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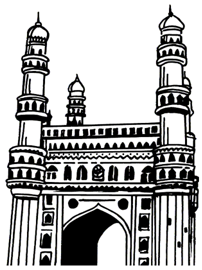

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SYLLABUS

UNIT - I

VECTOR ANALYSIS :

Scalar and vector fields, gradient of a scalar field and its physical significance. Divergence and curl of a vector field and related problems. Vector integration, line, surface and volume integrals. Stokes, Gauss and Greens theorems-simple applications.

UNIT - II

MECHANICS OF PARTICLES :

Laws of motion, motion of variable mass system, motion of a rocket, multi-stage rocket, conservation of energy and momentum. Collisions in two and three dimensions, concept of impact parameter, scattering cross-section

MECHANICS OF RIGID BODIES :

Definition of Rigid body, rotational kinematic relations, equation of motion for a rotating body, angular momentum and inertial tensor. Euler's equation, precession of a top, Gyroscope

UNIT - III

CENTRAL FORCES :

Central forces – definition and examples, conservative nature of central forces, conservative force as a negative gradient of potential energy, equation of motion under a central force, gravitational potential and gravitational field, motion under inverse square law, derivation of Kepler's laws.

SPECIAL THEORY OF RELATIVITY :

Galilean relativity, absolute frames, Michelson-Morley experiment, Postulates of special theory of relativity. Lorentz transformation, time dilation, length contraction, addition of velocities, mass-energy relation. Concept of four vector formalism.

UNIT - IV

OSCILLATIONS :

Simple harmonic oscillator, and solution of the differential equation– Physical characteristics of SHM, torsion pendulum measurements of rigidity modulus, compound pendulum, measurement of g, combination of two mutually perpendicular simple harmonic vibrations of same frequency and different frequencies, Lissajous figures.Damped harmonic oscillator, solution of the differential equation of damped oscillator. Energy considerations, logarithmic decrement, relaxation time, quality factor, differential equation offorced oscillator and its solution, amplitude resonance, velocity resonance.

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Frequently Asked & Important Questions

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---

2. Define gradient of a scalar field funct-ion. Explain the physical significance for the gradient of a scalar field.

Ans : **(Dec.-19(MGU), Dec.-18, Dec.-18(KU), Dec.-17(MGU), Dec.-16)**

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---

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---

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Ans : **(Imp.)**

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3. Define torsion pendulum? How do you determine modulus of rigidity using torsion pendulum?

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4. Discuss the combination of two mutually simple harmonic vibrations of same frequencies with neat diagrams?

Ans : **(Imp.)**

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5. What are damped osccillations? Solve the differential Equation of damped harmonic oscillator ?

Ans : **(July-21)**

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---

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Ans : **(Imp.)**

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Ans : **(Imp.)**

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UNIT I

VECTOR ANALYSIS :

Scalar and vector fields, gradient of a scalar field and its physical significance. Divergence and curl of a vector field and related problems. Vector integration, line, surface and volume integrals. Stokes, Gauss and Greens theorems-simple applications.

1.1 Vector Analysis

Q1. Explain the representation and notation of vectors.

Ans :

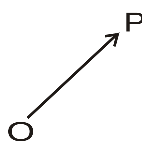
Definitions

There are two kinds of physical quantities. The first are quantitities which have only magnitude and which are not related to any fixed direction in space. Such quantities are called scalars. Examples of scalar are mass, length, density, volume etc. If the unit of measurement is fixed, then a real number is sufficient enough to represent a scalar quantity.

Second kind of quantities are those which have magnitude as well as direction. Such quantities are called vectors. Examples of vectors are velocity, displacement, force, acceleration etc.

Representation of Vectors

We shall represent vectors by directed line segments. Let O be any arbitrary fixed point in the space and P be any other point. Then the straight line OP has magnitude as well as direction. Therefore the directed line segment OP is capable of representing a vector quantity. We denote this vector by $\overrightarrow{OP}$ or simply by OP and read it as vector OP. The length OP represents the magnitude of the vector OP. The point O is called the origin or the initial point of the vector OP while P is called the terminal point.



Notation of Vectors

Vectors ate generally represented by clarendon letters (letters in bold faced type) and their magnitudes by the corresponding italic letters. Thus we may denote $\overrightarrow{OP}$ by a and its magnitude by a. Since it is very inconvenient to show the difference between italic and bold faced letters in writing, we may use the Greek letters α, β, χ, δ etc. to represent vectors and the letters a, b, c, d etc. to represent their magnitudes. However, it is more convenient to represent vectors by $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ etc. and their magnitudes by a, b, c, d etc.

Modulus of a vector. The non-negative number which is the measure of the magnitude of a vector is called is modulus or module. Thus the length of the line segment OP is the modulus of $\overrightarrow{OP}$. The modulus a of a vector a is sometimes written as $+|a|$.

Q2. What are different kinds of vectors ?

Ans :

(i) Zero or Null Vector

The zero or the null vector is a vector whose mudulus is zero, and the whose direction is indeterminate. The null vector is represented by the symbol 0 (printed in bold faced typed). In the case of the null vector the initial and terminal points coincide. Thus $\overrightarrow{AA}$, $\overrightarrow{OO}$, etc. are null vectors.

(ii) Unit Vector

A vector whose, modulus is unity, is called a unit vector. The unit vector in the direction of vector a is represented by $\hat{a}$. It is read as 'a Cap'.

(iii) Like and Unlike Vectors

Vectors having the same direction are called like vectors and those having opposite directions are called unlike vectors.

(iv) Collinear or Parallel Vector

Vectors having the same line of action or having the lines of action parallel to one another are called collinear or parallel vectors.

(v) Equal Vectors

Two vectors are said to be equal if, and only if, they are parallel, have the sense of direction, and the same are called like vectors and those having opposite directions are called unlike vectors.

(vi) Collinear or Parallel Vectors

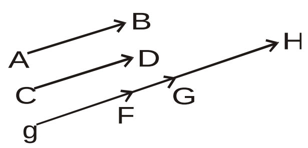
Vectors having the same line of action or having the lines of action parallel to one another are called collinear or parallel vectors.

(vii) Equal Vectors

Two vectors are said to be equal if, and only if, they are parallel, have the same sense of direction, and the same magnitude. The starting points of the vectors are immaterial. It is the direction of magnitude which are important. To denote the equality of vectors, the usual equality sign(=) is used. Thus, if a and b are equal vectors, we write $a = b$.

If the lines AB, CD and EFGH are parallel and AB = CD = EF = GH, then we have

$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{GH}$.



The equality of two vectors, are discussed above, does not mean that the quantities represented by a and b are equivalent in all respects. For example, if two equal forces (in the same directions) are applied at different points of a rigid body, they may have different medchanical effects.

(viii) Negative Vector

The vector which has the same modulus as the vector a but opposite direction, is called the negative of a.

The negative of a is represented by $-a$. Thus if $\overrightarrow{AB} = a$, then $\overrightarrow{BA} = -a$.

(ix) Co-initial Vectors

The vectors which have the same initial point are called co-initial vectors.

(x) Coplanar Vectors

The vectors which are parallel to the same plane or which lie in the same plane are said to be coplanar.

(xi) Localised and free vectors

A vector which is drawn parallel to a given vector through a specified point in space is called as localised vector. There can be one and only one such vector. But if the origin of vectors is not specified, the vectors are said to be free vectors.

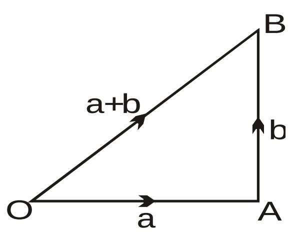
Q3. What are the properties of vector addition?

Ans :

Let a and b be any two given vectors. If three points O, A, B are taken such that $\overrightarrow{OA} = a$, $\overrightarrow{AB} = b$, then the vector $\overrightarrow{OB}$ (= say c) is called the vector sum or the resultant of the given vectors a and b and we write

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} \quad \text{or} \quad c = a + b$$

It should be noted that the terminal point of vector a is the initial point of vector b and the resultant vector c is the join of the initial point of a to the terminal point of b.



The above law of addition is known as the triangle law of addition.

Q4. Explain the Properties of Vector Addition.

Ans :

(i) Vector addition is commutative i.e., $a+b=b+a$, where a and b are any two vectors

Proof :

Let a and b be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{AB}$

Then by definition of addition of two vectors, we have

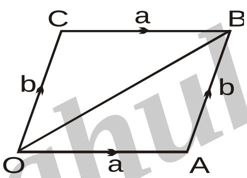
$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = a+b \qquad ...(1)$$

Complete the parallelogram OABC

We have by definition of equality of two vectors,

$$\overrightarrow{OC} = \overrightarrow{AB} = b$$

and $\overrightarrow{CB} = \overrightarrow{OA} = a$



Now by definition of addition of two vectors, we have

$$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = b+a$$

$\therefore$ From (1) and (2), we have

$a+b = b+a$, which proves the statement.

(ii) Vector addition is associative i.e,

$(a+b)+c = a+(b+c)$,

where a, b, c are any three vectors

Proof :

Let $\overrightarrow{OA} = a$, , $\overrightarrow{AB} = b$ and $\overrightarrow{BC} = c$

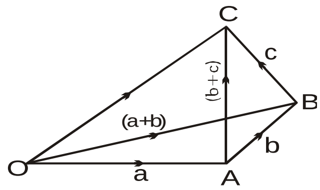
Complete the quadrilateral

OABC

We have, by definition of addition of two vectors,

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$

But by definition of addition of two vectors, we have



$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = b + c$$

$\therefore$ $\overrightarrow{OC} = \overrightarrow{OA} + (\overrightarrow{AB} + \overrightarrow{BC}) = a+(b+c)$ ...(1)

Again we have by definition of addition of two vectors,

$$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$$

But $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ [by definition of addition of two vectors]

$= a+b.$

$\therefore$ $\overrightarrow{OC} = (a+b)+c$ ...(2)

Hence from (1) and (2), we have

$a+(b+c) = (a+b+c$, which proves the statement.

Note :

From the above property we notice that the sum of three vectors a, b and c is independent of the order in which they are added. Hence this sum can be written as $a+b+c$ without any doubt.

(iii) For every vector a, $a+0=a$, where 0 is the zero vector

Proof :

Let $\overrightarrow{OA} = a$ and $\overrightarrow{AA} = 0$

We have by definition of addition of two vectors

$$\overrightarrow{OA} = \overrightarrow{OA} + \overrightarrow{AA} = a+0$$

$\therefore$ $a = a+0$, which proves the statement

(iv) To every vector a, there corresponds the vector –a such that $a+(-a)=0$, where 0 is the zero vector

Proof :

Let $\overrightarrow{OA} = a$; then $\overrightarrow{AO} = -a$

We have by definition of addition of two vectors

$$\overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO}$$

$\therefore$ $a+(-a) = 0$. Hence the result

Substraction of vectors

If a, b, be any two given vectors, then we write $a+(-b) = a - b$, and call the operation subtraction. Thus to subtract the vector b from a, reverse the direction of b and add it to a.

Q5. Explain about multiplication of a vector by a scalar.

Ans :

Let m be a scalar and a be a vector, then ma is defined as a vector whose modulus is $|m|$ times the modulus of the vector a and whose direction is that of the vector a or opposite to the vector a according as m is positive or negative.

1. From the above definition it is obvious that if two non-zero a and b are collinear, there exists a non-zero scalar m such that a = mb.

   Conversely if there exists a relation of the type a = mb between two non-zero vectors a and b, then the vectors a and b must be collinear.

2. If a denotes the modulus of a non-zero vector a, then the unit vector $\hat{a}$ in the direction of a is given by

$$\hat{a} = \frac{a}{a} \text{ or } \hat{a} = \frac{a}{|a|}$$

Since $a \neq 0$, therefore $|a| \neq 0$ and so $\frac{1}{|a|}$ is a positive real number. Now by the definition of multiplication of a vector by a scalar, $\frac{1}{|a|}a$ is a vector whose direction is that of the vector a. Again, by the same definition, modulus of the vector $\frac{1}{|a|}a = \frac{1}{|a|}$ times the modulus of a = $\frac{1}{|a|}a = 1$. Thus $\frac{1}{|a|}a$ is unit vector in the direction of the vector a.

Hence in order to obtain a unit vector in the direction of any given non-zero vector we are to divide that vector by its modulus i.e, we are to multiply that vector by the reciprocal of its modulus.

Q6. State the Properties of Multiplication of Vector by Scalars.

Ans :

(i) The scalar multiple of a vector satisfies associative law, i.e.,

$$m\ (na) = (mn)\ a = n\ (ma).$$

(ii) The scalar multiple of a vector satisfies the distributive laws;

i.e., $(m+n)\ a = ma+na$ ...(1)

and $m\ (a+b) = ma + mb$, ...(2)

where m and n are scalars and a and b are vectors.

To prove (1) let $m+n$ be positive. Then the L.H.S. of (1) represents a vector whose modulus is $(m+n)$ times the modulus of a and which points in the same direction as a. The R.H.S. of (1) represents the sum of two vectors of magnitudes $|m||a|$ and $|n|\ |a|$ pointing in the directions of a or opposite to a. The sum of these two vectors is a vector of magnitude $(m+n)\ |a|$ and pointing in the direction of a.

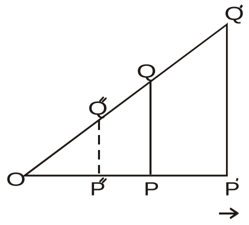
If $(m+n)$ is negative, then both the sides of (i) represent a vector of magnitude I $(m + n)|a|$ and pointing in the direction opposite to a.

Hence the result (1) follows.

Now we shall prove the result (2).

Let us first take m positive.

Let $\overrightarrow{OP} = a$ and $\overrightarrow{PQ} = b$



Then, by definition of addition of two vectors, we have

$$\overrightarrow{OQ} = a + b$$

Produce OP to P' such that mOP = OP'.

Through P' draw a line parallel to PQ in the sense of $\overrightarrow{PQ}$ to meet OQ produced in Q'. Then since the triangles OPQ and OP'Q' are similar, we have

$$\frac{OP'}{OP} = \frac{OQ'}{OQ} = \frac{P'Q'}{PQ} = m \quad [\because \ OP' = mOP]$$

$\therefore$ OQ' = mOQ and thus

$$\overrightarrow{OQ'} = m\,\overrightarrow{OQ} = m(a+b)$$

and $P'Q' = mPQ$ and thus

$$\overrightarrow{P'Q'} = m\,\overrightarrow{PQ} = mb$$

Now in triangle OP'Q', we have by definition of addition of two vectors

$$\overrightarrow{OQ'} = \overrightarrow{OP'} + \overrightarrow{P'Q'}.$$

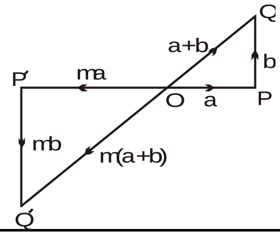
$\therefore$ $m(a+b) = ma + mb.$

In the above case we have taken m to be positive and >1. If m is positive and less than 1, then P'Q' will be nearer to O than PQ as shown in the figure by dotted line P"Q".

It case m is negative, we take points P' and Q' on PO and QO produced such that

$$\overrightarrow{OP'} = m\overrightarrow{OP} \text{ and } \overrightarrow{OQ'} = m\overrightarrow{OQ}.$$

Now the result can be proved as above.



Q7. Describe scalar product of two vectors.

Ans :

The scalar product or dot product of two vectors A and B is defined as the product of the magnitudes of two vectors (i.e., A and B) and the cosine of the angle between them.

$$A \cdot B = A\,B \cos\theta$$

where θ is the angle between two vectors

$$\therefore \quad \cos\theta = \frac{A \cdot B}{AB}$$

The product is a scalar quantity. The product may be positive or negative depending upon the angle θ. The product is negative when θ is between $\pi/2$ and $3\pi/2$.

$$A \cdot B = A\,B\cos\theta = A\,(B\cos\theta) = (A\cos\theta)\,B$$

Hence, the scalar product of two vectors will be equal to the product of the length of one vector and the length of the component of the second vector along the first vector direction.

Properties

1. The scalar product is commutative, i.e.,

   $A \cdot B = B \cdot A$

2. The scalar product follows the distributive law, i.e.,

   $A \cdot (B + C) = A \cdot B + A \cdot C$

3. The scalar product of a vector by itself is equal to the square of the scalar magnitude, i.e.,

   $A \cdot A = A^2$

4. The scalar product of two vectors vanishes when the vectors are at right angles, i.e.,

   $A \cdot B = A\,B\cos 90^\circ = 0$

5. When the vectors are parallel, the scalar product of two vectors is equal to the product of their scalar magnitudes

   $A \cdot B = A\,B\cos\theta = A\,B$

6. The scalar product of unit orthogonal vectors, i, j and k have the following relations.

$$i \cdot i = j \cdot j = k \cdot k = 1$$

$$i \cdot j = j \cdot k = k \cdot i = 0$$

7. The scalar product of two vectors is equal to the sum of the products of their corresponding x, y and z components.

If $A = i A_x + j A, k A$

and $B - i B_s + j B_y + k B_2$

then $A \cdot B = Ax\,Bx + Ay\,By + A_z B_z$

Examples :

1. if $A = B$, then $A \cdot B = A \cdot A = A^2 = |A|^2$

   Thus the vector dotted with itself gives the square of its magnitude. This provides a method of finding the magnitude of a vector.

2. If $A \cdot B = 0$ but $A \neq 0$ and $B \neq 0$.

   then the two vectors i.e., A and B are perpendicular to each other or the two vectors are orthogonal.

Example :

Deduce the angle made by the vector 4i – 3j + 5k with Z axis.

Sol :

By the definition of dot product, the angle θ between two vectors is given by

$$\cos\theta = \frac{A \cdot B}{AB}$$

Let the vector B be along the Z axis. Then B = B k

Now,

$$A \cdot B = (4i - 3j + 5k) \cdot (Bk)$$

$$= 4B\, i \cdot k - 3B\, j \cdot k + 5B\, k \cdot k$$

$$= 0 - 0 + 5B = 5B$$

$$A = |A| = \sqrt{(A_x^2 + A_y^2 + A_z^2)}$$

$$= \sqrt{[(14)^2 + (-3)^2 + (5)^2]} = 5\sqrt{2}$$

$$= [(0)^2 + (0)^2 + (B)^2] = B$$

$$\therefore \quad \cos\theta = \frac{5B}{(5\sqrt{2})(B)} = \frac{1}{\sqrt{2}}$$

$$\text{or} \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^o.$$

Q8. Discuss about vector product of two vectors.

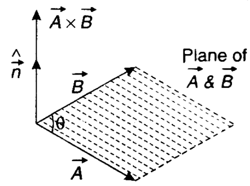
Ans :

The vector product or cross product of two vectors is defined as a vector having magnitude equal to the product of the magnitude of two vectors and the sine of the angle between them. The direction being perpendicular to the plane containing the two vectors.

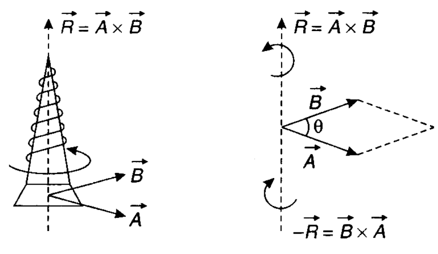
If A and B are two vectors then their vector product is A × B which can be expressed as

$$A \times B = AB \sin 0\ \hat{n}$$

where *A* and *B* are magnitudes of A and B, 0 is the angle between them and n is a unit vector perpendicular to the plane of A and B. The direction of A × B is given by the right hand rule.



According to this rule, if a right handed screw is imagined to be placed at the common tail point of the two vectors [whose vector product is to be obtained fig. (a) and is given a rotation from first vector A to second vector B. then the advancement of the screw gives the direction of resultant vector R = A × B. It is obvious from fig. (b), that if the screw is rotated from B to A. then the direction of advancement of the screw will be just opposite. Therefore



(a) (b)

B × A = vector having magnitude of R but opposite in direction

= – R.

Thus A × B ≠ B × A

Q9. Explain the Properties of Vector Product.

Ans :

i) The vector product is not commutative

A × B ≠ B × A

ii) The vector product is distributive

A × (B + C) = A × B + A × C

iii) The vector product of two parallel vectors is a null vector i.e.

$A \times B = AB \sin\theta\,\hat{n} = AB \sin 0\,\hat{n} = 0$

iv) The vector product of a vector by itself is a null vector (zero)

$A \times A = A, A \sin\theta\,\hat{n} = AA \sin 0\,\hat{n} = 0$

v) The magnitude of the vector product of two vectors mutually at right angle is equal to the product of the magnitude of the vectors.

$A \times B = AB \sin\theta\,\hat{n} = AB \sin 90°\,\hat{n} = AB\,\hat{n}$

vi) The vector product of unit orthogonal vectors, i, j and k have the following relations :

i × i = j × j = k × k = 0

i × j = –j × i = k,

j × v k = –k × j = i,

and k × i = –i × k = j

vii) The vector product of two vectors in terms of their x, y and z components can be expressed in the form of determinant.

If $A = iA, + j A_y + kA_Z$

and $B = i B_x + j B_y + k B_z$

then $A \times B = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

viii) The vector product of two vectors A and B gives the area of the parallelogram formed by the vectors. Let two vectors A and B are inclined at an angle θ as shown in figure. OPMQ is the parallelogram formed by the vectors A and B. QN is the perpendicular drawn from Q on OP. The area of parallelogram

= OP × perpendicular distance QN

= OP × OQ sin 6

= AB sin 0

= A × B.

Q10. Define scalar triple product of two vectors.

Ans :

The scalar product of a vector A with the vector product of two other vector B and C is termed as scalar triple product. This is written as

$A \cdot (B \times C)$

The scalar product $A \cdot (B \times C)$ gives a scalar. This gives the volume of the parallelopiped formed by vectors A, B and C.

Expressing vectors in their components forms, we have

$A = i A_x + j A_y + k A_z$

$B = i B_x + j B_y + k B_z$

and $C = i C_x + j C_y + k C_2$

$\therefore \quad A \cdot (B \times C) = A_x (B_y C_z - B_z C_y)$

$+ A_y (B_z C_x - B_x C_z)$

$+ A_z (B_x C_y - B_y C_x)$

In compact form, the above expression can be expressed as

$$A \cdot (B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Properties of Scalar Triple Product

1. If the scalar triple product is zero, then the three vectors are coplanar.

2. The following cylic relation hold gold in scalar triple product

$$A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B)$$

3. The dot and cross can be interchanged in a scalar triple product

$$A\cdot(B\times C) = (A\times B)\cdot C$$

4. In scalar triple product

$$A\cdot(B\times C) = -A\cdot(C\times B)$$

Q11. Define vector triple product. Write the properties of vector triple product.

Ans :

The vector product of a vector A with the vector produce of two other vector B and C is termed as vector triple produc t. This is written as

$$A \times (B\times C)$$

By the property of vectors products, $A\times(B\times C)$ is a vector perpendicular to A and also to $(B\times C)$. Further $(B\times C)$ is perpendicular to plane of B and C. Hence $A\times B(\times C)$ must be in the plane of B and C.

$$A \times (B\times C) = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ (B_y C_z - B_z C_y) & (B_z C_x - B_x C_z) & (B_x C_y - B_y C_x) \end{vmatrix}$$

Properties

i) $A\times(B\times C) \neq (A\times B)\times C$

The associative law does not apply to vector products.

ii) $A\times(B\times C) = B\,(A\cdot C) - C\,(A\cdot B)$

iii) $A\times(B\times C) = -(B\times C)\times A$

iv) $A\times(B\times C) + B\times(C\times A) + C\times(A\times C) = 0$

Q12. Explain the concept of vector area.

Ans :

We notice that, the vector product of two vectors A and B is another vector C and

$$C = A \times B = (AB\sin\theta)\,\hat{n}$$

where, θ is the angle between A and B and $\hat{n}$ is the unit vector in a direction perpendicular to the plane defined by A and B.

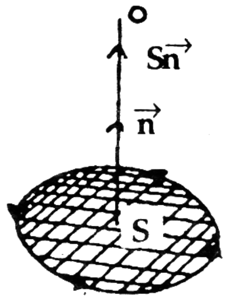
AB Sin θ is the magnitude of C and we know that this mag-nitude is equal to the area of the parallelogram formed with sides as A and B, having an angle 8 between them. The details that can be inferred from the vector product about the parallelogram are

a) The location of the parallelogram in space.

b) Relative position of one side with respect to the other, and

c) The area which has got a numerical value equal to the magnitude (length) of the vector C

Any plane area of magnitude. AB sin θ, with its positive unit normal n can be visualised to represent the vector product A x B. This idea leads us to the concept of vector area.



Any plane area such as S shown in the Fig. above can be considered to have got the Vector Properties, i.e., both magnitude and direction.

The magnitude is equal to just the area and the direction is that of the normal to the plane of the area. The sign of the vector area is by convention, defined with respect to the order in which the area is traced out when viewed from an external point as O. If the area is traced in the anticlockwise direction, the positive direction of vector area will be along the unit normal n. The direction of tracing the area S and the unit normal n are related to by the Right Handed Screw Rule.

The equation S = S n defines the vector area S

Vector areas can also be resolved into components and added together, just like all the other vector quantities.

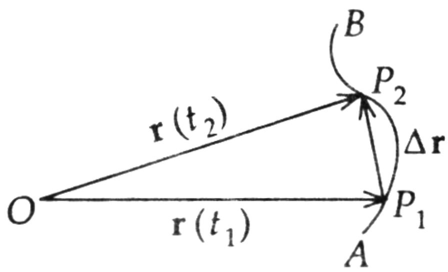
Vector areas are quite useful in physics. In enunciating the Gauss's Law in electro statics or relating the macroscopic current i to the microscopic current density J, the vector area concept is quite useful.

Q13. What is vector differentiation? Discuss the rules of vector differentiation.

Ans :

Any change in a vector involves both the change in the magnitude of the vector and also the change in its direction. Hence, a vector derivative is basically different from the ordinary derivative of a (physical) scalar quantity.

Let us consider a particle moving along the curved path AB. Let teh particle be at a point P_1 on the curve at time t_1, and be at P_2 at time t_2. With respect to the origin O, the position vector of P_1 is $\overrightarrow{OP}_1 = r(t_1)$ and to P_2 is $\overrightarrow{OP}_2 = r(t_2)$



The displacement of the particle in the time interval

$\Delta t = t_2 - t_1$ will be $\overrightarrow{\Delta r} = r(t_2) - r(t_1)$

This follows from the law of addition of vectors and nothing that as Δt is very small, Δr is also quite small. $\frac{\Delta r}{\Delta t}$ gives the rate of change of r. The direction of this ratio will be along P_1, P_2, that is, along the direction of Δr.

When Δt is very very small, that is $\Delta t \to 0$, Δr will be represented by dr and Δt by dt and $\frac{\Delta r}{\Delta t} \to \frac{dr}{dt}$ and is called the vector derivative of r. We know that dr/dt is nothing else but, the velocity v of the particle.

Hence $v = \frac{dr}{dt}$

We should note that, dr is a vector increment and hence involves both a change in magnitude, as well as change in its direction.

If r is expressed in its rectangular components as

$r = xi + yj + zk$

then the velocity $v = \frac{dt}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$ ...(1)

and acceleration $a = \frac{dv}{dt} = \frac{d^2r}{dt^2} = \left(\frac{d^2x}{dt^2}\right)i + \left(\frac{d^2y}{dt^2}\right)j + \left(\frac{d^2z}{dt^2}\right)k$ ...(2)

Rules

1. Differentitation of sum of two vectors

If C = A + B, then

$$\frac{dC}{dt} = \frac{dA}{dt} + \frac{dB}{dt}$$

i.e., $\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt}$

Vector differentiation is distributive

2. Differentiation of the scalar product of two vectors

If C = A · B, then

$$\frac{dC}{dt} = \left(\frac{dA}{dt}\right) \cdot B + A \cdot \left(\frac{dB}{dt}\right)$$

i.e., $\frac{d}{dt}(A \cdot B) = \left(\frac{dA}{dt}\right) \cdot B + A \cdot \left(\frac{dB}{dt}\right)$

Vector differentiation is commutative

Let us consider a special case where A is a constant vector that is, the magnitudeof A is a constant and B = A

Then $\frac{d}{dt}(A \cdot A) = \left(\frac{dA}{dt}\right) \cdot A + A \cdot \left(\frac{dA}{dt}\right)$

$$\frac{d}{dt}(A^2) = 2\left(\frac{dA}{dt}\right)\cdot A$$

But, as A is a constant, $\frac{d}{dt}(A^2) = 0$

Hence, $2\left(\frac{dA}{dt}\right) \cdot A = 0$ or $\left(\frac{dA}{dt}\right)\cdot A = 0$

That is, $\left(\frac{dA}{dt}\right)$ is perpendicular to A.

A is a constant defines the position of a point on the surface of a sphere of a radius a.

dA/dt gives the velocity and it is perpendicular to A. That is, when a particle moves on the surface of a sphere, its velocity will always be normal to the radius vector. This is obvious in circular motion.

3. Differentiation of vector product of two vectors

If $C = A \times B$, then

$$\frac{dC}{dt} = \left(\frac{dA}{dt}\right) \times B + A \times \left(\frac{dB}{dt}\right)$$

(i.e.,) $\frac{d}{dt}(A \times B) = \left(\frac{dA}{dt}\right) \times B + A \times \left(\frac{dB}{dt}\right)$

4. Differentiation of scalar tirple product

If $E = A \cdot (B \times C)$ (E is a scalar)

$$\frac{dE}{dt} = \left(\frac{dA}{dt}\right)\cdot(B\times C) + A \cdot \left(\frac{dB}{dt}\times C\right) + A\cdot\left(B\times\frac{dC}{dt}\right)$$

i.e., $\frac{d}{dt}[A\cdot(B\times C)] = \left(\frac{dA}{dt}\right)\cdot(B\times C) + A\cdot\left(\frac{dB}{dt}\times C\right) + A\cdot\left(B\times\frac{dC}{dt}\right)$

5. Differentiation of vector triple product

If $D = A \times (B\times C)$ then

$$\frac{d}{dt}[A\times(B\times C)] = \left(\frac{dA}{dt}\right)\times(B\times C) + A\times\left(\frac{dB}{dt}\times C\right) + A\times\left(B\times\frac{dC}{dt}\right)$$

In general, vector differentiation follows the same rules as of ordinary differential calculus.

But, here we should be cautious enough to note that a vector product foes not follow the commutative law.

1.2 Scalar and Vector Fields

Q14. Explain in detail about scalar and vector fields.

(OR)

What are scalar and vector fileds?

Ans : **(Dec.-19, June-18(KU), Dec.-19(KU)**

We know that a physical quantity can be expressed as a continuous function of the position of a point in the region of space. For example when a rod is heated at one end, then there is a variation of temperature along the length of the rod. The physical quantity temperature at any point (x, y, z) can be expressed by a continuous function T (x, y, z). Such a function is termed as a point function or function of position. The region specifying that physical quantity is labelled as its field. Depending upon the nature of physical quantity the field may be scalar or a vector.

I) Scalar Field

If a scalar physical quantity is assigned to each point in space then we have a scalar field in that region of space. The scalar field in three dimensions can be represented by a scalar point function ϕ(x, y, z).

Example

The electronic potential due to a single positive charge q columns depends on the position of the point from the charge. Then $V_0(x_0, y_0, z_0)$ and $V(x_1, y_1, z_1)$ are the scalar point functions at (x_0, y_0, z_0) and (x_1, y_1, z_1). Now the region is a scalar field.

In a scalar field, there are a number of surfaces are known as level surfaces.

The level surfaces, equipotential (gravitational or electrostatic) etc.

For example, if we imagine sphere of different radii taking a point charge q as the centre, then we get equipotential level surfaces. These are spherical in nature and at any level surface the scalar point function V (x, y, z) has a constant value. If the level surface are parallel to each other, then the scalar field is called as stationary scalar field.

The concept of a scalar field can easily be understood with the help of the following examples:

i) Consider a solid block of material whose faces are maintained at different temperatures. Now the temperature of the body will vary from point to point, i.e., temperature will be a function of position coordinates x, y, z in rectangular coordinate system. Hence, temperature is a scalar field.

ii) The density at any point inside a body occupying given region is a scalar field. The electrical potential is differeent at different points. Hence, electric potential is scalar field.

II) Vector Field

When a vector physical quantity is expressed from point to point in the region of space by a continuous vector function A(x, y, z) then the region is a vector field. The example of vector field are gravitiational, magnetic, electric intensity.

The vector point function at any point in the field is given by a vector having unique value for a magnitude and direction.

Both magnitude and direction change continuously from point to point throughout the field region. Starting from any desired point in the field and processing throughout infinitesimal distances from point to point in the direction of field, we obtain a curved line. So the field can be mapped out by curved lines known as flux lines or lines of flow.

The tangent at any point of a line gives the direction of A at the point. The magnitude of A at any point of on a flux line is given by the number of flux line crossing unit area perpendicular to their direction drawn at any point.

The number of lines passing through unit area of the surface perpendicular to their direction is called as magnetic flux. The magnetic flux depends on the distance of the point from the magnetic pole. In this way, the magnetic flux at that point. When the flow lines are parallel to each other, then the vector field is called as stationary vector field.

1.2.1 Gradient of a Scalar Field and Physical significance

Q15. Define gradient of a scalar field funct-ion. Explain the physical significance for the gradient of a scalar field.

(OR)

Explain gradent of a scalarfield.

(OR)

Define gradient of a scalars filed obtain an expression for it.

Ans : **(Dec.-19(MGU), Dec.-18, Dec.-18(KU), Dec.-17(MGU), Dec.-16)**

In order to consider the gradient of a scalar, let $\phi(x, y, z)$ be a scalar function of position of a scalr point of coordinates (x, y, z). The partial derivatives of ϕ along the three coordinate axes are

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \text{ and } \frac{\partial\phi}{\partial z}$$

The gradient of a scalar function ϕ is defined as

$$\text{grad } \phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \quad \text{...(1)}$$

We know that vector differential operator ∇ (del) is defined as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$\therefore \quad \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \quad \text{...(2)}$$

The equation (2) is the same as equation (1). It is obvious from equation (2) that del operator (∇) is a vector operator and when operated with a scalr (ϕ) converts the scalar into a vector. The vector $(\nabla\phi)$ is called the gradient of the scalar.

The gradient is a different operator by means of which we can associate a vector field with a scalar field. For example, the intensity of electric field E, (a vector quantity) is the gradient of potential V (a scalar quantity) with a negative sign, i.e.,

$$E = -\text{grad } V$$

The negative sign indicates that the direction of field intensity is opposite to the direction of increasse of potential.

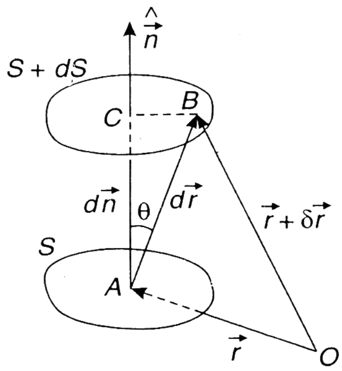
Let S(x, y, z) be a scalar point function depending on the three cartesian coordinates in space. Suppose $\partial S/\partial x, \partial S/\partial y$ and $\partial S/\partial z$ be the partial derivatives along the three perpendicular axes respectively. Now the gradient of the scalar functioon S can be expressed as

$$\text{grad } S = i\frac{\partial S}{\partial x} + j\frac{\partial S}{\partial y} + k\frac{\partial S}{\partial z}$$

$$\text{or} \quad \text{grad } S = \nabla S \text{ where } \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Physical significance for the gradient of a scalar field

Consider two such surfaces, very close to each other, as shown in fig. (1). These surfaces-are specified by constant scalar function S and S + dS respectively. Let A and B are two points on these surfaces and O is a point outside the surfaces. Let r and r + dr be the radius vectors of points A and B with respect to origin O respectively. The vector drawn from A to B will be dr ($\because$ AB = OB – OA = r+dr – r = dr). Let the normal drawn from the point A of surface S is AC and $\hat{n}$ is the unit vector along the normal. Now certainly AC will be minimum distance between the two surfaces. Suppose the angle between AB and AC is θ.



Now the magnitude of the rate of increase of S at point A in the direction AB is $\partial S/\partial r$. Similarly, the rate of increase of S in the direction AC is $\partial S/\partial n$. From ΔABC, AC = AB cos θ or $\partial n = \partial n\ \partial r$ cos θ

$$\therefore \quad \frac{\partial S}{\partial n} = \frac{\partial S}{\partial r\cos\theta} \quad \text{or} \quad \frac{\partial S}{\partial r} = \frac{\partial S}{\partial n}\cos\theta$$

It is obvious from eq. (1) that $\partial S/\partial r$ is maximum when $\theta = 0$ because now cos θ=1. So the maximum value of $\partial S/\partial r$ is $\partial S/\partial n$. In this way the maximum rate of increase of a scalar function S at any point in a scalar field is given, in magnitude and direction by the vector

$$\frac{\partial S}{\partial n}\hat{n}$$

where $\hat{n}$ is the unit normal vector at that point. The vector is defined as the gradient of the scalar field S at that point and is written as

$$\text{grad } S = \frac{\partial S}{\partial n}\hat{n}$$

Thus the gradient of a scalar field S is a vector whose magnitude at any point is equal to the maximum rate of increase of s at that point and whose direction is along the normal to the level surface at that point. This gives the physical significance of gradient of scalar field.

1.2.2 Divergence of a Vector Field

Q16. What is called divergence? Derive expression for divergence of a vector field.

(OR)

Explain divergence of a vector field and its physical significance.

(OR)

Show that the dNA $\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$ where A is a vector filed.

Ans : **(Aug.-21, June-18(KU), June-17, Dec.-19(KU), Dec.-16)**

The operator ∇ can be involved in the multiplication with a vector. The scalar or dot product of operator ∇ with a vector A (i.e., $\nabla \cdot A$) is called as divergence. The divergence of a vector field at any point is defined as the amount of flux per unit volume diverging from that point. The divergence is a scalar.

Let A be a vector function differentiable at each point (x, y, z) in a region of space. Now the divergence of A is given by

$$\nabla \cdot A = \left(i\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(i\,A_x + j\,A_y + k\,A_z)$$

$$= \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$

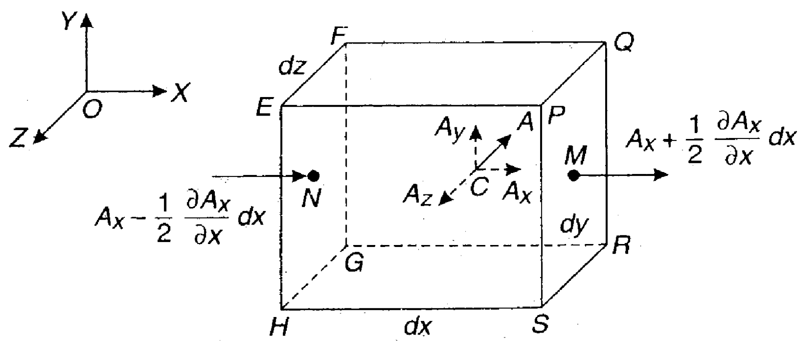
$$\therefore \quad \text{div } A = \nabla \cdot A = \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$

This is the expression of divergence in cartesian coordinates.

Expression for Divergence of a Vector Field

Consider a small rectangular parallelopiped in vector field as shown in fig. below. Let dx, dy and dz be the lengths of the sides of parallelopiped parallel to the coordinate X, Y and Z axes respectively.

Let a vector A represents the velocity of a fluid centre C with components A_x, A_y and A_z along the three axes respectively. Let $\partial A_x/\partial x$ represents the rate of change of A_x along the X-axis. Similarly $\partial A_y/\partial y$ and $\partial A_z/\partial z$ will be the rate of change of A_y and A_z along Y and Z-axes respectively.



The value of A_x at the centre M of face PQRS

= value of A_x at the centre C + increase in magnitude from C to M

= value of A_x at centre + rate of change × distance

$$= A_x + \frac{\partial A_x}{\partial x} \times \frac{dx}{2} = A_x + \frac{1}{2}\frac{\partial A_x}{\partial x}dx$$

Similarly the magnitude at the centre N of face EFGH

$$= A_x - \frac{1}{2}\frac{\partial A_x}{\partial x}dx$$

The negative sign is taken because N is towards left of C.

We know that the value of the fluid flowing per unit time through a face is equal to the product of the area of the face and normal component of the vector upon it. This is known as flux through the face. Hence

Flux entering the face EFGH $= \left(A_x - \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right) dy\,dz$

where dy dz is the area of face EFGH

and Flux leaving the face PQRS $= \left(A_x + \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right) dy\,dz$

The excess of flux leaving the parallelopiped over that entering in x-direction is given by

$$= \left(A_x + \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right) dy\,dz - \left(A_x - \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right) dy\,dz$$

$$= \frac{\partial A_x}{\partial x}dx\,dy\,dz$$

Similarly, the next flux leaving the paralleopiped in Y and Z-directions are

$$\frac{\partial A_y}{\partial y}dx\,dy\,dz \text{ and } \frac{\partial A_z}{\partial z}dx\,dy\,dz$$

$\therefore$ Total flux leaving or diverging from paralleopiped

$$= \frac{\partial A_x}{\partial x}dx\,dy\,dz + \frac{\partial A_y}{\partial y}dx\,dy\,dz + \frac{\partial A_z}{\partial z}dx\,dy\,dz$$

$$= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dx\,dy\,dz$$

Here dx dy dz is the volume of the elementary parallelopiped. Hence the amount of flux diverging per unit volume

$$= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

This is defined as divergence of A.

