160	130	100
N_2	0.20	200	210	180	150
N_3	0.30	200	210	230	190
N_4	0.35	200	210	230	250

From this given matrix calculate conditional opportunity loss values and EOL.

Sol :

Calculating Conditional Opportunity Loss (COL) and Expected Opportunity Loss (EOL) values.

States of nature	Probabilities (1)	Course of action Conditional opportunity loss (COL)				Courses of action Expected opportunity loss (EOL)			
		A_1 (2)	A_2 (3)	A_3 (4)	A_4 (5)	A_1 (1)×(2)	A_2 (1)×(3)	A_3 (1)×(4)	A_4 (1)×(5)
N_1	0.15	0 [200 – 200]	40 [200 – 160]	70 [200 – 130]	100 [200 – 100]	0	6	10.5	15
N_2	0.20	10 [210 – 200]	0 [210 – 210]	30 [210 – 180]	60 [210 – 150]	2	0	6	12
N_3	0.30	30 [230 – 200]	20 [230 – 210]	0 [230 – 230]	40 [230 – 190]	9	6	0	12
N_4	0.35	50 [250 – 200]	40 [250 – 210]	20 [250 – 230]	0 [250 – 250]	17.5	14	7	0
Expected opportunity Loss (EOL) →						28.5	26	23.5	39

The minimum expected opportunity loss is 23.5 which corresponds to course of action A_3. The optimum course of action is A, as per EOL criterion.

Q6. Indicate the difference between decision making under risk and uncertainty in statistical decision theory.

Ans : **(Sep.-15)**

Differences between decision making under risk and uncertainity,

S.No.	Basis	Decision Making Under Risk	Decision Making Under Uncertainty
1.	**Meaning**	In case of decision making under risk, decision maker possesses knowledge regarding what probabilities can be assigned to the occurrence of each outcome or state of nature.	In case of decision making under uncertainty, decision maker does not have the knowledge about the probabilities to be assign to the occurrence of each event or state of nature.
2.	**Payoffs**	Decision maker can compute expected payoffs with the help of knowledge regarding probabilities.	Decision maker cannot compute expected payoffs due to lack of knowledge about probabilities.
3.	**Criteria**	The criteria or techniques used for Decision making under risk includes- Expected Monetary Value (EMV) and Expected Opportunity Loss (EOL).	The criteria or techniques used for decision making under uncertainty includes pessimism, realism, optimism, regret and equiprobable criterion.
4.	**Complexities**	It involves level of complexities.	It involves high level of complexities.
5.	**Example**	Investing in stock market is one of the example of decision making under risk.	Launching new product is an example of decision making under uncertainty.

4.3 DECISION MAKING UNDER PERFECT INFORMATION

Q7. What do you understand by decision making under perfect information? What are the underlying assumptions in such decision making process?

Ans : **(Sep.-16)**

Decision Making Under Perfect Information

Information reduces the extent of uncertainty in decision-making. Assuming that the decision maker has perfect information about the market variables, decision-making takes a different turn.

Accurate, current and relevant information available in required details at the time of decision-making improves decision-making. For instance, a firm wants to increase the price of its products (decision to be made). However, this decision depends on the price strategy of the competitors (chance factors). Now, supposing that the firm is positive that the competitor will not raise their product prices, that is, the firm has perfect information about the occurrence of an outcome, the firm can choose its optimal strategy with confidence. The maximum expected monetary value that could be achieved by using perfect information is called Expected Profit with Perfect Information (EPPI).

Before obtaining any information the decision maker should know the significance of that information. Is the information required? What is the value of that information in decision-making? For this, compare the expected marginal profit you can gain by using the perfect information in question (EPPI) with the decisions made without the information. For instance, if an inventory manager knows the exact requirement of stock for each day, he can stock the same level of inventory and save on costs. The sales forecast is thus of great significance to the inventory manager.

Assumptions

Following are some of the important assumptions of decision making under perfect information,

1. It assumes that, decision maker should have complete information about the all decision related alternatives and consequences.
2. It assumes that, decision maker should be able to evaluate the decision making alternatives from the available information.
3. It assumes that, decision makers should seek to maximize expected utility or profits.
4. It assumes perfect forecast of possible decisions.

6. A_1, A_2, A_3 are the acts and S_1, S_2, S_3 are the states of nature. Also known that $P(S_1) = 0.5$, $P(S_2) = 0.4$ and $P(S_3) = 0.1$.Calculate EVPI.

Pay off table is as given below :

State of natute	Pay off table		
	A_1	A_2	A_3
S_1	30	25	22
S_2	20	35	20
S_3	40	30	35

Sol :

EMV for A_1 = (0.5 × 30) + (0.4 × 20) + (0.1 × 40) = 15 + 8 + 4 = 27

EMV for A_2 = (0.5 × 25) + (0.4 × 35) + (0.1 × 30) = 12.5 +14 + 3 = 29.5

EMV for A_3 = (0.5 × 22) + (0.4 × 20) + (0.1 × 35) = 11 + 8 + 3.5 = 22.5

The highest EMV is for the strategy A_2 and it is 29.5

Now to find EVPI, work out expected value for maximum pay off under all states of nature.

	Max.profit of each state	Probability	Expected value (= Prob. × Profit)
S_1	30	0.5	1.5
S_2	35	0.4	14
S_3	40	0.1	4

∴ Expected pay off with perfect information = 33

∴ Thus the expected value of perfect information (EVPI) = Expected Value with Perfect Information – Maximum EMV = 33 – 29.5 = 3.5

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7. A grocery with a bakery department is faced with the problem of how many cakes to buy in order to meet the day's demand. The grocer does not want to sell day-old goods in competition with fresh products, leftover cakes are therefore, a complete loss. On the other hand, if a customer desires a cake and all of them have been sold, the disappointed customer will buy elsewhere and the sales will be lost. The grocer has, therefore, collected information on the past sales on a selected 100 day period as shown below,

Sales Per Day	**No. of Days**	**Probability**
25	**10**	**0.1**
26	**30**	**0.3**
27	**50**	**0.5**
28	**10**	**0.1**

A cake costs ` 80 and sells for ` 100. Construct the pay-off table and the opportunity loss table. What is the optimum number of cakes that should be bought each day?

Sol :

Marginal profit = 100 – 80 = 20

Marginal loss = 80

Construction of Payoff Table

Conditional profit (Payoff) = (Market profit × Cakes sold) – (Market loss × Cakes not sold)

= ` (100 – 80 × Cakes sold) – (` 80 × Cakes not sold)

$$= \begin{Bmatrix} 20\text{ D} & ; & \text{D} \geq \text{S} \\ 20\text{ D} - 80(\text{S} - \text{D}) & ; & \text{D} < \text{S} \end{Bmatrix}$$

Note: D = Number of units demanded

S = Number of units stocked.

Computation of Conditional Profit Values (Payoffs)

State of Nature (Daily Likely Demand)	Probability	Conditional Payoff (`)			
		Course of Action (Stocking Cakes)			
		A_1 - 25	A_2 - 26	A_3 - 27	A_4 - 28
E_1 - 25	0.10	500	420	340	260
E_2 - 26	0.30	500	520	440	360
E_3 - 27	0.50	500	520	540	460
E_4 - 28	0.10	500	520	540	560
	EMV	500	510	490	420

Since, the course of action A_2 has maximum EMV, so the store should select A_2, i.e., it should stock 26 cakes.

Calculation of Expected Opportunity Loss (EOL)

State of Nature (Demand)	Probability	Conditional Opportunity Loss (`)			
		Course of Action (Stock)			
		A_1 – 25	A_2 – 26	A_3 – 27	A_4 – 28
E_1 - 25	0.10	0	80	160	240
E_2 - 26	0.30	20	0	80	160
E_3 - 27	0.50	40	20	0	80
E_4 - 28	0.10	60	40	20	0
	EOL	32	22	42	112

Since, A_2 course of action has minimum EOL, so the store should select A_2 i.e, it should stock 26 cakes.

8. A departmental store with a bakery section is faced with the problem of how many cakes to buy in order to meet the day's demand. The departmental store prefers not to sell day old cakes in competition, left over cakes are therefore a complete loss. On the other hand, if a customer desires a cake and all of them have been sold, the customer will buy elsewhere and the sales will, be lost. The store has therefore collected information on the past sales based on selected 100-day period as shown in the table:

Sales per day	**15**	**16**	**17**	**18**
Number of days	**20**	**40**	**30**	**10**
Probability	**0.20**	**0.40**	**0.30**	**0.10**

Construct the conditional profit and the opportunity loss tables. What is the optimal number of cakes that should be bought each day? A cake costs ` 2 and sells for ` 2.50.

Sol : **(Sep.-15)**

Calculation of conditional profit,

Marginal profit = Selling price – Cost price = 2.50 – 2.00

= 0.5

Marginal loss (U.C) = Loss on an unsold cake

= 2.00

Conditional profit (pay off) = (Marginal profit × Cake sold) – (Marginal loss × Cake not sold).

Computation of Conditional Profit or Values (Payoffs)

Demand	Probability	Conditional Payoff / Course of Action (Stocking cakes)			
		A_1 – 15	A_2 – 16	A_3 – 17	A_4 – 18
15	0.20	7.5	5.5	3.5	1.5
16	0.40	7.5	8.0	6.0	4.0
17	0.30	7.5	8.0	8.5	6.5
18	0.40	7.5	8.0	0.5	9
	EMV	**7.5**	**7.5**	**6.5**	**4.75**

Since, the course of action A_1 and A_2 has maximum EMV, so the store should select either A_1 or A_2 i.e., it should stock 15 or 16 cakes.

Computation of Expected Opportunity Loss (EOL)

		Conditional Opportunity Loss Courses of Action (Stock)				Expected Loss			
Demand	**Probability (1)**	**A_1–15 (2)**	**A_2–15 (3)**	**A_3–17 (4)**	**A_4–18 (5)**	**A_1 (1×2)**	**A_2 (1×3)**	**A_3 (1×4)**	**A_4 (1×5)**
15	0.20	0	2	4	6	0	0.4	0.8	1.2
16	0.40	0.5	0	2	4	0.2	0	0.8	1.6
17	0.30	1	0.5	0	2	0.3	0.15	0	0.6
18	0.10	1.5	1	0.5	0	0.15	0.10	0.05	0
		EOL				**0.65**	**0.65**	**1.65**	**3.4**

The minimum expected loss can be used by two stock (i.e.,) (A -15), (A - 16). Hence, departmental store can either stock 15 or 16 cakes.

4.4 Decision Tree

Q8. Explain briefly about Decision Tree Analysis.

Ans : **(Nov.-20)**

The diagrammatic representation of logical relationship between the parts of a complex situation of a decision making problem is called a decision tree. It specifies the choices, risks, objectives & monetary gains involved in a business problem.

However, situations may arise when a decision-marker to revise his previous decisions due to availability of additional information. Thus he intends to make a sequence of interrelated decisions over several future periods. Such a situation is called a *sequential or multi period decision process*. For example, in the process of marketing a new product, a company usually first go for 'Test Marketing' and other alternative courses of action might be either 'Intensive Testing' or 'Gradual Testing'. Given the various possible consequences - good, fair, or poor, the company may be required to decide between redesigning the product, an aggressive advertising campaign or complete withdrawal of product, etc. Based on this decision there might be an outcome that leads to another decision and so on.

A decision tree analysis involves the construction of a diagram that shows, at a glance, when decisions are expected to be made - in what sequence, their possible outcomes, and the corresponding payoffs.

A decision tree consists of nodes, branches, probability estimates, and payoffs. There are two types of nodes:

1. **Decision (or act Node)**

   A decision node is represented by a square and represents a point of time where a decision-maker must select one alternative course of action among the available. The courses of action are shown as branches or arcs emerging out of decision node.

2. **Chance (or event) Node**

   Each course of action may result in a chance node. The chance node is represented by a circle and indicates a point of time where the decision - maker will discover the response to this decision.

Branch emerge from and connect various nodes and represent either decisions or states of nature. There are two types of branches :

1. **Decision Branch**

   It is the branch leading away from a decision node and represents a course of action that can be chosen at a decision point.

2. **Chance Branch**

   It is the branch leading away from a chance node and represents the state of nature of a set of chance events. The assumed probabilities of the states of nature are written alongside their respective chance branch.

3. **Terminal Branch**

   Any branch that makes the end of the decision tree (not followed by either a decision or chance node), is called a *terminal branch*. A terminal branch can represent either a course of action. The terminal points of a decision tree are supposed to be mutually exclusive points so that exactly one course of action will be chosen.

Q9. Explain advantages and disadvantages of decision tree.

Ans : **(Nov.-20)**

Advantages of Decision Tree

The decision tree approach has several advantages like :

a) This approach facilitates investment decisions in a scientific way.

b) This approach gives an overall view of all the possibilities associated with a project, helps the management to take decisions keeping the entire situations in the mind.

c) As this technique links the probable outcomes of a decision one after an interrelated manner along with probabilities assigned to each sequential outcome, it is very useful in tacking investment situations requiring decisions to be taken in a sequence.

Disadvantages of Decision Tree

The various disadvantages of the decision tree approach are as follows .

a) A prime decision may have a number of sequential decision points and each one of such decision points may have numerous decision branches (or) decision alternatives.

b) The decision tree analysis becomes very complex, when a project has a life more than two years.

4.5 CONSTRUCTION OF DECISION TREE

Q10. Write about the method and steps involved in construction of decision tree.

Ans :

Method of Constructing Decision Tree

1. Identification of all the possible courses of action.
2. List the possible results i.e., 'states of nature' of each course of action specified.
3. Calculation of payoff of each possible combination of courses of action and results. Payoff will be in monetary terms usually.
4. Assigning probabilities to the different possible results for each given course of action. Likelihood of occurrence of a particular event is being indicated by the probability.
5. At last choose that course of action which gives the maximum payoff.

Steps Involved in the Construction of Decision Tree

Decision tree is a diagrammatic representation of alternative courses of action and sequence of states of nature. Courses of action and states of nature (or outcomes) are arranged as branches of a tree.

The various steps involved in decision tree analysis are listed down below,

Step 1:

Determine the number of decisions to be taken and the alternative strategies available for each decision in a sequential manner.

Step 2:

Determine the outcome (or event) which may occur from each alternative strategy (course of action).

Step 3:

Construct a tree diagram representing the order in which decisions are taken and outcomes are occurring. The decision tree diagram begins from left side and move towards right side.

Step 4:

Determine the probabilities of occurrences of each state of nature.

Step 5:

Determine the pay off values for each pair (or combination) of state of nature and course of action.

Step 6:

Calculate expected pay off value for each course of action starting from right side of the decision tree.

Step 7:

Select the course of action (or alternative strategy) with the best expected pay off value.

Step 8:

Work backwards from last decision point to first decision point and at each decision point repeat the steps from step 4 to step 7.

9. A glass factory that specializes in crystal is developing a substantial backlog and for this the firm's management is considering three courses of action: To arrange for subcontracting (S_1), to begin overtime production (S_2), and to construct new facilities (S_3). The correct choice depends largely upon the future demand, which may be low, medium, or high. By consensus, management ranks the respective probabilities as 0.10, 0.50 and 0.40. A cost analysis reveals the effect upon the profits. This is shown in the table below :

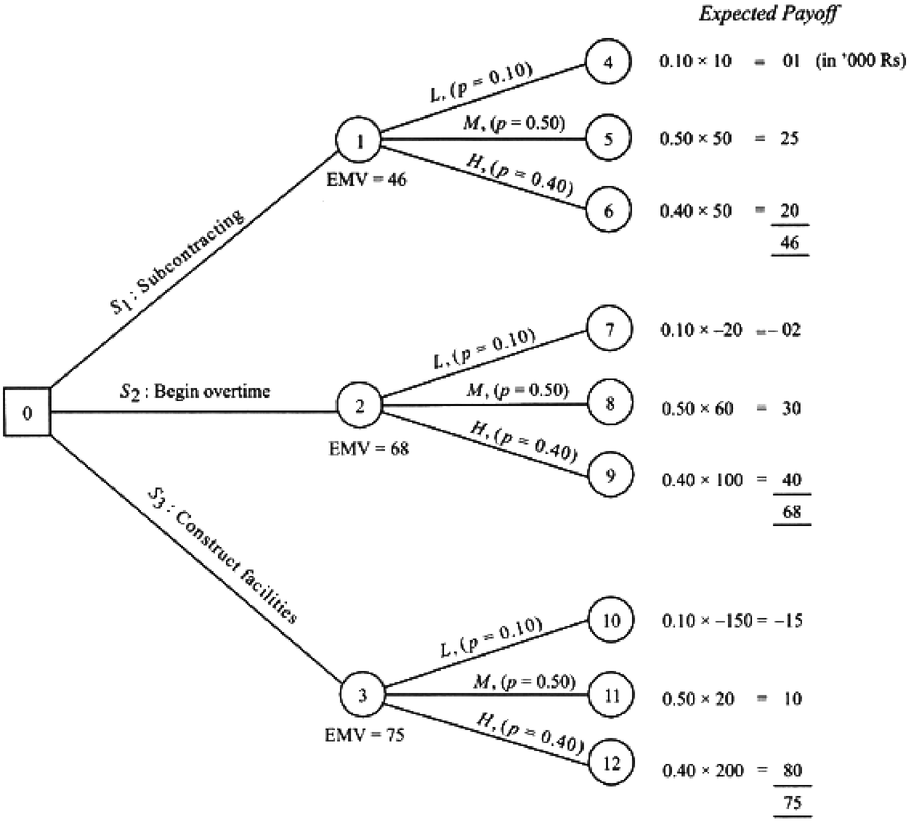
Demand	Probability	Course of Action		
		S_1 (Subcontracting)	S_2 (Begin Overtime)	S_3 (Construct Facilities)
Low (L)	0.10	10	-20	-150
Medium (M)	0.50	50	60	20
High (H)	0.40	50	100	200

Show this decision situation in the form of a decision tree and indicate the most preferred decision and its corresponding expected value.

Sol :

A decision tree that represents possible courses of action and states of nature is shown in Fig. below. In order to analyze the tree, we start working backwards from the end branches.

The most preferred decision at the decision node O is found by calculating the expected value of each decision branch and selecting the path (course of action) that has the highest value.



Since node 3 has the highest EMV, therefore, the decision at node 0 will be to choose the course of action S_3, i.e. construct new facilities.

10. A businessman has to independent investment portfolios A and B, available to him, but he lacks the capital to undertake both of them simultaneously. He can either choose A first and then stop, or if A is not successful, then take, B or vice versa. The probability of success of A is 0.6, while for B it is 0.4. Both investment schemes require an initial capital outlay of Rs. 10,000 and both return nothing if the venture proves to be unsuccessful. Successful completion of A will return Rs. 20,000 (over cost) and successful completion of B will return Rs 24,000 (over cost). Draw a decision tree in order to determine the best strategy.

Sol :

The decision tree based on the given information is shown in figure. The evaluation of each chance node and decision is given in table.

Decision point	Outcomes (Rs)	Probability	Conditional value	Expected value
D_3 (i) Accept A	Success	0.6	20,000	12,000
	Failure	0.4	– 10,000	– 4,000
				8,000
(ii) Stop	-	-	-	0
D_2 (i) Accept B	Success	0.4	24,000	9,600
	Failure	0.6	– 10,000	– 6,000
				3,600
(ii) Stop	-	-	-	0
D_1 (i) Accept B	Success	0.6	20,000 + 3,600 = 23,600	14,160
	Failure	0.4	– 10,000	– 4,000
				10,160
(ii) Accept B	Success	0.4	24,000 + 8,000 = 32,000	12,800
	Failure	0.4	– 10,000	– 6,000
				6,800
(iii) Do nothing	-	-	- 0	

Table : Evaluation of Decision and Chance Nodes

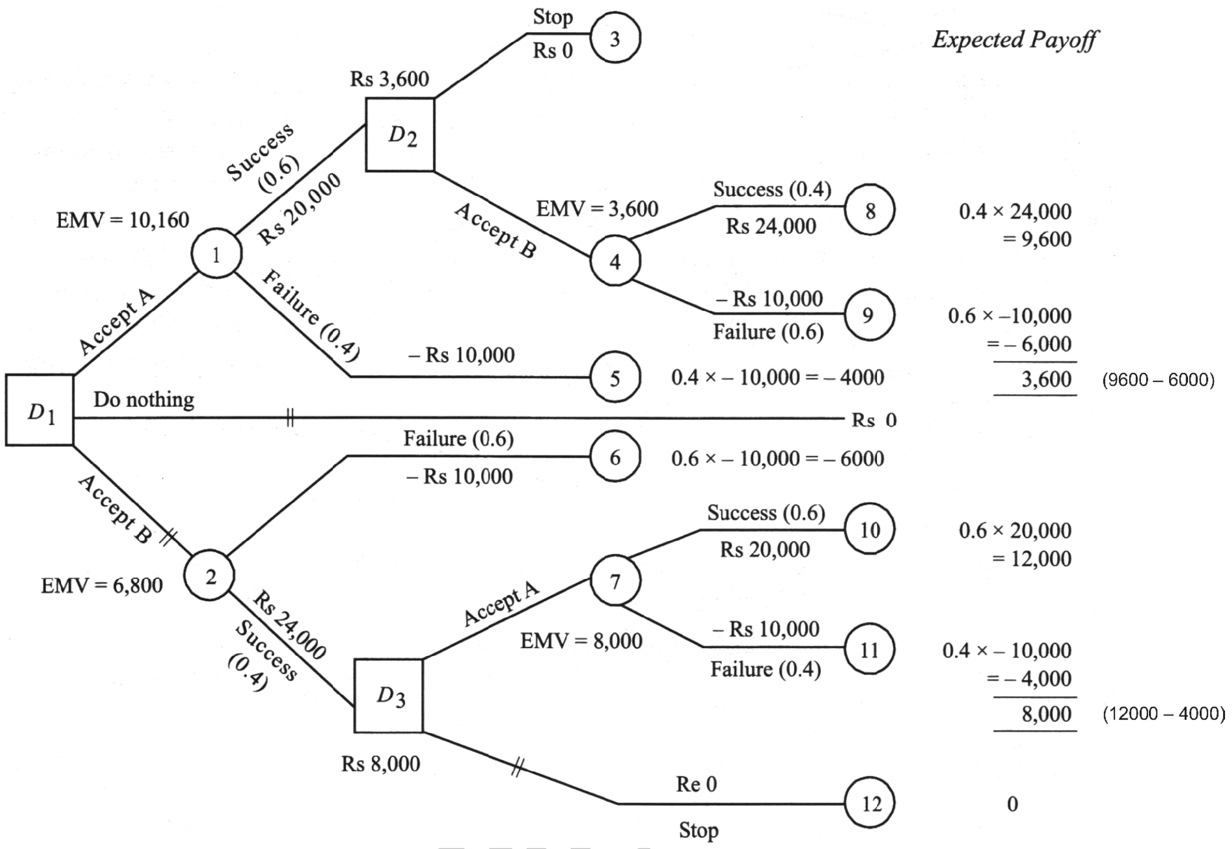


Figure : Decision Tree

Since the EMV = Rs 10,160 at node D1 is highest, therefore the best strategy is to accept course of action A first and if A is successful, then accept B.

4.6 NETWORK ANALYSIS

Q11. What is Network Analysis ? Explain the features of Network Analysis.

Ans :

Network Analysis

Network analysis is the analysis of a network, which is the graphic depiction of 'activities' and 'events'. It is done for planning, scheduling and controlling a project. The techniques used for network analysis are PERT and CPM.

Feature of Network Analysis

The following are the salient features of network analysis.

(i) Network analysis is a technique which is used to estimate the time and resources required for the completion of a project. This helps firm in planning, scheduling and controlling the projects.

(ii) The information of network analysis and the linkages of events represents the relationship and sequence existing between the activities of project.

(iii) The events of a network are meaningful and easily identifiable.

(iv) The estimates of time for each event are associated with uncertainties.

(v) Network analysis provide optimum solution to minimize the utilities of resources.

(vi) Through network analysis, the criticality of each event of a project can be determined.

Advantages of Using Networks

1. Networks give a logical representation of layout and sequence of difficult project.
2. Critical activities or events of the complete project (can be identified through networks).
3. Networks are the focus point for co-ordination and action.
4. The base of the project like cost, revenue, working out times etc., are provided by networks.
5. Networks help in planning and controlling complex projects.

4.6.1 Network Diagram

Q12. What is network diagram ? State the rules for drawing network. What are the steps involved in developing a network diagram ?

Ans : **(Imp.)**

Network Diagram

Network diagram can be defined as the technique of arranging the activities in proper sequence which shows the relationship and dependency of each activity with one another. It usually shows a process or sequence as how a task to be performed effectively. The activities under network diagram represents events using arrows, circles, arcs and nodes respectively.

Rules for Network Construction

1. Each activity is represented by only one arrow in the network.
2. Network should be developed on the basis of logical or technical dependencies between various activities of the project.
3. The arrow representing activities are indicative of the logical precedence only.
4. The arrow direction indicates the general progression in time.
5. When a number of activities terminate at one event, it indicates that no activity emanating from that event may start unless all activities terminating there have been completed.
6. Events (or) nodes are identified by numbers.
7. The activities are identified by the numbers of their starting and ending events.
8. A network should have only one initial and one terminal node.
9. An event may be a merge, burst or merge and burst event.
10. Dummy activities have to be used, if parallel activities between two events exist without intervening events.
11. Dummy activities have to be used when two or more activities have some of their immediate predecessor activities in common.
12. Looping is not permitted in a network.
13. Errors like dangling and redundancy are to be avoided.

Steps to Develop a Network and Its Components

In project scheduling, the first step is to sketch an arrow diagram called network diagram which shows interdependencies and the p.ecedence relationship among activities of the project.

To develop a network, the following steps are followed,

1. Break down the project into smaller activities.
2. Define the activities of the project.
3. Estimate the duration of each activity.
4. Identity the precedence relationships among various activities.
5. Draw the network using the above information in the form of circle connected by arrows.

Note that events are represented by circles and activities are represented by arrows.

Numbering a Network Diagram

After constructing the network, we know the preceding and succeeding activities of the project and the milestones. The milestones or key events have to be numbered understanding the progress of a project. For the purpose of numbering the events or nodes in a network diagram, we use the "I-J" rule.

The steps involved in numbering the network diagram is as follows,

Step-1

The start of a project is the starting event or the starting node of the network diagram and it is Burst event as we know Burst event is one which has arrows emerging from it but no arrow is entering into it. So find the start node and number the event as 1.

Step-2

Delete all the arrows emerging from the start node 1 and it results in one or more events or nodes as Burst events i.e. start event as described in step 1.

Step-3

Number one or more events obtained as start events from step 2. Note the numbering should be done in a chronological order and numbering can be done from top to bottom, i.e., case I, it results into two start events, then assign the event numbers 2 and 3.

Case-I

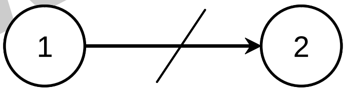


Fig. : Case of a Single Start Event Resulting from Step 2 and Numbering as per Step 3

Case-II

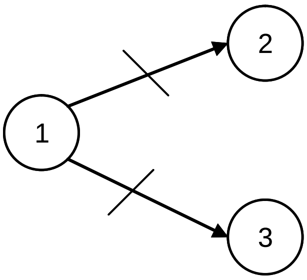


Fig. : Case of Two Start Events Resulting from Step 2 and Numbering as per Step 3

Step-4

Repeat step 2 and 3 until the completion event of the project is reached i.e., till you reach the final node indicated in the network diagram.

Q13. Explain the basic terminology are used in Network.

Ans :

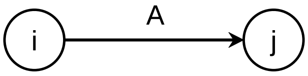
i) Network

It is the graphic representation of logically and sequentially connected arrows and nodes representing activities and events of a project. Networks are also called arrow diagram.

ii) Activity

An activity represents some action and is a time consuming effort necessary to complete a particular part of the overall project. Thus, each and every activity has a point of time where it begins and a point where it ends.

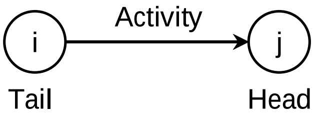
It is represented in the network by an arrow



Here A is called the activity.

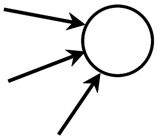
iii) Event

The beginning and end points of an activity are called events or nodes. Event is a point in the time and does not consume any resources. It is represented by a numbered circle. The head event called the jth event has always a number higher than the tail event called the ith event.

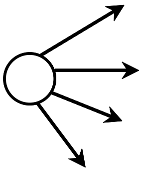


iv) Merge and Burst Events

It is not necessary for an event to be the ending event of only one activity but can be the ending event of two or more activities. Such event is defined as a merge event.



If the event happens to be the beginning event of two or more activities it is defined as a burst event.



v) Preceding, Succeeding and Concurrent Activities

- Activities which must be accomplished before a given event can occur are termed as preceding activities.
- Activities which cannot be accomplished until an event has occurred are termed as succeeding activities.
- Activities which can be accomplished concurrently are known as concurrent activities.

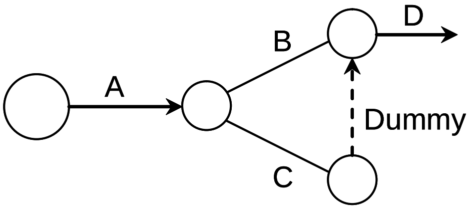
This classification is relative, which means that one activity can be preceding to a certain event and the same activity can be succeeding to some other event or it may be a concurrent activity with one or more activities.

vi) Dummy Activity

Certain activities which neither consumes time nor resources but are used simply to represent a connection or a link between the events are known as dummies. It is shown in the network by a dotted line. The purpose of introducing dummy activity is

(i) To maintain uniqueness in the numbering system as every activity may have distinct set of events by which the activity can be identified.

(ii) To maintain a proper logic in the network.



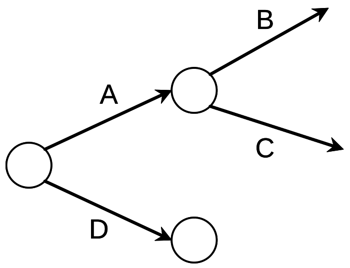
PROBLEMS ON CONSTRUCTION OF NETWORK

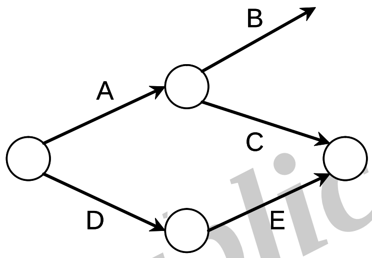
11. Construct a network for the project whose activities and their precedence relationships are as given below :

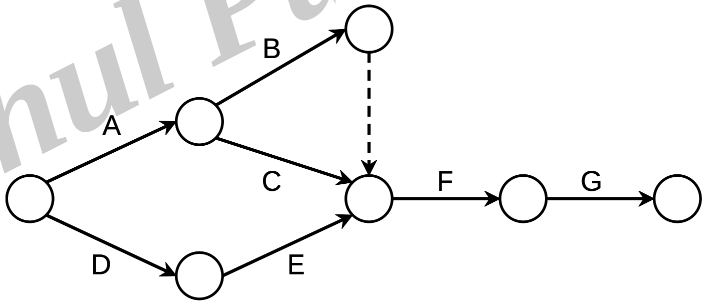
Activities	A	B	C	D	E	F	G	H	I
Immediate Predecessor	–	A	A	–	D	B,C,E	F	D	G,H

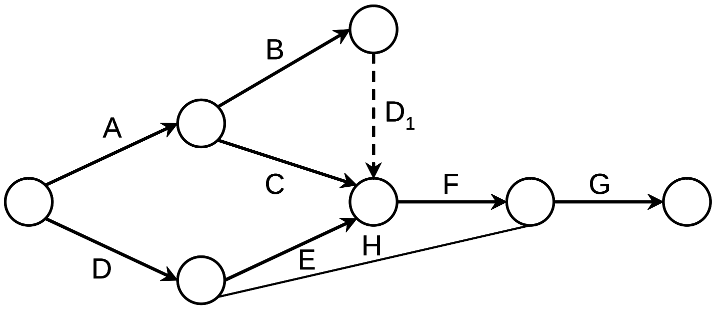
Sol :

From the given constraints, it is clear that A, D are the starting activity and I the terminal activity. B, C are starting with the same event and are both the predecessors of the activity F. Also E has to be the predecessor of both F and H. Hence, we have to introduce a dummy activity.



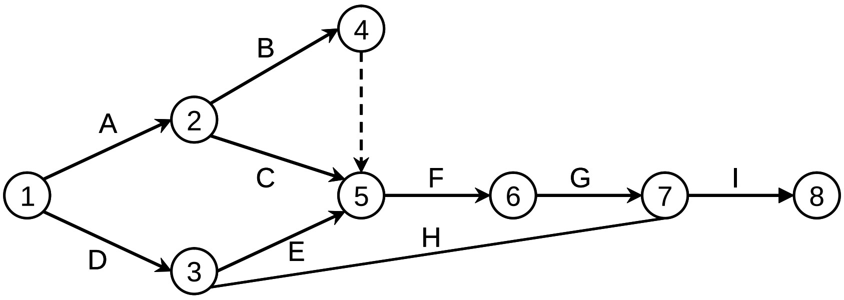






D_1 is the dummy activity.

Finally we have the following network.

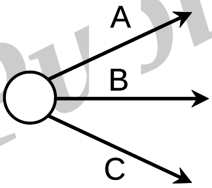


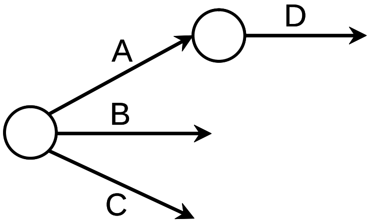
12. Construct a network for each of the projects whose activities and their precedence relationships are given below :

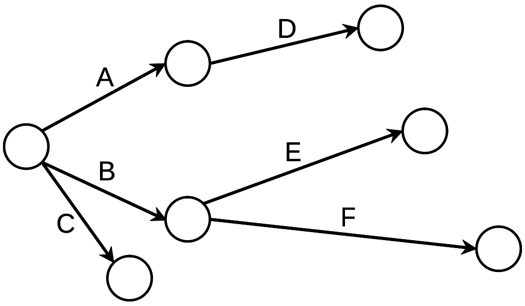
Activity	A	B	C	D	E	F	G	H	I	J	K
Predecessor	–	–	–	A	B	B	C	D	E	H,I	F,G

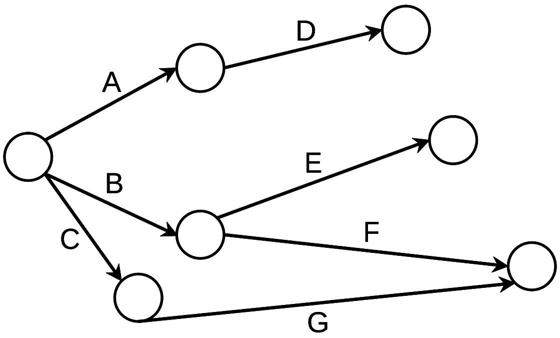
Sol :

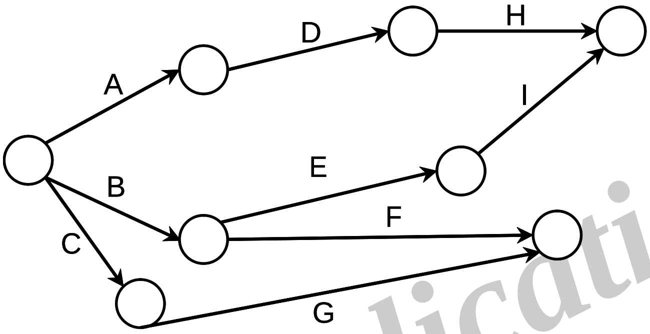
A, B, C are the concurrent activities as they start simultaneously. B becomes the predecessor of activity E and F. Since the actigities J, K have 2 preceeding activities dummy may be introduced (if possible).

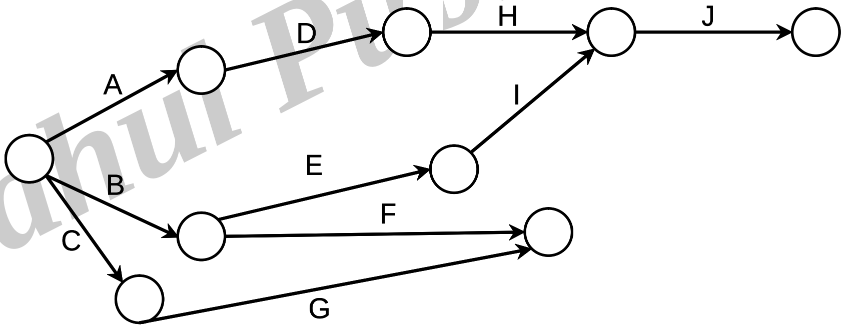




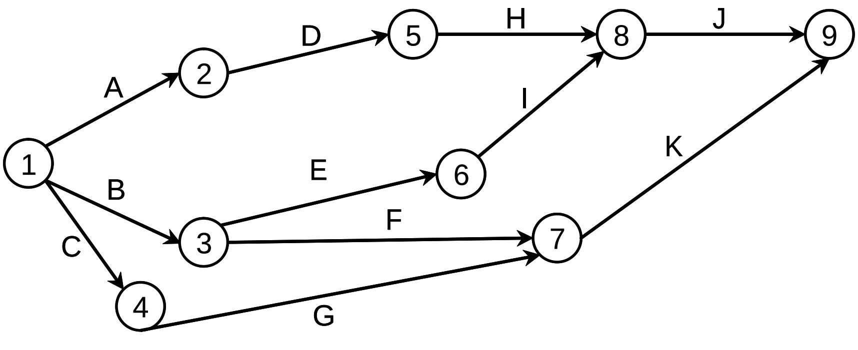








Finally, we have



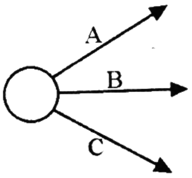
13. A, B, C can start simultaneously

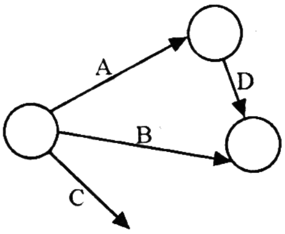
A < D, I; B < G, F; D < G, F; C < E; E < H, K; F < H, K; G, H < J

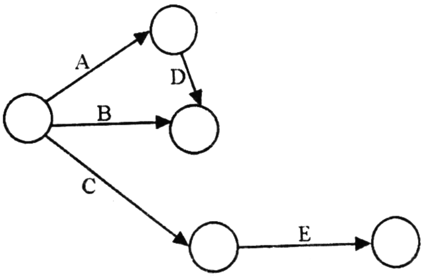
Sol :

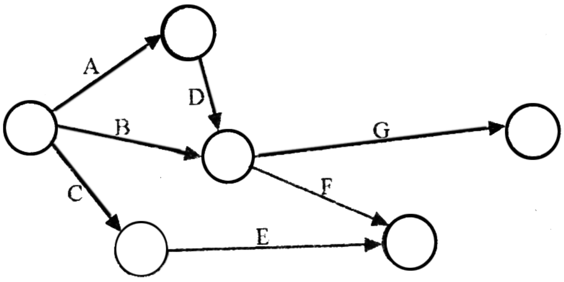
The above constrains can be formatted into a table.

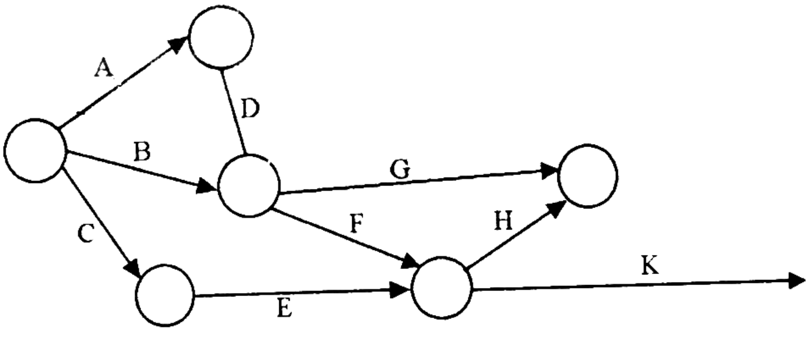
Activity	A	B	C	D	E	F	G	H	I	J	K
Predecessor Activity	–	–	–	A	C	B, D	B, D	E,F	A	G,	E,F

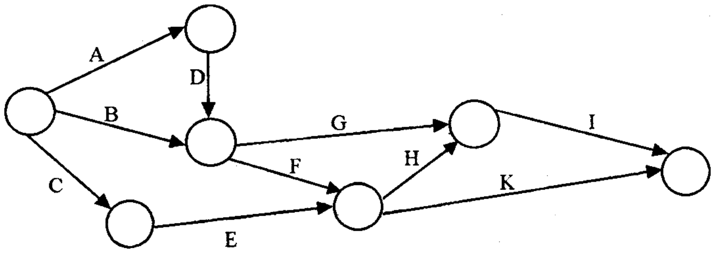


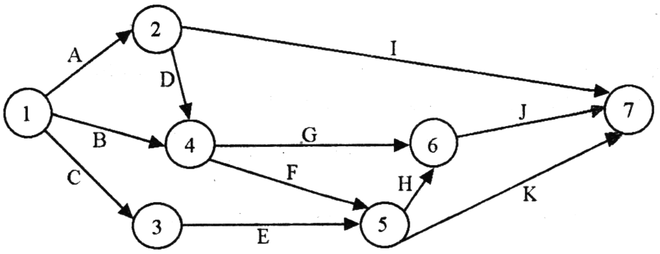












4.6.2 PERT

Q14. What is PERT? What are the three estimates needed for PERT analysis?

Ans :

PERT technique was developed to help the US Navy's Polaris Missile programme in its planning and scheduling. Since then, this technique has proved to be useful for all jobs which have an element of uncertainty in the matter of estimation of duration as in the case with new types of project.

PERT is concerned with estimating the time for different stages in such a programme or project and find out what the critical path is, i.e., which consumes the maximum resources.

PERT provides the framework with which a project can be described, scheduled and then controlled.

The project specification is the description of a project so that all interested parties know what is planned and what the outcome should be. The project is scheduled by the project costs, benefits and risks associated with it. Constraints are imposed to control the projects.

This probability distribution of activity times is based upon three afferent time estimates made for each activity. These are as follows.

i) Optimistic time estimate

ii) Most likely time estimate

iii) Pessimistic lime estimate

i) Optimistic time estimate

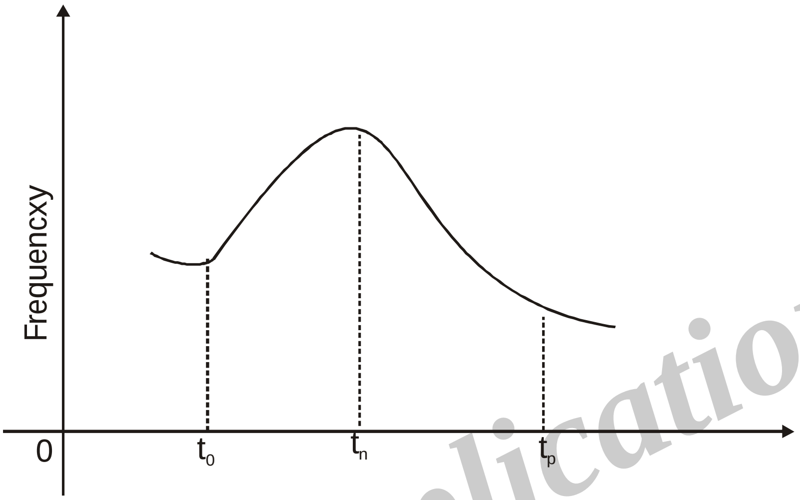
It is the smallest time taken to complete the activity if everything goes on well. There is very little chance that activity can be done in time less than the optimistic time. It is denoted by t_0 or a.

ii) Most likely time estimate

It refers to the estimate of the normal me the activity would take. This assumes normal delays. It is the mode the probability distribution. It is denoted by t_m or (m).

iii) Pessimistic time estimate

It is the longest time that an activity would take if everything goes wrong. It is denoted by t_p or b. These three time values are shown in the following figure.



Time distribution curve

From these three time estimates, we have to calculate the expected time of an activity. It is given by the weighted average of the three time estimates

$$\boxed{t_e = \frac{t_0 + 4t_m + t_p}{6}}$$

[β distribution with weights of 1,4,1, for t t_m, and t_p estimates respectively]

Variance of the activity is given by

$$\sigma^2 = \left[\frac{t_p - t_0}{6}\right]^2$$

The expected length (duration), denoted by to T_p of the entire project is the length of the critical path (ie) the sum of the t_e's of all the activities along the critical path.

The main objective in the analysis through PERT is to find the com-pletion for a particular event within specified date T_S given by P ($Z \leq$ D) where

$$D = \frac{\text{Due date} - \text{Expected date of completion}}{\sqrt{\text{Pr oject variance}}}$$

Where Z stands for standard normal variable.

Q15. State the steps involved in PERT.

Ans :

Step 1

Draw the project network.

Step 2

Compute the expected duration of each activity using formula formula.

$$t_e = \frac{\tau_0 + 4\ t_m + t_p}{6}$$

Also calculate the expected variance σ^2 of each activity.

Step 3

Compute the earliest start, earliest finish, latest start, latest finish and total float of each activity.

Step 4

Find the critical path and identify the critical activities.

Step 5

Compute the project length variance σ^2 which is the sum of the variance of all the critical activities and hence find the standard deviation of the project length σ.

Step 6

Calculate the standard normal variable $Z = \frac{T_s - T_e}{\sigma}$ where T_s is the scheduled time to complete the project.

T_e = Normal expected project length duration.

σ = Expected standard deviation of the project length.

Using the normal curve, we can estimate the probability of completing the project within a specified time.

4.6.3 Critical Path Method (CPM)

Q16. Define critical path. State the steps involved in determining critical path.

Ans : **(May-19)**

The sequence of critical activities in a network is called the critical path. The critical path is the longest path in the network from the starting event to ending and defines the minimum time required to complete the project.

Featurs of Critical Path

(i) If the project has to be shortened, then some of the activities on that path must also be shortened. The application of additional resources on other activities will not give the desired result unless that critical path is shortened first.

(ii) The variation in actual performance from the expected activity duration time will be completely reflected in one-to-one fashion in the anticipated completion of the whole project.

The iterative procedure of determining the critical path is as follows :

Step 1:

List all the jobs and then draw arrow (network) diagram. Each job is indicated by an arrow with the direction of the arrow showing the sequence of jobs. The length of the arrows has no significance. The arrows are placed based on the predecessor, successor, and concurrent relation within the job.

Step 2:

Indicate the normal time (t_{ij}) for each activity (i, j) above the arrow which is deterministic.

Step 3:

Calculate the earliest start time and the earliest finish time for each event and write the earliest time E_i for each event i in the ▭. Also calculate the latest finish and latest start time. From this we calculate the latest time L_j for each event j and put in the Δ.

Step 4:

Tabulate the various times namely normal time, earliest time and latest time on the arrow diagram.

Step 5:

Determine the total float for each activity by taking the difference between the earliest start and the latest start time.

Step 6:

Identify the critical activities and connect them with the beginning event and the ending event in the network diagram by double line arrows. Which gives the critical path.

Step 7:

Calculate the total project duration.

Note: The earliest start, finish time of an activity, and the latest start, finish time of an activity are shown in the table. These are calculated by using the following hints.

To find the earliest time we consider the tail event of the activity. Let the starting time of the project namely $ES_1 = 0$. Add the normal time with the starting time to get the earliest finish time. The earliest starting time for the tail event of the next activity is given by the maximum of the earliest finish time for the head event of the previous activity.

Similarly, to get the latest time, we consider the head event of the activity.

The latest finish time of the head event of the final activity is given by the target time of the project. The latest start time can be obtained by subtracting the normal time of that activity. The latest finish time for the head event of the next activity is given by the minimum of the latest start lime for the tail event of the previous activity.

Q17. What are the differences between PERT and CPM.