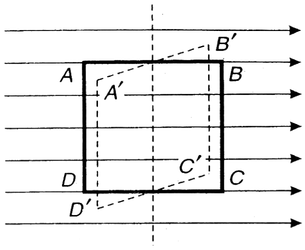
Thus $\text{div } A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$.

1.2.3 Curl of a Vector Field

Q17. What is curl of a vector field ? Obtain expression for curl of a vector field.

Ans : **(Dec.-19, Dec.-19(MGU), Dec.-18(KU)**

Consider two areas ABCD and A' B' C' D' in a uniform electric field which is represented by straight parallel lines as shown in fig. The area A' B' C' D' is perpendicular to lines of force. So the contribution of line integrals is zero. For area ABCD, the line integrals along AD and BC are zero while the line integrals along AB and CD are not zero. This shows that there is a certain orientation of the area for which the line integral is maximum.



The curl of a vector field is defined as the maximum line integral of the vector per unit area. It is essentially a vector quantity. The direction is normal to the area.

If A is a vector function differentiate at each point (x, y, z) in a region of a space, then the curl (or rotation) of A expressed by the cross product of ∇ and A, i.e.,

$$\text{Curl A} = \nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Physical Significance of a Curl

The curl is a closed line integral per unit area as the area shrinks to a point. It gives the circulation per unit area i.e., circulation density of a vector about a point at which the area is going to shrink. Thus curl of a vector at a point quantifies the circulation of a vector around that point. In general if there is no rotation, there is no curl while large angular velocities means greater values of curl. The curl also gives the direction, which is along the axis through a point at which curl is defined.

The magnetic field lines produced by the current carrying conductor are rotating in the form of concentric circles around the conductor. Thus there exists a curl of magnetic field intensity which we have defined as $\nabla \times \bar{H}$. The direction of curl is along the axis about which rotation of a vector field exists and the proper direction is to be obtained by right handed screw rule. If the direction of rotation of vector field about a point reverses, the sign of the curl also reverses.

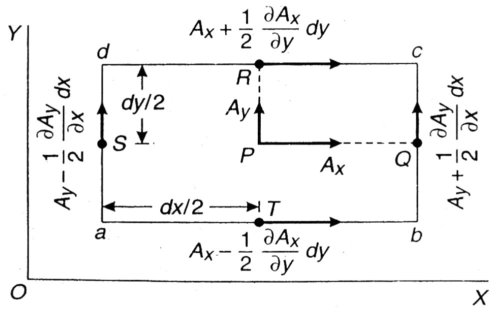
The water velocity in a river which increases linearly towards the surface, the magnetic field lines due to current carrying conductor, the body rotating about a fixed axis are few examples of a curl.

Thus if curl of a vector field exists then the field is called rotational. For irrotational vector field, the curl vanishes i.e. curl is zero.

Another physical interpretation of a curl is about a rigid body rotating about a fixed axis with uniform angular velocity. Thus if *v* is its linear velocity then its angular velocity (ω) is half the curl of its linear velocity. The curl *v* represents the net rotation of a body about the axis.

Expression for Curl of a Vector Field

Consider a small rectangular area abcd in a vector field along X-Y plane as shown in fig. Let dx and dy be the sides of the rectangular area parallel to X and Y axes respectively. Further let the value of vector field at the centre P be A and its components are A_x and A_y along X and Faxes respectively.



Now we shall consider the values of these components at the middle points Q, R, S and T. It should be remembered that the rate of change of A_x along y-axis can be represented by $\partial A_x/\partial y$ and similarly the rate of change of A_y along x-axis can be represented by $\partial A_y/\partial x$.

The increase in magnitude of A_x in going from P to R will be

$$= \frac{\partial A_x}{\partial y}(PR) = \frac{\partial A_x}{\partial y}\left(\frac{1}{2}dy\right) = \frac{1}{2}\frac{\partial A_x}{\partial y} dy$$

The value of A_x at $R = V_x + \frac{1}{2}\frac{\partial A_x}{\partial y} dy$ ... Eqn. (1)

Similarly,

The value of A_x at $T = A_x - \frac{1}{2}\frac{\partial A_x}{\partial y} dy$ ... Eqn. (2)

The value of A_y at $T = A_y + \frac{1}{2}\frac{\partial A_y}{\partial x} dx$ ... Eqn. (3)

The value of A_y at $S = A_y - \frac{1}{2}\frac{\partial A_y}{\partial x} dx$ ... Eqn. (4)

Now the line integral along the boundary abcd

= ab × (component of the vector along ab) + bc × (component of the vector along be) + cdx (component of the vector along cd) + da × (component of vector along da).

$$= \left(A_x - \frac{1}{2}\frac{\partial A_x}{\partial y}dy\right)dx + \left(A_y + \frac{1}{2}\frac{\partial A_y}{\partial x}dx\right)dy - \left(A_x + \frac{1}{2}\frac{\partial A_y}{\partial y}\right)dx - \left(A_y - \frac{1}{2}\frac{\partial A_y}{\partial x}dx\right)dy$$

The last two line integrals are taken as negative because the components of the vectors are in opposite directions.

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)dx\,dy \qquad \text{... Eqn. (5)}$$

The line integral per unit area $= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$ ... Eqn. (6)

($\because$ area of the rectangle is dx dy)

By definition, eq.(6) represents the magnitude of the component of curl A taken about Z-axis. Hence, we may write

$$\text{curl}_z\, A = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)k \qquad \text{... Eqn. (7)}$$

where k is unit vector along the Z-axis.

Similarly, the components of curl A about Y and X axes shall be

$$\text{curl}_y\, A = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)j \qquad \text{... Eqn. (8)}$$

and $$\text{curl}_x\, A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)i \qquad \text{... Eqn. (9)}$$

Adding eqs. (7),(8),(9) we get

$$\text{curl } A = i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \qquad \text{... Eqn. (10)}$$

In the determinant form eq.(10) can be expressed as

$$\text{curl } A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}.$$

Curl in Various Coordinate Systems

As the curl of $\bar{A}$ i.e, $\nabla \times \bar{A}$ is a cross product it can be expressed in determinant from in various co-ordinate systems.

1. Cartesian Co-ordinate System

$$\nabla \times \bar{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \begin{matrix} \leftarrow \nabla \\ \\ \leftarrow \bar{A} \end{matrix}$$

$$= \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\bar{a}_z$$

2. Cylindrical co-ordinate system

$$\nabla \times \bar{A} = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_x & rA_\phi & A_z \end{vmatrix}$$

$$= \left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\bar{a}_\phi + \left[\frac{1}{r}\frac{\partial (rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right]\bar{a}_z$$

3. Spherical co-ordinate system

$$\nabla \times \bar{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a} & r\sin\theta\bar{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\nabla \times \bar{A} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial A_\phi \sin\theta}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right]\bar{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial (r\,A_\phi)}{\partial r}\right]\bar{a}_\theta + \frac{1}{r}\left[\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\bar{a}_\phi$$

Q18. What are the properties of a curl?

Ans :

The various properties of curl are,

1. The curl of a vector is a vector quantity
2. $\nabla \times (\bar{A} + \bar{B}) = \nabla \times \bar{A} + \nabla \times \bar{B}$
3. $\nabla \times \nabla \times \bar{A} = \nabla (\nabla . \bar{A}) - \nabla^2 \bar{A}$
4. The divergence of a curl is zero

   $\nabla * (\nabla \times \bar{A}) = 0$
5. The curl of a scalar makes no sense

   $\nabla \times \alpha =$ No sense if α is scalar.
6. The curl of gradient of a vector is zero.

   $\nabla \times \nabla V = 0$
7. $\nabla \times \bar{A} \times \bar{B} = \bar{A}(\nabla * \bar{B}) - \bar{B}(\nabla * \bar{A}) * (\bar{B} * \nabla) A - (\bar{A} * \nabla)\bar{B}$

1.3 VECTOR INTEGRATION

1.3.1 Line, Surface and Volume Integrals

Q19. What are line, surface and volume integrals? Explain.

(OR)

Write a short note on vector integration

(OR)

Explain line, surface and volume.

Ans : **(Dec.-19, Dec.-18, Dec.-(MGU), Dec.-18(KU))**

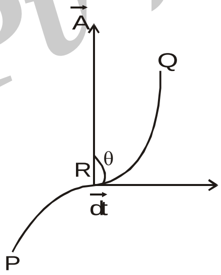
The integrals which are commonly used are :

1. Line integral
2. Surface integral and
3. Volume integral

1. Line Integral

Integral $\int_P^Q A \cdot dl$ is defined as the line integral of A along the curve PQ.

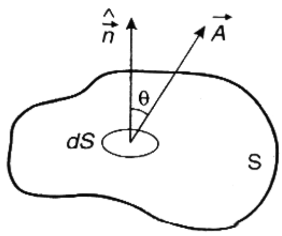
If A denotes the electric field intensity at any point, then the line integral represents the potential difference between P and Q.



2. Surface Integral

Consider a simple surface S in a vector field bounded by a curve as shown in fig. Let dS be an infinitesimal element of the surface. This surface element of area dS can be represented by area vector dS. If $\hat{n}$ be a unit positive vector (drawn outward the surface) in the direction of dS, then

$$dS = \hat{n}\ dS$$



Let A be a vector at middle of the element dS in the direction making an angle θ with $\hat{n}$. Now the scalar product

$$A \cdot dS = A \cdot n\, dS = A\, dS \cos\theta$$

is called the flux of vector field A across the area element dS. The total flux of the vector field across the entire surface area S is given by

$$\iint_s A \cdot dS = \iint_s A \cdot n\, dS = \iint_s A \cos\theta\, dS$$

This is defined as the surface integral.

3. Volume Integral

The integral evaluated over a three dimensional domain is known as volume integral.

Consider a closed surface in space enclosing a volume V. If A be a vector point function at a point in a small element of volume dV, then the integral

$$\iiint_V A\, dV$$

is called the volume integral of vector A over the surface.

1.4 STOKE'S THEOREM

Q20. State and prove stoke's theorem.

Ans : **(June-19, June-18, Dec.-16)**

Statement

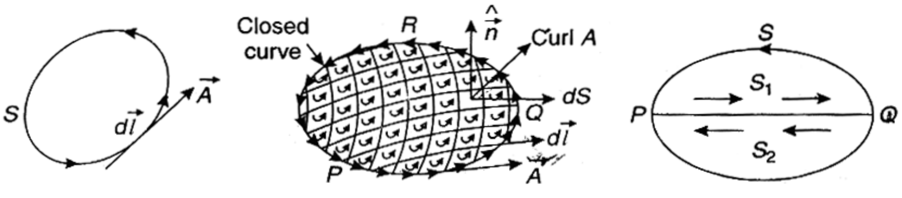
The line integral of a vector field A around a closed curve is equal to the surface integral of the curl of A taken over the surface S surrounded by the closed curve, i.e.,

$$\oint_C A \cdot dl = \iint_S \text{curl } A \cdot dS = \iint_S (\nabla \times A)\, dS$$

Proof

Consider a surface S enclosed in a vector field A. The line integral of A around the curve PQR traced counter - clockwise is

$$\oint_C A \cdot dl$$



Let the entire surface be divided into a large numbr of square loops. Suppose $\hat{n}$ be a unit positive outward normal upon dS. The vector area of the element is

$$\hat{n}\, dS = dS$$

So the line integral of A around the boundry of the area dS is

$$\text{curl } A \cdot dS$$

This applies to all surface element. Hence the sum of the line integrals of A around the boundaries of all the area elements is given by

$$\iint_S \text{curl } A \cdot dS$$

This is the surface integral of A.

The sum of the line integrals on the boundary line of the curve is given by eq. (2) This is also given by eq. (1). Hence

$$\oint_C A \cdot dl = \iint_S \text{curl } A \cdot dS = \iint_S (\nabla \times A) \cdot dS$$

This is Stoke's theorem.

From eq. (3), Stokes theorem gives a method to convert a surface integral into a line integral and vice versa. When curl A is zero, the line integral of A over the closed path is zero and hence the field is conservative.

1.5 Gauss's Theorem

Q21. State and prove Gauss's divergence theorem.

Ans : **(Aug.-21, Dec.-19, June-19, June-18(KU, June-17)**

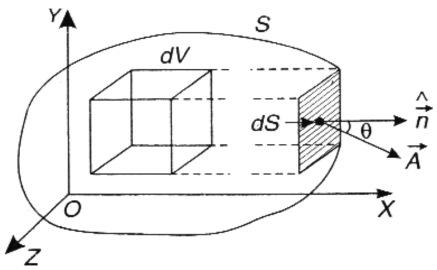
Statement :

The surface integral of the normal component of vector A taken over a closed surface S is equal to the volume integral of the divergence of vector A over the volume V enclosed by the surface S, i.e.,

$$\iint_S A \cdot dS = \iiint_V \text{div } A \, dV = \iiint_V (\nabla \cdot A) \, dV$$

Proof :

Let us consider a closed surface S of any arbitrary shape drawn in a vector field A as shown in figure below.



Let the surface encloses a volume V.

We know that div A represents the amount of flux diverging per unit volume and hence the flux diverging from the element of volume dV will be div A dV.

So the total flux coming out from the entire volume is given by

$$\iiint_V \text{div } A \, dV$$

Now we consider a small element of area dS on the surface S as shown in fig. Let $\hat{n}$ represents the unit vector drawn normal to area dS. It should be remembered that outward drawn normal on a surface is taken as positive. If the field vector A and outward normal $\hat{n}$ are at an angle θ, then the component of A along $\hat{n}$ is

$$A\cos\theta = A \cdot \hat{n}$$

The flux of A through the surface element dS is given by

$$(A \cdot \hat{n})\, dS = A \cdot dS$$

So the total flux through the entire surface S is given by $\iint_S A \cdot dS$ ... (2)

This must be equal to the total flux diverging from the whole volume V enclosed by the surface S. Hence from eqs. (1) and (2) we get

$$\iint_S A \cdot dS = \iiint_V \text{div } A\, dV \qquad \text{... (3)}$$

This is Gauss theorem of divergence and may also be written as

$$\iint_S (A \cdot \hat{n})\, dS = \iiint_V (\nabla \cdot A)\, dV \qquad \text{... (4)}$$

(i) Gauss divergence theorem provides a relation between surface and volume integrals.

(ii) This theorem is applicable for closed surface only.

1.6 Green's Theorem

Q22. State and explain Green's Theorem applications.

Ans : **(June-17, Dec.-16)**

Statement :

If ϕ and ψ are two scalar point functions such that these functions and their first derivatives are continuously differentiable, in a region bounded by a closed surface S, then we have

$$\iiint_V (\phi\nabla^2\psi + \nabla\phi.\nabla\psi)\, dV = \iint_S (\phi\nabla\psi).dS \qquad \text{... (1)}$$

and

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \iint_V (\phi\nabla\psi - \psi\nabla\phi).ds \qquad \text{... (2)}$$

These equations are known as first and second form of Green's theorem.

Proof :

Let us take the following mathematical from Gauss's divergence theorem

$$\iiint_V \text{div } A\, dV = \iint_V A.dS$$

Because $\nabla\psi$ will also be a vector quantity, $\phi\nabla\psi$ will be a vector quantity.

Let this is represented by vector A. Thus

$$A = \phi\nabla\psi$$

or $$i A_x + j A_y + k A_z = \phi\left(i\frac{\partial\psi}{\partial x} + j\frac{\partial\psi}{\partial y} + k\frac{\partial\psi}{\partial z}\right)$$

From this equation we can see that $A_x = \phi\frac{\partial\psi}{\partial x}$, $A_y = \phi\frac{\partial\psi}{\partial y}$ and $A_z = \phi\frac{\partial\psi}{\partial z}$

Now $$\text{div } A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Substituting the values of A_x, A_y and A_z, we get,

$$\text{div } A = \frac{\partial}{\partial x}\left(\phi\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\phi\frac{\partial\psi}{\partial z}\right)$$

$$= \left(\phi\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\phi}{\partial x}.\frac{\partial\psi}{\partial x}\right) + \left(\phi\frac{\partial^2\psi}{\partial y^2} + \frac{\partial\phi}{\partial y}.\frac{\partial\psi}{\partial y}\right) + \left(\phi\frac{\partial^2\psi}{\partial z^2} + \frac{\partial\phi}{\partial z}.\frac{\partial\psi}{\partial z}\right)$$

$$= \phi\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}.\frac{\partial^2\psi}{\partial z^2}\right) + \frac{\partial\phi}{\partial x}.\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}.\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial z}.\frac{\partial\psi}{\partial z} = \phi\nabla^2\psi + \nabla\phi.\nabla\psi \quad ..(4)$$

Substituting the value of div A from equation (4) in equation (3), we get

$$\iiint_V (\psi\nabla^2\phi + \nabla\psi.\nabla\phi)dV = \iint_S (\phi\nabla\psi).\nabla S \quad \text{... (5)}$$

This is known as Green's first theorem.

Interchanging ϕ and ψ in equation . (a), we have

$$\iiint_V (\psi\nabla^2\phi + \nabla\psi.\nabla\phi)dV = \iint_S (\psi\nabla\phi).\nabla S \quad \text{... (a)}$$

Subtracting equation (5) from equation (a), we get,

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi).dS \quad \text{... (b)}$$

This is known as Green's second theorem.

PROBLEMS

1. Compute A × B and B × A, when A = 10i – 6j and B = –4i + 3j

Sol :

We know that

$$A\times B = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} i & j & k \\ 10 & -6 & 0 \\ -4 & 3 & 0 \end{vmatrix}$$

$$= k\,[10 \times 3 - (-6)(-4)] = 6\,k$$

Further $A \times B = -B \times A$

$\therefore$ $B \times A = -A \times B = -6\,k$

---

2. If A and B are two vectors given by $A = i\,A_x + j\,A_y + k\,A_z$

$B = i\,B_x + j\,B_y + k\,B_z$

show that, A × B = – B × A

Sol :

We know that

$$A \times B = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= -\begin{vmatrix} i & j & k \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix}$$

($\because$ Interchange of rows in determinant produces a negative sign)

$= -B \times A.$

---

3. Prove that $|A\times B|^2 + |A \cdot B|^2 = A^2B^2$

Sol :

We know that

$|A \times B| = A\,B \sin\theta$

and $|A \cdot B| = A\,B\,\cos\theta$

$\therefore$ $|A\times B|^2 + |A\cdot B|^2 = (A\,B \sin\theta)^2 + (A\,B\cos\theta)^2$

$= A^2\,B^2\,[\sin^2\theta + \cos^2\theta] = A^2\,B_2$

4. Express the scalar triple product A·(B×C) in terms of the cartiesian components of the vectors A, B and C.

Sol :

First of all, we express the vectors A and B×C in their cartesian components. Thus

$$A\cdot(B\times C) = (i\,A_x + j\,A_y + k\,A_z)\cdot \begin{vmatrix} i & j & k \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (i\,A_x + j\,A_y + k\,A_z)\cdot [i\,(B_y C_z - B_z C_y) + j\,(B_z C_x - B_x C_z) + k\,(B_x C_y - B_y C_x)]$$

$$= [A_x (B_y C_z - B_z C_y) + A_y (B_z C_x - B_x C_z)] + A_z (B_x C_y - B_y C_x)$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

5. A paraellopiped has edges described by the vectors i + 2j, 4j and j+3k. Find its volume.

Sol :

Volume of parallelopiped = A · (B×C)

Further, $$A\cdot(B\times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Given that, A = i + 2j, i.e., $A_x = 1, A_y = 2$ and $A_z = 0$

B = 4j i.e, $B_x = 0, B_y = 4$ and $B_z = 0$

C = j + 3k i.e., $C_x = 0$, $C_y = 1$ and $C_z = 3$

$$\therefore \quad A\cdot(B\times C) = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 12$$

6. Prove that if the scalar triple product of three vectors vanishes, the vectors are coplanar.

Sol :

A · (B×C) = 0, i.e.,

Given that, A · (B×C) = 0, i.e.,

Volume of parallelopiped = 0

So, the vectors A, B and C must be in the same plane.

7. If A = 2i – 3j + k, B = 3i + 4j and C = 3i + 6k, deduce the values of :
i) A · (B×C) and ii) (A×B) · C

Sol :

i) $A \cdot (B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

$= \begin{vmatrix} 2 & -3 & 1 \\ 3 & 4 & 0 \\ 3 & 0 & 6 \end{vmatrix}$

$= 2\,(24 - 0) - (-3)(18 - 0) + 1\,(0 - 12) = 90$

ii) We know that

$(A \times B) \cdot C = A \cdot (B \times C)$

$(A \times B) \cdot C = 90$

8. Express the scalar triple product A · (B × C) in terms of the cartiesian components of the vectors A, B and C.

Sol :

$A \cdot (B \times C) = (i\,A_x + j\,A_y + k\,A_z) \cdot \begin{vmatrix} i & j & k \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

$= (i\,A_x + j\,A_y + k\,A_z) \cdot [i\,(B_y C_z - B_z C_y) + j\,(B_z C_x - B_x C_z) + k\,(B_x C_y - B_y C_x)]$

$= [A_x (B_y C_z - B_z C_y) + A_y (B_z C_x - B_x C_z)] + A_z (B_x C_y - B_y C_x)$

$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

9. A parallelopiped has edges described by the vectors i + 2j, 4j and j + 3 k. Find its volume.

Sol :

Volume of parallelopiped = A · (B × C)

Further $A \cdot (B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

Given that, $A = i + 2j$, i.e., $A_x = 1, A_y = 2$ and $A_z = 0$

$B = 4j$ i.e., $B_x = 0, B_y = 4$ and $B_z = 0$

$C = j + 3k$ i.e., $C_x = 0, C_y = 1$ and $C_z = 3$

$$\therefore \quad A \cdot (B \times C) = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 12$$

10. Prove that if the scalar triple product of three vectors vanishes, the vectors are coplanar.

Sol :

$A \cdot (B \times C)$ = Volume of parallelopiped formed by them

Given that, $A \cdot (B \times C) = 0$, i.e.,

Volume of parallelopiped = 0

So, the vectors A, B and C must be in the same plane.

11. If $A = 4i - 5j + 3k$, $B = 2i - 10j - 7k$ and $C = 5i + 7j - 4k$, then deduce,

i) $A \times (B \times C)$ and ii) $(A \times B) \times C)$

Sol :

(i) $(A \cdot C) = 20 - 35 - 12 = -27$

and $(A \cdot B) = 8 + 50 - 21 = 37$

$\therefore \quad A \times (B \times C) = (2i - 10j - 7k) \times (-27) - (5i + 7j - 4k) \times (37)$

$= (-54i + 270j + 189k) - (185i + 259j - 148k)$

$= 239i + 11j + 337k$

(ii) $(A \times B) \times C) = -C \times (A \times B)$

$= [-A(C \cdot B) - B(C \cdot A)]$

$= B(A \cdot C) - A(B \cdot C)$

$(A \cdot C) = (20 - 35 - 12) = -27$

$(B \cdot C) = (10 - 70 + 28) = -32$

$\therefore \quad (A \times B) \times C) = (2i - 10j - 7k)(-27) - (4i - 5j + 3k)(-32)$

Solving we get $= 74i + 110j + 285k$.

12. Find the directional derivatives of a scalar point function f in the direction of coordinate axes.

Sol :

The grad f at any point (x, y, z) is the vector.

But grad $f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

The directional derivative of f in the direction of i.

$$= \text{grad } f \,.\, i$$

$$= \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \right) i = i^2 \frac{\partial f}{\partial x}$$

$$= \frac{\partial f}{\partial x}$$

Similarly, the directional derivatives of f in the direction of j and k are $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ respectively.

13. Find the directional derivative of $f(x, y, z) = x^2yz + 4xz^2$ at the point (1, –2, 1) in the direction of the vector 2i – j – 2k.

Sol :

$f(x, y, z) = x^2yz + 4xz^2$

The grad f at any point (x, y, z) is the vector and is given by

$$\text{grad } f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i\,(2xyz + 4z^2) + j\,x^2z + k\,(x^2y + 8xz)$$

At the point (1, –2, –1)

$$\text{grad } f = i\,(4 + 4) + j\,(-1) + k\,(-2 - 8)$$

$$= 8i - j - 10k$$

If a be the unit vector in the direction of the vector 2i – j – 2k, then

$$a = \frac{2i - j - 2k}{\sqrt{4+1+4}} = \frac{2}{3}i - \frac{1}{3}j - \frac{2}{3}k$$

Hence the required directional derivative is

$$\frac{df}{ds} = \text{grad } f \;\; a = (8i - j - 10k)\left(\frac{2}{3}i - \frac{1}{3}j - \frac{2}{3}k\right)$$

$$= \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

As this is + ve, f is increasing in this direction.

14. In what direction from the point (1, 1, –1) is the directional derivative of $f = x^2 - 2y^2 + 4z^2$ is a maximum ? Also find the value of this maximum directional derivative.

Sol :

The function f is given by $f = x^2 - 2y^2 + 4z^2$

$$\text{grad } f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i(2x) + j(-4y) + k(8z) = 2xi - 4yj + 8zk$$

At the point (1, 1, –1)

$$\text{grad } f = 2i - 4j - 8k$$

The directional derivative of f is maximum in the direction of

$$\text{grad } f = 2i - 4j - 8k$$

The maximum value of this directional derivative is given by

$$|\text{grad } f| = |2i - 4j - 8k| = \sqrt{4+16+64}$$

$$= \sqrt{84} = 2\sqrt{21}$$

15. If $S = S(x, y, z) = x^2 - x^2 y + x y^2 z^2$, then find the value of grad S at the point (2, –1, –3)

Sol :

We know that grad $S = \nabla S$ where $\nabla \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$$\therefore \quad \text{grad } S = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 - x^2 y + x y^2 z^2)$$

$$= i\frac{\partial}{\partial x}(x^2 - x^2 y + x y^2 z^2) + j\frac{\partial}{\partial y}(x^2 - x^2 y + x y^2 z^2) + k\frac{\partial}{\partial z}(x^2 - x^2 y + x y^2 z^2)$$

$$= i(2x - 2xy + y^2 z^2) + j(-x^2 + 2xyz^2) + k(2zxy^2)$$

Substituting the given values, we get

$$\text{grad } S = i\{2 \times 2 - 2 \times 2(-1) + (-1)^2(-3)^2\} + j\{-(2)^2 + 2 \times 2(-1)(-3)^2\} + k\{(-3)(2)(-1)^2\}$$

$$= i(17) - j(40) - k(12).$$

16. If $\phi = 4x^3 + 3yz^2 - z^3$, find $\nabla^2 \phi$ at (1, –1, –1).

Sol :

$$\nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(4x^3 + 3yz^2 - z^3)$$

$$= \frac{\partial^2}{\partial x^2}(4x^3 + 3yz^2 - z^3) + \frac{\partial^2}{\partial y^2}(4x^3 + 3yz^2 - z^3) + \frac{\partial^2}{\partial z^2}(4x^3 + 3yz^2 - z^3)$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(4x^3 + 3yz^2 - z^3)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(4x^3 + 3yz^2 - z^3)\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(4x^3 + 3yz^2 - z^3)\right]$$

Now

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(4x^3+3y\,z^2-z^3)\right] = \frac{\partial}{\partial x}(12x^2) = 24 \times \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(4x^3+3y\,z^2-z^3)\right] = \frac{\partial}{\partial y}(3z^2) = 0$$

and $$\frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(4x^3+3y\,z^2-z^3)\right] = \frac{\partial}{\partial z}(6yz - 2z^2) = 6y - 6z$$

$\therefore \quad \nabla^2 \phi = 24x + 6y - 6z$

Substituting the given values, we get

$\nabla^2 \phi = 24 \times 1 + 6(-1) - 6(-1) = 24 - 6 + 6 = 24$

17. If r is the position vector of a point, then evaluate

(1) grad r and (2) grad (1/r)

Sol :

1. We know that $r = i\,x + j\,y + k\,z$

$r = (x^2 + y^2 + z^2)^{1/2}$

Now grad

$$r = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{1/2} + i\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} + j\frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2}$$

$$+ k\frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}$$

$$= i\frac{\frac{1}{2}\times 2x}{(x^2+y^2+z^2)^{1/2}} + j\frac{\frac{1}{2}\times 2y}{(x^2+y^2+z^2)^{1/2}} + k\frac{\frac{1}{2}\times 2z}{(x^2+y^2+z^2)^{1/2}}$$

$$= \frac{ix+jy+kz}{(x^2+y^2+z^2)^{1/2}} = \frac{r}{r}$$

Let $\hat{r}$ be a unit vector along r, then $r = \hat{r}\,r$ or $r/r = \hat{r}$

$\therefore \quad$ grad $r = \hat{r}$

2. $$\text{grad}\left(\frac{1}{r}\right) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{1/2}$$

$$= i\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + j\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= i\frac{-\frac{1}{2}\times 2x}{(x^2+y^2+z^2)^{3/2}} + j\frac{-\frac{1}{2}\times 2y}{(x^2+y^2+z^2)^{3/2}} + k\frac{-\frac{1}{2}\times 2z}{(x^2+y^2+z^2)^{3/2}}$$

$$= -\frac{ix+jy+kz}{(x^2+y^2+z^2)^{3/2}} = -\frac{r}{r^3} = -\frac{\hat{r}}{r^2}$$

18. If r is the position vector, such that ϕ = log r, find grad ϕ.

Sol :

The position vector $r = i\,x + j\,y + k\,z$

and $r^2 = x^2 + y^2 + z^2$ or $r = (x^2 + y^2 + z^2)^{1/2}$

$\therefore$ $\log r = \log (x^2 + y^2 + z^2)^{1/2}$

$$= \frac{1}{2} \log (x^2 + y^2 + z^2)$$

Now grad ϕ = grad log r = ∇ log r

$$= \frac{1}{2} \nabla \log (x^2 + y^2 + z^2)$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x} \log (x^2 + y^2 + z^2)\, i + \frac{\partial}{\partial y} \log (x^2 + y^2 + z^2)\, j + \frac{\partial}{\partial z} \log (x^2 + y^2 + z^2)\, k\right]$$

$$= \frac{1}{2}\left[\frac{2x}{(x^2 + y^2 + z^2)} i + \frac{2y}{(x^2 + y^2 + z^2)} j + \frac{2z}{(x^2 + y^2 + z^2)} k\right]$$

$$= \frac{xi + yj + zk}{(x^2 + y^2 + z^2)} = \frac{r}{r^2}$$

19. If the electric potential at any point due to a charge q is given by V = q/r then show that the electric intensity E is given by $E = \frac{q}{r^3}\, r$

Sol :

We know that E = – grad V

$$\therefore \quad E = -\text{grad}\left(\frac{q}{r}\right) \qquad \left(\because V = \frac{q}{r}\right)$$

$$= -q\,\text{grad}\left(\frac{1}{r}\right)$$

But $\text{grad}\left(\frac{1}{r}\right) = \left(\frac{-r}{r^3}\right)$

$$\therefore \quad E = -q \times \left(\frac{-r}{r^3}\right) = \frac{q}{r^3} r$$

20. If $A = iy + j(x^2 + y^2) + k(yz + zx)$, then find div A at (1, –2, 3).

Sol :

The vector A is given by $A = iy + j(x^2 + yz) + k(yz + zx)$

But div $A = \nabla . A$

where $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$$\because \quad \text{div } A = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \quad [iy + j(x^2 + xz) + k(yz + zx)]$$

$$= \frac{\partial}{\partial x} y + \frac{\partial}{\partial y}(x^2 + yz) + \frac{\partial}{\partial z}(yz + zx)$$

$$= 0 + 2y + (y + x) = 3y + x$$

At the point (1, –2, 3)

div $A = 3(-2) + 1 = -6 + 1 = -5$

21. If $A = i\,y + j(x^2 + y^2) + k(yz + zx)$ then find div A at point (1, –2, 3).

Sol :

We know that the div $A = \nabla \bullet A$

where $\nabla \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$$\therefore \quad \text{div } A = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) . \{i\,y + j(x^2 + y^2) + k(yz + zy)\}$$

$$= \frac{\partial}{\partial x} y + \frac{\partial}{\partial y}(x^2 + y^2) + \frac{\partial}{\partial z}(yz + zx)$$

$$= 0 + 2\,y + (y + x) = (3y + x)$$

At point (1, –2, 3)

div $A = 3 \times (-2) + 1 = -5$.

22. Evaluate div F, where $F = 2x^2 z\,i - x\,y^2 z\,j + 3y^2 x\,k$

Sol :

$$\nabla \bullet F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \bullet (2x^3 z\,i - x\,y^2 z\,j + 3y^2 x\,k)$$

$$\therefore \quad \nabla \bullet F = \frac{\partial}{\partial x}(2x^3 z) - \frac{\partial}{\partial y}(x\,y^2 z) + \frac{\partial}{\partial z}(3y^2 x)$$

$$= 6x^2 z - 2xyz + 0$$

$$= 6x^2 z - 2xyz.$$

23. If r is the position vector of a point, then show that

(a) div r = 3 and (b) ∇ (r • A) = A

Sol :

(a) The position vector r is written as

$$r = i\,x + j\,y + k\,z$$

so that $r^2 = x^2 + y^2 + z^2$

Now $\text{div } r = \nabla \cdot r$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (i\,x + j\,y + k\,z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1 = 3$$

(b) $r \cdot A = (i\,x + j\,y + k\,z) \cdot (i\,A_x + j\,A_y + k\,A_z)$

$$= x\,A_x + y\,A_y + z\,A_z$$

$$\therefore \quad \nabla\,(r \cdot A) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x\,A_x + y\,A_y + z\,A_z)$$

$$= i\,A_x + j\,A_y + k\,A_z = A$$

24. Find the value of constant c for which the vector A = i (x + 3 y) + j (y – 2 z) + k (x + c z) is solenoidal.

Sol :

The vector A is solenoidal if the divergence is zero. We know that

$$\text{div } A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(x + 3\,y) + \frac{\partial}{\partial y}(y - 2\,z) + \frac{\partial}{\partial z}(x + c\,z)$$

$$= 1 + 1 + c = 2 + c$$

$$2 + c = 0 \quad \text{or} \quad c = -2$$

25. If a is a constant vector, find div (r × a).

Sol :

The vector $r = i\,x + j\,y + k\,z$

Let $a = i\,a_1 + j\,a_2 + k\,a_3$

$$r \times a = \begin{vmatrix} i & j & k \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= i\,(a_3\,y - a_2\,z) + j\,(a_1\,z - a_3\,z) + k\,(a_2\,x - a_1\,y)$$

Now $\quad \text{div}\,(r \times a) = \dfrac{\partial}{\partial x}(a_3\,y - a_2\,z) + \dfrac{\partial}{\partial y}(a_1\,z - a_3\,x) + \dfrac{\partial}{\partial z}(a_2\,x - a_1\,y)$

$$= 0 + 0 + 0 = 0$$

26. If A = i (3 x y) – j (y^2) evaluate $\oint_C$ A • dr, where C is a closed curve in x, y plane, y = $2x^2$ from (0, 0) to (1, 2).

Sol :

Here integration is performed in x, y plane, hence z = 0 and r = i x + j y.

$$\therefore \quad \oint_C A \cdot dr = \oint_C [i\,(3\,x\,y) - j\,y^2] \cdot [i\,dx + j\,dy]$$

$$= \oint_C (3\,x\,y\,dx - y^2\,dy)$$

$$= \int_0^1 [3x\,(2\,x^2)\,dx - (2x^2)\,4x\,dx] \qquad (\because y = 2x^2 \text{ and } dy = 4\,x\,dx)$$

$$= \int_0^1 [6x^3\,dx - 16\,x^5\,dx]$$

$$= \left[\frac{6x^4}{4} - \frac{16\,x^6}{6}\right]_0^1 = \left[\frac{6}{4} - \frac{16}{6}\right] = -\frac{7}{6}$$

27. Evaluate the line integral of function F = 6 x i + 4 y j between the points (0, 0) and (2, 2) in X-Y plane.

Sol :

The line integral is given by $\int_P^Q F \cdot dl = \int_P^Q (F_x\,dx + F_y\,dy + F_z\,dz)$

Here F is confined to X-Y plane, hence $F_z = 0$

$$\therefore \quad \int_P^Q F \cdot dl = \int_P^Q (F_x\,dx + F_y\,dy)$$

$$= \int_{(0,0)}^{(2,2)} (6\,x\,dx + 4\,y\,dy)$$

$$= \int_0^2 6\,x\,dx + \int_0^2 4\,y\,dy$$

$$= [3\,x^2]_0^2 + [2\,y^3]_0^2 = 12 + 8$$

$$= 20.$$

28. If r the position vector of a point, prove that curl r = 0.

Sol :

When r is the position vector of (x, y, z), then

$$r = ix + jy + kz$$

Now curl r $= \nabla \times r$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (ix + jy + kz)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$\therefore \quad \text{curl } r = i\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + j\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + k\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= i(0 - 0) + j(0 - 0) + k(0 - 0) = 0$$

29. If $v = x^2zi - 2y^3z^3j + xy^2zk$ then find curl v at the point (1, –1, + 1)

Sol :

Curl v $= \nabla \times v$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (x^2zi - 2y^3z^2j + xy^2zk)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -zy^3z^2 & xy^2z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(xy^2z) - \frac{\partial}{\partial z}(-2y^3z^2)\right] - j\left[-\frac{\partial}{\partial z}(xy^2z) + \frac{\partial}{\partial x}(xy^2z)\right] + k\left[\frac{\partial}{\partial x}(-2y^3z^2) - \frac{\partial}{\partial y}(x^2z)\right]$$

or, curl $v = i\,[2xyz + 4y^3z] - j\,(-x^2 + y^2z) + k\,[0 - 0]$

$\therefore$ At the point (1, – 1, 1) we have curl

Curl $v = i\,[2(1)(-1)(1) + 4(-)^3(1)] - j\,[-1^2 + (-1)^2(1)]$

$= -6i + j(0)$

$\therefore$ Curl $v = -6i$.

30. If $f = x^2y\, i - 2xz\, j + 2yz\, k$, find :

i) div ii) curl f and iii) curl curl f

Sol :

The vector f is given by $f = x^2yi - 2xz\, i + 2yz\, K$

i) But div $f = \nabla . f$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (x^2yi - 2xz\, j + 2yz\, k)$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(2yz)$$

$$= 2xy + 0 + 2y = 2xy + 2y$$

$$= 2y(x + 1)$$

ii) curl $f = \nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right] i + \left[\frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial z}(2yz)\right] j + \left[\frac{\partial}{\partial z}9 - 2xz\frac{\partial}{\partial y}(x^2y)\right]k$$

$$= (2z + 2x)i + (0 - 0)\, j + (-2z - x^2)\, k$$

$$= (2x + 2z)i - (x^2 + 2z)\, k$$

iii) curl curl $f = \nabla \times (\nabla \times i)$

$$= \nabla \times ((2x + 2z)\, i - (x^2 + 2z)\, k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-x^2 - 2z) - \frac{\partial}{\partial z}(0)\right] i + \left[\frac{\partial}{\partial z}(2x - 2z) - \frac{\partial}{\partial x}(-x^2 - 2z)\right] j + \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(2x + 2z)\right] k$$

$$= 0i + (2 + 2x)\, k + (0 - 0)\, k = (2x + 2)k$$

31. Show that $A = e^{-x}(-yz\,i + zj + yk)$ is irrotational

Sol :

We known that when $\nabla \times A = 0$, the function is irrotational.

Given that $A = -yze^{-x}\,i + ze^{-x} + j + ye^{-x}\,k$

$$\therefore \quad \nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yze^{-x} & ze^{-x} & ye^{-x} \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(ye^{-x}) - \frac{\partial}{\partial z}(ze^{-x})\right] - j\left[\frac{\partial}{\partial x}(y - x) - \frac{\partial}{\partial z}(-yze^{-x})\right] + k\left[\frac{\partial}{\partial x}(ze^{-x}) - \frac{\partial}{\partial y}(-yze^{-x})\right]$$

$$= i(e^{-x} - e^{-x}) - j(-ye^{-x} + ye^{-x}) + k(-ze^{-x} + ze^{-x})$$

$$= 0$$

So, the function is irrotational.

32. If V is a constant vector, show that (i) div V = 0 and (ii) curl V = 0

Sol :

V is a constant vector.

i) $\text{div } V = \left(i\frac{\partial V}{\partial x} + j\frac{\partial V}{\partial y} + k\frac{\partial V}{\partial z}\right)$

$= i.0 + j.0 + k.0 = 0$

ii) $\text{curl } V = \left(i \times \frac{\partial V}{\partial x} + j \times \frac{\partial V}{\partial y} + k \times \frac{\partial V}{\partial z}\right)$

$= i \times 0 + j \times + k \times 0 = 0$

33. If $A = iy + j(x^2 + y^2) + k(yz + zx)$, then find the value of curl A at (1, –1, 1).

Sol :

We know that curl $A = \nabla \times A$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times [iy + j(x^2 + y^2) + k(yz + zx)]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & (x^2 + y^2) & (yz + zx) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial x}(yz + zx) - \frac{\partial}{\partial z}(x^2 + y^2)\right] + j\left[\frac{\partial}{\partial z}y - \frac{\partial}{\partial z}(yz + zx)\right] + k\left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}y\right]$$

$= i(z + 0) + j(0 - z) + k(2x - 1)$

$= iz - jz + k(2x - 1)$

At the point (1, – 1, 1)

Curl A $= i \times 1 - j \times 1 + k(2 \times 1 - 1) = i - j + k$

---

34. If a rigid body is rotating with angular velocity ω prove that

$\omega = \frac{1}{2}$ curl V where v is the linear velocity.

Sol :

Curl v $= \nabla \times v$

Now $v = (\omega \times r)$

$\therefore$ $v = i(\omega_y z - y\omega_z) + j(\omega_z x - z\omega_x) + k(\omega_x y - x\omega_y)$

Now $\nabla \times v = \nabla \times (\omega \times r)$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (\omega_y z - y\omega_z) & (\omega_z x - z\omega_x) & (\omega_x y - x\omega_y) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(\omega_x y - x\omega_y) - \frac{\partial}{\partial z}(\omega_z x - z\omega_x)\right] + j\left[\frac{\partial}{\partial y}(\omega_x z - y\omega_z) - \frac{\partial}{\partial z}(\omega_x y - x\omega_y)\right]$$

$$+ k\left[\frac{\partial}{\partial y}(\omega_z x - z\omega_x) - \frac{\partial}{\partial z}(\omega_y z - x\omega_z)\right]$$

$= i(\omega_x + \omega_x) + j(\omega_y + \omega_y) + k(\omega_z + \omega_z)$

$= 2[i\omega_x + j\omega_y + k\omega_z] = 2\omega$

$\therefore \quad \omega = \frac{1}{2}$ curl v.

---

35. Find the value of curl (a × r), where a is a constant vector.

Sol :

$$a \times r = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ x & y & z \end{vmatrix}$$

$= i(a_y z - a_z y) + j(a_z x - a_x z) + k(a_x y - a_y x)$

Now, curl (a × r) = $\nabla \times (a \times r)$

$$\therefore \quad \nabla \times (a \times r) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial x & \partial/\partial z \\ (a_y z - a_z y) & (a_z x - a_x z) & (a_x y - a_y x) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(a_x y - a_y x) - \frac{\partial}{\partial z}(a_z x - a_x z)\right] + j\left[\frac{\partial}{\partial z}(a_y z - a_z x) - \frac{\partial}{\partial x}(a_x y - a_y x)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(a_z x - a_x z) - \frac{\partial}{\partial x}(a_y z - a_z y)\right]$$

$= i[a_x + a_x] + j[a_y + a_y] + k[a_z + a_z]$

$= 2[ia_x + ja_y + ka_z] = 2$ a.

36. Obtain the values of the following :

(i) curl grad S, (ii) grad div A and (iii) div curl A

Sol :

i) curl grad S = $\nabla \times \nabla S$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \left(i\frac{\partial S}{\partial x} + j\frac{\partial S}{\partial y} + k\frac{\partial S}{\partial z}\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} & \frac{\partial S}{\partial z} \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\left(\frac{\partial S}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial S}{\partial y}\right)\right] + j\left[\frac{\partial}{\partial z}\left(\frac{\partial S}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial S}{\partial z}\right)\right] + k\left[\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial S}{\partial z}\right)\right]$$

$= 0 + 0 + 0 = 0$

ii) grad div A = $\nabla (\nabla . A)$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= i\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + j\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + k\frac{\partial}{\partial k}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

iii) div curl A = ∇ (∇ . A)

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) . \left[i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \left(\frac{\partial^2 A_z}{\partial x\,\partial y} - \frac{\partial^2 A_y}{\partial x\,\partial z}\right) + \left(\frac{\partial^2 A_x}{\partial y\,\partial z} - \frac{\partial^2 A_z}{\partial y\,\partial x}\right) + \left(\frac{\partial^2 A_y}{\partial z\,\partial x} - \frac{\partial^2 A_x}{\partial z\,\partial y}\right)$$

$$= 0 \left[\left(\because \frac{\partial^2}{\partial x\,\partial y} = \frac{\partial^2}{\partial y\,\partial x}, \frac{\partial^2}{\partial y\,\partial z} = \frac{\partial^2}{\partial z\,\partial y}, \frac{\partial^2}{\partial z\,\partial x} = \frac{\partial^2}{\partial x\,\partial z}\right)\right]$$

37. Prove that

i) ∇ × (A + B) = ∇ × A + ∇ × B or

curl (A + B) = curl A + curl B

ii) ∇ ×(A × B) = A(∇ .B) – B(∇ ·A) or

curl (A × B) = A div B – B div A

Sol :

i) ∇ × (A + B)

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times [i(A_x + B_x) + j(A_y + B_y) + k(A_z + B_z)]$$

$$= k\frac{\partial}{\partial x}(A_y + B_y) - j\frac{\partial}{\partial x}(A_z + B_z) + k\frac{\partial}{\partial x}(A_x + B_x)] + i\frac{\partial}{\partial y}(A_z + B_z) + j\frac{\partial}{\partial z}(A_x + B_x)$$

$$- i\frac{\partial}{\partial z}(A_z + B_z)$$

$$= i\left[\frac{\partial}{\partial y}(A_z + B_z) - \frac{\partial}{\partial z}(A_y + B_y)\right] + j\left[\frac{\partial}{\partial z}(A_x + B_x) - \frac{\partial}{\partial x}(A_z + B_z)\right] + k\left[\frac{\partial}{\partial x}(A_y + B_y) - \frac{\partial}{\partial y}(A_x + B_x)\right]$$

$$= i\left[\frac{\partial A_z}{\partial y} + \frac{\partial B_z}{\partial y} - \frac{\partial A_y}{\partial z} - \frac{\partial B_y}{\partial z}\right] + j\left[\frac{\partial A_x}{\partial z} + \frac{\partial B_x}{\partial z} - \frac{\partial A_x}{\partial x} - \frac{\partial B_z}{\partial x}\right] + k\left[\frac{\partial A_x}{\partial x} + \frac{\partial B_y}{\partial x} - \frac{\partial A_x}{\partial y} - \frac{\partial B_x}{\partial y}\right]$$

$$= \left[i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + i\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\right] + \left[j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + j\left(\frac{\partial B_x}{\partial y} - \frac{\partial B_z}{\partial z}\right)\right] + \left[k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + k\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\right]$$

$$= \left[i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] + \left[i\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + j\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + k\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_y}{\partial y}\right)\right]$$

$$= \nabla \times A + \nabla \times B = \text{curl } A + \text{curl } B$$

$$\text{curl } (A + B) = \text{curl } A + \text{curl } B$$

ii) We know that $A \times (B \times C) = B(A.C) - C(A.B)$

Hence, $\nabla \times (A \times B) = A(\nabla .B) - B(\nabla . A)$

$= A \text{ div } B - B \text{ div } A$

$\text{curl } (A \times B) = A \text{ div } B - B \text{ div } A$

38. Prove that curl curl A = grad div A $- \nabla^2$ A

Sol : **(June-18)**

curl curl A $= \nabla \times (\nabla \times A)$

We know that $A \times (B \times C) = B(A \cdot C) - C (A \cdot B)$

$$\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - (\nabla \cdot \nabla)A$$

$$= \text{grad (div A)} - \nabla^2 A$$

So curl curl A $=$ grad (div A) $- \nabla^2$ A

The value can also be obtained in the following way :

$$(\nabla \times A) = \left[i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$\therefore \nabla \times (\nabla \times A) = \nabla \times \left[i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$= \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{bmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] + j\left[\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$+ k\left[\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]$$

$$= i\left[\frac{\partial^2 A_y}{\partial y\,\partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z\,\partial x}\right] - i\frac{\partial^2 A_x}{\partial x^2} + i\frac{\partial^2 A_x}{\partial x^2} + j\left[\frac{\partial^2 A_z}{\partial z\,\partial y} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial x\,\partial y}\right]$$

$$- j\frac{\partial^2 A_y}{\partial y^2} + j\frac{\partial^2 A_y}{\partial y^2} + k\left[\frac{\partial^2 A_x}{\partial x\,\partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial y\,\partial z}\right] - k\frac{\partial^2 A_z}{\partial z^2} + k\frac{\partial^2 A_z}{\partial z^2}$$

$$= \left[-i\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right) - j\left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right) - k\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\right]$$

$$+ \left[i\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial z\,\partial x} + \frac{\partial^2 A_x}{\partial z\,\partial x}\right) + j\left(\frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_x}{\partial x\,\partial y} + \frac{\partial^2 A_y}{\partial y\,\partial z}\right) + k\left(\frac{\partial^2 A_z}{\partial z^2} + \frac{\partial^2 A_x}{\partial x\,\partial z} + \frac{\partial^2 A_y}{\partial y\,\partial z}\right)\right]$$

$$= \left[-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(i\,A_x + j\,A_y + k\,A_z)\right] + \left[i\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + j\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\right.$$

$$\left. + k\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\right]$$

$$= -\nabla^2 A + \nabla\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) = -\nabla^2 A + \nabla\,(\nabla\,.\,A)$$

Then curl curl A = grad div A – ∇^2 A

39. By stokes theorem, prove that $\oint_C \mathbf{r.d}l = 0$, where r is position vector.

Sol :

By stokes theorem, we have

$$\oint_C A\,.\,dl = \iint_S \text{curl } A\,.\,dS$$

Replacing the vector A by position vector r, we get

$$\oint_C r.dl = \iint_S \text{curl } r\,.\,dS$$

$$= \iint_S 0.dS \qquad (\because \text{ curl } r = 0)$$

Hence, $\oint_C r.dl = 0$

40. Stoke's theorem, prove that curl grad ϕ **= 0.**

Sol :

According to Stoke's theorem, we have

$$\oint_C A \, . \, dl = \iint_S \text{curl } A \, . \, dS$$

Let A = rad ϕ, then $\oint_C \text{grad } \phi . \, dl = \iint_S \text{curl grad } \phi . \hat{n} \, dS$

$$\text{Now, grad } \phi \, . \, dl = \left(i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right) + (i \, dx + j \, dy + k \, dz)$$

$$= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi$$

$\therefore$ $\oint_C d\phi = [\phi]_A^A$ where A is any point on C

Hence, $\iint_S \text{curl grad } \phi . \hat{n} \, dS = 0$

This is true for all surface elements S, i.e, curl grad ϕ = 0

41. Evaluate by Stoke's theorem $\oint_C (y\,z\,dx + x\,z\,dy + x\,y\,dz)$ where C is the curve $x^2 + y^2 = 1$ and $z = y^2$.

Sol :

$$\oint_C (y\,z\,dx + x\,z\,dy + x\,y\,dz)$$

$$= \oint_C [(i\,y\,z + j\,z\,x + k\,x\,y) \, . \, (i\,dx + j\,dy + k\,dz)$$

$$= \oint_C [(i\,y\,z + j\,z\,x + k\,x\,y) \, . \, dr = \oint_C A \, . \, dr$$

$\therefore$ A = i y z + j z x + k x y

$$\text{Now} \quad \text{curl } A = \begin{vmatrix} i & j & k \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ y\,z & x\,z & x\,y \end{vmatrix}$$

$$= i \left(\frac{\partial}{\partial y} xy - \frac{\partial}{\partial z} x\,z \right) + j \left(\frac{\partial}{\partial z} y\,z - \frac{\partial}{\partial x} xy \right) + k \left(\frac{\partial}{\partial x} x\,z - \frac{\partial}{\partial y} y\,z \right)$$

$$= i\,(x - x) + j(y - y) + k\,(z - z) = 0$$

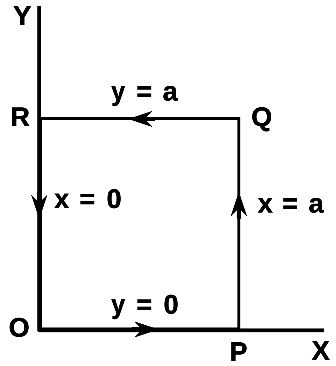
By Stokes theorem $\oint_C A \, . \, dl = \iint_S \text{curl } A \, . \, \hat{n} \, dS = 0$

$\therefore$ Given integral is zero.

42. Verify Stoke's theorem for the vector A = x(ix + jy) integrated round the square, in the plane z = 0, whose sides are along the lines x = 0, y = 0, x = a, y = a.

Sol :

The situation is shown in figure below.



Here $\oint_C A \cdot dr = \int_{OP} A.dl + \int_{PQ} A.dl + \int_{QR} A.dl + \int_{RO} A.dl$

where $\int_{OP} A.dl = \int_0^a x(ix + jy) \cdot i\, dx = \int_0^a x^2 dx = \frac{a^3}{3}$

$$\int_{PQ} A \bullet dl = \int_0^a x(ix + jy) \bullet j\, y) \bullet j\, dy = \int_0^a a\, y\, dy = \frac{a^3}{2}$$

$$\int_{QR} A \bullet dl = \int_0^a x(i\,x + j\,y) \bullet i\, dx = \int_0^a x^2 dx = \frac{a^3}{3}$$

and $\int_{RO} A \cdot dl = \int_0^a x(i\,x + j\,y) \cdot j\, dy = \int_0^a 0(ix + jy) \cdot j\, dy = 0$

$$\therefore \quad \oint_C A \cdot dr = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2} \quad \text{... (1)}$$

Further, $\iint_S \text{curl } A \cdot dS = \iint_S \text{curl } x(ix + jy) \cdot dS$

$$= \int_0^a \int_0^a k\, y \cdot k\, dx\, dy \quad (\because \text{curl } x\,(i\,x + j\,y) = k\,y)$$

$$= \int_0^a \int_0^a y\, dx\, dy = \int_0^a dx \int_0^a y\, dy = a \times \frac{a^2}{2} = \frac{a^3}{2} \quad \text{... (2)}$$

From equation (1) and (2) Stoke's theorem stands verified.

43. Evaluate $\iint \mathbf{r.\hat{n}\, dS}$, where S is a closed surface.

Sol :

By Gauss's theorem

$$\iint_S r\cdot\hat{n}\,dS = \iiint_V \text{div } r\, dV = \iiint_V \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(ix+jy+kz)\,dV$$

$$= \iiint_V \left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\right)dV = 3\iiint_V dV = 3\,V$$

where V is the volume enclosed by surface S.

44. If F = i ax + jby + kcz, where a, b, c are constants show that $\iint_S F.\hat{n}\,dS = \frac{4}{3}\pi(a+b+c)$ **where S is the surface area of a unit sphere.**

Sol :

According to Gauss's theorem $\iint_S F\cdot\hat{n}\,dS = \iiint_V \text{div.}F\,dV$

where V is the volume enclosed by S.

Now $\iint_S F\cdot\hat{n}\,dS = \iiint_V \nabla\cdot(i\,a\,x + j\,b\,y + k\,c\,z)dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(a\,x)+\frac{\partial}{\partial y}(b\,y)+\frac{\partial}{\partial z}(c\,z)\right]dV$$

$$= \iiint_V (a+b+c)\,dV = (a+b+c)\iiint_V dV = (a+b+c)\cdot\frac{4}{3}\pi$$

[$\because$ volume V is enclosed by spehre of radius one $= \frac{4}{3}.\pi.(1)^3 = \frac{4}{3}\pi$]

$$= \frac{4}{3}\pi(a+b+c)$$

45. Find $\iint_S \mathbf{F\cdot\hat{n}\,dS}$ where F = i 4 x z – j y2 + k y z and S is the surface of a cube bounded by x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.

Sol :

According to Gauss' theorem $\iint_S F\cdot\hat{n}\,dS = \iiint_V \text{div } F\,dV$

$$= \iiint_V \left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)dx\,dy\,dz$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz)+\frac{\partial}{\partial y}\left(-y^2\right)+\frac{\partial}{\partial z}(y\,z)\right]dx\,dy\,dz$$

$$= \iiint_V (4z - 2y + y)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 [2z^2 - yz]_0^1 \; dx\, dy = \int_0^1 \int_0^1 (2 - y)\, dy\, dx$$

$$= \int_0^1 \left[2y - \frac{y^2}{2}\right]_0^1 dx = \int_0^1 \frac{3}{2} dx = \left[\frac{3}{2} x\right]_0^1 = \frac{3}{2}$$

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46. If $A = 2xz^2\,\hat{i} - yz\,\hat{j} + 3xz^3\,\hat{k}$. Find ∇ .A at (1, 1, 1).

Sol :

Given,

$A = 2xz^2\,\hat{i} - yz\,\hat{j} + 3xz^3\,\hat{k}$

$(x, y, z) = (1, 1, 1)$ ...(1)

The expression for calculating the gradient of A is given by,

$$\nabla A = \hat{i}\frac{\partial A}{\partial x} + \hat{j}\frac{\partial A}{\partial y} + \hat{k}\frac{\partial A}{\partial z} \quad ...(2)$$

Substituting equation (1) in equation (2),

$$\nabla A = \hat{i}.\frac{\partial}{\partial x}(2xz^3\,\hat{i} - yz\,\hat{j} + 3xz^2\,\hat{k}) + \hat{j}\frac{\partial}{\partial y}(2xz^2\,\hat{i} - yz\,\hat{j} + 3xz^3\,\hat{k}) + \hat{k}\frac{\partial}{\partial z}(2xz^2\,\hat{i} - yz\,\hat{j} + 3xz^3\,\hat{k})$$

$$= \left[\frac{\partial}{\partial z}(2xz^2)(\hat{i}.\hat{j}) + \frac{\partial}{\partial x}(3xz^3)(\hat{i}.\hat{k})\right] + \left[\frac{\partial}{\partial y}(2xz^2)(\hat{j}.\hat{i}) + \frac{\partial}{\partial y}(-yz)(\hat{i}.\hat{j}) + \frac{\partial}{\partial y}(3xz^3)(\hat{j}.\hat{k})\right]$$

$$= \left[\frac{\partial}{\partial x}(2xz^2)(\hat{k}.\hat{i}) + \frac{\partial}{\partial z}(-yz)(\hat{k}.\hat{j}) + \frac{\partial}{\partial z}(3xz^3)(\hat{k}.\hat{k})\right]$$

$$= \frac{\partial}{\partial x}(2xz^2(\hat{i}.\hat{i}) + \frac{\partial}{\partial z}(-yz)(\hat{j}.\hat{j}) + \frac{\partial}{\partial z}3xz^3(\hat{k}.\hat{k}) \qquad [\because \hat{p}.\hat{q} = 0]$$

$$= \frac{\partial}{\partial x}(2xz^2) - \frac{\partial}{\partial y}yz + \frac{\partial}{\partial z}3xz^3 \qquad [\because \hat{p}.\hat{p} = 1]$$

$= 2z^2 - z + 3x(3z^2)$

$\Rightarrow \quad \nabla A = 2z^2 - z + 3x(3z^2)$

Substituting (1, 1, 1) in above equation,

$\Rightarrow \quad \nabla A = 291)^2 - (1) + 9(4)(1)^2$

$= 2 - 1 + 9$

$= 10$

$\therefore \quad \nabla A = 10$

Short Question and Answers

1. Define gradient of a scalars filed obtain an expression for it.

Ans :

In order to consider the gradient of a scalar, let $\phi(x, y, z)$ be a scalar function of position of a scalr point of coordinates (x, y, z). The partial derivatives of ϕ along the three coordinate axes are

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \text{ and } \frac{\partial\phi}{\partial z}$$

The gradient of a scalar function ϕ is defined as

$$\text{grad } \phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \quad ...(1)$$

We know that vector differential operator ∇ (del) is defined as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$\therefore \quad \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \quad ...(2)$$

The equation (2) is the same as equation (1). It is obvious from equation (2) that del operator (∇) is a vector operator and when operated with a scalr (ϕ) converts the scalar into a vector. The vector $(\nabla\phi)$ is called the gradient of the scalar.

The gradient is a different operator by means of which we can associate a vector field with a scalar field. For example, the intensity of electric field E, (a vector quantity) is the gradient of potential V (a scalar quantity) with a negative sign, i.e.,

$$E = -\text{ grad } V$$

The negative sign indicates that the direction of field intensity is opposite to the direction of increasse of potential.

Let S(x, y, z) be a scalar point function depending on the three cartesian coordinates in space. Suppose $\partial S/\partial x, \partial S/\partial y$ and $\partial S/\partial z$ be the partial derivatives along the three perpendicular axes respectively. Now the gradient of the scalar functioon S can be expressed as

$$\text{grad } S = i\frac{\partial S}{\partial x} + j\frac{\partial S}{\partial y} + k\frac{\partial S}{\partial z}$$

or

$$\text{grad } S = \nabla S \text{ where } \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

2. Show that $\bar{F} = (y^2 - x^2) + \hat{i} + 2xy\hat{j}$ conservative.

Ans :

Given force is,

$$\vec{F} = (y^2 - x^2)\hat{i} + 2xy\hat{j}$$

A force is said to be conservative, if the curl of the force is zero.

i.e., curl $\vec{F} = 0$

$$\therefore \quad \text{Curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - x^2) & 2xy & 0 \end{vmatrix}$$

$$= \left|\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2xy)\right| - j\left|\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial x}(y^2 - x^2)\right| + k\left|\frac{\partial}{\partial z}(2xy) - \frac{\partial}{\partial z}(y^2 - x^2)\right|$$

$$= i|0 - 0| - j|0 - 0| + k|2y - 2y| = 0$$

$$\therefore \quad \text{Curl } F = \nabla \times F = 0$$

Thus, the given force is conservative in nature.

3. Explain divergence of a vector field and its physical significance.

Ans :

The operator ∇ can be involved in the multiplication with a vector. The scalar or dot product of operator ∇ with a vector A (i.e., $\nabla \cdot A$) is called as divergence. The divergence of a vector field at any point is defined as the amount of flux per unit volume diverging from that point. The divergence is a scalar.

Let A be a vector function differentiable at each point (x, y, z) in a region of space. Now the divergence of A is given by

$$\nabla \cdot A = \left(i\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (i A_x + j A_y + k A_z)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

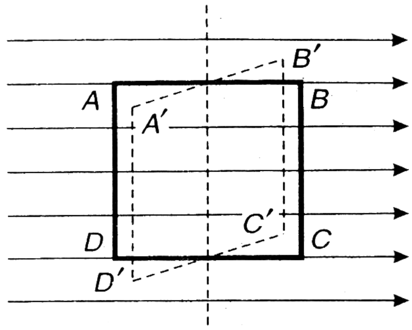
$$\therefore \quad \text{div } A = \nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

This is the expression of divergence in cartesian coordinates.

4. Curl of a Vector Field.

Ans :

Consider two areas ABCD and A' B' C' D' in a uniform electric field which is represented by straight parallel lines as shown in fig. The area A' B' C' D' is perpendicular to lines of force. So the contribution of line integrals is zero. For area ABCD, the line integrals along AD and BC are zero while the line integrals along AB and CD are not zero. This shows that there is a certain orientation of the area for which the line integral is maximum.



The curl of a vector field is defined as the maximum line integral of the vector per unit area. It is essentially a vector quantity. The direction is normal to the area.

If A is a vector function differentiate at each point (x, y, z) in a region of a space, then the curl (or rotation) of A expressed by the cross product of ∇ and A, i.e.,

$$\text{Curl } A = \nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

5. Prove that Curl of a gradient is zero.

Ans :

Ley ϕ be any scalar point function.

$$\text{grad } \phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi$$

$$\Rightarrow \quad \text{grad } \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$\text{Curl (grad } \phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right) - \bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = \bar{i}\,(0) + \bar{j}\,(0) + \bar{k}\,(0)$$

$\Rightarrow$ Curl (grad ϕ) = 0

$\therefore$ Curl of a gradient is zero

6. Gauss's divergence theorem.

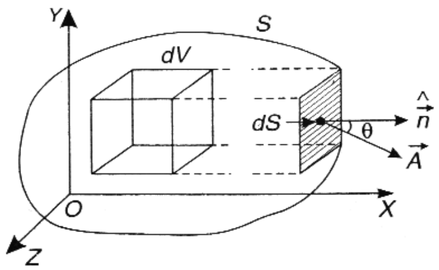
Ans :

The surface integral of the normal component of vector A taken over a closed surface S is equal to the volume integral of the divergence of vector A over the volume V enclosed by the surface S, i.e.,

$$\iint_S A \cdot dS = \iiint_V \text{div } A \, dV = \iiint_V (\nabla \cdot A) \, dV$$

Proof :

Let us consider a closed surface S of any arbitrary shape drawn in a vector field A as shown in figure below.



Let the surface encloses a volume V.

We know that div A represents the amount of flux diverging per unit volume and hence the flux diverging from the element of volume dV will be div A dV.

So the total flux coming out from the entire volume is given by

$$\iiint_V \text{div } A \, dV$$

Now we consider a small element of area dS on the surface S as shown in fig. Let $\hat{n}$ represents the unit vector drawn normal to area dS. It should be remembered that outward drawn normal on a surface is taken as positive. If the field vector A and outward normal $\hat{n}$ are at an angle θ, then the component of A along $\hat{n}$ is

$$A \cos\theta = A \cdot \hat{n}$$

The flux of A through the surface element dS is given by

$$(A \cdot \hat{n})\, dS = A \cdot dS$$

So the total flux through the entire surface S is given by $\iint_S A \cdot dS$ ... (2)

This must be equal to the total flux diverging from the whole volume V enclosed by the surface S. Hence from eqs. (1) and (2) we get

$$\iint_S A \cdot dS = \iiint_V \text{div } A \, dV \qquad ... (3)$$

This is Gauss theorem of divergence and may also be written as

$$\iint_S (A \cdot \hat{n})\, dS = \iiint_V (\nabla \cdot A)\, dV \qquad ... (4)$$

(i) Gauss divergence theorem provides a relation between surface and volume integrals.

(ii) This theorem is applicable for closed surface only.

7. State and explain Green's theorem.

Ans :

Statement :

If ϕ and ψ are two scalar point functions such that these functions and their first derivatives are continuously differentiable, in a region bounded by a closed surface S, then we have

$$\iiint_V (\phi\nabla^2\psi + \nabla\phi.\nabla\psi)\, dV = \iint_S (\phi\nabla\psi).dS \qquad ... (1)$$

and

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \iint_V (\phi\nabla\psi - \psi\nabla\phi).\, ds \qquad ... (2)$$

These equations are known as first and second form of Green's theorem.

Proof :

Let us take the following mathematical from Gauss's divergence theorem

$$\iiint_V \text{div } A\, dV = \iint_V A\,.dS$$

Because $\nabla\psi$ will also be a vector quantity, $\phi\nabla\psi$ will be a vector quantity.

Let this is represented by vector A. Thus

$$A = \phi\nabla\psi$$

or $$i A_x + j A_y + k A_z = \phi\left(i\frac{\partial\psi}{\partial x} + j\frac{\partial\psi}{\partial y} + k\frac{\partial\psi}{\partial z}\right)$$

From this equation we can see that $A_x = \phi\frac{\partial\psi}{\partial x}$, $A_y = \phi\frac{\partial\psi}{\partial y}$ and $A_z = \phi\frac{\partial\psi}{\partial z}$

Now $$\text{div } A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Substituting the values of A_x, A_y and A_z, we get,

$$\text{div } A = \frac{\partial}{\partial x}\left(\phi\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\phi\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\phi\frac{\partial\psi}{\partial z}\right)$$

$$= \left(\phi\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\phi}{\partial x}.\frac{\partial\psi}{\partial x}\right)+\left(\phi\frac{\partial^2\psi}{\partial y^2}+\frac{\partial\phi}{\partial y}.\frac{\partial\psi}{\partial y}\right)+\left(\phi\frac{\partial^2\psi}{\partial z^2}+\frac{\partial\phi}{\partial z}.\frac{\partial\psi}{\partial z}\right)$$

$$= \phi\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}.\frac{\partial^2\psi}{\partial z^2}\right)+\frac{\partial\phi}{\partial x}.\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}.\frac{\partial\psi}{\partial y}+\frac{\partial\phi}{\partial z}.\frac{\partial\psi}{\partial z} = \phi\nabla^2\psi+\nabla\phi.\nabla\psi \quad ..(4)$$

Substituting the value of div A from equation (4) in equation (3), we get

$$\iiint_V (\psi\nabla^2\phi+\nabla\psi.\nabla\phi)dV = \iint_S (\phi\nabla\psi).\nabla S \quad ... (5)$$

This is known as Green's first theorem.

Interchanging ϕ and ψ in equation . (a), we have

$$\iiint_V (\psi\nabla^2\phi+\nabla\psi.\nabla\phi)dV = \iint_S (\psi\nabla\phi).\nabla S \quad ... (a)$$

Subtracting equation (5) from equation (a), we get,

$$\iiint_V (\phi\nabla^2\psi-\psi\nabla^2\phi)dV = \iint_S (\phi\nabla\psi-\psi\nabla\phi).dS \quad ... (b)$$

This is known as Green's second theorem.

8. Prove that $\nabla.(A\times r) = r.(\nabla\times A)$

Ans :

Let A and r be the vectors given as,

$A = A_x i + A_y j + A_z k$

$r = xi + yj + zk$

Consider,

$$A\times r = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ x & y & z \end{vmatrix}$$

$= i[A_y z - A_z y_y] - j[A_x z - A_z x] + k[A_x y - A_y x]$

$\therefore \quad A\times r = i[A_y z - A_z y] + j[A_z x - A_x z] + k[Z_x y - A_y x]$

Applying divergence on both sides,

$$\nabla.(A\times r) = \nabla.[i[A_yz - A_zy] + j[A_zx - A_xz] + k[A_zy - A_yx]]$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).[i[A_yz - A_zy] + j[A_zx - A_xz] + k(A_xy - A_yx)]$$

$$= i.i\frac{\partial}{\partial x}[A_yz - A_zy] + j.j\frac{\partial}{\partial y}[A_zx - A_xz] + k.k\frac{\partial}{\partial z}[A_xy - A_yx]$$

$$= z\frac{\partial A_y}{\partial x} - y\frac{\partial A_y}{\partial x} + \frac{x\partial A_z}{\partial y} - z\frac{\partial A_x}{\partial y} + y\frac{\partial A_x}{\partial_z} - x\frac{\partial A_y}{\partial z}$$

$$\therefore \quad \nabla(A\times r) = x\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_y}{\partial z}\right] + y\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial z}\right] + z\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] \quad ...(1)$$

Consider, $\nabla \times A = \text{curl } A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$$= i.ix\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z}\right) - j\left(\frac{\partial A_z}{\partial z} - \frac{\partial A_x}{\partial z}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Consider $r.\text{curl } A = (xi + yi + zk).\left[i\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial_z} - \frac{\partial A_z}{\partial x}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$

$$= i.ix\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j.jy\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k.kz\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\therefore \; r.\text{curl } A = x\left[\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z}\right] + y\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + z\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] \quad ...(2)$$

Comparing equations (1) and (2),

$$\nabla.(A \times r) = r.\text{curl } A$$

$$\Rightarrow \quad \nabla.(A \times r) = r.(\nabla \times A)$$

$$\therefore \quad \nabla.(A \times r) = r.(\nabla \times A)$$

9. What are scalar and vector fileds?

Ans :

We know that a physical quantity can be expressed as a continuous function of the position of a point in the region of space. For example when a rod is heated at one end, then there is a variation of temperature along the length of the rod. The physical quantity temperature at any point (x, y, z) can be expressed by a continuous function T (x, y, z). Such a function is termed as a point function or function of position. The region specifying that physical quantity is labelled as its field. Depending upon the nature of physical quantity the field may be scalar or a vector.

I) Scalar Field

If a scalar physical quantity is assigned to each point in space then we have a scalar field in that region of space. The scalar field in three dimensions can be represented by a scalar point function $\phi(x, y, z)$.

Example

The electronic potential due to a single positive charge q columns depends on the position of the point from the charge. Then $V_0(x_0, y_0, z_0)$ and $V(x_1, y_1, z_1)$ are the scalar point functions at (x_0, y_0, z_0) and (x_1, y_1, z_1). Now the region is a scalar field.

In a scalar field, there are a number of surfaces are known as level surfaces.

The level surfaces, equipotential (gravitational or electrostatic) etc.

For example, if we imagine sphere of different radii taking a point charge q as the centre, then we get equipotential level surfaces. These are spherical in nature and at any level surface the scalar point function V (x, y, z) has a constant value. If the level surface are parallel to each other, then the scalar field is called as stationary scalar field.

The concept of a scalar field can easily be understood with the help of the following examples:

i) Consider a solid block of material whose faces are maintained at different temperatures. Now the temperature of the body will vary from point to point, i.e., temperature will be a function of position coordinates x, y, z in rectangular coordinate system. Hence, temperature is a scalar field.

ii) The density at any point inside a body occupying given region is a scalar field. The electrical potential is differeent at different points. Hence, electric potential is scalar field.

II) Vector Field

When a vector physical quantity is expressed from point to point in the region of space by a continuous vector function A(x, y, z) then the region is a vector field. The example of vector field are gravitiational, magnetic, electric intensity.

The vector point function at any point in the field is given by a vector having unique value for a magnitude and direction.

10. What are line, surface and volume integrals?

Ans :

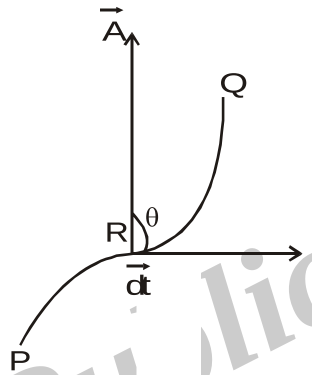
The integrals which are commonly used are :

1. Line integral
2. Surface integral and
3. Volume integral

1. Line Integral

Integral $\int_P^Q A \cdot dl$ is defined as the line integral of A along the curve PQ.

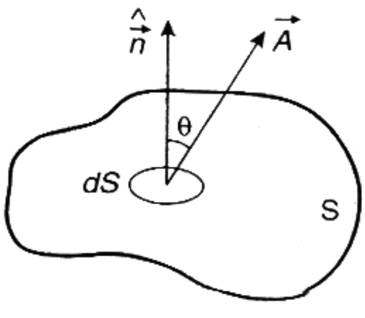
If A denotes the electric field intensity at any point, then the line integral represents the potential difference between P and Q.



2. Surface Integral

Consider a simple surface S in a vector field bounded by a curve as shown in fig. Let dS be an infinitesimal element of the surface. This surface element of area dS can be represented by area vector dS. If $\hat{n}$ be a unit positive vector (drawn outward the surface) in the direction of dS, then

$$dS = \hat{n}\ dS$$



Let A be a vector at middle of the element dS in the direction making an angle θ with $\hat{n}$. Now the scalar product

$$A \cdot dS = A \cdot n\ dS = A\ dS \cos\theta$$

is called the flux of vector field A across the area element dS. The total flux of the vector field across the entire surface area S is given by

$$\iint_s A \cdot dS = \iint_s A \cdot n\ dS = \iint_s A \cos\theta\ dS$$

This is defined as the surface integral.

3. Volume Integral

The integral evaluated over a three dimensional domain is known as volume integral.

Consider a closed surface in space enclosing a volume V. If A be a vector point function at a point in a small element of volume dV, then the integral

$$\iiint_V A\, dV$$

is called the volume integral of vector A over the surface.

11. If f = $(x^2 + y^2 + z^2)^{1/2}$ then find grad ϕ.

Ans :

Given,

$$\phi = (x^2 + y^2 + z^2)^{1/2}$$

$$\text{Grand } \phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\cdot\frac{\partial}{\partial x}[(x^2 + y^2 + z^2)^{1/2}] + \bar{j}.\frac{\partial}{\partial y}[(x^2 + y^2 + z^2)^{1/2} + \bar{k}.\frac{\partial}{\partial z}[(x^2 + y^2 + z^2)^{1/2}]$$

$$= \bar{i}.\frac{1}{2}(x^2+y^2+z^2)^{\frac{1}{2}-1}(2x) + \bar{j}.\frac{1}{2}(x^2+y^2+z^2)^{\frac{1}{2}-1}(2y) + \bar{k}.\frac{1}{2}(x^2+y^2+z^2)^{\frac{1}{2}-1}(2z)$$

$$= \bar{i}(x^2+y^2+z^2)^{\frac{1}{2}-1}(x) + \bar{j}(x^2+y^2+z^2)^{\frac{-1}{2}}(y) + \bar{k}(x^2+y^2+z^2)^{\frac{-1}{2}}(z)$$

$$= \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}\left[x\bar{i} + y\bar{j} + z\bar{k}\right]$$

$$\text{Grad } \phi = \frac{x\bar{i} + y\bar{j} + z\bar{k}}{(x^2+y^2+z^2)^{1/2}}$$

12. What are different kinds of vectors?

Ans :

(i) Zero or Null Vector

The zero or the null vector is a vector whose mudulus is zero, and the whose direction is indeterminate. The null vector is represented by the symbol 0 (printed in bold faced typed). In the case of the null vector the initial and terminal points coincide. Thus $\overrightarrow{AA}$, $\overrightarrow{OO}$, etc. are null vectors.

(ii) Unit Vector

A vector whose, modulus is unity, is called a unit vector. The unit vector in the direction of vector a is represented by $\hat{a}$. It is read as 'a Cap'.

(iii) Like and Unlike Vectors

Vectors having the same direction are called like vectors and those having opposite directions are called unlike vectors.

(iv) Collinear or Parallel Vector

Vectors having the same line of action or having the lines of action parallel to one another are called collinear or parallel vectors.

(v) Equal Vectors

Two vectors are said to be equal if, and only if, they are parallel, have the sense of direction, and the same are called like vectors and those having opposite directions are called unlike vectors.

(vi) Collinear or Parallel Vectors

Vectors having the same line of action or having the lines of action parallel to one another are called collinear or parallel vectors.