Ans : **(Dec.-19)**

S.No.	PERT	CPM
1.	A Probabilistic model with uncertainty in activity duration. The duration of each activity is normally computed from multiple time estimates with a view to take into account time uncertainty. These estimates are ultimately used to arrive at the probability of achieving any given scheduled date of project completion.	A deterministic model will well known activity (single) times based upon the past experience.
2.	It is said to be event oriented as the results of analysis are expressed in terms of events or distinct points in time indicative of progress.	It is activity oriented as the results of calculations are considered in terms of activities or operations of the project.
3.	The use of dummy activities is required for representing the proper sequencing.	The use of dummy activities is not necessary.
4.	It is used for repetitive jobs.	It is used for non-repetitive jobs.
5.	It is applied mainly for planning and scheduling research programmes.	It is used for construction and business problems.
6.	PERT analysis does not usually consider costs.	CPM deals with costs of project schedules and their minimization. The concept of crasing is applied mainly to CPM Models.
7.	PERT is an important control device too, for it assists the management in controlling a project by calling attention as a result of constant review to such delays in activities which might cause a delay in the project's completion date.	It is difficult to use CPM as a controlling device for the simple reason that one must repeat the entire evaluation of the project each time, the changes are introduced into the network.

4.6.4 Critical Path Determination

Q18. Discuss the various ways of determining critical path.

Ans :

Identification of Critical Path

There are two ways of determining the critical path as given below,

(i) Complete Enumeration Method

(a) Compute earliest and latest times for each of the activities in the project.

(b) Identify various paths for the initial node to the terminal node..

(c) List down the activities for each path.

(d) Determine the length of each path by adding up the time estimates of the activities in the path.

(e) The path which has the longest duration is called the critical path.

If there are more than one paths with longest duration, then all such paths are called critical paths. Thus, a network may have more than one critical path. The above method may be tedious for complex large projects. Hence, another method given below is usually followed.

(ii) Critical Path Determination Method

An algorithm for determination of the critical path in a network in terms of the scheduling times of the activities is given below.

Following rules apply in locating the critical path of a network,

(a) If for an activity, the ES time equals the LF time at the head of the arrow and ES time equals the LF time at the tail of the arrow, the activity is possibly a critical activity, lying on the critical path. This is the first necessary condition for criticality.

(b) If the first condition is met and the difference between ES time at the head of the arrow and ES time at the tail of the arrow is equal to the duration of the activity, then the activity is critical and lies on the critical path. This represents a sufficient condition for a critical activity.

(c) The sequence of such critical activities from the initial event to the terminal event forms the critical path.

4.6.5 Project Completion Time

Q19. Explain how do you calculate the probability of completing project within given time.

Ans :

We calculate the probability of project completion within the specified time, because in reality the total project may not be completed within time due to variability in the activity times as the project progresses.

By using the probability distribution, the probability of completing the project by schedule time (T_S) is given as,

$$\text{prob.}\left[Z \le \frac{T_S - T_e}{\sigma_e}\right]$$

and the value of Z is given by,

$$Z = \frac{T_S - T_e}{\sigma_e}$$

Where

T_S = Scheduled time

T_e = Expected completion time of the project (the value at last node of network diagram)

s_e = Variance along critical path.

Find the value of,

$$\frac{T_S - T_e}{\sigma_e} \text{ from normal distribution table.}$$

Similarly, to find the duration of the project that will have ‘X’ percentage chance of being completed can be calculated as given below.

$$P\left[z \le \frac{T_S - T_e}{\sigma_e}\right] = \frac{'X'}{100}$$

Here obtain the value of $\frac{'X'}{100}$ from normal distribution table, then solving the equation.

Value of $\frac{'X'}{100}$ from normal distribution table = $\frac{T_S - T_e}{\sigma_e}$ we can obtain the value of T_c (the duration of the project with 'X' percentage of chance of being completed.)

PROBLEMS

14. The following table shows the jobs of a network along with their time estimates.

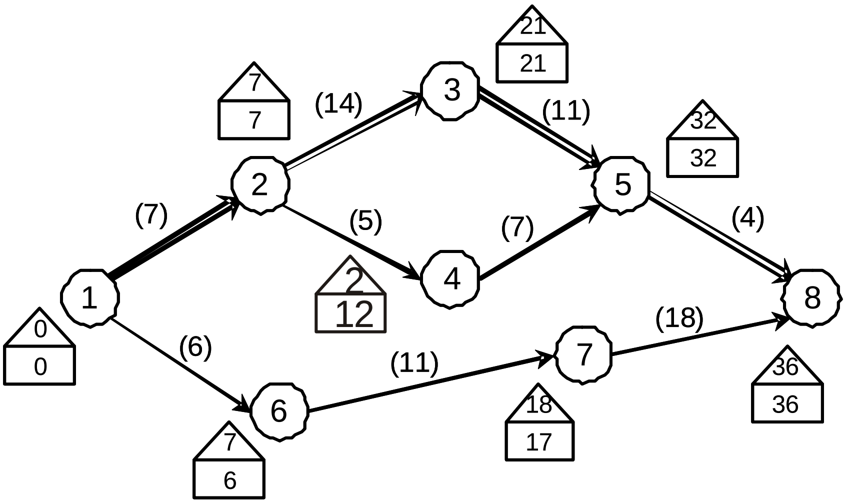
Job	1 – 2	1 – 6	2 – 3	2 – 4	3 – 5	4 – 5	6 – 7	5 – 8	7 – 8
a (days)	1	2	2	2	7	5	5	3	8
m (*)	7	5	14	5	10	5	8	3	17
b (*)	13	14	26	8	19	17	29	9	32

Draw the project network and find the probability that the project is completed in 40 days.

Sol :

First we calculate the expected time and standard deviation for each activity.

Activity	$\mathbf{t_e = \frac{t_a + 4t_m + t_p}{6}}$	$\mathbf{\sigma^2 = \left(\frac{t_p - t_a}{6}\right)^2}$
1 – 2	$\frac{1+4\times7+13}{6} = 7$	$\left(\frac{13-1}{6}\right)^2 = 4$
1 – 6	$\frac{2+4\times5+14}{6} = 6$	$\left(\frac{14-2}{6}\right)^2 = 4$
2 – 3	$\frac{2+4\times14+26}{6} = 14$	$\left(\frac{26-2}{6}\right)^2 = 16$
2 – 4	$\frac{2+5\times4+8}{6} = 5$	$\left(\frac{8-2}{6}\right)^2 = 1$
3 – 5	$\frac{7+4\times10+19}{6} = 11$	$\left(\frac{19-7}{6}\right)^2 = 4$
4 – 5	$\frac{5+5\times4+17}{6} = 7$	$\left(\frac{17-5}{6}\right)^2 = 4$
6 – 7	$\frac{5+8\times4+29}{6} = 11$	$\left(\frac{29-5}{6}\right)^2 = 16$
5 – 8	$\frac{3+3\times4+9}{6} = 4$	$\left(\frac{9-3}{6}\right)^2 = 1$
7 – 8	$\frac{8+4\times7+32}{6} = 18$	$\left(\frac{32-8}{6}\right)^2 = 16$



Expected project duration = 36 days

Critical path 1 – 2 – 3 – 5 – 8

Project length variance σ^2 = 4 + 16 + 4 + 1

= 25

σ = 5

Probability that the project will be completed in 40 days is given by

$P(Z \leq D)$

$$D = \frac{T_s - T_e}{\sigma} = \frac{40-36}{5} =- \frac{4}{5} = 0.8$$

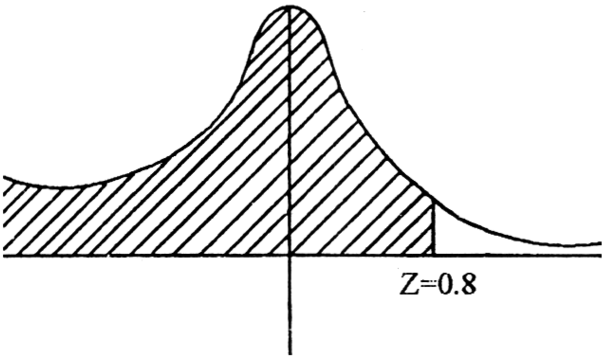
Area under the normal curve for Δ = 0.8

$P(Z \leq 0.8)$

$= 0.5 + \phi\,(0.8)$ [$\phi(8)$ = 0.2.881 (from table)]

= 0.5 + 0.2881 = 0.7881

= 78.81%



Conclusion : If the project is performed 100 times under the same conditions, there will be 78.81 occasions for this job to be completed in 40 days.

15. A small project is composed of seven activities whose time estimates are listed in the table as follows:

Activity	Estimated duration (weeks)		
	Optimistic	Most Likely	Pessimistic
1 – 2	1	1	7
1 – 3	1	4	7
2 – 4	2	2	8
2 – 5	1	1	1
3 – 5	2	5	14
4 – 6	2	5	8
5 – 6	3	6	15

You are required to:

(a) Draw the project network.

(b) Find the expected duration and variance of each activity.

(c) Calculate the early and late occurrence for each event and the expected project length.

(d) Calculate the variance and standard deviations of project length.

(e) What is the probability that the project will be completed—

(i) 4 weeks earlier than expected.

(ii) Not more than 4 weeks later than expected.

(iii) If the project due date is 19 weeks, what is the probability of meeting the due date.

Sol :

The expected time and variance of each activity is computed as shown in the table below :

Activity	a	m	b	$t_e = \frac{a+4t_m+b}{6}$	$s^2 = \left(\frac{b-a}{6}\right)^2$
1 – 2	1	1	7	2	1
1 – 3	1	4	7	4	1
1 – 4	2	2	8	3	1
2 – 5	1	1	1	1	0
3 – 5	2	5	14	6	4
4 – 6	2	5	8	5	1
5 – 6	3	6	15	7	4

The earliest and the latest occurrence time for each is calculated as below:

$E_1 = 0$; $E_2 = 0 + 2 + = 2$

$E_3 = 0 + 4 = 4$

$E_4 = 0 + 3 = 3$

$E_5 = \text{Max } (2 + 1, 4 + 6) = 10$

$E_6 = \text{Max } (10 + 7, 3 + 5) = 17.$

To determine the latest expected time we start from E_6 being the last event and move backwards subtracting t_e from each activity. Hence, we have

$L_6 = E_6 = 17$

$L_3 = L_6 - 7 = 17 - 7 = 10$

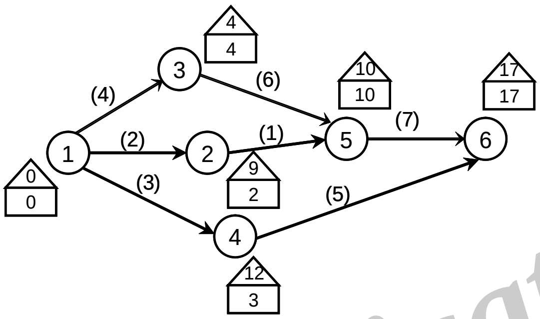
$L_4 = 17 - 5 = 12$

$L_3 = 10 - 6 = 4$

$L_2 = 10 - 1 = 9$

$L_1 = \text{Min}\ (9 - 2, 4 - 4, 12 - 3) = 0$

Using the above information we get the following network, where the critical path is shown by the double line arrow.



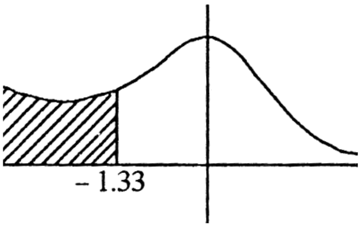
We observe the critical path of the above network as 1 $\rightarrow$ 3 $\rightarrow$ 5 – 6.

The expected project duration is 17 weeks i.e. $T_e = 17$ weeks.

The variance of the project length is given by

$a^2 = 1 + 4 + 4 = 9.$

(i) The probability of completing the project within 4 weeks earlier than expected is given by



$$P(Z \leq D) \text{ where } D = \frac{T_s - T}{s}$$

$$D = \frac{\text{Due date} - \text{Expected date of completion}}{\sqrt{\text{Project variance}}}$$

$$D = \frac{17 - 4 - 17}{3} = \frac{13 - 17}{3} = \frac{-4}{3}$$

$$= -1.33$$

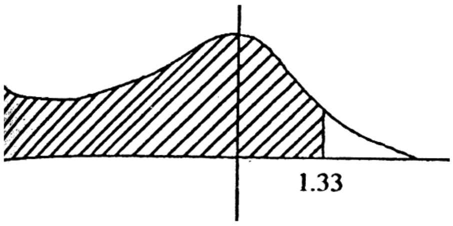
$\therefore$ $P(Z \leq -1.33) = 0.5 - \phi(1.33)$

$= 0.5 - 0.4082$ (from the table)

$= 0.918 = 9.18\%$

Conclusion : If the project is performed 100 times under the same conditions, then there will be 9 occassions for this job to be completed in 1 weeks earlier than expected.

(ii) The probability of completing the project not more than 4 weeks later than expected is given by



P(Z $\leq$ D) where

$$D = \frac{T_s - T_e}{\sigma}$$

Here $T_s = 17 + 4 = 21$

$$0 = \frac{21-17}{3} = \frac{4}{3} = 1.33$$

P(Z $\leq$ 1.33)

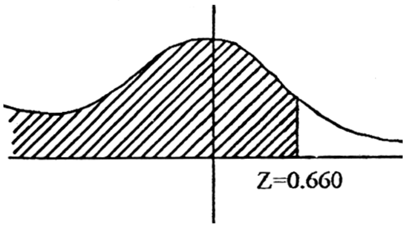
= 0.5 + ϕ(1.33)

= 0.5 + 0.4082 (from the table)

= 0.9082 = 90.82%

Conclusion : If the project is performed 100 times under the same conditions, then there will be 90.82 occasions when the job will be complated not more than 4 weeks later than expected

(iii) The probability of completing the project within 19 weeks, is given by



P(Z $\leq$ D) where $D = \frac{19-17}{3} = \frac{2}{3}$ [$\because T_s = 19$]

= 0.666

i.e., P(Z $\leq$ 0.666) = 0.5 + ϕ(0.666).

= 0.5 + 0.2514 (from the table)

= 0.7514 = 74.14%

Conclusion : If the project is performed 100 times, under the same conditions, then there will be 75.14 occasions for this job to be completed in 19 weeks.

16. A small maintaince project consists of following jobs whose precedence relationships is given below.

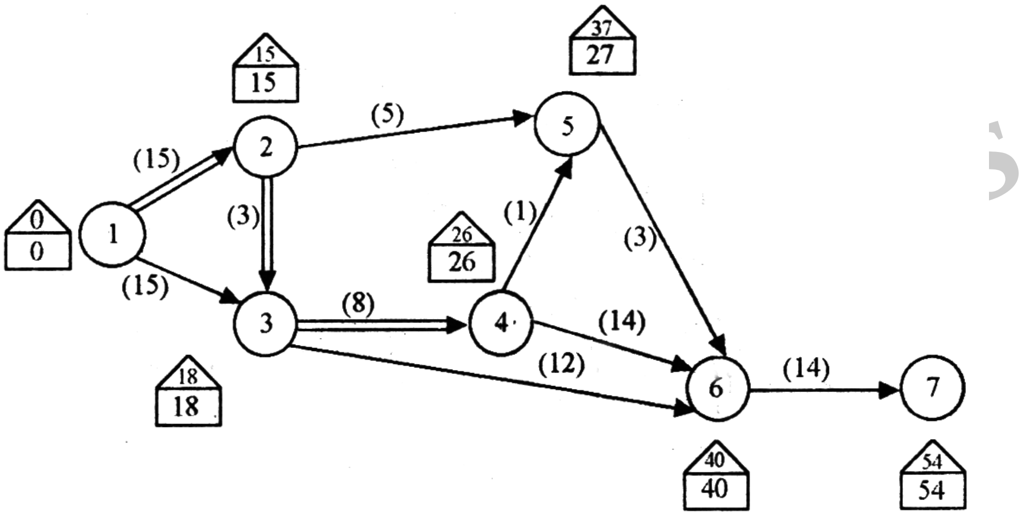
Job	1 – 2	1 – 3	2 – 3	2 – 5	3 – 4	3 – 6	4 – 5	4 – 6	5 – 6	6 – 7
Duration(days)	15	15	3	5	8	12	1	14	3	14

(a) Draw an arrow diagram representing the project.

(b) Find the total float for each activity.

(c) Find the critical path and the total project duration

Sol :



Foorward pass calculation

In this we estimate the earliest start and the earliest finish time ES_1 given by

ES_j = Max ($ES_i + t_{ij}$) where ES_i is the earliest start time and t_{ij} is the normal time for the activity (i, j).

$ES_1 = 0$

$ES_2 = ES\ i + t_{15} = 0 + 15 = 15$

$ES_3 = \text{Max}(ES_2 + t_{23}, ES_1 + t_{13})$

$= \text{Max}(15 + 3, 0+15) = 18$

$ES_4 = ES_3 + t_{34} = 18 + 8 = 26$

$ES_5 = \text{Max}(ES_2 + t_{25}, ES_4 + t_{45})$

$= \text{Max}(15 + 12, 26+1) = 27$

$ES_6 = \text{Max}(ES_3 + t_{3b}, ES_4 + t_{46}, ES_5 + t_{56})$

$= \text{Max}(18 + 12, 26 + 14, 27 + 3)$

$= 40$

$ES_7 = ES_6 + t_{67} = 40 + 14 = 54.$

Backward pass calculation

In this we calculate the latest finish and latest start time Lf_i given by $LF_i = Min_i (LF_j – t_{ij})$ where LF_j is the latest finish time for the event j.

$LF_i = 54$

$LF_b = LF_7 – f_{67} = 54 – 14 = 40$

$LF_5 = LS_6 – t_{56} = 40 – 3 = 37$

$LF_4 = Min (LS_5 – t_{45} , LS_6 – t_{46})$

$= Min (37 – 1.40 – 14) = 26$

$LF_3 = Min (LF_4 – t_{34}, LF_6 – t_{36})$

$= Min (26 – 8, 40 – 12)= 18$

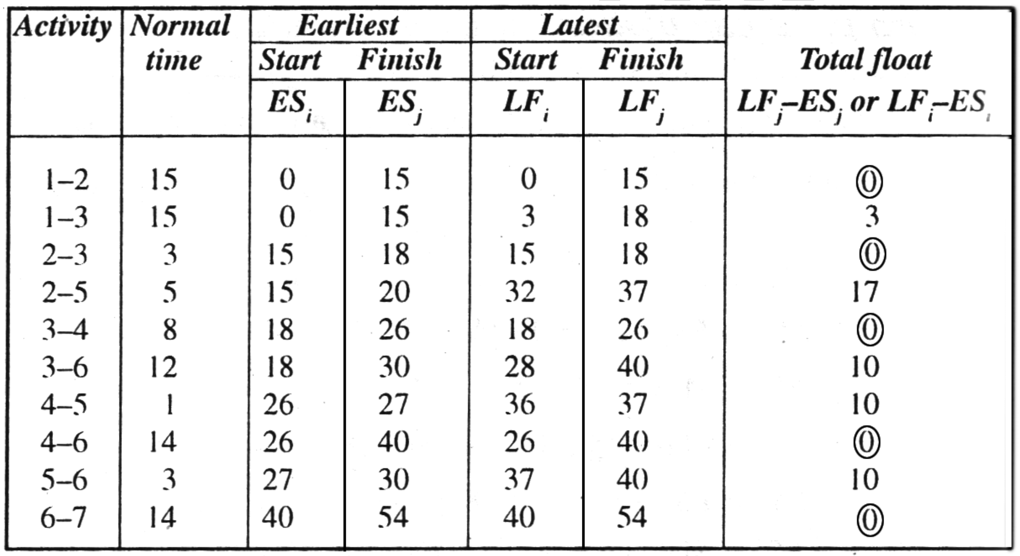
$LF_2 = Min (LF_5 – t_{25}, LF_3 – t_{23})$

$= Min (37 – 5, 18 – 3) = 15$

$LF_1 = Min (LF_3 – t_{13} , LF_2 – t_{12})$

$= Min (18 – 15, 15 – 15) = 0$

The following table gives the calculation for critical path and total float.



Activity	*Normal time*	*Earliest*		*Latest*		*Total float*
		Start	*Finish*	*Start*	*Finish*	
		ES_i	ES_j	LF_i	LF_j	$LF_j–ES_j$ or $LF_i–ES_i$
1–2	15	0	15	0	15	⓪
1–3	15	0	15	3	18	3
2–3	3	15	18	15	18	⓪
2–5	5	15	20	32	37	17
3–4	8	18	26	18	26	⓪
3–6	12	18	30	28	40	10
4–5	1	26	27	36	37	10
4–6	14	26	40	26	40	⓪
5–6	3	27	30	37	40	10
6–7	14	40	54	40	54	⓪

From the above table we observe that the activities 1 – 2, 2 – 3, 3 –- 4 – 6, 6 – 7 are the critical activities and the critical path is given by 1 – 2 – 3 – 4 – 6 – 7

The total project completion is given by 54 days.

4.6.6 Project Crashing

Q20. What is Project Crashing ?

Ans :

The time-cost trade-off problem is based on the conception that the duration of some of the activities of a project can be cut down, if some additional resources-men, material and/or equipment-are employed on them. For example, assigned to it and only 8 days with 5 people assigned. Of course, as additional personnel are assigned to an activity, the cost of performing that task normally increases. In our example, the cost of completing the activity with 3 people assigned is 36 man-days while with 5 people it jumps to 40 man-days.

Diseconomies of scale may be exhibited with too much labour on the job with inefficiency increasing very rapidly. In general, the more the time by which an activity time is required to be cut, the greater the amount of resources required to be employed on it. Thus, higher amounts of direct activity cost would be associated with smaller activity duration times, while longer duration time would involve comparatively lower direct cost. Such deliberate reduction of activity times by putting in extra effort is called **crashing**.

It is significant to note that for technical reasons, the duration of an activity cannot be reduced indefinitely. The crash time represents the fully expedited or the minimum activity duration time that is possible, and any attempts to further 'crash' would only raise the activity direct costs without reducing the time. The activity cost corresponding to the crash time is called the **crash cost** which equals the minimum direct cost required to achieve the crash performance time. These are in contrast to the normal activity time and the normal activity cost.

The time-cost relationships can be visualized graphically in the form of a time versus cost curve which is, for a limited portion at least, sloping downward, as shown in the graph. The widely accepted convex shape of the curve between the crash time-cost point and the normal time-cost point.

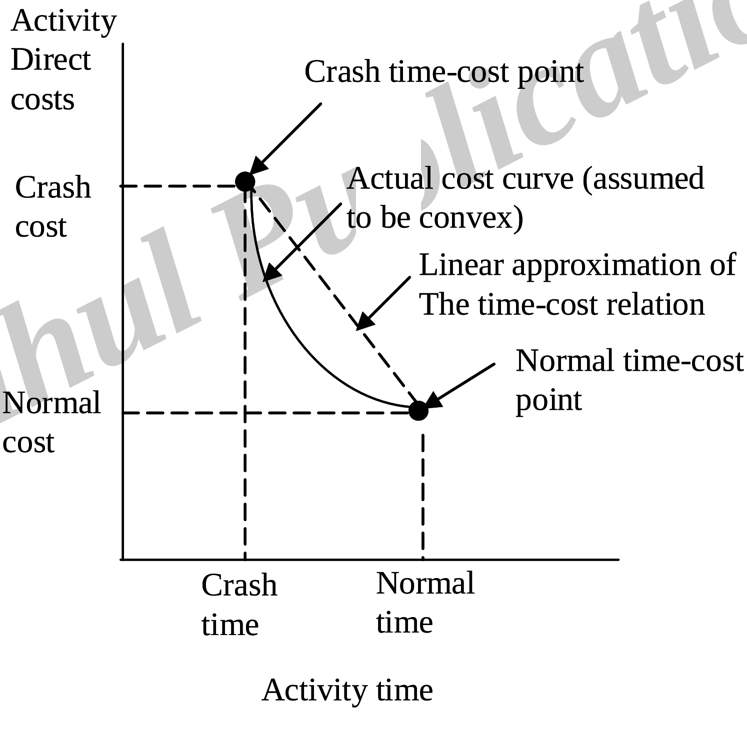


Figure : Time-Cost Curve

indicates that it is marginally costlier to induce the last percentages of reduction in activity time duration than the first percentages. The linear approximation represents a linear incremental cost per unit of time saved under conditions of crashing. The incremental cost for an activity can be determined using the following equation :

$$\text{Incremental Cost} = \frac{\text{Crash Cost-Normal Cost}}{\text{Normal time-Crash time}}$$

Thus, for an activity, if crash cost and time are Rs. 700 and 8 days, and if the normal cost and time are Rs. 500 and 12 days, we have

$$\text{Incremental cost per day} = \frac{700-500}{12-8} = \frac{200}{4}$$

$$= \text{Rs. } 50.$$

The time-cost trade-off analysis comprises the following stepts.

Step 1 :

The first step is to identify and crash the critical activity that has the minimum incremental cost of crashing. In the event of multiple critical paths, an activity from each such path is chosen. Of the various combinations available, the one with the least cost is selected. In particular, it may be economical to consider joint critical activities-activities that are common to two or more critical paths.

In each case, the crashing is done for one time unit-by a day if the activity times are given in days.

Step 2 :

In the second step, the network is revised by adjusting the time and cost of the crashed activity. The critical path(s) is identified again, and we revert to the step 1. This process is continued till no more crashing of the project is possible.

Now the optimal duration of the project can be determined. It would be the time duration corresponding to which the total cost-direct cost plus indirect cost-is the minimum.

PROBLEMS

17. The following is a table showing details of a project,

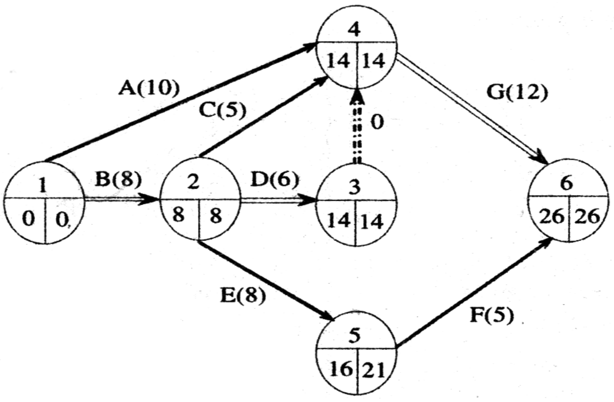
Task	Immediate Predecessor	Normal time in Weeks	Normal cost in Thousands `	Crash time in Weeks	Crash Cost in Thousands `
A	-	10	20	7	30
B	-	8	15	6	20
C	B	5	8	4	14
D	B	6	11	4	15
E	B	8	9	5	15
F	E	5	5	4	8
G	A,D,C	12	3	8	4

Indirect cost in ` 400 per day. Find the optimum duration and the associated cost of the project.

Sol :

Time Cost Trade off

Optimal project duration is the one which has the least total cost



The earliest and latest times for each activity were computed using the formulae,

$E_j = \text{Max}\ [E_i + t_{ij}]$

$L_i = \text{Min}\ [L_j - t_{ij}]$

Critical activities: 1 – 2, 2 – 3, 3 – 4, 4 – 6

Critical path : 1-2-3-4-6

Project duration: 26 weeks.

$$\boxed{\text{Total cost} = \text{Direct normal cost} + \text{Indirect cost for 26 weeks}}$$

$= 71{,}000 + (26 \times 7 \times 400)$

$= 71{,}000 + 72{,}800$

= ` 1,43,800.

Incremental cost due to crashing is,

$$\text{Cost Slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$$

Cost Slopes

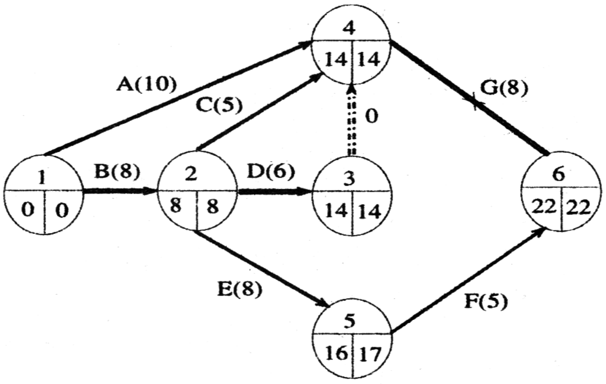
Critical Activity	**Crash Cost Per Week (`)**
B(1 – 2)	$\frac{20-15}{8-6} = 2.5$
D(2 – 3)	$\frac{15-11}{6-4} = 2$
G(4 – 6)	$\frac{4-3}{12-8} = 0.25$

Iteration 1

The critical activity G has the lowest cost slope (i.e., ` 0.25 × 1000) = ` 250 per week and thus is crashed by 4 weeks.

Crashing Activity G(4 - 6) 4 Weeks

Recompute the earliest time and latest time of each activity with duration of activity G(4 – 6) as 8 instead of 12.



Critical activities: 1 – 2, 2 – 3, 3 – 4, 4 – 6

Critical path: 1 - 2 - 3 - 4 - 6

Project duration: 22 weeks.

Total Cradled cost = Direct Normal Cost + Increased Direct Cost due to Crashing of G + Indirect Cost for 22 weeks

= 71,000 + (0.25 × 1000 × 4) + (22 × 7 × 400)

= 71,000 + 1000 + 61600

= ` 1,33,600.

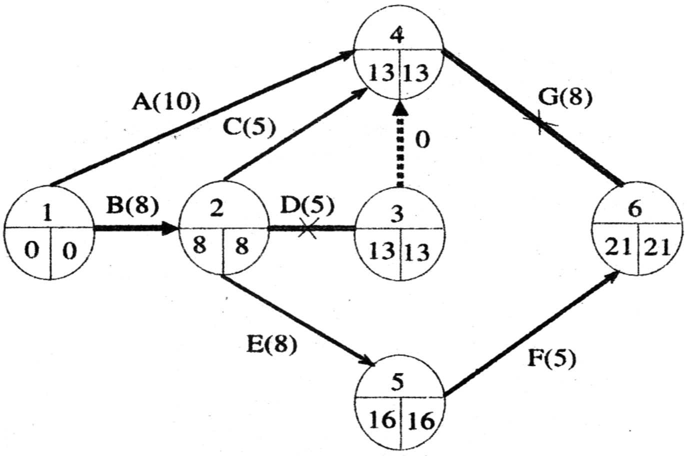
This cost (` 1,33,600) is less than cost before crashing (` 1,43,800). Try crashing further.

Iteration 2

The next least cost activity on the critical path is activity D(2 – 3). Crash D(2 – 3) only by 1 week though it can be crashed by 2 weeks, because the critical path has duration of 22 weeks and next longest path is 21 weeks and the difference is 1 week.

Crashing Activity D (2 – 3) by 1 Week

Recompute the earliest time and latest time with duration of activity D(2 – 3) as 5 instead of 6.



Critical activities: 1 – 2, 2 – 3, 3 – 4, 4 – 6

Critical paths : 1 - 2 - 5 - 6
: 1 - 2 - 3 - 4 - 6
: 1 - 2 - 4 - 6

Project duration : 21 weeks.

Total Crashed Cost = Direct Normal Cost + Increased Direct Cost due to Crashing of D + Indirect cost for 21 weeks

= 71,000 + 1000 + (2 × 1000 × 1) + (21 × 7 × 400)
= 71,000 + 1000 + 2000 + 58,800
= ` 1,32,800.

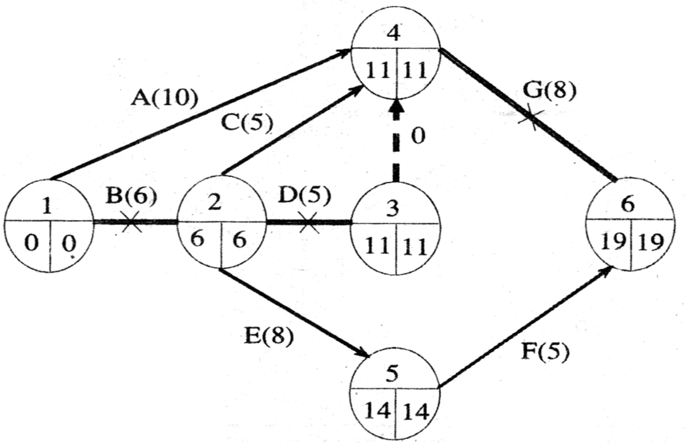
This cost is less than the previous cost. Try crashing further.

Iteration 3

As activity B is common in all three critical paths, it can be crashed to a maximum of 2 weeks.

Crashing Activity B(1 – 2) by 2 Weeks

Recompute times with duration of B(1 – 2) as 6 instead of 8.



Critical activities : 1 - 2, 2 - 3, 3 - 4, 4 - 6

Critical paths: 1 - 2 - 5 - 6
: 1 - 2 - 3 - 4 - 6
: 1 - 2 - 4 - 6

Project duration: 19 weeks.

Total crashed cost = Direct Normal cost + Increased Direct cost of due to Crashing of B, D and G + Indirect cost for 19 weeks

= 71,000 + (1,000 + 2000 + 2.5 × 1,000 × 2) + (19 × 7 × 400)
= 71,000 + (3000 + 5000) + 53,200
= 71,000 + 8000 + 53,200
= ` 1,32,200.

This cost is less than the previous cost. Try further.

Cost Slopes

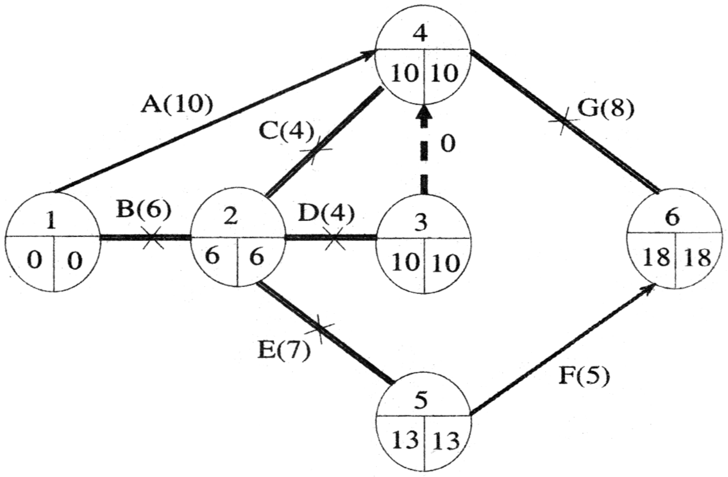
Critical Activity	Crash Cost Per Week (`)
C(2 – 4)	$\frac{14-8}{5-4} = 6$
D(2 – 3)	$\frac{15-11}{6-4} = 2$
E(2 – 5)	$\frac{15-9}{8-5} = 0.25$

Iteration 4

As activities B and G cannot be crashed anymore, hence crashing activity C in path 1 - 2 - 4 - 6, activity D in path 1 - 2 - 3 - 4 - 6 and activity E in path 1 - 2 - 5 - 6, can be crashed by 1 week as to minimize the project duration to 18 weeks which is equivalent to the project time of non-critical path 1-4-6.

Crashing activities C(2 - 4), D(2 - 3) and E(2 - 5) by 1 Week

Recompute the earliest time with duration of activities C(2 - 4) as 4 instead of 5, D(2 - 3) as 4 instead of 5 and E(2 - 5) as 7 instead of 8.



Critical activities: 1 - 2, 2 - 3, 3 - 4, 4 - 6

Critical Paths: 1 - 2 - 4 - 6

: 1 - 2 - 3 - 4 - 6

: 1 - 2 - 5 - 6

Project duration: 18 weeks.

Total Crashed Cost = Direct Normal cost + Increased Direct cost due to Crashing of Activities C, D and + E Indirect cost for 18 weeks

$$= 71,000 + (1,000 + 2000 + 5000) + (6 \times 1000 \times 1) + (2 \times 1000 \times 1)$$

$$+ (2 \times 1000 \times 1) + (18 \times 7 \times 400)$$

= 71,000 + 8000 + 6000 + 2000 + 2000 + 50,400

= ` 1,39,400.

The project cost for 18 weeks is more than the project cost for 19 weeks. As the total cost has increased, stop the iteration and the least cost is the previous iteration result.

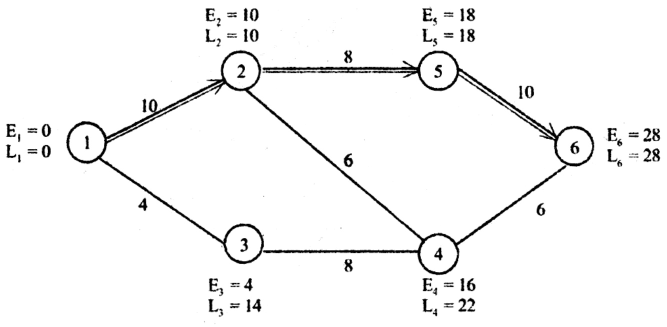
Thus, the optimum project duration is 19 weeks and the associated least cost is ` 1,32,200.

Q21. Given the following data, workout the minimum duration of the project and corresponding cost.

Activity	Job		Time		Cost
A	1-2	10	6	400	600
B	1-3	4	2	100	140
C	2-4	6	4	360	440
D	34	8	4	600	900
E	2-5	8	6	840	1100
F	4-6	6	2	200	300
G	5-6	10	8	1200	1400

Ans :

Step 1: Construct the Network



Critical path is 1 - 2 - 5 - 6.

Project duration is 28 days.

Total cost = 3700.

Step 2: Incremental Costs due to Crashing

$$\text{Cost slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$$

$$1 - 2 = \frac{600 - 400}{10 - 6} = 50$$

$$1 - 3 = \frac{140 - 100}{4 - 2} = 20$$

$$2 - 4 = \frac{440 - 360}{6 - 4} = 40$$

$$3 - 4 = \frac{900 - 600}{8 - 4} = 75$$

$$2 - 5 = \frac{1100 - 840}{8 - 6} = 130$$

$$4 - 6 = \frac{300 - 200}{6 - 2} = 25$$

$$5 - 6 = \frac{1400 - 1200}{10 - 8} = 100$$

Step 3: Crashing

Iteration -1

Critical path = 1 - 2 - 5 - 6.

Critical activities = 1 - 2, 2 - 5, 5 - 6.

The critical activity having the least cost slope is 1 - 2.

The various paths and their total time are as follows,

1 - 2 -5 - 6 = 10 + 8 + 10 = 28

1 - 2 - 4 - 6 = 10 + 6 + 6 = 22

1 - 3 - 4 - 6 = 4 + 8 + 6 = 18

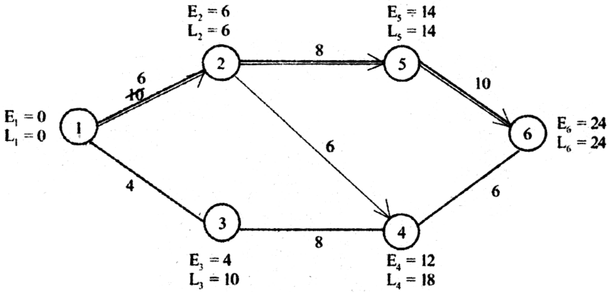
Since the difference between the critical path and the next longest path is 6, activity (1 - 2) can be crashed for a maximum of 6 units.

Activity 1 - 2 = Normal time – Crash time

= 10 - 6 = 4

But it is possible to crash only by 4 units.

Crash Activity 1 - 2 by 4 Units



Total cost = Normal direct cost + Direct cost due to crashing

= 3700 + (4 × 50)

= 3900.

Iteration - 2

Critical path = 1 - 2 - 5 - 6.

The critical activity having the next least cost slope is 5 - 6.

The various paths and their total time are as follows.

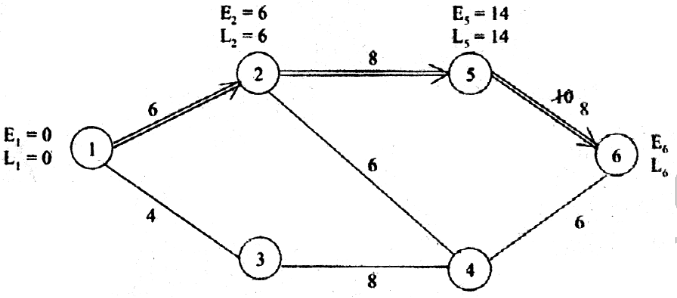
1 - 2 - 5 - 6 = 6 + 8 +10 = 24

1 - 2 - 4 - 6 = 6 + 6 + 6 = 18

1 - 3 - 4 - 6 = 4 + 8 + 6 = 18

The difference between critical path and the next longest path is 24 -18 = 6. But it is possible to crash only by 2 units.

Crash Activity 5 - 6 by 2 Units



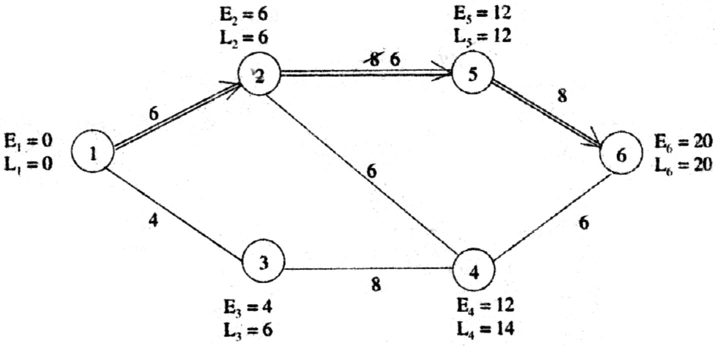
Total cost = (3700 + 4 × 50) + (2 × 100)

= 4100

Iteration - 3

The critical activity having the least cost now is 2-5.

Crash Activity 2 - 5 by 2 Units



Since, all critical activities has been fully crashed therefore further crashing is not possible. Iteration steps.

Total cost = 3700 + (4 × 50) + (2 × 100) + (2 × 130)

= 4360.

The maximum duration of the project is 20 days.

Total cost = 4360.

Exercises Problems

1. A small project consits of 11 activities A, B, C....K. The preceedence relationship A,B can start simultaneously. Given A < C, D, I; B < G, F; D < G,F;

Activity	A	B	C	D	E	F	G	H	I	J	K
Duration (days)	5	3	10	2	8	4	5	6	12	8	9

Draw the network of the project. Summarise the CPM calculations in a tabular form computing total, and free floats of activities and hence determine the critical path

[Ans : Critical path A - D - F - H - J - E]

2. A small maintenance project consists of the following 12 jobs.

Job	1 – 2	2 – 3	2 – 4	3 – 4	3 – 5	4 – 6	5 – 8
Duration(days)	2	7	3	3	5	3	5
Job	6 – 7	6 – 10	7 – 9	8 – 9	9 – 10		
Duration	8	4	4	1	7		

Determine the following critical path

[Ans : Critical path 1 – 2 – 3 – 4 – 5 – 6 – 7 – 9 – 10]

3. Consider the following data for activities in a given project.

Activity	A	B	C	D	E	F
Predecessor	–	A	–	B, C	C	D, E
Time(days)	5	4	7	3	4	2

Draw the arrow diagram for the project.Compute the earliest and the latest event times. What is the Minimum project completion time? List the activities on the critical path.

[Ans : A → B → E → dummy → F15 days]

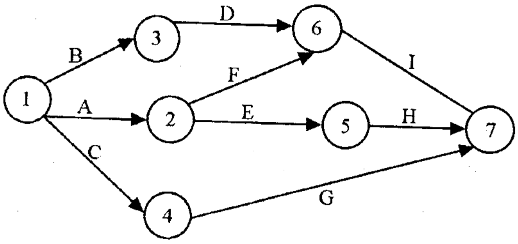
4. For the following project, determine the critical path and its duration ?

Activity	A	B	C	D	E	F	G	H
Predecessors	–	A	A	B	B	D, E	D	C, F, G
Time(days)	2	4	8	3	2	3	4	8

[Ans : 1 – 2 – 3 – 4 – 6 – 7; 21days]

5. A project is represented by the network shown below and has the following table :

Task	A	B	C	D	E	F	G	H	I
Least time	5	18	26	16	15	6	7	7	3
Greatest	10	22	40	20	25	12	12	9	5
Most likely time	8	20	33	18	20	9	10	8	4



Determine the following :

i) Expected tasks time and their variance

ii) The earliest and the latest expected time to reach each mode

iii) The critical path

iv) The probability of completing the project within 41.5 weeks.

[Ans : Critical path 1 → 4 → 7 Project duration = 42.8 weeks Probability of completing the project within 41.5 weeks = 0.30]

Short Question and Answers

1. Define decision theory.

Ans :

The success or failure that an individual or organization experiences, depends to a large extent, on the ability of making acceptable decisions on time. To arrive at such a decision, a decision-maker needs to enumerate feasible and viable courses of action (alternatives or strategies), the projection of consequences associated with each course of action, and a measure of effectiveness (or an objective) to identify the best course of action.

Decision theory is both descriptive and prescriptive business modeling approach to classify the degree of knowledge and compare expected outcomes due to several courses of action. The degree of knowledge is divided into four categories: complete knowledge (i.e. certainty), ignorance, risk and uncertainty as shown in Fig. below.

Ignorance	Uncertainty	Risk	Certainty

Increasing knowledge →

A decision problem or a decision-making situation includes two components, viz., the decision or several possible acts and the actual events that may occur in the future known as states of nature. While making the decisions the manager is uncertain which states of nature will occur in the future and also he has no control over them.

2. What are the different decision making environments?

Ans :

1. Decision-making Under Uncertainty

Decision-making under uncertainty illustrates a situation where in more than two outcomes/ events may occur from one decision point and decision maker has no knowledge regarding the probabilities to be assigned to occurrence of each event or state of nature. The lack of information regarding the probabilities of occurrences of events make the decision making process very complicated. Under uncertainty situations, decision maker cannot compute expected pay offs for each course of action due to lack of probabilities. Launching of new product into the market, setting up of new plant etc., can be taken as examples of uncertainty situation. The selection of one best course of action relies upon the nature of decision-maker and rules of the organisation.

The following choices are available before the decision maker in situ-ations of uncertainty.

(a) Maximax Criterion

(b) Minimax Criterion

(c) Maximin Criterion

(d) Laplace Criterion (Criterion of equally likelihood)

(e) Hurwicz Alpha Criterion (Criterion of Realism)

2. Decision-making Under Risk

Often business decisions are taken under risk conditions. Like uncertainty, two or more outcomes/events may occur from one single decision but decision maker possesses required knowledge regarding what probabilities can be assigned to each state of nature. The information regarding the probabilities of each state of nature can be acquired either from historical records or from personal judgements of the decision maker. As probabilities of each state of nature are known under conditions of risk, the course of action (or alternative strategy) with highest expected value is chosen as a best course of action.

The criterias used for selecting the best course of action under conditions of risk are,

(a) Expected Monetary Value (EMV)

(b) Expected Opportunity Loss (EOL)

3. Decision-making Under Certainty

Decision making under certainty pertains to a situation wherein the decision maker knows with certainty the outcome of each course of action. Each decision has only one state of nature and decision maker chooses one best pay off among the available alternative strategies. The state of nature is arraigned with the probability equal to ' 1' as only one state of nature occurs from each course of action (alternative strategy). Eventhough stage of nature is only one, courses of action may be many. We rarely find managerial decision problems with complete information regarding future outcomes.

3. Expected Monetary Value.

Ans :

When the probabilities ran be assigned to the various states of nature, it is possible to calculate the statistical expectation of gain for each course of action.

The conditional value of each event in the pay off table is multiplied by its probability and the product is summed up. The resulting number is the EMV for the act. The decision maker then selects from the available alternative actions, the action that leads to the maximum expected gain (that is the action with highest EMV).

4. Expected Opportunity Loss.

Ans :

The difference between the greater pay off and the actual pay off is known as opportunity loss. Under this criterion the strategy which has minimum Expected Opportunity Loss (EOL) is chosen. The calculation of EOL is similar to that of EMV.

Consider the following Example: Given below is an opportunity loss table. A_t and A_2 are the strategies and S_1 and S_2 are the states of nature.

	A_1	A_2
S_1	0	10
S_2	2	–5

Let the probabilities for two states be 0.6 and 0.4

EOL for A, = (0 × 0.6) + (2 × 0.4) = 0.8

EOL for A_2 = (10 × 0.6)+ (–5 × 0.4) = 6 – 2 = 4

EOL for A is least. Therefore the strategy A_1 may be chosen.

5. Decision Making Under Perfect Information.

Ans :

Information reduces the extent of uncertainty in decision-making. Assuming that the decision maker has perfect information about the market variables, decision-making takes a different turn.

Accurate, current and relevant information available in required details at the time of decision-making improves decision-making. For instance, a firm wants to increase the price of its products (decision to be made). However, this decision depends on the price strategy of the competitors (chance factors). Now, supposing that the firm is positive that the competitor will not raise their product prices, that is, the firm has perfect information about the occurrence of an outcome, the firm can choose its optimal strategy with confidence. The maximum expected monetary value that could be achieved by using perfect information is called Expected Profit with Perfect Information (EPPI).

Before obtaining any information the decision maker should know the significance of that information. Is the information required? What is the value of that information in decision-making? For this, compare the expected marginal profit you can gain by using the perfect information in question (EPPI) with the decisions made without the information. For instance, if an inventory manager knows the exact requirement of stock for each day, he can stock the same level of inventory and save on costs. The sales forecast is thus of great significance to the inventory manager.

6. Decision Tree Analysis.

Ans :

The diagrammatic representation of logical relationship between the parts of a complex situation of a decision making problem is called a decision tree. It specifies the choices, risks, objectives & monetary gains involved in a business problem.

However, situations may arise when a decision-marker to revise his previous decisions due to availability of additional information. Thus he intends to make a sequence of interrelated decisions over several future periods. Such a situation is called a *sequential or multi period decision process*. For example, in the process of marketing a new product, a company usually first go for 'Test Marketing' and other alternative courses of action might be either 'Intensive Testing' or 'Gradual Testing'. Given the various possible consequences - good, fair, or poor, the company may be required to decide between redesigning the product, an aggressive advertising campaign or complete withdrawal of product, etc. Based on this decision there might be an outcome that leads to another decision and so on.

7. Advantages of Decision Tree .

Ans :

The decision tree approach has several advantages like :

a) This approach facilitates investment decisions in a scientific way.

b) This approach gives an overall view of all the possibilities associated with a project, helps the management to take decisions keeping the entire situations in the mind.

c) As this technique links the probable outcomes of a decision one after an interrelated manner along with probabilities assigned to each sequential outcome, it is very useful in tacking investment situations requiring decisions to be taken in a sequence.

8. Disadvantages of Decision Tree

Ans :

The various disadvantages of the decision tree approach are as follows .

a) A prime decision may have a number of sequential decision points and each one of such decision points may have numerous decision branches or decision alternatives.

b) The decision tree analysis becomes very complex, when a project has a life more than two years.

9. Method of Constructing Decision Tree.

Ans :

1. Identification of all the possible courses of action.
2. List the possible results i.e., 'states of nature' of each course of action specified.
3. Calculation of payoff of each possible combination of courses of action and results. Payoff will be in monetary terms usually.
4. Assigning probabilities to the different possible results for each given course of action. Likelihood of occurrence of a particular event is being indicated by the probability.
5. At last choose that course of action which gives the maximum payoff.

10. What is Network Analysis ?

Ans :

Network analysis is the analysis of a network, which is the graphic depiction of 'activities' and 'events'. It is done for planning, scheduling and controlling a project. The techniques used for network analysis are PERT and CPM.

Feature of Network Analysis

The following are the salient features of network analysis.

(i) Network analysis is a technique which is used to estimate the time and resources required for the completion of a project. This helps firm in planning, scheduling and controlling the projects.

(ii) The information of network analysis and the linkages of events represents the relationship and sequence existing between the activities of project.

11. Rules for Network Construction.

Ans :

1. Each activity is represented by only one arrow in the network.
2. Network should be developed on the basis of logical or technical dependencies between various activities of the project.
3. The arrow representing activities are indicative of the logical precedence only.
4. The arrow direction indicates the general progression in time.
5. When a number of activities terminate at one event, it indicates that no activity emanating from that event may start unless all activities terminating there have been completed.
6. Events or nodes are identified by numbers.
7. The activities are identified by the numbers of their starting and ending events.
8. A network should have only one initial and one terminal node.
9. An event may be a merge, burst or merge and burst event.
10. Dummy activities have to be used, if parallel activities between two events exist without intervening events.
11. Dummy activities have to be used when two or more activities have some of their immediate predecessor activities in common.
12. Looping is not permitted in a network.
13. Errors like dangling and redundancy are to be avoided.

12. What is PERT ?

Ans :

PERT technique was developed to help the US Navy's Polaris Missile programme in its planning and scheduling. Since then, this technique has proved to be useful for all jobs which have an element of uncertainty in the matter of estimation of duration as in the case with new types of project.

PERT is concerned with estimating the time for different stages in such a programme or project and find out what the critical path is, i.e., which consumes the maximum resources.

PERT provides the framework with which a project can be described, scheduled and then controlled.

The project specification is the description of a project so that all interested parties know what is planned and what the outcome should be. The project is scheduled by the project costs, benefits and risks associated with it. Constraints are imposed to control the projects.

This probability distribution of activity times is based upon three afferent time estimates made for each activity. These are as follows.

i) Optimistic time estimate

ii) Most likely time estimate

iii) Pessimistic lime estimate

13. Define critical path.

Ans :

The sequence of critical activities in a network is called the critical path. The critical path is the longest path in the network from the starting event to ending and defines the minimum time required to complete the project.

14. Featurs of Critical Path

Ans :

(i) If the project has to be shortened, then some of the activities on that path must also be shortened. The application of additional resources on other activities will not give the desired result unless that critical path is shortened first.

(ii) The variation in actual performance from the expected activity duration time will be completely reflected in one-to-one fashion in the anticipated completion of the whole project.

15. What are the differences between PERT and CPM.

Ans :

S.No.	PERT	CPM
1.	A Probabilistic model with uncertainty in activity duration. The duration of each activity is normally computed from multiple time estimates with a view to take into account time uncertainty. These estimates are ultimately used to arrive at the probability of achieving any given scheduled date of project completion.	A deterministic model will well known activity (single) times based upon the past experience.
2.	It is said to be event oriented as the results of analysis are expressed in terms of events or distinct points in time indicative of progress.	It is activity oriented as the results of calculations are considered in terms of activities or operations of the project.
3.	The use of dummy activities is required for representing the proper sequencing.	The use of dummy activities is not necessary.
4.	It is used for repetitive jobs.	It is used for non-repetitive jobs.
5.	It is applied mainly for planning and scheduling research programmes.	It is used for construction and business problems.
6.	PERT analysis does not usually consider costs.	CPM deals with costs of project schedules and their minimization. The concept of crasing is applied mainly to CPM Models.
7.	PERT is an important control device too, for it assists the management in controlling a project by calling attention as a result of constant review to such delays in activities which might cause a delay in the project's completion date.	It is difficult to use CPM as a controlling device for the simple reason that one must repeat the entire evaluation of the project each time, the changes are introduced into the network.

16. What is Project Crashing ?

Ans :

The time-cost trade-off problem is based on the conception that the duration of some of the activities of a project can be cut down, if some additional resources-men, material and/or equipment-are employed on them. For example, assigned to it and only 8 days with 5 people assigned. Of course, as additional personnel are assigned to an activity, the cost of performing that task normally increases. In our example, the cost of completing the activity with 3 people assigned is 36 man-days while with 5 people it jumps to 40 man-days.

Diseconomies of scale may be exhibited with too much labour on the job with inefficiency increasing very rapidly. In general, the more the time by which an activity time is required to be cut, the greater the amount of resources required to be employed on it. Thus, higher amounts of direct activity cost would be associated with smaller activity duration times, while longer duration time would involve comparatively lower direct cost. Such deliberate reduction of activity times by putting in extra effort is called **crashing**.

It is significant to note that for technical reasons, the duration of an activity cannot be reduced indefinitely. The crash time represents the fully expedited or the minimum activity duration time that is possible, and any attempts to further 'crush' would only raise the activity direct costs without reducing the time. The activity cost corresponding to the crash time is called the **crash cost** which equals the minimum direct cost required to achieve the crash performance time. These are in contrast to the normal activity time and the normal activity cost.

17. Briefly explain the difference between expected opportunity loss and expected value of perfect information.

Ans :

The following are the differences between EOL and EVPI,

Points	Expected Opportunity Loss (EOL)	Expected Value of Perfect Information (EVPI)
1. Meaning	The expected opportunity loss (EOL) refers to the expected difference between the conditional value or profits of the best course of action and the course of action and the course of action taken.	The expected value of perfect information is the maximum cost a decision maker is agreeable to pay to acquire the perfect information about the happening or non-happening of an event.
2. Alternative	EOL is an alternative to EMV (Expected Monetary Value)	EVPI is not an alternative to EMV
3. Formulae	EOL is symbolically expressed as, EOL (State of nature, N_i) $= \sum_{i=0}^{n} P(O)_i L_{ij}$	The formula for calculating EVPI can be expressed as, EVPI = Expected profit with perfect information expected monetary Value.

18. What is Cost Slope ?

Ans :

The cost slope indicating the increase in cost per unit reduction in time is defined as

$$\text{Cost slope} = \frac{\text{Crash Cost - Normal Cost}}{\text{Normal time} - \text{Crash time}} = \frac{C_C - C_N}{T_N - T_C}$$

(i.e.) the cost slope represents the rate of increase in the cost of performing the activity per unit reduction in time and is called Cost/time trade off. It varies from activity to activity. The total project cost is the sum total of the project direct and indirect cost.

UNIT V

Queuing Theory: Queuing Structure and basic component of an Queuing Model, Distributions in Queuing Model, Different in Queuing Model with FCFS, Queue Discipline, Single and Multiple service station with finite and infinite population. Game theory, Saddle point, Value of the Game.

5.1 QUEUING THEORY

Q1. Define the term queue. State the various examples of queus.

Ans :

A flow of customers from finite/infinite population towards the service facility forms a queue (waiting line) on account of lack of capability to serve them all at a time. In the absence of a perfect balance between the service facilities and the customers, waiting time is required either for the service facilities or for the customer's arrival.

The arriving unit that requires some service to be performed is called customer. The customer may be persons, machines, vehicles, etc. Queue (waiting line) stands for the number of customers waiting to be serviced. This does not include the customer being serviced. The process or system that performs services to the customer is termed by service channel or service facility.

Examples:

(a) Industrial Production Process

- i) Facilities required to keep a batch of machines in economic operation. Supply of raw materials, despatch of finished products.
- ii) Costly items in stock (inventory).
- iii) Assembly lines.
- iv) Tool room service.
- vi) Storage/Dumps
- vii) Computer centres.

(b) Transport

- i) No. of bus terminals/bus stops.
- ii) No. of siding/platforms.
- iii) No.of runways/checking counters in airports.
- iv) Shipping - No. of births/pilots.

(c) Communication

$\left.\begin{array}{l}\text{(i) Trunk calls}\\ \text{(ii) Telephones}\end{array}\right\}$ No. of booths/line

(d) Public Service Industry

- i) Hospital Wards/Out Patients Deptt. required in a Hospital.
- ii) Level crossings/Tool booths required/Ticket counters.
- iii) Banks/Insurance Companies.

(e) Others

- i) (Human relations) / Co-ordination - No. of subordinates to an Executive.
- ii) Waiting for promotion.
- iii) Waiting to die from birth (stages).
- iv) Theatres/Hours for arranging screening of pictures/wedding/meetings.
- v) Waiting for ticket to see picture.

Q2. Explain briefly about queuing system.

(or)

What are the features of queuing syetem.

Ans : **(May-19, March-15)**

The essential features are

1. Input Source or Population

The input source or population is a set that contains the probable potential customers to come out for service.

2. Arrival Pattern

The customers are expected to arrive at their own convenience and conditions. The patterns of arrival times often follow one of the following probability distribution.

i) Poisson distribution (represented by M)

ii) Exponential distribution (represented by M)

iii) Erlang distribution (represented by E_r)

3. Limit of Queue

(Restricted/unrestricted queues). The limit of queue is some times restricted buy server's behaviour. If the limit is imposed by the server, the arrivals will be limited for a given period. If there is no restriction, the queue is said to be unlimited or infinite. The limited queue is denoted by N while unlimited queue by.

4. Queue Discipline

It is the order in which the customer is selected from the queue for service. There are numerous ways in which customers in queue can be served of which some are listed below.

(a) First In First Out (FIFO) or First Come first serve (FCFS) : It is the discipline in which the customers are served in the chronological order of their arrivals.

Eg. Tickets at a cinema hall; sales at a grocery shop, trains on a (single line) platform etc.

(b) Last In First Out (LIFO) or Last Come First Serve (LCFS) : If the service is made in opposite order of arrivals of customers, i.e, who ever comes last is served first and first obviously goes to last, it is called LIFO or LCFS system.

Eg. Stack of plates: Loading and unloading a truck or go-down; office filing of papers in chronological orders; wearing socks and shoes; dressing a shirt and coat over it, packing systems etc.

(c) Service In Random Order (SIRO) : By this rule, the customer for service is picked up at random, irrespective of their arrivals.

Eg. Lottery system from which one is picked up, the dresses waiting in a ward robe from which one is to be chosen, food stuffs in a buffet, sales counter of commodities or vegetables etc.

(d) Priority Service : Under this rule, the server gives priority to certain customer (s) due to some importance or prestigeous or high cost group of the customers.

Eg. A telephone urgent call given to a customer is charged at higher price, a separate counter for cheques at a electricity bill payment counter.

(e) Pre-emptive Priority Rule : Under this rule, highest priority is given to certain customer(s) irrespective of their arrival and costs.

Eg. An emergency case arriving at a doctor's clinic who is attending to a regular out-patient. (The doctor will stop his service to the regular patient and immediately rushes to emergency case).

(f) Non Pre-emptive Priority Rule : There is also a rule by priority to the special customer with the priority will not pre-emptive the current service. The service to the special customer starts immediately after the completion of current service.

Eg. A medical representative will be given appointment immediately after the current service to an out-patient at a doctor's clinic.

Q3. Define Kendall's Notation.

Ans :

Generally, queueing model may be completely specified in the following symbol form (a//b/c): (d/ e) where

a = probability law for the arrival (inter-arrival) time.

b = probability law according to which the customers are being served.

c = number of channels (or service stations)

d = capacity of the system, i.e., the maximum number allowed in the system (in service and waiting).

e = queue discipline.

5.2 QUEUING STRUCTURE AND BASIC COMPONENT OF AN QUEUING MODEL

Q4. Discuss in detail queuing structure and basic components of a queuing model.

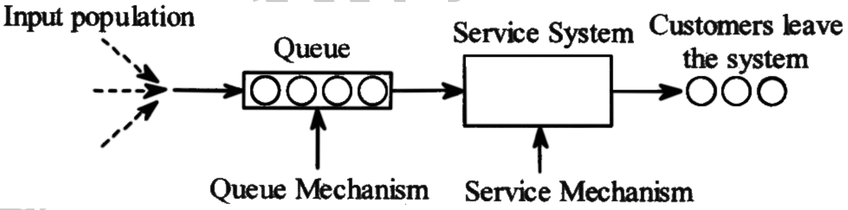
OR

Depict the general structure of a queuing system. Discuss the elements of a queuing system.

Ans : **(Nov.-20, May-19)**

Queuing Structure

The general structure of a queuing system is shown in the figure below,



The basic process assumed by most queuing model (as depicted in the above figure) is as follows,

1. Customers requiring service are generated over time by an input source.
2. These customers enter the queuing system and join a queue.
3. At certain time, a member of the queue is selected for service by some rule called queue discipline.
4. The required service is then provided to the customer by the service mechanism.
5. The customer leaves the queuing system.

Elements/Components of a Queuing Model

Queuing system is a complete system comprising customers arrival, waiting in queue, picked up for service as per specific discipline being served and the departure.

The elements of a queuing system are,

(i) Arrival process

(ii) Service system

(iii) Queue structure.

(i) Arrival Process

Arrival process refers to the way in which customers arrive at the service facility (individually or in batches) as per the scheduled or un-scheduled time. Classifica-tion of arrivals from input population are as follows,

(a) According to Size

The size is the total number of customers who require service from time to time. The size may be assumed to be either infinite or finite.

Example

- **Finite Population**

  Machines waiting for repair.

- **Infinite Population**

  Calls arriving at a telephone exchange.

(b) According to Numbers

Customers may arrive for service individually or in group.

Example

Individually - Planes landing at the airport.

In groups - Families visiting hotels.

(c) According to Time

Customers may arrive in the system at known regular times or they might arrive in a random way. The respective models are called deterministic and probabilistic models.

(ii) Service System/Process

The service process or mechanism refers to the way in which customers are serviced and leave the service system or facility.

(a) Structure of the Service System/Process

The service system consists of one or more facilities each of which contains one or more parallel service channels, called servers.

- **Single Service Facility**

  It consists of a single server.

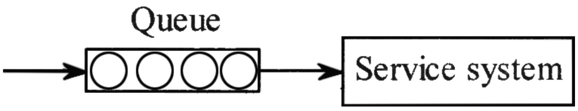
- **Multiple Parallel Facilities**

  It consists of multiple servers with single or multiple queue.

- **Service Facilities in Series**

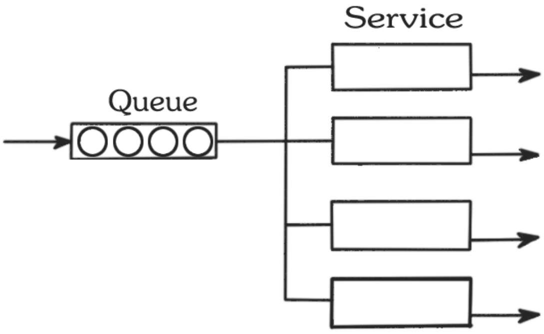
  It consists of multiple servers which are connected in series.

Single Server-Single Q



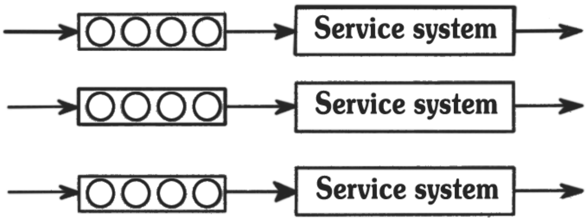
Figure

Multiserver-Parallel-Single Q



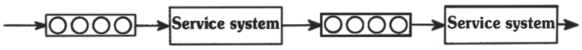
Figure

Multiserver-Parallel-Multi Q



Figure

Multiserver-Series



Figure

(b) Speed of Service

The speed with which the service is provided is given as service rate or a service time. Service rate is the number of customers serviced during a particular time period. Service time is the time elapsed from the commencement of service to its completion for a customer at a service facility.

(iii) Queue Structure

It consist of the following,

(a) Queue Discipline

This refers to the order in which the members of the queue are selected for service. It may be first-come- first served, last-come-first served, service-in-random order or priority service.

(b) Queue Behaviour

The attitude or behaviour of the customers entering the queuing system.

- Customer with patience waits till the service is offered.
- Impatient Customer does one of the following defections,

(i) Jockeying

Customers switch to other queues in the hope of reducing waiting time and thereby getting service quickly.

(ii) Reneging

Customers wait in the queue for sometime and then leaves without getting the service.

(iii) Balking

Customers do not join the queue now and decide to join at a later time or may give up the idea of getting service.

5.3 DISTRIBUTIONS IN QUEUING MODEL

Q5. Explain distributions in queuing model.

Ans :

The distributions in queuing model includes the following models,

1. Probabilistic Queueing Model

The queuing model which is widely used is probabilistic queuing model. The models commonly used under this classification are defined in terms of the arrival and service process, number of servers and population type.

The arrival distribution, in general follows the Poisson distribution and the service process follows the exponential distribution. There are other distributions that may be followed by the arrival and service process like Enlarge distribution.

Types of Probabilistic Queuing Models

(a) **Poisson:** Exponential, Single Server Model - Infinite Population.

(b) **Poisson:** Exponential, Single Server Model - Finite Population.

(c) **Poisson:** Exponential, Multiple Server Model - Infinite Population.

2. Deterministic Queuing Model

A deterministic queuing model is the one in which customers arrive in the queuing system at regular intervals and the service time for each customer is known as constant.

Examples

(i) If trucks arrive at a go-down every 10 minutes, then the interval between the arrival of any two successive trucks is exactly 10 minutes. Let us assume that the truck unloader takes exactly 10 minutes to unload the truck items. The arrival and service rates are equal to each 6 trucks per hour. In such a situation, there shall be no queue and the service person (truck unloader > is always busy.

(ii) If the service time is reduced to 8 minutes, then the truck unloader will be idle for some time i.e.. 10-8 = 2 minutes. In such situations also there will be no queue.

(iii) If the service time is increased to 14 minutes, then the truck unloader will be completely busy and the trucks will be waiting for service. This results in the formation of queue. But, this queue will keep growing and eventually the system leads to an explosive situation. To overcome this, additional service stations can be provided i.e., another truck unloader.

Symbolically,

Let, λ = Arrival rate

μ = Service rate.

Then, if $\lambda > m \rightarrow$ Waiting line formed, service facility always busy, system fails eventually.

If $\lambda < \mu \rightarrow$ No waiting time, proportion of time service facility idle is $1 - \frac{\lambda}{\mu}$.

$$\frac{\lambda}{\mu} = \rho,$$

ρ = Average utilization or traffic intensity.

Thus,

If $\rho > 1$ system would ultimately fail

If $\rho \leq 1$ system works and p is the proportion of time it is busy.

Such deterministic nature of arrival and service is rare and may exist in highly automated plants.

5.4 DIFFERENT IN QUEUING MODEL WITH FCFS

5.4.1 Single Server Queuing Models

Q6. Explain briefly about Single Server Queuing Models.

Ans : **(Mar.-15)**

Model I [(M/M/1) : (∞/FCFS)] (Exponential Services – Unlimited Queue)

In this model, three events are possible when a small change is seen in time interval i.e. Time is denoted by 't' and the number of customers by 'n'.

1. The system remains in n state, when there is no new arrival or no departure of customers.
2. If the system is in 'n + 1' state, it represents the departure of a single customer by reducing the total number of customers to be 'n'.
3. If the system is in 'n – 1' state, it represents a single arrival with no departures of customers.

Assumptions

(a) Arrival rate of a customer is derived from Poisson distribution and it emerges from an infinite population.

(b) Queue discipline follows a principle of "first come first service" basis.

(c) On an average, service rate is greater than the arrival rate of customers. Mathematically, it is represented by SR > AR.

(d) Average arrival rate is constant and does not change with time as customer arrival is independent of other factor.

Model II [(M/M/1) : (∞/SIRO)]

This queueing model is similar to Model I except with a single difference in the nature of a queueing discipline. With a single difference in the nature of queueing discipline. This model follows the discipline of SIRO (Service In Random Order). As the value of P_n is not dependent on any specific queue discipline, it is given by a formula,

$$\boxed{P_n = (1-\rho)\rho^n \text{(Where, } n = 1, 2, 3, \text{...........} n)}$$

In this model, other results of a queue will change only when 'P_n' changes, which represents that the results of SIRO model is directly related to the value of 'P_n'.

Assumptions

The assumptions of this model are same as that of Model I[M/M 1 : ∞ /FCFS].

Model III [(M/M/1) : (N/FCFS)] (Exponential Service – Finite or Limited Queue)

This is a queueing system in which the number of arrivals is described by a Poisson distribution, the service time is described by an exponential distribution with a single server and limited population. It is different from Model I with respect to the capacity of a system is limited. Only 'N' customers can be accommodated in the system due to various constraints.

Example, due to limited number (N) of beds in a hospital, limited seats in an election and so on. Equations derived in this model will be equal to that of model I as long as n is less than N (i.e., n < N).

In this model, steady state differential equation given by.

$\lambda P_0 = \mu P_1$ (Where, n = 0)

$(l + \mu)P_n = lP_{n-1} + \mu P_{n+1}$ (Where, n = 1, 2, ..., N – 1)

$\lambda P_{N-1} = \mu P_N$ (Where, n = N)

1. The queue length is found to be limited, i.e., number of customers in a queue is limited and is given by 'AT.
2. In this model, steady state is obtained when the service rate is equal to or less than the arrival rate of a customer i.e., ($\mu > \lambda$).
3. Except for the above mentioned assumptions, the assumptions are same as that of model I.

5.4.2 Multi-server Queuing Models

Q7. Explain briefly about Multi-server Queuing Models.

Ans :

Model IV [(M/M/s) : (∞ /FCFS)] (Exponential Service - Unlimited Queue)

In this model, there are multiple but identical servers (s) for handling the customer queues. Example, bank counters. Certain formulae of this model includes,

$\lambda_n = \lambda$ (Where, $n \geq 0$)

$\mu_n = n\mu$ (Where, $n < s$)

$\mu_n = s\mu$ (Where, $n \geq s$)

Assumptions

1. The customer arrival follows a Poisson distribution (at an average rate of customers per unit of time).
2. The service queue discipline is on a First Come First Served (FCFS) basis.
3. Service time is distributed exponentially.
4. It is assumed that only one queue is formed.

In this model, if there are customers in the queue, two cases are possible.

Here, n = Number of customers in the system and

s = Number of servers in the system.

Case 1

If $n < s$: The result will be, there is no queue. In this scenario, the service rate can be mathematically represented as,

$\mu_n = n\mu$ (if $n < s$)

Case 2

If $n > s$: The result will be, all the servers will be busy serving the customers. In this scenario, the service rate can be mathematically represented as,

$\mu_n = s\mu$ (if $n < s$)

Model V [(M/M/s) : (N/FCFS)] (Exponential Service - Limited (Finite) Queue)

This model is a modified form of Model IV. It consists of multiple servers serving limited or finite queue (or number of customers). If the number of customers exceeds the queue length then they have to leave the system service. Example, if all the seats in a plane are booked, booking will be closed as no further seat will be available. Thus, while conducting a cost benefit analysis, the service cost, arrival cost and even a cost of losing a customer must be considered. To arrive at the results,

$\lambda_n = \lambda$ (When $n \leq N$)

$\lambda_n = 0$ (Where, $n > N$)

$\mu_n = n\mu$ (If $n \geq N$) and

$\mu_n = s\mu$ (If $s \leq n \leq N$)

Assumptions

The assumptions are the same as Model IV with a single exception wherein the queue length is limited.

5.4.3 Finite Calling Population Queuing Models

Q8. Explain briefly about Finite Calling Population Queuing Models

Ans :

Model VI [(M/M/1) : (M/GD)] (Single Server - Finite Population (Source) of Arrival)

In this model, there is a single server and limited (finite) population of arrivals. This model is identical to Model I except for the number of customers (input source) being limited. Thus, no new customers are allowed to enter the system when the server is serving the existing range of customers.

Example

A mechanic repairing a machine. Other machines in the workshop are waiting customers and the mechanic is the servers. Thus, the service rate in this model is,

$$\lambda_n = \lambda(M - n)\,\lambda \text{ (Where, n = 1, 2, ..., M)}$$

$$\lambda_n = 0 \text{ (If n > N)}$$

$$\mu_n = \mu \text{ (Where, 1, 2, ..., M)}$$

Assumptions

The assumptions for this model are same as that of Model I.

Model VII [(M/M/s) : (M/GD)] (Multi - Server Finite Population (Source) of Arrivals)

This queuing model consists of multiple servers with an arrival from a finite population. If the number of servers is more than one 0 > 1), then the steady state equations can be derived in the same manner as in Models IV and V. In this model, the arrival rate and service rate would be,

$$\lambda_n = (M - n)\,\lambda \text{ (Where, } 0 \le n < M)$$

$$\lambda = 0 \text{ (If } n \ge M)$$

$$\mu_n = n\mu \text{ (Where, } 0 \le n < s)$$

$$\mu_n = s\mu \text{ (If } n \ge s)$$

5.4.4 Multi-Phase Service Queuing Model

Q9. Explain briefly about Multi-Phase Service Queuing Model.

Ans :

Model VIII [(M/E/l) : (∞ /FCFS)] (Erlang Service Time Distribution)

A.K. Erlang was the first to study such a distribution. He has divided the service into *k* phases, each phase taking $\frac{1}{k\mu}$ (average service time) for its completion. The total service time of a system is a sum of individual times of all phases. Each customer is served in *k* phases and will be completed only when all the 'k' phases are completed. Thus, each new arrival increase the number of phases by *k.* Therefore, the total number of phases in the system (waiting) will be,

$$n = mk + s$$

The steady state distribution equation is given by the formula,

$$\lambda P_0 = k\mu P_l \text{ (If n = 0) (Where, } P_0 = 1 - \rho k)$$

$$(\lambda + k_u)P_n = \lambda P_{n-k} + k\mu P_{n+1} \text{ (Where, } n \ge 1)$$

Assumptions

1. Time taken in each phase is according to exponential distribution.
2. Service is divided into *k* phases.
3. Each phase takes $\frac{1}{k\mu}$ of average service time.
4. The mean service time remains same, even though different distribution are obtained for the different values.

5.4.5 Special Purpose Queuing Model

Q10. Explain briefly about Special Purpose Queuing Model.

Ans :

Model IX [(M/D/1 : (∞ FCFS)]

This model consist of a single server and constant service time (D) with an unlimited queue (∞). For serving each customer, if service time is constant (i.e., l/μ.) then the variance will be zero ($\sigma^2 = 0$).

Assumptions

1. The service time is assumed to be constant for all the customers, which maintains a lower bound on the average queue length.
2. By assuming exponential services, the upper bound of the mean queue length is obtained.

5.5 QUEUE DISCIPLINE

Q11. What is queue discipline? Explain the ways in which customers in the queue are served With examples.

OR

What are the various types of queuing disciplines? Give suitable examples with their managerial implications.

Ans : **(May-19, Apr.-16, Mar.-15)**

Queue Discipline

Queue discipline refers to the order in which the members in a queue are selected for service.

Ways of Queue Discipline

The different ways in which customers are served include the following,

I. Static Queue Disciplines

These disciplines are based on status of individual customer in a queue. Some of the static queue disciplines includes,

(a) First-Come, First-Served (FCFS)

In this discipline, customers are served according to the arrival. Example: At airports, taxi is engaged on a FCFS discipline.

(b) Last-Come, First Served (LCFS)

In this discipline, the order of service is opposite to FCFS. Customers or items arrived at last are served first in LCFS.

Example: In production process, the last item which has arrived at work place is processed first than the item which has arrived first.

II. Dynamic Queue Disciplines

These disciplines are based on attribute of individual customers. They are,

(a) Service-in-Random Order (SIRO)

Under this, selection of customer for service is done randomly regardless of their arrivals.

Example: Hospitals, clinics etc.

(b) Priority Service

Customers are categorized as per priority like urgency or service time etc. In each category, FCFS rule is followed for providing service.

Example: Electricity or telephone bills payment through cash or cheque.

(c) Pre-emptive Priority (or Emergency)

In this discipline, most important customer is served first by interrupting the service of the less priority customer, Example: Electronics Showrooms.

(d) None Pre-emptive Priority

Under this rule, customer with more priority is served but only after the completion of the service under process. Example: Cloth Stores.

5.6 SINGLE SERVICE STATION WITH FINITE POPULATION

Q12. Explain the Single Server Model with Finite Population.

Ans :

Single Channel, Unrestricted Queue with Finite Population Model

This single server model considers infinite capacity in the system (i.e., unrestricted queue) along with finite number of customers in the calling population.

The Kendall's notation for this model is,

$$(M/M/1) : (FCFS/\infty /N)$$

Relationships used in this Model

Let, there be 'M' units in the source population

$\frac{1}{\lambda}$ = Average inter-arrival time between successive arrivals.

μ = Service rate.

1. Probability that the system shall be idle,

$$P_0 = \left[\sum_{i=0}^{M} \left[\frac{M!}{(M-n)!} P_0 \left(\frac{\lambda}{\mu} \right)^i \right] \right]^{-1}$$

2. Probability that there are 'n' customers (units) in the system,

$$P_n = P_0 \left(\frac{\lambda}{\mu} \right)^n \frac{M!}{(M-n)!}, \quad \text{if } 0 < n \le M$$

$$P_n = 0 \text{ if } n > M$$

3. Expected length of queue,

$$L_q = M - \frac{\lambda + \mu}{\lambda}(1 - P_0)$$

4. Expected number of customers (units) in the system,

$$L_s = M - \frac{\mu}{\lambda}(1 - P_0)$$

5. Expected waiting time of a customer in the queue,

$$W_q = \frac{1}{\mu}\left(\frac{M}{1-P_0} - \frac{\lambda+\mu}{\lambda}\right)$$

6. Expected time a customer spends in the system,

$$W_s = \frac{1}{\mu}\left(\frac{M}{1-P_0} - \frac{\lambda+\mu}{\lambda} + 1\right)$$

(or)

$$W_q + \frac{1}{\mu}.$$

PROBLEMS

1. A television repairman finds that the time spent on his jobs has an exponential distribution with a mean of 30 minutes. If he repairs the sets in the order in which they came in, and if the arrival of sets follows a Poisson distribution with an approximate average rate of 10 per 8-hour day, what is the repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

Sol :

From the data of the problem, we have:

λ = 10/8 = 5/4 sets per hour; and

μ = (1/30) 60 = 2 sets per hour

(a) Expected idle time of repairman each day

Since number of hours for which the repairman remains busy in an 8-hour day (traffic intensity) is given by:

(8) (λ/μ) = (8) (5/8) = 5 hours

Therefore, the idle time for a repairman in an 8-hour day will be: (8 – 5) = 3 hours

(b) Expected (or average) number of TV sets in the system

$$L_s = \frac{\lambda}{\mu-\lambda} = \frac{5/4}{2-(5/4)} = \frac{5}{3} = 2 \text{ (aaprox.)}$$

TV sets

2. In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the interarrival time follows an exponential distribution and the service time (the time taken to hump a train) distribution is also exponential with an average of 36 minutes. Calculate:

Sol :

From the data of the problem, we have

λ = 30/60 × 24 = 1/48 trains per minute and μ = 1/36 trains per minute.

The traffic intensity then is, $\rho = \lambda/\mu$ = 36/48 = 0.75

(a) Expected queue size (line length):

$$L_s = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3 \text{ trains}$$

(b) Probability that the queue size exceeds 10:

$p(n \geq 10) = p^{10} = (0.75)^{10} = 0.06$

If the input increases to 33 trains per day, then we have λ = 33/60 × 24 = 11/48 trains per minute and μ = 1/36 trains per minute.

Thus, traffic intensity, $\rho = \frac{\lambda}{\mu} = \left(\frac{11}{480}\right)(36)$ = 0.83

Hence, recalculating the values for (i) and (ii)

$$L_s = \frac{\rho}{1-\rho} = \frac{0.83}{1-0.83}$$

= 5 trains (approx.), and

$p(n \geq 10) = \rho^{10} = (0.83)^{10} = 0.2$ (approx.)

3. Arrivals at telephone booth are considered to be Poisson with an average time of 10 minutes between one arrival and the next. The length of phone calls is assumed to be distributed exponentially, with a mean of 3 minutes.

(a) What is the probability that a person arriving at the booth will have to wait?

(b) The telephone department will install a second booth when convinced that an arrival would expected waiting for at least 3 minutes for a phon call. By how much should the flow of arrivals increase in order to justify a second booth?

(c) What is the average length of the queue that form from time to time?

(d) What is the probability that it will take a customer more than 10 minutes altogether to wait for the phone and complete his call?

Sol :

From the data of the problem, we have

$\lambda = 1/10 = 0.10$ person per minute and

$\mu = 1/3 = 0.33$ person per minute

(a) Probability that a person has to wait at the both.

$P(n > 0) = 1 - P_0 = \lambda/\mu = 0.10/0.33 = 0.3$

(b) The installation of second booth will be justified only if the arrival rate is more than the waiting time. Let λ' be the increased arrival rate. Then the expected waiting time in the queue will be

$$W_q = \frac{\lambda'}{\mu(\mu - \lambda')}$$

$$3 = \frac{\lambda'}{0.33(0.33 - \lambda')} \quad \text{or} \quad \lambda' = 0.16$$

where $W_q = 3$ (given) and $\lambda = \lambda'$ (say) for second booth. Hence, the increase in the arrival rate is 0.16– 0.10 = 0.06 arrivals per minute.

(c) Average length of non-empty queue:

$$L = \frac{\mu}{\mu - \lambda} = \frac{0.33}{0.23}$$

= 2 customers (approx.)

(d) Probability of waiting for 10 minutes or more is given by:

$$P(t \geq 10) = \int_{10}^{\infty} \frac{\lambda}{\mu}(\mu - \lambda)e^{-(\mu-\lambda)t}\,dt$$

$$= \int_{10}^{\infty} (0.3)\,(0.23)e^{-0.23t}\,dt$$

$$= 0.069 \left[\frac{e^{-0.23t}}{-0.23}\right]_{10}^{\infty}$$

$$= 0.03$$

This shows that on an average 3 per cent of the arrivals will have to wait for 10 minutes or more before they can use the phone.

4. A warehouse has only one loading dock manned by a three person crew. Trucks arrive at the loading dock at an average rate of 4 trucks per hour and the arrival rate is Poisson distributed. The loading of a truck takes 10 minutes on an average and can be assumed to be exponentially distributed. The operating cost of a truck is Rs. 20 per hour and the members of the loading crew are paid Rs. 6 each per hour. Would you advise the truck owner to add another crew of three persons?

Sol :

From the data of the problem, we have

$\lambda = 4$ per hour, and $\mu = 6$ per hour.

For Existing Crew

Total hourly cost = Loading crew cost + Cost of waiting time

= {(Number of loaders) × (Hourly wage rate)}

+ {(Expected waiting time per truck, W_s) (Expected arrival per hour, λ) × (Hourly waiting cost)}

$$= 6 \times 3 + \frac{1}{6-4} \times 4 \times 20$$

= Rs. 58 per hour.

After Proposed Crew

$$\text{Total hourly cost} = 6 \times 6 + \frac{1}{12-4} \times 4 \times 20$$

= Rs. 46 per hour.

5.7 SINGLE SERVICE STATION WITH INFINITE POPULATION

Q13. Explain the Single Server Model with infinite Population.

Ans :

The various formulae used in Single Channel Queuing Model (Infinite Population) are as follows,

Let,

λ = Mean or expected number of arrivals per time period (mean arrival rate).

μ = Mean or expected number of customers served per time period (mean service rate).

1. Probability that the service facility is busy.

$$\rho = \frac{\lambda}{\mu}, \ \rho = \text{Utilization parameter}$$

2. Probability that the service facility is idle.

$$P_0 = 1 - \frac{\lambda}{\mu}$$

3. Probability that there are 'n' units in the system.

$$P_n = 1 - \left(\frac{\lambda}{\mu}\right)^n$$

4. Probability that there are more than or equal to 'k' units in the system i.e., queue $\geq$ 'k'.

$$P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^k$$

$$\text{Also, } P(n > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$$

$$P(n > k) = \left(\frac{\lambda}{\mu}\right)^k \left(1 - \frac{\lambda}{\mu}\right)$$

5. Expected or average number of units in the system (system length).

$$L_s = \frac{\lambda}{\mu(\mu - \lambda)}$$

7. Expected or average length of non-empty queue.

$$L_q = \frac{\mu}{\mu - \lambda}$$

Note

L_q = Average length of all queues including empty queues.

L'_q = Average length non-empty queues only.

8. Expected or average waiting time in system.

$$W_s = \frac{1}{\mu - \lambda}$$

9. Expected or average waiting time in the queue.

$$W_q = \frac{\lambda}{\mu}\left[\frac{1}{\mu - \lambda}\right]$$

10. Probability that the waiting time in a queue is $\geq$ t.

$$P(\text{waiting time} \geq t) = \frac{\lambda}{\mu} e^{-(\mu-\lambda)r}$$

PROBLEMS

5. Consider a self-service store with one cashier. Assume Poisson arrivals and exponential service times. Suppose that 9 customers arrive on an average every 5 minutes and the cashier can serve 10 customers in five minutes. Compute the following,

(a) Average number of customers queueing for service.

(b) Probability of having more than 10 customers in the system.

(c) Probability that a customer has to queue for more than 2 minutes.

(d) If the service can be speeded up to 12 in 5 minutes by using a different ash register, what will be the effect on quantities (a), (b) and (c)?

Sol :

(a) Average Number of Customers Queueing for Service.

Given that,

$$\lambda = \frac{9}{5} \text{ customer per minute}$$

$$\mu = \frac{10}{5} \text{ customer per minute}$$

$$\boxed{P = \frac{\lambda}{\mu}}$$

$$P = \frac{\frac{9}{5}}{\frac{10}{5}}$$

$$= \frac{1.8}{2} = 0.9$$

$P = 0.9$

Now,

$\boxed{1-P}$

$\Rightarrow 1 - 0.9 = 0.1$

$$\boxed{L_s = \frac{P}{1-P}}$$

$$= \frac{0.9}{0.1}$$

= 9 customers.

(b) Probability of having more than 10 Customers in the system

$$P(n \geq 10) = P^{10} = (0.9)^{10}$$

$$= 0.349$$

(c) Probability that a customer has to Queue for more than 2 minutes.

$$P(\text{waiting} \geq 2) = \int_2^\infty \frac{\lambda}{\mu}(\mu - \lambda)\, e^{-(\mu - \lambda)t}\, dt$$

$$P(t \geq 2) = \int_2^\infty 0.9\,(0.2)\,[e^{-(0.2)t}\ dt]$$

$$= \int_2^\infty 0.18\,[e^{-0.2t}\ dt] \quad \left[\therefore \int e^{-atdt} = \frac{e^{-at}}{-a}\right]$$

$$= 0.18 \int_2^\infty \frac{e^{-0.2t}}{-0.2}$$

$$= 0.18 \left[\frac{e^{-0.2t}}{-0.2}\right]_2^\infty$$

$$= 0.18 \left[\frac{e^{-0.2(\infty)} - e^{-0.2(2)}}{-0.2}\right]$$

$$= 0.18 \left[\frac{0 - 0.670}{-0.2}\right] \quad [\because\ e^{-\infty} = 0]$$

$$= 0.18 \left[\frac{-0.670}{-0.2}\right]$$

$$= 0.18 \times 3.35$$

$$= 0.603$$

(d) If the service can be speeded upto 12 in 5 minutes by using different cash register, what will be the effect on quantities, (a), (b) and (c)?

Effect on (a)

$$\lambda = \frac{9}{5} = 1.8 \text{ customer per minute}$$

$$\mu = \frac{12}{5} = 2.4 \text{ customer per minute}$$

$$P = \frac{\lambda}{\mu} = \frac{1.8}{2.4} = 0.75$$

Now, $\boxed{1-P}$

$\Rightarrow 1 - 0.75 = 0.25$

$$L_s = \frac{P}{1-P}$$

$$= \frac{0.75}{0.25} = 3 \text{ customers}$$

Effect on (b)

$P(n \geq 10) = P^{10} = (0.75)^{10}$

$= 0.056.$

Effect on (c)

$$P(\text{waiting } (w) \geq 2) = \int_2^\infty \frac{\lambda}{\mu}(\mu-\lambda)\, e^{-(\mu-\lambda)t} dt$$

$$P(t \geq 2) = \int_2^\infty \frac{1.8}{2.4}(2.4 - 1.8)\, [e^{-(2.4-1.8)t}\, dt]$$

$$= 0.75\ (0.6)\ [e^{-(0.6)t}\ dt]$$

$$= \int_2^\infty 0.45\, [e^{-0.6t} dt]$$

$$\left[\because \int e^{-atdt} = \frac{e^{-at}}{-a}\right]$$

$$= 0.45 \int_2^\infty \frac{e^{-0.6t}}{-0.6}$$

$$= 0.45 \left[\frac{e^{-0.6t}}{-0.6}\right]_2^\infty$$

$$= 0.45 \left[\frac{e^{-0.6(\infty)} e^{-0.6(2)}}{-0.6}\right]$$

$$= 0.45 \left[\frac{0 - 0.30}{-0.6}\right] \left[\because e^{-\infty} = 0\right]$$

$$= 0.45 \left[\frac{-0.30}{-0.6}\right]$$

$= 0.45 \times 0.5$

$= 0.225.$

---

6. A car hiring firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as Poisson variate with mean 1.5. Calculate the Proportion of days on which,

(a) Neither car is used and

(b) Some demand is refused.

Sol : **(Feb.-15)**

Since the number of demands for a cari is distributed as Poisson distributed with mean 1 = 1.5.

$\therefore$ Proportion of days on which neither car is used

= Probability of there being no demand for the car

= e – 1.5

= 0.2231

Proportion of days on which some demand is refused.

= Probability for the number of demands to be more than two.

$= 1 - P(x < 2)$

$$= 1 - \left(e^{-\lambda} + \frac{\lambda e^{-\lambda}}{\angle 1} + \frac{\lambda^2 e^{-\lambda}}{\angle 2}\right)$$

$$= 1 - e^{-1.5}\left(1 + 1.5 + \frac{(1.5)^2}{2}\right)$$

$= 1 - 0.2231(1 + 1.5 + 1.125)$

$= 1 - 0.2231 \times 3.625$

$= 1 - 0.8087375$

$= 0.1912625$

5.8 MULTIPLE SERVICE STATION WITH INFINITE POPULATION

Q14. Explain the multiple service station with infinite population.

Ans :

Multi Service Station Models with Infinite Population : (M/M/C) : (FCFS/∞ / ∞)

In multi service station several service stations work in parallel fashion wherein the customers waiting in a queue can be served by one or more than one service station. The same type of services are provided to the customers by each service station. The customers are an serviced on FCFS (First Come First Served) basis. The two rates i.e., the arrival rate (λ) and the service rate (μ) represents the mean values of customers that can be derived from Poisson distribution and exponential distribution.

The various properties of multichannel system are as follows,

Properties of Multichannel System

Properties	Explanation of Properties	Formula used
1.	Expected/Average number of in the system	$L_S = \frac{\lambda \cdot \mu \left(\frac{\lambda}{\mu}\right)^c}{(c-1)!(c\mu-\lambda)^2} P_0 + \frac{\lambda}{\mu}$
2.	Expected/Average number of customers waiting in a queue.	$L_q = L_S =$ Average number being served $= L_S - c \cdot \frac{\lambda}{c\mu} = L_s = \frac{\lambda}{\mu}$ $= \frac{\lambda\mu \cdot \left(\frac{\lambda}{\mu}\right)^c}{(c-1)!(c\mu-\lambda)^2} \cdot P_0$
3.	Average time a customer spends	$W_s = \frac{L_s}{\lambda} = \frac{\mu\left(\frac{\lambda}{\mu}\right)^c}{(c-1)!(c\mu-\lambda)^2} \cdot P_0 + \frac{1}{\mu}$
4.	Average waiting time of a	$W_q = \frac{L_q}{\lambda} = \frac{\mu\left(\frac{\lambda}{\mu}\right)^c}{(c-1)!(c\mu-\lambda)^2} \cdot P_0$
5.	Probability that a customer has to wait queue.	$P(n \geq c) = \frac{\mu\left(\frac{\lambda}{\mu}\right)^c}{(c-1)!(c\mu-\lambda)} \cdot P_0$
6.	Probability that a customer enters	$1 - P(n \geq c) = 1 - \frac{\mu \cdot \left(\frac{\lambda}{\mu}\right)^c}{(c-1)!(c\mu-\lambda)} \cdot P_0$
7.	Average number of idle servers	= c – (Average number of customer served)
8.	Utilization rate	$\rho = \frac{\lambda}{C\mu}$
9.	Efficient of M/M/C model	$= \frac{\text{Average number of customers served}}{\text{Total number of customers}}$

PROBLEMS

7. The Taj Service Station has five mechanics each of whom can service a scooter in 2 hours on the average. The scooters are registered at a single counter and then sent for servicing to different mechanics. Scooters arrive at the service at an average rate of 2 scooters per hour. Assuming that the scooter arrivals are Poisson distribution and the servicing times are distributed exponentially determine,

(a) Utilization factor

(b) The probability that the system shall be idle

(c) The probability that there shall be 3 scooters in the service centre

(d) The expected number of scooter waiting in the queue and in the system.

(e) The average waiting time in the queue and

(f) The average time spent by a scooter in waiting and getting serviced.