Choose the Correct Answers

1. Two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other if [a]

   (a) $\vec{A}\vec{B}=0$ (b) $\vec{A}\times\vec{B}=0$

   (c) $\vec{A}\cdot\vec{B}=1$ (d) $\vec{A}\times\vec{B}=1$

2. If l, m, n are the direction cosines of a vector, then [b]

   (a) $l^2 + m^2 + n^2 = 0$ (b) $l^2 + m^2 + n^2 = 1$

   c) $l^2 + m^2 + n^2 > 0$ (d) $l^2 + m^2 + n^2 < 1$

3. Two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other if ______. [a]

   (a) $\vec{A}\cdot\vec{B}=0$ (b) $\vec{A}\times\vec{B}=0$

   (c) $\vec{A}\cdot\vec{B}=1$ (d) $\vec{A}\times\vec{B}=1$

4. Three vectors $\vec{A}, \vec{B}$ and $\vec{C}$ will form a triangle if [c]

   (a) $\vec{A}+\vec{B} > \vec{C}$ (b) $\vec{A}+\vec{B}<\vec{C}$

   (c) $\vec{A}+\vec{B}=\vec{C}$ (d) $\vec{A}+\vec{B}+\vec{C}=0$

5. Volume of parallelopiped formed by $\vec{A}, \vec{B}$ and $\vec{C}$ is ______. [d]

   (a) $\vec{A}\times(\vec{B}\cdot\vec{C})$ (b) $\vec{A}(\vec{B}\times\vec{C})$

   (c) $\vec{A}\cdot(\vec{B}\cdot\vec{C})$ (d) $\vec{A}\cdot(\vec{B}\times\vec{C})$

6. The magnitude of the vector drawn in a direction perpendicular to the surface $x^2+2y^2+z^2 = 7$ at the point (1, –1, 2) is [d]

   (a) $\frac{2}{3}$ (b) $\frac{3}{2}$

   (c) 3 (d) 6

7. The directional derivative of $\phi = xyz$ at the point (1, 1, 1) in the direction $\hat{i}$ is [c]

   (a) –1 (b) $-\frac{1}{3}$

   (c) 1 (d) $\frac{1}{3}$

8. If $\vec{f} = ax\,\vec{i} + by\,\vec{j} + cz\,\vec{k}$, a, b, c are constants, then $\iint F.ds$ where S is the surface of a unit sphere is [b]

(a) $\frac{\pi}{3}(a+b+c)$ (b) $\frac{4}{3}\pi(a+b+c)$

(c) $2\pi\,(a+b+c)$ (d) $\pi\,(a+b+c)$

9. The line integral $\int_c x^2dx + y^2dy$, where c is the boundary of the region $x^2+y^2<a^2$ equals [b]

(a) 0 (b) 0

(c) πa^2 (d) $\frac{1}{2}\pi a^2$

10. If $\vec{R} = xi + yi + zk$ and $\vec{A}$ is a constant vector, curl $(\vec{A}\times\vec{R})$ is equal to [d]

(a) $\vec{R}$ (b) $2\vec{R}$

(c) $\vec{A}$ (d) $2\vec{A}$

11. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $(\vec{r}) = r$, then $div\,\vec{r}$ is [ ]

(a) 2 (b) 3

(c) –3 (d) –2

12. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ then $\nabla\phi(r)$ is [c]

(a) $\phi'(r)\vec{r}$ (b) $\frac{\phi'(r)\vec{r}}{r}$

(c) $\frac{\phi'(r)\vec{r}}{r}$ (d) none of these

13. Two vectors are such that $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$. The angle between the vectors is [d]

(a) 0^0 (b) 30^0

(c) 60^0 (d) 90^0

14. Pict out the scalar quantity out of the following : [b]

(a) Force (b) Electric potential

(b) Momentum (d) Intensity of electric field

15. If two forces are equal and their resultant is also equal to one of them, then the angle between the two forces is [b]

(a) 60^0 (b) 120^0

(c) 90^0 (d) 0^0

16. Which of the following is scalar quantity ? [a]

(a) work (b) acceleration

c) electric field d) displacement

17. Which is vector quantity ? [a]

(a) Flux density (b) Magnetic flux

c) Intensity of magnetic flux (d) Magnetic potential

18. A vector $\vec{A}$ points vertically upward and $\vec{B}$ points towards north. The vector product $\vec{A} \times \vec{B}$ is : [b]

(a) zero (b) along west

(c) along east (d) vertically downward

19. Surface is _______. [d]

(a) scalar (b) vector

(c) neither scalar nor vector (d) both scalar and vector

20. The vector sum of two forces is perpendicular to their vector difference. In that case, the forces [c]

(a) are not equal to each other in magnitude

(b) cannot be predicted

(c) are equal to each other

(d) are equal to each other in magnitude

Fill in the Blanks

1. The line integral $\int_c x^2dx + y^2dy$, where c is the boundary of the region $x^2+y^2<a^2$ equals _______.

2. A force field $\vec{F}$ is said to be conservative if _______.

3. If $\vec{F}$ is the velocity of a fluid particle then $\int_c \vec{F}.\vec{dr}$ represents _______.

4. If $\vec{A}$ is such that $\nabla \times \vec{A} = 0$ then $\vec{A}$ is called _______.

5. If $\vec{F}$ is a conservative force field, thent he value of curl $\vec{F}$ is _______.

6. The unit vectors $\hat{r}, \hat{\theta},$ and $\hat{\phi}$ are _______.

7. A particle is moving in a plane, its velocity $\vec{v}$ is given by _______.

8. Total vector surface area of a closed volume is _______.

9. Two vectors $\vec{A}$ and $\vec{B}$ are collinear if _______.

10. If $\phi(x, y, z)$ be a scalar function then $\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial y}{\partial z}$ is called _______.

11. The integration of a vector along a curve is called its _______.

12. If A be a vector point function at a point in a small element of volume dv, then the integral $\iiint_v A\, dv$ is called _______.

13. The _______ of a vector field is defined as the maximum line integra of the vector per unit area.

14. The scalar product or dot product of two vectors A and B is defined as _______.

15. If vector r, is a function of a scalar variable t, then we write $\vec{r} =$ _______.

16. The magnitude of a vector cannot be _______.

17. The angle between vectors $(\vec{A} \times \vec{B})$ and $(\vec{B} \times \vec{A})$ is _______.

18. If $\hat{n}$ is the unit vector in the direction of $\vec{A}$, then $\hat{n} =$ _______.

ANSWERS

1. a
2. Curl $\vec{F} = 0$
3. circulation
4. irrotational
5. 0
6. perpendicular
7. $\dot{r}\,\hat{r} + r\,\dot{\theta}\hat{\theta}$
8. null vector
9. $\vec{A} \times \vec{B} = 0$
10. gradient of the scalar function ϕ.
11. line integral
12. the volume integral of vector A.
13. Curl
14. the product of the magnitudes of two vectors.
15. $\vec{r}(t)$.
16. negative
17. π
18. $\frac{\vec{A}}{|\vec{A}|}$

UNIT II	**MECHANICS OF PARTICLES:** Laws of motion, motion of variable mass system, motion of a rocket, multi-stage rocket, conservation of energy and momentum. Collisions in two and three dimensions, concept of impact parameter, scattering cross-section **MECHANICS OF RIGID BODIES :** Definition of Rigid body, rotational kinematic relations, equation of motion for a rotating body, angular momentum and inertial tensor. Euler's equation, precession of a top, Gyroscope

2.1 Mechanics of Particles

Q1. Explain the concept of mechanics of particles.

Ans :

Introduction

A particle is considered to be an object having mass and having neither any size nor any internal structure at all. It may be defined as a point mass. If the relative positions of different particles change with time, then such an arrangement is referred to as a system of particles. A rigid body stands as an example for a system of particles in which there will be a continuous distribution of mass and the relative distance between any two particles will not change even when the body is subjected to an external force.

In mechanics we treat a body as just a point mass (a particle) as an approximation. But this approximation is not always valid. Any body is an aggregate of particles (a system of particles). The body may be a solid or a fluid. It is possible to determine the general laws of motion of bodies like an Indian club (a rigid body), flowing water or galaxies (flexible systems) without considering the mutual influence between various particles of the system. We consider only the external influence - that is, the external force.

The branch of physics that deals with the motion of one particle (or a body) or a system of particles (or bodies) without any reference to the forces acting on, is called Kinematics. In this unit we are concerned with the motion (change of coordinates of particle or particles among themselves and with time) of one particle (or body) or many particles (or bodies) with reference to the forces acting on. This branch of physics is called Particle Dynamics. First we shall deal with a single particle.

2.1.1 Laws of Motion

Q2. State and explain Newton's Law of Motion.

Ans : **(Aug.-21)**

Newton's laws are applied to objects which are idealised as single point masses, in the sense that the size and shape of the object's body are neglected to focus on its motion more easily. This can be done when the object is small compared to the distances involved in its analysis, or the deformation and rotation of the body are of no importance. In this way, even a planet can be idealised as a particle for analysis of its orbital motion around a star.

In their original form, Newton's laws of motion are not adequate to characterise the motion of rigid bodies and deformable bodies. Leonhard Euler in 1750 introduced a generalisation of Newton's laws of motion for rigid bodies called Euler's laws of motion, later applied as well for deformable bodies assumed as a continuum. If a body is represented as an assemblage of discrete particles, each governed by Newton's laws of motion, then Euler's laws can be derived from Newton's laws. Euler's laws can, however, be taken as axioms describing the laws of motion for extended bodies, independently of any particle structure.

Newton's laws hold only with respect to a certain set of frames of reference called Newtonian or inertial reference frames. Some authors interpret the first law as defining what an inertial reference frame is; from this point of view, the second law only holds when the observation is made from an inertial reference frame, and therefore the first law cannot be proved as a special case of the second. Other authors do treat the first law as a corollary of the second. The explicit concept of an inertial frame of reference was not developed until long after Newton's death.

In the given interpretation mass, acceleration, momentum, and (most importantly) force are assumed to be externally defined quantities. This is the most common, but not the only interpretation of the way one can consider the laws to be a definition of these quantities.

Newtonian mechanics has been superseded by special relativity, but it is still useful as an approximation when the speeds involved are much slower than the speed of light.

Q3. Describe Newton's Law of Motion.

Ans :

I. Newton's First Law of Motion

Every particle or system will continue its initial state till the force acts on it. If no external force is acting, the particle in rest will remain in rest and the particle moving with some uniform velocity will continue with the same velocity. This law tells that, w.r.t. to center of mass, the center of mass of an isolated system will remain at rest or moves with uniform velocity. As a matter of fact every particle or system will oppose the change of state. This law introduces a quantity called inertia.

Inertia

It is the inherent property of a particle to oppose the change of state. It can be measured with the mass of particle.

Examples

1. A passenger standing in a jerk train will get a when the train suddenly moves. Sometimes it may happen that passenger falls in a direction opposite to the movement of train. This is due to inertia of rest.
2. Similarly if a bus takes a sudden turn while going, then passenger sitting in the bus may bend away from the turn. This is due to directional inertia.
3. Bicyclist going with more speed suddenly used the front brakes to stop it, then he may fall. This is due to motional inertia.

Force

From the first law of Newton, force can be defined. Force is an agency to change or trying to change the state of a system.

Sometimes two or more particles in a system under the action of internal forces. But these internal forces can not change the state. Only the external forces can change the state. We can understand this by the following examples.

1. An object on a table can not be moved without applying a force on it. That is the property of inertia may helps it to be in same state. But it changes its position only by applying the force.
2. A driver should apply accelerator to improve the speed of a bus, similarly he has to apply the brakes to bring it to rest. So only on the application of force, we can change the state of a system.

II. Newton's Second Law of Motion

This law based on the concept of momentum. The quantity momentum is the product of mass and velocity, which is a vector. Its unit in S.I system is kg ms^{-1}. Its dimensional formula MLT^{-1}.

Second law of Newton states that "rate of change of momentum is equal to the external force acting on it". The change in momentum happens only in the direction of force. From this law, force and acceleration can be related.

Let us consider a particle of mass 'm' is moving with initial velocity 'u'. A force 'F' is applied for a time 't' sec. The final velocity of the particle is v.

Initial momentum $P_i = mu$.

Final momentum $P_f = mv$

Change in momentum $P_f - P_i = m(v-u)$

$$\text{Rate of change of momentum} = \frac{m(v-u)}{t}$$

But from the definition of acceleration

$$a = \frac{(v-u)}{t}$$

$\therefore$ Rate of change of momentum = ma.

From the second law, force applied.

$F \alpha\, ma$

and F = k.ma, K is constant of proportionality. Its value depends on the measure of force. Unit force is one which can produce an acceleration of one m/s^2 in a one kg object. i.e., if m = 1, a = 1 then F = 1 for which K = 1.

$$\therefore \quad F = ma.$$

In the S.I system, units of force is Newton and in C.G.S system unit of force is Dyne.

$$1N = 10^5 \text{ Dynes.}$$

Newton's Third Law of Motion

This law will not explain the motion of a particle but explains the mutual interacting forces among the particles. The law states that "for every action there is an equal and opposite reaction" present. So no force can be applied on a particles without the reaction, since action and reaction observed on two different particles.

Let an object 'A' applied a force F_{AB} on B another object 'B'. Simultaneously 'B' applies a fore F_{BA} on the object 'A'. But from the third law of Newton F_A = - F_{BA}. The force F_{AB} is action and F_{BA}. is reaction.

Examples

1. Book placed on a table applies its weight on the table which is action in turn table applies the source force on the book in opposite direction which is reaction.
2. Revolving moon attracted by the earth and at the same time then moon also attracts the earth with the same force.
3. Even in an atom the coulombic forces between the nucleus and electron also constitute action and reaction pair.

Q4. What are the physical quantities involved in the dynamics of a particle?

Ans :

In the dynamics of a particle we come across with several physical quantities as force, linear momentum, impulse, angular momentum etc., in addition to the familiar physical quantities of length, mass, time, velocity and accleration. We have already come across force in figure.

Let us consider some of these physical quantities.

(a) Linear Momentum (p)

We have already seen that the product of the mass 'm' and velocity 'v' of a particle is defined as its momentum 'p' usually called the linear momentum.

$$p = mv$$

As, velocity v is a vector, momentum p is also a vector quantity. As is evident from the above equation, a heavy mass moving with a small velocity may have the same momentum as a lighter mass moving with a greater velocity. It is momentum the product of the mass and velocity of a body, that gives an estimation of the force required to stop the body.

(b) Impulse

In any given impact, the total change in the linear momentum of any body due to the action of a force is defined as the impulse of the force.

$$\text{Impulse} = \Delta p = p_2 - p_1 = m(v_2 - v_1) \quad ....(1)$$

where v_1, p_1 refer to initial velocity and initial momentum and v_2, p_2 refer to final velocity and final momentum.

The impulse is a vector quantity.

As $F = \frac{dp}{dt}$ we can write

$$\Delta p = p_2 - p_1 = \int_{t_1}^{t_2} dp = \int_{t_1}^{t_2} F.dt$$

when we apply a constant force F,

$$\text{Impulse } \Delta p = F.\Delta t \quad ...(2)$$

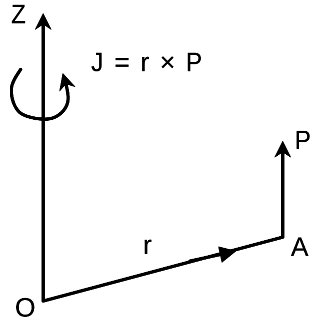
Impulse is hence defined as the product of the average force multiplied with the duration of the impact. Impulse plays an important role when very large forces are acting for a very short interval of time. Examples are hitting a nail into a wall, hitting a cricket ball with bat.

(c) Angular momentum (L or J)

The angular momentum J of a particle of mass 'm', revolving around an axis is defined by.

$J = r \times p$ ...(3)

where p is the linear momentum of the particle A, and r is the radius vector from the axis of rotation to the particle, as shown in the figure.



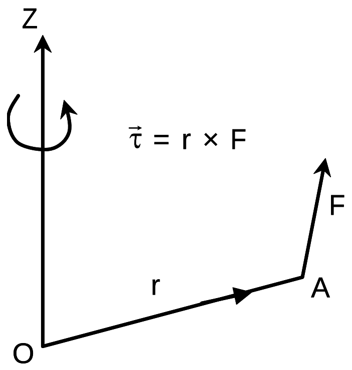
It is also the moment of momentum. It is a pseudo vector (as it is the cross product of two real vectors). The concept of angular momentum is quite useful in describing rotatory motion of objects in a way just as the linear momentums plays an important role in the linear motion.

(d) Torque ($\vec{\tau}$)

The torque $\vec{\tau}$ of a force 'F' around an axis is defined by

$\vec{\tau} = r \times F$ ...(3)

where $\vec{F}$ is the force acting on the particle A, and r is the radius vector from the axis of rotation to the particle as shown in figure.



It is also called the moment of force around the axis. It is a pseudo vector.

It plays the same role in rotatory motion as the force F does in the linear motion. If the force F acts along the radius rector r then there will be only translatory motion and there will be no rotatory motion for the particle. ($r \times F = 0$ in such a case)

Relation between Angular Momentum and Torque :

Angular momentum $J = r \times p$ and hence

$$\frac{dJ}{dt} = r \times \frac{dp}{dt} + \frac{dr}{dt} \times p$$

But $$\frac{dr}{dt} \times p = \frac{dr}{dt} \times mv = m.\frac{dr}{dt} \times \frac{dr}{dt} = 0$$

$$\therefore \quad \frac{dJ}{dt} = r \times \frac{dp}{dt} = r \times F = \vec{\tau}$$

$$\vec{\tau} = \frac{dJ}{dt} \quad ...(4)$$

Thus, we can define torque as the rate of change of angular momentum just as we defined force as the rate of change of linear momentum.

From equation (4) it is evident that,

If $\vec{\tau} = 0$, then $\frac{dJ}{dt} = 0$

i.e., J is a constant.

It means that, in the absence of an external torque, the angular momentum of a body is conserved. This is called the law of conservation of angular momentum.

(e) Work (W)

Work is said to be done by a force (F) when its point of application undergoes a change and is measured by the product of the force and the displacement in the direction of the force.

If, on the application of a force F, a body undergoes a displacement r in a direction making an angle q with F, then work done by the force is given by

$$dW = F(dr \cos \theta) = F.dr \qquad ...(5)$$

and the total work done is

$$W = \int_{r_1}^{r_2} F.dr \qquad ...(6)$$

Evidently, work is a scalar quantity.

If work done (W) by a force is positive, it means that the force actually does work on the body.

If W turns out to be negative, it means that the box i is doing work against the force (for example, when brakes are applied to a car, the car does work against the braking force an ultimately comes to rest).

(f) Energy (W or E)

The energy associated with a body is defined as the capacity of the body to do work. The body is not doing any work, but it can do work when it is necessary. In mechanics we have energy in two different forms - Kinetic Energy and Potential Energy.

Kinetic Energy is the energy possessed by a body by virtue of its motion. Potential Energy is the energy possessed by a body by virtue of its position or state in a force field (like gravitational field or electrical field etc.)

(g) Kinetic Energy of a body and the Work-Energy Theorem

Let us consider a body of mass 'm', moving with a velocity 'v'. By virtue of its motion, the body possesses kinetic energy. This will be equal to the work that the body can do.

Kinetic energy of a body of mass m, moving with a velocity v is given by

$$K.E. = \frac{1}{2}mv^2 \quad \text{or} \quad K = \frac{1}{2}mv^2 \qquad ...(7)$$

Work - Energy Theorem

Let us consider that a body of mass 'm' is under the continuous influence of (an unbalanced or resultant) a force 'F' and as a result its velocity change form v_i to v_f when it moves from position r_1 to position r_2.

The work done by the force is given by

$$W = \frac{1}{2} mv^2{}_f - \frac{1}{2} mv_i^2 \qquad ... (8)$$

$$\text{(i.e.) } W = K_{final} - K_{initial} \qquad ... (9)$$

That is, the work done by an unbalanced or resultant force on a body is equal to the change in kinetic energy of the body. This is called the Work-Energy theorem.

(h) Potential Energy of a Body (P.E.)

The potential energy of a body at a point in a filed is defined as the work, that can be done by the body in moving from the given position in the field to the reference point of zero potential energy.

The capacity of the body to do work because of its position arises due to the existence of a force field, such as gravitational field, electric field etc.

For example, let us consider a body of mass m, at a height 'h' from the surface of the earth.

The potential energy of a body of mass m, at a height h from the ground is given by

$$P.E. = mgh \qquad ...(10)$$

From U(h) = mgh we can notice that,

$$-\frac{\partial U(h)}{\partial h} = -mg = |F(h)|$$

In general, are can write as

$$|F(x)| = -\frac{\partial U(x)}{\partial x} \qquad ...(11)$$

or, in the most generalized from

$$F(r) = \vec{\nabla} U(r) \qquad ...(12)$$

i.e., Force is the negative gradient of potential energy.

Q5. Define linear momentum of a particle and system of particles.

(OR)

State law of conservation of linear momentum and explain with two examples.

Ans :

1. Linear Momentum of a Particle

Let the mass of a particle is 'm' and its velocity is $\vec{v}$, then

Linear momentum $\vec{P} = m\vec{v}$

The measure of every quantity depends on the frame of reference. While mentioning a quantity one have to mention a frame of reference. From newton's second law, the rate of change in momentum is equal to the external force acting on it. The direction of change in momentum can be expected along the direction of force.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d}{dt}\vec{v} = m\vec{a}$$

Since $\frac{d\vec{v}}{dt} = \vec{a}$, $\vec{a}$ is the acceleration.

So $\vec{F} = m\vec{a}$

2. Linear Momentum of System of Particles

Consider a system of N particles. Their masses are $m_1, m_2, ... m_n$. And their velocities are $v_1, v_2, ...$ vn respectively.

Total linear momentum of the system

$$\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + ... \quad ...(1)$$

But $\vec{p} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3$ ....

From the definition of centre of mass total linear momentum of a system is equal to the momentum of centre of mass.

$\vec{p} = M\vec{v}_{cm}$ ($\vec{v}_{cm}$ is the velocity of centre of mass, M is total mass.)

Differentiating equation (1) w.r.t time.

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{cm}}{dt}$$

$$\frac{d\vec{p}}{dt} = M\vec{a}_{cm}$$

$$\frac{d\vec{P}}{dt} = \vec{F}_{ext} \quad ...(2)$$

$\vec{F}_{ext}$ is the external force acting on the system of particles. In a system one can expect the internal force amount the particles, but these forces will cancel in terms of action and reaction pairs.

3. Conservation of Linear Momentum

If no external force is acting on the system ($\vec{F}_{ext} = 0$) from equation (2) we have

$$\therefore \frac{d\vec{P}}{dt} = 0$$

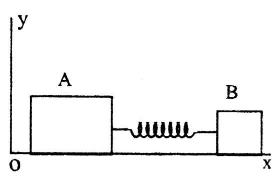
$\vec{P}$ is a constant

If no external force is acting on a system of particles, the total linear momentum of the system of particulars remains same which imply the conservation of linear momentum.

Example

Two masses A, B are with masses m_A and m_B respectively. They are connected by a spring as shown in figure, and placed on a table. Masses are pulled apart and released. Two masses are vibrating along the length of the spring.

Two masses are experiencing internal mutual attractive force. Treating two masses as a whole, no external force acting on it before the masses are pulled and after their relax, system should follow conservation if linear momentum.



Initial momentum of the system = Final momentum of the system.

$$0 = m_A v_A + m_B v_B$$

where v_A, v_B are the velocities, after the masses were released.

So $m_A v_A + m_{BvB}$

or $v_A = \frac{-m_B v_B}{m_A}$

Kinetic energy of mass A is

$$K_A = \frac{1}{2} m_A v_A^2 = \frac{(m_A v_A)^2}{2m_A}$$

Kinetic energy of mass B is

$$K_B = \frac{1}{2} m_B v_B^2$$

From the above $\frac{K_A}{K_B} = \frac{m_A}{m_B} \times \frac{v_A^2}{v_B^2}$

Since $\frac{v_A}{v_B} = -\frac{m_B}{m_A}$

Hence $\frac{K_A}{K_B} = \frac{m_A}{m_B} \times \frac{m_B^2}{m_A^2} = \frac{m_B}{m_A}$

Due to conservation of linear momentum, the ratio of kinetic energies are equal to the reciprocal ratio of their masses.

2.1.2 Motion of Variable Mass System

Q6. Derive the equation of motion of variable mass system.

(OR)

Derive an expression for the motion of variable mass system

Ans : **(June-21, Dec.-19, Dec.-16)**

In the systems dealt with sofar, we always assumed that the total mass M of the system remains constant. Such systems are usually referred to as Isolated systems. But sometimes the mass may be changing with time. Either mass may enter into the system (dm/dt is positive) or mass may leave the system (dm/dt is negative) with time. For example, let us take the flight of a rocket. Most of the mass of a rocket consists of its fuel. After firing, all of the fuel will eventually be burnt and ejected out from the nozzle of the rocket engine.

Figure below shows a system of mass M whose centre of mass is moving with velocity v as seen from a particular reference at any instant t. At a latter instant $t + \Delta t$, a mass ΔM has been ejected from the system and its centre of mass moves with a velocity u as seen by an observer.

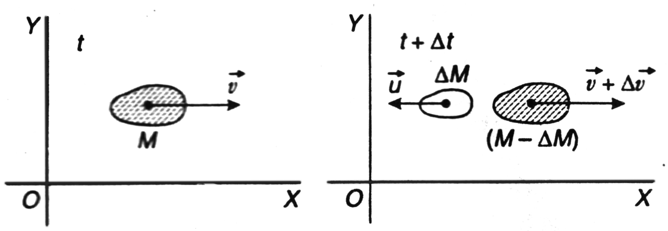


Fig.:

Now the system mass is reduced to M – ΔM and the velocity v of the centre of mass of the system is changed to v + Δv. The system represents a motion like of a rocket.

Such kinds of problems may be solved by applying Newton's second law to the combination of two masses i.e., the remaining mass (M – ΔM) and the ejected mass ΔM combined together as a whole system. It should be remembered that Newton's second law can not be applied separately either to mass (M – ΔM) alone or to mass ΔM.

From Newton's second law $F_{ext} = \dfrac{dP}{dt}$

For small interval of time $F_{ext} \simeq \dfrac{\Delta P}{\Delta t} = \dfrac{P_f - P_i}{\Delta t}$

Considering both the parts of masses ΔM and M – ΔM as forming one and the same system, we can write

$$F_{ext} = \frac{P_f - P_i}{\Delta t} = \frac{[(M - \Delta M)(v + \Delta v) + \Delta M u] - [Mv]}{\Delta t} \quad \text{... (1)}$$

where P_f and P_i are the final and initial momentum of the system respectively.

$$F_{ext.} = M\frac{\Delta v}{\Delta t} - v\frac{\Delta M}{\Delta t} - \Delta v\frac{\Delta M}{\Delta t} + u\frac{\Delta M}{\Delta t} \quad \text{... (2)}$$

If Δt approaches zero, the configuration on right hand side of figure, approaches that left hand side i.e., Δv/Δt approaches dv/dt, the acceleration of the body. Since there is a decrease in the mass with time hence ΔM/Δt is replaced by –dM/dt as Δt approaches zero. Finally, Δv goes to zero as Δt approaches zero. Making these changes, equation (2) leads to

$$F_{ext} = M\frac{dv}{dt} + v\frac{dM}{dt} - u\frac{dM}{dt}$$

or
$$F_{ext} = \frac{d}{dt}(Mv) - u\frac{dM}{dt} \quad \text{... (3)}$$

Equation (3) expresses Newton's second law as applied to a body of variable mass. It is obvious from this equation that d/dt(Mv) is not equal to the external force acting on the system unless the ejected mass comes out with zero velocity.

Equation (3) can also be expressed in the following form :

$$F_{ext} = M\frac{dv}{dt} + v\frac{dM}{dt} - u\frac{dM}{dt}$$

or $$M\frac{dv}{dt} = F_{ext.} + (u - v)\frac{dM}{dt} \qquad ... (4)$$

The last term of equation (4) is the rate of change of momentum of the system due to mass leaving it. This can be regarded as the reaction force exerted on the system by the leaving mass. For a rocket, this term is called the thrust and it is the rocket designer's aim to make it as large as possible. Now we can write

$$M\frac{dv}{dt} = F_{ext.} + F_{reaction.} \qquad ... (5)$$

2.1.3 Motion of a Rocket

Q7. Explain the principle and motion of a rocket.

(OR)

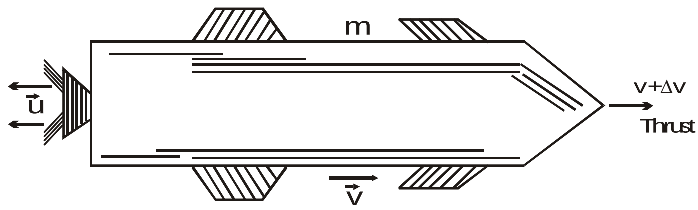
Derive an equation of motion of a rocket.

(OR)

Describe the principle of motion of a rocket as system of variable mass.

Ans : **(Dec.-17)**

A moving rocket in which the fuel gets burnt and comes out in the form of exhaust gases is a good example of a variable mass system. The rocket consists of a combustion chamber in which a liquid or a solid is burnt. When the fuel is burnt, the pressure inside the composition chamber rises very high. Due to the high pressure, the hot gases (burnt liquid or solid fuel) are expelled from the nozzle at the tail of the rocket). These expelled gases will be in the form of a jet having a very high exhaust velocity. This is the action. Consequently, as a reaction, the rocket moves in a direction opposite to the direction of the out coming gases. Thus, the rocket works on the principle of Newton's third law of motion or the law conservation of momentum which is a consequence of the Newton's third law.



Originally at time t, the rocket of mass M (including fuel) is moving with a velocity V in the laboratory frame of reference. Let us suppose that in a time interval Dt (that is at time t + Dt) an amount of mass DM is ejected from the rocket out in the form of exhaust gas jet. Let us suppose that – U is the velocity of exhaust gas jet with respect to the rocket. Then the velocity of the gas jet in the laboratory frame will be (V – U). Let us write.

$$V - U = V_{rel}$$

The time interval Δt may be made as small as possible, and in the limiting case, $\Delta t \to 0$

$$\frac{\Delta M}{\Delta t} \to \frac{dM}{dt}$$

$$\begin{matrix}\text{The rate of change of momentum of} \\ \text{gas} - \text{jet coming out of the rocket}\end{matrix} = \text{force acting on the jet}$$

$$= \text{(rate of change of mass the rocket)} \times \text{velocity}$$

$$= \frac{dM}{dt}(V - U)$$

According to Newton's third law of motion, this is equal in magnitude and opposite in direction to the thrust acting on the rocket responsible for the forward motion of the rocket.

$$\text{Thrust on the rocket } T = \frac{dM}{dt}(V - U) \qquad \text{... (1)}$$

Let $F_{external}$ be the external force acting on the rocket, which is due to gravitation.

Hence, $F_{ext} = -Mg$ ... (2)

$\therefore$ Force acting on the rocket in the forward direction will be

$$F_{Resultant} = \frac{dM}{dt}(V - U) - Mg \qquad \text{... (3)}$$

But, according to Newton's second law of motion, this force will be equal to the rate of change in the momentum (p = MV) of the rocket. That is

$$F_N = \frac{d}{dt}(MV) = M\frac{dV}{dt} + V\frac{dM}{dt} \qquad \text{... (4)}$$

From (3) and (4)

$$M\frac{dV}{dt} + V\frac{dM}{dt} = \frac{dM}{dt}(V - U) - Mg$$

$$= .\, V\frac{dM}{dt} - U\frac{dM}{dt} - Mg$$

or $$M\frac{dV}{dt} = -U\frac{dM}{dt} - Mg \qquad \text{... (5)}$$

This is the first rocket equation.

Now, let us go back to the stage when the rocket is just about to be fired when the gases just start burning. Initially, at t = 0, let the rocket has got a velocity V_0 and has got total mass M_0.

In the present state, let at t = t, the velocity the be V and mass M.

From equation (5)

$$\frac{dV}{dt} = -U\frac{dM}{dt} - g$$

or $$dV = -U\frac{dM}{M} - g\,dt$$

Integrating,

$$\int_{V_0}^{V} dV = -U\int_{V_0}^{V}\frac{dM}{M} - g\int_{V_0}^{V} dt = -U\int_{M_0}^{M}\frac{dM}{M} - g\int_{t_0}^{t} dt$$

$$[V]_{V_0}^{V} = -U\,[\text{Log}_e\,M]_{V_0}^{V} - g\,[t]_{t_0}^{t}$$

$$V - V_0 = -U\,\text{Log}_e\left(\frac{M}{M_0}\right) - g^t$$

or $$V - V_0 = U\,\text{Log}_e\left(\frac{M}{M_0}\right) - g^t$$

or $$V = V_0 + U\,\text{Loge}\left(\frac{M_0}{M}\right)$$

$$V = V_0 + U\text{Log}_e\left(\frac{M_0}{M}\right) - g_t \qquad \text{... (6)}$$

This is the second rocket - equation.

Let the rate of decrease of mass of the rocket be taken as α

(i.e.,) $$\frac{dM}{dt} = \alpha \qquad \text{... (7)}$$

If the exhaust gases flow out at a uniform rate, then α is a constant

and $$M = M_0 - \alpha t = M_0\left[1 - \frac{\alpha}{M_0 T}\right]$$

Let $$\beta = \frac{\alpha}{M_0} \qquad \text{... (8)}$$

Then $$M = M_0[1 - \beta t] \qquad \text{... (9)}$$

Equation (9) will be modified as

$$V - V_0 = U\,\text{Log}_e\left(\frac{1}{1-\beta t}\right) - g_t$$

or $$V = V_0 - U\log_e(1 - \beta t) - g^t \qquad \text{... (10)}$$

Special Cases :

(a) If mass of the rocket is ignored, we can neglect the external force due to gravitation – (i.e.,) the term involving ‘g’ and hence, From equation (6) we get,

$$V = V_0 + U \, Log_e \frac{M_0}{M} \quad \ldots (11)$$

and $$M \frac{dV}{dt} = -U \frac{dM}{dt} \quad \ldots (12)$$

(b) And further, if the initial velocity V_0 of the rocket is zero, then,

$$V = U \, Log_e \left(\frac{M_0}{M} \right) \quad \ldots (13)$$

(c) We can get the distance travelled by the rocket just by integrating the equations giving the velocity of the rocket (i.e., equation 6, 10, 11, 12).

2.1.4 Multistage Rocket

Q8. What are the various stages of the rocket(multistage rocket) in motion?

(OR)

Explain the working of multistage rocket.

Ans : **(Dec.-19, Dec.-18(MGU))**

A rocket is the vehicle employed for space journey. It works on the principle of jet propulsion. The principle of jet propulsion depends on the law of conservation of momentum, according to which the momentum of the jet emerging in the backward direction makes the rocket to move in the forward, direction.

According to the type of fuel used, rockets are classified as (i) liquid fuel rockets and (ii) solid fuel rockets.

A rocket to have maximum velocity at its final stage, insists.

1. Relative velocity of gases to be maximum.
2. Final mass of the rocket M is very much less than initial mass of the rocket M_0.

Relative velocity of gases coming out of rocket depends on temperature, pressure within the chamber. It also depends on area of crossection of the nozzle. With the presently using fuel, temperature that develops in the chamber is 3000^0C. Due to this temperature and depending on the crossesction of nozzle, the maximum relative velocity can be expected only 2 km/sec.

From the optimum design of the fuel chamber (for liquid fuels) presently the value of $\frac{M_0}{M}$ is maintained at nearly 10, and for solid fuels this value further low.

Therefore even on neglecting the gravitational force, the maximum velocity that a rocket can attain, starting from rest is from equation :

$$\bar{v} = \bar{u}_{rel} \log_e m - gt + c$$

$$\bar{v} = 0 + 2 \log_e (10) - 0$$

$$\bar{v} = 2 \times 2.3 = 4.6 \text{ km/sec.}$$

This velocity is very much less than the orbital velocity of a rocket i.e., (11.2 km/sec). Due to this reason, in order to launch a satellite multistage rockets are used. At the end of first stage, the rocket may attain a velocity nearly 4.6 km/sec and later second stage that begin to work and first stage of rocket detaches from the rocket. The velocity is adding up and finally, with all different stages the rocket attains the required velocity.

Q9. State and prove work-energy theorem.

Ans :

Statement :

The workdone by a force on a particle is equal to the change in the kinetic energy of the particle.

We know that the quantity $F \bullet dr = F\, dr \cos\theta$ is defined as the work done by force F on the particle during small displacement dr. When the force F acts on the particle during a finite displacement, the work done is obtained as

$$W = \int F \bullet dr = \int F \cos\theta \, dr \qquad ...(1)$$

Proof :

Consider that a body of mass m is acted upon by a resultant acceleration force F along the X-axis. Suppose the body moves from a position x_1 to position x_2 along X-axis. Let the velocity of the body increases from v_1 to v_2. The workdone by the force in the displacement is

$$W = \int_{x_1}^{x_2} F\, dx \qquad ... (2)$$

But according to Newton's second law

$$F = m\,a = m\frac{dv}{dt} = m\frac{dv}{dx} \cdot \frac{dx}{dt} = mv\frac{dv}{dx}$$

Substituting this value in equation (2), we get

$$W = \int_{x_1}^{x_2} m\,v = \frac{dv}{dx}\, dx = m\int_{v_1}^{v_2} v\, dv$$

$$= m\left[\frac{v^2}{2}\right] = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$= K_2 - K_1 \qquad ... (3)$$

where the quantity $\frac{1}{2} m\, v^2$ is defined as the knienetic energy of the body, K_2 and K_1 are the final and initial kinetic energies of the body.

If ΔK be the change in kinetic energy, then $\Delta K = K_2 - K_1$

$$\therefore \quad W = \Delta K \qquad ... (4)$$

So, whenever a body is acted upon by a number of forces, then the work-done by the resultant force is equal to change in the kinetic energy of the body. This is known as work-energy theorem.

2.1.5 Conservation of Energy

Q10. Explain about the law of conservation of energy.

Ans : **(June-17)**

Law of Conservation of Energy

Suppose conservative forces operate on a system of particles. If U_f and U_i be the potential energies of the system and W be the workdone then

$$U_f - U_i = -W$$

and $W = K_f - K_i$ (work energy theorem)

Then $U_f - U_i = -(K_f - K_i)$

$$(U_f + K_f) = (U_i + K_i) \quad ...(1)$$

The sum of potential energy and kinetic energy is called as total mechanical energy. Equation (1) shows that the total mechanical energy of the system remains constant when conservative forces are acting on the system. This is called as conservation of mechanical energy.

According to law of conservation of mechanical energy "energy can neither be created nor destroyed". It can be transformed from one form to another form.

Let us consider the case of a body of mass m at a height h above the ground as shown in figure (a). At position A, the kinetic energy of the body is zero while its potential energy is m g h. The sum of two energies is m g h. Now suppose that the body falls through a distance x where the velocity of the body becomes v. Using the formula, $v^2 = u^2 + 2gh$, we have the velocity as $v = \sqrt{(2gx)}$ because u = 0. In this case, the kinetic energy of the body is

$$\frac{1}{2}mv^2 = \frac{1}{2}m(2gx) = mgx$$

Potential energy of the body = m g (h – x).

$\therefore$ Total energy = m g x + m g (h – x) = m g h

Suppose the body reaches to ground where its potential energy is zero. Here kinetic energy of the body is m g h. So, the sum of kinetic energy and potential energy remains constant. So, the principle of conservation of energy may also be stated as "the total energy in any system always remains constant."

The variation of energy is shown in figure.

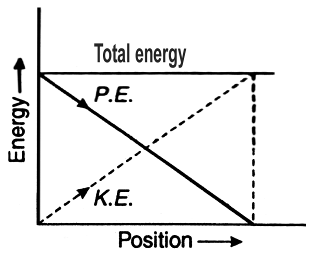


Fig.: (b)

It is obvious from figure that total energy remains constant (m g h) throughout.

2.1.6 Conservation of Momentum

Q11. State and prove the law of conservation of linear momentum. What is its importance in physics?

Ans :

Law of conservation of linear momentum : If the resultant external forces acting on a system of particles is zero, the total linear momentum of the system remains constant.

Proof :

Consider a system of n particles whose masses are $m_1, m_2, ... m_n$. The particles are in fixed positions so as to form a rigid body. Suppose the particles of the system are interacting with each other and are also acted on by external forces, so that they acquiare velocities $\nu_1, \nu_2, ... \nu_n$ respectively. Then their total momentum P of the system is the vector sum of the momenta $p_1, p_2, ... p_n$ of the individual particles.

i.e., $P = p_1 + p_2 + ... p_n$

$= m_1\nu_1 + m_2\nu_2 + ... m_n\nu_n$

Differentiating with respect to time t, we have

$$\frac{dP}{dt} = \frac{dp_1}{dt} + \frac{dp_2}{dt} + ... + \frac{dp_n}{dt}$$

or $$\frac{dP}{dt} = \frac{d}{dt}(p_1 + p_2 + ... + p_n)$$

or $F = F_1 + F_2 + ... F_n$

where $F_1, F_2, ... F_n$ are the forces acting on the particles of masses $m_1, m_2, ...m_n$ respectively.

The internal forces along cannot bring about any changes in the momentum of the body since they form pairs of equal and opposite forces according to III law of motion and balance each other. So they do not contribute anything to the total force. The above forces $F_1, F_2, ... F_n$ thus represent only the external force acting on the system and F their resultant.

If the resultant external force is zero i.e., $F = 0$ then

$$\frac{dP}{dt} = 0$$

i.e., $P = p_1 + p_2 + ... p_n = \text{constant}$

This is the law of conservation of linear momentum and may be stated as follows:

Statement :

If the resultant external forces acting on a system of particles is zero, the total linear momentum of the system remains constant.

Importance :

1. The law is a fundamental one and an exact law in nature. No violations of it have been observed.
2. In mechanics, the law is useful in solving many problems.

3. The law helps in the investigation of fundamental particles.
4. The law is applicable to atomic and nuclear physics.
5. The law holds good where Newtonian mechanics fails.
6. The law is applicable even at relativistic velocities.

Q12. State and prove the law of conservation of angular momentum with examples.

Ans :

Law of Conservation of Angular Momentum

The law of conservation of angular momentum states that if no external torque acts on a body rotating about a fixed point, the angular momentum of the body remains constant.

If a number of particles, each free to move independently of the other or attached to one another to form a rigid body, then the angular momentum of the different particles of the system about any point is the vector sum of the angular momenta of the individual particles about that point.

Thus if J_1, J_2, ... are the angular momenta of the different particles of the system about the given point, then the total angular momentum J of the whole system about that point is given by

$$J = J_1 + J_2 + ...$$
$$= (r_1 + mv_1) + (r_2 + mv_2) + ...$$
$$= \Sigma (r \times mv) = \Sigma (r \times p)$$

The torque τ is given by

$$\tau = \frac{dj}{dt} = \Sigma r \times \frac{d}{dt}(mv)$$
$$= \Sigma r \times F$$

In the above summation, the moments of the internal forces which form collinear action-and-reaction pairs of equal and opposite forces, having equal and opposite moments about the given point balance each other and hence they produce no effect.

Thus, the rate of change of angular momentum of the system about any fixed point is the sum of the torques about that point of all the external forces acting on the system.

If the external torque $\tau = 0$, then

$$\frac{dJ}{dt} = 0 \text{ or } J = \text{constant}$$

i.e., $J = J_1 + J_2 + ... = \text{constant}$

Thus when the external torque (or sum of the external torques) acting on a system of particles is zero, the total angular momentum of the system remains constant.

The above is the principle or law of conservation of angular momentum.