Sol :

Given that,

Mean arrival rate, $\lambda = 2$ scooters per hour

Average service rate, $\mu = \frac{1}{2}$

$= 0.5$ scooter per hour

Number of mechanics, $S = 5$

(a) Utilization Factor

$$P = \frac{\lambda}{S\mu} = \frac{2}{5 \times 0.5}$$

$$= \frac{2}{2.5} = 0.8$$

(b) The probability that the system shall be idle,

$$P_0 = \left[\sum_{n=0}^{S-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \left[\frac{S\mu}{S\mu - \lambda}\right]\right]^{-1}$$

(or)

$$P_0 = \left[\sum_{n=0}^{S-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^S}{S!(1-P)}\right]^{-1}$$

$$= \left[\sum_{n=0}^{5-1} \frac{\left(\frac{2}{0.5}\right)^n}{n!} + \frac{\left(\frac{2}{0.5}\right)^5}{5!(1-0.8)}\right]^{-1}$$

$$\sum_{n=0}^{5-1} \frac{\left(\frac{2}{0.5}\right)^n}{n!} = \frac{4^0}{0!} + \frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!}$$

$$= \frac{103}{3}$$

$$\frac{\left(\frac{2}{0.5}\right)^5}{5!(1-0.8)} = \frac{4^5}{120 \times 0.2} = \frac{1024}{24} \text{ or } \frac{128}{3}$$

$$\therefore P_0 = \left[\frac{103}{3} + \frac{128}{3}\right]^{-1}$$

$$= [77]^{-1} = 0.013$$

(c) The probability that there shall be 3 scooters in the service centre,

$$P_n = \frac{(\lambda/\mu)^n}{n!} \times P_0$$

$\therefore$ When $n \leq k$

$$P_3 = \frac{\left(\frac{2}{0.5}\right)^3}{3!} \times 0.013$$

$$= \frac{64}{6} \times 0.013$$

$$= 10.67 \times 0.013 = 0.1387$$

(d) The expected number of scooters waiting in the queue and in the system,

$$L_q = \frac{(\lambda/\mu)^S.P}{S!(1-P)^2} \times (P_0)$$

$$= \frac{\left(\frac{2}{0.5}\right)^5.(0.8)}{5!(1-0.8)^2} \times 0.013$$

$$= \frac{1024 \times 0.8}{120 \times 0.04} \times 0.013$$

$$= \frac{819.2}{4.8} \times 0.013$$

= 2.22 Scooters.

$$L_S = L_q + \frac{\lambda}{\mu}$$

$$= 2.22 + \frac{2}{0.5}$$

$$= 2.22 + 4$$

= 6.22 scooters

(e) The average waiting time in the queue,

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{2.22}{2} = 1.11 \text{ hours}$$

(f) The average time spent by a scooter in waiting and getting serviced,

$$W_S = W_q + \frac{1}{\mu}$$

$$= 1.11 + \frac{1}{0.5}$$

$$= 1.11 + 2$$

= 3.11 hours

8. In a maintenance shop, the inter-arrival times at a tool station are exponential with an average time of 10 minutes. The length of the service time (amount of time taken by the tool station operator to meet the needs of the maintenance man) is assume to be exponentially distributed, with mean 6 minutes. Find,

(a) The probability that a person arriving at the tool station will have to wait.

(b) Average length for the queue that forms and the average time that an operator spends in the Q-system.

(c) The manager of the shop will set up a second tool station when an arrival would have to wait 10 minutes or more for the service. By how much must the rate of arrival be increased in order to justify a second tool station?

(d) The probability that an arrival will have to wait for more than 12 minutes for service and a obtain its tools.

(e) Estimate the fraction of the day that tool station operator will be idle.

(f) The probability that there will be six or more operators waiting for the service.

Sol : **(Feb.-16)**

Given that,

$$\lambda = \frac{60}{10}$$

= 6 per hour

= 10 per hour

(a) A person will have to wait if the service facility is not idle.

$$\therefore \quad \text{Probability of waiting} = p = \frac{\lambda}{\mu}$$

$$= \frac{6}{10}$$

$$= 0.6$$

(b) $$Lq = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$= \frac{6^2}{10(10-6)} = \frac{6^2}{100-60}$$

$$= \frac{36}{40} = \frac{9}{10}$$

or $= 0.9$

$$L_S = L_q + \frac{\lambda}{\mu}$$

$$= 0.9 + 0.6$$

$$= 1.5$$

$$\therefore W_s = \frac{L_s}{\lambda}$$

$$= \frac{1.5}{6}$$

$$= \frac{1}{4} \text{ hours}$$

(c) $$W_q = \frac{L_q}{\lambda}$$

$$= \frac{0.9}{6hrs} = 0.15$$

$$= 0.15 \times 60 \text{ minutes}$$

$$= 9 \text{ minutes}$$

Let 'λ' be the arrival rate when a second tool is justified, i.e.,

$W_q \geq$ 10 minutes.

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{10}{60}$$

$$6\lambda = 10(10-\lambda)$$

$$6\lambda = 100 - 10\lambda$$

$$6\lambda + 10\lambda = 100$$

$$6\lambda = 100$$

$$\lambda = \frac{100}{16}$$

$$\lambda = 6.25$$

∴ If the arrival rate exceeds 6.25 per hour, then the second tool will be justified.

(d) Probability of waiting for 12 minimum or more is given by,

Problem (waiting time ≥ 12)

$$= \int_{12}^{\infty} \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)t}dt$$

$$= \int_{12}^{\infty} (0.6)\,(4)[e^{-4t}\ dt]$$

$$= \int_{12}^{\infty} 2.4[e^{-4t}dt] \quad \left[\because \int e^{-atdt} = \frac{e^{-at}}{-a}\right]$$

$$= 2.4\int_{12}^{\infty} \frac{e^{-4t}}{-4}$$

$$= 2.4\left[\frac{e^{4t}}{-4}\right]_{0.2}^{\infty} \quad \left[\because \frac{12}{60} = \frac{1}{5} \text{ or } 0.2\right]$$

$$= 2.4\left[\frac{e^{-4(\infty)}e^{-4(0.2)}}{-4}\right]$$

$$= 2.4 \left[\frac{0 - 0.45}{-4}\right] \quad [\because e - \infty = 0]$$

$$= 2.4 \left[\frac{-0.45}{-4}\right]$$

$$= 2.4 \times 0.1125 = 0.27$$

(e) $\mathbf{P = 1 - \frac{\lambda}{\mu}}$ **or 1 – P**

$$= 1 - \frac{6}{10}$$

$$= 1 - 0.6 = 0.4$$

∴ 40% of the time of tool crib operator is idle.

(f) Probability of six or more operators waiting for the service,

$$= P^6 \qquad \left(\text{i.e.,} \left(\frac{\lambda}{\mu}\right)^6\right)$$

$$= (0.6)^6.$$

5.9 GAME THEORY

Q15. Define game theory. Explain the characteristic of game theory.

Ans :

In the competitive world, it is essential for an executive to study or at least guess the activities or actions of his competitor. Moreover, he has to plan his course of actions or reactions or counter actions when his competitor uses certain technique. Such war or game is a regular feature in the market and the competitors have to make their decisions in choosing their alternatives among the predicted outcomes so as to maximize the profits or minimizing the loss.

Characteristics of Game Theory.

There can be various types of games. They can be classified on the basis of the following characteristics.

i) **Chance of Strategy :** If in a game activities are determined by skill, it is said to be a game of strategy; if they are determined by change, it is a game of chance. In general, a game may involve game of strategy as well as a game of chance.

ii) **Number of Persons :** A game is called an n-person game if the number of persons playing in n. The person means an individual or a group aiming at a particular objective.

iii) **Number of Activities :** These may be finite or infinite.

(iv) **Number of Alternatives (choices) Available to Each Person** in a particular activity may also be finite or infinite. A finite game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is said to be infinite.

(v) **Information to the players about the past activities of other players** is completely available, partly available, or not available at all.

(vi) **Payoff :** A quantitative measure of satisfaction a person gets at the end of each play is called a payoff. It is a real-valued function of variables in the game. Let v_i be the payoff to the player P_i, $1 \leq i \leq n$, in an n-person game. If Σ_i^n v = 0 then the game is said to be a zero-sum game.

Q16. State the basic terminology are used in game theory.

Ans :

1. **Game Theory :** A mathematical theory based on which strategy steps are employed to win a game played in a conflicting situation to maximise the benefits (or profit) or to minimise the damage (or loss).

2. **Game :** A competitive situation having the following characteristics:

(i) The situation is competitive.

(ii) There are a finite number of competitors (or players).

(iii) Each player has a finite number of strategies available to him or her.

(iv) The game is said to be played when both competitors initiate actions based on their chosen strategies.

(v) Every game results in an outcome.

(vi) Every outcome has stakes, i.e., payment given or taken.

3. **Number of players :** A game involving only two players (competitors) is called a two-person game. If the number of players exceeds two, then the game is known as "'n'-person game" where 'n' denotes the number of players.

4. **Sum of gains and losses :** If in a game the gains of one player are exactly the same as the losses to another player, such that the sum of the gains and losses equals zero, then the game is said to be a "zero-sum game". Otherwise it is said to be "non-zero sum game".

5. **Strategy :** It is a "plan of action" conceived and carefully executed by each party to the game. It involves a list of all possible actions (or moves or courses of action) that a player will take for every outcome (pay-off) that might arise. The rules governing the choices are known in advance to the players. Also the outcome resulting from a particular choice is also known to the player in advance and is expressed in terms of numerical values. The players need not have a definite information about each other's strategies.

6. **Optimal strategy :** A particular strategy by which a player optimises his gains or losses without knowing the competitor's strategies is called "optimal strategy".

7. **Pure strategy :** This is a predetermined plan of action based on which the games are played and which does not change during the game. It is a decision rule which is always used by the player to select the particular course of action. Each player knows in advance of all strategies available to each and out of which he or she always selects only one particular strategy irrespective of the strategy the opponent may choose. Pure strategy is used by a player to achieve the objective of maximising the gains or minimising the losses.

8. **Mixed strategy :** It is a plan of action which is changed while the game is in progress, when both players are guessing as to which course of action to be selected on a particular occasion with some fixed probability. Thus, there is a probabilistic situation and objective of the players is to maximise expected gains or to minimise expected losses by making a choice among pure strategies with fixed probabilities.

9. **Pay-off :** The outcome of the game is known as "pay-off'.

10. **Pay-off Matrix :** A table (in the form of a matrix) showing the outcome of the game (in terms of gains or losses) when different strategies are adopted by the players.

11. **Fair game :** A game is said to be fair when the value of the game is zero.

12. **Value of the game :** The maximum guaranteed expected outcome per play when players follow their optimal strategy is called the "value of the game".

13. **Solution of a game :** When the best strategies of both players are determined and the value of the game is determined, we say, the game is solved or the solution of the game is obtained.

14. Maximin : The maximum value of the minimum pay-offs in each row.

15. Minimax : The minimum value of the maximum pay-offs in each column.

16. Saddle point : The game value is called the saddle point in which each player has a pure strategy. The saddle point is the lowest numerical value in a row and the largest numerical value in e column, which are equal to each other.

17. Strictly determinable game : A game is said to be strictly determinable if the maximin value is equal to the minimax value. In other words for the optimal strategy for both players, the pay-off for both players will be the same, i.e., the gain of one player equals the loss of another."

Q17. State the Assumptions of game Theory

Ans :

Assumptions of game theory

The underlying assumptions, the rules of the game as given as follows :

1. The player act rationally and intelligently.
2. Each player has available to him a finite set of possible courses of action.
3. The player attempt to maximize gains and minimize losses.
4. All relevant information is known to each players.
5. The players make individual decisions without direct communication.
6. The players simultaneously select their respective courses of action.
7. The pay off is fixed and determined in advance.

Q18. Explain the advantages and disadvantages of game theory

Ans :

Advantages of Game Theory

1. Game theory keeps deep insight to few less known aspects, which arise in situations of conflicting interests.
2. Game theory creates a structure for analysis of decision-making in various situations like interdependence of firms etc.
3. For arriving at optimal strategy, game theory develops a scientific quantitative technique for two person zero-sum games.

Disadvantages of Game Theory

1. The highly unrealistic assumption of game theory is that the firm has prior knowledge about its competitor's strategy and is able to construct the payoff matrix for possible solutions, which is not correct. The main fact is that any firm is not exactly aware of its competitor's strategy. He can only make guesses about its strategy.
2. The hypothesis of maximin and minimax clearly shows that players are not risk lover and have whole knowledge about the strategies but the fact is that it is not possible.
3. It is totally impractical to understand that the several strategies followed by the rival player against others lead to an endless chain.
4. Most economic problems occur in the game if many players are involved in comparison to two-person constant sum game, which is not easy to understand. For example, the number of sellers and buyers is quite large in monopolistic competition and the game theory does not provide any solution to it.
5. In real market situations, it is doubtful to find the use of mixed strategies for making non zero-sum games.

Q19. Explain briefly about pay off matrix.

Ans :

Payoff is the outcome of playing the game. A payoff matrix is a table ‘showing the amount received by the player named at the left hand side after all possible plays of the game. The payment is made by the player named at the top of the table.

If a player A has m-courses of action and player B has n-courses, then a payoff matrix may be constructed using the following steps.

(i) Row designations for each matrix are the course of action available to A

(ii) Column designations for each matrix are the course of action avail-able to B

(iii) With a two person zero sum game, the cell entries in B’s payoff matrix will be the negative of the corresponding entries in A’s pay-off matrix and the matrices will be as shown below.

$$\text{Player A} \quad \overset{\text{Player B}}{\begin{bmatrix} 1 & 2 & 3 & \cdots & j & \cdots & n \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & & & \cdots & & \cdots & \\ m & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}}$$

A's payoff matrix

$$\text{Player A} \quad \overset{\text{Player B}}{\begin{array}{c|ccccccc|} & 1 & 2 & 3 & \cdots & j & \cdots & n \\ 1 & -a_{11} & -a_{12} & a_{13} & \cdots & -a_{1j} & \cdots & -a_{1} \\ 2 & -a_{21} & -a_{22} & a_{23} & \cdots & -a_{2j} & \cdots & a_{2} \\ \vdots & \vdots & & & & & & \\ i & -a_{i1} & -a_{i2} & -a_{i3} & \cdots & -a_{1j} & \cdots & a_{in} \\ \vdots & \vdots & & & \cdots & & \cdots & \\ m & -a_{m1} & -a_{m2} & -a_{m3} & \cdots & -a_{mj} & \cdots & a_{mn} \end{array}}$$

Q20. State the different types of games.

Ans :

1. **Two-person games and n-person games:** In two person games the players may have many possible choices open to them for each play of the game but the number of players remain only two. Hence it is called a two person game. In case of more than two persons, the game is generally called n-person game.

2. **Zero Sum Game:** A zero sum game is one in which the sum of the payment to all the competitors is zero for every possible outcome of the game is in a game if the sum of the points won equals the sum of the points lost i.e.

3. **Two person zero sum game:** A game with two players, where the gain of one player equals the loss to the other is known as a two person Zero sum game. It is also called a rectangular game because their payoff matrix is in the rectangular foirn. The characteristics of such a game are

   (i) Only two players participate in the game

   (ii) Each player has a finite number of strategies to use

   (iii) Each specific strategy results in a payoff

   (iv) Total payoff to the two players at the end of each play is zero.

5.9.1 Saddle Point

Q21. Define saddle point.

Ans :

A saddle point is a position in the payoff matrix where the maximum of row minima coincides with the minimum of column maxima. The payoff at the saddle point is called the value of the game.

We shall denote the maximin value by γ, the minimax value of the game by $\overline{\gamma}$ and the value of the game by γ.

Note

(i) A game is said to be fair if

maximin value = minimax value = 0, i.e., if $\overline{\gamma} = \underline{\gamma} = 0$

(ii) A game is said to be strictly determinable if maximin value = minimax value $\neq 0$, $\underline{\gamma} = \gamma = \overline{\gamma}$.

9. Solve the game whose pay off matrix is given by.

$$\begin{array}{cc} & \textbf{Player B} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \textbf{Player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

Sol :

$$\begin{array}{ccc} & \text{Player B} & \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} & \text{Row minima} \\ \text{Player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} & \begin{array}{c} 1 \\ -4 \\ -1 \end{array} \\ \text{Column maxima} & \begin{array}{ccc} 1 & 5 & 1 \end{array} & \end{array}$$

Maxi (minimum) = Max (1, –4, –1) = 1

Mini (maximum) = Min (1, 5, 1) = 1.

i.e., Maximin value $\underline{\gamma} = 1$ = Minimax value $\overline{\gamma}$

$\therefore$ Saddle point exists. The value of the game is the saddle point which is 1. The optimal strategy is the position of the saddle point and is given by $(A_1 B_1)$.

10. For what value of 1, the game with the following matrix is strictly determinable?

$$\begin{array}{cc} & \textbf{Player B} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \textbf{Player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} \end{array}$$

Sol :

$$\begin{array}{ccc} & \text{Player B} & \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} & \text{Row minima} \\ \text{Player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} & \begin{array}{c} 2 \\ -7 \\ -2 \end{array} \\ \text{Column maxima} & \begin{array}{ccc} -1 & 6 & 2 \end{array} & \end{array}$$

The game is strictly determinable, if

$\underline{\gamma} = \gamma = \overline{\gamma}$. Hence $\overline{\gamma} = 2$, $\overline{\gamma} = -1$

$\Rightarrow -1 \leq \lambda 2$.

11. Determine which of the following two person zero sum games are strictly determinable and fair. Given the optimum strategy for each player in the case of strictly determinable games.

(a)
$$\text{Player A}\ \begin{array}{c|cc} & \multicolumn{2}{c}{\text{Player B}} \\ & B_1 & B_2 \\ A_1 & -5 & 2 \\ A_2 & -7 & -4 \end{array}$$

(b)
$$\text{Player A}\ \begin{array}{c|cc} & \multicolumn{2}{c}{\text{Player B}} \\ & B_1 & B_2 \\ A_1 & 1 & 1 \\ A_2 & 4 & -3 \end{array}$$

Sol :

(a)
$$\text{Player A}\ \begin{array}{c|cc|c} & \multicolumn{2}{c}{\text{Player B}} & \\ & B_1 & B_2 & \text{Row minima} \\ A_1 & -5 & 2 & -5 \\ A_2 & -7 & -4 & -7 \\ \text{Column maxima} & -5 & 2 & \end{array}$$

Since $\underline{\gamma} = \overline{\gamma} - 5 = 0,$ the game is strictly determinable. There exists as saddle point = –5. Hence the value of the game is –5. The optimal strategy is the position of the saddle point given y (A_i, B_i).

(b)
$$\text{Player A}\ \begin{array}{c|cc|c} & \multicolumn{2}{c}{\text{Player B}} & \\ & B_1 & B_2 & \text{Row minima} \\ A_1 & 1 & 1 & 1 \\ A_2 & 4 & -3 & -3 \\ \text{Column maxima} & 4 & 2 & \end{array}$$

Maxi (minimum) = $\underline{\gamma}$ = Max (1, –3) = 1.

Mini (maximum) = $\overline{\gamma}$ = Min (4, 1) = 1.

Since $\underline{\gamma} = \overline{\gamma} = 1 \neq 0,$ the game is strictly determinable. The value of game is 1. The optimal strategy is (A_2, B_2).

12. Solve the game whose payoff matrix is given below.

$$\begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix}$$

Sol :

$$\text{Player A}\ \begin{array}{c|ccccc|c} & B_1 & B_2 & B_3 & B_4 & B_5 & \text{Row minima} \\ A_1 & -2 & 0 & 0 & 5 & 3 & -2 \\ A_2 & 3 & 2 & 1 & 2 & 2 & 1 \\ A_3 & -4 & -3 & 0 & -2 & 6 & -4 \\ A_4 & 5 & 3 & -4 & 2 & -6 & -6 \\ \text{Column maxima} & 5 & 3 & 1 & 5 & 6 & \end{array}$$

Maxi (minimum) = $\underline{\gamma}$ = Max (–2, 1, –4, –6)

= 1.

Mini (maximum) = γ = Min (5, 3, 1, 5, 6)

= 1.

Since $\underline{\gamma}$ = $\overline{\gamma}$ $\neq$ 1, there exists a saddle point. The value of the game is 1. The position of the saddle point is the optimal strategy and is given $[A_2. B_3]$.

Q22. Explain briefly about game without saddle point.

Ans :

2 × 2 Games without saddle point

Consider a 2 × 2 two-person zero sum game without any saddle point having the payoff matrix for player A.

$$\begin{array}{c|cc} & B_1 & B_2 \\ A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

The optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \text{ and } S_B \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

where $p_1 = \dfrac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$,

$p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$

$q_1 = \dfrac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

$q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$

The value of the game is,

$$r = \frac{a_{11}\, a_{22} - a_{12}\, a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

PROBLEMS

13. Solve the following payoff matrix, determine the optimal strategies and the value of game.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

Sol :

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array} \text{ Let this be}$$

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

The optimum mixed strategies

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \text{ and } S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

where $p_1 = \dfrac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

$$= \frac{4 - 3}{(5 + 4) - (1 + 3)} = \frac{1}{5}$$

$p_2 = 1 - p_1 \Rightarrow p_2 = 1 - \dfrac{1}{5} = \dfrac{4}{5}$

$q_1 = \dfrac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

$$= \frac{4 - 1}{(5 + 4) - (1 + 3)} = \frac{3}{5}$$

$q_2 = 1 - q_1 \Rightarrow q_2 = \dfrac{1}{1} - \dfrac{3}{5} = \dfrac{2}{5}$

Value of game $\gamma = \dfrac{(5 \times 4) - (1 \times 3)}{(5 + 4) - (1 + 3)} = \dfrac{17}{5}$

$\therefore$ The optimum mixed strategies

$$S_A = \left(\frac{1}{5}, \frac{4}{5}\right);\ S_B = \left(\frac{3}{5}, \frac{2}{5}\right)$$

Value of game $= \dfrac{17}{5}$.

14. Solve the following game and determine the value of the game.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array}$$

Sol :

It is clear that the pay off matrix odes not posses anysaddle point. The players will use mixed stategies. The optimum mixed strategy for the players are,

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix},\ p_1 + p_2 = 1$$

$$S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}, q_1 + q_2 = 1.$$

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{4 - (-4)}{4 + 4 - (-4 - 4)} = \frac{8}{16} = \frac{1}{2}$$

$$p_2 = 1 - p_1 \Rightarrow p_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{4 - (-4)}{4 + 4 - (-4 - 4)} = \frac{8}{16} = \frac{1}{2}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{2} = \frac{1}{2}.$$

∴ The optimum strategies is,

$$S_A = \left(\frac{1}{2}, \frac{1}{2}\right); S_B = \left(\frac{1}{2}\frac{1}{2}\right)$$

The value of the game is

$$\gamma = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})},$$

$$= \frac{(4 \times 4) - [-4 \times (-4)]}{(4 + 4) - [-4 + (-4)]} = 0$$

Q23. Explain briefly about dominance property.

Ans :

Sometimes it is observed that one of the pure strategies of either player is always inferior to atleast one of the remaining ones. The superior strategies are said to dominate the inferior ones. In such cases of dominance, we reduce the size of the payoff matrix by deleting those strategies which are dominated by others. The general rule for dominance are:

(i) If all the elements of a row, say Kth row, are less than or equal to the corresponding elements of any other row say rth row, then Kth row is dominated by the rth row.

(ii) If all the elements of a column, say Kth column, arc greater than or equal to the corresponding elements of any other column, say rth column, then the Kth column is dominated by the rth column.

(iii) Dominated rows and columns may be deleted to reduce the size of the pay-off matrix as the optimal strategies will remain unaffected.

(iv) If some linear combinations of some rows dominates ith row, then the ith row will be deleted. Similar arguments follow for column.

PROBLEMS

15. Solve the following game

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{bmatrix} \end{array}$$

Sol :

Since all the elements in the third row are less than or equal to the corresponding elements of second row. Therefore, row III is dominated by row II. Delete this dominated row. The reduced payoff matrix is given by,

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \end{bmatrix} \end{array}$$

The elements of 3rd column is greater than or equal to the corresponding elements of the first column which gives that the column 3 is dominated by the column 1. This dominated column is deleted and the reduced payoff matrix is given by,

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix} \end{array}$$

The reduced payoff matrix is a 2 × 2 matrix. The optimal strategy for player A and B is given by,

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & 0 \end{pmatrix} p_1 + p_2 = 1$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & 0 \end{pmatrix} q_1 + q_2 = 1$$

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$p_1 = \frac{2-6}{2+1-(7+6)} = \frac{-4}{-10} = \frac{2}{5}$$

$$p_2 = 1 - \frac{2}{5} = \frac{3}{5}$$

$$q_1 = \frac{2-6}{2+1-(7+6)} = \frac{-5}{-10} = \frac{1}{2}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

Value of the game $\gamma = \frac{2 \times 1 - 7 \times 6}{2+1-(7+6)}$

$$= \frac{-40}{10} = 4$$

The optimal strategy is given by

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ \frac{2}{5} & \frac{3}{5} & 0 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Value of game is $\gamma = 4$.

16. Is the following two person zero sum game stable?

$$\begin{array}{cc} & \text{Player B} \\ \text{Palyer A} & \begin{bmatrix} 5 & -10 & 9 & 0 \\ 6 & 7 & 8 & 1 \\ 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{bmatrix} \end{array}$$

Sol :

Since the game has no saddle point it is not a stable one. All the elements of the first row and a second row are ≤ to the corresponding elements of third row. Hence, these two rows are dominated rows. Deleting these two rows from the payoff matrix, the reduced payoff matrix is given by,

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{bmatrix} \end{array}$$

In this modified payoff matrix, we observe that all the elements of the second column are ≥ to the corresponding elements of the fourth column. Hence, this dominated column (2nd column) is deleted from the payoff matrix. The reduced payoff matrix is given by,

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{pmatrix} 8 & 15 & 1 \\ 3 & -1 & 4 \end{pmatrix} \end{array}$$

Now we observe that no row or column dominates another row or column. However, we note that a convex combination of 2nd and 3rd column is given by

$$15 \times \frac{1}{2} + 1 \times \frac{1}{2} = 8 \leq 8$$

$$-\frac{1}{2} \times \frac{1}{2} + 4 \times \frac{1}{2} = \frac{3}{2} \leq 3$$

and hence the elements of the first column is greater than or equal to the corresponding elements of this combination. Deleting this dominated column, the reduced payoff matrix is given by

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{pmatrix} 15 & 1 \\ -1 & 4 \end{pmatrix} \end{array}$$

$$S_A = \begin{bmatrix} A_3 & B_4 \\ p_1 & p_2 \end{bmatrix}, p_1 + p_2 = 1$$

$$p_1 = \frac{4-(-1)}{15+4-(1+1-1)} = \frac{5}{19}$$

$$q_2 = 1 - \frac{5}{19} = \frac{14}{19}$$

$$S_B = \begin{bmatrix} B_3 & B_4 \\ q_1 & q_2 \end{bmatrix}, q_1 + q_2 = 1$$

$$p_1 = \frac{4-1}{19} = \frac{3}{19}$$

$$q_2 = 1 - \frac{3}{19} = \frac{16}{19}$$

The optimum strategy of the given payoff matrix is given by,

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & \frac{5}{19} & \frac{15}{19} \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & \frac{3}{19} & \frac{16}{19} \end{pmatrix}$$

and the value of game is,

$$\frac{(4 \times 15) - (1 \times -1)}{19} = \frac{61}{19}$$

17. Following is the pay-off matrix for Player A

$$\begin{bmatrix} \mathbf{3} & \mathbf{5} & \mathbf{4} & \mathbf{2} \\ \mathbf{5} & \mathbf{6} & \mathbf{2} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} & \mathbf{4} & \mathbf{0} \\ \mathbf{3} & \mathbf{3} & \mathbf{5} & \mathbf{2} \end{bmatrix}$$

Using the dominance properly, obtain the optimum strategies for both the players and determine the value of the game.

Sol :

Since all elements of row 1 dominates row 3, i.e., 3 > 2, 5 > 1, 4 = 4, 2 > 0

Therefore row 3 can be eliminated. The matrix reduces to

$$\begin{bmatrix} 3 & 5 & 4 & 2 \\ 5 & 6 & 2 & 4 \\ 3 & 3 & 5 & 2 \end{bmatrix}$$

All elements in column 2 dominates column 1. Therefore column 2 can be eliminated. The matrix reduces to

$$\begin{bmatrix} 3 & 4 & 2 \\ 5 & 2 & 4 \\ 3 & 5 & 2 \end{bmatrix}$$

Column 1 dominates column 3, so column 1 can be eliminated to give

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \\ 5 & 2 \end{bmatrix}$$

Row 3 dominates row 1, so row 1 can be eliminated. The game reduces to 2 × 2 sub game

$$\begin{bmatrix} 2 & 4 \\ 5 & 2 \end{bmatrix}$$

The optimum strategies for Player A and Player B are

$$\left(0, \frac{3}{5}, 0, \frac{2}{5}\right) \text{ and } \left(0, 0, \frac{2}{5}, \frac{3}{5}\right)$$

The volume of the game is $= \frac{16}{5}$.

18. Use dominance to reduce the size of the following game to 2 × 2 game and hence find the optimal strategies and value of the game.

Player B

		I	**II**	**III**	**IV**
	I	**3**	**2**	**4**	**0**
Player A	**II**	**3**	**4**	**2**	**4**
	III	**4**	**2**	**4**	**0**
	IV	**0**	**4**	**0**	**8**

Sol :

Step 1 :

Since all elements of the third row are greater than or equal to the corresponding entries in the first row, the first row is dominated by third row and row 1 is deleted.

$$\begin{array}{c|cccc|} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline \text{I} & 3 & 4 & 2 & 4 \\ \text{II} & 4 & 2 & 4 & 0 \\ \text{IV} & 0 & 4 & 0 & 8 \\ \hline \end{array}$$

Step 2 :

Since all elements of column 1 are greater than or equal to corresponding elements in the column II. So delete column I.

$$\begin{array}{c|ccc|} & \text{II} & \text{III} & \text{IV} \\ \hline \text{II} & 4 & 2 & 4 \\ \text{III} & 2 & 4 & 0 \\ \text{IV} & 4 & 0 & 8 \\ \hline \end{array}$$

Step 3 :

The first column is dominated by average of column III and IV i.e.,

$$4 > \frac{3+4}{2};\ 2 = \frac{4+0}{2};\ 4 = \frac{0+8}{2}.$$

So column II is deleted. We get

$$\begin{array}{c|cc|} & \text{II} & \text{IV} \\ \hline \text{II} & 2 & 4 \\ \text{III} & 4 & 0 \\ \text{IV} & 0 & 8 \\ \hline \end{array}$$

Step 4 :

Row II is equal to the average of rows III and IV. Delete Row II. We get

$$\begin{array}{c|cc|} & \text{III} & \text{IV} \\ \hline \text{III} & 4 & 0 \\ \text{IV} & 0 & 8 \\ \hline \end{array}$$

$$p_1^* = \frac{8-0}{(4+8)-(0+0)}$$

$$= \frac{2}{3};\ p_2 = 1 - p_1$$

$$= \frac{1}{3}$$

and value of the game is given by

$$V = \frac{4\times 8 - 0\times 0}{(4+8)-(0+0)}$$

$$= \frac{8}{3}$$

Hence, the optimal strategy for A is $\left(0\,0\,\frac{2}{3}\,\frac{1}{3}\right)$, for B $\left(0\,0\,\frac{2}{3}\,\frac{1}{3}\right)$ and value of game is = $\frac{8}{3}$.

Q24. Explain the graphical method for 2 × n or m × 2 games.

Ans :

Principle of dominance is applied to rectangular games without saddle point so that the size of the matrix can be reduced and appropriate solution method can be employed. After applying the dominance principle, if the size of the matrix is reduced to 2 × 2, then the algebraic method can be used to solve the game similar to Type-II rectangular games without saddle point.

But if the size reduces to 2 × n or m × 2, then a new solution methodology called graphical method is used.

By using graphical approach, it is aimed to reduce the game to the size of 2 × 2 by identifying and eliminating dominated strategies and then solve it by the analytical method i.e., (algebraic method).

Graphical Method

The steps of the graphical method are summarized as follows,

Ensure the game has no saddle point.

Step 1

Using dominance rule, reduce the size of the m × n matrix to 2 × n or m × 2 matrix.

Step 2

Draw two vertical parallel axis of distance one unit each axis representing the two strategies of player A, in case of 2 × n matrix and player B, in case of m × 2 matrix.

Step 3

Plot the strategies on the graph, one line for each strategy. Thus, there will be n lines for 2 × n matrix and m lines form m × 2 matrix.

Step 4

Identify the,

(i) Maximum ordinate of the convex set bounded above (lower envelope) for 2 × n matrix and

(ii) Minimum ordinate of the convex set bounded below (upper envelope) for m × 2 matrix.

Step 5

This ordinate gives the value of the game.

Step 6

Identify the 2 strategies, out of the *m* or *n* strategies which, bounds the convex set. The 2 strategies lead to reduction of the size of the payoff matrix to 2 × 2.

Step 7

Solve the 2 × 2 matrix by algebraic method.

PROBLEMS

19. Solving the game whose payoff matrix is,

$$\begin{array}{cc|cccc|} & & & B & & \\ & & I & II & III & IV \\ \cline{3-6} A & I & 1 & 4 & -2 & -3 \\ & II & 2 & 1 & 4 & 5 \\ \cline{3-6} \end{array}$$

Sol :

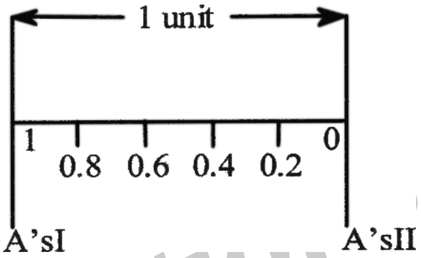
The game has no saddle point

Step 1

The size of the mtrix is already of the required size 2 × n to use graphical method.

Step 2

Draw two parallel lines to represent the 2 strategies of A with a distance of one unit.



Figure

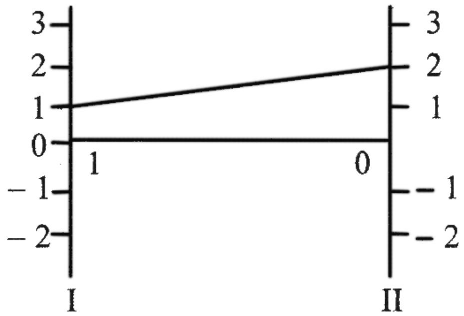
Step 3

Plot the strategies of B on the graph.

For example,

$$\begin{array}{cc|cc} & & B & \\ & & I & II \\ \hline A & I & 1 & \\ & II & 2 & \end{array}$$

B's 1 strategy is plotted as (1, 2) i.e, 1 on As'1 and 2 on A's II and join the two points to form a straight line.

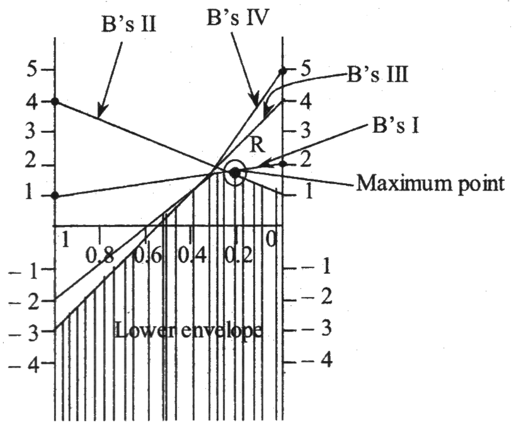


Figure

Similarly, plot the other strategies

Step 4

Identify the maximum value of the lower envelope as it is a 2 × n matrix.



Step 5

Maximum point is R i.e., value of game $\cong$ 2.

Step 6

The two strategies of B passing through the maximum point are B's I and B's II. Thus the reduced 2 × 2 matrix is,

$$\begin{array}{cc|cc|} & & \multicolumn{2}{c}{B} \\ & & I & II \\ \cline{3-4} A & I & 1 & 4 \\ & II & 2 & 1 \\ \cline{3-4} \end{array}$$

Step 7

Using algebraic method, the given matrix is solved.

a_{11} = 1, a_{12} = 4, a_{21} = 2 and a_{22} = 1

(i) Let the player of select strategy I with probability x and strategy II with probability 1 – x.

(ii) Let the player B of select strategy I with probability y and strategy II with probability 1 – y.

Value of x

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{1 - 2}{(1+1) - (4+2)}$$

$$= \frac{-1}{2-6} = \frac{-1}{-4} = \frac{1}{4}$$

$$1 - x = 1 - \frac{1}{4} = \frac{3}{4}$$

Value of y

$$y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{1 - 4}{(1+1) - (4+2)} = \frac{-3}{-4} = \frac{3}{4}$$

$$1 - y = 1 - \frac{3}{4} = \frac{1}{4}$$

Value of Game (v)

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(1 \times 1) - (4 \times 2)}{(1+1) - (4+2)}$$

$$= \frac{-7}{-4} - \frac{7}{4}$$

Optimal Strategy

Players	Strategy	Probability
For player A	I	$\frac{1}{4}$
	II	$\frac{3}{4}$
For player B	I	$\frac{3}{4}$
	II	$\frac{1}{4}$
	III	0
	IV	0

Value of game, v = $\frac{7}{4}$

PROBLEMS

20. Solving the following game using graphical method.

$$\begin{array}{cc} & \textbf{B's Strategy} \\ & \begin{array}{cc} \mathbf{b_1} & \mathbf{b_1} \end{array} \\ \textbf{A's Strategy} & \begin{array}{c} \mathbf{a_1} \\ \mathbf{a_2} \\ \mathbf{a_3} \\ \mathbf{a_4} \end{array} \begin{bmatrix} -7 & 6 \\ 7 & -4 \\ -4 & -2 \\ 8 & -6 \end{bmatrix} \end{array}$$

Sol :

Step 1

The given matrix is of required size m × 2 to use graphical method.

Step 2

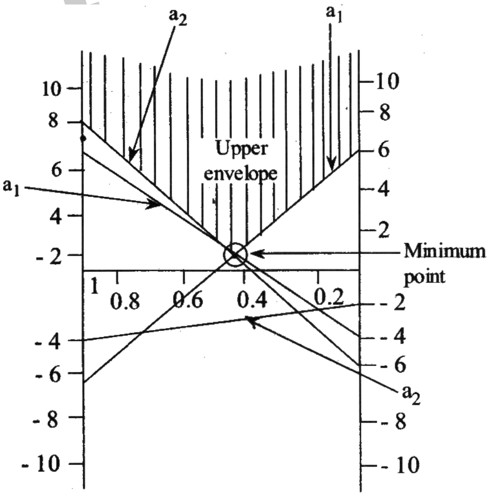
Draw two parallel lines to represent b_1 and b_2 strategies.

Step 3

Plot the strategies of A.

Step 4

Identify the minimum point of upper envelope.



Figure

The two strategies passing through the minimum point are a_x and a_r

Thus, the reduced payoff matrix is,

$$\begin{array}{c|cc} & b_1 & b_2 \\ \hline a_a & -7 & 6 \\ a_2 & 7 & -4 \end{array}$$

Where,

$a_{11} = -7, a_{12} = 6, a_{21} = 7$ and $a_{22} = -4$

Value of x

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{-4 - 7}{(-7 - 4) - (6 + 7)}$$

$$= \frac{11}{24}$$

$$1 - x = 1 - \frac{11}{24} = \frac{13}{24}$$

Value of y

$$y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{-4 - 6}{(-7 - 4) - (6 + 7)} = \frac{5}{12}$$

$$1 - y = 1 - \frac{5}{12} = \frac{7}{12}$$

Value of Game v

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(-7 \times -4) - (6 \times 7)}{(-7 - 4) - (6 + 7)} = \frac{7}{12}$$

Optimal Strategy

Players	Strategy	Probability
For player A	a_1	$\frac{11}{24}$
	a_2	$\frac{13}{24}$
	a_3	0
	a_4	0
For player B	b_1	$\frac{5}{12}$
	b_2	$\frac{5}{12}$

Value of game, $v = \frac{7}{4}$

---

21. Solve the following game graphically.

$$\begin{bmatrix} \mathbf{-6} & \mathbf{0} & \mathbf{6} & \mathbf{-3/2} \\ \mathbf{7} & \mathbf{3} & \mathbf{-8} & \mathbf{2} \end{bmatrix}$$

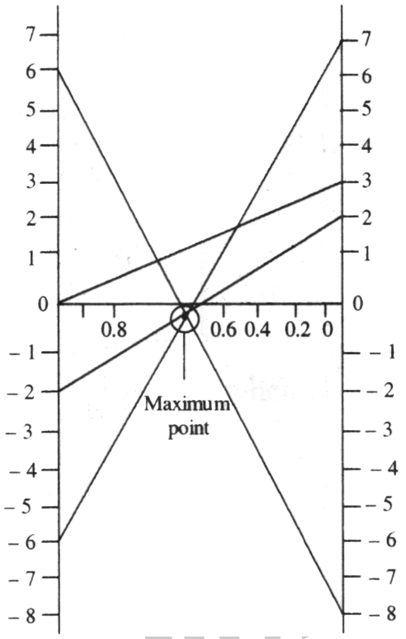
Sol :

Step 1

The required size of 2 × n is already given to use graphical method

Step 2

Draw two parallel lines representing the two strategies with a distance of 1 unit.



Step 3

Plot the stategies of the columns player.

Step 4

Identify the maximum point of the lower envelope. 2 × 2 matrix from strategies passing through maximum point i.e., strategies of column I and II

$$\begin{array}{cc} & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ \begin{array}{c} \text{I} \\ \text{II} \end{array} & \begin{bmatrix} -6 & 6 \\ 7 & -8 \end{bmatrix} \end{array}$$

Thus $a_{11} = -6$, $a_{12} = 6$, $a_{21} = 7$ and $a_{22} = -8$

Value of x

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{-8-7}{(-6-8)-(6+7)}$$

$$= \frac{-15}{-14-13} = \frac{-15}{-27} = \frac{5}{9}$$

$$1 - x = 1 - \frac{5}{9} = \frac{4}{9}$$

Value of y

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{-8 - 6}{(-6 - 8) - (6 + 7)}$$

$$= \frac{-14}{-14 - 13} = \frac{-14}{-27} = \frac{14}{27}$$

$$1 - y = 1 - \frac{14}{27} = \frac{13}{27}$$

Value of Game V

$$v_1 = \frac{(a_{11} \times a_{22}) - (a_{12} \times a_{21})}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(-6 \times -8) - (6 \times 7)}{(-6 - 8) - (6 + 7)}$$

$$= \frac{48 - 42}{48 - 42} = \frac{6}{6} = 1$$

$$v = \frac{(a_{11} \times a_{22}) - (a_{12} \times a_{21})}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(-6 \times -3) - (0 \times 7)}{(-6 - 3) - (0 + 7)}$$

$$= \frac{18}{-16} = \frac{-9}{-8}$$

Optimal Strategy

Royal player $\left[\frac{5}{9}, \frac{4}{9}\right]$

Column player $\left[\frac{14}{27}, \frac{13}{27}, 0, 0\right]$

Value of the game, v = 1 (ie., row player losses 1 point).

22. Solve the following 2 × 2 game graphically

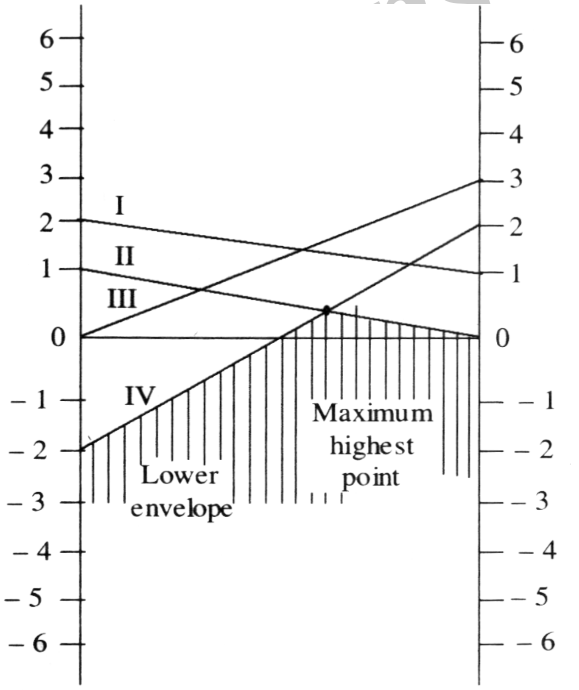
Player B

	2	**1**	**0**	**–2**
Player A	**1**	**0**	**3**	**2**

Sol :

Column player pure strategies Row players (A/s) expected payoff

$$\begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} \begin{bmatrix} 2x_1 & + & 1x_2 \\ 1x_1 & + & 0x_2 \\ 0x_1 & + & 3x_2 \\ -2x_1 & + & 2x_2 \end{bmatrix}$$



The highest point on lower envelope appears at the intersection of the lines represented by column II and IV strategies.

$1x_1 + 0x_2$ and $-2x_1 + 2x_2$

The required 2 × 2 matrix is,

Player B

$$\text{Player A} \quad \begin{array}{c|cc} & q_1 & q_2 \\ \hline P_1 & 1 & 0 \\ P_2 & -2 & 2 \end{array}$$

1. Let A play strategy P_1 with ‘x’ as probability and P_2 with (1 – x) as probability.
2. Let B play strategy q_1 with ‘y’ as probability and q_2 with (1 – y) as probability.

Valuve of x = $\frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{2-(-2)}{(1+2)-(0+(-2))} = \frac{4}{(3)-(-2)} = \frac{4}{5}$

$1 - x = 1 - \frac{4}{5} = \frac{1}{5}$

Value of y = $\frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

$$= \frac{2-(0)}{(1+2)-(0+(-2))} = \frac{2}{(3)-(-2)} = \frac{2}{5}$$

$1 - y = 1 - \frac{2}{5} = \frac{3}{5}$

Value of the game

$$\nu = \frac{(a_{11} \times a_{22}) - (a_{12} \times a_{21})}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(1 \times 2) - (0 \times (-2))}{(1+2) - (0 + (-2))}$$

$$= \frac{2-(-0)}{3-(-2)} = \frac{2}{5}$$

Optimal Strategy

Row player $\left[\frac{4}{5}, \frac{1}{5}\right]$

Column player $\left[\frac{2}{5}, \frac{3}{5}, 0, 0\right]$

Value of the game, $\nu = \frac{2}{5}$

Exercise Problems

1. In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time (the time taken to dump a train) distribution is also exponential with an average of 36 minutes, calculate (i) expected queue size (line length), (ii) probability that the queue size exceeds 10. (iii) If the input of trains increase to an average of 33 per day, what will be the change in (i) and (ii).

   [**Ans :** (i) 3 trains, (ii) 0.83, (iii) the change in (i) 5 trans (approx.) and (ii) 0.155 (approx.)]

2. Arrivals at a telephone booth are considered to be Poisson with an average time of 10 minutes between one arrival to the next. The length of phone call is assumed to e distributed exponentially, which mean 3 minutes.

   (i) What is the probability that a person arriving at the booth will have to wait.

   (ii) The telephone expect waiting for at least 3 minutes for a phone call. By how much the flow of arrivals should increase in order to justify a second booth.

   (iii) What is the average length of queue that forms time to time.

   (iv) What is the probability that it will take him more than 10 minutes altogether to wait for the phone call and completed his call ?

   [**Ans :** (i) 0.3, (ii) 0.06 arrivals per min., (iii) 1.43 min., (iv) 0.03]

3. A milk plant distributes its products by trucks, loaded at the loading dock. It has its own fleet plus the trucks of a transport company are used. This company has complained that sometimes the trucks have to wait in the queue and thus the company loses money. The company has asked the equivalent to witing time. The data available is as follows :

   Average arrival rate = 3 per hour.

   Average service rate = 4 per hours.

   The transport company has provided 40% of the total number of trucks.

   Determine.

   (a) The probability that a truck has to wait.

   (b) The waiting time of a truck

   (c) Expected waiting time for transport company trucks per day.

   [**Ans :** (a) 0.75, (b) 1 hr., (c) 21.6 hrs / day.]

4. A road transport company has one reservation clerk on duty at a time. She handles information of bus schedules and makes reservations. Customers arrive at a rate of 8 per hour and the clerk can service 12 customers on the average per hour. After stating your assumptions, answer the following :

   (i) What is the average number of customers a waiting for the service of the clerk ?

   (ii) What is the average time a customer has to wait before getting service ?