2.2 Collisions

Q13. Explain the concept of collisions. What are the types of collision?

Ans :

Meaning

Collision between two bodies mean their coming into contact with each other. The term at present, however, is not confined only to the actual contact. Two bodies which exert forces on each other and affect each other's motion, though not in actual contact are said to collide or interact with each other. For example, interaction between astronomical bodies through the gravitational force can be regarded as a collision. Another example being the interaction i.e., scattering of protons (or α-particles) by heavy nucleus through electrostatic forces can also be regarded as a collision.

Types

Motion of bodies after collision depends upon their degree of elasticity. Hence, collisions can be divided into two categories viz., inelastic collision and elastic collision.

(i) Inelastic collision

When two bodies after collision, move as one and do not have any tendency to separate as they were before collision, the collision is said to be perfectly inelastic.

For example, a collision between a bullet, and its target is completely inelastic, if the bullet remains embedded in the target after collision. The relative velocity between the bullet and the target after collision is zero.

The kinetic energy in all such collisions is not conserved. The linear momentum is, however, conserved.

(ii) Elastic collision

When two bodies after collision, separate by virtue of their elastic properties, the collision is said to be perfectly elastic.

Examples of such collisions are inter atomic collisions or collisions between subatomic particles. In practical life, collisions between" two ivory balls or two glass balls can be regarded as perfectly elastic.

No kinetic energy or momentum is lost in elastic collision i.e., conservation laws of both kinetic energy as well as momentum holds good in the case of perfectly elastic collisions.

In fact, no collision is perfectly elastic or perfectly inelastic. Only the degree of elasticity of collision varies. The degree of elasticity of collisions is measured in terms of the coefficient of restitution.

Definition

The coeff of restitution is the ratio of the relative velocity of separation of the two bodies after collision to the relative velocity of approach before collision.

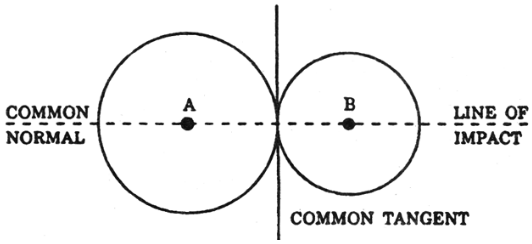
i.e., coefficient of restitution (e) $= \dfrac{\text{Rel vel of separation after collision}}{\text{Rel vel of approach before collision}}$

The value of e depends upon the nature of the colliding bodies. For example, $e = 0.5$, for two glass balls and $e = 0.20$, for two lead balls.

In the case of perfect elastic collision, $e = 1$.

Line of Impact

When two bodies collide, their surfaces will be in contact during collision. The line perpendicular to the common tangent to the surfaces at the point of contact is called the line of impact.



Direct impact

If the centres of mass of the colliding bodies are initially moving along the line of impact, then the impact is called head on or direct. After the direct impact, bodies continue to move along the line of impact.

Oblique impact

If the centres of mass of the colliding bodies are not initially moving along the line of impact, then the impact is called oblique.

Q14. Distinguish between elastic and inelastic collisions. Show that in one dimensional elastic collision between two particles

(i) the relative velocity of approach before collision is equal to the relative velocity of separation after collision.

(ii) if the particles are of equal mass, they simply exchange velocities during collision.

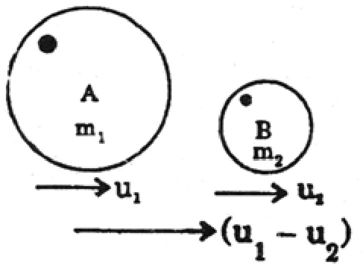
Ans : **(Aug.-21, Dec.-21)**

S.No.	Elastic Collisions	S.No.	Inelastic Collisions
1.	Elastic collisions conserve kinetic energy.	1.	Inelastic collisions do not conserve kinetic energy.
2.	These type of collisions do not occur in common and can never be observed at macroscopic scales.	2.	These type of collisions occur at atomic level.
3.	Coefficient of restitution of elastic collisions is equal to unity (i.e., e = 1).	3.	Coefficient of restitution of inelastic collisions lies between 0 and 1 (i.e., 0 < e < 1).
4.	The most common example of elastic collision is the collision between two billiard balls.	4.	The common example of inelastic collision is the collision between two vehicles wherein the colliding, vehicles get lock together with each other.

The kinetic energy in all such collisions is not conserved. The linear momentum is, however, conserved.

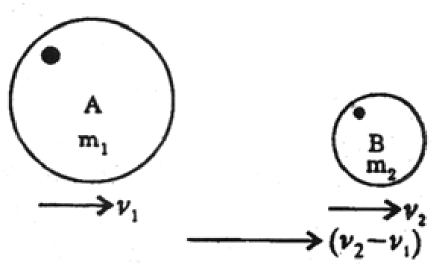
Elastic collision in one dimension

Consider two smooth and non-rotating spheres A and B of masses m, and m_2 moving in the direction shown in [Fig. (1) (a)] with velocities u_1 and u_2 respectively before collision. Let them collide headon without any rotation. During collision, the balls get depressed at the region of mutual contact and move together momentarily. A part of the kinetic energy is stored in the balls , as their potential energy of deformation: As the collision is perfectly elastic, the spheres tend to acquire the original state. In so doing, they separate and the potential energy of deformation reappears as their kinetic energy. Let u_1 and u_2 be their respective velocities after collision in the direction shown in [Fig. (1) (b)].



Before Collision

1 (a)



Before Collision

1 (a)

According to the principle of conservation of linear momenta

$$\left.\begin{matrix}\text{Momenta of A and B} \\ \text{before collision}\end{matrix}\right\} = \begin{pmatrix}\text{Momenta of A and B} \\ \text{after collision}\end{pmatrix}$$

$$\therefore \quad m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$\text{or} \quad m_1(u_1 - v_1) = m_2(v_2 - u_2) \qquad \text{... (1)}$$

Applying the principle of conservation of K.E., we have

$$\left.\begin{matrix}\text{Total K.E. of A and B} \\ \text{before collision}\end{matrix}\right\} = \begin{pmatrix}\text{Total K.E. of A and B} \\ \text{after collision}\end{pmatrix}$$

$$\therefore \quad \frac{1}{2} m_1u_1^2 + \frac{1}{2} m_2u_2^2 = \frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2$$

$$\text{or} \quad \frac{1}{2} m_1(u_1^2 - v_1^2) = \frac{1}{2} m_2(v_2^2 - u_2^2) \qquad \text{... (2)}$$

Eq (2) $\div$ Eq. (1), we have

$$\frac{m_1(u_1^2 - v_1^2)}{m_1(u_1 - v_1)} = \frac{m_2(v_2^2 - u_2^2)}{m_2(v_2 - u_2)}$$

$$\text{or} \quad u_1 + v_1 = v_2 + u_2$$

$$\text{or} \quad u_1 + v_1 = v_2 + u_2 \qquad \text{... (3)}$$

Thus in an elastic one dimensional collision, the relative velocity of approach before collision is equal to the relative velocity of separation after collision.

$$\left[\text{or } \frac{(v_2 - v_1)}{(u_1 - u_2)} = 1\right]$$

$$\text{i.e.,} \quad \frac{\text{Rel. vel of separation after collision}}{\text{Rel. vel of approach before collision}} = 1$$

This means that the coefficient of restitution is unity.

Calculation of velocities after collision :

$$v_2 - v_1 = u_1 - u_2 \qquad \text{... Eq. (3)}$$

or $\quad v_2 = u_1 - u_2 + v_1$

Substituting this value of v_2 in Eq. (1), we get

$$m_1 (u_1 - v_1) = m_2 (v_2 - u_2)$$

$$= m_2 (u_1 - u_2 + v_1 - u_2)$$

$$= m_2 (u_1 - 2u_2 + v_1)$$

or $\quad m_1u_1 - m_1v_1 = m_2u_1 - 2m_2u_2 + m_2v_1$

or $\quad u_1(m_1 - m_2) + 2m_2u_2 = v_1 (m_1 + m_2)$

$$\therefore \quad v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 + \left(\frac{2m_2}{m_1 + m_2}\right) u_2 \qquad \text{... (5)}$$

Substituting this value of v_1 in Eq. (4), we get

$$u_2 = u_1 - u_2 + v_1 \qquad \text{... Eq. (4)}$$

$$= u_1 - u_2 + \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 + \left(\frac{2m_2}{m_1 + m_2}\right) u_2$$

$$v_2 = u_1 \left[1 + \frac{m_1 - m_2}{m_1 + m_2}\right] + u_2 \left[\frac{2m_2}{m_1 + m_2} - 1\right]$$

$$= \left(\frac{2m_1}{m_1 + m_2}\right) u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) u_2 \qquad \text{... (6)}$$

Special Cases :

(i) When the colliding bodies are of equal masses i.e., $m_1 = m_2$. Putting m_1 for m_2 in Eq. (5), we get

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 + \left(\frac{2m_1}{m_1 + m_2}\right) u_2 \qquad \text{... (5)}$$

$$= \left(\frac{m_1 - m_1}{m_1 + m_1}\right) u_1 + \left(\frac{2m_1}{m_1 + m_2}\right) u_2$$

$$= 0 + \frac{2m_1}{2m_1} u_2$$

i.e., $\quad v_1 = u_2$

Similarly, putting m_1 for m_2 in Eq. (6), we get

$$v_2 = \left(\frac{2m_1}{m_1+m_2}\right) u_1 + \left(\frac{m_2-m_1}{m_1+m_2}\right) u_2 \quad \text{... (6)}$$

$$= \left(\frac{2m_1}{m_1+m_1}\right) u_1 + \left(\frac{m_1-m_1}{m_1+m_1}\right) u_2$$

$$= \frac{2m_1}{2m_1} u_1 + 0$$

i.e., $v_2 = u_1$

Thus $v_1 = u_2$ and $v_2 = u_1$ i.e., the velocities of two perfectly elastic bodies of the same mass are interchanged after collision.

(ii) When one of the colliding bodies is at rest. Suppose the sphere B is at rest i.e., $u_2 = 0$. Putting $u_2 = 0$ in Eq. (5) and (6), we get

$$v_1 = \left(\frac{m_1-m_2}{m_1+m_2}\right) u_1 \quad \text{... (7)}$$

and $$v_2 = \left(\frac{2m_1}{m_1+m_2}\right) u_1 \quad \text{... (8)}$$

Let us further consider three sub-cases

(a) When both the spheres are of the same mass i.e., $m_1 = m_2$, then

$$v_1 = \left(\frac{m_1-m_1}{m_1-m_1}\right) u_1 = \frac{0}{2m_1} u_1 = 0 \text{ from Eq. (7)}$$

$$v_2 = \left(\frac{2m_1}{m_2+m_1}\right) u_1 = \frac{2m_1}{2m_1} u_1 = u_1 \text{ from Eq. (8)}$$

Hence, after collision, the sphere A stops and B moves with the initial velocity of A i.e., the velocities of A and B get interchanged after collision. This result has one of the most important applications in nuclear physics i.e., in slowing down the neutrons. As neutrons and protons have equal masses, most of the energy of the fast neutrons is reduced and transferred to protons by passing them through a substance like water which contains a number of protons.

(b) When m_1 is negligible as compared to m_2, then

$$v_1 = \left(\frac{0-m_2}{0+m_2}\right) u_1 = u_1 \quad \text{... from Eq. (7)}$$

... from Eq. (8)

and v_2 becomes very small $\to 0$

Hence the velocity of the sphere A after impact is equal and opposite to its velocity before impact. The velocity of the sphere B, after impact is almost equal to zero.

Thus when a smooth elastic sphere strikes a smooth elastic plane fixed rigidly to earth, it rebounds back nearly with the same speed.

(c) When m_2 is negligible as compared to m_1, then

$v_1 = u_1$ (approximately) ... from Eq. (7)

and $v_2 = 2u_2$ (approximately) ... from Eq. (8)

Thus, the velocity of the sphere A, after impact, is nearly equal to its velocity before impact and the velocity of the sphere B, after impact, is nearly double the velocity of the sphere A before impact.

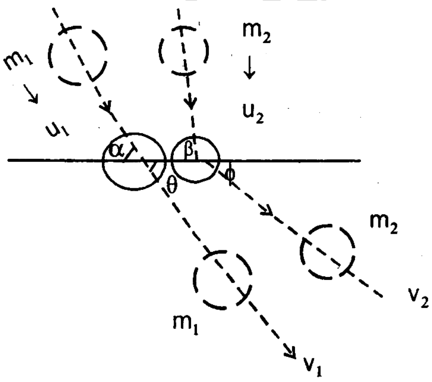
2.2.1 Collision in two and Three Dimensions

Q15. Explain in detail about collisions in two and three dimensions.

Ans : **(Dec.-19(MGU), June-19, May-18, Dec.-17, June-17)**

If the centres of mass of the colliding bodies are not initially moving along the line of impact, then the impact is called oblique or collision in two dimensions.

In a collision process, particles participating in a collision may approach in any direction before the collision, and they may recede in any direction after the collision. If the particles are confined to a plane, such collisions are called two dimensional collisions. If the particles are moving in space such collisions are called three dimensional collisions. Depending on the way how a collision takes place, a collision categorized as head on collision or oblique collision.



Before the collision if particles are approaching along the line joining their centres is called head on collision, if not it is an oblique collision. Let us discuss two dimensional obique collision.

Let two spheres of masses m_1 and m_2 are colliding with initial velocities u_1 and u_2 making on angles of α, β respectively with the line joining their centres. After the collision the final velocities are v_1 and v_2 making an angles of θ, ϕ, respectively as shown in fig above. In the oblique collision, the impact force during the collision will act only along the line joining their centres, but not in a direction normal to their line joining centres. Therefore the normal velocity components do not change during the collision.

$\therefore \quad v_1 \sin\theta = u_1 \sin\alpha$ ... Eqn. (1)

$\therefore \quad v_2 \sin\phi = u_2 \sin\beta$ ... Eqn. (2)

From the conservation of linear momentum

$m_1u_1 \cos\alpha + m_2u_2 \cos\beta = m_1 v_1 \cos\theta + m_2 v_2 \cos\phi$ ... Eqn. (3)

From the conservation of kinetic energy

$$\frac{1}{2}m_1u_1^2\cos^2\alpha + \frac{1}{2}m_2u_2^2\cos^2\beta = \frac{1}{2}m_1v_1^2\cos^2\theta + \frac{1}{2}m_2v_2^2\cos^2\phi \quad ... (4)$$

By solving the above equations, the values of v_1, v_2, θ, ϕ can be determined.

From equation (3)

$$m_1(u_1\cos\alpha - v_1\cos\theta) = m_2(v_2\cos\phi - u_2\cos\beta) \quad ... (5)$$

Similarly from equation (4)

$$m_1(u_1^2\cos^2\alpha - v_1^2\cos^2\theta) = m_2(v_2^2\cos^2\phi - u_2^2\cos^2\beta) \quad ... (6)$$

By dividing equation (6) with (5)

$$u_1\cos\alpha + v_1\cos\theta = v_2\cos\phi + u_2\cos\beta \quad ... (7)$$

Substituting the above values in equation (5), we have

$$v_1\cos\theta = \frac{m_1 - m_2}{m_1 + m_2}u_1\cos\alpha + \frac{2m_1}{m_1 + m_2}u_2\cos\beta \quad ... (8)$$

$$v_2\cos\phi = \frac{m_1 - m_2}{m_1 + m_2}u_2\cos\beta + \frac{2m_1}{m_1 + m_2}u_1\cos\alpha \quad ... (9)$$

Special Cases

1. If $u_2 = 0$ then from equation (2)

   $v_2\sin\phi = 0$ and $v_2 \neq 0$

   $\therefore$ $\phi = 0$ i.e., after the collision the mass m_2 will move along the line joining the two centres.

2. If $m_1 = m_2$, from equation (8)

   $v_1\cos\theta = u_2\cos\beta$

   Similarly from equation (9)

   $v_2\cos\phi = u_1\cos\alpha$

   The velocities will interchange along the line joining the centres of the spheres.

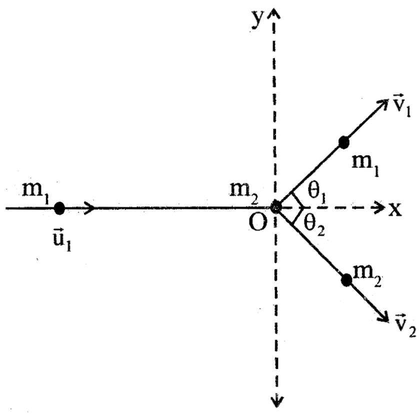
3. If $m_1 = m_2$ and then $u_2 = 0$

   $v_1\cos\theta = 0 \quad \therefore \cos\theta = 0 \quad \therefore \theta = 90^o$

   From equation (2) $\sin\phi = 0 \quad \therefore \phi = 0$

   After the collision, the two spheres will move in mutually perpendicular directions.

Let us discuss a special two dimensional elastic collision. This collision normally we see in nuclear reactions.



Consider a mass m, is moving with a velocity u_1. It hits another stationary mass m_2 in a head on collision. After the collision, the incident particle i.e m, moves with a velocity v_1 making an angle of θ_1 with the line joining the centres. And the target particle m_2 moves with a velocity v_2, making an angle θ_2.

The angle θ_1 is called scattering angle and the angle θ_2 is called recoil angle. Resolving the velocities along the X and Y axes, from conservation of linear momentum along X-axis.

$$m_1u_1 + 0 = m_1v_1 \cos\theta_1 + m_2v_2 \cos\theta_2 \quad \text{... (10)}$$

From the conservation of linear momentum along Y – axis, we have

$$0 + 0 = m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2$$

$$\therefore \quad m_1v_1 \sin\theta_1 - m_2v_2 \sin\theta_2 \quad \text{... (11)}$$

From conservation of kinetic energy

$$\frac{1}{2} m_1 u_1^2 + 0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad \text{... (12)}$$

There are three equations, 10, 11 and 12. But the unknowns are v_1, v_2, θ_1, θ_2. It is not possible to determine the four unknowns from three equations. But the unknows can be determinal in a special case. If $m_1 = m_2$ then from equation (10), we have

$$u_1 = v_1 \cos\theta_1 + v_2 \cos\theta_2 \quad \text{... (13)}$$

$$v_1 \sin\theta_1 = v_2 \sin\theta_2 \quad \text{... (14)}$$

$$u_1^2 = v_1^2 + v_2^2 \quad \text{... (15)}$$

From equation (13), we have

$$v_2 \cos \theta_2 = u_1 - v_1 \cos\theta_1$$

$$v_2^2 \cos^2 \theta_2 = u_1^2 + v_1^2 \cos^2 \theta_1 - 2u_1v_1 \cos\theta_1 \quad \text{... (16)}$$

From equation (14), we have

$$v_2^2 \sin^2 \theta_2 = v_1^2 \sin^2 \theta_1 \quad \text{... (17)}$$

Adding equations (16) and (17)

$$v_2^2 = u_1^2 - v_1^2 - 2u_1 v_1 \cos\theta_1 \quad \text{... (18)}$$

From equation (15)

$$v_2^2 = u_1^2 - v_1^2 \quad \text{... (19)}$$

From equations (18) and (19)

$$u_1^2 - v_1^2 = u_1^2 + v_1^2 - 2u_1 v_1 \cos\theta_1$$

$$-v_1^2 = v_1^2 - 2u_1 v_1 \cos\theta_1$$

$$-2v_1^2 = -2u_1 v_1 \cos\theta_1$$

or $v_1 = u_1 \cos\theta_1$ ... (20)

Substituting this in equation (10)

$$v_2^2 = u_1^2 - u_1^2 \cos^2\theta_1$$

$$v_2^2 = u_1^2 (1 - \cos^2\theta_1)$$

$$v_2^2 = u_1^2 \sin^2\theta_1$$

or $v_2 = u_1 \sin\theta_1$ .. (21)

The final velocities v_1, v_2 are normal components of u_1. Therefore $\theta_1 + \theta_2 = 90^o$.

Therefore in a two dimensional elastic head on collision, if the incident particle hits with a target particle initially at test, then after the collision two particles move in mutually perpendicular directions.

2.2.2 Concept of Impact Parameter, Scattering Cross Section

Q16. Explain the terms impact parameter and scattering cross-section.

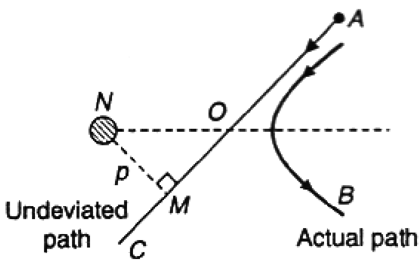
Ans : **(Dec-19(KU), Dec.-19(MGU), Dec.-18, June-18)**

Rutherford observed that when a sharp beam of α–particles falls on a photographic plate in vaccum, a sharp image is obtained. If, however – a thin foil of metal is placed in the path of the beam, the image becomes diffused (or scattered). The diffusion increases with both thickness and the atomic weight of the obstacle. The phenomenon is called scattering of α–particles.

Most of the scattered α – particles are deviated through angles of the order of 2^o or 3^o from the direction of the incident beam; but a small number, say about 1 in every 10,000 scattered through angles more than 90^o. A few of the α–particles wee even scattered directly in the backward direction i.e., angle of scattering = 180^o.

Impact Parameter

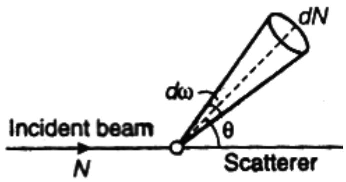
Consider a positive particle, like a proton or an α-particle, approaching a massive nucleus N of an atom, as shown in figure.



Due to coulombic force of repulsion, the particle follows a hyperbolic path AB with nucleus N as its focus. In the absence of the repulsive force, the particle would have followed the straight line path AC. As shown in figure, p is the perpendicular distance from the nucleus N to the original direction AC of the particle. The distance (NM = p) is called the impact parameter. Thus impact parameter is defined as the closest distance between nucleus and positively charged particle projected towards it. This is also known as collision parameter.

Scattering Cross Section

When α-particles are projected into a thin metal foil, they are deflected or scattered in different directions. Let N be the incident intensity (number of incident particles crossing per unit time a unit surface placed perpendicular to the direction of propagation). Suppose dN be the number of particles scattered per unit time into solid angle dω located in the direction θ and ϕ [Fig. below] with respect to the bombarding direction. The ratio dN / N is called scattering cross-section.



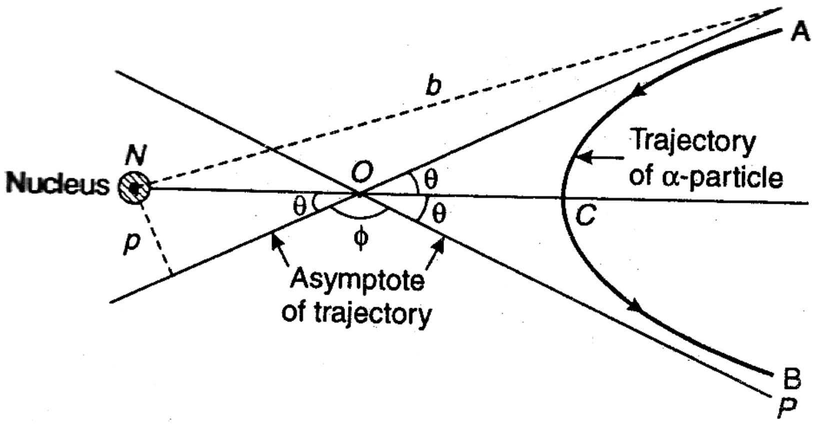
Thus the scattering cross section in a given direction is defined as the ratio of number of scattering particles into solid angle dω per unit time to the incident intensity.

$\therefore$ Scattering cross-section, $\sigma_{sc} = \dfrac{dN}{N}$

Q17. What is Rutherford's Scattering? Obtain the equation for the angle of scattering of α particle in Rutherford Scattering.

Ans :

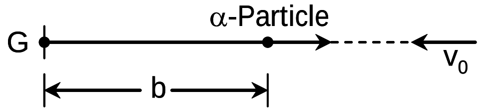
Let us examine the scattering of 'α' particle with gold nucleus. Rutherford conducted the 'α' scattering experiment in the year 1911. Radium source emits 'α' rays. These 'α' rays after passing through narrow slits collimated into a narrow beam. These 'α' rays will collide gold nucleus. The analyse the problem in a simple way let us consider this collision as a two particle collision between an 'α' particle and gold nucleus. Because both the particles are positive, the repulsive force acting between them is



$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2} \quad \text{... (1)}$$

q_1, q_1 are the charges of 'α' particle and gold nucleus respectively. To derive an expression for scattering crossection let us follow the following assumptions.

1. The collision between 'α' particle and gold nucleus is a elastic collision, neglecting wave nature of the particle.
2. Gold nucleus and α– particles are to be treated as point masses, neglecting the particle size.
3. Treat the gold nucleus as heavy and at rest.



Head on collision p = 0

Fig. : (4)

4. The α–particle with mass M, moving towards the nucleus G with an initial velocity v_0. Consider 'p' be the impact parameter for the 'α' particle. As the α particle approaching the nucleus its velocity is reducing and follow an hyperbolic path int he repulsive field of nucleus. As the 'α' particle reaching point 'c', its velocity becomes minimum and from there it returns back out of the field in a hyperbolic path. Initial direction of 'α' particle is x,y. After scattering its final direction is POQ. Let $y\hat{o}p = \phi$, scattering angle.

Consider the head on collision of 'α' particle with an impact parameter p=0. Here the 'α' particle will reach a point 'c' where its velocity becomes zero. From that point the 'α' particle retrace its pat with a scattering angle $\phi = 180^o$. Here Gc = b, which is called distance of closest approach.

If the atomic number of gold nucleus is Z

Then the potential at point 'c' due to the nucleus $= \frac{1}{4\pi\varepsilon_0}\frac{Ze}{b}$

Potential energy at point 'c' is $= \frac{2Ze^2}{4\pi\varepsilon_0 b}$ ... (2)

where '2e' is the charge on 'α' particle. From conservation of energy, the initial kinetic energy of 'α' particle is equal to potential energy.

$$\therefore \quad \frac{1}{2}mv_0^2 = \frac{2Ze^2}{4\pi\varepsilon_0 b}$$

$$\therefore \quad mv_0^2 = \frac{4Ze^2}{4\pi\varepsilon_0 b}$$

$$\therefore \quad b = \frac{Ze^2}{\pi\varepsilon_0 mv_0^2} \quad \text{... (3)}$$

Generally this situation is possible with impact parameter p = 0. This kind of scattering occurs rarely.

But the case $p \neq o$ is very general to observe. Here the 'α' particle moves on hyperbolic path ACD.

The scattering angle $\phi < 180^o$.

The velocity of 'α' particle at the vertex of hyperbolic path 'c' is v.

Initial angular momentum of 'α' particle at point A = mv_0p.

Final angular momentum of 'α' particle at point c is

= mvd (GC = d as in fig.)

From conservation of angular momentum

$$mv_0p = mvd$$

$$v_0p = vd$$

$$\therefore \quad v = \frac{v_0p}{d} \qquad \text{... (4)}$$

The value of 'p' can be determined from conservation of energy.

Kinetic energy of 'α' particle at point A = $\frac{1}{2}mv_0^2$

Kinetic energy of 'α' particle of point c = $\frac{1}{2}mv^2$

Potential energy of 'α' particle at point c = $\frac{2Ze^2}{4\pi\varepsilon_0 d}$

From energy conservation

$$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv^2 + \frac{2Ze^2}{4\pi\varepsilon_0 d}$$

$$mv_0^2 = mv^2 + \frac{Ze^2}{\pi\varepsilon_0 d}$$

$$v_0^2 = v^2 + \frac{Ze^2}{\pi m\varepsilon_0 d}$$

$$v^2 = v_0^2 - \frac{Ze^2}{\pi m \in_0 d}$$

$$v^2 = v_0^2 - \frac{bv_0^2}{d} \qquad \because \; b = \frac{Ze^2}{\pi \in_0 mv_0^2}$$

$$v^2 = v_0^2 \left(1-\frac{b}{d}\right) \qquad ... (5)$$

From equation (4), substitute the value of v

$$\frac{v_0^2 p^2}{d^2} = V_0^2 \left(1-\frac{b}{d}\right)$$

$$\frac{p^2}{d^2} = 1 - \frac{b}{d}$$

$$p^2 = d(d - b) \qquad ... (6)$$

From the property of hyperbola, the eccentricity $e = \sec\theta$, the length of semi latus rectrum.

$GO = a \sec\theta$. (Oc = a from fig above.

From ΔGNO $\quad GO = p.\text{cosec}\,\theta$

$\therefore \quad a \sec\theta = p\,\text{cosec}\,\theta$

$a = p \cot\theta$

From fig. (3)

$GC = GO + OC$

$$d = a\sec\theta + a = a(\sec\theta + 1)$$

$$= p\cot\theta\,(1 + \sec\theta)$$

$$= p[\cot\theta + \cot\theta\sec\theta]$$

$$= p\left[\frac{\cos\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\frac{1}{\cos\theta}\right]$$

$$d = p\left[\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}\right] = p\left[\frac{\cos\theta + 1}{\sin\theta}\right]$$

$$d = p\,\frac{2.\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\left(\frac{\theta}{2}\right)}$$

$$d = p\cot\left(\frac{\theta}{2}\right)$$

Substituting this value in equation (6)

$$p^2 = p\cot\left(\frac{\theta}{2}\right)\left[p\cot\left(\frac{\theta}{2}\right) - b\right]$$

$$p = \cot\left(\frac{\theta}{2}\right)\left[p\ \cot\left(\frac{\theta}{2} - b\right)\right]$$

$$p = p\cot^2\left(\frac{\theta}{2}\right) - b\cot\left(\frac{\theta}{2}\right)$$

$$b\cot\left(\frac{\theta}{2}\right) = p\cot^2\left(\frac{\theta}{2}\right) - p$$

$$b = \frac{p\left[\cot^2\left(\frac{\theta}{2}\right) - 1\right]}{\cot\left(\frac{\theta}{2}\right)}$$

$$b = \frac{p\left[\dfrac{\cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} - 1\right]}{\dfrac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}} = \frac{p\left[\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]}{\sin^2\left(\frac{\theta}{2}\right)} \times \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$$

$$b = \frac{2p\cos\theta}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = \frac{2p\cos\theta}{\sin\theta}$$

$$b = 2p\cot\theta$$

$$b = 2p\cot\left(\frac{\pi - \phi}{2}\right), \text{ from fig (3) } \theta = \frac{\pi - \phi}{2}$$

$$b = 2p\cot\left(\frac{\pi}{2} - \frac{\phi}{2}\right)$$

$$b = 2p\tan\left(\frac{\phi}{2}\right)$$

$$\therefore \tan\left(\frac{\phi}{2}\right) = \frac{b}{2p}$$

Substituting the value of ‘b’ from equation (3)

$$\tan\left(\frac{\phi}{2}\right) = \frac{Ze^2}{\pi\varepsilon_0 m v_0^2 2p} \quad \text{or}$$

$$\therefore \quad \tan\left(\frac{\phi}{2}\right) = \frac{Ze^2}{2\pi\varepsilon_0 m v_0^2 p} \qquad \text{... (7)}$$

This is an equation for scattering angle of ‘α’ particle for an impact parameter p. As the impact parameter is decreasing, scattering angle will increase.

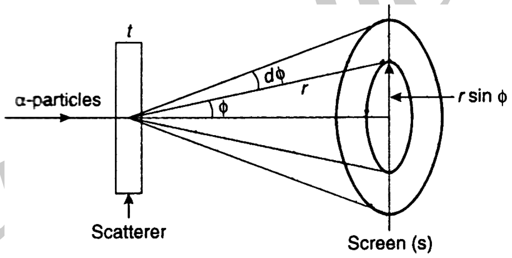
Q18. Explain about Rutherford’s cross-section. Obtain an expression for the Rutherford’s scattering cross-section and also number of scattered particles per unit area.

(OR)

Derive an expression for Rutherford’s scattering cross sections.

Ans : **(Dec.-18, June-18)**

Let n be the number of atoms per unit volume of the scatterer of thickness t. The scatered particles are detected by means of scintillations produced by them on a fluorescent screen S. Let Q be the total number of α-particles that strike the unit area of the scatterer. Our aim is to calculate the number of particles N that are scattered through an angle ϕ and strike unit area of screen S as a distance r.



It is clear that any α-particle whose initial velocity would bring it within a distance p of the nucleus will be deflected through an angle ϕ. In order to determine the probability that n α-particle would come within this distance, we imagine a circle of radius p drawn around each nucleus. The area occupied by all such circles in unit area of foil is πp^2 nt. The probable number of α-particles coming within the distance of an impact parameter p of the nucleus is given by

$$\pi p^2 \, tnQ$$

Hence, the number of α-particules having an impact parameter between p and $p + dp$ is given by

$$d\,(\pi p^2\, tnQ) = 2\pi\, p\, tnQ\, dp$$

Thus, the number of α-particles scattered between angle ϕ and $\phi + d\phi$ is

$$2\pi\, p\, t\, n\, Q\, dp \qquad \text{...(8)}$$

Eq.(8) will be used in calculating Rutherford’s scattering formula.

Now the scattering cross section σ is give by

$$\sigma = \frac{\text{Number of } \alpha \text{ particles scattered into solid angle per unit time}}{\text{Incident intensity}}$$

Solid angle between ϕ and $\phi + d\phi$, $dw = 2\pi \sin\phi \, d\phi$

The number of α-particles scattered into solid angle $d\omega$ is given by

$$s \, I \, d\omega = \sigma \, I \, 2\pi \sin\phi \, d\phi$$

This should be equal to the number of incident a-particles having impact parameters between p and (p + dp). The area between circles of radii (p + dp) and p is given by

$$d(\pi p^2) = 2\pi \, p \, dp$$

$\therefore$ Number of incident particles = $2\pi p \, dp \, I$ ... (10)

From equation (9) and (10), we get

$$\sigma \, I \, 2\pi \sin\phi \, d\phi = -2\pi p \, dp \, I$$

I = incident intensity. Negative sign is used to show that an increase in p causes a decrease in ϕ. Hence

$$\therefore \sigma = \frac{-2\pi \, p \, dp \, I}{2\pi \sin\phi \, d\phi \, I} = -\frac{p \, dp}{\sin\phi \, d\phi}$$

$$p = \frac{Ze^2}{2\pi\varepsilon_0 m v_0^2} \cot\frac{\phi}{2} \qquad \text{[Using equation (7)]} \qquad ... (11)$$

$$dp = \frac{Ze^2}{2\pi\varepsilon_0 m v_0^2}\left(-\frac{1}{2}\operatorname{cosec}^2\frac{\phi}{2} \, d\phi\right)$$

Substituting these value in equation (11), we get

$$\sigma = \frac{\left(\frac{Ze^2}{2\pi\varepsilon_0 m v_0^2}\right)^2 \cot\frac{\phi}{2} \cdot \frac{1}{2}\operatorname{cosec}^2\frac{\phi}{2} \, d\phi}{\sin\phi \, d\phi}$$

or

$$\sigma = \frac{Z^2 e^4 \cot\frac{\phi}{2} \operatorname{cosec}^2\frac{\phi}{2}}{8\pi^2\varepsilon_0^2 m^2 v_0^4 \cdot 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}$$

$$\therefore \qquad \sigma = \frac{Z^2 e^4}{16\pi^2 \varepsilon_0^2 m^2 v_0^4 \sin^4 \phi/2} \qquad ... (13)$$

This represents the Rutherford's scattering cross section.

Rutherford Scattering Formula

The number of 'α' particles, scatters between scattering angles ϕ and $\phi + d\phi$ is = $2\pi ptnQdp$

But $p = \frac{b}{2} \cot\left(\frac{\phi}{2}\right)$

$$p = \frac{Ze^2}{2\pi \in_0 mv_0^2} \cot\left(\frac{\phi}{2}\right)$$

$$dp = \frac{Ze^2}{2\pi \in_0 mv_0^2} \left(\left(-\frac{1}{2}\ \text{cosec}^2\left(\frac{\theta}{2}\right)\right)\right) d\phi$$

Substituting the value of dp in the above, we have

$$= 2\pi\ t\ n\ Q \left[\frac{Ze^2 \cot\left(\frac{\phi}{2}\right)}{2\pi \in_0 mv_0^2}\right] \left[\frac{Ze^2}{2\pi \in_0 mv_0^2}\left(-\frac{1}{2}\right) \text{cosec}^2\left(\frac{\phi}{2}\right) d\phi\right]$$

All these scattered particles will pass through an area on the screen

$dA = (2\pi\ r \sin\phi) r d\phi.$

$dA = 2\pi r \sin\phi\ r\ d\phi$

$dA = 2\pi \sin\phi\ r^2\ d\phi$

$$dA = 4\pi \sin\left(\frac{\phi}{2}\right) \cos\frac{\phi}{2}\ r^2\ d\phi$$

No. of scattered particles pass through a unit area

$$N = \frac{2\pi\ tnQ\left[\frac{Ze^2 \cot\left(\frac{\phi}{2}\right)}{2\pi \in_0 mv_0^2}\right]\left[\frac{Ze^2}{2\pi \in_0 mv_0^2}\right]\left[\frac{1}{2}\ \text{cosec}^2\left(\frac{\phi}{2}\right) d\phi\right]}{4\pi \sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) r^2 d\phi}$$

$$N = \frac{Q\ n\ t\ Z^2 e^4}{16\pi^2 \in_0^2\ m^2\ v_0^4}\ \frac{1}{\text{Sin}^4(\phi/2)}\ \frac{1}{r^2} \qquad \text{... (15)}$$

The above equation represents Rutherford scattering formula. The value of N is

1. Proportional to 't' thickness of target
2. Proportional to Z^2, square of the atomic number.
3. Inversely proportional to v_0^4
4. Inversely proportional to $\sin^4\left(\frac{\phi}{2}\right)$
5. Inversely proportional to square of the distance.

2.3 Definition of Rigid Body

Q19. What is rigid body and Inertia? Obtain expression for the kinetic energy of a rotating body.

Ans : **(Dec.-17(MGU)**

Rigid Body

A rigid body is an assembly of a large number of particles in which the number particle distance remains the same when it is acted upon by an external force or torque.

The shape of the body, therefore, remains unaltered during its motion which may be translational or rotational or combination of the two.

Interia

According to Newton's first law of motion a body at rest will remain at rest and a body moving with uniform velocity in a straight line will continue to do so unless an external force is applied to it. This property of a body by virtue of which it is unable to change its state of rest or of uniform motion in a straight line by itself is known as "inertia"

For translatory motion the value of inertia depends only on the mass of the body. The greater is the mass greater is the inertia.

Kinetic Energy of Rotation

For translatory motion kinetic energy depends upon mass m and velocity v and is given by $\frac{1}{2}mv^2$.

When a body rotates about an axis, the kinetic energy of its rotation is determined not only by its mass m and angular velocity ω, but also depends upon the position of the axis about which it rotates and the distribution of mass about this axis.

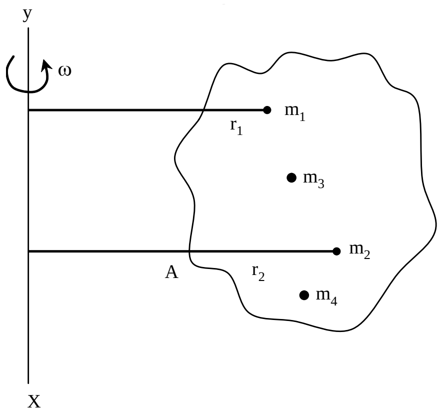
If a body A rotates about an axis xy with an angular velocity w, all its particles have the same angular velocity, but as they are at different distances from the axis of rotation, the linear velocities are different distances from the axis of rotation, their liner velocities are different. Let the linear velocities of the particles of mass m_1, m_2 ... ; distant r_1, r_2 ... from the axis of rotation be v_1, v_2, ... respectively. The kinetic energy of the body is therefore, equal to the sum of the kinetic energies of the various particles and is given by

$$\text{Total K.E.} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ...$$

Since v = rw

$$\therefore \quad \text{The K.E} = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + ...$$

$$= \frac{1}{2}(\Sigma mr^2)\omega^2 = \frac{1}{2}\omega^2\Sigma mr^2 = \frac{1}{2}I\omega^2$$



This expression is similar to the kinetic energy of a body in translation motion $\left(\frac{1}{2}mv^2\right)$. Here in rotational motion m is replaced by I and v is replaced by ω. It should be noted that m has a constant value whereas I depends upon the axis of rotation.

Q20. Explain moment of Inertia and Radius of Gyration.

Ans :

Moment of inertia of a body about an axis is defined as the sum of the product of the mass and the square of the distance of the different particles of the body from the axis of rotation.

The moment of inertia of the body is expressed as Σmr^2.

The K.E. of rotation = $\frac{1}{2}I\omega^2$

If ω = 1, then I = 2 × kinetic energy

Hence moment of inertia may also be defined as twice the kinetic energy of rotation of a body when its angular velocity is unity.

Radius of Gyration

If the entire mass of the body is supposed to be concentrated at a point such that the kinetic energy of rotation is the same as that of the body itself, then the distance of that point from the axis of rotation is called the radius of gyration of the body about that axis. If k denotes the radius of gyration and m the mass of the body supposed to be concentrated at that point, then we have

$$K.E = \frac{1}{2}I\omega^2 = \frac{1}{2} Smr^2 \omega^2 = \frac{1}{2}Mk^2\omega^2$$

$$\therefore \quad Mk^2 = \Sigma mr^2 = mn\left[\frac{r_1^2 + r_2^2 + ...}{n}\right]$$

where n is the number of particle each of mass m into which the given mass m is divided.

$$\text{Now } M = mn \quad \therefore \quad K = \left[\frac{r_1^2 + r_2^2 + ...}{n}\right]^{1/2}$$

According to the definition of radius of gyration given above the dimensions of k are those of length [L'] alone.

Now moment of inertia $I = Mk^2$

$\therefore$ Dimensions of $I = [M^1L^2]$

In S.I. units of moment of inertia is expressed as $kg\text{-}m^2$

Q21. What is the physical significance of moment of inertia?

Ans :

Physical Significance

Moment of inertia plays the same role in rotatory motion as mass does in linear motion, i.e., moment of inertia is an analogue of mass in linear motion.

According to Newton's First law of motion, a body continues in its state of rest or of uniform motion in a straight line unless some external force acts upon it. This property of matter is known as inertia. A body always resists an external force tending to change its state of rest of linear motion. Greater the mass of the body greater is the force required to produce its state of rest or of linear acceleration.

Similarly bodies possess rotational inertia, i.e., a body free to rotate about an axis opposes any change in its state of rest or of rotation. Greater the moment of inertia of a body greater is the couple required to produce a given angular acceleration.

The moment of inertia depends not only on the mass of a body but also on the distribution of mass about the axis of rotation.

If a solid disc and a wheel have the same mass of a body but also on the distribution of mass about axis of rotation.

If a solid disc and a wheel have the same mass, wheel will have a greater moment of inertia as the mass in it is distributed at larger distances from the axis of rotation passing through the centre. The analogy between the moment of inertia in rotational motion and mass in linear motion and mass in linear motion will be clear from the similarity in the relation for momentum, force, impulse, energy and work as illustrated below.

S.No.	Translation Motion	S.No.	Rotatory Motion
1.	Linear momentum = mV	1.	Angular momentum = $I\omega$
2.	Force = ma	2.	Torque or moment of the couple = I × angular acceleration = $I \propto$
3.	Impulse = $m(v_2 - v_1)$	3.	Angular impulse = $I(\omega_2 - \omega_1)$
4.	Kinetic energy = $\frac{1}{2}mv^2$	4.	Rotational K.E = $\frac{1}{2}I\omega^2$
5.	Work = Force × distance	5.	Work = Couple × angular displacement

Q22. State and prove the theorem of perpendicular axis for moment of inertia.

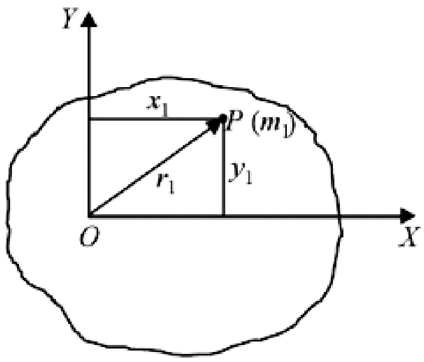
Ans :

This theorem states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about the two axes at right angles to each other, in its own plane intessecting each other at the point where the perpendicular axis passes through it.

To put the above in mathematical from let I_x and I_y be moments of inertia about the two axes perpendicular to each other in the plane of the lamina then the moment of inertia I about a line passing through the point of intensection and perpendicular to its plane is given by $I = I_x + I_y$.

Let OX and OY be the two perpendicular axes in the plane of the lamine. Let m_1 be the mass of a particle distant r_1 from an axis through 0 perpendicular to the plane XOY. The distance of this particle from the y-axis is x and that from the x-axis is y.

Moment of inertia of this particle about the x-axis $= m_1y_1^2$ and moment of inertia of this particle about the y-axis $= m_1x_1^2$.



If we divide the whole lamina into a number of particles of masses m_1, m_2, m_3, .. etc. at distances r_1, r_2, r_3, ... etc. So that the corresponding distance are y_1, y_2, y_3 ... from the x-axis and x_1, x_2, x_3 ... from the y-axis, then

Moment of inertia of the lamina about x-axis $I_x = m_1y_1^2 + m_2y_2^2 + ... = \Sigma my^2$ and the moment of inertia of the lamina about the y-axis

$$I_y = m_1x_1^2 + m_2x_2^2 + ... = \Sigma mx^2$$

$\therefore$ Moment of inertia of the lamina about a perpendicular axis through o

$$I = m_1r_1^2 + m_2r_2^2 + ... = \Sigma mr^2$$

$$= m_1(x_1^2 + y_1^2) + m_2(x_2^2 + y_2^2) + ...$$

$$= \Sigma mx^2 + \Sigma my^2 + I_x + I_y$$

Q23. State and prove theorem of parallel axes for moment of inertia.

Ans :

Theorem of parallel axes states that the moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis through its centre of gravity and the product of its mass and the square of the distance between the two axes.

Let CD be an axis in the plane of the paper and AB a parallel axis through G the centre of mass of the body. The perpendicular distance between the two axes is h. Let M be the mass of the body and m_1 the mass of the element at p distant x_1 and from AB.

Moment of inertia of m, about CD $= m_1(x_1 + h)^2$

$$= m_1(x_1^2 + h^2 + 2x_1h)$$

$$= m_1 x_1^2 + m_2h^2 + 2m_1x_1h$$

Moment of inertia of the body about CD

$$I = \Sigma m_1x_1^2 + \Sigma m_1h^2 + 2\Sigma m_1x_1h$$

If I_g is the moment of inertia of the body about AB, an axes through G, then $Sm_1x_1^2 = Ig$

$\therefore \quad I = I_g + Mh^2 + 2h\,\Sigma m_1x_1$

Now Σm_1x_1 is the sum of the moments of all the particles about AB passing through G the centre of gravity. Since the body is balanced about the centre of mass G, therefore the algebraic sum of all the moments abut G is zero.

$\therefore \quad \Sigma m_1x_1 = 0$

Hence $I = I_g + Mh^2$

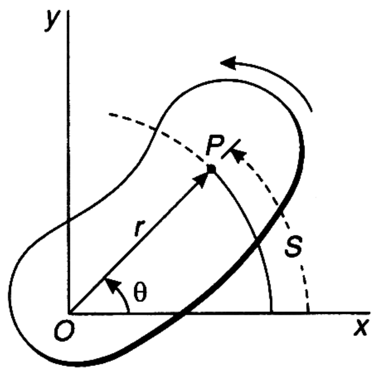
2.3.1 Rotational Kinetic Relations

Q24. Explain in detail about rotational kinetic relations.

Ans :

Let the axis of rotation be passing through O and is perpendicular to the plane of the paper, i.e., z-ais is the axis of rotation.

In figure (a), a plane of the rigid body that is at right angles to the axis of rotation is shown. We choose a particle P in this plane. To locate this particle, we require two coordinates namely position vector r and angle θ. For different particles in this plane or in the rigid body, values of r and θ will be different.



1. Angular Position

With **r** fixed if θ is varied, we arrive at rotational motion, *i.e.,* particle *P* rotates in a circle of radius r. θ is increasing in anticlockwise direction and is taken as positive whereas θ in clockwise direction is taken as negative. Obviously

$$\theta = \frac{s}{r} \text{ in radians.}$$

2. Angular velocity

Let the body be rotating in counter-clockwise direction. At time t_1, the particle is at $P(t_1)$ while at time t_2 it is at $P(t_2)$: the angular positions are θ_1, and θ_2. Therefore, the angular displacement is

$$\theta_2 - \theta_1 = \Delta\theta,$$

during the time interval

$$t_2 - t_1 = \Delta t,$$

so that average angular speed $\overline{\omega}$ of the particle in this time interval will be

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t}$$

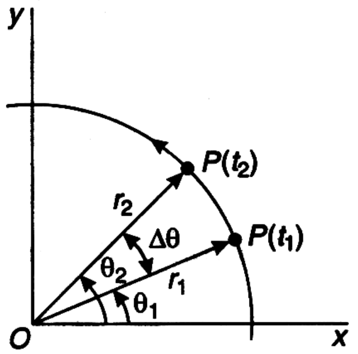


Figure (b)

The instantaneous angular speed ω is defined as limit to which ratio $\frac{\Delta\theta}{\Delta t}$ approaches as Δt tends to zero, i.e.,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Since in a rigid body interparticle distance is fixed, therefore if one of the particles undergoes an angular displacement dq in time dt so must the others which implies that ratio $\frac{d\theta}{dt}$ or angular speed w will be same for every particle in the body. This means that rather than speaking the angular velocity for single particle we can state for the whole rigid body rotates with an angular velocity w or it is the characteristic of the body as whole. Since $d\theta$ has no dimensions, w has the dimensions of an inverse time (T^{-1}). The unit of w is commonly taken as rad./sec or rev./sec.

3. Angular Acceleration

If angular speed ω of particle P is not constant, i.e., at time t_1 angular speed is ω_1 and at time t_2, angular speed is ω_2 then this variation in angular speed in time interval $(t_2 - t_1)$ gives rise to angular acceleration. The average angular acceleration is defined as

$$\overline{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

whereas the instantaneous angular acceleration is defined as

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}.$$

Since co is same for all particles in the rigid body, a must also be the same as a consequence of above relations. Thus α, like co, is a characteristic of the body as a whole. Its dimensions are T^{-2} and units are taken to be rad./sec^2.

2.3.2 Equation of Motion for Rotating Body

Q25. Obtain the equation of motion of a rotating body.

Ans :

Equation of a Motion of a Rotating Body

Consider a rigid body capable of rotatining with an angular velocity w about an axis AB passing through a fixed point O. The direction of angular velocity ω will be along the axis of rotation AB.

At any instant, let r be the position vector of a particle P of the body. If the length of the perpendicular drawn from P on AB is PC = r_0, then C will be the centre of the circle, described by the point P. The speed of the particle is given by

Speed = $\omega r_0 = \omega r \sin\theta$ where $\theta = \angle POC$

The velocity of the particle P is given by $v = \vec{\omega} \times r$

The direction of velocity v at any instant will be perpendicular to the position vector r and tangential to the circular path.

The angular momentum J_p of the particle P about the point O is given by

$$J_p = r \times mv_p$$

and its direction is perpendicular to r and v. The angular momentum J of the entire body about the point O is given by

$$J = \Sigma r \times mv_p = \Sigma mr \times (\vec{\omega} \times r)$$

$$= \Sigma m[(r.\, r)\vec{\omega} - (r.\vec{\omega})r] = \Sigma m[r^2 - \omega - (\omega r \cos\theta)\, r]$$

The magnitude of r along ω will be r cos θ. The magnitude of the component (J_0) of the angular momentum J along the axis of rotation will be given by

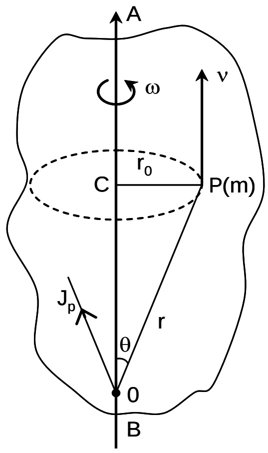
$$J_0 = \Sigma m[r^2\vec{\omega} - \vec{\omega}\, r \cos\theta\, r\cos\theta]$$

$$= \Sigma mr^2\vec{\omega}\,(1 - \cos^2\theta) = \Sigma mr^2\vec{\omega}\ \sin^2\theta$$

$$= \vec{\omega}\,\Sigma mr_0$$

$\Sigma m\, r_0^2$ is the moment of inertia of the body about the axis of rotation and is represented by I.

$$\therefore J_0 = I\,\vec{\omega} \text{ or } j = I\omega$$



The torque τ about a point is the rate of change of angular momentum about that point.

$$\therefore \quad \tau = \frac{dJ}{dt}$$

This is the general equation for a rotating body. For a body rotating about the axis of symmetry.

$$\tau = \frac{d\vec{J}}{dt} = \frac{d}{dt}(I\vec{\omega}) = I\frac{d\vec{\omega}}{dt}$$

where $d\vec{\omega}/dt$ is the angular momentum. If the axis of rotation and the axis of symmetry are not the same, $\vec{J}$ and $\vec{\omega}$ may not be along the same direction. The torque τ_0 about the axis of rotating is given by

$$\tau. = \frac{d\vec{J}_0}{dt} = \frac{d}{dt}(I\vec{\omega})$$

In case, the axis of rotation is fixed relative to the body, I will be constant so that the torque τ about the axis of rotation is given by

$$\tau = I\frac{d\vec{\omega}}{dt}$$

In the absence of external torque ($\tau = 0$), the angular momentum ($I\vec{\omega}$) about the axis of rotation is conserved.

i.e., $I\vec{\omega}$ = constant.

Q26. (a) Prove from first principles that out of an infinite number of straight lines which may be drawn parallel to a given direction the moment of inertia of a body is least about the one passing through its centre of gravity.

(b) Determine the moment of inertia of a diatomic molecule.

Ans :

(a) According to the principle of parallel axes the moment of inertia I of a body about an axis is equal to the sum of its moment of inertia about a parallel axis through its centre of gravity I_g and the product of its mass M and the square of the distance h between the two axes i.e., $I = I_g + Mh^2$

Hence for a number of axes which are all parallel to each other at distances h_1, h_2, h_3 etc. from the axis AB passing through the centre of gravity the moment of inertia is respectively given by

$(I_g + Mh_1^2)$, $(I_g + Mh_2^2)$

and so on. The value of h^2 is always positive whether h is towards the left or right of AB. Hence Mh^2 is a positive quantity.

The least value of I is obtained when h = 0 i.e., when the axis passes through the centre of gravity.

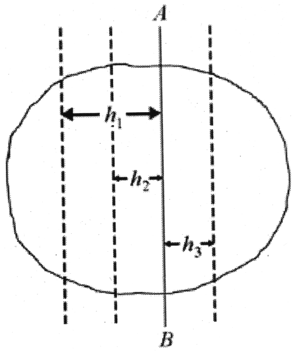


Fig. (a)

(b) Moment of inertia of a diatomic molecule (about its centre of mass): A diatomic molecule consists of two atoms (similar or dissimilar) separated by a distance greater than the atomic dimensions. The familiar examples of diatomic molecules are H_2, O_2, HCl etc.

To find the moment of inertia of a diatomic molecule like HCl about an axis passing through its centre of mass, let m_1 and m_2 be the masses of the atoms separated by a distance r (the inter-nuclear distance).

If the centre of mass of the molecule lies at O at a distance r_1 from m_1 and r_2 from m_2, then

$$m_1 r_1 = m_2 r_2$$

or $$m_1(r - r_2) = m_2 r_2$$

$\therefore$ $$r_2 = \frac{m_1 r}{m_1 + m_2}$$

Similarly $$r_1 = \frac{m_2 r}{m_1 + m_2}$$

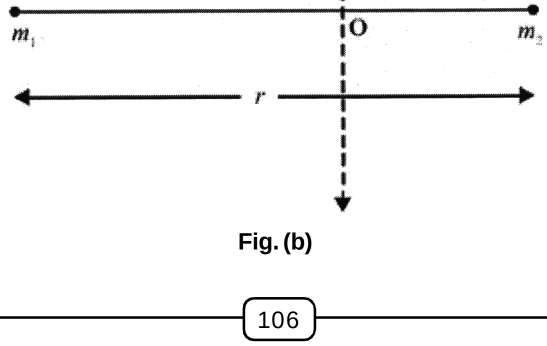


Fig. (b)

The moment of inertia of the molecule about an axis, passing through O, the centre of mass and perpendicular to the line joining the two nuclei is given by

$$I = m_1 r_1^2 + m_2 r_2^2$$

$$= m_1 \left(\frac{m_2 r}{m_1 + m_2}\right)^2 + m_2 \left(\frac{m_1 r}{m_1 + m_2}\right)^2$$

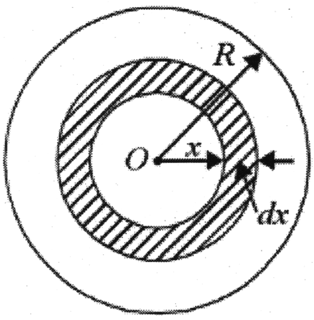
$$= \frac{m_1 m_2}{m_1 + m_2} r^2 \left[\frac{m_2}{m_1 + m_2} + \frac{m_1}{m_1 + m_2}\right]$$

where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is known as the reduced mass of the molecule

Q27. Determine the M.I. of a plane circular disc about an axis through its centre perpendicular to its plane.

Ans :

Moment of inertia of a circular disc about an axis through its centre perpendicular to its plane. Let M be the mass of the disc and R its radius. Consider an elementary ring of radius x and width dx as shown in fig. Its area is equal to the product of the circumference and width i.e., $2\pi x\,dx$.



$$\text{Mass per unit area} = \frac{M}{\pi R^2}$$

$$\therefore \quad \text{Mass of the element} = \frac{M}{\pi R^2} 2\pi x dx = \frac{2M}{R^2} x dx$$

Moment of inertia of the element about an axis through its centre perpendicular to its plane $= \dfrac{2M}{R^2}$ $x dx\ x^2 = \dfrac{2M}{R^2} x^3\, dx$

Hence moment of inertia of the whole disc about this axis

$$I = \frac{2M}{R^2} \int_0^R x^3\, dx = \frac{2M}{R^2} \left[\frac{x^4}{4}\right]_0^R = \frac{1}{2} MR^2$$

Q28. Determine the moment of inertia of a circular disc about its diameter.

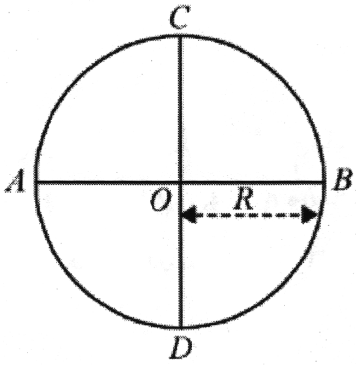
Ans :

Moment of inertia of a disc about its diameter. The moment of inertia of a circular disc about an axis perpendicular to its plane and passing through its centre is given by

$$I = \frac{1}{2} MR^2$$

where M is the mass and R its radius.

Now consider two perpendicular diameters AB and CD of the circular disc as in fig. Since all the diameters are symmetrical the moment of inertia of the disc about one diameter is the same as that about any other diameter.



If I_1 and I_2 are the moment of inertia of the disc about two axes perpendicular to each other, then applying the principle of perpendicular axis, the moment of inertia I of the disc about an axis perpendicular to the plane of the disc through O.

$$I = I_1 + I_2$$

Since the two diameters are symmetrical with respect to the disc $I_1 = I_2$

$$\therefore \quad I = 2I_1 \text{ or } I_1 = \frac{I}{2} = \frac{MR^2}{2} \times \frac{1}{2} = \frac{MR^2}{4}$$

2.3.3 Angular Momentum and Inertia Tensor

Q29. Define rigid body. Derive an expression for the angular moments of a rigid body and hence define inertia tensor.

Ans : **(Dec.-19(KU), June-18(KU, Dec.-16)**

Rigid Body :

A rigid body is defined as a system of particles in which the relative distance between its constituent particles remains constant and unchanged during its transational or rotational motion.

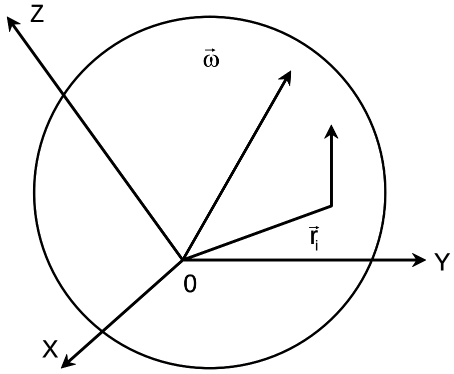
Suppose a rigid body is made up of a large number of particles. Let (x_i, y_i, z_i) and (x_j, y_j, z_j) be the co-ordinates of the i^{th} and j^{th} particle of the body and r_{ij} the distance between them. Then

$$r_{ij} = \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i + z_j)^2\right]^{1/2}$$

For a rigid body r_{ij} = a constant during any translational or rotational motion of the body. The translational motion may be, one, two or three dimensional. The rotational motion can only be either two or three dimensional.

Angular Momentum of a Rigid Body :

Consider a rigid body rotating about a fixed point with angular velocity $\vec{w}$. Take the origin 0 at this fixed point and the three co-ordinate axes X, Y and Z as shown in fig below.



The linear velocity of a particle i, having position vector $\vec{r}_i$

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

If m_i is the mass of this particle then the angular momentum of the particle i about the fixed point o.

$$\vec{l}_i = \vec{r}_i \times m_i \vec{v}_i$$

$$= \vec{r}_i \times m_i (\vec{\omega} \times \vec{r}_i) = m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

Then, total angular momentum of the rigid body

$$\vec{L} = \sum_i \vec{l}_i = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \qquad \text{... (1)}$$

where

$\sum_i$ represents summation over all the particles of the rigid body.

Using the vector identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}), \text{ we get}$$

$$\vec{L} = \sum_i m_i [(\vec{\omega} (\vec{r}_i \cdot \vec{r}_i) - \vec{r}_i (\vec{r}_i \cdot \vec{\omega}_i)]$$

$$= \sum_i m_i [\vec{\omega} r_i^2 - \vec{r}_i (\vec{r}_i \cdot \vec{\omega})] \qquad \text{... (2)}$$

If (x_i, y_i, z_i) are the cartesian co-ordinates of the particle i and $(\omega_x, \omega_y, \omega_z)$. The components of angular velocity $\vec{w}$ along the three co-ordinate axes, then

$$\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k} \text{ and } \therefore r_1^2 = x_1^2 + y_1^2 + z_1^2$$

and $$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

$$\therefore \quad \vec{r}_i \cdot \vec{w} = (x_i\hat{i} + y_i\hat{j} + z_i\hat{k}) \cdot (\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k})$$

$$= x_i\,\omega_x + y_i\,\omega_y + z_i\,\omega_z$$

Substituting the values of ω, r_i^2 and $\vec{r}_i \cdot \vec{\omega}$ in component form in eqn. (2) we get

$$\vec{L} = \sum_i m_i\,[(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k})\,(x_i^2 + y_i^2 + z_i^2) - (x_i\hat{i} + y_i\hat{j} + z_i\hat{k})\,(x_i\omega_x + y_i\omega_y + z_i\omega_z)]$$

$$= \sum_i m_i\,[\hat{i}\,(\omega_x x_i^2 + \omega_x y_i^2 + \omega_x z_i^2 - \omega_y x_i y_i - \omega_z x_i z_i)$$

$$+ \hat{j}\,(\omega_y x_i^2 + \omega_y y_i^2 + \omega_y z_i^2 - \omega_x x_i y_i - \omega_y y_i^2 - \omega_z y_i z_i)$$

$$+ \hat{k}\,(\omega_z x_i^2 + \omega_z y_i^2 + \omega_z z_i^2 - \omega_x x_i z_i - \omega_y y_i z_i - \omega_z z_i^2)]$$

If L_x, L_y and L_z are components of $\vec{L}$ along the three co-ordinate axes, then

$$L_x = \sum_i m_i\,(y_1^2 + z_1^2)\,\omega_x - \sum_i m_i\,x_i\,y_i\,\omega_y - \sum_i m_i\,x_i\,z_i\,\omega_z \quad \text{... (3)}$$

$$L_y = \sum_i m_i\,y_i\,x_i\,\omega_x + \sum_i m_i\,(z_1^2 + x_i^2)\,\omega_y - \sum_i m_i\,y_i\,z_i\,\omega_z \quad \text{... (4)}$$

and $$L_z = \sum_i m_i\,z_i\,x_i\,\omega_x - \sum_i m_i\,z_i\,y_i\,\omega_y + \sum_i m_i\,(x_i^2 + y_i^2)\,\omega_z \quad \text{... (5)}$$

We now substitute

$$\sum_i m_i\,(y_i^2\ \ z_1^2) = I_{xx}\ ;\ -\sum_i m_i\,x_i\,y_i = I_{xy}\ ;\ -\sum_i m_i\,x_i\,z_i = I_{xz}$$

$$-\sum_i m_i\,y_i\,x_i = I_{yx}\ ;\ -\sum_i m_i\,(z_1^2 + x_i^2) = I_{xy}\ ;\ -\sum_i m_i\,y_i\,z_i = I_{yz}$$

and $$-\sum_i m_i\,z_i\,x_i = I_{zx}\ ;\ -\sum_i m_i\,z_i\,y_i = I_{zy}\ ;\ \sum_i m_i\,(x_i^2 + y_i^2) = I_{zz}$$

Equations (3), (4) and (5) now become

$$L_x = I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z \quad \text{... (6)}$$

$$L_y = I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z \quad \text{... (7)}$$

and $$L_z = I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z \quad \text{... (8)}$$

$$\therefore \quad \vec{L} = \hat{i}\,(I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z) + \hat{j}\,(I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z) + \hat{k}\,(I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z)$$

This equation shows that the angular momentum vector $\vec{L}$ is, in general, not in the same direction as the angular velocity vector $\vec{\omega}$ nor it is in the direction of axis of rotation.

In the matrix form equations (6), (7) and (8) may be expressed as under

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Moment of Inertia Tensor

In vector notation the result stated above in matrix form may be expressed as

$$\vec{L} = \overleftrightarrow{I}\,\vec{w} \text{ where } \overleftrightarrow{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

is called the moment of inertia tensor or simply inertial tensor. It is a tensor of second rank, which has nine components.

Q30. Define principal moments of inertia, products of inertia, and principal axes of a rigid body. Why are they important?

Ans : **(Dec.-16)**

Principal moments of inertia

In vector notation the angular momentum $\vec{L}$ of a rigid body may be expressed as

$$\vec{L} = \overleftrightarrow{I}\,\vec{\omega} \text{ where } \overleftrightarrow{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

is called the moment of inertia tensor.

The nine quantities I_{xx}, I_{xy}, I_{xz} ; I_{yx}, I_{yy}, I_{yz} ; and I_{zx} I_{zy}, I_{zz} are the components of the moment of inertia of the body about the fixed X, Y and Z axes.

The diagonal elements I_{xx}, I_{yy} and I_{zz} are the moments of inertia of the rigid body about X-axis, Y-axis and Z-axis respectively and are called principal moments of inertia (or principal moments).

Products of inertia

The off diagonal elements I_{xy}, I_{xz}, I_{yx}, I_{yz} ; and I_{zx}, I_{zy} are called products of inertia. These occur in symmetric pairs i.e.,

$$I_{xy} = I_{yx} \; ; I_{yz} = I_{zy} \; ; \text{and } I_{xy} = I_{zx}$$

The rotational behaviour of a rigid body about a given point is determined by a set of six quantities, the three principal moments of inertia and the three products of inertia.

Principal axes of inertia

A set of three mutually perpendicular axes drawn through a point in the rigid body taken as origin, such that the products of inertia (I_{xy}, I_{yx}; I_{yz}; I_{zy} ; I_{xz}, I_{zx}) about then vanish i.e. each is equal to zero whereas (I_{xx}, I_{yy}, I_{zz}) the principal moments of inertia are non zero are called principal axes of inertia or simply principal axes.

In terms of principal axes, the angular momentum of a rigid body is given by

$$\vec{L} = \overleftrightarrow{I}_{xx}\,\vec{\omega}_x\,\hat{i} + I_{yy}\,\omega_y\,\hat{j} + I_{zz}\,\omega_z\,\hat{k}$$

Q31. Explain the statement 'Inertia tensor is symmetric'.

(OR)

State the properties of moment of inertia tensor.

Ans :

(a) Inertia tensor is symmetric

The moment of inertia tensor is given by

$$\overleftrightarrow{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

It is called symmetric because its off diagonal elements known as products of inertia are equal i.e.

$$I_{xy} = I_{yx} \; ; \; I_{xz} = I_{zx} \; ; \; I_{yz} = I_{zy}$$

(b) Properties of moment of inertia tensor

1. The moment of inertia tensor is a symmetric tensor i.e. its off diagonal elements are equal

$$\therefore \qquad I_{xy} = I_{yx} \; ; \; I_{xz} = I_{zx} \; ; \; I_{yz} = I_{zy}$$

As a result of this, there are only six independent components

$$I_{xx}, I_{yy}, I_{zz} \text{ and } I_{xy}, I_{yz}, I_{zx}$$

As the products of inertia about the three principal axes are zero, i.e.

$$I_{xy} = I_{yx} + I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$$

Only three components are left, I_{xx}, I_{yy} and I_{zz} which are sometimes written I_x, I_y, I_z.

2. Spherical top : A rigid body for which $I_{xx} = I_{yy} = I_{zz}$ is called a spherical top. In a spherical top all the axes are symmetric. A sphere is an example of a spherical top.

3. Symmetric top : A rigid body for which

$$I_{xx} = I_{yy} \neq I_{zz}$$

is called a symmetric top. A cylinder satisfies this condition. If the axis of the cylinder is taken as principal Z-axis, then X and Y-axes are symmetric axes. But a cylinder is not called a symmetric top. On the other hand all rigid bodies which do not have cylindrical shape but satisfy the condition given above are considered as a symmetric top. The earth flattened at the poles and bulging at the equator satisfies the above condition and is taken to be a symmetrical top.

4. Asymmetric top : A rigid body for which

$$I_{xx} \neq I_{yy} \neq I_{zz}$$

is called an asymmetric top. A rigid body, in general is an asymmetric top.

5. Rotor : A rigid body for which

$$I_{xx} = I_{yy} \text{ and } I_{zz} = 0$$

is called a rotor. Example, a diatomic molecule.

2.3.4 Euler's Equation

Q32. Derive Euler's equations of rotation of a rigid body about a fixed point.

(OR)

Obtain Euler's equations of a rigid body rotating about a fixed point.

(OR)

Derive Euler's equation for a rigid body.

Ans : **(Dec.-19(MGU), Dec.-18, June-18, Dec.-17(MGU))**

Euler's equations

The time rate of change of angular momentum of a rigid body about a fixed point is equal to the resultant external torque acting on the body about that fixed point.

If $\vec{\tau}$ is the torque and $\vec{L}$ the angular momentum, then

$$\vec{\tau} = \frac{d\vec{L}}{dt} \quad \text{... (i)}$$

This equation holds good if the system of axes has a fixed orientation in space i.e. the inertial frame is fixed in space.

In order to study the rotation of a rigid body, the system of axes should be fixed in the body itself and the origin should be coincident with the fixed point about which the body is rotating so that as the body rotates the co-ordinate axes also rotate with the body.

The time rate of change of any vector in a fixed frame can be transferred to the time rate of change of the same vector in a rotating frame using operator equation

$$\left(\frac{d}{dt}\right)_S (-) = \left(\frac{d}{dt}\right)_R (-) + \vec{\omega} \times (-)_R$$

where

$\left(\frac{d}{dt}\right)_S$ represents the time rate of change in stationary frame,

$\left(\frac{d}{dt}\right)_R$ the time rate of change in the rotating frame

and (–) the rotating vector.

Applying this operator equation to equation (i), we get

$$\vec{\tau}_R = \left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_R + \left(\vec{\omega} \times \vec{L}\right)_R$$

or $$\vec{\tau} = \frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} \quad \text{... (iii)}$$

Now, the angular momentum of a rigid body rotating with angular vector $\vec{\omega}$ about a fixed point is given by

$$\vec{L} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$$

$$= (I_{xx}\,\omega_x + L_{xy}\omega_y + L_{xz}\omega_z)\,\hat{i} + (I_{yx}\,\omega_x + I_{yy}\omega_y + I_{yz}\,\omega_z)\,\hat{j}$$

$$+ (I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z)\,\hat{k}$$

If $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the principal axes of inertia at the fixed point in the rigid body about which it is rotating. Then

$$I_{xy} = I_{yx} = I_{zx} = I_{xz} = I_{yz}\, I_{zy} = 0$$

and we get $\vec{L} = I_{xx}\,\omega_x\,\hat{i} + I_{yy}\,\omega_y\,\hat{j} + I_{zz}\,\omega_z\,\hat{k}$

$$\therefore \quad \frac{d\vec{L}}{dt} = I_{xx}\frac{d\omega_x}{dt}\,\hat{i} + I_{yy}\frac{d\omega_y}{dt}\,\hat{j} + I_{zz}\frac{d\omega_z}{dt}\,\hat{k} \quad \text{... (iii)}$$

This gives the first term on the right hand side of eq. (ii).

To find the value of second term, we have

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ I_{xx}\omega_x & I_{yy}\omega_y & I_{zz}\omega_z \end{vmatrix}$$

$$= \hat{i}\,(I_{zz}\,\omega_y\,\omega_z - I_{yy}\,\omega_y\,\omega_z) + \hat{j}\,(I_{xx}\,\omega_x\,\omega_z - I_{zz}\,\omega_x\,\omega_z)$$

$$+ \hat{k}\,(I_{yy}\,\omega_x\,\omega_y - I_{xx}\,\omega_x\,\omega_y)$$

$$= \hat{i}\,\omega_y\,\omega_z\,(I_{zz} - I_{yy}) + \hat{j}\,\omega_x\,\omega_z\,(I_{xx} - I_{zz})$$

$$+ \hat{k}\,\omega_x\,\omega_y\,(I_{yy} - I_{xx}) \quad \text{... (iv)}$$

Substituting the value of $\frac{d\vec{L}}{dt}$ from Eq. (iii) and $\vec{\omega} \times \vec{L}$ from Eq. (iv) in Eq. (ii), we get

$$\vec{\tau} = I_{xx}\frac{d\omega_x}{dt}\,\hat{i} + I_{yy}\frac{d\omega_y}{dt}\,\hat{j} + I_{zz}\frac{d\omega_z}{dt}\,\hat{k}$$

$$+ \hat{i}\,\omega_y\,\omega_z\,(I_{zz} - I_{yy}) + \hat{j}\,\omega_x\,\omega_z\,(I_{xx} - I_{zz}) + \hat{k}\,\omega_x\,\omega_y\,(I_{yy} - I_{xx}) \quad \text{... (v)}$$

Now $\vec{\tau} = \tau_x \hat{i} + \tau_y \hat{j} + \tau_z \hat{k}$ ... (vi)

Comparing (v) and (vi), we get

$$\tau_x = I_{xx} \frac{d\omega_x}{dt} + \omega_y \omega_z (I_{zz} - I_{yy}) \quad \text{... (vii)}$$

$$\tau_y = I_{yy} \frac{d\omega_y}{dt} + \omega_x \omega_z (I_{xx} - I_{zz}) \quad \text{... (viii)}$$

$$\tau_z = I_{zz} \frac{d\omega_z}{dt} + \omega_x \omega_y (I_{yy} - I_{xx}) \quad \text{... (ix)}$$

Eq. numbers (vii), (viii) and (ix) for τ_x, τ_y and τ_z are known as Euler's equations for the motion of the rigid body. These equations give the values of components of torque $\vec{\tau} = \frac{d\vec{L}}{dt}$ relative to the rotating principal axes, in terms of angular velocity of the principle axes and principal moments of inertia.

2.3.5 Precessional of a Symmetric Top

Q33. What is a symmetric top ? Derive an expression for the angular velocity of precession of a symmetric top.

OR

Discuss the motion of a top and obtain an expression for the precessional frequency.

Ans : **(Aug-21, Dec.-19, June-19, Dec.-18(MGU), Dec.-17)**

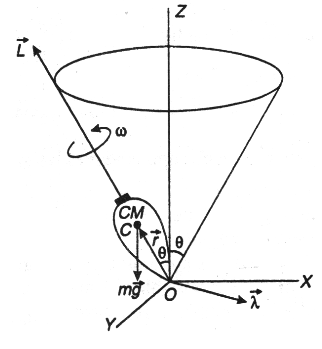
Symmetric top

A symmetrical body spinning about an axis which is fixed at one point is called a top.

Precession of a top spinning in Earth's gravitational field

Consider a top spinning about its axis of symmetry with an angular velocity ω. Its tip being a fixed point coinciding with the origin O of an inertial frame of reference.

The angular velocity vector $\vec{\omega}$ and the angular momentum vector L will be pointed along the axis of rotation according to the right hand rule. Let the axis of rotation make an angle θ with the vertical at the instant considered.



The forces acting on the top are

(i) its weight mg acting vertically downwards at its centre of mass C.

(ii) the upward force on the tip or pivot O. This force exerts no torque about O as it passes through O.

Hence the weight mg exerts a torque τ about O given by

$$\tau = r \times mg$$

where r is the position vector of C with respect to O. τ is directed perpendicular to the plane containing r and mg according to the right hand rule.

The magnitude of $|\tau|$ is given by

$$|\tau| = r\, mg \sin(180 - \theta)$$
$$= r\, mg \sin\theta \quad \text{... (1)}$$

when a torque τ acts on a rigid rotating body, it changes the angular momentum L of the body, given by

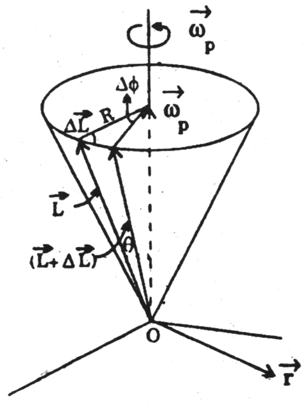
$$\tau = \frac{\Delta L}{\Delta t}$$

or $$\Delta L = \tau\, \Delta t \quad \text{... (2)}$$

In the present case τ is perpendicular to L. Hence the change in L i.e., ΔL (being in the direction of τ) will also be perpendicular to L.

If L be the angular momentum of the top at time t, the angular momentum L + ΔL, after a time interval Δt is given by the vector sum of L and ΔL. As ΔL is perpendicular to L and is very small, the new angular momentum vector has the same magnitude as the old but a different direction so that the angular momentum vector of the top moves around a horizontal circle. i.e., L precesses around the vertical axis and sweeps out a cone. This is called precessional motion of the top. The angular velocity of precession ω_p (or precessional frequency) is given by

$$\omega_p = \frac{\Delta\phi}{\Delta t} \quad \text{... (3)}$$



where $\Delta\phi$ is the angle through which the vector L rotates in time Δt. As is clear from the figure that $\Delta L << L$, hence

$$\Delta L = R\, \Delta\phi$$

or $$\Delta\phi = \frac{\Delta L}{R} = \frac{\Delta L}{L \sin\theta}$$

Substituting the value of $\Delta\phi$ in eq. (3), we have

$$\omega_p = \frac{\Delta L}{L \sin\theta \times \Delta t} = \frac{\Delta L}{\Delta t} \times \frac{1}{L \sin\theta}$$

But $$\frac{\Delta L}{\Delta t} = \tau \quad \text{from Eq. (2)}$$

$$\therefore \quad \omega_p = \frac{\tau}{L \sin\theta}$$

and $$\tau = r\, mg \sin\theta \quad \text{from Eq. (1)}$$

Hence $$\omega_p = \frac{r\, mg \sin\theta}{L \sin\theta} = \frac{mgr}{L}$$

Thus the precessional angular velocity is independent of θ and is inversely proportional to the magnitude of the angular momentum L.

If L is large ω_p the precessional angular velocity will be small i.e., the faster the top spins about its own axis, the more slowly it precesses about the vertical axis.

2.3.6 Gyroscope

Q34. What is Gyroscope? Give its uses advantages and applications.

(OR)

Describe Gyroscope.

Ans : **(June-18)**

A gyroscope is a heavy symmetrical body (top) in the form of a heavy circular disc or fly wheel rotating at a very high speed about is axle.

Gyroscopes have two basic properties : precession. Those are defined as follows:

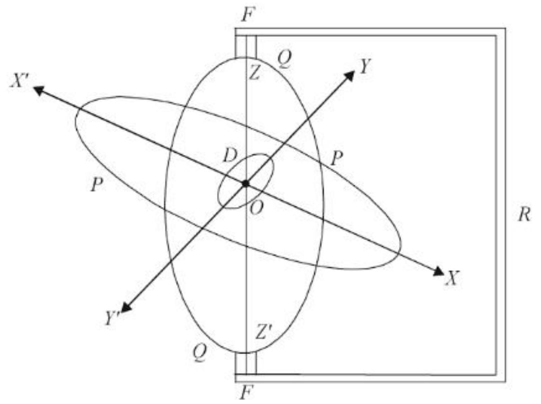
1. **RIGIDITY :** The axis of rotation (spin axis) of the gyro wheel tends to remain in a fixed direction in space if no force is applied to it.
2. **PRECESSION :** The axis of rotation has a tendency to turn at a right angle to the direction of an applied force.