   (iii) The management is contemplating to install a computer system to handle the information and reservations. This is expected to redue the service time from 5 to 3 minutes. The additional cost of having the new system works out to Rs. 50/- per day. If the cost of goodwill of having a customer to wait is estimated to be 12 paise per minute spent waiting before being served, should be the company installa the computer system ? Assume 8-hour working day.

   [**Ans :** (i) 1.33 customers, (iii) 10 min, (iii) Instal the system.]

5. Solve the following 2 × 2 games :

   (i) $\begin{pmatrix} -4 & 6 \\ 2 & -3 \end{pmatrix}$

   (Ans : (1/3, 2/3), (3/5, 2/5), V = 0)

   (ii) $\begin{pmatrix} 6 & -3 \\ -3 & 0 \end{pmatrix}$

   [**Ans :** (1/4, 3/2), (1/4, 3/4), V = 3/4]

Saddle Points :

Find the saddle point (or points) and hence solve the following games :

6. Player B

$$\text{Player A}\quad \begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 15 & 2 & 3 \\ A_2 & 6 & 5 & 7 \\ A_3 & -7 & 4 & 0 \end{array}$$

[**Ans :** (A_2, B_2), v = 5]

7. Player B

$$\text{Player A}\quad \begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 1 & 7 & 3 & 4 \\ A_2 & 5 & 6 & 4 & 5 \\ A_3 & 7 & 2 & 0 & 3 \end{array}$$

[**Ans :** (A_2, B_2), v = 4]

8. Player B

$$\text{Player A}\quad \begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline \text{I} & -5 & 2 & 1 & 20 \\ \text{II} & 5 & 5 & 4 & 6 \\ \text{III} & 4 & -2 & 0 & -5 \end{array}$$

[**Ans :** (II, III), v = 4]

9. Player B

$$\text{Player A}\quad \begin{array}{c|ccccc} & \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\ \hline \text{I} & 9 & 3 & 1 & 8 & 0 \\ \text{II} & 6 & 5 & 4 & 6 & 7 \\ \text{III} & 2 & 4 & 4 & 3 & 8 \\ \text{IV} & 5 & 6 & 2 & 2 & 1 \end{array}$$

[**Ans :** (II, III), v = 4]

10.

$$\begin{array}{c|ccc} & \text{I} & \text{II} & \text{III} \\ \hline \text{I} & 6 & 8 & 6 \\ \text{II} & 4 & 12 & 2 \end{array}$$

[**Ans :** (I, I), (I, III), v = 6]

11.

$$\begin{array}{c|ccc} & C_1 & C_2 & C_3 \\ \hline R_1 & 3 & 0 & -3 \\ R_2 & 2 & 3 & 1 \\ R_3 & -4 & 2 & -1 \end{array}$$

[**Ans :** (R_2, C_3), v = 1]

Graphical Method :

Use graphical method to reduce the following games and hence solve :

12. B

$$A\ \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix}$$

[**Ans :** A (4/11, 7/11), B(0, 7/11, 4/11), v = –5/11]

13. B

$$A\ \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \end{bmatrix}$$

[**Ans :** (3/7, 4/7), (2/7, 0, 5/7), v = 8/7]

14. Player B

$$\text{Player A}\ \begin{bmatrix} 2 & 7 \\ 3 & 5 \\ 11 & 2 \end{bmatrix}$$

[**Ans :** (9/14, 0, 5/14), (5/14, 9/14), v = 73/14]

15. Player B

$$\text{Player A}\ \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ -7 & -9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix}$$

[Hint : Upper boundary is the line (5 → 6)]

[**Ans :** (0, 1, 0, 0); (1, 0), v = 5]

16. Solve the following 2 × 4 games graphically :

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 3 \end{bmatrix} \end{array}$$

[**Ans :** (2/5, 3/5), (0, 4/5, 0, 1/5), v = 2/5]

17. Obtain the optimal strategies for both persons and the value of the game for zero-sum two-person game whose pay-off matrix is given as follows :

$$\begin{array}{cc} & \text{Player A} \\ \text{Player B} & \begin{bmatrix} 1 & 3 & -1 & 4 & 2 & -5 \\ -3 & 5 & 6 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

[**Ans :** (0, 3/5, 0, 2/5, 0, 0), (4/5, 1/5), v = 17/5].

Short Question and Answers

1. Define the term queue.

Ans :

A flow of customers from finite/infinite population towards the service facility forms a queue (waiting line) on account of lack of capability to serve them all at a time. In the absence of a perfect balance between the service facilities and the customers, waiting time is required either for the service facilities or for the customer's arrival.

The arriving unit that requires some service to be performed is called customer. The customer may be persons, machines, vehicles, etc. Queue (waiting line) stands for the number of customers waiting to be serviced. This does not include the customer being serviced. The process or system that performs services to the customer is termed by service channel or service facility.

2. Queue Discipline.

Ans :

It is the order in which the customer is selected from the queue for service. There are numerous ways in which customers in queue can be served of which some are listed below.

(a) First In First Out (FIFO) or First Come first serve (FCFS)

It is the discipline in which the customers are served in the chronological order of their arrivals.

Eg. Tickets at a cinema hall; sales at a grocery shop, trains on a (single line) platform etc.

(b) Last In First Out (LIFO) or Last Come First Serve (LCFS)

If the service is made in opposite order of arrivals of customers, i.e, who ever comes last is served first and first obviously goes to last, it is called LIFO or LCFS system.

Eg. Stack of plates: Loading and unloading a truck or go-down; office filing of papers in chronological orders; wearing socks and shoes; dressing a shirt and coat over it, packing systems etc.

(c) Service In Random Order (SIRO)

By this rule, the customer for service is picked up at random, irrespective of their arrivals.

Eg. Lottery system from which one is picked up, the dresses waiting in a ward robe from which one is to be chosen, food stuffs in a buffet, sales counter of commodities or vegetables etc.

(d) Priority Service

Under this rule, the server gives priority to certain customer (s) due to some importance or prestigeous or high cost group of the customers.

Eg. A telephone urgent call given to a customer is charged at higher price, a separate counter for cheques at a electricity bill payment counter.

(e) Pre-emptive Priority Rule

Under this rule, highest priority is given to certain customer(s) irrespective of their arrival and costs.

Eg. An emergency case arriving at a doctor's clinic who is attending to a regular out-patient. (The doctor will stop his service to the regular patient and immediately rushes to emergency case).

3. Define Kendall's Notation.

Ans :

Generally, queueing model may be completely specified in the following symbol form (a//b/c): (d/e) where a = probability law for the arrival (inter-arrival) time.

b = probability law according to which the customers are being served.

c = number of channels (or service stations)

d = capacity of the system, i.e., the maximum number allowed in the system (in service and waiting).

e = queue discipline.

4. Queue Behaviour.

Ans :

The attitude or behaviour of the customers entering the queuing system.

- Customer with patience waits till the service is offered.
- Impatient Customer does one of the following defections,

(i) Jockeying

Customers switch to other queues in the hope of reducing waiting time and thereby getting service quickly.

(ii) Reneging

Customers wait in the queue for sometime and then leaves without getting the service.

(iii) Balking

Customers do not join the queue now and decide to join at a later time or may give up the idea of getting service.

5. Probabilistic Queueing Model.

Ans :

The queuing model which is widely used is probabilistic queuing model. The models commonly used under this classification are defined in terms of the arrival and service process, number of servers and population type.

The arrival distribution, in general follows the Poisson distribution and the service process follows the exponential distribution. There are other distributions that may be followed by the arrival and service process like Enlarge distribution.

Types of Probabilistic Queuing Models

(a) **Poisson:** Exponential, Single Server Model - Infinite Population.

(b) **Poisson:** Exponential, Single Server Model - Finite Population.

(c) **Poisson:** Exponential, Multiple Server Model - Infinite Population.

6. Deterministic Queuing Model.

Ans :

A deterministic queuing model is the one in which customers arrive in the queuing system at regular intervals and the service time for each customer is known as constant.

Examples

(i) If trucks arrive at a go-down every 10 minutes, then the interval between the arrival of any two successive trucks is exactly 10 minutes. Let us assume that the truck unloader takes exactly 10 minutes to unload the truck items. The arrival and service rates are equal to each 6 trucks per hour. In such a situation, there shall be no queue and the service person (truck unloader > is always busy.

(ii) If the service time is reduced to 8 minutes, then the truck unloader will be idle for some time i.e.. 10-8 = 2 minutes. In such situations also there will be no queue.

(iii) If the service time is increased to 14 minutes, then the truck unloader will be completely busy and the trucks will be waiting for service. This results in the formation of queue. But, this queue will keep growing and eventually the system leads to an explosive situation. To overcome this, additional service stations can be provided i.e., another truck unloader.

Symbolically,

Let, λ = Arrival rate

μ = Service rate.

Then, if $\lambda > m \rightarrow$ Waiting line formed, service facility always busy, system fails eventually.

If $\lambda < \mu \rightarrow$ No waiting time, proportion of time service facility idle is $1 - \frac{\lambda}{\mu}$.

$$\frac{\lambda}{\mu} = \rho,$$

ρ = Average utilization or traffic intensity.

Thus,

If $\rho > 1$ system would ultimately fail

If $\rho \leq 1$ system works and p is the proportion of time it is busy.

Such deterministic nature of arrival and service is rare and may exist in highly automated plants.

7. Define game theory.

Ans :

In the competitive world, it is essential for an executive to study or at least guess the activities or actions of his competitor. Moreover, he has to plan his course of actions or reactions or counter actions when his competitor uses certain technique. Such war or game is a regular feature in the market and the competitors have to make their decisions in choosing their alternatives among the predicted outcomes so as to maximize the profits or minimizing the loss.

8. Characteristics of Game Theory.

Ans :

There can be various types of games. They can be classified on the basis of the following characteristics.

i) **Chance of Strategy :** If in a game activities are determined by skill, it is said to be a game of strategy; if they are determined by change, it is a game of chance. In general, a game may involve game of strategy as well as a game of chance.

ii) **Number of Persons**

A game is called an n-person game if the number of persons playing in n. The person means an individual or a group aiming at a particular objective.

iii) **Number of Activities**

These may be finite or infinite.

(iv) **Number of Alternatives (choices) Available to Each Person** in a particular activity may also be finite or infinite. A finite game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is said to be infinite.

(v) **Information to the players about the past activities of other players** is completely available, partly available, or not available at all.

9. State the Assumptions of game Theory

Ans :

Assumptions of game theory

The underlying assumptions, the rules of the game as given as follows :

1. The player act rationally and intelligently.
2. Each player has available to him a finite set of possible courses of action.
3. The player attempt to maximize gains and minimize losses.
4. All relevant information is known to each players.
5. The players make individual decisions without direct communication.
6. The players simultaneously select their respective courses of action.
7. The pay off is fixed and determined in advance.

10. Advantages of Game Theory

Ans :

1. Game theory keeps deep insight to few less known aspects, which arise in situations of conflicting interests.
2. Game theory creates a structure for analysis of decision-making in various situations like interdependence of firms etc.
3. For arriving at optimal strategy, game theory develops a scientific quantitative technique for two person zero-sum games.

11. Disadvantages of Game Theory

Ans :

1. The highly unrealistic assumption of game theory is that the firm has prior knowledge about its competitor's strategy and is able to construct the payoff matrix for possible solutions, which is not correct. The main fact is that any firm is not exactly aware of its competitor's strategy. He can only make guesses about its strategy.
2. The hypothesis of maximin and minimax clearly shows that players are not risk lover and have whole knowledge about the strategies but the fact is that it is not possible.
3. It is totally impractical to understand that the several strategies followed by the rival player against others lead to an endless chain.
4. Most economic problems occur in the game if many players are involved in comparison to two-person constant sum game, which is not easy to understand. For example, the number of sellers and buyers is quite large in monopolistic competition and the game theory does not provide any solution to it.

12. Define saddle point.

Ans :

A saddle point is a position in the payoff matrix where the maximum of row minima coincides with the minimum of column maxima. The payoff at the saddle point is called the value of the game.

We shall denote the maximin value by γ, the minimax value of the game by $\overline{\gamma}$ and the value of the game by γ.

Note

(i) A game is said to be fair if

maximin value = minimax value = 0, i.e., if $\overline{\gamma} = \underline{\gamma} = 0$

(ii) A game is said to be strictly determinable if maximin value = minimax value $\neq 0$, $\underline{\gamma} = \gamma = \overline{\gamma}$.

Internal Assessment (Mid Examinations)

The pattern of Mid Exams or Continuous Internal Evaluation (CIE) prescribed by the JNTU-H as per the Regulations 2019 (R19) for all the semesters is as follows,

- There would be two Mid Exams or Continuous Internal Evaluation (CIE) for each semester,
  - The Ist **Mid Term Examinations** would be conducted during the Middle of the Semester.
  - The IInd **Mid Term Examinations** during the last week of instructions.
- The Mid Exam I and II would have the same pattern of question paper which would carry **25 Marks** each and the time duration for conducting each Mid exam would be 120 min.
- The pattern of Mid Exam Question Paper would consist of two parts i.e., **Part-A** and **Part-B.**
  - **Part-A** consist of 5 compulsory questions each carries 2 marks (i.e 5 × 2 = 10 marks).
  - **Part-B** consist of 5 questions out of which 3 questions should be answered, each question carries 5 marks (i.e 5 × 3 = 15 marks).
- The average of the two Mid exams will be added with the 75 marks of External end examination which equals to 100 marks (i.e 25 + 75 = 100).

UNIT - I

Part - A

1. Define Operations Research. (Refer Unit-I, SQA-1)
2. Decision Making (Refer Unit-I, SQA-4)
3. Define a model ? (Refer Unit-I, SQA-7)
4. Explain limitations of operation research. (Refer Unit-I, SQA-8)
5. Explain different types of decision makings. (Refer Unit-I, SQA-5)

Part - B

1. Explain briefly about Nature and Scope of Operations Research. (Refer Unit-I, Q.No. 2)
2. State the evolution of operations research. (Refer Unit-I, Q.No. 5)
3. State the applications of operations research. (Refer Unit-I, Q.No. 6)
4. Explain the process for developing an operations research model. (Refer Unit-I, Q.No. 11)
5. Explain the different types of models. (Refer Unit-I, Q.No. 15)
5. Outline the general principles used in model building within the context of OR. Briefly explain the scientific method in OR. (Refer Unit-I, Q.No. 16)

UNIT - II

Part - A

1. What are the Assumptions of LPP ? (Refer Unit-II, SQA-2)
2. What is Graphical Method LPP ? (Refer Unit-II, SQA-4)
3. Define Simplex Method. (Refer Unit-II, SQA-6)
4. Define Multiple Optimal Solution. (Refer Unit-II, SQA-9)
5. Define Infeasible Solution. (Refer Unit-II, SQA-10)

Part - B

1. Elucidate the various assumptions of LPP ? (Refer Unit-II, Q.No. 3)

2. A firm manufacturers 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has 2 machines and given below is the required processing time in minutes for each machine on each product.

		Product		
		A	B	C
Machines	C	4	3	5
	D	2	2	4

Machine C and D have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 A's 200 B's and 50 C's but no more than 150 A's. Setup an LP problem to maximize the profit. (Refer Unit-II, Prob. 5)

3. Solve the LPP

Minimize Z = $4x_1 + x_2$

Subject to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0.$ (Refer Unit-II, Prob. 17)

4. Define duality principle. What is primal dual relationship. Write the rules for converting primal to dual. (Refer Unit-II, Q.No. 22)

5. What are the advantages and limitations of LPP ? (Refer Unit-II, Q.No. 23)

UNIT - III

Part - A

1. What is an Assignment Problem (Refer Unit-III, SQA-1)

2. When does Multiple Optimal Solutions are said to be operated in assignment problem. (Refer Unit-III, SQA-3)

3. What is Vogels Approximation Method (VAM)? (Refer Unit-III, SQA-8)

4. Define Transportation Problem. (Refer Unit-III, SQA-6)

5. What is degeneracy in transportation problem? (Refer Unit-III, SQA-10)

Part - B

1. Discuss the steps involved in the Hungarians Method used to find optimal solution to an Assignment Problem. (Refer Unit-III, Q.No. 4)

2. Solve the following assignment problem by Hungarian assignment method.

Time (in minutes)			
Worker	Job 1	Job 2	Job 3
A	4	2	7
B	8	5	3
C	4	5	6

(Refer Unit-III, Prob. 1)

3. Solve the following assignment problem of minimizing total time for doing all the jobs:

Job / Operator	I	II	III	IV	V
1	6	2	5	2	6
2	2	5	8	7	7
3	7	8	6	9	8
4	6	2	3	4	5
5	9	3	8	9	7
6	4	7	4	6	8

(Refer Unit-III, Prob. 5)

4. A salesmen must travel from city to city to maintain his accounts. This week he has to leave his home base and visit each other city and return home. The table shows the distances (in kilometers) between the various cities. The home city is city A. Use the assignment method to determine the tour that will minimize the total distances of visiting all cities and returning home.

		To City				
		A	B	C	D	E
From city	A	–	375	600	150	190
	B	375	–	300	350	175
	C	600	300	–	350	500
	D	160	350	350	–	300
	E	190	175	500	300	–

Suggest an optimal assignment and the total maximum sales increase per month. If for certain reasons sales representative 'B' cannot be assigned to sales territory III, will the optimal assignment schedule be different ? If so find that schedule and the effect on total sales. (Refer Unit-III, Prob. 9)

5. Differences between Assignment and Transportation Problem. (Refer Unit-III, Q.No. 21)

UNIT - IV

Part - A

1. What are the different decision making environments? (Refer Unit-IV, SQA-2)
2. Advantages of Decision Tree. (Refer Unit-IV, SQA-7)
3. Decision Tree Analysis. (Refer Unit-IV, SQA-6)
4. Define critical path. (Refer Unit-IV, SQA-13)
5. What is Project Crashing ? (Refer Unit-IV, SQA-16)

Part - B

1. What are the different decision making environments? (Refer Unit-IV, Q.No. 4)
2. Describe some methods which are useful for decision making under uncertainity (Refer Unit-IV, Q.No. 5)
3. Write a note on decision making under risk. Explain briefly about various criteria involved in the process of decision-making under risk. (Refer Unit-IV, Q.No. 6)
4. Write about the method and steps involved in construction of decision tree. (Refer Unit-IV, Q.No. 10)
5. What is network diagram ? State the rules for drawing network. What are the steps involved in developing a network diagram ? (Refer Unit-IV, Q.No. 12)
6. What are the differences between PERT and CPM. (Refer Unit-IV, Q.No. 17)

UNIT - V

Part - A

1. Define the term queue. State the various examples of queus. (Refer Unit-V, SQA-1)
2. Queue Behaviour. (Refer Unit-V, SQA-4)
3. Define game theory. (Refer Unit-V, SQA-7)
4. Advantages of Game Theory (Refer Unit-V, SQA-10)
5. Define saddle point. (Refer Unit-V, SQA-12)

Part - B

1. What are the features of queuing syetem. (Refer Unit-V, Q.No. 2)
2. What are the various types of queuing disciplines? Give suitable examples with their managerial implications. (Refer Unit-V, Q.No. 11)
3. A car hiring firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as Poisson variate with mean 1.5. Calculate the Proportion of days on which,
   (a) Neither car is used and
   (b) Some demand is refused. (Refer Unit-V, Prob. 6)
4. Define game theory. Explain the characteristic of game theory. (Refer Unit-V, Q.No. 15)
5. Explain briefly about dominance property. (Refer Unit-V, Q.No. 23)

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY HYDERABAD

MBA II-Semester Examinations

October / November - 2020

R19

QUANTITATIVE ANALYSIS FOR BUSINESS DECISIONS

Time : 2 Hours] [Max. Marks : 75

Note : Answer any five questions
All questions carry equal marks.

1. Explain the applications of OR in different Managerial Area **(Unit - I, Q.No.6)**

2. Solve the following LPP by Graphical Method :

Maximize Z = $50 X_1 + 60 X_2$

Subject to : $2X_1 + X_2 \leq 300$;

$4X_1 + 8X_2 \leq 900$; And $X_1 , X_2 \geq 0$

Ans :

Convert the inequalities into equations

$2 x_1 + x_2 = 300$ .... (1)

put $x_1 = 0, x_2 = 300$

$x_2 = 0, x_1 = 300/2 \Rightarrow \boxed{x_1 = 150}$

x_1	0	150
x_2	300	0

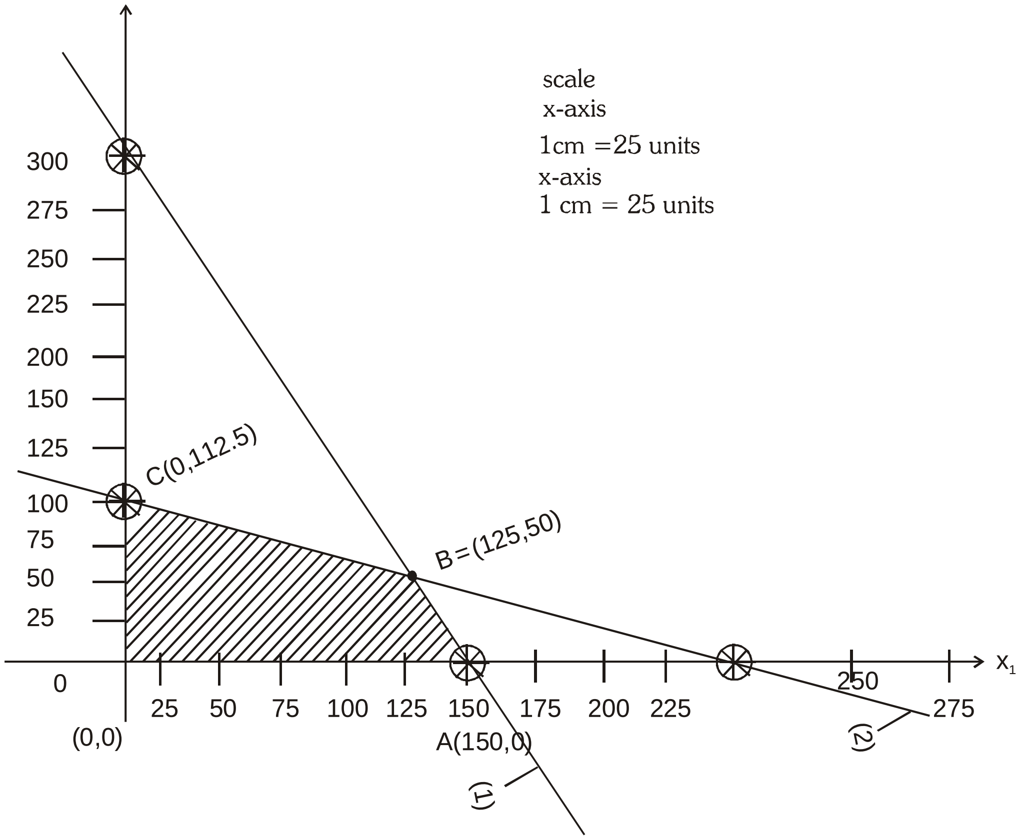
$4x_1 + 8x_2 = 900$ ....(2)

Put $x_1 = 0, x_2 = 900/8 = 112.5$

$x_2 = 0, x_1 = 900/4 = 225$

x_1	0	225
x_2	112.5	0

Plot these values on a graph paper



B is the intersection of equation (1) & (2) solve (1) & (2) to get (x_1,x_2), coordinates at 'B'

$$4\cancel{x}_1 + 8x_2 = 900.........(2)$$

$$2\times(1)\ \cancel{4}x_1 + 2x_2 = 600$$

$$\underline{-\qquad -\qquad -}$$

$$\underline{6x_2 = 300}$$

x_2 = 300/6

$\Rightarrow$ x_2 = 50

Substitute x_2 = 50 inequation (2)

At 'B' (x_1, x_2) are (125,50)

$2x_1 + x_2 = 300$

$2x_1 + 50 = 300$

$x_1 = 250/2$

$\boxed{x_1 = 125}$

Substitute the corner points of the feasible region OABC in the objective function

$z = 50\, x_1 + 60\, x_2$

at O (0,0) $z = 50(0) + 60(0) = 0$

at A(150,0) $z = 50(150) + 60(0) \Rightarrow = \text{Rs.}7500$

at B(125,50) $z = 50(125) + 60(50)$

$= 6250 + 3000 = \text{Rs.}9{,}250$ Maximum

at C(0, 112.5) $z = 50(0) + 60(112.5)$

$\therefore z = \text{Rs. } 6{,}750$

---

3. What is an Assignment Problem? Discuss its Objectives ? **(Unit - III, Q.No.1)**

4. What is Decision Tree ? Discuss the Advantages and Disadvantages of Decision Tree Approach ? **(Unit - IV, Q.No.8,9)**

5. Explain the structure and components of a Queuing Model. **(Unit - V, Q.No.4)**

6. Solve the following Assignment Problem : -

Hospitals→ / Doctors →	A	B	C	D
P	7	13	4	12
Q	5	3	8	2
R	9	12	11	8
S	10	6	8	7

Ans :

Apply Hungarian Method

Step 1 : Identify the smallest cost element in each row and subtract it from the corresponding elements of that row.

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline P & 3 & 9 & 0 & 8 \\ Q & 3 & 1 & 6 & 0 \\ R & 1 & 4 & 3 & 0 \\ S & 4 & 0 & 2 & 1 \\ \hline \end{array}$$

Step 2 : From the above table, identity the smallest cost element in each column and subtract it from the corresponding elements of that column.

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline P & 2 & 9 & 0 & 8 \\ Q & 2 & 1 & 6 & 0 \\ R & 0 & 4 & 3 & 0 \\ S & 3 & 0 & 2 & 1 \\ \hline \end{array}$$

Step 3 : Make assignments

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline P & 2 & 9 & \boxed{0} & 8 \\ Q & 2 & 1 & 6 & \boxed{0} \\ R & \boxed{0} & 4 & 3 & \cancel{0} \\ S & 3 & \boxed{0} & 2 & 1 \\ \hline \end{array}$$

The optimum assignment is $P \rightarrow C, \; Q \rightarrow D, \; R \rightarrow A, \; S \rightarrow B$

The total minimum assignment cost

= 4 + 2 + 9 + 6

= 21 Rs.

---

7. Explian the Opportunities and Short - comings of using an OR Model. **(Unit - I, Q.No.18)**

8. The Mean Arrival Rate at a Service Centre at a Bank is 4 hour. The Mean Service Rate is 8 minutes. Assuming Poisson Arrival Rate and Exponential Servicing Time, determine the following :

(a) Utilization Factor ;

(b) Probability that 2 persons are in the System;

(c) Expected no. of persons in the Queue.

Ans :

(i) $\rho = \lambda / \mu = \frac{4}{15/2} = 0.53$

(ii) $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)}$

$$= \frac{(4)^2}{\frac{15}{2}\left(\frac{15}{2} - 4\right)}$$

$$= \frac{16}{\frac{15}{2} \times \frac{7}{2}} = \frac{16}{26.25}$$

L_q = 0.609 customer

(iii) Probability that 2 persons are in the system

$$P_n = \left[1 - \frac{\lambda}{4}\right]\left[\frac{\lambda}{4}\right]^n$$

$$P_2 = \left[1 - \frac{4}{7.5}\right]\left[\frac{4}{7.5}\right]^2$$

$$= \left[\frac{7.5 - 4}{7.5}\right]\left[\frac{16}{56.25}\right]$$

$$= \frac{3.5}{7.5} \times \frac{16}{56.25}$$

$$= 0.13$$

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY HYDERABAD

M.B.A II - Semester Examination

R17

December - 2019

QUANTITATIVE ANALYSIS FOR BUSINESS DECISIONS

Time : 3 Hours] [Max. Marks : 75

Note : This question paper contains two parts A and B.

Part A is compulsory which carries 25 marks. Answer all questions in Part A, Part B consists of 5 units. Answer any one full question from each unit. Each question carries 10 marks and may have a, b, c as sub questions.

ANSWERS

PART - A (5 × 5 = 25 Marks)

1. (a) What is a model ? What purpose does it serve ? **(Unit - I, SQA-7)**

(b) What is duality in Linear Programming problem ? **(Unit - II, SQA-22)**

(c) What is an unbalanced assignment problem ? Illustrate your answer with example. **(Unit - III, SQA-7)**

Ans :

For example (3m)

Cost per Repair (Rs. Lakh)

Contractors / Road

	R_1	R_2	R_3	R_4
C_1	9	14	19	15
C_2	7	17	20	19
C_3	9	18	21	18
C_4	10	12	18	19
C_5	10	15	21	16

Since the number of rows ≠ number of column, the given matrix is not a square matrix.

Introduce a dummy column with zero cost elements to make it a square matrix and then apply HAM.

	R_1	R_2	R_3	R_4	R_5
C_1	9	14	19	15	0
C_2	7	17	20	19	0
C_3	9	18	21	18	0
C_4	10	12	18	19	0
C_5	10	15	21	16	0

(d) (i) What is cost slope ? **(Unit - IV, SQA-16)**

(ii) Distinguish between PERT & CPM. **(Unit - IV, SQA-15)**

(e) For the GI/G/I, FCS model, using the basic definitions and relationships, verify the following relationships :

(i) $L = L_q + (1 - P_0)$ (ii) $L = L_q + \rho$ (iii) $P_0 = 1 - \rho$

Ans :

(i) $L = L_q + (1 - \rho_0)$

$$\frac{\lambda}{\mu-\lambda} = \frac{\lambda^2}{\mu(\mu-\lambda)} + \lambda/\mu$$

$$\frac{\lambda}{\mu-\lambda} = \frac{\mu\lambda^2 + \lambda\mu(\mu-\lambda)}{\mu^2.(\mu-\lambda)}$$

$$\frac{\lambda}{\mu-\lambda} = \frac{\cancel{\mu\lambda^2} + \mu^2\lambda - \cancel{\mu\lambda^2}}{\mu^2(\mu-\lambda)}$$

$$= \frac{\cancel{\mu^2}\lambda}{\cancel{\mu^2}(\mu-\lambda)}$$

$\frac{\lambda}{\mu-\lambda} = \frac{\lambda}{\mu-\lambda}$ Hence proved L.H.S = R.H.S

(ii) $L = L_q + \rho$

$$\frac{\lambda}{\mu-\lambda} = \frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{\mu\lambda^2 + \mu\lambda(\mu-\lambda)}{\mu^2(\mu-\lambda)}$$

$$= \frac{\cancel{\mu\lambda^2} + \mu^2\lambda - \cancel{\mu\lambda^2}}{\mu^2(\mu-\lambda)}$$

$$= \cancel{\mu^2}\lambda / \cancel{\mu^2}(\mu-\lambda)$$

$$\frac{\lambda}{\mu-\lambda} = \frac{\lambda}{\mu-\lambda}$$

$\therefore L = L_q + \rho$ Hence proved

(iii) $L_s = 1 - \rho$

$$\frac{\mu - \lambda}{\mu} = 1 - \frac{\lambda}{\mu}$$

$$\frac{\mu - \lambda}{\mu} = \frac{\mu - \lambda}{\mu}$$

$\therefore$ $L_s = 1 - \rho$ Hence proved

PART - B (5 × 10 = 50 Marks)

2. (a) Explain briefly the steps (perhaps overlapping) involved in an Operations Research Study.

(Unit - I, Q.No -11)

(b) A company having a mechanical workshop has recently discontinued production of an unprofitable product. It has resulted in a considerable spare capacity. The company has decided to use this capacity to the maximum extent to produce three products which are profitable. The productivity coefficient in machine hours per unit and available machine time is given below :

Machine type	Product 1	Product 2	Product 3	Time Available Machine Hours per week
Milling Machine	9	3	5	500
Lathe	5	4	0	350
Grinder	3	0	2	150

The sales departments has indicated that the demand for Products 1 and 2 exceeds the maximum production rate whereas sales potential for Product 3 is 20 units per week. The profits for the three products have been estimated respectively as Rs. 3500. Rs. 1400, and Rs. 1750 for the three products. The company wants to decide the optimum level of production to maximize its profit.

Formulate this problem as a mathematical model.

Ans :

Decision vaiables

Let x_1, x_2, x_3 be the numbers of products 1,2 and 3.

Objective Function

Since the profits for the three products 3500, 1400, 1750

Max $Z = 3500\, x_1 + 1400\, x_2 + 1750\, x_3$

Constraints

There are 3 constraints of different machines of 3 products, product 1, product 2, and product 3.

$9x_1 + 3x_2 + 5x_3 \leq 500$

$5x_1 + 4x_2 + 0x_3 \leq 350$

$3x_1 + 0\, x_2 + 2x_3 \leq 150$

The maximum potential sales for product 3 is 20 units per week

$x_3 \leq 20$

$\therefore$ Max Z = $3500\,x_1 + 1400\,x_2 + 1750\,x_3$

Subject to constraints

$9x_1 + 3\,x_2 + 5x_3 \leq 500$

$5x_1 + 4x_2 + 0x_3 \leq 350$

$3x_1 + 0x_2 + 2x_3 \leq 150$

The maximum potential sales for product 3 is 20 units per week $x_3 \leq 20$

$\therefore$ Max z = $3500x_1 + 1400x_2 + 1750\,x_3$

Subject to constraints

$9x_1 + 3x_2 + 5x_3 \leq 500$, and $3x_1 + 0x_2 + 2x_3 \leq 150$

$5x_1 + 4x_2 + 0x_3 \leq 350$, $x_3 \leq 20$

$x_1, x_2, x_3 \geq 0$

OR

3. (a) Write a short note on sensitivity analysis and its importance.

Ans :

Sensitivity Analysis and its Importance Sensitivity Analysis

Sensitivity analysis is the study of sensitivity of the optimal solution of an LP problem due to discrete variations (changes) in its parameters.

The degree of sensitivity of the solution due to these variations can range from no change at all to a substantial change in the optimal solution of the given LP problem.

Theprocess of studying the sensitivity of the optimal solution of an LP problem is also called 'Post-optimality analysis' because it is done after an optimal solution, assuming a given set of parameters, has been obtained for the model.

Importance of sensitivity Analysis

- It is used to know how sensitivity the optimal solution is to the changes in the original input data values
- It provides the sensitive ranges (both lower and upper limits) within which the LP model parameters can vary without changing the optimality of the current optimal solution.

(b) Consider the following problem :

Minimize $Z = 5x_1 + 7x_2$

$2x_1 + 3x_2 \geq 42,$

$2x_1 + 4x_2 \geq 60,$

$x_1 + x_2 \geq 18,$

and $x_1 \geq 0, x_2 \geq 0$

Solve this problem graphically and explain briefly the steps involved in reaching the optimum solution.

Ans :

Min Z = $5x_1 + 7x_2$

subject to the

$2x_1 + 3x_2 \geq 42$

$3x_1 + 4x_2 \geq 60$

$x_1 + x_2 \geq 18$

$x_1 , x_2 \geq 0$

Convert inequalities into equations

$2x_1 + 3x_2 = 42$ ......(1)

when $x_1 = 0$, $x_2 = 14$

$x_2 = 0$, $x_1 = 21$

$3x_1 + 4x_2 = 60$ ......(2)

when $x_1 = 0$, $x_2 = 15$

$x_2 = 0$, $x_1 = 20$

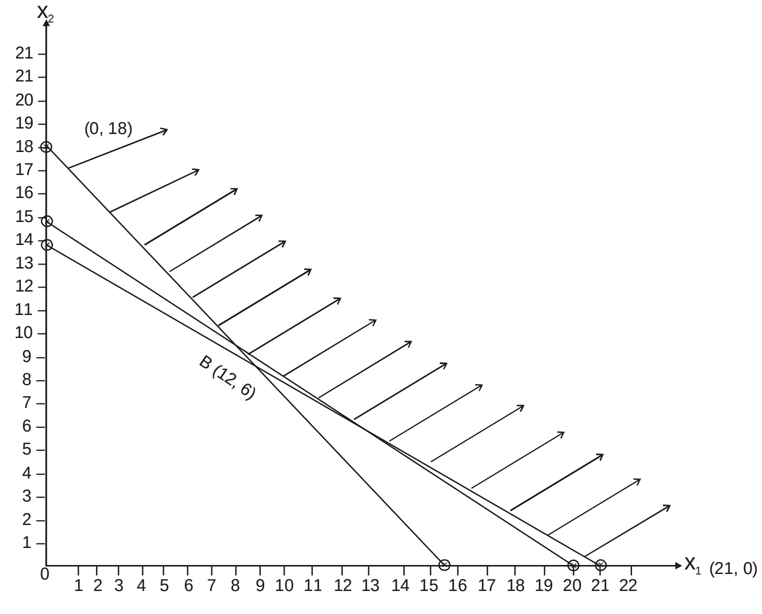
$x_1 + x_2 = 18$ ....(3)

when $x_1 = 0$, $x_2 = 18$

$x_2 = 0$, $x_1 = 18$

Solve Eqns (1) & (3)

$$\begin{array}{l} \cancel{2x_1} + 3x_2 = 42 \\ \Rightarrow \cancel{2x_1} + 2x_2 = 36 \quad \text{....... (3)} \\ \quad - \quad - \quad - \\ \hline x_2 = 6 \\ \hline \end{array}$$



Substitute $x_2 = 6$ in equn (3)

$x_1 + 6 = 18, x_1 = 18 - 6 = 12$

Min $Z = 5x_1 + 7x_2$

A (0,18) = 5(0) + 7(18) = 126

B(12,6) = 5(12) + 7(6) = 102

C(21,0) = 5(21) + 7(0) = 105

At B(12,6) Minimum value

4. David, LaDenna and Lydia are the sole partners and workers in a company which products fine clocks. David and LaDenna each are available for 40 hours per week at the company, which Lydia is available to work for a maximum of 20 hours per week. The company makes two different types of clocks : a grand-father clock and a wall clock. To make a clock, David (a mechanical engineer) assembles the inside mechanical parts of the clock while LaDenna (Woodworker) produces the hand carved wood casings. Lydia is responsible for taking orders and shipping the clocks. The amount of time required for each of these tasks is shown below:

	Time required	
	Grand father Clock	Wall Clock
Assemble clock mechanism	6 hours	4 hours
Carve wood casing	8 hours	4 hours
Shipping	3 hours	3 hours

Each grandfather clock built and shipped yields a profit of \$300, while each wall clock yields a profit of \$200.

Three partners now want to determine how many clocks of each type should be produced per week to maximize the total profit.

(a) Formulate a linear programming model in algebraic form for the problem.

Ans :

Formulation of LPP :

Let x_1 be the number of units of Grand father clock

x_2 be the no. of units of wall clock on one unit of grand father clock, the company earn \$ 300 and that of wall clock a profit of \$ 200.

The objective of the company is to maximize the total profit.

Max $Z = 300\,x_1 + 200\,x_2$

S.T. $6x_1 + 4x_2 \leq 40$ hours

$8x_1 + 4x_2 \leq 40$ hours

$3x_1 + 3x_2 \leq 20$ hours

$x_1, x_2 \geq 0$

(b) Solve Graphically

Convert inequalities into equations

$$6x_1 + 4x_2 = 40 \text{ ....(1)}$$

When $x_1 = 0, x_2 = 10$

$x_2 = 0, x_1 = 6.67$ (or) $\frac{\overset{20}{\cancel{40}}}{\cancel{6}_{3}} = {}^{20}\!/_{3}$

x_1	0	$^{20}/_{3}$
x_2	10	0

$$8x_1 + 4x_2 = 40 \text{ ......(2)}$$

When $x_1 = 0, x_2 = 10$

$x_2 = 0, x_1 = 5$

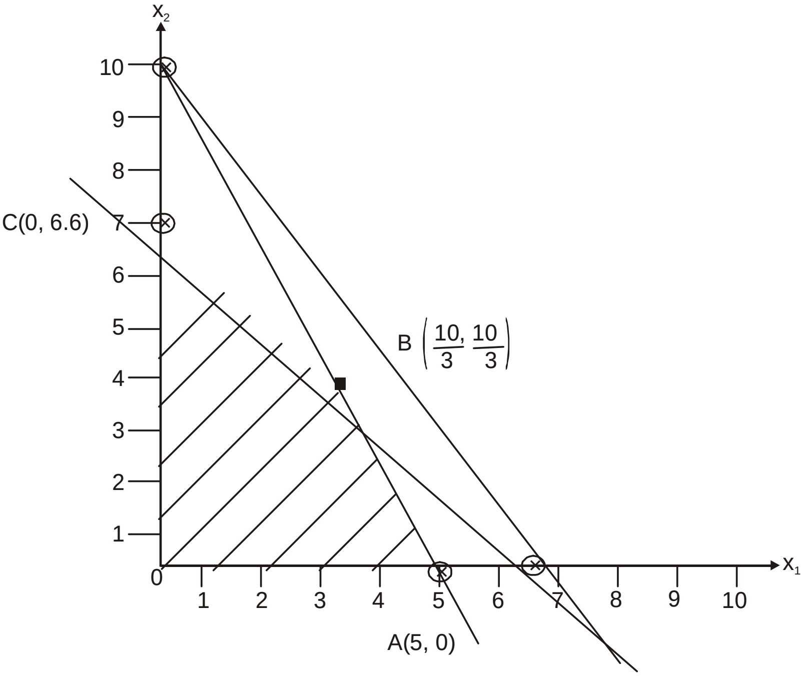
x_1	0	5
x_2	10	0

$$3x_1 + 3x_2 = 20 \text{ ....(3)}$$

When $x_1 = 0, x_2 = {}^{20}\!/_{3}$

$x_2 = 0, x_1 = {}^{20}\!/_{3}$

x_1	0	$^{20}/_{3}$
x_2	$^{20}/_{3}$	0



Solve eqns (2) & (3)

$$(2\times3) \Rightarrow \cancel{24x_1} + 12x_2 = 120$$
$$(3\times8) \Rightarrow \cancel{24x_1} + 24x_2 = 160$$
$$\quad - \qquad - \qquad -$$
$$+12x_2 = +\;40$$

$$x_2 = \frac{\overset{\overset{10}{\cancel{20}}}{\cancel{40}}}{\underset{\underset{3}{\cancel{6}}}{\cancel{12}}} = {}^{10}\!/_{3}$$

x_2 = ${}^{10}\!/_{3}$ in equn (3)

$$3x_1 + \cancel{3}\left(\frac{10}{\cancel{3}}\right) = 20$$

$3x_1$ = 10

x_1 = ${}^{10}\!/_{3}$

Max Z = 300 x_1 + 200 x_2

At (0,0) Z = 300 (0) + 200(0)$\Rightarrow$z = 0

At A (5,0) Z = 300 (5) + 200 (0) $\Rightarrow$ z = 1500

At B $\left(\frac{10}{3},\frac{10}{3}\right)$ Z = $\overset{100}{\cancel{300}}\left(\frac{10}{\cancel{3}}\right) + 200\left(\frac{10}{3}\right)$

= 1000 + 666.67 = 1666.67

At C(0,6.6) Z = 300(0) + 200(6.6) = 1320

At B$\left({}^{10}\!/_{3},{}^{10}\!/_{3}\right)$ the value of Z is maximum

OR

5. There are 3 sources and four destinations with the costs of transportation are as shown in the following table.

Source	Destination					Supply
	1	2	3	4	5	
1	8	6	3	7	5	20
2	5	∞	8	4	7	30
3	6	3	9	6	8	30
	25	25	20	10	20	

Balance this transportation matrix and solve for optimal solution.

Ans :

	D_1	D_2	D_3	D_4	D_5	Supply (a_i)
S_1	8	6	$\boxed{20}$ 3	7	$\boxed{0}$ 5	20
S_2	$\boxed{5}$ 5	∞	8	$\boxed{10}$ 4	$\boxed{15}$ 7	30
	6	$\boxed{25}$ 3	8	6	$\boxed{5}$ 8	30
S_3	$\boxed{20}$ 0	0	0	0	0	20
Demand(b_j)	25	25	20	10	20	100

Since $\Sigma a_i \neq \Sigma b_j$ the given TP is unbalanced one.

Add a dummy row with zero cost elements.

Apply VAM.

							Row Penalities
S_1	8	6	3	7	5	20	(2)
S_2	5	∞	8	4	7	30	(1)
S_3	6	3	9	6	8	30	(3)
S_4	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	20	(0)
	~~25~~5	25	20	10	20		
Column Penalities	(5)	(3)	(3)	(4)	(5)		

	D_1	↑ D_2	D_3	D_4	D_5		Row Penalities
S_1	8	6	$\boxed{20}$ 3	7	5	~~20~~ 0	(2)
S_2	5	∞	8	4	7	30	(1)
S_3	6	3	9	6	8	30	(3)
	5	25	20	10	20	80	Revise the penalties
Column Penalities	(1)	(3)	(5)	(2)	(2)		

	D_1	D_2	↑ D_4	D_5		Row Penalities
S_1	8	6	7	5	0	(1)
S_2	5	∞	4	7	30	(1)
S_3	6	$\boxed{25}$ 3	6	8	~~30~~ 5	(3)
	5	~~25~~	10	20	60	
Column Pentalities	(1)	(3)	(2)	(2)		

	D_1	D_4	D_5		R.P.
S_1	8	7	5	0	(2)
S_2	5	$\boxed{10}$ 4	7	~~30~~ 20	(1)
S_3	6	6	8	5	(0)
	5	10	20	35	
	(1)	(2)	(2)		

Revise the matrix

	D_1 ↑	D_5		R.P.
S_1	~~8~~	~~$\boxed{0}$ 5~~	0	(3)
S_2	5	7	20	(2)
S_3	6	8	5	(2)
	5	20	25	
	(1)	(2)		

	D_1	D_5		R.P.
S_2	$\boxed{5}$ 5	7	~~20~~ 15	(2)←
S_3	6	8	5	(2)
	5	20	25	
C.P.	(1)	(1)		

	D_5		R.P.
S_2	7	15	(7)
S_3	~~$\boxed{5}$ 8~~	~~5~~	(8)←
	~~20~~ 15	20	
C.P.	(1)		

	D_5	
S_2	$\boxed{15}$ 7	15
	15	

The final Transportation Table is

	D_1	D_2	D_3	D_4	D_5	
S_1	8	6	$\boxed{20}$ 3	7	$\boxed{0}$ 5	20
S_2	$\boxed{5}$ 5	∞	8	$\boxed{10}$ 4	$\boxed{15}$ 7	30
S_3	6	$\boxed{25}$ 3	9	6	$\boxed{5}$ 8	30
S_4	$\boxed{20}$ 0	0	0	0	0	20
Demand	25	25	20	10	20	100

Supply

T.C. = 20 × 3 + 0 × 5 + 5 × 5 + 10 × 4 + 15 × 7 + 25 × 3 + 5 × 8

= 60 + 0 + 25 + 40 + 105 + 75 + 40

= Rs. 345

To get optimal solution, introduce u_i's to rows and v_j's to columns and assume any one of u_i's or v_j's as zero.