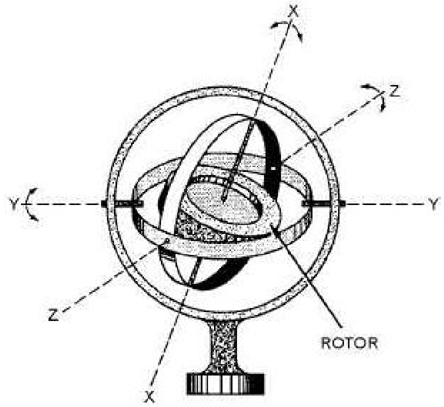
The gyroscope is mounted in gimbals so that the disc and axle are both free to turn as a whole about any one of the three perpendicular axes XX', YY' and ZZ' which intersect at a common point O. Each gimbal is mounted in the next gimbal with jewelled bearings which are made up of a very hard material like agate or saphire to reduce frictional torque.

The spinning disc D is fixed in a ring PP which is free to rotate about its axle coinciding with the axis of symmetry XX'. The ring PP in turn is fixed in another ring QQ which is free to rotate along YY' axis perpendicular to XX'. Further the ring QQ is fixed in a rigid frame work FF along the axis ZZ'. In this rigid frame work the gyroscope possesses three degree of freedom and can rotate about any of the three axes. The motion of the gyroscope consists of rotation, precession and nutation.

When a torque is applied to the axis of rotation of the disc it give rise to the precession of the axis of rotation. The rate of precession $\Omega = \frac{\tau}{I\omega}$

As Ω is inversely proportional to I and ω, larger the value of I - the moment of inertia of the disc and greater the angular velocity ω smaller will be the rate of precession. But a gyroscope, to be useful and effective, must have a large value of angular momentum, which is possible only with a heavy disc rotating at a very high speed.





Advantages

1. They really make smaller stabilized system
2. They impart greater Stabilization
3. They are Accurate and Easy to understand.

Disadvantage

1. The Gyroscopes are really expensive, but not in the terms of camera stabilization.
2. They are noisy if you are concerned about sound.
3. Pan and tilt speed is limited.
4. They take too much time to get up the speed.
5. They require another cable, battery and an inverter to work.

Uses

(i) Gyrocompass

The gyroscopes are used in ships and aeroplanes to give a continuous indication of the north - south direction. For this purpose the gyroscope is set along the magnetic meridian. Such a gyroscope is known as a gyrocompass.

For this purpose, a large and heavy gyroscope is used. The gyroscope is mounted in such a way that it can spin at a very high speed about a vertical axile, which can tilt forward or backward. The shaft of the gyroscope disc is supported in bearings fixed to the ship. When the ship rolls up and down the gyroscope is automatically tilted forward or backward so that its precession gives rise to a torque which acts in a direction opposite to that of the rolling ship. This torque brings the ship back to its stable position. Such a gyroscope is called a gyrostabiliser.

Q35. Explain the theory of action of a gyroscope?

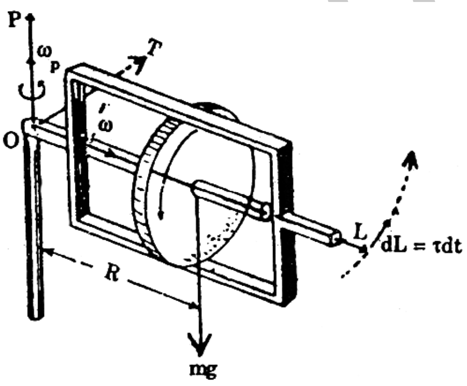
Ans :

Gyroscope

A top is a symmetrical body spinning about an axis which is fixed at one point. If the fixed point about which a symmetrical body is spinning about its axis coincides with the centre of gravity of the body, then it is called a gyroscope.

Theory and Action of a Gyroscope

The gyroscope consists of a heavy circular disc of large moment of inertia free to rotate at high speed is arranged writhing the rectangular frame. The disc rotates about its axis. The angular velocity vector ω will be along the axis from the fixed pivot O. The axis of gyroscope can itself rotate about O. For such a rotation of the angular velocity vector, the tip of the angular momentum vector L also moves in a horizontal circle with time (shown by dotted curve) making the gyroscope precess about O. But the precessional velocity ω_p is small compared to the angular vector of the circular disc within the frame work.



Let P be the upward force at the gyroscope mg is acting vertically downward at a distance R from O. The force P has a moment due to mg about O, which is given by

$$\vec{\tau} = mg\,R$$

and this torque acts in the direction perpendicular to $\vec{\omega}$ and P. This direction also moves in a circular path around O as the direction of ω changes.

$$\therefore \quad \vec{\tau} = \frac{dL}{dt}$$

Let $d\theta$ be the angle through which L turns in a time dt, then.

$$L\,d\theta = dL$$

$$\text{or} \quad d\theta = \frac{dL}{L}$$

Hence the precession velocity ω_p is given by

$$\omega_p = \frac{d\theta}{dt} = \frac{dL}{L} \times \frac{1}{dt} = \frac{\vec{\tau}}{L}$$

Thus the precessional velocity is inversely proportional to the angular momentum i.e., when the angular velocity of the gyroscope is large, the precessional velocity of the angular momentum is small. Thus gyroscope is a device characterized by the greater stability of its axis of rotation.

Q36. Write in detail about precession of equinoxes.

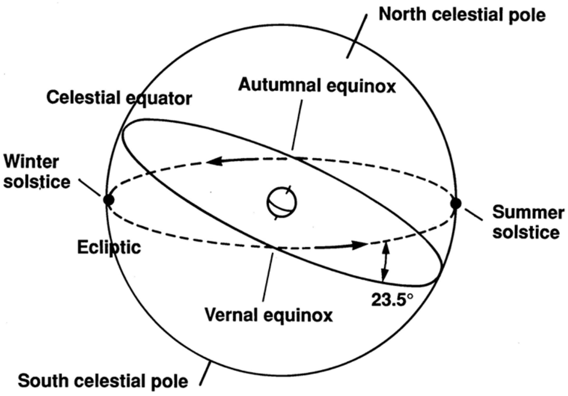
Ans :

Axial precession is gravity-induced, slow, and continuous change in the orientation of an astronomical body's rotational axis. In particular, it can refer to the gradual shift in the orientation of Earth's axis of rotation, which, similar to a wobbling top, traces out a pair of cones joined at their apices in a cycle of approximately 26,000 years. The term " precession" typically refers only to this largest part of the motion; other changes in the alignment of Earth's axis-nutation and polar motion– are much smaller in magnitude.

Earth's precession was historically called the precession of the equinoxes, because the equinoxes moved westward along the ecliptic relative to the fixed stars, opposite to the yearly motion of the Sun along the ecliptic.

The discovery of the precession of the equinoxes is mostly attributed in the west to Hellenisticera (2nd country BC) astronomer Hipparchus, although there are alternative suggestions claiming earlier discovery such as in Indian text vedanga Jyotisha from 700 BCE. With improvements in the ability to calculate the gravitational force between and among planents during the first half of the nineteenth century, it was recognized that the ecliptic itself moved slightly, which was named planetary precession, as early as 1863, while the dominant component was named lunisolar precession. Their combination was named general precession, instead of precession of the equinoxes.

Precession of Equinoxes is due to precessional motion of earth's rotational axis. The orbit plane of the earth, revolving around the sun and plane passing through the equator of earth are inclined to each other by an angle 23.5°. These two planes are intersecting at points A & B. These points are shown in figure. The point A is called vernal equinox and points A & B is called line of Equinoxe. While the earth revolving around the sun, it may reach point A on 21st march and point 'B' on 22nd September. For these two days the length of day and night will be equal.



At the equinoxe points the rotational axis of the earth, undergoes precessional motion. The internal torque created on the earth due to two reasons 1. shape of the earth is not a perfect sphere. It is bulged at the equator and flattended at the poles. This shape gives a symmetry of top 2. The attractive force experienced by the earth from sun and moon are not equal.

The unequal attractive fore due to sun and moon will create an internal torque on the earth. This leads to precessional motion of earth's rotational axis. The rotational axis of the earth's will complete a cone with respect to pole star. The precessional motion of rotational axis of earth will bring a change in the direction of line of equinoxe. This phenomenon is called precision of equinoxes. As the torque acting on the earth is small, the rotational axis will take nearly 25,800 years to complete a total cone due to precessional motion. For this small precessional angular velocity of the axis, the vega star will be fixed as pole star for nearly 1200 years.

PROBLEMS

1. The burnt fuel in a rocket is ejecting out with velocity of 1.6 km/sec. If the rocket starts from rest, then show that mass ratio of fuel to mass of empty rocket is 1100, for gaining an escape velocity of 11.2 km/sec ?

If the fuel burnt rate is 1/10 of initial mass, then find the time to attain the velocity of 11.2 km/sec?

Sol :

Initial velocity of rocket $v_i = 0$

Velocity of gases coming out of rocket $v_i = 1.6$ km/sec

The final velocity of rocket $v = 11.2$ km/sec.

Equation for the final velocity of rocket is

$$v = v_0 + u_{rel} \log_e \frac{M_0}{M}$$

$$11.2 = 0 + 1.6 \log_e \frac{M_0}{M}$$

$$\log_e \frac{M_0}{M} = \frac{11.2}{1.6} = 7$$

$\frac{M_0}{M} = e^7 = 1101$ ($\because$ $M = M_e$, mass of the empty rocket, the maximum velocity attains only when fuel is completely exhausted)

$$\frac{M_0}{M_e} = \frac{M_e + M_f}{M_e} = 1 + \frac{M_f}{M_e}$$

$$\frac{M_f}{M_e} = \frac{M_0}{M} - 1$$

$$\frac{M_f}{M_e} = 1101 - 1 = 1100$$

(b) Fuel burnt rate $\frac{dM}{dt} = \alpha = \frac{1}{10} M_0$

$$M = M_0 - \alpha t = M_0 \left(1 - \frac{\alpha t}{M_0}\right)$$

$$M = M_0 \left(1 - \frac{t}{10}\right) \qquad \therefore \; 1 - \frac{t}{10} = \frac{M}{M_0}$$

$$\frac{t}{10} = 1 - \frac{M}{M_0} = 1 - \frac{1}{1101} \qquad \because \; \frac{M_0}{M} = 1101$$

$$= \frac{1100}{1101}$$

$$t = \frac{1100}{1101} \times 10$$

t = 9.99 sec.

2. (a) Find the momentum of an electron with kinetic energy 100 eV ?

(b) The car and truck masses are 4000 kg and 10,000 kg respectively ?

(i) At what speed of truck does its momentum equal to momentum of car travelling with a speed of 30 m/s ?

(ii) At what speed of truck does its kinetic energy equal to kinetic energy of car travelling with a speed of 30 m/s ?

Sol :

(a) Mass of the electron $m = 9 \times 10^{-28}$ gram.

Kinetic energy $= \frac{1}{2} mv^2 = 100$ eV

$$\frac{1}{2} mv^2 = 100 \times 1.6 \times 10^{-12} \text{ erg.}$$

$$\frac{P^2}{2m} = 1.6 \times 10^{-10}$$

$$\therefore \; P^2 = 1.6 \times 10^{-10} \times 2 \times 9 \times 10^{-28}$$

$$= 1.6 \times 18 \times 10^{-38}$$

$$P = \sqrt{1.6 \times 18 \times 10^{-38}} = 5.37 \times 10^{-19} \text{ gm.cm.sec}^{-1}$$

(b) (i) If the velocity of truck is 'v' then

$10{,}000 \times v = 4000 \times 30$

$v = \frac{4000 \times 30}{10{,}000}$ $v = 12$ m/sec

(ii) If the velocity of truck is 'v_1' then

$$\frac{1}{2} \times 10{,}000 \times v_1^2 = \frac{1}{2} \times 4000 \times 30 \times 30$$

$$v_1^2 = 360$$

$$v_1 = 18.97 \text{ m/sec.}$$

3. A rocket burns 0.02 kg of fuel per second ejecting it as a gas with a velocity of 10,000 m/sec. What force does the gas exert on the rocket?

Sol :

The thrust ($F_{reaction}$) exerted by the escaping gas on the rocket is given by

$$F_{reaction} = u \frac{dM}{dt}$$

Here, $u = 10{,}000$ m/sec and $\left(\frac{dM}{dt}\right) = 0.02$ kg.

$\therefore \quad F_{reaction} = (10{,}000) \times (0.02)$

$= 200$ N

4. An empty rocket weights 6000 kg and contains 44000 kg of fuel. If the exhaust velocity of gases is 1 km/s, find the maximum velocity attained by the rocket.

Sol :

$$\nu_{max} = u \log_e \frac{M_0}{M}$$

$$= u \times 2.3 \log_{10} \frac{M_0}{M}$$

$$= 1 \times 2.3 \log_{10}\left(\frac{50{,}000}{6000}\right) \quad (\because M_0 = 44000 + 6000 = 50{,}000)$$

$$= 1 \times 2.3 \times 8.33$$

$$= 19.16 \text{ km/s}$$

5. A rocket having an initial mass M_0 starts from rest. When it attains a velocity ν, its mass becomes M. What is the ratio of (M_0 / M) when the velocity of exhaust gases is equal to ν (numerically).

Sol :

The velocity ν of the rocket at any instant of time t is given by

$$\nu = \nu_0 + u \log_e\left(\frac{M_0}{M}\right)$$

Given that $\nu_0 = 0$, $u = \nu$

$$\therefore \quad \nu = 0 + \nu \log_e\left(\frac{M_0}{M}\right)$$

or $$\log_e\left(\frac{M_0}{M}\right) = 1$$

or $$\frac{M_0}{M} = e = 2.717.$$

6. In a two stage rocket, the masses of first and second stages 300 kg and 30 kg respectively, the fuel filled is 2400 kg and 270 kg respectively. If the exhaust velocity of gases is 2 km/sec then what is the final velocity of rocket?

Sol :

The velocity attained by the rocket at the end of first stage

$$v = u \log_e \frac{M_0}{M}$$

But here $M_0 = 300 + 2400 + 30 + 270$

$= 3000$ kg

$M = 300 + 30 + 270 = 600$ kg

$u = 2$ km/sec

$$\therefore \quad v = 2 \times \log_e \frac{3000}{600} = 2 \times \log_e 5$$

$v = 3.22$ km / sec

The velocity v = 3.22 km/sec will be the initial velocity to second stage.

Thus by the end of second stage the final velocity

$v_0 = 3.22$ km/sec

$M_0 = 30 + 270 = 300$ kg

$M = 30$ kg

$$V = v_0 + u \log_e \frac{M_0}{M}$$

$$V = 3.22 + 2 \times \log_e \frac{300}{30}$$

$= 3.22 + 2 \times \log_e 10$

$V = 7.82$ km/sec

7. Find the length contraction of 1.5 m rod moving with a speed of 0.95 c (c is the velocity of light).

Sol :

$$l^1 = \frac{1}{\sqrt{1 - v^2 / c^2}}$$

$l = 1.5$ m

$v = 0.95$ c

$$l' = \frac{1.5}{\sqrt{1-(0.95)^2}} = \frac{1.5}{\sqrt{1-(0.9025)}} = \frac{1.5}{\sqrt{0.0975}} = \frac{1.5}{0.3122}$$

$l' = 4.8046$ m

8. A rocket of mass 10,000 kg has got a full of mass 30,000 kg inside it. Fid the exhaust velocity of the gases in 2 km/sec, find the maximum velocity of the rocket that can be obtained.

Sol :

Maximum velocity of rocket $v_{max} = u \log_e \left(\frac{M_0}{M}\right)$

M = Mass of rocket = 10,000 kg

M_0 = Mass of rocket + fuel mass = 10,000 + 30,000 = 40,000 kg

u = Exhaust velocity of gases

u = 2 km/s

$\therefore \quad v_{max} = 2 \log_e \left(\frac{40,000}{10,000}\right)$

$= 2 \log_e 4 = 2 \times 2.3 \log_{10} 4$

$= 2 \times 2.3 \times 0.60206$

$v_{max} = 2.7694$ m/s

9. Find the magnitudes of gravitational field and potential at a distance of 10 cm from the center of a solid sphere having a mass of 1 kg and a radius of 5 cm.

Sol :

M = 1 kg, a = 5 cm = 5×10^{-2} m

G = 6.67×10^{-11} N

G. Potential $V = -\frac{GM}{a} = \frac{6.67 \times 10^{11} \times 1}{5 \times 10^{-2}}$

$= 1.33 \times 10^{-9}$

G. field $f = \frac{-GM}{a^2} = \frac{-6.67 \times 10^{-11} \times 1}{(5 \times 10^{-2})^2}$

$= 0.266 \times 10^{-7}$

10. If the duration of a day is 28 hours, measured through a clock placed in spaceship with respect to a stationary observer, find the velocity of the spaceship.

Sol :

$\Delta t' = 2 \Delta t = 24$; $C = 3 \times 10^8$ m/s

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$28 = \frac{24}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$1-\frac{v^2}{c^2} = \left(\frac{24}{28}\right)^2 = 0.735$$

$$\frac{v^2}{c^2} = 0.265$$

$$v = \sqrt{0.265} \times C$$

$$= \sqrt{0.265} \times 3 \times 10^8 = 1.54 \times 10^8 \text{ m/sec}$$

11. The speed of a body of mass 20 kg moving along a circle of radius 1.5 m increases at the constant rate of 0.5 m/sec. Find the torque acting on the body.

Sol :

m = 20 kg; r = 1.5 m; V = 0.5 m/sec

$$\tau = \frac{dL}{dt} = \frac{d(mVr)}{dt}$$

$$\tau = \frac{mvr}{t} = \frac{20\times0.5\times1.5}{1} = 1.5 \text{ N}-\text{m}$$

12. A rocket of mass 20 kg has 180 kg fuel. The exhaust velocity of the fuel is 1.6 km/sec. Calculate the minimum rate of consumption of fuel so that the rocket may rise from the ground.

Sol :

Mass M = 20 + 180 = 200 kg

u = 1.6 km/sec., = 1.6 × 1000 m/sec.

$$u\frac{dM}{dt} = Mg$$

$$\frac{dM}{dt} = \frac{Mg}{u} = \frac{200\times9.8}{16\times1000}$$

= 1.225 km/sec.

13. The total mass of a rocket is 8000 kg. The exhaust velocity of gases 800 m/sec

(a) Find the burnt rate of fuel, just to lift the rocket.

(b) Find the burnt rate of fuel, to give an upward acceleration of 30 m/s^2

Sol :

(a) The thrust acting on the rocket must be equal to weight of the rocket.

Thrust on rocket

$$v_r = \frac{dM}{dt} = Mg$$

$$800 \times \frac{dM}{dt} = 8000 \times (9.8 + 30)$$

$$\frac{dM}{dt} = 98 \text{ kg/sec}$$

(b) To give an upward acceleration 'a', the required thrust to be act on rocket

$$v_r = \frac{dM}{dt} = m(g + a)$$

$$800 \frac{dM}{dt} = 8000(9.8 + 30)$$

$$\frac{dM}{dt} = \frac{8000 \times 39.8}{800}$$

$$= 398 \text{ kg/sec.}$$

14. The initial mass of a rocket is M_0. Rocket starts from rest. The final mass of rocket is M_f, where the final velocity is v_f. Then prove that

$$\frac{M_f}{M_0} = e^{-\frac{v_f}{u_{rel}}}$$

Sol :

The final velocity of rocket is given by

$$v = v_0 + u_{rel} \log_e \frac{M_0}{M} - gt$$

Neglecting the 'g' at higher altitudes, we have

$$\therefore \quad v = u_{rel} \log_e \frac{M_0}{M} \quad (\because v_0 = 0)$$

If $M = M_f$ then $v = v_f$

$$\log_e \frac{M_0}{M_f} = \frac{v_f}{u_{rel}}$$

Taking antilogarithm on both sides

$$\frac{M_0}{M_f} = e^{\frac{v_f}{u_{rel}}} \quad \text{or} \quad \frac{M_0}{M_f} = e^{-\frac{v_f}{u_{rel}}}$$

15. In a rocket the fuel burnt at a rate of 0.02 kg/sec. Exhaust velocity of gases 10,000 m/sec. Find the thrust acting on the rocket.

Sol :

Thrust acting on Rocket $F_{reaction} = u_r \frac{dM}{dt}$

$u_{rel} = 10,000 \text{ ms}^{-1}$

$\therefore \quad F_{reaction} = u_{rel} \frac{dM}{dt}$

$= 10,000 \times 0.02$

$F_{reaction} = 200 \text{ N}$

16. An empty rocket weights 5000 kg and contains 40,000 kg of fuel. If the exhaust velocity of the fuel is 2.0 km/sec., find the maximum velocity gained by the rocket. (Given that $\log_e 10 = 2.3$, $\log_{10} 3 = 0.4771$).

Sol :

Ignoring gravity effect, the velocity ν of a rocket at any time t is given by

$$\nu = \nu_0 + u \log_e \frac{M_0}{M}$$

where M_0 is the initial mass of the rocket (at t = 0) plus the mass of the fuel and M is the remaining mass at a time t. The velocity ν attains the maximum value when all the fuel is burnt. Then M is the mass of empty rocket.

According to the given problem

$\nu_0 = 0$, $M_0 = 5000 + 40,000 = 45,000$ kg, $M = 5000$ kg

and $u = 2.0$ km/sec.

$\therefore \quad \nu_{max} = (2.0 \text{ km/sec}) \log_e \frac{45000}{5000}$

$= (2.0 \text{ km/sec}) \log_e (3)^2$

$= (2.0 \text{ km/sec}) \, 2 \log_e 3$

$= (2.0 \text{ km/sec}) \, 2 \times 2.3 \times 0.4771$

$= 4.4$ km/sec.

17. From the nucleus of a radioactive element an election and nutron are disintegrated and travel in mutually perpendicular directions. The momenta are 9.22×10^{-21}, 5.33×10^{-21} $kgms^{-1}$ respectively. Then

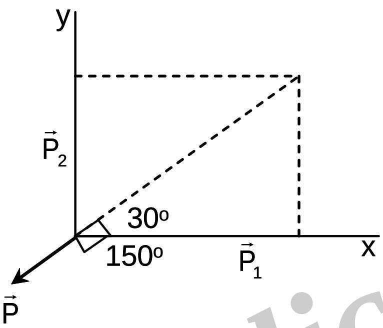
(a) Find the recoil velocity of nucleus

(b) Final direction of momentum of nucleus

(c) If the mass of nucleus is 3.9×10^{-25} kg, then find the kinetic energy.

Sol :

Let $\vec{P}_1$, $\vec{P}_2$ are the momenta of electron and proton respectively. $\vec{p}$ is the recoil momentum of nucleus.



From the conservation of momentum

$$0 = \vec{P}_1 + \vec{P}_2 + \overline{P}$$

$$\overline{P} = -(\vec{P}_1 + \vec{P}_2) = -(9.22 \times 10^{-21}\, \vec{i} + 5.33 \times 10^{-21}\, \vec{j})$$

$$|\vec{P}| = 10^{-21}\sqrt{(9.22)^2 + (5.33)^2}$$

$$= 1.065 \times 10^{-20} \text{ kg ms}^{-1}.$$

(a) Velocity of nucleus

$$V = \frac{|\vec{P}|}{m} = \frac{1.065 \times 10^{-20}}{3.9 \times 10^{-25}} = 0.27 \times 10^{5} \text{ m/s.}$$

(b) $\tan\theta = \dfrac{P_2}{P_1} = -\left(\dfrac{5.33}{9.22}\right)$

$$\theta = \tan^{-1}\left(\frac{-5.33}{9.22}\right) = 150^{o}$$

(c) Kinetic energy $= \dfrac{P^2}{2m} = \dfrac{(1.065 \times 10^{-20})^2}{2 \times 3.9 \times 10^{-25}}$

$$= \frac{1.134 \times 10^{-40}}{7.8 \times 10^{-25}} = 0.145 \times 10^{-15} \text{ J}$$

18. A machine gun fires 50 gm bullets at a speed of 1000 m/sec. The gum man holding the machine gun in his hands, can exert an average force of 180 newtons against the gun. Determine the maximum number of bullets he can fire in a minute.

Sol :

$$\text{Momentum of the bullet} = \left(\frac{50}{1000}\right) \times 100$$

$$= 50 \text{ kg-m/sec}$$

Let x be the maximum number of bullets fired in one minute.

$$\text{Momentum imparted per sec.} = 50 \times \left(\frac{x}{60}\right) \text{ kg-m sec}^2.$$

Now, Thrust exerted by gun man = rate at which momentum is imparted to the bullet

$$50 \times \left(\frac{x}{60}\right) = 180 \text{ newton}$$

$$x = \frac{180 \times 60}{50} = 216.$$

19. Two steel spheres of radii 2 cm and 3 cm move with velocities of 24 cm/sec. in opposite directions and collide head on. If the collision is elastic, calculate the velocities after impact.

Sol :

According to eq. (c) of article 2.16, we have

$$\nu_1 = \frac{2\,m_2\,u_2}{(m_1 + m_2)} + \frac{(m_1 + m_2)u_1}{(m_1 + m_2)}$$

Here the masses of steel spheres are proportional to the cube of their radii, i.e., 2^3 and 3^3 i.e., 8 and 27. So $m_1 = 8$ and $m_2 = 27$, $u_1 = 24$ cm/sec. and $u_2 = -24$ cm/sec. Hence,

$$\nu_1 = \frac{2 \times 27}{8 + 27} \times (-24) + \frac{(8 - 27)}{(8 + 27)} \times 24$$

Solving we get $\nu_1 = -50.05$ cm/sec.

This shows that after impact, the 2 cm radius sphere moves in a direction opposite to the direction of motion before impact.

Now according to equation (b) of article 2.16, we have

$$\nu_2 = \frac{2\,m_1\,u_1}{(m_1 + m_2)} + \frac{(m_2 - m_1)u_2}{(m_1 + m_2)}$$

$$= \frac{2 \times 8 \times 24}{(8 + 27)} + \frac{(27 - 8)(-24)}{8 + 27}$$

Solving we get $\nu_2 = -2.06$ cm/sec.

The negative sign shows that 3 cm radius sphere moves in its own direction after impact.

20. A 0.03 kg mass travelling at 0.08 m/s makes an elastic collision with a 0.05 kg mass at rest. Find the speed of each mass after collision.

Sol :

Given $u_1 = 0.08$ m/s, $m_1 = 0.03$ kg, $u_2 = 0$ and $m_2 = 0.05$ kg

Here we have $m_1 u_1 + m_2 u_2 + m_1 \nu_1 + m_2 \nu_2$

$\therefore \quad 0.03 \times 0.08 + 0.05 \times 0 = 0.03\,\nu_1 + 0.05\,\nu_2$

Solving we get $3\nu_1 + 5\nu_2 = 0.24$ ... (1)

From Newton's experimental law

$$\frac{\nu_2 - \nu_1}{u_2 - u_1} = -e = -1$$

$$\therefore \quad \frac{\nu_2 - \nu_1}{0 - 0.08} = -1$$

or $3\nu_2 - 3\nu_1 = 0.24$ ... (2)

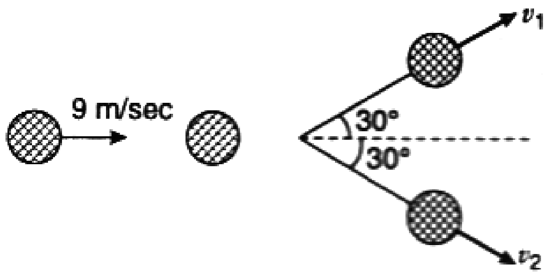
Solving eqs. (1) and (2) for ν_1 and ν_2, we get

$\nu_2 = 0.06$ m/s and $\nu_1 = -0.02$ m/s.

21. A ball moving with a speed of a 9 m/s strikes an identical stationary ball such that after the collision the direction of each ball makes an angle of 30° with the original line of motion. Find the speeds of the two balls after the collision. Is kinetic energy conserved in this collision process ?

Sol :

The situation is shown in fig.



Initial momentum of balls $= m \times 9 + m \times 0 = 9\,m$ ... (1)

where m is the mass of each ball. Let after collision their velocities be ν_1 and ν_2 respectively. Final momentum of the balls after collision along the same line

$$= m\,\nu_1 \cos 30 + m\,\nu_2 \cos 30$$

$$= \frac{m\,\nu_1(\sqrt{3})}{2} + \frac{m\,\nu_2(\sqrt{3})}{2} = \frac{m\,\nu_2(\sqrt{3})}{2}(\nu_1 + \nu_2) \quad ...(2)$$

According to the law of conservation of momentum

$$9m = \frac{m\sqrt{(3)}}{2}(\nu_1 + \nu_2)$$

or $$\frac{9 \times 2}{\sqrt{(3)}} = (\nu_1 + \nu_2) \qquad \text{... (3)}$$

The initial momentum of the balls along perpendicular direction = 0

Final momentum of balls along perpendicular direction

$$= m\,\nu_1 \sin 30 - m\,\nu_2 \sin 30 = \frac{m}{2}(\nu_1 + \nu_2)$$

According of the law of conservation of momentum

$$\frac{m}{2}(\nu_1 - \nu_2) = 0 \quad \text{or} \quad \nu_1 - \nu_2 = 0 \quad \text{or} \quad \nu_1 = \nu_2 \qquad \text{... (4)}$$

Solving eqs. (3) and (4), we get

$$\nu_1 = 3\sqrt{3} \text{ m/sec.} \quad \text{and} \quad \nu_2 = 3\sqrt{3} \text{ m/sec.}$$

According to the law of conservation of energy

Energy before collision = Energy after collision

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 \nu_1^2 + \frac{1}{2} m_2 \nu_2^2$$

$$\frac{1}{2} m (9)^2 + 0 = \frac{1}{2} m (3\sqrt{3})^2 + \frac{1}{2} m (3\sqrt{3})^2$$

$$\frac{81m}{2} = \frac{54m}{2}$$

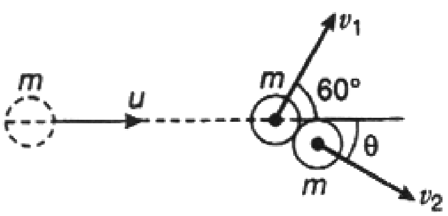
or L.H.S. $\neq$ R.H.S.

This shows that energy is not conserved in this collision i.e., this is case of inelastic collision.

22. A ball moving at a speed of 2.2 m/sec. strikes an identical stationary ball. After collision one ball moves at 1.1 m/sec. at 60° angle with the original line of motion. Find the velocity of the other ball.

Sol :

Let m be the mass of each ball and m, the initial velocity of the first ball. Let ν_1 and ν_2 be the final velocities of the balls respectively after collision as shown in figure.



Applying the law of conservation of momentum along original direction of motion, we have

$$m\,u = m\,\nu_1 \cos 60^o + m\,\nu_2 \cos\theta$$

or $u = \nu_1 \cos 60^o + \nu_2 \cos\theta$

$2.2 = 1.1\,(0.5) + \nu_2 \cos\theta$ $(\because \cos 60^o = 0.5)$

$\therefore$ $\nu_2 \cos\theta = 2.2 - 0.55 = 1.65$ ... (1)

Now applying conservation of momentum perpendicular to the original direction of motion, we have

$$0 = m\,\nu_1 \sin 60^o - m\,\nu_2 \sin\theta$$

or $0 = \nu_1 \sin 60^o - \nu_2 \sin\theta$

or $\nu_2 \sin\theta = 1.1\,(0.866)$ $(\because \sin 60^o = 0.866)$

or $\nu_2 \sin\theta = 0.953$

Squaring and adding eqs. (1) and (2), we get

$$\nu_2^2 = (1.65)^2 + (0.953)^2$$

or $\nu_2 = \sqrt{[(1.65)^2 + (0.953)^2]} = 1.9$ m/sec.

Dividing eq. (2) by eq. (1), we get

$$\tan\theta = \frac{0.953}{1.65} = 0.577$$

$$\theta = \tan^{-1}(0.577) = 30^o$$

23. Alfa particles of energy 4 MeV are scattered back from a gold foil Z = 79. Calculate the maximum volume in which the positive charge of the atom is likely to be concentrated.

Sol :

The α-particle will be scattered back a a point where its kinetic energy is converted into potential energy. Let b be the distance of closest approach, then

$$\text{K.E.} = \frac{1}{2} m\,\nu_0^2 = \frac{2\,Ze^2}{4\pi\varepsilon_0 b}$$

According to the given problem,

$$\text{K.E.} = 4 \text{ MeV} = 4 \times 10^6 \text{ eV}$$

$$= (4 \times 10^6)(1.6 \times 10^{-19}) \text{ joule.}$$

$$Z = 79 \text{ and } e = 1.6 \times 10^{-19} \text{ C}$$

$$\therefore \quad (4 \times 10^6)(1.6 \times 10^{-19}) = \frac{2 \times 79 \times (1.6 \times 10^{-19})^2}{\{1/(9 \times 10^9)\} \times b}$$

$$b = (9 \times 10^9)\,\frac{2 \times 79 \times (1.6 \times 10^{-19})}{(4 \times 10^6)}$$

$= 5.688 \times 10^{-14}$ m

Maximum volume $= \frac{4}{3} \pi b^3$

$= \frac{4}{3} \times (3.14) \times (5.688 \times 10^{-14})^3$

$= 7.72 \times 10^{-40}$ m^3

24. An α-particle with K.E. 6.0 × 10^{-14} joule is scattered at an angle of 60° by Coulomb field of a stationary nucleus. Find the impact parameter.

Sol :

The impact parameter is given by

$$p = \frac{Ze^2}{2 \pi \varepsilon_0 \, m v_0^2 \tan(\phi / 2)}$$

$$= \frac{2 \times (1.6 \times 10^{-19})^2}{2 \times 3.14 \times (8.85 \times 10^{-12}) \times (12 \times 10^{-14}) \times \tan 30^{\circ}}$$

$= 0.013 \times 10^{-12}$

$= 13 \times 10^{-15}$ m.

25. A circular disc of mass 100 kg and radius 50 cm is mounted axially and is rotating. Calculate the K.E. when its executes 25 rev./min.

Sol :

$$\text{K.E.} = \frac{1}{2} I \omega^2 = \frac{1}{2}\left(\frac{MR^2}{2}\right)\omega^2$$

M = 100 kg, R = 0.5 m and $\omega = 2\pi n$

n = number of rev./sec. $= \frac{25}{60}$

$$\omega = \frac{(2\pi \times 25)}{60} = \frac{5\pi}{6} = 2.6166 \text{ rad/s}$$

So, $$\text{K.E.} = \frac{1}{2}\left[\frac{100 \times (0.5)^2}{2}\right](2.6166)^2 = 42.79 \text{ J}$$

26. The speed of a particle of mass 20 kg that is moving along a circular radius is 1.5m increasing at the rate of 0.5 m/s for every second. Find the torque acting on it.

Sol :

We know that, $L = m v r$

Now, $\tau = \frac{dL}{dt} = \frac{d}{dt}(mvr) = mr\left(\frac{dv}{dt}\right)$

Here, m = 20kg, r = 1.5 m and $\left(\frac{dv}{dt}\right)$ = 0.5 m/s.

$\therefore \quad \tau = 20 \times 1.5 \times 0.5 = 15$ N–m

27. The kinetic energy of a metal disc rotaing at a constant speed of 5 revolutions per second is 100 joule. Find the angular momentum of the disc.

Sol :

$$L = I\omega$$

$$\omega = 2\pi n = 2\times 3.14 \times 5 = 31.4 \text{ sec}^{-1}$$

Now, $K.E. = \frac{1}{2} I\omega^2$

or $I = \left(\frac{2K.E.}{\omega^2}\right) = \frac{2\times 100}{(31.4)^2} = 0.2028$ kg-m^2

$\therefore \quad L = 0.2028 \times 31.4 = 6.368$ kg-m^2 sec^{-1}

28. A top is spinning at 30 rev./sec. about an axis making an angle of 30° with the vertical. Its mass is 1 kg and its rotational inertia is 5×10^{-4} kg-m². The centre of mass is 4 cm from the pivot point. If the spin is clockwise as seen from the above, what is the magnitude and direction of the angular velocity of precession ?

Sol :

The angular velocity of precession is given by

$$\omega_p = \frac{\tau}{L} = \frac{rmg}{Iw}$$

Here, r = 0.04 m, m = 1kg, $I = 5\times 10^{-4}$ kg-m^2

and $\omega = 2\pi n = 2\pi \times 30 = 60\,\pi$ rad/s.

$$\omega_p = \frac{0.04\times 1\times 9.8}{(5\times 10^{-4})(60\,\pi)} = 4.16 \text{ rad/sec}$$

The spin is clockwise as seen from above and hence the angular velocity of precession is in the downward direction.

29. A Fly wheel of mass M and radius 'R' is free to rotate on axel. A thread is rounded about the wheel and at the end of the thread a force 'F' is applied. Then find the angular acceleration and tangential acceleration of the point on the wheel ?

Sol :

Torque acting on wheel $\vec{\tau} = \vec{F} \times \vec{R}$

Moment of inertia of wheel $I = \frac{1}{2} MR^2$

But $\tau = I\alpha$, α-angular acceleration

$FR = \frac{1}{2} MR^2\alpha$

$\therefore \quad \alpha = \frac{2F}{MR}$

Tangential acceleration $\alpha_\tau = R\alpha$

$$\alpha_T = \frac{2FR}{MR} = \frac{2F}{M}$$

30. A symmetric top is spinning about its rotational axis with a speed of 30 revolutions per second. Axis of rotation making an angle of 30° with vertical. Mass of the top 0.5 kg. Its moment of inertia 5 × 10^{–4} kg.m². The distance of centre of mass from pivot is 4 cm. If we look on to the top, its rotational sense is clockwise. Find the magnitude and direction of precessional angular velocity of the top?

Sol :

Angular velocity $\omega = 30 \times 2\pi$ rad/sec

Moment of inertia $I = 5.0 \times 10^{-4}$ kg.m^2

Angular momentum $J = 5.0 \times 10^{-4} \times 30 \times 2\pi$

$J = 942 \times 10^{-4}$ kg-m^2/sec

Torque $\tau = mgr$

Mass m = 0.5kg

$\tau = 0.5 \times 9.8 \times 0.04 = 0.196$ Nm

Precessional angular velocity $\omega_p = \frac{\tau}{J} = \frac{0.196}{942 \times 10^{-4}}$

$\omega_p = 2$ rad/sec

The direction of precessional angular velocity is in clockwise.

31. A wheel is rotating with an angular velocity of 500 revolutions/minute on an axel. Another wheel same as the first at rest is joined to the axel. Both the wheels are rotating with a common speed. Find their common speed ?

Sol :

From the conservation of angular momentum

$$I\omega = I_1\omega_1 + I_2\omega_2$$

$I\omega$ = final angular momentum

$I_1\omega_1 + I_2\omega_2$ = initial angular momentum

$$\omega = \frac{I_1\omega_1 + I_2\omega_2}{I} - \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

$$\omega = \frac{I \times 500 + I \times 0}{I + I} = 250 \text{ revolutions/min.}$$

32. A circular ring of mass 100 grams and radius 10 cm is rotating about an axis passing through centre and ⊥ to the plane of ring with an angular velocity 10 revolutions per second. Then find

a) Moment of inertia of the ring

b) Angular momentum

c) The required torque to increase angular momentum to 10^5 eng.sec in one second.

Sol :

a) Moment of inertia of the ring

$I = MR^2$

$1 = 100 \times 10 \times 10 = 10^4$ gram.cm^2

b) Angular momentum $L = I\omega$

$L = 10^4 \times 10 \times 2\pi = 6.28 \times 10^5$ erg.sec

c) Torque $\tau = \frac{dL}{dt} = \frac{10^5}{10} = 10^4$ dyne.com.

33. A symmetric top is rotating with an angular velocity 18 revolutions/sec. It is inclined by an angle of 20° with the vertical. The radius of gyration of the top is 6 cm. If the distance of c.m from pivot is 5 cm. then find precessional angular velocity?

Sol :

Precessional angular velocity $\omega_p = \frac{mgr}{J} = \frac{gr}{k^2\omega}$

$$\omega_p = \frac{980 \times 5}{6 \times 6 \times 2 \times 3.14 \times 18}$$

$\omega_p = 1.2$ radians/sec.

34. A 500 gm stone is revolved at the end if a 0.4 m long string at the rate of 12.5 rad/s. Find its angular momentum.

Sol :

Given,

A store revolves at the end of a string

Mass of stone, m = 500 gm = 0.5 kg

length of string, $l = r = 0.4$ m

Angular velocity of stone, w = 12.5 rad/s

The angular momentum of stone is obtained as,

Angular momentum $= mr^2 \omega$

$= 0.5 \times (0.4)^2 \times 12.5$

$= 1$ kg m^2/s

$\therefore$ Angular momentum of stone = = 0 1 kg $m^2 s^{-1}$

Short Question and Answers

1. Define Torques. prove that the rate of change of angular momentum is equal to torque.

Ans :

Torque

The restoring force that acts on a rotating object is known as Torque and it is denoted by τ.