	D_1	D_2	D_3	D_4	D_5	a_j	a_i
S_1	(+5) 8	(+6) 6	$\boxed{20}$ 3	(+5) 7	$\boxed{0}$ 5	20	$\mu_1 = -2$
S_2	$\boxed{5}$ 5	∞	(+3) 8	$\boxed{10}$ 4	$\boxed{15}$ 7	30	$\mu_2 = 0$
S_3	0 6	$\boxed{25}$ 3	(+3) 9	(+1) 6	$\boxed{5}$ 8	30	$\mu_3 = 1$
S_4	$\boxed{20}$ 0	(+3) 0	0 0	(+1) 0	(+2) 0	20	$\mu_4 = -5$
b_j	25	25	20	10	20	100	

$v_1 = 5$, $v_2 = 5$, $v_3 = 5$, $v_4 = 4$, $v_5 = 7$

Values for basic cells, using the formula

$c_{ij} = \mu_i + v_j$

$c_{13} = \mu_1 + v_3$

$3 = -2 + v_3 \Rightarrow v_3 = 5$

$c_{15} = u_1 + v_5$

$5 = \mu_1 + 7 \Rightarrow \mu_1 = -2$

$c_{21} = \mu_2 + v_1$

$5 = 0 + v_1 \Rightarrow v_1 = 5$

$c_{24} = \mu_2 + v_4$

$4 = 0 + v_4 \Rightarrow v_4 = 4$

$c_{25} = \mu_2 + v_5$

$7 = 0 + v_5 \Rightarrow v_5 = 7$

$c_{32} = \mu_3 + v_2$

$3 = 1 + v_2 \Rightarrow v_2 = 2$

$c_{35} = \mu_3 + v_5$

$8 = \mu_3 + 7 \Rightarrow \mu_3 = 1$

values for non-basic cells

$\Delta_{ij} = c_{ij} - (\mu_i + v_j)$

$\Delta_{11} = c_{11} - (\mu_1 + v_1)$

$\Rightarrow= 8 - (-2 + 5) = 5$

$\Delta_{12} = c_{12} - (\mu_1 + v_2)$

= 6 –(–2 + 2) = 6

$\Delta_{14} = c_{14} - (\mu_1 + v_4)$

= 7–(–2+ 4) = 5

$\Delta_{23} = c_{23} - (\mu_2 + v_3)$

= 8 – (0 +5)= 3

$\Delta_{31} = c_{31} - (\mu_3 + v_1)$

= 6 – (1 + 5) = 0

$\Delta_{33} = c_{33} - (\mu_3 + v_3)$

=9 – (1 + 5) = 3

$c_{41} = \mu_4 + v_1$

$0 = 44 + 5 \Rightarrow \mu_4 = -5$

$\Delta_{34} = c_{34} - (\mu_3 + v_4)$

$= 6 - (1 + 4) = 1$

$\Delta_{42} = c_{43} - (\mu_4 + v_2)$

$= 0 -(-5 + 2) = +3$

$\Delta_{43} = c_{43} - (\mu_4 + v_3)$

$= 0 - (-5 + 5) = 0$

$\Delta_{44} = c_{44} - (\mu_4 + v_4)$

$= 0 - (-5 + 4) = 1$

6. For the following problem begin with Hungarian Method and using iterations solve for optimal solution

	Task			
Assignee	1	2	3	4
A	4	6	5	5
B	7	4	5	6
C	4	7	6	4
D	5	3	4	7

Ans :

Task

Assignee	1	2	3	4
A	4	6	5	5
B	7	4	5	6
C	4	7	6	4
D	5	3	4	7

Cost is given so minimization assignment problem

1. Row Operation

	1	2	3	4
A	0	2	1	1
B	3	0	1	2
C	0	3	2	0
D	2	0	1	1

2. Column Operation

$$\begin{array}{c|c|c|c|c|}
 & 1 & 2 & 3 & 4 \\ \hline
A & 0 & 2 & 0 & 1 \\ \hline
B & 3 & 0 & 0 & 2 \\ \hline
C & 0 & 3 & 1 & 0 \\ \hline
D & 2 & 0 & 0 & 4 \\ \hline
\end{array}$$

3. Making Assignments

$$\begin{array}{c|c|c|c|c|}
 & 1 & 2 & 3 & 4 \\ \hline
A & \boxed{0} & 2 & \cancel{0} & 1 \\ \hline
B & 3 & \boxed{0} & \cancel{0} & 2 \\ \hline
C & \cancel{0} & 3 & 1 & \boxed{0} \\ \hline
D & 2 & \cancel{0} & \boxed{0} & 4 \\ \hline
\end{array}$$

4. Conclusion

Optimality test

No. of Rows / Columns = No. of assignment

A = 4

Allocation	Min. Cost
A → 1	4
B → 2	4
C → 4	4
D → 3	4
Total Cost	`. 16

7. Consider the assignment problem with following cost matrix :

Person	Job		
	1	2	3
A	5	7	4
B	3	6	5
C	2	3	4

Formulate this problem as a linear programming problem and solve.

Ans :

Person	1	2	3
A	5	7	4
B	3	6	5
C	2	3	4

(i) Given assignment problem is balanced

(ii) Row Operations

	1	2	3
A	1	3	0
B	0	3	2
C	0	1	2

(iii) Column Operation

	1	2	3
A	1	2	0
B	0	2	2
C	0	0	2

(iv) Making assignment

	1	2	3
A	1	2	$\boxed{0}$
B	$\boxed{0}$	2	2
C	$\cancel{0}$	$\boxed{0}$	2

Optimality test

Allocation	Min. Cost
A → 3	4
B → 1	3
C → 2	3
Total Cost	`. 16

5. Conclusion :

No. of Row / Columns = No. of assignments

3 = 3

8. An investment of $10,000 in a high risk venture has a 50-50 chance over next year of increasing to $14,000 or decreasing to 8,000. Thus the net return can be either $4,000 or $2,000.

(a) Assuming a risk-neutral investor and a utility scale from 0 to 100, determine the utility of $0 net return on investment and associated indifference probability.

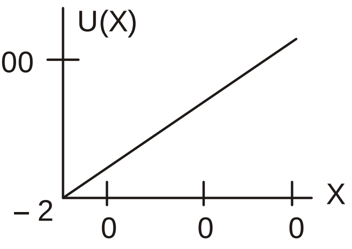
(b) Suppose the two investors And B have exhibited the following indifference probabilities:

	Indifference Probability	
Net Returns (S)	Investor A	Investor B
–2000	1.00	1.00
–1000	0.30	0.90
0	0.20	0.80
1000	0.15	0.70
2000	0.10	0.50
3000	0.05	0.40
4000	0.00	0.00

Graph the utility functions for investors A and B, and categorize each investors as either a risk averse person or a risk seeker.

Ans :

(a)



$$\frac{U(0)}{U(4)} = \frac{0-(-2)}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

$$U(0) = \frac{1}{3}(100) = 33.33$$

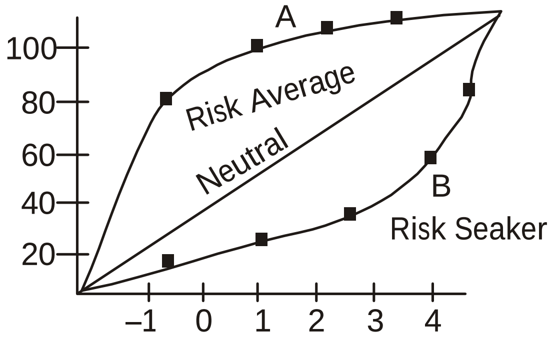
Now, $U(0) = P((-2) + (1-P)\,U(4)$

$= 100(1-P)$

Thus for $U(0) = 33.33,\ P = 0.6667$

(b)

X	$U(X)_A$	$U(X)_B$
– 2	0	0
– 1	70	10
0	80	20
1	85	30
2	90	50
3	95	60
4	100	100



(c) **Venture I :**

$U_A(3000) = 95, \quad U_A(-1000) = 70$

$EU(I) = 0.4 \times 98 + 0.6(70) = 80$

Venture II :

$U_A(2000) = 90, \quad U_A(0) = 80$

$EU(II) = 0.4 \times 90 + 0.6 \times 80 = 84$

Decision : Select II

$$E\{\$ \text{ Venture II}\} = \frac{84-80}{85-80} = \frac{x-0}{1-0} x = 0.8$$

(d) **Venture I :**

$U_B(3000) = 60, \quad U_B(-1000) = 10$

$EU(I) = 0.6 \times 60 + 0.4 \times 10 = 40$

Venture II :

$U_B(2000) = 50, \quad U_B(0) = 20$

$EU(II) = 0.6 \times 50 + 0.4 \times 20 = 38$

Decision : Select I

$E\{\$ \text{ Venture I}) = \$1500$

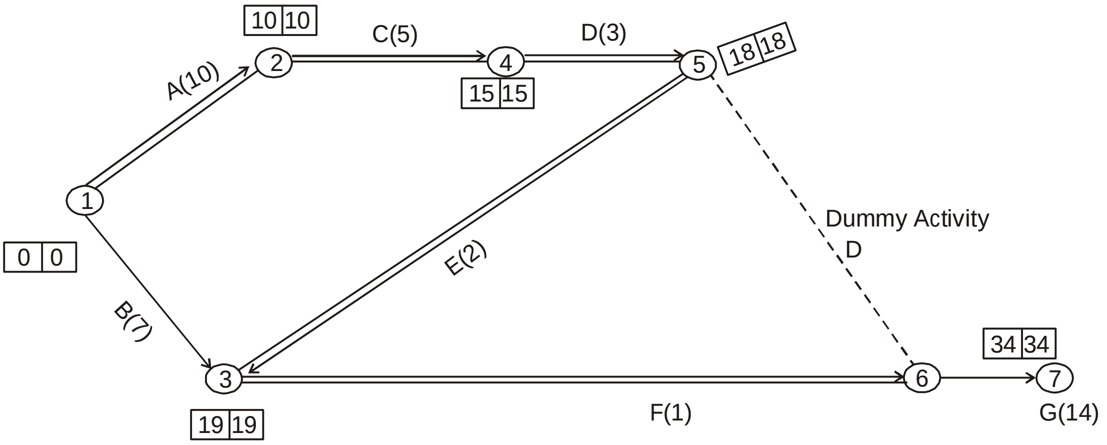
OR

9. A company is in the process of preparing a budget for launching a new product. The following table provides the associated activities and their durations. Construct the project network.

	Activity	Predecessors	Duration (days)
A:	Forecast Sales volume	-	10
B:	Study competitive market	-	7
C:	Design item and facilities	A	5
D:	Prepare production schedule	C	3
E:	Estimate cost of production	D	2
F:	Set sales price	B, E	1
G:	Prepare Budget	E, F	14

Ans :

Activity	**Predecessors**	**Durations**
A	–	10
B	–	7
C	A	5
D	C	3
E	D	2
F	B, E	1
G	E, F	14



Time Duration (T) = 34 hours

Critical Path : ① → ② → ③ → ④ → ⑤ → ⑥

10. On an average 96 patients per 24-hours day require the service of an emergency clinic. Also on an average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in this average lime would cost Rs. 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from $1\frac{1}{3}$ patients to $\frac{2}{3}$ patient.

Ans :

From the data of the problem are have

$$\lambda = \frac{96}{24 \times 60} = \frac{1}{15}$$

and $$\mu = \frac{1}{10}$$

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{15}}{\frac{1}{10}} = \frac{10}{15} = \frac{2}{3} \text{ patients per min.}$$

(i) Average number of patients in the queue

$$Lq = \frac{\left(\frac{\lambda^2}{\lambda}\right)}{1-\left(\frac{\lambda}{\mu}\right)} = \frac{\left(\frac{2}{3}\right)^2}{1-\frac{2}{3}}$$

$$= \frac{\frac{4}{9}}{\frac{3-2}{3}} = \frac{\frac{4}{\cancel{9}\ 3}}{\frac{1}{\cancel{3}}}$$

$$\therefore \quad \boxed{Lq = \frac{4}{3}}$$

(ii) Fraction of the time for which there no patients.

$$p_0 = 1 - p = 1 - \frac{2}{3} = \frac{3-2}{3}$$

$$\therefore \quad \boxed{p_0 = \frac{1}{3}} \text{ min}$$

(iii) When the average queue size is decreased from 4/3 patient, the new service rate μ is determined as :

$$Lq = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\left(\frac{1}{15}\right)^2}{\mu\left(\mu-\frac{1}{15}\right)^2}$$

$$\mu = \frac{2}{15} \text{ patients per minute}$$

Average rate of treatment required is

$$\frac{1}{\mu} = \frac{15}{2} = 7.5 \text{ min.}$$

A decrease in the average rate of treatment is 2.5 minutes (10 – 7.5)

Budget per patient = Rs(100 + 2.5 × 10)

= Rs. 125 per patient.

OR

11. (a) Explain two-person zero-sum game by giving suitable example **(Unit - V, Q.No. 20)**

(b) Explain Minimax and Maxmin principles used in solving games. **(Unit - V, Q.No. 16, 14th,15th)**

(c) What is the saddle point? **(Unit - V, Q.No. 21)**

(d) Define the value of a game. **(Unit - V, Q.No. 16, 12th)**

(e) Solve the following game by giving optimum strategies for each player and value of the game:

		Player B			
		B1	B2	B3	B4
Player A	A1	3	3	1	10
	A2	5	5	4	6
	A3	4	–2	0	–5

Ans :

		B_1	B_2	B_3	B_4		
	A_1	3	3	1	10	1	
Player A	A_2	5	5	4	6	$\boxed{4}$	Maximin
	A_3	4	–2	0	–5	–5	
Column maximum		5	5	$\boxed{4}$	10		
				↑ Minimax			

Since maximin = minimax exist saddle point

∴ Maximin = Minimax

∴ Value of game = 4

Optimal strategies : A – A_2 and player B – B_3

JAWAHARLAL TECHNOLOGICAL UNIVERSITY HYDERABAD

MBA II - Semester Examinations

R17

April / May - 2019

QUANTITATIVE ANALYSIS FOR BUSINESS DECISIONS

Time : 3 Hours   Max. Marks : 75

Note : This question paper contains two parts A and B.
Part - A is compulsory which carries 25 marks. Answer all questions in Part - A
Part - B consists of 5 Units. Answer any one full question from each unit.
Each question carries 10 marks and may have a, b, c as sub questions.

PART - A (5 × 5 = 25 Marks)

ANSWERS

1. Write about the following

   (a) Meaning and any two definitions of Operations Research. **(Unit-I, SQA-1)**

   (b) Characteristics of Transportation Problem. **(Unit-III, Q.No. 13)**

   (c) Assignment Problem and its characteristics. **(Unit-III, Q.No. 1)**

   (d) Types of Decision Making Environments. **(Unit-IV, Q.No. 4)**

   (e) Components of a Queuing system. **(Unit-V, Q.No. 4)**

PART - B (5 × 10 Marks)

2. Briefly describe the application of operations research in different management areas. **(Unit-I, Q.No. 6)**

OR

3. Describe the steps involved in processing for developing an Operations Research Model. **(Unit-I, Q.No. 16)**

4. Find the Dual of the following :

Minimize $Z = 8X_1 + 10X_1$

Subject : $2X_1 + 3X_2 \geq 8,$

$5X_1 + 6X_2 \geq 18,$

$X_1 + 2X_2 \geq 13,$

$2X_1 + 3X_2 \geq 10$ and X1, X2 ≥ 0.

Ans :

Dual of LPP

Let P_1, P_2, P_3, P_4 be the decision variables of dual LPP.

Max $Z^* = 8P_1 + 18P_2 + 13P_3 + 10P_4$

Subject to

$2P_1 + 5P_2 + P_3 + 2P_4 \leq 8$

$3P_1 + 6P_2 + 2P_3 + 3P_4 \leq 10$

and $P_1, P_2, P_3, P_4 \geq 0$

Introducing surplus variables (S_1, S_2)

Max $Z^* = 8P_1 + 18P_2 + 13P_3 + 10P_4 + 0S_1 + 0S_2$

Subject to

$2P_1 + 5P_2 + P_3 + 2P_4 + S_1 = 8$

$3P_1 + 6P_2 + 2P_3 + 3P_4 + S_2 = 10$

$P_1, P_2, P_3, P_4, S_1, S_2 \geq 0$

$S_1 = 8 \quad S_2 = 10$

Iteration 1

	C_{ij}		8	18	13	10	0	0	Min
C_B	B_V	X_B	P_1	P_2	P_3	P_4	S_1	S_4	Ratio = $\frac{X_B}{P_2}$
0	S_1	8	2	5	1	2	1	0	$\frac{8}{5} = 1.6$
0	S_2	10	3	6	2	3	0	1	$\frac{10}{6} = 1.67$
	Z_j		0	0	0	0	0	0	
	$C_j - Z_j$		–8	–18	–13	–10	0	0	
				↑					

P_2 is entering variables, S_1 is a leaving variable key element is '5'

$$\text{New Row Values} = \frac{\text{Old Row Value}}{\text{Key Element}}$$

X_B	P_1	P_2	P_3	P_4	S_1	S_2
$\frac{8}{5}$	$\frac{2}{5}$	1	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0

Values of other Row

R_2 (New) = R2 (Old) – 6 R_1 (New)

X_B	P_1	P_2	P_3	P_4	S_1	S_2
$\frac{2}{5}$	$\frac{3}{5}$	0	$\frac{4}{5}$	$\frac{3}{5}$	$-\frac{6}{5}$	1

Iteration 2

	Cj		8	18	13	10	0	0	Min
C_B	B_V	X_B	P_1	P_2	P_3	P_4	S_1	S_4	Ratio = $\frac{X_B}{P_3}$
18	P_2	$\frac{8}{5}$	$\frac{2}{5}$	1	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	$\frac{8/5}{1/5} = 8$
0	S_2	$\frac{2}{5}$	$\frac{3}{5}$	0	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{-6}{5}$	1	$\frac{2/5}{4/5} = 0.5$
	Z_j		$\frac{36}{5}$	18	$\frac{18}{5}$	$\frac{36}{5}$	$\frac{18}{5}$	0	
	$C_j - Z_j$		$\frac{4}{5}$	0	$\frac{47}{5}$	$\frac{14}{5}$	$-\frac{18}{5}$	0	

P_3 is entering variable, S_2 is leave variable key element is $\frac{4}{5}$.

$$R_2 \text{ (New)} = R_2 \text{ (Old)} \div \frac{4}{5}$$

X_B	P_1	P_2	P_3	P_4	S_1	S_2
$\frac{1}{2}$	$\frac{3}{4}$	0	1	$\frac{3}{4}$	$-\frac{3}{2}$	$\frac{5}{4}$

$$R_1 \text{ (New)} = R_1 \text{ (Old)} - \frac{1}{5} R_2 \text{ (New)}$$

X_B	P_1	P_2	P_3	P_4	S_1	S_2
$\frac{3}{2}$	$\frac{1}{4}$	1	0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$

Iteration 3

	C_{ij}		8	18	13	10	0	0	Min
C_B	B_V	X_B	P_1	P_2	P_3	P_4	S_1	S_4	Ratio = $\frac{X_B}{S_1}$
18	P_2	$\frac{3}{2}$	$\frac{1}{4}$	1	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{-1}{4}$	$\frac{3/2}{1/2} = 3$
13	P_3	$\frac{1}{2}$	$\frac{3}{4}$	0	1	$\frac{3}{4}$	$\frac{-3}{2}$	$\frac{5}{4}$	
	Z_j		$\frac{57}{4}$	18	13	$\frac{57}{4}$	$-\frac{2}{2}$	$\frac{97}{4}$	
	$C_j - Z_j$		$-\frac{25}{4}$	0	0	$-\frac{17}{4}$	$\frac{21}{2}$	$-\frac{17}{4}$	
							↑		

S_1 is entering variable P_2 is leaving variable key element is $1/2$

$$R_1 \text{ (New)} = R_1 \text{ (Old)} \div \frac{1}{2}$$

X_B	P_1	P_2	P_3	P_4	S_1	S_2
3	$1/2$	2	0	$1/2$	1	$\frac{-1}{2}$

$$R_2 \text{ (New)} = R_2 \text{ (Old)} + \frac{3}{2} R_1 \text{ (New)}$$

X_B	P_1	P_2	P_3	P_4	S_1	S_2
5	$3/2$	3	1	$3/2$	0	$1/2$

Iteration 4

	C_{ij}		8	18	13	10	0	0	Min
C_B	B_V	X_B	P_1	P_2	P_3	P_4	S_1	S_4	Ratio
0	S_1	3	$\frac{1}{2}$	2	0	$\frac{1}{2}$	1	$\frac{-1}{2}$	
13	P_3	5	$\frac{3}{2}$	3	1	$\frac{3}{2}$	0	$\frac{1}{2}$	
	Z_j		$39/2$	39	13	$\frac{39}{2}$	0	$\frac{13}{2}$	
	$C_j - Z_j$		$-23/2$	–21	0	$\frac{-19}{2}$	0	$\frac{-13}{2}$	

Since $C_j - Z_j \geq 0$ the solution is optimal maximum $Z = 65$ (13×5)

$P_1 = 0 \quad P_2 = 0 \quad P_3 = 5 \quad P_4 = 0$

Optimal solution is

$x_1 = 0 \qquad x_2 = 13/2$

OR

5. Prove a Mathematical Model of Transportation Problem. What is Degeneracy in Transportation Problem? **(Unit-III, Q.No. 11, 20)**

6. What is mathematical formulation of an Assignment Problem ? Give certain variations of the Assignment Problem. **(Unit-III, Q.No. 3, 5, 6, 7)**

7. Solve the following Assignment Problem.

Jobs→ Workers ↓	1	2	3
A	8	6	5
B	8	6	2
C	6	6	3

Note : The cost involved for each worker to his concerned Job is given in `. Find the optimum solution in the above problem by Hungarian Method.

Ans :

Step 1 : Row reduction :

Select minimum number from each row and subtract it from each element in row.

Workers	J_1	J_2	J_3
A	3	1	0
B	6	4	0
C	3	3	0

Step 2 : Column Reduction

Select minimum number from each column and subtract it from each element of column.

Workers	J_1	J_2	J_3
A	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$
B	3	3	$\boxed{0}$
C	$\boxed{0}$	2	$\cancel{0}$

No. of assignment are equals to order of matrix.

Worker A → J_2 = 6

Worker B → J_3 = 2

Worker C → J_1 = 6

Minimum Cost = 14

8. What is Critical Path in Network. Analysis ? What are its advantages ? **(Unit-IV, Q.No. 16)**

Advantages :

- It demonstrates the graphical view of any project.
- It helps to identify the most important tasks that you have to manage.
- It helps to save your time and reduce timelines.
- It helps to compare planned and actual progress.
- It helps to make dependencies visible and clear.
- It helps to plan, schedule and control your projects.
- It helps to identify all critical activities that need your special attention.

OR

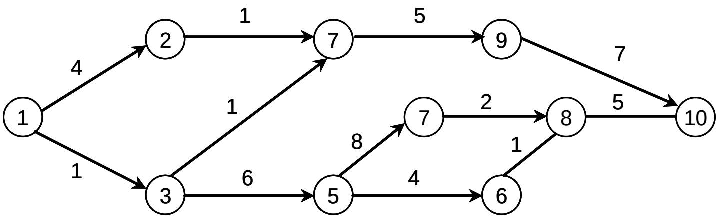
9. For the following given problem,

(a) Construct the Network Diagram and

(b) Determine the Critical Path and Project Duration.

Activity	1 – 2	1 – 3	2 – 4	3 – 4	3 – 5	4 – 9	5 – 6	5 – 7	6 – 8	7 – 8	8 – 10	9 – 10
Time (Days)	4	1	1	1	6	5	4	8	1	2	5	7

Ans :



Critical Path = 1, 3, 5, 7, 8, 10

Project duration is

1 + 6 + 8 + 2 + 5 = 22 days.

10. Discuss the structure of Queuing System and Queue Discipline. **(Unit-V, Q.No. 11, 2)**

OR

11. In a MBA college, for finger print attendance, students arrive at the machine in Poisson distribution, forming a single waiting line. Their average arrival time in 10 minutes and average time to complete the option is 5 - minutes.

Determine : (a) Average no. of students in the System, (b) Average no of students in the Queue, (c) Average time a student spends in the Queue, and (d) Average time a student spends in the System.

Ans :

Arrival Rate = $\frac{1}{10}$

Service Rate = $\frac{1}{5}$

(a) Average No. of students in the systems

$$= \frac{\lambda}{4 - \lambda}$$

$$= \frac{\frac{1}{10}}{\frac{1}{5} - \frac{-1}{10}} = \frac{\frac{1}{10}}{\frac{1}{10}}$$

$$= \frac{1}{10} \times \frac{10}{1} = \frac{10}{10} = 1 \text{ student.}$$

(b) Average no. of students in the queue.

$$= \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$= \frac{\left(\frac{1}{10}\right)^2}{\frac{1}{5}\left(\frac{1}{5} - \frac{1}{10}\right)}$$

$$= \frac{\frac{1}{100}}{\frac{1}{5} \times \frac{1}{10}} = \frac{1}{100} \times \frac{50}{1}$$

$$= \frac{50}{100} = \frac{1}{2}$$

(c) Average time a students spends in the queue.

$$= \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$= \frac{\left(\frac{1}{10}\right)}{\frac{1}{5}\left(\frac{1}{5} - \frac{1}{10}\right)}$$

$$= \frac{\frac{1}{10}}{\frac{1}{5} \times \frac{1}{10}} = \frac{1}{10} \times \frac{50}{1} = \frac{50}{10} = 5$$

(d) Average time a student spends in the system

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{5} - \frac{1}{10}} = \frac{1}{\frac{1}{10}} = 10 \text{ minutes}$$

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY HYDERABAD

M.B.A II - Semester Examination

R17

December - 2018

QUANTITATIVE ANALYSIS FOR BUSINESS DECISIONS

Time : 3 Hours] [Max. Marks : 75

Note : This question paper contains two parts A and B.

Part A is compulsory which carries 25 marks. Answer all questions in Part A, Part B consists of 5 units. Answer any one full question from each unit. Each question carries 10 marks and may have a, b, c as sub questions.

ANSWERS

PART - A (5 × 5 = 25 Marks)

1. (a) How do Operations Research methods help executives take better business decisions? **(Unit-I, SQA-4)**

(b) What is degeneracy and cycling in linear programming? How does one resolve this?

Ans :

Degeneracy

Consider the problem

Max f(X)

s.t. A $\leq$ b

X ≥ 0

- There are n variables, m inequality constraints and n non-negativity constraints and hence there are (m + n) hyperplanes associated with the constraints. If more than n hyperplanes pass through an extreme point of the feasible region, then such a point is called a degenerate extreme point . The excess number of planes over n is called the order of degeneracy.
- The LP is written in standard form by adding slack variables. If B is a square, non-singular sub - matrix of the constraint matrix (in standard form after adding the slack variables), then the solution $X = \begin{bmatrix} X_B \\ X_N \end{bmatrix}$ where $X_B = B^{-1}$ b and $X_N = 0$ is the basic solution. If XB 0, then X is called the basic feasible solution. If $X_B > 0$, then X is called the nondegenerate basic feasible solution. If at least on component of $X_B = 0$, then X is a degenerate basic feasible solution.
- When there exists more than one basis representing an extreme point, then this extreme point is degenerate. The converse is not necessarily true.

Cycling

In the simplex method, a step in which one change s from a basis to an adjacent basis; both representing the same extreme point solution is called a degenerate iteration. Performing a sequence of degenerate iterations, all representing the same extreme point with the objective function value remaining unc hanged is called cycling. It is possible that we may stay at a non -optimal point and cycle through a sequence of associated bases over and over again without reaching the optimal solution.

(c) How does transportation problem differ from assignment problem? **(Unit-III, Q.No.21)**

(d) If a company has several independent investment opportunities, each of which has an equal chance of gaming Rs. 1,00,000 or losing Rs. 60,000. What is the probability that the company will lose money in two such investments and in three investments?

Ans :

An investment has equal chance of Gaining ` 1,00,000

Lossing ` 60,000

Probability of gaining = $\frac{1}{2}$

Probability of losing = $\frac{1}{2}$

Two investment = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ = 0.25

Three investment = $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ = 0.125

(e) Player A paid Rs.8 if two coins turn both heads and Rs.10 if two coins turn both tails. Player B is paid Rs. 3 when two coins do not match. Given the choice of being A or B which one would you choose and what would be your strategy?

Ans :

	H	T	Raw minimum
H	8	–3	–3
T	–3	10	–3

Column

Maxima

No saddle point

$a_{11} = 8$ $a_{12} = -3$, $a_{21} = -3$ $a_{22} = 10$

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} - a_{21})}$$

$$= \frac{10 - (-3)}{(8 + 10) - (-3 - 3)}$$

$$= \frac{13}{18 - (-6)}$$

$$= \frac{13}{24}$$

$$y = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{10 + 3}{(8 + 10) - (-3 - 3)}$$

$$= \frac{13}{18 - (-6)}$$

$$= \frac{13}{24}$$

Value of game

$$= \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{8 \times 10 - (-3)(-3)}{(8 + 10) - (-3 - 3)}$$

$$= \frac{80 - 9}{18 + 6} = \frac{71}{24}$$

Optimal strategies A and B are

	Player A		Player B	
Strategy	1	2	1	2
Probability	$\frac{13}{24}$	$\frac{1-13}{24} = \frac{11}{24}$	$\frac{13}{24}$	$\frac{1-13}{24} = \frac{11}{24}$

PART - B (5 × 10 = 50 Marks)

2. (a) What are the opportunities and shortcomings of operations research approach? **(Unit-I, Q.No.18)**

(b) What is mathematical model? What is its relevance in OR ? **(Unit-I, Q.No.16)**

OR

3. (a) What is the scope for application of OR in production, inventory management and distribution? **(Unit-I, Q.No.6)**

(b) How do you test and validate a model? **(Unit-I, Q.No.14)**

4. A company produces two types of pens A mid B. Pen A is of superior quality and pen B of inferior quality. Profit on pen A and pen B are Rs.5 and Rs.3 per pen respectively. Raw material required for each pen A is twice as that of pen B. The supply of raw material is sufficient only for I (KM) pens of B per day. Pen A requires a special clip and only 400 clips are available per day. For pen B only 700 clips are available per day. Find graphically the product mix so that the company can make maximum profit.

Ans :

x_1 = No.of type A pens

x_2 = No. of types B pens

Max $z = 5x_1 + 3x_2$

Subject to $2x_1 + x_2 \leq 1000$

$x_1 \leq 400$

$x_2 \leq 700$

$x_1, x_2 \geq 0$

Convert inequality constraints into euqility constraints

$2x_1 + x_2 = 1000$ ...(1)

$x_1 = 100$ ...(2)

$x_2 = 700$ ...(3)

$x_1 = 0$ substitute in equation (1)

$2x_1 + x_2 = 1000$

$0 + x_2 = 1000$

$x_2 = 1000$

Assume $x_2 = 0$ substitute in equation (1)

$2x_1 + x_2 = 1000$

$2x_1 + 0 = 1000$

$2x_1 = 1000$

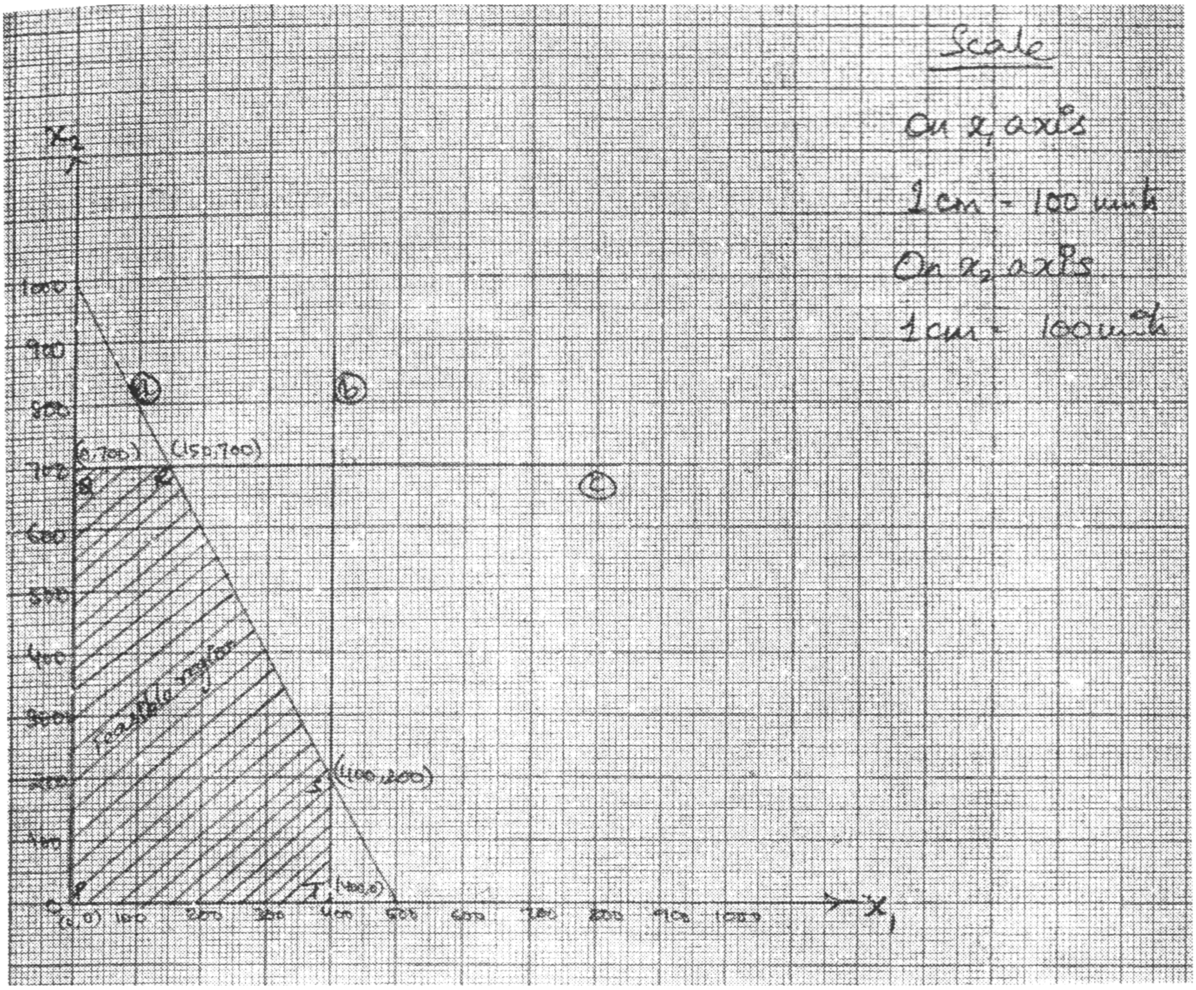
$$x_1 = \frac{1000}{2} = 500$$

A = (500, 1000)

Equation (2) and (3) has only one variable

$B = (x_1, x_2) = (400, 0)$

$C = (x_1, x_2) = (0, 700)$



Step 4:

The coordinates of extreme point of feasible region are P(0, 0) Q(0, 700) R (150,700) T(400,200) Y(400,0)

Extreme point	Coordinates	$Z = 5x_1 + 3x_2$
P	(0, 0)	5(0) + 3(0) = 0
Q	(0, 700)	5(0) + 3(700) = 2100
R	(150, 700)	5(150) + 3(700) = 2850
T	(400, 200)	5(400) + 3(200) = 2600
Y	(400, 0)	5(400) + 3(0) = 2000

Maximum profit is ` 2850 when product mix involves $x_1 = 150$, $x_2 = 700$

5. A firm manufacturing a single product has three plants I, II and III. They have produced 60, 35 and 40 units respectively during the month. The firm had made a commitment to sell 22 units to customer A, 45 units to customer B, 20 units to customer C, 18 units to customer D and 30 units to customer E. Find the minimum possible transportation cost of shifting the manufactured product to the five customers. The net unit cost of transporting from the three plants to the five customers is given below :

Customers →

Plants	**A**	**B**	**C**	**D**	**E**
I	4	1	3	4	4
II	2	3	2	2	3
III	3	5	2	4	4

Ans :

Voge's approximation method

Plants	Customers						RP$_1$	RP$_2$	RP$_3$	RP$_4$
	A	B	C	D	E					
I	~~4~~	$\boxed{45}$ ~~1~~	$\boxed{15}$ ~~3~~	~~4~~	~~4~~	~~60~~ ~~15~~ 0	2	2	1	–
II	$\boxed{17}$ ~~2~~	~~3~~	~~2~~	$\boxed{18}$ ~~2~~	~~3~~	~~35~~ ~~17~~ 0	–	–	–	–
III	$\boxed{5}$ 3	~~5~~	$\boxed{5}$ 5	~~4~~	$\boxed{30}$ 4	40	1	1	1	1
	~~22~~ ~~5~~ 0	~~45~~ 0	~~20~~ ~~5~~ 0	~~18~~ 0	30	135				

CP$_1$	1	2	–	2	1
CP$_2$	1	2	–	–	1
CP$_3$	1	–	–	–	1
CP$_4$	1	–	–	–	1

Total

Transportation cost

$(45\times1) + (15\times3) + (17\times2) + (18\times2) + (5\times3) + (5\times2) + (30\times4)$

$= 305$

MODI is checking IBFS

No.of occupied cells = 7

$m + n - 1 = 3 + 5 - 1 = 7$

$C_{ij} = u_i + v_i$ Assume $u_1 = 0$

$C_{12} = u_1 + v_2 = 1 \Rightarrow 0 + v_2 = 1 \Rightarrow v_2 = 1$

$C_{13} = u_1 + v_3 = 3 \Rightarrow 0 + v_3 = 3 \Rightarrow v_3 = 3$

$C_{21} = u_2 + v_1 = 2 \Rightarrow u_2 + 4 = 2 \Rightarrow u_2 = -2$

$C_{24} = u_2 + v_1 = 2 \Rightarrow -2 + v_4 = 2 \Rightarrow v_4 = 4$

$C_{31} = u_3 + v_1 = 3 \Rightarrow -1 + v_1 = 3 \Rightarrow v_1 = 4$

$C_{33} = u_3 + v_3 = 2 \Rightarrow u_3 + 3 = 2 \Rightarrow u_3 = -1$

$C_{35} = u_3 + v_5 = 4 \Rightarrow -1 + v_5 = 4 \Rightarrow v_5 = 5$

$D_{ij} = C_{ij} - (u_i + v_i)$

$\Delta_{11} = 4 - (0 + 4) = 0$

$\Delta_{14} = 4 - (0 + 4) = 0$

$\Delta_{15} = 4 - (0 + 5) = -1$

$\Delta_{22} = 3 - (-2 + 1) = 4$

$\Delta_{23} = 2 - (-2 + 3) = 1$

$\Delta_{25} = 3 - (-2 + 5) = 0$

$\Delta_{32} = 5 - (-1 + 1) = 5$

$\Delta_{34} = 4 - (-1 + 4) = 1$

Since opportunity cost for one cell is negative. Draw a loop from (1, 5) which opportunity cost (–1)

Plants	Customers				
	A	B	C	D	E
I	4	$\boxed{45}$ 1	$\boxed{15}$ 3	4	4
II	$\boxed{17}$ 2	3	2	$\boxed{18}$ 2	3
III	$\boxed{5}$ 3	5	$\boxed{5}$ 2	4	$\boxed{30}$ 4

(1, 3) = 15 – 15 = 0

(1, 5) = 0 + 15 = 15

(3, 3) = 5 + 15 = 20

(3, 5) = 30 – 15 = 15

Plants	Customers				
	A	B	C	D	E
I	4	$\boxed{45}$ 1	3	4	$\boxed{15}$ 4
II	$\boxed{17}$ 2	3	2	$\boxed{18}$ 2	3
III	$\boxed{5}$ 3	5	$\boxed{20}$ 2	4	$\boxed{15}$ 4

$(45 \times 1) + (15 \times 4) + (17 \times 2) + (18 \times 2) + 5 \times 3 + 20 \times 2 + 15 \times 4$

$= 45 + 60 + 34 + 36 + 15 + 40 + 60 = 290$

OR

6. Casually Medical Officer in a hospital has received four requests from Ambulance van facility. Currently, six vans are available for assignment and the estimated response time in minutes are shown in the table below :

Incidents	**Van 1**	**Van 2**	**Van 3**	**Van 4**	**Van 5**	**Van 6**
I	16	15	13	14	15	18
II	18	16	12	13	17	16
III	14	14	17	16	15	15
IV	13	17	19	18	14	17

Determine which van should, and what will the average response time.

Sol :

The given matrix has 4 rows and 6 columns thus it is an unbalanced AP converting unbalanced AP into balanced AP by adding two dummy Rows with zero.

	Van 1	Van 2	Van 3	Van 4	Van 5	Van 6
I	16	15	13	14	15	18
II	18	16	13	13	17	18
III	14	14	17	16	15	15
IV	13	17	19	18	14	17
D_1	0	0	0	0	0	0
D_2	0	0	0	0	0	0

Column Reduction

It is not required as eacj column has a zero.

	Van 1	Van 2	Van 3	Van 4	Van 5	Van 6
I	3	2	$\boxed{0}$	1	2	5
II	6	4	$\cancel{0}$	1	5	4
III	$\cancel{0}$	$\boxed{0}$	3	2	1	1
IV	$\boxed{0}$	4	6	5	1	4
D_1	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$
D_2	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$

Hungarian Rule

	Van 1	Van 2	Van 3	Van 4	Van 5	Van 6
I	2	1	$\boxed{0}$	$\cancel{0}$	1	4
II	5	3	$\cancel{0}$	$\boxed{0}$	4	3
III	$\cancel{0}$	$\boxed{0}$	4	2	1	1
IV	$\boxed{0}$	4	7	5	1	4
D_1	$\cancel{0}$	$\cancel{0}$	1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$
D_2	$\cancel{0}$	$\cancel{0}$	1	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$

OR

7. A travelling salesman has to visit 5 cities. He wishes to start from a particular city, visit each city once and then return to his starting point. The travelling cost in (` '000) of each city from a particular city is given.

From	**A**	**B**	**C**	**D**	**E**
A	–	2	5	7	1
B	6	–	3	8	2
C	8	7	–	4	7
D	12	4	6	–	5
E	1	3	2	8	–

What is the sequence of visit of the salesman so that the cost is minimized?

Sol :

Step 1: Raw Reduction

	A	B	C	D	E
A	–	1	4	6	0
B	4	–	1	6	0
C	4	3	–	0	3
D	8	0	2	–	1
E	0	2	1	7	–

Step 2: Column Reduction

	A	B	C	D	E
A	–	1	3	6	0
B	4	–	0	6	0
C	4	3	–	0	3
D	8	0	1	–	1
E	0	2	0	7	–

Solving by Hungarian method

	A	B	C	D	E
A	–	1	3	6	$\boxed{0}$
B	4	–	$\boxed{0}$	6	$\cancel{0}$
C	4	3	–	$\boxed{0}$	3
D	8	$\boxed{0}$	1	–	1
E	$\boxed{0}$	2	$\cancel{0}$	7	–

allocation are

A → E, B → C, C → D, D→ B, E → A

It is not a feasible solution it violates conditions that salesmen can visit city only once.

First Situation

Shift the assignment of (A – E) to (A – B)

	A	B	C	D	E
A	–	$\boxed{1}$	3	6	$\cancel{0}$
B	4	–	$\boxed{0}$	6	$\cancel{0}$
C	4	3	–	$\boxed{0}$	3
D	8	$\cancel{0}$	1	–	$\boxed{1}$
E	$\boxed{0}$	2	$\cancel{0}$	7	–

A → B → C → D → E → A

Cost = 2000 + 3000 + 4000 + 500 + 1000 = 15000

Second Situation

Shift the assignment of (D – B) to (D – E)

	A	B	C	D	E
A	–	$\boxed{1}$	3	6	$\xcancel{0}$
B	4	–	$\xcancel{0}$	6	$\xcancel{0}$
C	4	3	–	$\boxed{0}$	3
D	8	$\xcancel{0}$	$\xcancel{1}$	–	$\boxed{1}$
E	$\boxed{0}$	2	$\xcancel{0}$	7	–

A – B – C – D – E – A

= 2000 + 3000 + 4000 + 5000 + 1000 = 15000

Minimum cost = 15000

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8. Ramana often files from Chennai to Hyderabad. He can use the airport bus which cost ` 250 but if the takes it, there is a 0.08 chance he will miss the flight. The stay in a hotel costs ` 2,700 with a 0.96 chance of being on time for the flight. For ` 3,500 he can use a taxi which will make 99 percent chance of being on time for the flight. If Ramana catches the plane on time he will conclude a business transaction which will produce a profit of ` 1,00,000. Otherwise he will lost it. Which mode of transport should Ramana use? Answer on the basic of EMV criterion.

Ans :

Calculations of EMV in different situations

Situation 1:

Ramana uses airport bus

	Cost	Probability	Expected value
Catches the flight	1,00,000 – 250	99750 × 0.92	91,770
Misses the flight	– 250	– 250 × 0.08	– 20
		EMV	91,750

Situation 2:

Ramana stays in Hostel

	Cost	Probability	Expected value
Catches the flight	1,00,000–2700	9300 × 0.96	93,408
Misses the flight	– 2700	– 2700 × 0.96	– 108
		EMV	93,300

Situation 3:

Ramana uses taxi

	Cost	Probability	Expected value
Catches the flight	1,00,000–2700	96,500 × 0.99	95,535
Misses the flight	– 3500	– 3500 × 0.01	– 35
		EMV	95,500

After all considerations we can conclude that Ramana should use taxi because it has highest EMV

OR

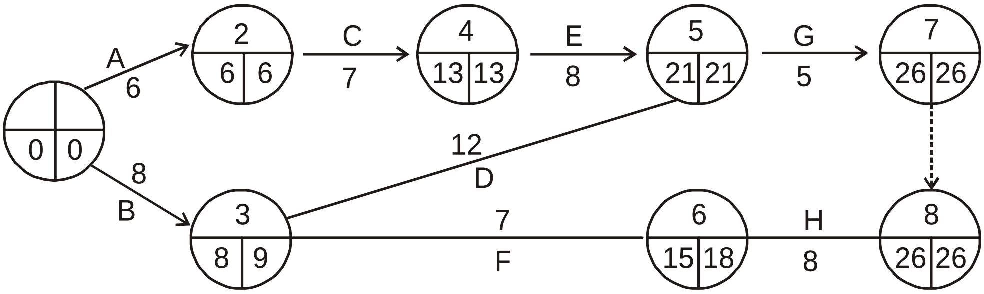
9. The required data for a small project consisting of different activities are given below :

Activity	Predecessor Activity	Normal Duration in Days	Normal Cost in `	Crash Duration in Days	Crash Cost in `
A	–	6	300	5	400
B	–	8	400	6	600
C	A	7	400	5	600
D	B	12	1000	4	1400
E	C	8	800	8	800
F	B	7	400	6	500
G	D, E	5	1000	3	1400
H	F	8	500	5	700

(a) Draw the network and find out the normal project length and minimum project length.

(b) If the project is to be completed in 21 days with minimum crash cost which activities should be crashed to how many days ?

Ans :



Normal cost = 4800

Critical path 1 – 2 – 4 – 5 – 7 – 8

$$\text{Cost slope} = \frac{\text{crash cost} - \text{normal cost}}{\text{normal time} - \text{crash time}}$$

1 – 2 = 100

1 – 3 = 100

2 – 4 = 100

3 – 5 = 50

3 – 6 = 100

5 – 7 = 200

8 – 8 = 200/3

4 – 5 = 0

Two critical activities with least

Slope is 1 – 2 and 2 – 4

Various path of network and total time

$1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 7 \rightarrow 8 = 26$

$1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 8 = 25$

$1 \rightarrow 3 \rightarrow 6 \rightarrow 8 = 23$

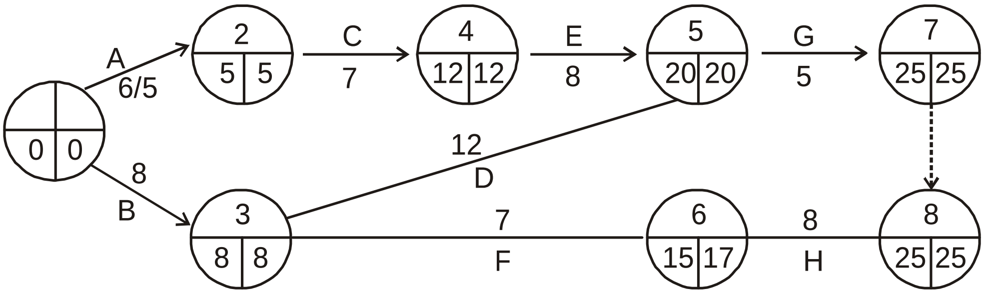
longest duration is 25 which is 1 less than critical path crash activity 1 – 2 to maximum possible extent

cost after crashing = 4800 + 100

= 4800 + 100

= 4900

Duration = 25 days



Critical path

$1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 7 \rightarrow 8$ and

$1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 8$

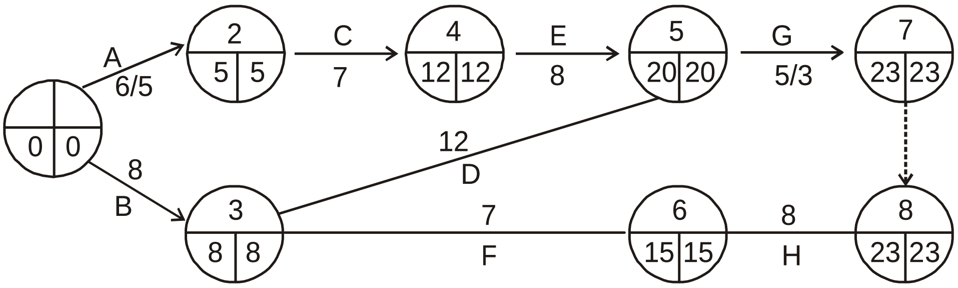
Crash common activity G(5 – 7)

Maximum extent 5 – 3 = 2 days

Crash cost = 4900 + (2 × 200)

= 4900 + 400 = 5300

Project duration = 23 days



Critical path 1 – 2 – 4 – 5 – 7 – 8

1 – 3 – 5 – 7 – 8

1 – 3 – 6 – 8

$$\text{Total cost} = 5300 + (100\times 2) + (50\times 2) + \left(\frac{200}{3}\times 2\right)$$

$$= 5300 + 200 + 100 + 133.33$$

$$= 5733.33$$

10. In a railway marshalling yard, goods train arrive a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distributions and the service-time (the time taken to hump the train) distribution is also exponential with an average of 36 minutes. Calculate,

(a) Expected queue size

(b) Probability that queue size exceeds 10

(c) If the input of trains increases to an average of 33 per day, what will be the change in a and b.

Ans :

$$\lambda = \frac{30}{60\times 24}$$

$$= \frac{1}{48}\text{ trains per minutes}$$

36 minutes is taken service time $\mu = \frac{1}{36}$ trains per minutes

$$P = \frac{\lambda}{4} = \frac{\frac{1}{48}}{\frac{1}{36}} = \frac{3}{4}$$

(a) Expected Quene size

$$L_S = \frac{P}{1-P} = \frac{\frac{3}{4}}{1-\frac{3}{4}} = \frac{\frac{3}{4}}{\frac{1}{4}}$$

$$= \frac{3}{\cancel{4}}\times\frac{\cancel{4}}{1} = \frac{3}{1} = 3$$

(b) Probability that the Quene size exceeds 10

$$p(n \geq 10) = P^{10} - \left(\frac{3}{4}\right)^{10}$$

$$= 0.056$$

(c) If the input of trains increases to an average 33 per day

$$\lambda = \frac{33}{60 \times 24} = \frac{11}{480}$$

$$\lambda = \frac{1}{36}$$

$$P = \frac{\lambda}{\cancel{4}} = \frac{\frac{11}{480}}{\frac{1}{36}} = \frac{33}{40}$$

$$L_s = \frac{P}{1-P} = \frac{\frac{33}{40}}{1-\frac{33}{40}} = 4.71$$

OR

11. Two firms F1 and F2 make colour and black and white television sets. F1 can make either 300 colour sets in a month or an equal number of black and white sets, and make a profit of ` 2,000 per colour sets and 1,500 per black and white set, F2 can on the other hand, make either 600 colour sets or 300 colour and 300 black & white sets or 600 black and white sets per month. It also has the same profit margin on the two sets as F1. Each month there is a market of 300 colour sets and 600 black and white sets and the manufacturers would share market in the proportion in which they manufacture a particular type of sets.

Write the payoff matrix of F1 and F2 per month. Obtain F1 and F2's optimal strategies and the value of the game.

Ans :

Strategies of F_1

A_1 = Make 300 colour sets

A_2 = Make 300 black and white sets

Strategies of F_2

B_1 = Make 600 colour sets

B_2 = Make 300 colour sets & 300 black sets

B_3 = Make 600 black & white sets

Profit on colour = 2000

Profit on black and white = 1500

Market of colour sets P.M = 300

Market of black and white P.M = 600

$$F_1 = \frac{300}{300+600} = \frac{1}{3}$$

$$A_1\,B_1 = \frac{1}{3} \times 300 \times 2000 = 2{,}00{,}000$$

$$A_1\,B_2 = \frac{1}{3} \times 450 \times 2000 = 3{,}00{,}000$$

$$A_1\,B_3 = \frac{1}{3} \times 900 \times 2000 = 6{,}00{,}000$$

$$A_2\,B_1 = \frac{1}{3} \times 900 \times 1500 = 4{,}50{,}000$$

$$A_2\,B_2 = \frac{1}{3} \times 900 \times 1500 = 4{,}50{,}000$$

$$A_2\,B_3 = \frac{1}{3} \times 600 \times 1500 = 3{,}00{,}000$$

Payoff matrix of F_1

$$\begin{array}{cccc} & B_1 & B_2 & B_3 \\ \left[\begin{array}{c} A_1 \\ A_2 \end{array}\right. & \begin{array}{c} 2{,}00{,}000 \\ 4{,}50{,}000 \end{array} & \begin{array}{c} 3{,}00{,}000 \\ 4{,}50{,}000 \end{array} & \left.\begin{array}{c} 6{,}00{,}000 \\ 3{,}00{,}000 \end{array}\right] \end{array}$$

Payoff matrix of F_2

$$\begin{array}{cccc} & B_1 & B_2 & B_3 \\ \left[\begin{array}{c} A_1 \\ A_2 \end{array}\right. & \begin{array}{c} -2{,}00{,}000 \\ -4{,}50{,}000 \end{array} & \begin{array}{c} -3{,}00{,}000 \\ -4{,}50{,}000 \end{array} & \left.\begin{array}{c} -6{,}00{,}000 \\ -3{,}00{,}000 \end{array}\right] \end{array}$$

F_2 payoff matrix will be negative of F_1's payoff matrix in a zero sum two person game

	B_1	B_2	B_3	
A_1	2,00,000	3,00,000	6,00,000	2,00,000
A_2	4,50,000	4,50,000	3,00,000	3,00,000
Column Maximum	4,50,000	4,50,000 ↑ Minimum	6,00,000	→ Maximum

F_1 Strategy

$$P_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{3,00,000 - 4,50,000}{(2,00,000 + 3,00,000) - (6,00,000 + 4,50,000)}$$

$$= \frac{-1,50,000}{-5,50,000} = \frac{3}{11}$$

$$P_2 = 1 - P_1$$

$$1 - \frac{3}{11} = \frac{8}{11}$$

F_2 Strategy

$$q_1 = \frac{a_{11} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{3,00,000 - 6,00,000}{(2,00,000 + 3,00,000) - (6,00,000 + 4,50,000)}$$

$$= \frac{-3,00,000}{-5,50,000} = \frac{6}{11}$$

$$q_2 = 1 - q_1$$

$$1 - \frac{6}{11} = \frac{5}{11}$$

Valume of the game

$$= \frac{(2,00,000 \times 3,00,000) - (6,00,000 \times 4,50,000)}{(2,00,000 + 3,00,000) - (6,00,000 + 4,50,000)}$$

$$= \frac{210000000000}{5,50,000}$$

= 3, 81, 818 (app)

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY HYDERABAD

M.B.A II - Semester Examination

R17

JUNE/JULY-2018

QUANTITATIVE ANALYSIS FOR BUSINESS DECISIONS

Time : 3 Hours] [Max. Marks : 75

PART - A (5 × 5 = 25 Marks)

[Short Answer type]

ANSWERS

1. a) Choose the right alternative :

i) When did the operations research originated and applied practically ? **[c]**

a) 1990s b) 1970s c) 1940s d) 1920s

ii) What is the core assumption in linear programming ? **[d]**

a) Linear results b) Linear variables

c) Linear data d) Linear Relationship

iii) Where does the operations research cannot be applied ? **[d]**

a) Optimization b) the distribution of material

c) Performance analysis d) None of the above

iv) What is 'finding the maximum or the minimum of an objective function of a set of decision variables', called as ? **[b]**

a) Performance analysis b) Optimization

c) None of the above d) the distribution of material

v) What is not an application of operations research ? **[d]**

a) facility planning b) scheduling

c) yield management d) none of the above

---

b) Use graphical method to solve the problem :

Minimize $Z = 3x_1 + 2x_2$

Subject to $x_1 + 2x_2 \leq 12$

$2x_1 + 3x_2 = 12$

$2x_1 + x_2 \geq 8$

And

$2x_1 + x_2 \geq 8$

Ans :

Step - 1

Objective Function

Maximize $Z = 3x_1 + 2x_2$

Subject to,

$x_1 + 2x_2 \leq 12$

$$2x_1 + 3x_2 = 12$$

$$2x_1 + x_2 \geq 8$$

And

$$2x_1 + x_2 \geq 8$$

Step-2

Converting inequality constraints into equality constraints.

$x_1 + 2x_2 = 12 \rightarrow (1)$

$2x_1 + 3x_2 = 12 \rightarrow (2)$

$2x_1 + x_2 = 8 \rightarrow (3)$

Equation-I

By solving equation (1), we get the value of 'A'

Put x_1, = 0 in equation (1)

$x_1 + 2x_2 = 12$

$0 + 2x_2 = 12$

$$\boxed{x_2 = 6}$$

Put x_2 = 0 in equation (1)

$x_1 + 2x_2 = 12$

$x_1 + 2(0) = 12$

$$\boxed{x_1 = 12}$$

$A = (x_1, x_2) = (12, 6)$

Equation-II

By solving equation (2), we get the value of 'B'

Put x_1 = 0 in equation (2)

$2x_1 + 3x_2 = 12$

$2(0) + 3x_2 = 12$

$3x_2 = 12$

$$\boxed{x_2 = 4}$$

Put x_2 = 0 in equation (2)

$2x_1 + 3x_2 = 12$

$2x_1 + 3(0) = 12$

$2x_1 = 12$

$$\boxed{x_1 = 6}$$

$B = (x_1, x_2) = (6, 4)$

Equation-III

By solving equation (3), we get the value of 'C'

Put $x_1 = 0$ in equation (3)

$2x_1 + x_2 = 8$

$2(0) + x_2 = 8$

$\boxed{x_2 = 8}$

Put $x_2 = 0$ in equation (3)

$2x_1 + x_2 = 8$

$2x_1 + 0 = 8$

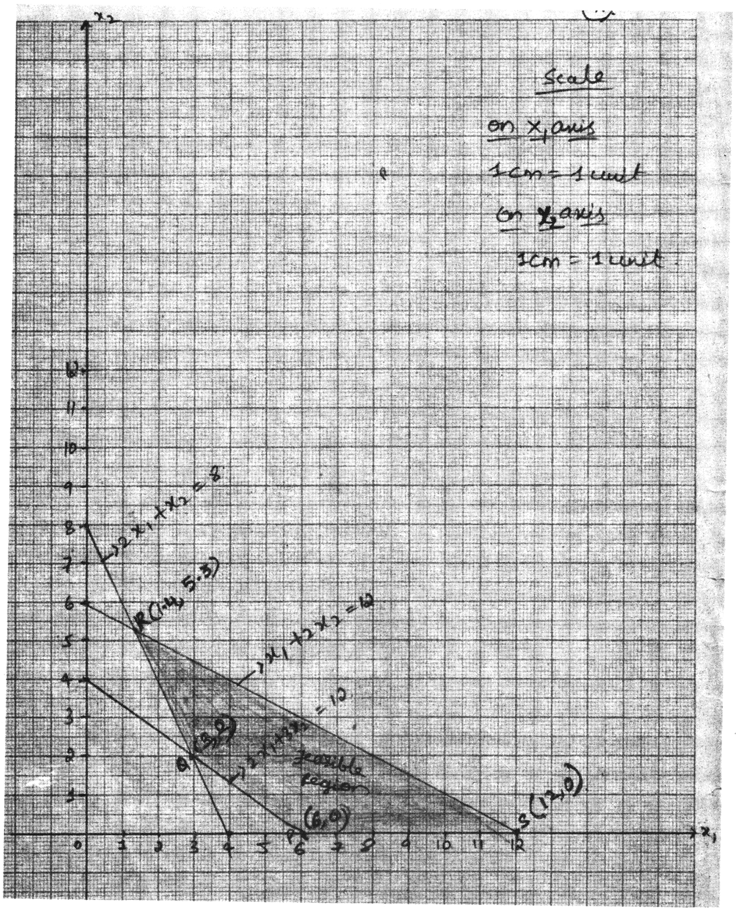
$2x_1 = 8$

$\boxed{x_1 = 4}$

$C = (x_1, x_2) = (4, 8)$

Step-3

Plot the values of A, B and C on graph to obtain feasible region.



Step-4

The co-ordinates of the extreme points of the feasible region are,

P(6,0), Q(3, 2), R(1.4, 5.3) and 5 = (12, 0)

Substitute these values in objective function.

Extreme Point	**Coordinates** (x_1, x_2)	**Objective Function Value** $Z = 3x_1 + 2x_2$
P	(6, 0)	$3(6) + 2(0) = 18 + 0 \Rightarrow 18$
Q	(3, 2)	$3(3) + 2(2) = 9 + 4 \Rightarrow 13$
R	(1.4, 5.3)	$3(1.4) + 2(5.3) = 4.2 + 10.6 \Rightarrow 14.8$
S	(12,0)	$3(12) + 2(0) = 36 + 0 \Rightarrow 36$

The minimum value of objective function is Z = 13 which occurs at the extreme point Q(3,2). Therefore, the optimal solution of the given LP problem is $x_1 = 3$, $x_2 = 2$ and min Z = 13.

---

c) A small airline operating seven day a week, serves three cities Nasik, Pune and Aurangabad, according to the schedule in table. The layover, cost per stop is roughly proportional to the square of layover time. How should planes be assigned so as to minimize total lay over cost ?

Flight No.	From	Time of Departure	To	Time of arrival
1.	Pune	9 : 00 AM	Nasik	Noon
2.	Pune	10 : 00 AM	Nasik	1:00 PM
3.	Pune	3 : 00 AM	Nasik	6:00 PM
4.	Pune	8 : 00 AM	Aurangabad	Midnight
5.	Pune	10 : 00 PM	Aurangabad	2:00 AM
6.	Nasik	4 : 00 AM	Pune	7:00 AM
7.	Nasik	11 : 00 AM	Pune	2:00 PM
8.	Nasik	3 : 00 PM	Pune	6:00 PM
9.	Aurangabad	7 : 00 AM	Pune	11:00 AM
10.	Aurangabad	3 : 00 PM	Pune	7:00 PM

Ans :

To solve this problem, two assumptions are made,

(a) A plane can make maximum 2 trips (to and from)

(b) A plane flying from a specific city should be back within 24 hours for the next scheduled trip from the city.

From the given information, it can be observed that there is no route between ‘Nasik and Aurangabad’. Thus, the given problem can be sub-divided into two,

(i) Routes between ‘Pune and Aurangabad’ and

(ii) Routes between ‘Pune and Nasik’.

(i) Constructing the Cost Matrix for Routes Connecting Cities 'Pune and Aurangabad'

Table-1

Flight Number	9	10
4	130	226
5	146	178

Calculation of Elements in Table-1 For Route 4 – 9

For route 4 – 9, a plane taking flight number 4 from 'Pune to Aurangabad' and number 9 from 'Aurangabad to Pune', would have a layover time of 9 horns (11:00 AM to 8:00 PM) at city Pune and 7 hours (midnight to 7:00 AM) at city Aurangabad. Therefore, layover cost for the route 4 – 9 would be $(9)^2$ + (7)2 = 81 + 49 = 130 units.

For Route 5 – 9

For route 5 – 9, a plane taking flight number 5 from 'Pune to Aurangabad' and number 9 from 'Aurangabad to Pune', would have a layover time of 11 hours (11:00 AM to 10:00 PM) at city Pune and 5 hours (2:00 AM to 7:00 AM) at city Aurangabad. Therefore, layover cost for the route 5-9 would be $(11)^2$ + (5)2 = 121 + 25 = 146 units.

For Route 4 –10

For route 4 – 10, a plane taking flight number 4 from 'Pune to Aurangabad' and number 10 from 'Aurangabad to Pune', would have a layover time of 1 hours (7:00 PM to 8:00 PM) at city Pune and 15 hours (midnight to 3:00 PM) at city Aurangabad. Therefore, layover cost for the route 4 – 10 would be $(1)^2$ + (15)2 = 226.

For Route 5 – 10

For route 5 – 10, a plane taking flight number 5 from 'Pune to Aurangabad' and flight number 10 from 'Aurangabad to Pune', would have a layover time of 3 hours (7:00 PM to 10:00 PM) at city Pune and 13 hours (2:00 AM to 3:00 PM) at city Aurangabad. Therefore, layover cost for the route 5 – 10 would be (3)2 + (13)2 = 9 + 169 = 178 units.

The optimal assignment can be obtained by applying Hungarian method of assignment.

Table-2

Flight Number	9	10
4	130	226
5	146	178

Row Reduction Matrix

In Row-1, minimum value is 130, so subtracting 130 from all elements of R_1

In Row-2, minimum value is 146, so subtracting 146 from all elements of R_2.

We get,

Table-3

Flight Number	9	10
4	0	96
5	0	32

Column Reduction Matrix

In column-1, all values are ‘0’, so no need of column reduction for column-1.

In column-2, minimum value is 32, so subtract 32 from all the values of column-2.

We get,

Table-4

Flight Number	9	10
4	0	64
5	0	0

Making assignments as follows,

Table-5

Flight Number	9	10
4	$\boxed{0}$	64
5	$\xcancel{0}$	$\boxed{0}$

Optimal assignment: 4 – 9 → 130

5 – 10 → 178

Total Cost → 308 Units.

(ii) Constructing the Cost Matrix for Routes Connecting Cities ‘Pune and Nasik’

Table-6

Flight Number	6	7	8
1	260	M	234
2	234	M	260
3	164	290	M

Calculation of Elements in Table-6

For Route 1-6

For route 1 – 6, a plane taking flight number 1 from ‘Pune to Nasik’ and flight number 6 from ‘Nasik to Pune’ would have a total layover time of 2 hours (7:00 AM to 9:00 AM) at city Pune and 16 hours (noon to 4:00 AM) at city Nasik. Therefore, layover cost for the route 1-6 would be $(2)^2 + (16)^2 = 4 + 256 = 260$ units.

For Route 2 – 6

For route 2 – 6, a plane taking flight number 2 from ‘Pune to Nasik’ and flight number 6 from ‘Nasik to Pune’ would have a total layover time of 3 hours (7:00 AM to 10:00 AM) at city Pune and 15 hours (1:00 PM to 4:00 AM) at city Nasik. Therefore, layover cost for the route 2 – 6 would be $(3)^2 + (15)^2 = 9 + 225 = 234$.

For Route 3 – 6

For route 3 – 6, a plane taking flight number 3 from ‘Pune to Nasik’ and flight number 6 from ‘Nasik to Pune’ would have a total layover time of 8 hours (7:00 AM to 3:00 PM) at city Pune and 10 hours (6:00 PM to 4:00 AM) at city Nasik. Therefore, layover cost for the route 3 – 6 would be $(8)^2 + (10)^2 = 64 + 100 = 164$.

For Route 1 – 7

As plane taking flight number 1 cannot return for flight number 7 due to assumption (b), the layover cost related with this flight is considered very high say M.

For Route 2 – 7

For route 2 – 7, a plane taking flight number 2 from ‘Pune to Nasik’ and flight number 7 from ‘Nasik to Pune’, would have a layover time of 20 hours (2:00 PM to 10:00 AM) at city Pune and 22 hours (1:00 PM to 11:00 AM) at city Nasik. As plane taking flight number 2 cannot return for flight number 7 due to assumption (b), the layover cost related with this flight is considered very high say M.

For Route 3 – 7

For route 3 – 7, a plane taking flight number 3 from ‘Pune to Nasik’ and flight number 7 from ‘Nasik to Pune’ would have a layover time of 1 hour (2:00 PM to 3:00 PM) at city Pune and 17 hours (6:00 PM to 11:00 AM) at city Nasik. Therefore, layover cost for the route 3 – 7 would be $(1)^2 + (17)^2 = 1 + 289 = 290$.

For Route 1– 8

For route 1 – 8, a plane taking flight number 1 from ‘Pune to Nasik’ and number 8 from ‘Nasik to Pune’ would have a layover time of 15 hours (6:00 PM to 9:00 AM) at city Pune and 3 hours (Noon to 3:00 PM) at city Nasik. Therefore, layover cost for the route 1 – 8 would be $(15)^2 + (3)^2 = 225 + 9 = 234$.

For Route 2 – 8

For route 2 – 8, a plane taking flight number 2 from ‘Pune to Nasik’ and flight number 8 from ‘Nasik to Pune’ would have a total layover time of 16 hours (6:00 PM to 10:00 AM) at city Pune and 2 hours (1:00 PM to 3:00 PM) at city Nasik. Therefore, layover cost for the route 2-8 would be $(16)^2 + (2)^2 = 256 + 4 = 260$.

For Route 3 – 8

For route 3 – 8, a plane taking flight number 3 from ‘Pune to Nasik’ and flight number 8 from ‘Nasik to Pune’ would have a total layover time of 21 hours (6:00 PM to 3:00 PM) at city Pune and 21 hours (6:00 PM to 3:00 PM) at city Nasik. As plane taking flight number 3 cannot return for flight number 8 due to assumption (b), the layover cost related with this flight is considered very high say M.

The optimal assignment can be obtained by applying Hungarian method of assignment.

Table-7

Flight Number	6	7	8
1	260	M	234
2	234	M	260
3	164	290	M

Row Reduction Matrix

In Row-1, minimum value is 234, so subtracting 234 from all elements of R_1.

In Row-2, minimum value is 234, so subtracting 234 from all elements of R_2.

In Row-3, minimum value is 164, so subtracting 164 from all elements of R_3.

We get,

Table-8

Flight Number	6	7	8
1	26	M	0
2	0	M	26
3	0	126	M

Column Reduction Matrix

Column-1 has ‘0’, no need of column reduction for column-1.

In column-2, minimum value is 126, so subtract 126 from the values of such column.

In column-3, minimum value is 0, so no need of column reduction for column-3.

We get,

Table-9

Flight Number	6	7	8
1	26	M	$\boxed{0}$
2	$\boxed{0}$	M	26
3	$\cancel{0}$	$\boxed{0}$	M

Optimal Assignment

1 – 8 → 234

2 – 6 → 234

3 – 7 → 290

Total Cost → 758 Units.

Therefore, from the two optimal solutions obtained above, the complete flight schedule of planes is arrived at and is shown in Table - 9.

Plane Number	Flight Number	Departure		Arrival	
		City	Time (hrs)	City	Time (hrs)
1	1	A	09 : 00 AM	B	12 : 00 (Noon)
	8	B	03 : 00 PM	A	06 : 00 PM
2	2	A	10 : 00 AM	B	01 : 00 PM
	6	B	04 : 00 PM	A	07 : 00 AM
3	3	A	03 : 00 PM	B	06 : 00 PM
	7	B	11 : 00 AM	A	02 : 00 PM
4	4	A	08 : 00 PM	C	Midnight
	9	C	07 : 00 AM	A	11 : 00 AM
5	5	A	10 : 00 PM	C	12 : 00 AM
	10	C	03 : 00 AM	A	07 : 00 PM

The total minimum layover cost is 308 + 758 = 1066 units.

d) i) What is a zero-sum game ? **(Unit-V, Q.No. 20, 2^{nd} Point)**

ii) Find the solution to the following game :

–5	2	0	7
5	6	4	8
4	0	2	–3

Ans :

Maximin Value

Step-1: Identify the minimum value in each row.

				Row Minima
–5	2	0	7	–5
5	6	4	8	4
4	0	2	–3	–3

Step-2 : Select the maximum of the minimum value of each row and enclose it in a rectangle.

				Row Minima
– 5	2	0	7	– 5
5	6	4	8	$\boxed{4}$ → Maximin Value
4	0	2	– 3	– 3

Minimax Value

Step-3: Identify the maximum value in each column.

	– 5	2	0	7
	5	6	4	8
	4	0	2	– 3
Column Maxima	5	6	4	8

Step-4: Select the minimum of the maximum value of each column and enclose in a circle.

	– 5	2	0	7
	5	6	4	8
	4	0	2	-3
Column Maxima	5	6	(4)	8

Minimax Value

Saddle Point

Mark the maximin and minimax values in the payoff matrix.

$$\begin{array}{|cccc|ll} \hline -5 & 2 & 0 & 0 & -5 & \\ 5 & 6 & 4 & 8 & \boxed{4} & \text{Maximin Value} \\ 4 & 0 & 2 & -3 & -3 & \\ 5 & 6 & \boxed{4} & 8 & & \\ \hline \end{array}$$

Minimax Value

As the maximin and minimax value of the game coincides, it has a saddle point which is equal to '4'.

$\therefore$ Value of game = 4.

e) An insurance company has three claims adjusters in its branch office. People with claims against company are found to arrive in a Poisson fashion, at an average rate of 20 per 8 hour day. The amount of time that an adjuster spends with a claimant is found to have and exponential distribution, with mean service time 40 minutes. Claimants are processed in order of their appearance.

i) How many hours a week an adjuster expect to spend with claimants ?

ii) How much time, on the average, does a claimant spend in the branch office ?

Ans :

Given,

$$\lambda = \frac{20}{8} = \frac{5}{2} \text{ arrivals per hour}$$

$$\mu = \frac{1}{40} \text{ service per minute or } \frac{3}{2} \text{ per hour,}$$

$S = 3$

$$\frac{\lambda}{\mu} = \frac{5}{3}$$

$$P_0 = \left[\sum_{n=0}^{s=1} \frac{1}{n!}\left[\frac{\lambda}{\mu}\right]^n + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \frac{s\mu}{(s\mu-\lambda)}\right]^{-1}$$

$$= \left[1+\frac{5}{3}+\frac{1}{2}\left(\frac{5}{3}\right)^2+\frac{1}{6}\left(\frac{5}{3}\right)^3 \cdot \frac{3(3)}{3(3)-5}\right]^{-1}$$

$$= \left[1+\frac{5}{3}+\frac{1}{2}\left(\frac{25}{9}\right)+\frac{1}{6}\left(\frac{125}{27}\right)\frac{9}{4}\right]^{-1}$$

$$= \left[1+\frac{5}{3}+\frac{25}{18}+\frac{125}{162}+\frac{9}{4}\right]^{-1}$$

$$= \left[\left(1+\frac{5}{3}\right)+\frac{25}{18}+\frac{1125}{648}\right]^{-1}$$

$$= \left[1+\frac{1080+900+1125}{648}\right]^{-1}$$

$$= \left[1+\frac{3105}{648}\right]^{-1}$$

$$= \frac{648}{3753} = \frac{24}{139}$$

$$P_1 = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)P_0$$

$$= \frac{5}{3}\times\frac{24}{139}$$

$$P_1 = \frac{40}{139}$$

$$P_2 = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$= \frac{1}{2}\left(\frac{5}{3}\right)^2\times\frac{24}{139}$$

$$= \frac{1}{2}\left(\frac{5}{3}\right)^2\times\frac{24}{139} = \frac{25}{18}\times\frac{24}{139}$$

$$P_2 = \frac{100}{417}$$

(i) Expected number of idle adjusters at any given time are given by

$$3P_0 + 2P_1 + 1P_2 = 3\left(\frac{24}{139}\right) + 2\left(\frac{40}{139}\right) + \left(\frac{100}{417}\right)$$

$$= \frac{72}{139}+\frac{80}{139}+\frac{100}{417} = \frac{216+240+100}{417}$$

$$= \frac{4}{3}\text{ adjusters}$$

Therefore, the probability that no adjuster is idle = $1 - \left(\frac{4}{9}\right)$ i.e., anticipated weekly time an adjuster spends with claimants.

$$= \frac{5}{9} \times 40 \text{ (assuming working days in a week = 5)}$$

$$= 22.2 \text{ hours.}$$

(ii) The average time an arrival spends in the system is found from the equation

$$W_s = W_q + \frac{1}{\mu}$$

$$= \frac{\mu\left(\frac{\lambda}{\mu}\right)^3}{(s-1)\,(s\mu-\lambda)^2} P_0 + \frac{1}{\mu}$$

$$= \frac{\frac{3}{2}\left(\frac{5}{3}\right)^3 P_0}{(3-1)!\left(3\left(\frac{3}{2}\right)-\frac{5}{2}\right)^2} + \frac{2}{3}$$

$$= \frac{\frac{3}{2}\left(\frac{125}{27}\right)\times\frac{24}{139}}{2(4)} + \frac{2}{3}$$

$$\Rightarrow \frac{\frac{\cancel{3}_1}{\cancel{2}_1}\left(\frac{125}{\cancel{27}\,\cancel{9}_3}\right)\times\frac{\cancel{24}^{\cancel{12}^4}}{139}}{2(4)} + \frac{2}{3}$$

$$= \frac{\frac{125\times 4}{3\times 139}}{8} + \frac{2}{3} = \frac{\frac{500}{417}}{8} + \frac{2}{3}$$

$$= \frac{125}{834} + \frac{2}{3} = \frac{227}{278}\text{hrs} = 0.816 \times 60$$

$$= 48.96 \cong 49 \text{ mins.}$$

PART - B (5 × 10 = 50 Marks)

[Essay Answer type]

2. a) Write a brief not on the origin of Operations Research. **(Unit-I, Q.No. 5)**
   b) Write four different areas of application of Operations Research. **(Unit-I, Q.No. 6)**
   c) What are the limitations of Operations Research. **(Unit-I, Q.No. 18)**

Or

3. The following table summarizes the key facts about two products, A and B and resources Q, R and S, required to produce them.

Resource	Resource usage per unit produced		Amount
	Product A	Product B	Resource Available
Q	2	1	2
R	1	2	2
S	3	3	4
Profit per unit	3	2	

Formulate the model and solve it graphically.

Ans :

Step-1: Objective Function

Max Z = 3A + 2B

Subject to,

$2A + 1B \leq 2$

$1A + 2B \leq 2$

$3A + 3B \leq 4$

and $A \geq 0, B \geq 0$.

Step-2: Converting inequality constraints into equality constraints.

$2A + 1B = 2 \rightarrow (1)$

$1A + 2B = 2 \rightarrow (2)$

$3A + 3B = 4 \rightarrow (3)$

Step-3: Solving Equations

Equation 1

By solving equation (1), we get the values P and Q.

Case-I

If A = 0, then by solving equation (1), we get,

$2A + 1B = 2$

$2(0) + 1B = 2$

$0 + 1B = 2$

$\boxed{B = 2}$

$\therefore$ (0, 2)

Case-II

If B = 0, then by solving equation (1), we get,

$2A + 1B = 2$

$2A + 1\,(0) = 2$

$2A + 0 = 2$

$A = \frac{2}{2} \Rightarrow 1$

$\boxed{A = 1}$

$\therefore$ (1, 0)

Equation 2

By solving equation (2), we get the values of R and S.

Case-1

If A = 0, then by solving equation (2), we get,

$1A + 2B = 2$

$1(0) + 2B = 2$

$B = \frac{2}{2}$

$\boxed{B = 1}$

$\therefore$ (0, 1)

Case-II

Let B = 0, then by solving equation (2), we get,

$1A + 2B = 2$

$1A + 2(0) = 2$

$1A = 2$

$\boxed{A = 2}$

$\therefore$ (2, 0)

Equation 3

By solving equation (3), we get T & U points

Case-I

If A = 0, then by solving equation (3), we get,

$3A + 3B = 4$

$3(0) + 3B = 4$

$\boxed{B = \frac{4}{3}} \Rightarrow B = 1.33$

$\therefore$ (0, 1.33)

Case-II

If B = 0, then by solving equation (3), we get,

$3A + 3B = 4$

$3A + 3(0) = 4$

$3A + 0 = 4$

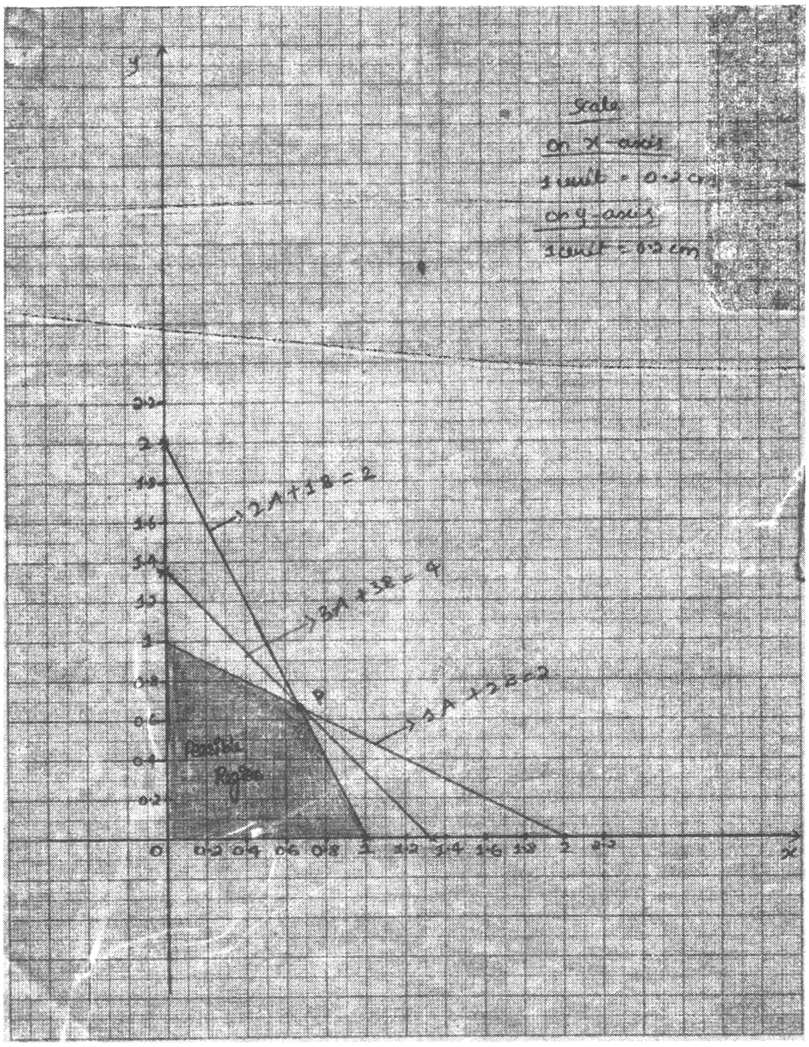
$A = \frac{4}{3}$

$\boxed{A = 1.33}$

$\therefore$ (1.33,0)

Step-4

Plot the values obtained by solving three equations on the graph to find out the feasible region.



In the above graph it is seen that only the two equations i.e., 2A + 1B = 2 and 1A + 2B = 2 is effecting the feasible region. By solving these two points, we can determine point 'p'.

$2A + 1B = 2 \quad \rightarrow (4)$

$1A + 2B = 2 \quad \rightarrow (5)$

Multiply equation (4) with 's' and (5) with '2', we get,

$2A + 1B = 2$

$2A + 4B = 4$

$- \quad - \quad -$

$-3B = -2$

$B = \frac{-2}{-3}$

$\boxed{B = \frac{2}{3}}$

Substitute $B = \frac{2}{3}$ in equation (5), we get,

$1A + 2B = 2$

$1A + 2\left(\frac{2}{3}\right) = 2$

$1A + \frac{4}{3} = 2$

$1A = 2 - \frac{4}{3}$

$1A = \frac{6-4}{3}$

$\boxed{A = \frac{2}{3}}$

Point 'p' $= \left(\frac{2}{3}, \frac{2}{3}\right)$

Substitute these A and B values in objective function.

Max Z = 3A + 2B

$= 3\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right)$

$$= \frac{6}{3} + \frac{4}{3}$$

$$= \frac{6+4}{3}$$

$$Z = \frac{10}{3}$$

$$\therefore \boxed{Z = 3.33}$$

---

4. Consider the following problem,

$$\text{Maximize } Z = 2x_1 + 4x_2 + 3x_3$$

Subject to

$$3x_1 + 4x_2 + 2x_3 \le 60$$
$$2x_1 + x_2 + 2x_3 \le 40$$
$$x_1 + 3x_2 + 2x_3 \le 80$$

And $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

Solve this problem using Simplex method.

Ans :

Given that,

Maximize $Z = 2x_1 + 4x_2 + 3x_3$

Subject to,

$3x_1 + 4x_2 + 2x_3 \le 60$

$2x_1 + x_2 + 2x_3 \le 40$

$x_1 + 3x_2 + 2x_3 \le 80$ and

$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

Standardization

Maximize $Z = 2x_1 + 4x_2 + 3x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to,

$3x_1 + 4x_2 + 2x_3 + S_1 = 60$

$2x_1 + x_2 + 2x_3 + S_2 = 40$

$x_1 + 3x_2 + 2x_3 + S_3 = 80$

$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$

Initial Solution

Let, $x_1 = 0, x_2 = 0, X_3 = 0$

We get,

$S_1 = 60, S_2 = 40, S_3 = 80$

Develop initial simplex table

Basic variables : S_1, S_2, S_3

Calculation of Z_j : $\Sigma C_B X_{ij}$

Index row : $C_j - Z_j$

C_j			**2**	**4**	**3**	**0**	**0**	**0**	**Minimum**
C_B	**B_V**	**X_B**	**x_1**	**x_2**	**x_3**	**S_1**	**S_2**	**S_3**	**Ratio** $\left(\frac{X_B}{X_2}\right)$
0	S_1	60	3	Key Element $\boxed{4}$	2	1	0	0	$\frac{60}{4} = 15 \leftarrow$
0	S_2	40	2	1	2	0	1	0	$\frac{40}{1} = 40$
0	S_3	80	1	3	2	0	0	1	$\frac{80}{3} = 26.67$
Z_j			0	0	0	0	0	0	
$C_j - Z_j$			2	4	3	0	0	0	

↑

The variable S_1 leave the basis and variable x_2 enters the basis.

Key element = 4

For key row,

$$\text{New Row Value} = \frac{\text{Old Row Value}}{\text{Key Element}}$$

$$1^{st} \text{ Rwo } (R_1) = \frac{60}{4}; \frac{3}{4}; \frac{4}{4}; \frac{2}{4}; \frac{1}{4}; \frac{0}{4}; \frac{0}{4}$$

$$= 15 ; 0.75; 1; 0.5; 0.25; 0; 0$$

For non key rows,

New Row Value (R_2) = Old R_2 Row Values – (Corresponding Key Element × New (R_t) Row Values)

$= 40 - (1 \times 15) = 40 - 15 \Rightarrow 25$

$= 2 - (1 \times 0.75) = 2 - 0.75 \Rightarrow 1.25$

$= 1 - (1 \times 1) = 1 - 1 \Rightarrow 0$

$= 2 - (1 \times 0.5) = 2 - 0.5 \Rightarrow 1.5$

$= 0 - (I \times 0.25) = 0 - 0.25 \Rightarrow - 2.5$

$= 1 - (1 \times 0) = 1 - 0 \Rightarrow I$

$= 0 - (1 \times 0) = 0 - 0 \Rightarrow 0$

New Key Row (R_3) = Old R_3 Row Values – (Corresponding Key Element × New R_t Row Values)

$= 80 - (3 \times 15) = 80 - 45 \Rightarrow 35$

$= 1 - (3 \times 0.75) = 1 - 2.25 \Rightarrow -1.25$

$= 3 - (3 \times 1) - 3 - 3 \Rightarrow 0$

$= 2 - (3 \times 0.5) = 2\text{-}1.5 \Rightarrow 0.5$

$= 0 - (3 \times 0.25) = 0 - 0.75 \Rightarrow -0.75$

$= 0 - (3 \times 0) = 0 - 0 \Rightarrow 0$

$= 1 - (3 \times 0) = 1 - 0 \Rightarrow 1$

C_j			2	4	3	0	0	0	Minimum Ratio
C_B	B_V	X_B	x_1	x_2	x_3	S_1	S_2	S_3	
4	x_2	15	0.75	1	0.5	0.25	0	0	$\frac{15}{0.5} = 30$
0	S_2	25	1.25	0	Key Element $\boxed{1.5}$	– 0.25	1	0	$\frac{25}{1.5} = 16.67$
0	S_3	35	– 1.25	0	0.5	– 0.75	0	1	$\frac{35}{0.5} = 70$
Z_j			3	4	2	1	0	0	
$C_j - Z_j$			– 1	0	1	– 1	0	0	

↑

Key Element = 1.5

To get the new values of the matrix for key row,

$$(R_2) = \frac{25}{1.5}; \frac{1.25}{1.5}; \frac{0}{1.5}; \frac{1.5}{1.5}; \frac{-0.25}{1.5}; \frac{1}{1.5}; \frac{0}{1.5}$$

$= 16.67; 0.83; 0; 1; -0.17; 0.67; 0$

For non key row (R_1) = Old R_1 Row Values – (Corresponding Key Element × New Key Row R_2)

$= 15 - (0.5 \times 16.67) = 15 - 8.34 \Rightarrow 6.66$

$= 0.75 - (0.5 \times 0.83) = 0.75 - 0.42 \Rightarrow 0.33$

$= 1 - (0.5 \times 0) = I - 0 \Rightarrow 1$

$= 0.5 - (0.5 \times 1) = 0.5 - 0.5 \Rightarrow 0$

$= 0.25 - (0.5 \times (-0.17)) = 0.25 - (-0.09) \Rightarrow 0.34$

$= 0 - (0.5 \times 0.67) = 0 - (0.34) = -0.34$

$= 0 - (0.5 \times 0) = 0 - 0 = 0$

For (R_3) = Old R_3 Row Values – (Corresponding Key Element × New Key Rows R_2)

$= 35 - (0.5 \times 16.67) = 35 - (8.34) \Rightarrow 26.66$

$= -1.25 - (0.5 \times 0.83) = -1.25 - (0.42) \Rightarrow -1.67.$

$= 0 - (0.5 \times 0) = 0 - 0 \Rightarrow 0$

$= 0.5 - (0.5 \times 1) = 0.5 - (0.5) \Rightarrow 0$

$= -0.75 - (0.5 \times -0.17)$

$= -0.75 - (-0.09) = -0.66$

$= 0 - (0.5 \times 0.67) = 0 - (0.34) \Rightarrow -0.34$

$= 1 - (0.5 \times 0) = 1 - 0 \Rightarrow 1$

C_j			**2**	**4**	**3**	**0**	**0**	**0**
C_B	**B_V**	**X_B**	**x_1**	**x_2**	**x_3**	**S_1**	**S_2**	**S_3**
4	x_2	6.66	0.33	1	0	0.34	– 0.34	0
3	x_3	16.67	0.83	0	1	– 0.17	0.67	0
0	S_3	26.66	–1.67	0	0	– 0.66	–0.34	1
Z_j			3.81	4	3	0.85	0.65	0
$C_j - Z_j$			– 1.81	0	0	– 0.85	– 0.65	0

Since, all '$C_j - Z_j$' values are $\leq$ 0, thus the optimality reached with,

$x_1 = 0; x_2 = 6.66; x_3 = 16.67$

Substituting the values of x_1 and x_2, x_3 in the objective function.

$$\begin{aligned} \text{Max } Z &= 2x_1 + 4x_2 + 3x_3 \\ &= 2(0) + 4(6.66) + 3(16.67) \\ &= 0 + 26.64 + 50.01 \\ &= 76.65 \end{aligned}$$

Or

5. Production manager of a company produces three types of spare parts for automobiles. The manufacturer of each part requires processing on each of to machines, with following processing times (in hours) :

	Part		
Machine	A	B	C
1	0.02	0.03	0.05
2	0.05	0.02	0.04

Each machine is available 40 hours per month. Each part manufactured will yield a unit profit as follows:

	Part		
	A	B	C
Profit	5000	4000	3000

The manager wants to determine the mix of spare-parts to produce in order to maximise total profit.

Ans :

Formulation of Problem into Linear Programming Model

Max Z = 5000 A + 4000 B + 3000 C

Subject to,

$0.02A + 0.03B + 0.05C \leq 40$

$0.05\,A + 0.02B + 0.04C \leq 40$

and $A \geq 0,\ B \geq 0,\ C \geq 0$

Solving LPP by Using Simplex Method

Standardization

Maximize $Z = 5000\,A + 4000\,B + 3000\,C + 0S_1 + 0S_2$

Subject to,

$0.02\,A + 0.03\,B + 0.05\,C + S_1 = 40$

$0.05\,A + 0.02\,B + 0.04\,C + S_2 = 40$

$A, B, C, S_1, S_2 \geq 0$

Setting up Initial Simplex Table

Table - 1

C_j			**5000**	**4000**	**3000**	**0**	**0**	**Minimum**
C_B	**B_V**	**X_B**	**A**	**B**	**C**	**S_1**	**S_2**	**Ratio**
0	S_1	40	0.02	0.03	0.05	1	0	$\frac{40}{0.02} = 2000$
0	S_2	40	$\boxed{0.05}$	0.02	0.04	0	1	$\frac{40}{0.05} = \xrightarrow{800}$
Z_j			0	0	0	0	0	
$C_j - Z_j$			5000	4000	3000	0	0	

↑

Entering variable is A

Droping variable is S_2

Iteration - 1

$$\text{New } R_2 = \frac{\text{Old } R_2}{\text{Key Element}} = \frac{\text{Old } R_2}{0.05} = 800,\ 1,\ \frac{2}{5},\ \frac{4}{5},\ 0,\ 20$$

$$\text{New } R_1 = \text{Old } R_1 = 0.02 \times \text{New } R_2 = 24,\ 0,\ \frac{11}{500},\ \frac{17}{500},\ 1,\ \frac{-2}{5}$$

Table - 2

C_j			**5000**	**4000**	**3000**	**0**	**0**	**Minimum Ratio**
C_B	**B_V**	**X_B**	**A**	**B**	**C**	**S_1**	**S_2**	
0	S_1	24	0	$\boxed{\frac{11}{500}}$	$\frac{17}{500}$	1	$\frac{-2}{5}$	$\frac{24}{\frac{11}{500}} = 1091 \rightarrow$
5000	A	800	1	$\frac{2}{5}$	$\frac{4}{5}$	0	20	$\frac{800}{\frac{2}{5}} = 2000$
Z_j			5000	2000	4000	0	1,00,000	
$C_j - Z_j$			0	2000	–1000	0	–1,00,000	

↑

Entering Variable is B

Droping Variable is S_1

Iteration - 2

$$\text{New } R_1 = \frac{\text{Old } R_1}{\text{Key Element}} = \frac{\text{Old } R_1}{\frac{11}{500}} = 1091, 0, 1, \frac{17}{11}, \frac{500}{11}, \frac{-200}{11}$$

$$\text{New } R_2 = \text{Old } R_2 - \frac{2}{5} \times \text{New } R_1 = 364, 1, 0, \frac{2}{11}, \frac{-200}{11}, \frac{300}{11}$$

Table - 3

C_j			**5000**	**4000**	**3000**	**0**	**0**	**Minimum Ratio**
C_B	**B_V**	**X_B**	**A**	**B**	**C**	**S_1**	**S_2**	
4000	B	1091	0	1	$\frac{17}{11}$	$\frac{500}{11}$	$\frac{-200}{11}$	
5000	A	364	1	0	$\frac{2}{11}$	$\frac{-200}{11}$	$\frac{300}{11}$	
Z_j			5000	4000	7091	90909	63,636	
$C_j - Z_j$			0	0	–4091	–90909	–63,636	

Since, all '$C_j - Z_j$' values are $\leq$ 0, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max Z = 5000 (364) + 4000 (1091) + 0

= 18,20,000 + 43,64,000 = 61,84,000

Production manager can maximize total profits by producing 364 units of A spare parts and 1091 units of B spare parts to get the profit of ` 61,84,000.

6. The coach of a swim team has five best swimmers. In a 200-yard medley there are four different strokes. The coach needs to assign four best swimmers with best of their ability in the respective strokes so that the team can give its best performance. The data for the five fastest swimmers and their best times in seconds for 50 yards is given below :

Stroke	Rajesh	Dhruv	Nishant	Amol	Brijesh
Backstroke	37.7	32.9	33.8	27.0	35.4
Breakstroke	43.4	33.1	42.2	34.7	41.8
Butterfly	33.3	28.5	38.9	30.4	33.6
Freestyle	29.2	26.4	29.6	28.5	31.1

a) Formulate this problem as assignment problem.

b) Obtain an optimal solution.

Ans :

The given problem is unbalanced assignment problem. So, convert this problem into balanced assignment problem by adding 1 column with zero profit of assignment.

Table

Stroke	Back Stroke	Breast Stroke	Butterfly	Freestyle	Dummy
Rajesh	37.7	43.4	33.3	29.2	0
Dhruv	32.9	33.1	28.5	26.4	0
Nishant	33.8	42.2	38.9	29.6	0
Amol	27.0	34.7	30.4	28.5	0
Brijesh	35.4	41.8	33.6	31.1	0

Row Reduction Matrix

Row reduction is not necessary as each row has a zero.

Column Reduction Matrix

Select the minimum number from each column and subtract it from each element of the column.

Table - 2

Stroke	Backstroke	Breaststroke	Butterfly	Free Style	Dummy
Rajesh	**10.7**	**10.3**	**4.8**	**2.8**	$\boxed{0}$
Dhruv	**5.9**	$\boxed{0}$	$\xcancel{0}$	$\xcancel{0}$	$\xcancel{0}$
Nishant	**6.8**	**9.1**	**10.4**	**3.2**	$\xcancel{0}$
Amol	$\boxed{0}$	**1.6**	**1.9**	**2.1**	$\xcancel{0}$
Brijesh	**8.4**	**8.7**	**5.1**	**4.7**	$\xcancel{0}$

As number of assignments $\neq$ Order of matrix, applying Hungarian rule.

Table - 3

Stroke	Back Stroke	Breast Stroke	Butterfly	Free Style	Dummy
Rajesh	10.7	10.3	4.8	2.8	$\boxed{0}$
~~Dhruv~~	~~5.9~~	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$
Nishant	6.8	9.1	10.4	3.2	$\cancel{0}$
~~Amol~~	$\boxed{0}$	~~1.6~~	~~1.9~~	~~2.1~~	$\cancel{0}$
Brijesh	8.4	8.7	5.1	4.7	$\cancel{0}$

Table - 4

Stroke	Back Stroke	Breast Stroke	Butterfly	Free Style	Dummy
Rajesh	7.9	7.5	2	$\boxed{0}$	$\cancel{0}$
Dhruv	5.9	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	2.8
Nishant	4	6.3	7.6	0.4	$\boxed{0}$
Amol	$\boxed{0}$	1.6	1.9	2.1	2.8
Brijesh	5.6	5.9	2.3	1.9	$\cancel{0}$

Table - 5

Stroke	Back Stroke	Breast Stroke	Butterfly	Free Style	Dummy
~~Rajesh~~	~~7.9~~	~~7.5~~	~~2~~	$\boxed{0}$	$\cancel{0}$
~~Dhruv~~	~~5.9~~	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	~~2.8~~
Nishant	4	6.3	7.6	0.4	$\boxed{0}$
~~Amol~~	$\boxed{0}$	~~1.6~~	~~1.9~~	~~2.1~~	~~2.8~~
Brijesh	5.6	5.9	2.3	1.9	$\cancel{0}$

Table - 6

Stroke	Back Stroke	Breast Stroke	Butterfly	Free Style	Dummy
Rajesh	7.9	7.5	2	$\boxed{0}$	0.4
Dhruv	5.9	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	3.2
Nishant	3.6	5.9	7.2	$\cancel{0}$	$\boxed{0}$
Amol	$\boxed{0}$	1.6	1.9	4	5.1
Brijesh	5.2	5.5	1.9	1.5	$\cancel{0}$

Table - 7

Stroke	Back Stroke	Breast Stroke	Butterfly	Free Style	Dummy
Rajesh	7.9	7.5	2	$\boxed{0}$	0.4
Dhruv	5.9	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	3.2
Nishant	3.6	5.9	7.2	$\cancel{0}$	$\boxed{0}$
Amol	$\boxed{0}$	1.6	1.9	2.1	3.2
Brijesh	5.2	5.5	1.9	1.5	$\cancel{0}$

Table - 8

Stroke	Back Stroke	Breast Stroke	Butterfly	Free Style	Dummy
Rajesh	6	5.6	0.1	$\boxed{0}$	0.4
Dhruv	5.9	$\boxed{0}$	$\cancel{0}$	1.9	5.1
Nishant	1.7	4	5.3	$\cancel{0}$	$\boxed{0}$
Amol	$\boxed{0}$	1.6	1.9	4	5.1
Brijesh	3.3	3.6	$\boxed{0}$	1.5	$\cancel{0}$

As the number of assignments = Order of matrix, the solution is said to be an optimal solution.

Allocations	
Rajesh → Free Style	29.2
Dhruv → Breast Stroke	33.1
Nishant → D (Dummy)	0
Amol → Back Stroke	27.0
Brijesh → Butterfly	33.6
	122.9

Or

7. Consider the following assignment problem of assigning four operators to four machines. The cost matrix is as follows :

Particulars		Machine			
		1	2	3	4
Operator	1	5	5	–	2
	2	7	4	2	3
	3	9	3	5	–
	4	7	2	6	7

Operator I cannot be assigned to 3. Similarly operator 3 cannot be applied to 4. Find the minimum cost.

Ans :

Step -1 : Row Reduction

Select the minimum number from each row and subtract it from each element of rwo. Then matrix will be as follows,

Machines Operator	1	2	3	4
1	3	3	8	0
2	5	2	0	1
3	6	0	2	7
4	5	0	4	5

Step-2 : Column Reduction Matrix (Column Reduction)

Select minimum number from each column and subtract it from each element of column. Then matrix will be as follows,

Machines Operator	1	2	3	4
1	3	3	8	0
2	5	2	0	1
3	6	0	2	7
4	5	0	4	5

Step-3: Assignment

Assignments are made to the above matrix as follows,

Machines Operator	1	2	3	4
1	$\boxed{0}$	3	8	$\cancel{0}$
2	2	2	$\boxed{0}$	1
3	3	$\boxed{0}$	2	7
4	2	$\cancel{0}$	4	5

As number of assignments are not equal to order of matrix, apply Hungarian rule.

Step-4 : Applying Hungarian Method

Machines Operator	1	2	3	4
1	~~$\boxed{0}$~~	~~3~~	~~8~~	~~$\cancel{0}$~~
2	~~2~~	~~2~~	~~$\boxed{0}$~~	~~1~~
3	3	$\boxed{0}$	2	7
4	2	$\cancel{0}$	4	5

Step -5

Machines / **Operator**	**1**	**2**	**3**	**4**
1	$\cancel{0}$	5	8	$\boxed{0}$
2	2	5	$\boxed{0}$	1
3	1	$\boxed{0}$	1	5
4	$\boxed{0}$	$\cancel{0}$	2	3

As number of assignments are equal to order of matrix, the solution is optimal

Optimal Solution

$1 \to 4$	2
$2 \to 3$	2
$3 \to 2$	3
$4 \to 1$	7
	14

$\therefore$ The minimum cost is `14/-

8. Daily demand for loaves of bread at grocery store are given by the following probability distribution.

x	100	150	200	250	300
p(x)	0.20	0.25	0.30	0.15	0.10

If a loaf is not sold the same day, it can be disposed of at Re 1 at the end of the day. Otherwise is price of a fresh loaf is Rs.11 The cost per loaf to the store is Rs.2.50 Assuming stock level is restricted to one of the demand levels, how many should be stocked daily ?

Ans :

Marginal Profit (MP) = Selling Price – Cost
= 11 – 2.50
= 8.50

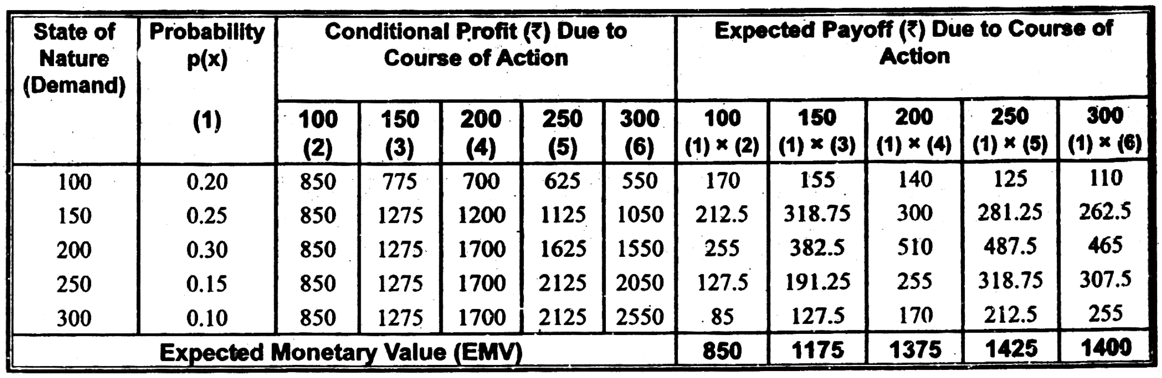
Marginal Loss (MC) = Cost Price – Cost of Unsold
= 2.50 – 1
= 1.50

Condition Profit (Payoff) = (Marginal Profit × Loaves Sold) – (Marginal Loss × Loaves Not Sold)
= (11 – 2.50) (Loaves Sold) – (2.50 – 1) (Loaves Unsold)

$$= \begin{Bmatrix} 8.50\ D & \text{...... if } D \geq N \\ (11-2.50)0-1.50(N-D)=100-1.5N & \text{...... if } D < N \end{Bmatrix}$$

In the table below, the resulting conditional profit values and corresponding expected payoffs are calculated.

Calculation of Conditional Profit Table and Expected Payoffs



State of Nature (Demand)	Probability p(x)	Conditional Profit (₹) Due to Course of Action					Expected Payoff (₹) Due to Course of Action				
	(1)	100 (2)	150 (3)	200 (4)	250 (5)	300 (6)	100 (1) × (2)	150 (1) × (3)	200 (1) × (4)	250 (1) × (5)	300 (1) × (6)
100	0.20	850	775	700	625	550	170	155	140	125	110
150	0.25	850	1275	1200	1125	1050	212.5	318.75	300	281.25	262.5
200	0.30	850	1275	1700	1625	1550	255	382.5	510	487.5	465
250	0.15	850	1275	1700	2125	2050	127.5	191.25	255	318.75	307.5
300	0.10	850	1275	1700	2125	2550	85	127.5	170	212.5	255
Expected Monetary Value (EMV)							**850**	**1175**	**1375**	**1425**	**1400**

As the maximum EMV of 1425 corresponds to the course of action 250, the retailer should stock 250 loaves every day.

---

9. a) Consider the game

		B		
		1	2	3
A	1	5	50	50
	2	1	1	0.1
	3	10	1	10

Verify that the strategies $\left(\frac{1}{6}, 0, \frac{5}{6}\right)$ for player A and $\left(\frac{49}{54}, \frac{5}{54}, 0\right)$ for player B are optimal and find the value of the game.

Ans :

Step-1 : Calculating Maximin Value and Minimax Value

Identify the minimum value in each row and select the maximum of the minimum value of each row and enclose it in a rectangle

		B			Row Minima
		1	2	2	
A	1	5	50	50	$\boxed{5}$
	2	1	1	0.1	0.1
	3	10	1	10	1

→ Maximin

Identify the maximum value in each column,

Select the minimum of these maximum values and enclose it in a rectangle.

		B			Row
		1	**2**	**3**	**Minima**
A	1	5	5	50	$\boxed{5}$ → Maxi min
	2	1	1	0.1	0.1
	3	10	1	10	1
	Column Maxima	$\boxed{10}$	50	50	

Mini max

∴ The game has no saddle point, since minimax value is not equal to maximin.

Step-2

Apply the dominance principle to reduce the size of payoff matrix.

(a) Row Dominance

Compare A' s 1st and 2nd Row

$5 > 1, 50 > 1, 50 > 0.1$

Compare A's 2nd and 3rd Row

$1 < 10, 1 = 1, 0.1 < 10$

Since row with lesser values is A_2, it is deducted

The reduced matrix is,

		B		
		1	**2**	**3**
A	1	5	50	50
	3	10	1	10

(b) Column Dominance

All the values of B_3 are '>' or '=' to the values of B_1 and B_2

Thus, strategy B_3 is superior and deleted from the matrix.

The reduced 2 × 2 matrix is,

		B	
		1	**2**
A	1	5	50
	3	10	1

Step - 3

Re-apply the maximin - minimax principle

		B	
		1	**2**	**Row Minima**
A	1	5	50	$\boxed{5}$ → Maxi min
	3	10	1	1
	Column Maxima	$\boxed{10}$	50	

Mini max

Since, minimax value $\neq$ maximin value, there is no saddle point. This implies that the game uses mixed strategies or algebraic method.

Step-4

As the matrix is of 2 × 2 size, apply algebraic method to get the solution

B

A		**1**	**2**
	1	a_{11}	a_{12}
	3	a_{21}	a_{22}

B

A		**1**	**2**
	1	5	50
	3	10	1

Thus, a_{11} = 5; a_{12} = 50

a_{21} = 10, a_{22} = 1

1. Let, player A selects a_1 strategy with probability x and strategy a_3 with probability 1 – x.
2. Let, player B selects b_1 strategy with probability y and strategy b_3 with probability 1 – y.

Value of A (a_1)

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{1 - 10}{(5 + 1) - (50 + 10)}$$

$$= \frac{-9}{(6) - (60)}$$

$$= \frac{-9}{-54}$$

$$\boxed{x = A(a_1) = \frac{1}{6}}$$

a_3 = 1 – x

$$= 1 - \frac{1}{6}$$

$$\boxed{a_3 = \frac{5}{6}}$$

Value of B (b_1),

$$y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{1 - 50}{(5 + 1) - (50 + 10)}$$

$$= \frac{-49}{(6) - (60)} = \frac{\cancel{-}49}{\cancel{-}54}$$

$$\boxed{y = \frac{49}{54}}$$

$$\boxed{y = B(b_1) = \frac{49}{54}}$$

$$b_2 = 1 - y$$

$$= 1 - \frac{49}{54}$$

$$= \frac{54 - 49}{54}$$

$$\boxed{b_2 = \frac{5}{54}}$$

Value of Game,

$$v = \frac{a_{11}.a_{22} - a_{12}.a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(5 \times 1) - (50 \times 10)}{(5 + 1) - (50 + 10)}$$

$$= \frac{5 - 500}{6 - 60}$$

$$= \frac{-495}{-54} = \frac{55}{6}$$

$$\boxed{v = \frac{55}{6}}$$

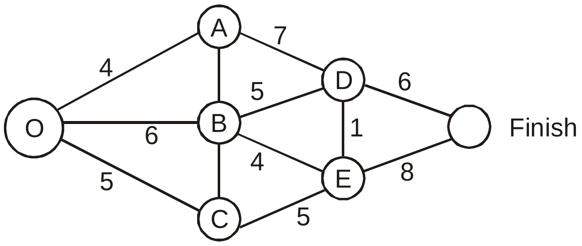
Optimal Strategy

	Strategy	Probability
For Player A	a_1	$\frac{1}{6}$
	a_2	0
	a_3	$\frac{5}{6}$
For Player B	b_1	$\frac{49}{54}$
	b_2	$\frac{5}{54}$
	b_3	0

$\therefore$ Value of Game $\boxed{v = \frac{55}{6}}$

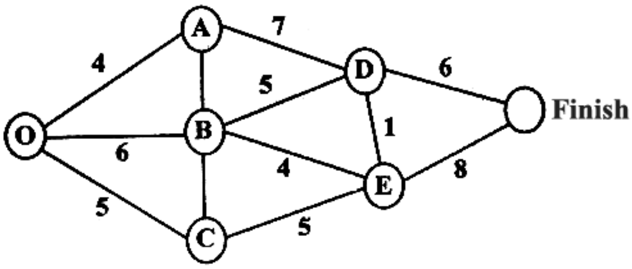
---

b)

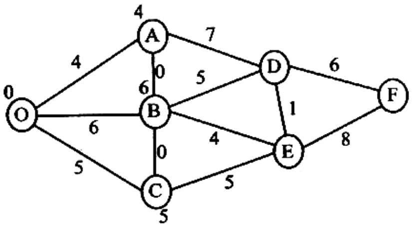


Find the shortest path.

Ans :



Note : In the network diagram, the distance from A→B and B→C is not given, so we are assuming that the distance from A→B and B→C as '0'.



From the network diagram, the following paths can be obtained :

O → A → D → F = 4 + 7 + 6 = 17

O → A→ B → D → F = 4 + 0 + 5 + 6 = 15

O → B → D → F = 6 + 5 + 6 = 17

O → B → D → E → F = 6 + 5 + 1 + 8 = 20

O → B → E → F = 6 + 4 + 8 = 18

O → B → C → E → F = 6 + 0 + 5 + 8 = 19

O → C →E → F = 5 + 5 + 8 = 18

The shortest path among the above mentioned paths is O → A → B → D → F

Duration : 4 + 0 + 5 + 6 = 15

---

10. a) Consider a typical barber shop. Demonstrate that it is a queuing system by describing its components. **(Unit-V, Q.No. 2)**

b) Consider a single server queuing system with any service time distribution of inter arrival times (GI/G/I) model. Use only basic definitions and relationships to verify the following relations :

i) $L = L_q + (1–P_0)$

ii) $L = L_q + p$

iii) $L_0 = 1 – p$

Ans :

(i) $L = L_q + (1 – P_0)$

L = Average number of units in the system

L_q = Average number of units in the queue

$1 – P_0 = \rho$ (i.e., probability that the service facility is busy)

We know that

$$L = \begin{Bmatrix} L_q \text{ When no Body is in the system} \\ L_q + 1 \text{ Otherwise} \end{Bmatrix}$$

∴ The value of L is given as follows,

$$L = P_0 L_q + (1 – P_0) (L_q + 1)$$

$$= \cancel{P_0 L_q} + L_q - \cancel{P_0 L_q} + 1 - P_0$$

$$= L_q + (1 – P_0)$$

$$\therefore \quad \boxed{L = L_q + (1 - P_0)}$$

The given condition i.e., $L = L_q + (1 – P_0)$ is verified proved.

(ii) $L = L_q + \rho$

We know that, $P_0 = 1 - \rho$

We have already proved in bit (i) that,

$$L = L_q + (1 - P_0)$$

If we substitute $P_0 = 1 - \rho$ in it, we get,

$$L = L_q + (1 - (1 - \rho)$$

$$L = L_q + (\cancel{1} - \cancel{1} + \rho)$$

$$\boxed{L = L_q + \rho}$$

$\therefore$ The given condition i.e., $L = L_q + \rho$ is verified/proved.

(iii) $P_0 = 1 - \rho$

[Note : There is a print mistake in the question. Instead of $P_0 = 1 - \rho$; $L_0 = 1 - \rho$ is given]

P_0 = Probability that the service facility is idle

ρ = Probability that the service facility is busy

If we want to know the probability that the service facility is idle, we need to do,

$$1 - \rho$$

$$\therefore \; P_0 = 1 - \rho$$

---

11. a) Explain why the utilization factor ρ for the server in a single-server queuing system must equal $1 - P_0$ where P_0 is the probability of having 0 customers in the system.

Ans :

Let,

(i) P_i = Average fraction of time when there are i customers in the system.

(ii) The average rate at which the process crosses the vertical line from left to right is $P_i \lambda$.

(iii) The average rate at which the process crosses the vertical line from right to left is $P_{i+1}\mu$

Therefore,

$$P_i \lambda = P_{i+1}\mu$$

$$P_{i+1} = \frac{\lambda}{\mu} P_i = \rho P_i$$

Where,

λ = Arrival Rate

μ = Service Rate

$i = 0, 1, 2, \ldots\ldots. \infty$

$P_1 = \rho P_0$

$P_2 = \rho P_1 = \rho^1 P_0$

$$P_3 = \rho P_2 = \rho^2 P_0$$

$$\vdots$$

$$P_n = \rho P_{n-1} = \rho^{n-1} P_0$$

In case if we know that P_0 (i.e., probability of an empty system), we would have able to calculate all of the other long term probabilities. To calculate so, we must note that, as the system should be in some state, the long term probabilities should be sum to one. This indicates the following,

$$1 - \sum_{n=0}^{\infty} P_x = \sum_{n=0}^{\infty} \rho^n P_0 = \frac{P_0}{1-\rho}$$

This shows that the probability of finding the system empty is,

$$P_0 = 1 - \rho$$

---

b) The jobs to be performed on a particular machine arrive according a Poisson input process with a mean rate of two per hour. Suppose that machine breaks down and will require 1 hour to be repaired. What is the probability that the number of new jobs that will arrive during this time is (i) 0, (ii) 2, (iii) 5 or more ?

Ans :

Step-1

Poisson process is an on-going process where the events takes place on a continual basis with mean rate $\propto$. It is denoted as [X(t); t $\geq$ 0]. By using the following formula, the probability of occurrence of new events is calculated.

$$P\left[X(t) = n\right] = \frac{(\propto t)^n e^{-\alpha t}}{n!} \text{ ....(1)}$$

Where,

X(t) = number of occurrences by time t

$\propto$ = arrival rate

n = number of events

P[X(t) = n] = Probability of arrival of new jobs.

Poisson distribution is observed with mean rate in which events takes place at the rate of $\propto$ = 2 per hour.

The mean of Poisson distribution is computed by E(x)

$$= \propto t = 2 \times 1 = 2$$

(i) Probability that the Number of New Jobs That will Arrive During this time is 'Zero' [i.e., P(X(t) = 0)]

The number of new jobs which arrive during time t is 0 i.e., n = 0. By substituting n = 0 and $\propto t = 2$ in equation (1), we get

$$= P[X\ (t)] = \frac{(2)^2 . e^{-2}}{2!}$$

$$= \frac{4.e^{-2}}{2 \times 1}$$

$$= \frac{4(0.1353)}{2}$$

$$= 2(0.1353)$$

$$P[X\ (t)] = 0.2706$$

Hence, the probability that 2 jobs will arrive in time t is 0.2706

(iii) Probability that the Number of New Jobs That will Arrive During time is '5' or more [i.e., $P(X(t) \geq 5)$].

P(new jobs will arrive during this time five or more) = $P(X(t) \geq 5)$.

In bit-(i) and (ii), we have already calculated $P(X(t) = 0)$ and $P(X(t) = 2)$. So, now we need to find out $P[X(t) = 1]$, $P[X(t) = 3]$ and $P[X(t) = 4]$.

Finding out P[X(t) = 1]

$$P[X(t) = 1] = \frac{(2)^1.e^{-2}}{1!}$$

$$= \frac{2 \times 0.1353}{1} = 0.2706$$

Finding out P[X(t) = 3]

$$P[X(t) = 3] = \frac{(2)^3.e^{-2}}{3!}$$

$$= \frac{8 \times 0.1353}{3 \times 2 \times 1} = \frac{8 \times 0.1353}{6} = 0.1804$$

Finding out P[X(t) = 4]

$$P[X(t) = 4] = \frac{(2)^4.e^{-2}}{4!}$$

$$= \frac{(2 \times 2 \times 2 \times 2) \times 0.1353}{4 \times 3 \times 2 \times 1} = \frac{16 \times 0.1353}{24} = 0.0902$$

$$P[X(t) \geq 5] = 1 - P[X(t) < 5]$$

$$= 1 - [PX(t) = 0] + [PX(t) = 1] + [PX(t) = 2] + [P\ X(t) = 3] + [P\ X(t) = 4]$$

$$= 1 - [0.1353 + 0.2706 + 0.2706 + 0.1804 + 0.0902]$$

$$= 1 - 0.9471$$

$$= 0.0529$$

$$P[X(t) \geq 5] = 0.0529$$

$\therefore$ The probability of new jobs will arrive during this time 5 or more [i.e., $P[X(t) \geq 5]$ is 0.0529.

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY HYDERABAD

M.B.A II - Semester Examination

February - 2017

R15

QUANTITATIVE ANALYSIS FOR BUSINESS DECISIONS

Time : 3 Hours] [Max. Marks : 75

Answer any five questions
All questions carry equal marks.

ANSWERS

PART - A (5 × 5 = 25 Marks)

1. (a) Define an OR model and state their advantages and limitations. **(Unit-I, Q.No. 14, 18)**

b) State the different steps are required in solving LPP by graphic method. **(Unit-II, Q.No. 9)**

(c) What do you understand by an assignment problem give a brief outline for solving it. **(Unit-III, Q.No. 1)**

(d) What are pay-off and regret functions? How can entries in a regret table be derived from a pay-off table. **(Unit-V, Q.No. 16)**

(e) What is a queuing problem? What are the basic characteristics of a queuing system? **(Unit-V, Q.No. 1, 2)**

PART - B (5 × 10 = 50 Marks)

2. State any five areas for the application of OR techniques in financial management with suitable examples? How it improves the performance of the organization. **(Unit-I, Q.No. 6)**

OR

3. Explain the nature, scope and significance of quantitative analysis. **(Unit-I, Q.No. 12)**

4. Solve the following transportation problem:

	Destination				
Source	1	2	3	4	Supply
1	15	18	22	16	30
2	15	19	20	14	40
3	13	16	23	17	30
Demand	20	20	25	35	100

Is the optimal solution obtained by you a unique one? If not, why? What are the alternate optima then?

Sol :

Finding IBFs by using VAM

Step 1

Calculate row and column penalty i.e., subtract lowest value and next lowest value.

	1	2	3	4	Supply	Row Penalty
1	15	18	22	16	30	1
2	15	19	20	14	40	1
3	⑬^{20}	16	23	17	30	3
Demand	20	20	25	35	100	
Column penalty	2	2	2	2		

Maximum penalty 3, occurred in low ‘3’. Maximum allocation in this column is 20. It satisfy demand ‘1’ and left supply is 10. Eliminate column ‘1’ because demand is 0.

Step 2

Calculate low and Column penalty for minimized matrix.

	2	3	4	Supply	Low Penalty
1	18	22	16	30	2
2	19	20	⑭^{35}	40	5
3	16	23	17	10	1
Demand	20	25	35		
Column Penalty	2	2	2		

Max. Penalty is 5 in low ‘2’. Allocate max. supply to column 4. left supply = 5. Eliminate column ‘4’ because demand = 0.

Step 3

Calculate low and column penalty for the minimized matrix.

	2	3	Supply	Low Penalty
1	18	22	30	4
2	19	20	5	1
3	⑯^{10}	23	10	7
Demand	20	25		
Column Penalty	2	2		

Max. Penalty is ‘7’ in low ‘3’. Allocate max. supply to column 2. left supply = 10. Eliminate low ‘3’ because supply = 0.

Step 4

Calculate low and Column penalty for the minimized matrix

	2	3	Supply	Low Penalty
1	$\textcircled{18}^{10}$	22	30	4
2	19	20	5	1
Demand	10	25		
Column Penalty	1	2		

Max. penalty = 4 in low '1'. Allocate Max. supply to demand in Column '2' left supply = 20. Eliminate column '2' because demand = 0.

Step 5

Allocate supply according to the demand in the minimized matrix.

	3	Supply
1	$\textcircled{22}^{20}$	20
2	$\textcircled{20}^{5}$	5
Demand	25	

Consolidated Allocation table:

	Destination				
Source	1	2	3	4	Supply
1	15	$\textcircled{18}^{10}$	$\textcircled{22}^{20}$	16	30
2	15	19	$\textcircled{20}^{5}$	$\textcircled{14}^{35}$	40
3	$\textcircled{13}^{20}$	$\textcircled{16}^{10}$	23	17	30
Demand	20	20	25	35	

Transportation cost = 13 × 20 + 18 × 10 + 16 ×10 + 20 × 5 + 14 × 35

= 260 + 180 + 160 + 100 + 700 + 22 × 20

= 1840.

Here, the number of allocations = 6 is equal to M + n – 1 = 3 + 4 – 1 = 6.

∴ This solution is non-degenerate.

Optimality Test by Using Modi Method

Find U_i and V_j for all occupied cells (i, j) where $C_{ij} = U_i + V_j$

$S_3D_1 = U_3 + V_1 = 13$

Assume $V_1 = 0$

$U_3 + V_1 = 13 \Rightarrow \boxed{U_3 = 13}$

$S_3D_2 = U_3 + V_2 = 16$

$\Rightarrow 13 + V_2 = 16$

$\boxed{\Rightarrow V_2 = 3}$

$S_1D_2 = U_1 + V_2 = 18$

$\Rightarrow U_1 + 16 = 18 = U_1 = 18 - 3$

$\boxed{\Rightarrow U_1 = 15}$

$S_1D_3 = U_1 + V_3 = 22$

$= 15 + V_3 = 22$

$\boxed{V_3 = 7}$

$S_2D_3 = U_2 + V_3 = 20$

$= U_2 + 7 = 20$

$\boxed{U_2 = 13}$

$S_2D_4 = U_2 + V_4 = 14$

$= 13 + V_4 = 14$

$\boxed{V_4 = 1}$

Step 2

Find d_{ij} for all unoccupied cells (i, j) where $d_{ij} = c_{ij} - (U_i + V_j)$

$S_1D_1 = C_{11} - (U_1 + V_1)$

$\Rightarrow 15 - (U_1 + 0) = 15 - (15 + 0) = 0$

$S_1D_3 = C_{13} - (U_1 + V_3)$

$\Rightarrow 22 - (15 + 7) = 0$

$S_2D_1 = C_{14} - (U_1 + V_4)$

$\Rightarrow 16 - (15 + 1) = 0$

$S_2D_2 = C_{22} - (U_2 + V_2)$

$\Rightarrow 19 - (13 + 3) = 3$

$S_3D_3 = C_{33} - (U_3 + V_4)$

$\Rightarrow 23 - (13 + 7) = 3$

$S_3D_4 = C_{34} - (U_3 + V_4)$

$\Rightarrow 17 - (13 + 1) = 3$

Since all $d_{ij} \geq 0$ so final optimal solution is arrived.

Source	Destination 1	2	3	4	Supply
1	15	⑱^{10}	㉒^{20}	16	30
2	15	19	⑳^{5}	⑭^{35}	40
3	⑬^{20}	⑯^{10}	23	17	30
Demand	20	20	25	35	

The minimum transportation cost = 1840.

5. Solve the following LP problem using Simplex method.

Maximize $Z = 10X_1 + 15\,X_2 + 20\,X_3$

Subject to

$$2\,X_1 + 4\,X_2 + 6\,X_3 \leq 24$$

$$3\,X_1 + 9\,X_2 + 6\,X_3 \leq 30$$

$$X_1,\ X_2 \text{ and } X_3 \leq 0$$

Sol :

Step 1

Convert inequalities to equalities by adding black variable S_1, S_2, in the constraints.

$$2\,X_1 + 4\,X_2 + 6\,X_3 + S_1 = 24$$

$$3\,X_1 + 9\,X_2 + 6\,X_3 + S_2 = 30$$

$$X_1,\ X_2,\ X_3,\ S_1,\ S_2,\ \geq 0$$

Simplex Table

Key column

Key row

		C_j	10	15	20	0	0	
C_B	B	X_B	X_1	X_2	X_3	S_1	S_2	Min. Ratio
0	S_1	24	2	4	6	1	0	$24/6 = 4$
0	S_2	30	3	9	6	0	1	$30/6 = 5$
	Zy	0	0	0	0	0	0	
	$C_j - Z_i$		10	15	20	0	0	

Make key element '6' as '1' and all other elements in key column as zero.

Iteration 1

		C_j	10	15	20	0	0	
C_B	B	X_B	X_1	X_2	X_3	S_1	S_2	Min. Ratio
20	X_3	4	$1/3$	$2/3$	1	$1/6$	0	12
0	S_2	6	1	5	0	–1	1	6
	Zy	80	$\frac{20}{3}$	$\frac{40}{3}$	20	$\frac{20}{6}$	0	
	$C_j - Z_i$		$\frac{+10}{3}$	$\frac{5}{3}$	0	$\frac{-20}{6}$	0	

Iteration 2

		C_j	10	15	20	0	0
C_B	B	X_B	X_1	X_2	X_3	S_1	S_2
20	X_3	2	0	–1	1	$1/2$	$-1/3$
10	X_1	6	1	5	0	–1	1
	Zy	100	10	30	20	0	10 / 3
	$C_j - Z_i$		10	–15	0	$-1/2$	–1

Here all the value of $C_j - Z_j \le 0$. Here, optimum solution is given by

Max $z = 100$ $X_1 = 6;$ $X_2 = 0$ $X_3 = 2$

6. Solve the following assignment problem by Hungarian assignment method.

	Time (in minutes)		
Worker	**Job 1**	**Job 2**	**Job 3**
A	4	2	7
B	8	5	3
C	4	5	6

(Unit-III, Prob. 1)

7. Explain the assignment model as a special case of the transportation model. **(Unit-III, QNo. 1)**

8. Discuss some methods which are useful for decision making under uncertainity **(Unit-IV, QNo. 5)**

9. List out and elaborate the approaches for decisions under risk and explain with appropriate examples. **(Unit-IV, QNo. 6)**

10. A TV mechanic finds that the time spent on his jobs has an exponential distribution with a mean 30 minutes. If he repairs sets on the first-come-first-served basis and if the arrival of sets is with an average rate of 10 per 8 hour day, what is maechanisc expected idle time each day? Also obtain average number of units in the system.

Sol :

Arrival rate = λ = 10 per day of 8 hours

= 10/8 = 1.25/hv

Service time = $(1/\mu)$ = 30 minutes = ½ hr.

$\therefore$ Service rate = μ = 2/hr.

(a) Expected idle time of repairment

Prob. system is busy = (λ/μ) = (1.25/2) = 0.625

$\therefore$ repairment is busy in a should day = 8 × 0.625 = 5hours.

$\therefore$ repairman is idle for 8 – 5 = 3 hrs.

(b) Number of sets ahead of the TV set brought in

Avg. number of customers in the system = $\left[\dfrac{\lambda}{(\mu - \lambda)}\right]$

$$= \frac{1.25}{[2 - 1.25]} = 1.667$$

$\therefore$ 1.667 TV sets in the system ahead of the TV

Set just brought in.

---

11. A car hiring firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as Poisson variate with mean 1.5. Calculate the proportion of days on which.

(a) Neither car is used, and

(b) Some demand is refused.

Ans :

(a) When both cars are not used, V = 0.

$$P(V = 0) = e^{-1.5} = 0.2231$$

Hence the proportional of days on which neither car is used is 22.31% Further, some used = 110 – 22.81 = 77.19%.

(b) Some demand is refused

$$\therefore P(V > 2) = 1 - P(V \le 2) = 1 - \sum_{r=0}^{2} e^{-1.5} \frac{(1.5)^r}{r!}$$

$$= 1 - 0.2231 \left[1 + 1.5 + \frac{(1.5)^2}{2!}\right]$$

$$= 0.1913$$

Hence the proportion of days is 19.13%.

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY HYDERABAD

M.B.A II - Semester Examination

Model Paper - I

R19

QUANTITATIVE ANALYSIS FOR BUSINESS DECISIONS

Time : 3 Hours] [Max. Marks : 75

Note : This question paper contains two parts A and B.

Part A is compulsory which carries 25 marks. Answer all questions in Part A.

Part B consists of 5 Units. Answer any one full question from each unit. Each question carries 10 marks and may have a, b, c as sub questions.

Part-A (5 × 5 = 25 Marks)
[Short Answer type]

ANSWERS

1. (a) Explain limitations of operation research. **(Unit-I, SQA-8)**

   (b) What are the assumptions of LPP. **(Unit-II, SQA-2)**

   (c) Define travelling salesman problem. **(Unit-III, SQA-5)**

   (d) Compare and contrast PERT and CPM. **(Unit-IV, SQA-15)**

   (e) Define saddle point. **(Unit-V, SQA-12)**

PART-B (5 × 10 = 50 Marks)
[Essay Answer type]

2. Explain the process for developing an operations research model. **(Unit-I, Q.No.11)**

OR

3. Explain briefly about Nature and Scope of Operations Research. **(Unit-I, Q.No.2)**

4. Solve using, two-phase simplex method

Solve the following LPP

Min $Z = 10X + 15Y$

S.T.C. ...,

$Y \geq 3$

$X - Y \geq 0$

$Y \leq 12;$

$X + Y \leq 30$

$X \leq 20$ and $X, Y \geq 0$ **(Unit-II, Prob.20)**

OR

5. (i) The manufacturer of patent medicines has proposed to prepare a production plan for medicines A and B. There are sufficient ingredients available to make 20,000 bottles of medicine A and 40,000 bottles of medicine B but there are only 45,000 bottles into which either of the medicines can be filled. Further, it makes three hours to prepare enough material to fill 100 bottles of medicine A and one hour to prepare enough material to fill 1000 bottles of medicine B and there are 66 hours available

for this operation. The profit is ` 8 bottle for medicine A and ` 7 per bottle for medicine B. Formulate this problem as a LPP and solve it by graphical method. **(Unit-II, Prob.8)**

(ii) Elucidate the various assumptions of LPP ? **(Unit-II, Q.No.3)**

6. A company has 4 machines available for assignment to 4 tasks. Any machine can be assigned to any task, and each task requires processing by one machine. The time required to set up each machine for the processing of each task is given in the table below.

What kind of assignment will allow the company to minimize the total setup time needed for the processing of all four tasks?

	TIME (Hours)			
	Task 1	Task 2	Task 3	Task 4
Machine 1	13	4	7	6
Machine 2	1	11	5	4
Machine 3	6	7	2	8
Machine 4	1	3	5	9

(Unit-III, Prob.2)

OR

7. A cement company has three factories which manufacture cement which is then transported to four distribution centres. The quantity of monthly production of each factory, the demand of each distribution centre and the associated transportation cost per quintal are given below :

Factory	Distribution Centres				Monthly Production (quintals)
	W	X	Y	Z	
A	10	8	5	4	7,000
B	7	9	15	8	8,000
C	6	10	14	8	10,000
Monthly demand (in quintals)	6,000	6,000	8,000	5,000	25,000

(a) Suggest the optimum transportation schedule.

(b) Is there any other transportation schedule which is equally attractive? If so, write that.

(c) If the company wants that atleast 5,000 quintals of cement are transported from factory C to distribution centre Y, will the transportation schedule be any different ? If so, what will be the new optimum schedule and the effect on cost? **(Unit-III, Prob.17)**

8. (i) Construct a network for each of the projects whose activities and their precedence relationships are given below :

Activity	A	B	C	D	E	F	G	H	I	J	K
Predecessor	–	–	–	A	B	B	C	D	E	H,I	F,G

(Unit-IV, Prob.12)

(ii) Describe some methods which are useful for decision making under uncertainity **(Unit-IV, Q.No.5)**

OR

9. (i) A food products' company is contemplating the introduction of a revolutionary new product with new packaging or replacing the existing product at much higher price (S_1). It may even make a moderate change in the composition of the existing product, with a new packaging at a small increase in price (S_2), or may mall a small change in the composition of the existing product, backing it with the word "New' and a negligible increase in price (S_3). The three possible states of nature or events are: (i) high increase in sales (N_1), (ii) no change in sales (N_2) and (iii) decrease in sales (N_3). The marketing department of the company worked out the payoffs in terms of yearly net profits for each of the strategies of three events (expected sales). This is represented in the following table :

Strategies	States of Nature		
	N_1	N_2	N_3
S_1	7,00,000	3,00,000	1,50,000
S_2	5,00,000	4,50,000	0
S_3	3,00,000	3,00,000	3,00,000

Which strategy should the concerned executive choose on the basis of

(a) Maximin criterion (b) Maximax criterion

(c) Minimax regret criterion (d) Laplace criterion **(Unit-IV, Prob.1)**

(ii) Define project crashing. **(Unit-IV, Q.No.20)**

10. (i) What are the various types of queuing disciplines? Give suitable examples with their managerial implications. **(Unit-V, Q.No.11)**

(ii) Consider a self-service store with one cashier. Assume Poisson arrivals and exponential service times. Suppose that 9 customers arrive on an average every 5 minutes and the cashier can serve 10 customers in five minutes. Compute the following,

(a) Average number of customers queueing for service.

(b) Probability of having more than 10 customers in the system.

(c) Probability that a customer has to queue for more than 2 minutes.

(d) If the service can be speeded up to 12 in 5 minutes by using a different ash register, what will be the effect on quantities (a), (b) and (c)? **(Unit-V, Prob.5)**

OR

11. (i) Define game theory ? Explain the advantages and disadvantages of game theory. **(Unit-V, Q.No.15,18)**

(ii) Solve the following payoff matrix, determine the optimal strategies and the value of game.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

(Unit-V, Prob.13)

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY HYDERABAD

M.B.A II - Semester Examination

Model Paper - II

R19

QUANTITATIVE ANALYSIS FOR BUSINESS DECISIONS

Time : 3 Hours] [Max. Marks : 75

Note : This question paper contains two parts A and B.

Part A is compulsory which carries 25 marks. Answer all questions in Part A.

Part B consists of 5 Units. Answer any one full question from each unit. Each question carries 10 marks and may have a, b, c as sub questions.

Part-A (5 × 5 = 25 Marks)

[Short Answer type]

			Answers
1.	(a)	What are the differences between quantitative analysis and qualitative analysis?	**(Unit-I, SQA-6**)
	(b)	Mathematical Formulation of LPP.	**(Unit-II, SQA-3)**
	(c)	Define Transportation Problem.	**(Unit-III, SQA-6)**
	(d)	What is Project Crashing?	**(Unit-IV, SQA-16)**
	(e)	Queue Behaviour.	**(Unit-V, SQA-4)**

PART-B (5 × 10 = 50 Marks)

[Essay Answer type]

2. State the applications of operations research. **(Unit-I, Q.No.6**)

OR

3. Explain the opportunities and Short comings of using an OR Model. **(Unit-I, Q.No.18**)

4. A company manufactures two products A and B. These products are processed in the same machine. It takes 10 minutes to process one unit of product A and 2 minutes for each unit of product B and the machine operates for a maximum of 35 hrs in a week. Product A requires 1 kg. and B 0.5 kg. of raw material per unit the supply of which is 600 kg. per week. Market constraint on product B is known to be 800 unit every week. Product A costs Rs. 5 per unit and sold at Rs. 10. Product B costs Rs.6 per unit and can be sold in the market at a unit price of Rs. 8. Determine the number of units of A and B per week to maximize the profit. **(Unit-II, Prob.2**)

OR

5. Solve the LPP

Max $Z = 3X_1 + 2X_2$

Subject to $4X_1 + 3X_2 \leq 12$

$4X_1 + X_2 \leq 8$

$4X_1 - X_2 \leq 8$

$X_1, X_2 \geq 0$ **(Unit-II, Prob.15**)

6. Solve the following assignment problem of minimizing total time for doing all the jobs:

Operator \ Job	I	II	III	IV	V
1	6	2	5	2	6
2	2	5	8	7	7
3	7	8	6	9	8
4	6	2	3	4	5
5	9	3	8	9	7
6	4	7	4	6	8

(Unit-III, Prob.5)

OR

7. Find the initial basic feasible solution for the following transportation problem by VAM.

Destination						
Origin		D_1	D_2	D_3	D_4	*Supply*
	O_1	11	13	17	14	250
	O_2	16	18	14	10	300
	O_3	21	24	13	10	400
	Demand	200	225	275	250	950

(Unit-III, Prob.14)

8. Given the following data, workout the minimum duration of the project and corresponding cost.

Activity	Job		Time		Cost
A	1-2	10	6	400	600
B	1-3	4	2	100	140
C	2-4	6	4	360	440
D	34	8	4	600	900
E	2-5	8	6	840	1100
F	4-6	6	2	200	300
G	5-6	10	8	1200	1400

(Unit-IV, Prob.21)

OR

9. Write about the method and steps involved in construction of decision tree. **(Unit-IV, Q.No.10)**

10. Discuss in detail queuing structure and basic L-components of a queuing model. **(Unit-V, Q.No.4)**

OR

11. Solve the following payoff matrix, determine the optimal strategies and the value of game.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

(Unit-V, Prob.13)

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY HYDERABAD

M.B.A II - Semester Examination

R19

Model Paper - III

QUANTITATIVE ANALYSIS FOR BUSINESS DECISIONS

Time : 3 Hours] [Max. Marks : 75

Note : This question paper contains two parts A and B.

Part A is compulsory which carries 25 marks. Answer all questions in Part A.

Part B consists of 5 Units. Answer any one full question from each unit. Each question carries 10 marks and may have a, b, c as sub questions.

Part-A (5 × 5 = 25 Marks)

[Short Answer type]

ANSWERS

1. (a) Problem Solving **(Unit-I, SQA-3)**

   (b) Define Unbounded Solution. **(Unit-II, SQA-8)**

   (c) Assumptions of an Assignment Problem. **(Unit-III, SQA-2)**

   (d) Rules for Network Construction. **(Unit-IV, SQA-11)**

   (e) Advantages of Game Theory **(Unit-V, SQA-10)**

PART-B (5 × 10 = 50 Marks)

[Essay Answer type]

2. Explain the process for developing an operations research model. **(Unit-I, Q.No.11)**

OR

3. Outline the general principles used in model building within the context of OR. Briefly explain the scientific method in OR. **(Unit-I, Q.No.16)**

4. State the various Applications of LPP. **(Unit-II, Q.No.4)**

OR

5. Use penalty method to

Maximize $Z = 3x_1 + 2x_2$

Subject to the constraints

$2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$ **(Unit-II, Prob.16)**

6. Solve the following transportation problem.

		Destination				
		P	*Q*	*R*	*S*	*Supply*
Source	*A*	21	16	25	13	11
	B	17	18	14	23	13
	C	32	17	18	41	19
	Demand	6	10	12	15	43

Origin\Dest	*P*	*Q*	*R*	*S*	*Supply*	P_I	P_{II}	P_{III}	P_{IV}	P_V	P_{VI}
A	$\boxed{21}$	$\boxed{16}$	$\boxed{25}$	$\boxed{13}$ ⑪	11	3	–	–	–	–	–
B	$\boxed{17}$ ⑥	$\boxed{18}$	$\boxed{14}$ ③	$\boxed{23}$ ④	13	4	4	4	4 ←	–	–
C	$\boxed{32}$	$\boxed{17}$ ⑩	$\boxed{18}$ ⑨	$\boxed{48}$	19	1	1	1	1	1	17
Demand	6	10	12	15	43						
P_I	4	1	4	10↑							
P_{II}	15	1	4	18↑							
P_{III}	15↑	1	4	–							
P_{IV}	–	1	4	–							
P_V	–	17	18↑	–							
P_{VI}	–	17↑	–	–							

(Unit-III, Prob.16)

OR

7. A company has three plants at locations A, B and C which supply to warehouses located at D, E, F, G and H. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in `) are given below,

		To				
		D	**E**	**F**	**G**	**H**
From	**A**	**5**	**8**	**6**	**6**	**3**
	B	**4**	**7**	**7**	**6**	**5**
	C	**8**	**4**	**6**	**6**	**4**

Determine an optimum distribution for the company in order to minimize the total transportation cost. **(Unit-III, Prob.19)**

8. A conditional pay off matrix is given below. By using this matrix calculate EMV for each course of action and determine the optimum EMV.

States of Nature	Probabilities	Conditional Pay off Matrix Course of Action				
		A_1	A_2	A_3	A_4	A_5
N_1	0.03	0	–30	–70	–110	–150
N_2	0.15	0	4	–25	–65	–110
N_3	0.20	0	4	9	–20	–60
N_4	0.50	0	4	9	14	–15
N_5	0.12	0	4	9	14	10

(Unit-IV, Prob.4)

OR

9. The following is a table showing details of a project,

Task	Immediate Predecessor	Normal time in Weeks	Normal cost in Thousands `	Crash time in Weeks	Crash Cost in Thousands `
A	-	10	20	7	30
B	-	8	15	6	20
C	B	5	8	4	14
D	B	6	11	4	15
E	B	8	9	5	15
F	E	5	5	4	8
G	A,D,C	12	3	8	4

Indirect cost in ` 400 per day. Find the optimum duration and the associated cost of the project. **(Unit-IV, Prob.17**)

10. The Taj Service Station has five mechanics each of whom can service a scooter in 2 hours on the average. The scooters are registered at a single counter and then sent for servicing to different mechanics. Scooters arrive at the service at an average rate of 2 scooters per hour. Assuming that the scooter arrivals are Poisson distribution and the servicing times are distributed exponentially determine,

(a) Utilization factor

(b) The probability that the system shall be idle

(c) The probability that there shall be 3 scooters in the service centre

(d) The expected number of scooter waiting in the queue and in the system.

(e) The average waiting time in the queue and

(f) The average time spent by a scooter in waiting and getting serviced. **(Unit-V, Prob.7**)

OR

11. Solve the following game graphically.

$$\begin{bmatrix} -6 & 0 & 6 & -3/2 \\ 7 & 3 & -8 & 2 \end{bmatrix}$$

(Unit-V, Prob.21)