Relation between Torque and Angular Momentum

Angular momentum of a body is given as,

$$L = I\omega$$

Differentiating the above equation with respect to time t,

$$\frac{dl}{dt} = \frac{d(I\omega)}{dt}$$

$$\frac{dl}{dt} = I\frac{d\omega}{dt} \qquad \left(I\frac{d\omega}{dt} = \pi\right)$$

Since, $\frac{d\omega}{dt}$ is angular acceleration,

$$I\frac{dl}{dt} = \tau$$

2. State and explain Newton's Law of Motion.

Ans :

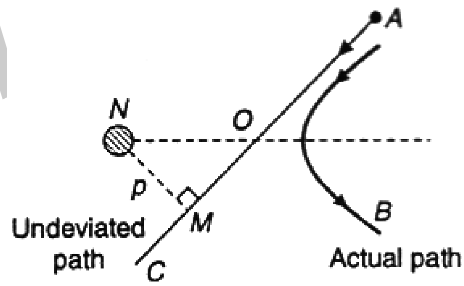
Newton's laws are applied to objects which are idealised as single point masses, in the sense that the size and shape of the object's body are neglected to focus on its motion more easily. This can be done when the object is small compared to the distances involved in its analysis, or the deformation and rotation of the body are of no importance. In this way, even a planet can be idealised as a particle for analysis of its orbital motion around a star.

In their original form, Newton's laws of motion are not adequate to characterise the motion of rigid bodies and deformable bodies. Leonhard Euler in 1750 introduced a generalisation of Newton's laws of motion for rigid bodies called Euler's laws of motion, later applied as well for deformable bodies assumed as a continuum. If a body is represented as an assemblage of discrete particles, each governed by Newton's laws of motion, then Euler's laws can be derived from Newton's laws. Euler's laws can, however, be taken as axioms describing the laws of motion for extended bodies, independently of any particle structure.

3. Impact Parameter

Ans :

Consider a positive particle, like a proton or an α-particle, approaching a massive nucleus N of an atom, as shown in figure.

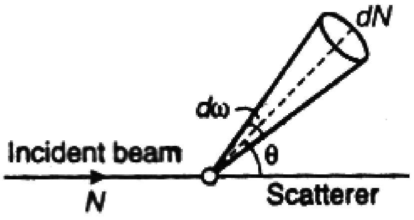


Due to coulombic force of repulsion, the particle follows a hyperbolic path AB with nucleus N as its focus. In the absence of the repulsive force, the particle would have followed the straight line path AC. As shown in figure, p is the perpendicular distance from the nucleus N to the original direction AC of the particle. The distance (NM = p) is called the impact parameter. Thus impact parameter is defined as the closest distance between nucleus and positively charged particle projected towards it. This is also known as collision parameter.

4. Scattering Cross Section

Ans :

When α-particles are projected into a thin metal foil, they are deflected or scattered in different directions. Let N be the incident intensity (number of incident particles crossing per unit time a unit surface placed perpendicular to the direction of propagation). Suppose dN be the number of particles scattered per unit time into solid angle $d\omega$ located in the direction θ and ϕ [Fig. below] with respect to the bombarding direction. The ratio dN / N is called scattering cross-section.



Thus the scattering cross section in a given direction is defined as the ratio of number of scattering particles into solid angle $d\omega$ per unit time to the incident intensity.

$$\therefore \quad \text{Scattering cross-section, } \sigma_{sc} = \frac{dN}{N}$$

5. Gyroscope.

Ans :

A gyroscope is a heavy symmetrical body (top) in the form of a heavy circular disc or fly wheel rotating at a very high speed about is axle.

Gyroscopes have two basic properties : precession. Those are defined as follows:

1. **RIGIDITY :** The axis of rotation (spin axis) of the gyro wheel tends to remain in a fixed direction in space if no force is applied to it.

2. **PRECESSION :** The axis of rotation has a tendency to turn at a right angle to the direction of an applied force.

The gyroscope is mounted in gimbals so that the disc and axle are both free to turn as a whole about any one of the three perpendicular axes XX', YY' and ZZ' which intersect at a common point O. Each gimbal is mounted in the next gimbal with jewelled bearings which are made up of a very hard material like agate or saphire to reduce frictional torque.

6. Describe the principle of motion of a rocket as system of variable mass.

Ans :

A moving rocket in which the fuel gets burnt and comes out in the form of exhaust gases is a good example of a variable mass system. The rocket consists of a combustion chamber in which a liquid or a solid is burnt. When the fuel is burnt, the pressure inside the composition chamber rises very high. Due to the high pressure, the hot gases (burnt liquid or solid fuel) are expelled from the nozzle at the tail of the rocket). These expelled gases will be in the form of a jet having a very high exhaust velocity. This is the action. Consequently, as a reaction, the rocket moves in a direction opposite to the direction of the out coming gases. Thus, the rocket works on the principle of Newton's third law of motion or the law conservation of momentum which is a consequence of the Newton's third law.

7. Calculate the thrust on a rocket.

Ans :

The expression for motion of a rocket is given by,

$$M\frac{dy}{dt} = F_{ext} + F_{reaction}$$

if the rocket moves away from the influence of gavitational force of the earth, the there is no external force acting on it.

$$\Rightarrow \quad F_{ext} = 0$$

The motion of a rocket becomes,

$$M\frac{dy}{dt} = 0 + F_{reaction}$$

$$= V_{rel}\frac{dM}{dt}$$

The term $V_{rel}\frac{dM}{dt}$ is termed as thrust of a rocket

$$\therefore \quad \text{Rocket thrust} = V_{rel}\frac{dM}{dt}$$

Where,

V_{rel} – Exhaust velocity

M – Mass of a rocket

8. Derive Euler's equation for a rigid body.

Ans :

The time rate of change of angular momentum of a rigid body about a fixed point is equal to the resultant external torque acting on the body about that fixed point.

If $\vec{\tau}$ is the torque and $\vec{L}$ the angular momentum, then

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad \text{... (i)}$$

This equation holds good if the system of axes has a fixed orientation in space i.e. the inertial frame is fixed in space.

In order to study the rotation of a rigid body, the system of axes should be fixed in the body itself and the origin should be coincident with the fixed point about which the body is rotating so that as the body rotates the co-ordinate axes also rotate with the body.

The time rate of change of any vector in a fixed frame can be transferred to the time rate of change of the same vector in a rotating frame using operator equation

$$\left(\frac{d}{dt}\right)_S (-) = \left(\frac{d}{dt}\right)_R (-) + \vec{\omega} \times (-)_R$$

where

$\left(\frac{d}{dt}\right)_S$ represents the time rate of change in stationary frame,

$\left(\frac{d}{dt}\right)_R$ the time rate of change in the rotating frame

and (–) the rotating vector.

9. Distinguish between elastic and inelastic collisions.

Ans :

S.No.	Elastic Collisions	S.No.	Inelastic Collisions
1.	Elastic collisions conserve kinetic energy.	1.	Inelastic collisions do not conserve kinetic energy.
2.	These type of collisions do not occur in common and can never be observed at macroscopic scales.	2.	These type of collisions occur at atomic level.
3.	Coefficient of restitution of elastic collisions is equal to unity (i.e., e = 1).	3.	Coefficient of restitution of inelastic collisions lies between 0 and 1 (i.e., 0 < e < 1).
4.	The most common example of elastic collision is the collision between two billiard balls.	4.	The common example of inelastic collision is the collision between two vehicles wherein the colliding, vehicles get lock together with each other.

10. Explain the theory of action of a gyroscope?

Ans :

A top is a symmetrical body spinning about an axis which is fixed at one point. If the fixed point about which a symmetrical body is spinning about its axis coincides with the centre of gravity of the body, then it is called a gyroscope.

Theory and Action of a Gyroscope

The gyroscope consists of a heavy circular disc of large moment of inertia free to rotate at high speed is arranged writhing the rectangular frame. The disc rotates about its axis. The angular velocity vector ω will be along the axis from the fixed pivot O. The axis of gyroscope can itself rotate about O. For such a rotation of the angular velocity vector, the tip of the angular momentum vector L also moves in a horizontal circle with time (shown by dotted curve) making the gyroscope precess about O. But the precessional velocity ω_p is small compared to the angular vector of the circular disc within the frame work.

11. Inelastic collision.

Ans :

When two bodies after collision, move as one and do not have any tendency to separate as they were before collision, the collision is said to be perfectly inelastic.

For example, a collision between a bullet, and its target is completely inelastic, if the bullet remains embedded in the target after collision. The relative velocity between the bullet and the target after collision is zero.

The kinetic energy in all such collisions is not conserved. The linear momentum is, however, conserved.

12. Elastic collision.

Ans :

When two bodies after collision, separate by virtue of their elastic properties, the collision is said to be perfectly elastic.

Examples of such collisions are inter atomic collisions or collisions between subatomic particles. In practical life, collisions between" two ivory balls or two glass balls can be regarded as perfectly elastic.

No kinetic energy or momentum is lost in elastic collision i.e., conservation laws of both kinetic energy as well as momentum holds good in the case of perfectly elastic collisions.

In fact, no collision is perfectly elastic or perfectly inelastic. Only the degree of elasticity of collision varies. The degree of elasticity of collisions is measured in terms of the coefficient of restitution.

13. Show that $\tau = 1\alpha$ for a rotating rigid body.

Ans :

The force acting on the particle produces an angular acceleration in the body, causing it to rotate about z-axis. The torque acting on the particle 'P' or the rigid body is given as,

$$T = \bar{\tau} = r \times \bar{F} \quad \text{...(1)}$$

The equation of motion of rigid rotating body can be obtained by estimating the relationship between torque applied and angular acceleration of the body. Consider the rotation of the body through an infinitesimal angle $d\theta$ within an infinitesimal time dt. Let the particle move from P(t) to P(t + dt) along the radius are 'r' in dt sec. Figure (ii) illustrates the diagrammatic representation of the rotating body with necessary notations.

14. Explain the working of multistage rocket.

Ans :

A rocket is the vehicle employed for space journey. It works on the principle of jet propulsion. The principle of jet propulsion depends on the law of conservation of momentum, according to which the momentum of the jet emerging in the backward direction makes the rocket to move in the forward, direction.

According to the type of fuel used, rockets are classified as (i) liquid fuel rockets and (ii) solid fuel rockets.

A rocket to have maximum velocity at its final stage, insists.

1. Relative velocity of gases to be maximum.
2. Final mass of the rocket M is very much less than initial mass of the rocket M_0.

Relative velocity of gases coming out of rocket depends on temperature, pressure within the chamber. It also depends on area of crossection of the nozzle. With the presently using fuel, temperature that develops in the chamber is 3000^0C. Due to this temperature and depending on the crossesction of nozzle, the maximum relative velocity can be expected only 2 km/sec.

From the optimum design of the fuel chamber (for liquid fuels) presently the value of $\frac{M_0}{M}$ is maintained at nearly 10, and for solid fuels this value further low.

Therefore even on neglecting the gravitational force, the maximum velocity that a rocket can attain, starting from rest is from equation :

$$\bar{v} = \bar{u}_{rel} \log_e m - gt + c$$

$$\bar{v} = 0 + 2 \log_{e-} (10) - 0$$

$$\bar{v} = 2 \times 2.3 = 4.6 \text{ km/sec.}$$

This velocity is very much less than the orbital velocity of a rocket i.e., (11.2 km/sec). Due to this reason, in order to launch a satellite multistage rockets are used. At the end of first stage, the rocket may attain a velocity nearly 4.6 km/sec and later second stage that begin to work and first stage of rocket detaches from the rocket. The velocity is adding up and finally, with all different stages the rocket attains the required velocity.

15. Moment of Inertia.

Ans :

Moment of inertia of a body about an axis is defined as the sum of the product of the mass and the square of the distance of the different particles of the body from the axis of rotation.

The moment of inertia of the body is expressed as Σmr^2.

The K.E. of rotation $= \frac{1}{2} I\omega^2$

If $\omega = 1$, then $I = 2 \times$ kinetic energy

Hence moment of inertia may also be defined as twice the kinetic energy of rotation of a body when its angular velocity is unity.

Choose the Correct Answers

1. If moment of inertia of a wheel, having radius of gyration 60 cm, is 360 kg m^2 then mass of the wheel is ______. [d]

   (a) 200 kg (b) 500 kg

   (c) 800 kg (d) 1000 kg

2. Angular momentum is the vector product of : [a]

   (a) linear momentum and radius vector

   (b) moment of inertia and angular acceleration

   (c) linear momentum and angular velocity

   (d) linear velocity and radius velocity

3. The orbit of an artificial satellite is ______. [c]

   (a) hyperbolic (b) parabolic

   (c) elliptic (d) none of these

4. M.I. of a solid sphere is [c]

   (a) $\frac{2}{3}m^2r^2$ (b) $\frac{3}{2}mr^2$

   (c) $\frac{2}{5}mr^2$ (d) $\frac{3}{5}mr^2$

5. If a satellite is launched into a circular orbit close to the earth. its velocity is [b]

   (a) $\sqrt{2gR}$ (b) $\sqrt{gR}$

   (c) gR (d) 2gR

6. Law of conservation of angular momentum is consequence of ______. [b]

   (a) homogeneity of space (b) isotropy of space

   (c) homogeneity of space and time (d) homogeneity of time

7. Homogeneity of time leads to conservation of ______. [c]

   (a) linear momentum (b) angular momentum

   (c) total energy (d) kinetic energy

8. Which is not explicit function of time ? [c]

   (a) velocity (b) acceleration

   (c) potential energy (d) momentum

9. A bicycle in motion does not fall because one of the following is conserved. [b]

   (a) linear momentum (b) angular momentum

   (c) kinetic energy (d) all of the above

10. Newton's law of motion are based on the assumption that space is ______. [c]

   (a) homogeneous (b) isotropic

   (c) homogeneous and isotropic (d) invariant under rotation

11. A body of mass m collides against a wall with a velocity v and rebounds with the same velocity. The change in memontum of the wall is ______. [c]

   (a) zero (b) mv

   (c) –2mv (d) –mv

12. Ratio of inertial mass to gravitational mass is [b]

   (a) 1 : 2 (b) 1 : 1

   (c) 2 : 1 (d) no fixed number

Fill in the Blanks

1. Moment of inertia is ______.
2. Units of M.I are ______.
3. The number of co-ordinates required to describe a collision in centre of mass frame is ______.
4. In elastic collision there is a conservation of ______.
5. The scatering cross-section has the dimensions of ______.
6. If ϕ is the angle of scattering in lab and θ in c.m. system, then for $m_1 = m_2$ we have ϕ = ______.
7. The path of an α-particle in Rutherford scattering is always ______.
8. When the velocities get intercharged after collision of two bodies, the collision is ______ elastic.
9. The minimum velocity with which a body may be projected to become a satellite of the earth is ______.
10. The value of escape velocity is ______ km/sec.
11. The time period of a geostationary satellite is ______ hours
12. Rocket works on the principle of conservation of ______.
13. If the force on a rocket moving with a velocity of 300 m/sec is 210H. then the rate of fuel combustion is ______ kg/sec.
14. Newton's second law gives the measure of ______.
15. A body which does not undergo any change in shape or size by the application of external forces is called ______.
16. Law of conservation of linear momentum is consequence of ______.
17. The unit of angular momentum is ______.
18. Number of dimensions space has is ______.

ANSWERS

1. $\frac{2kE}{w^2}$
2. $Kg.m^2$
3. 3
4. linear momentum
5. area
6. $\frac{\theta}{2}$

7. hyperbola
8. perfectly
9. 7.92 km/sec.
10. 11.2 km/sec
11. 24
12. linear momentum.
13. 0.7
14. force
15. a rigid body
16. homogeneity of space.
17. Kg m^2 s^{-1} or Joule second.
18. three

UNIT III

CENTRAL FORCES :
Central forces – definition and examples, conservative nature of central forces, conservative force as a negative gradient of potential energy, equation of motion under a central force, gravitational potential and gravitational field, motion under inverse square law, derivation of Kepler's laws.

SPECIAL THEORY OF RELATIVITY:
Galilean relativity, absolute frames, Michelson-Morley experiment, Postulates of special theory of relativity. Lorentz transformation, time dilation, length contraction, addition of velocities, mass-energy relation. Concept of four vector formalism.

3.1 Central Forces

3.1.1 Definition and Examples

Q1. Define central and non-central forces. What are the characteristics of central force?

Ans : **(June-21, Dec.-19, June-17)**

(i) Central Force

A central force is defined as a force, which always acts on a particle or body towards or away from a fixed, point and whose magnitude depends upon only on the distance from the fixed point. This fixed point is known as the centre of the force.

(ii) Non-Central Force

A non-central force is that force which does not simply depend upon the distance between the centres of the two interacting bodies but also on other parameters such as their spin and relative orientation.

Characteristics

The characteristics of non-central forces are :

i) They are short range forces i.e. the force acts only when the interacting particles are very close to each other.

ii) Non-central forces do not necessarily act along the line joining the centres of the two bodies.

iii) A non-central force is non-conservative and cannot be derived from some scalar potential i.e. they are not the gradient of some scalar function.

Examples

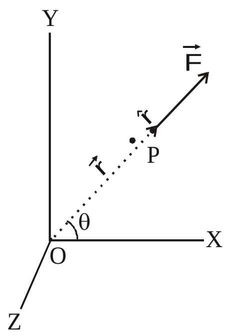
Familiar examples of non-central forces are :

(i) **Weak forces** called into play in b-decay and decay processes where the decay products are leptons (electrons, positrons, neutrinos, m-mesons etc.) arc non-central. Weak forces arc non-zero only when the interacting particles just overlap.

(ii) **Strong nuclear forces** between proton-proton (p-p interaction), proton-neutron (p-n interaction) and neutron-neutron (n-n interaction) are non-central as these are due to the exchange of p^+, p^- and p^0 mesons respectively.

Explanation

It is convenient to express the central force in polar coordinate system. Let O be the centre of force, which is taken as the origin of coordinate system. P is a particle whose polar coordinates are r and q. The central force on particle P is expressed by F. Mathematically, F is expressed as,



$$F = \hat{r}\ f(r) \qquad \text{... Eqn. (1)}$$

where f(r) is a function of the distance r of the particle from the fixed point and $\hat{r}$ is unit vector along the radius vector r of the particle with respect to that fixed point. In case of two particles the magnitude of central force depends upon the distance of separation of two particles and the direction being along the line joining the particles.

Examples :

1. The gravitional force exerted on a particle of mass m, by another stationary particle of mass m_2 is a central force and can be written as

$$F_{12} = -G\,\frac{m_1 m_2}{r^2}\,\hat{r}$$

where r is the distance between the two particles. Here negative sign indicates that the force is attractive.

But $F = f(r)\,\hat{r}$

$\therefore \quad f(r) = -G\,\frac{m_1 m_2}{r^2} = -\frac{C}{r^2}$

where $C = -G\,m_1 m_2$ is a constant.

Thus $f(r) \propto \frac{1}{r^2}$

which is the famous inverse square law.

Thus the earth moves around the sun under a central force which is always directed towards the sun.

2. The electrostatic force exerted on a charged particle q_x by another stationary charged particle q_2 is a central force and is given by

$$F_{12} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1 q_2}{r^2}\,\hat{r}$$

But $F = f(r)\,\hat{r}$

$\therefore \quad f(r) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1 q_2}{r^2} = \frac{C}{r^2}$

or $f(r) \propto \frac{1}{r^2}$

Thus, electron in a hydrogen atom moves under a central force which is always directed towards the nucleus.

3. A mass attached to one end of a spring whose other end is fixed is also an example of central force. The spring always pulls towards the fixed end of pushes way from it by an elastic force.

$F = -kx$

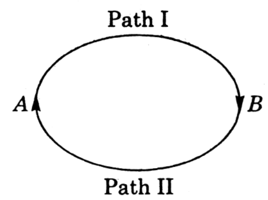
where 'x' is the distance of the mass from the unstretched position of the spring and k is spring constant.

3.2 Conservative Nature of Central Forces

Q2. Explain the conservative nature of central force?

Ans : **(June-18)**

A force is said to be conservative if the work done in moving a particle from one point to another point is independent of the path followed. But depends on the end points. Consider a particle that moves from A to B. It can follow path I or path II or any other path. If the force is conservative, the work done along these paths is constant. The points A and B are fixed.



$$\int_A^B \vec{F}.\overline{dr} = \int_B^A \vec{F}.\overline{dr}$$

path I path II

The work done in moving a particle along a closed curve is zero.

W_{AB} = work done in moving from A to B

$\int_A^B \vec{F}.\overline{dr}$ .... (1)

W_{BA} = work done in moving from B to A = $+W_{BA} = -W_{AB}$

$\therefore$ Net work done = $W_{AB} + W_{BA}$

$W_{AB} + W_{BA} = 0$ .... (2)

3.3 Conservative Force as a Negative Gradient of Potential Energy

Q3. Show that the conservative force is the negative gradient of potential energy.

Ans : **(Dec.-19, Dec-17)**

The work done against central force is equal to increase in potential energy of the body. The potential energy of a body on the earth is zero. When the body is lifted to certain height against gravitational force, an increase in potential energy takes place. Let U (x, y, z) is the potential energy function due to central force field, the change in potential energy is

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$

$$= \left(\vec{i}\frac{\partial U}{\partial x} + \vec{j}\frac{\partial U}{\partial y} + \vec{k}\frac{\partial U}{\partial z}\right)\left(\vec{i}\ dx + \vec{j}\ dy + \vec{k}\ dz\right) = dr \qquad ...(1)$$

If $\vec{F}$ is central force, the work done against central force is equal to the change in potential energy

$$dU = -\vec{F}.\vec{d}r$$

$$F = F_x d_x + F_y d_y + F_z d_z \qquad ...(2)$$

$$F_x = \frac{-\partial U}{\partial x},\ F_y = \frac{-\partial U}{\partial y},\ F_z = \frac{-\partial U}{\partial z}$$

From equation (1) and (2), one can obtain

$$\vec{F} = -\left(\vec{i}\frac{\partial U}{dx} + \vec{j}\frac{\partial U}{\partial y} + \vec{k}\frac{\partial U}{\partial z}\right)$$

$$= -\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)U$$

$$\vec{F} = -\vec{\nabla}U = -\text{ grad } U \qquad ...(3)$$

3.4 Equation of Motion Under a Central Force

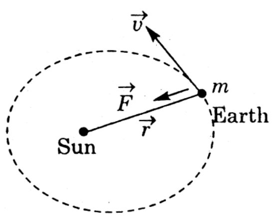
Q4. Prove that the motion under a central force takes place in fixed plane?

Ans :

Consider a particle in X-Y plane. Let r and υ are the position and velocity vectors of the particle. The angular momentum of the particle is

$$\vec{L} = \vec{r} \times \vec{p}$$

$$= \vec{r} \times m\vec{\upsilon} \qquad ...... (1)$$



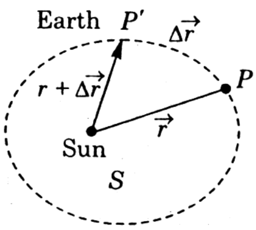
The direction of L is along Z-axis. Under a central force, L is constant and it is always along Z-axis. The vectors r and υ lie in X-Y plane.

Consider the motion of earth around the sun. The earth moves under the gravitational force extended by sun. As per law of conservation of angular momentum, the angular momentum L of the earth with respect to sun is constant, r and υ are always perpendicular to L, orbit of the earth lies in a plane perpendicular to L. This shows that the motion under a central force takes place in a fixed plane.

Areal Velocity Under Central Force

Consider the motion of the earth around the sun. Let r be the radius vector of earth with respect to sun. In a short interval of time Δt, the earth moves from P to P'.

Let ΔA is the area swept out by radius vector in a time interval Δt.



$$\Delta A = \text{area of SPP'} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$\Delta A = \frac{1}{2}\vec{r} \times \Delta\vec{r}$$

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\vec{r} \times \frac{\Delta\vec{r}}{\Delta t}$$

$$= \frac{1}{2}\vec{r} \times \frac{\overrightarrow{dr}}{dt}$$

$$= \frac{1}{2}\vec{r} \times \vec{\upsilon}$$

$$= \frac{1}{2m}\vec{r} \times m\upsilon$$

$$= \frac{L}{2m}$$

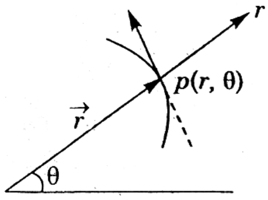
Under central force, angular momentum in constant.

$$\frac{dA}{dt} = \text{constant dt}$$

$\frac{dA}{dt}$ is called as areal velocity. The areal velocity under central force is constant. The radius vector sweeps out equal areas in equal intervals of time.

Radial and Centripetal Acceleration in Polar Coordinates

Consider a particle at the point P. Let r, θ be the polar co-ordinates of the particle at P. Let $\hat{r}$, $\hat{\theta}$ be the unit vectors along and perpendicular to the radius vector r. The trajectory of a point is a plane curve through the point.



Let r = r $\hat{r}$

The velocity of a particle is written as

$$V = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \qquad ...(1)$$

$$= \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} \qquad \left[\because \frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{\theta}\right]$$

$\frac{dr}{dt}\hat{r}$ is the radial component of the velocity of the particle. This occurs due to change of r when θ is constant. $r\frac{d\theta}{dt}\hat{\theta}$ is the transverse component of velocity of the particle. This occurs due to change in θ when r is constant.

The acceleration of the particle is

$$\alpha = \frac{d^2r}{dt^2} = \frac{d}{dt}\left[\frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}\right]$$

$$= \left(\frac{d^2r}{dt^2}.\hat{r} + \frac{dr}{dt}.\frac{d\hat{r}}{dt}\right) + \left(\frac{dr}{dt}.\frac{d\theta}{dt}.\hat{\theta} + \frac{rd^2\theta}{dt^2}\hat{\theta} + r\frac{d\theta}{dt}.\frac{d\hat{\theta}}{dt}\right)$$

$$= \left(\frac{d^2r}{dt^2}.\hat{r} + \frac{dr}{dt}.\frac{d\theta}{dt}\hat{\theta}\right) + \left(\frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} + \frac{r\,d^2\theta}{dt^2}\hat{\theta} - r\frac{d\theta}{dt}\frac{d\theta}{dt}\hat{r}\right) \qquad \left[\because \frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt}.\hat{r}\right]$$

$$= \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\hat{r} + \left\{r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}.\frac{d\theta}{dt}\right\}\hat{\theta} \qquad ...(2)$$

$$= a_r\hat{r} + a_t\hat{\theta}$$

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \text{Radial component of acceleration}$$

$$a_t = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}.\frac{d\theta}{dt}$$

$$a_t = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

Thus the radial and centripetal acceleration of a particle in polar coordinates is discussed.

Q5. Deduce Equation of motion of a particle under the action of a central force. Express the Equation in terms of total Energy.

Ans :

The acceleration of a particle under central force in polar coordinates is written as

$$\alpha = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\theta} \quad .... (1)$$

= radial component + transverse component

The central force is a radial force. Therefore the radial acceleration is present. The transverse acceleration is zero.

$$\therefore \quad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \pm\frac{F(r)}{m} \quad .... (2)$$

and
$$r\left(\frac{d^2\theta}{dt^2}\right) + 2\left(\frac{dr}{dt}\right).\left(\frac{d\theta}{dt}\right) = 0$$

i.e.,
$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \quad .... (3)$$

Where F(r) is the magnitude of the radial force, m is the mass of the particle.

Let $r^2\frac{d\theta}{dt} = h$ (Constant)

Let $r = \frac{1}{u}$

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{dt}.\frac{d\theta}{dt}$$

$$\frac{dr}{dt} = -r^2\left(\frac{d\theta}{dt}\right).\left(\frac{du}{d\theta}\right)$$

$$= -h\left(\frac{du}{d\theta}\right) \qquad \left[\because h = r^2\left(\frac{d\theta}{dt}\right)\right]$$

Differentiating once again one can obtain

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$$

$$= -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}.\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}.\frac{h}{r^2} \qquad \left[\because \frac{d\theta}{dt} = \frac{h}{r^2}\right]$$

$$= -h^2u^2\frac{d^2u}{d\theta^2} \qquad \text{... (5)}$$

Putting this value in equation (2), one can obtain

$$-h^2u^2\frac{d^2u}{d\theta^2} - r\left(\frac{h}{r^2}\right)^2 = \pm\frac{F(r)}{m} \qquad \left[\because \left(\frac{d\theta}{dt}\right) = \frac{h}{r^2}\right]$$

$$\frac{d^2u}{d\theta^2} + u = \pm\frac{F\left(\frac{1}{u}\right)}{mh^2u^2} \qquad \text{.... (6)}$$

This is the general equation of motion under central force.

Q6. Find the central force due to potential energy function u = – kr².

Ans :

Given that,

Potential energy function, $u = -kr^2$

The central force due to potential energy function is given as,

$$F = -\frac{du}{dr}$$

Substituting the corresponding values in above equation,

$$F = \frac{-d}{dr}(kr^2)$$

$$= -2krN$$

∴ Central force, F = – 2kr N.

3.5 Gravitational Potential and Gravitational Field

Q7. Define gravitational field and gravitational potential. Obtain Expression for gravitational potential due to a point mass.

Ans : **(June-19, Dec.-16)**

The region around a body where the gravitational force of attraction is present is called the gravitational field.

Gravitational attraction or the intensity of the gravitational field at a point in the field is the force experienced by a unit is placed at that point.

The intensity of the gravitational field is

$$F = G\frac{M \times 1}{r^2}$$

$$F = \frac{GM}{r^2}$$

Gravitational Potential

It is defined as the work done to bring unit mass from infinite distance to a distance r from mass M.

Force acting on a unit mass is $F = -\frac{GM}{r^2}$

If the amount of work done to displace a body to a distance dr in gravitational field is $d\omega$, then

$$d\omega = F(r)dr$$

$$= -\frac{Gm}{r^2}dr$$

$$\therefore \quad V(r) = -\int_{\infty}^{r} d\omega$$

$$= \int_{\infty}^{r} -\frac{Gm}{r^2}dr$$

$$V(r) = -\frac{Gm}{r}$$

$\therefore$ The potential energy of a body of mass m is

$$U(r) = -\frac{GMm}{r}$$

Where G = Gravitational constant

M = mass of earth

m = mass of a body

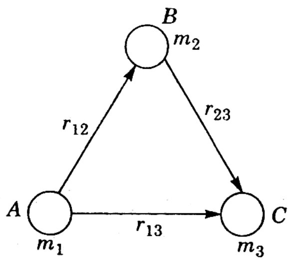
r = Distance between M and m

Gravitational Energy of Many Particle System

If there are more then two bodies, the gravitational potential energy is calculated using the principle of super position. Consider a system that has three masses m_1, m_2 and m_3.

The potential energy of the system with masses m_1 and m_2 is

$$U_{12} = \frac{-Gm_1m_2}{r_{12}} \qquad \text{... (1)}$$



The potential energy of the system with masses m_2 and m_3 is

$$U_{23} = \frac{-Gm_2m_3}{r_{23}}$$

The potential energy of the system with masses m_1 and m_3 is

$$U_{13} = \frac{-Gm_1m_3}{r_{13}} \qquad \text{... (3)}$$

The potential energy of the three masses is

$$U = U_{12} + U_{13} + U_{23}$$

$$= -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right) \qquad \text{... (4)}$$

This is also called as the self energy of the system.

3.6 Motion Under Inverse Square Law

Q8. Show that the inverse square law of gravitation leads to kepler's law?

Ans :

The equation of motion of a particle moves under the influence of central field force is

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(r)}{h^2u^2} \qquad \text{... (1)}$$

f(r) - force per unit mass.

The Newtons' gravitational force is

$$F = \frac{-GMm}{r^2}$$

The negative sign indicates that the force is attractive in nature.

M = mass of sun, m = mass of planet

Force acting on planet per unit mass

$$f(r) = \frac{-Gm}{r^2} \quad ...(3)$$

Putting this value in equation (1), one can obtain

$$\frac{d^2u}{d\theta^2} + u = +\frac{Gm}{h^2r^2u^2}$$

$$u = \frac{1}{r} \Rightarrow u^2r^2 = 1$$

$$\frac{d^2u}{d\theta^2} + u = \frac{Gm}{h^2}$$

But GM = μ

$$\therefore \quad \frac{d^2u}{d\theta^2} + u = \frac{u}{h^2}$$

$$\frac{d^2}{d\theta^2}\left(u - \frac{\mu}{h^2}\right) + \left(u - \frac{\mu}{h^2}\right) = 0 \quad .... (4)$$

This is equation of motion of planet under inverse square law.

3.7 Derivation of Kepler's Laws

Q9. State keplers laws of planetary motion.

Ans : **(Dec.-17)**

Kepler proposed three laws for planetary motion.

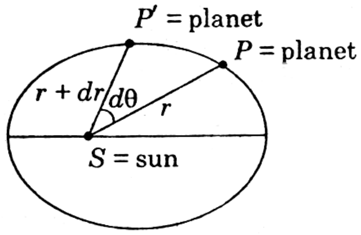
They are as follows:

1. First Law

Each planet revolves around the sun in an elliptical orbit around the sun. The sun is at the one of the foci of the ellipse [Fig. 7].

This law is called as the law of elliptical orbits.

This law gives us the shape of the orbit of a planet around the sun.



2. Second Law

The radius vector of any planet relative to the sun sweeps out equal areas in equal times. The areal velocity of the radius vector is constant.

This law is called as the law of areas. This law gives the relationship between the speed of the planet and its distance from the sun.

3. Third law

The square of the period of revolution of any planet around the sun is directly proportional to the cube of the length of semi-major axis of the elliptical orbit. This is called as harmonic law.

$$T^2 \propto a^3$$

T = period of revolution of planet around the sun

a = length of semi-major axis of the elliptical orbit.

Q10. Derive Keplers laws.

(OR)

State and obtain kepler's law motion planetary.

Ans : **(June-17, Dec.-16)**

Consider a planet of mass m. This moves under the gravitational field of the sun. As per Newtons' law of gravitation, the attractive force between the planet and the sun is

$$\vec{F} = \frac{-GMm}{r^2}\hat{r}$$

Where M = mass of the sun

m = mass of planet

r = distance between planet and sun

$\hat{r}$ = unit vector along the radius

The gravitational force is a central force. Therefore the angular momentum is conserved. The motion of the planet takes place in a fixed plane. The areal velocity of its radius vector is constant.

The areal velocity is

$$\frac{dA}{dt} = \frac{L}{2m}$$

$$\frac{h}{2} = \frac{1}{2m}\left(mr^2\frac{d\theta}{dt}\right)$$

$$h = r^2\frac{d\theta}{dt} \qquad \text{... (2)}$$

The radial force on the planet is

$$F = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] \qquad \text{... (3)}$$

Using equations (1) and (3), one can has

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = \frac{-GMm}{r^2}$$

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{-Gm}{r^2}$$

Using equation (2), one can has

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = \frac{-GM}{r^2}$$

Put $r = \frac{1}{u}$ ... (4)

$$\therefore \quad \frac{d^2r}{dt^2} - h^2u^3 = -GMu^2 \qquad \text{... (5)}$$

Differentiating equation (4), one can obtain

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}.\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\left(\frac{du}{d\theta}\right)(hu^2) \qquad \left[\because \frac{d\upsilon}{dt} = hu^2\right]$$

$$= -h\frac{du}{d\upsilon}$$

Differentiating once again,

$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}.\frac{d\theta}{dt}$$

$$= -hu^2\frac{d^2u}{d\theta^2} \qquad \left[\because \frac{d\theta}{dt} = hu^2\right]$$

Putting this value in equation (5), one can obtain

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -GMu^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

$$\frac{d^2u}{d\theta^2} + \left(u - \frac{GM}{h^2}\right) = 0 \qquad \text{... (6)}$$

$$\frac{d^2}{d\theta^2}\left(u - \frac{GM}{h^2}\right) + \left(u - \frac{GM}{h^2}\right) = 0 \qquad \text{... (7)}$$

The solution of this equation is

$$u - \frac{GM}{h^2} = -C\cos\theta$$

$$u = \frac{GM}{h^2} - C\cos\theta$$

$$= \frac{GM - Ch^2\cos\theta}{h^2}$$

$$h^2u = GM - h^2C\cos\theta$$

$$h^2u = GM\left[1 - \frac{h^2C}{GM}\cos\theta\right]$$

$$\frac{h^2/GM}{r} = 1 - \frac{h^2C}{GM}\cos\theta \qquad \text{... (8)}$$

This equation is of the form

$$\frac{l}{r} = 1 - e\cos\theta \qquad ...(9)$$

This equation shows a conic section of eccentricity

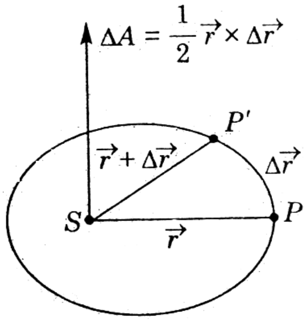
$e = \frac{h^2C}{GM}$ and semi-latus rectum $l = \frac{h^2}{GM}$.

The total energy of the planet is negative as the orbit of the planet around the sun is closed. The total energy is negative when e < 1. Therefore the orbit of the planet around the sun is ellipse. This is Keplers' first law.

Derivation of Second Law

Consider the planetary motion. Let S is the center of the sun. P is the center of the planet in its orbit. Let $\vec{r}$ is the radius vector of the planet with respect to S. The planet moves from P to P' in a small interval of time.

The vector area $\Delta\vec{A}$ swept by the radius vector in time interval Δt is



$$\Delta\vec{A} = \frac{1}{2}\vec{r} \times \Delta\vec{r} \qquad ... (1)$$

Dividing both sides by Δt, one can obtain

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\vec{r} \times \frac{\Delta\vec{r}}{\Delta t}$$

$$\frac{dA}{dt} = \frac{1}{2}\vec{r} \times \frac{d\vec{r}}{dt}$$

$$= \frac{1}{2}\vec{r} \times \vec{\upsilon}$$

$$= \frac{1}{2m}\vec{r} \times m\vec{\upsilon}$$

$$\frac{d\vec{A}}{dt} = \frac{1}{2m}\vec{L}$$

$\vec{L}$ is constant for a central force

$$\frac{d\vec{A}}{dt} = \text{constant}$$

Thus the areal velocity is constant. This proves the Keplers' second law of motion.

Third Law

Consider the semi- latus rectum l of the elliptical orbit. a and b are the semi- major and semi-minor axes of the ellipse, respectively.

$$l = \frac{b^2}{a} = \frac{h^2}{GM}$$

If t is the period of the planet around the sun,

$$T = \frac{\text{area of ellipse}}{\text{areal velocity}} = \frac{\pi ab}{h/2}$$

$$T^2 = \frac{4\pi^2a^2b^2}{h^2}$$

$$= \frac{4\pi^2a^2b^2}{GMb^2/a}$$

$$= \frac{4\pi^2a^3}{GM}$$

$$T^2 \propto a^3$$

Thus the square of the time period of revolution of planet around the sun is directly proportional to the length of the semi major axis of the elliptical orbit.

3.8 Special Theory of Relativity

Q11. Define postulates of special theory of relativity.

Ans : **(June-19)**

Einstein published the special theory of relativity in 1905. This theory was based on the postulates.

Postulate 1

The laws of physics have the same form in all inertial frames of references moving with a constant velocity relative to one another.

This is called as principle of relativity. This postulate defines the absence of universal frame of reference.

As per this postulate, it is impossible by any means to demonstrate absolute motion. The absolute motion is meaningless. The motion of bodies relative to one another has physical meaning. There is no absolute motion according to Einstein. Undetection of absolute motion implies undetection of ether. If the laws of physics were different for observers in different frames in relative motion, it could be determined from these differences which objects are stationary in space and which are moving. There is no universal frame of reference, this distinction between objects cannot be made.

Postulate 2

The speed of light in free space is the same in all inertial frames of references. This is called as the principle of the constancy of the speed of light.

This postulate follows directly from the result of Michelson-Morley experiment. As per this postulate, the speed of light is same in all directions. It is the greatest velocity.

3.9 Galilean Relativity

Q12. Explain the common terms in theory of relativity.

Ans :

While sitting in a moving vehicle, if you look at the distant objects like trees or anything stationary, they appear to move in a direction opposite to the direction of the motion of the vehicle. It is easy to realize that for a person standing on the ground, you will appear to be moving in a particular direction, while to you, the standing person would appear to move in a direction opposite to your direction of motion. In other words, what one observes is relative, it is not absolute and depends on the state of motion of the observer.

We take another example, a boy sitting in a moving vehicle throws a ball upward and the ball returns to his hand after sometime and it is being watched by a person standing outside. Now, imagine what is observed by the boy and the person standing outside. For the boy, the ball has moved up straight and come down straight into his hand, as if he had been stationary. What does the person standing outside observes ? He observes, that after the boy has thrown the ball and because the boy is continuously moving, the ball takes a parabolic path before finally coming to his hand. These two examples make it clear that what an observer observes, depends on his state or his frame of reference, as discussed below.

Newton. Galileo, Lorentz, Michelson, Morley, Einstein and others have made significant contributions in developing the subject of relativity, the understanding of which is necessary for knowing the mysteries of physics. We are going to see that length, mass, time, etc. are not absolute and their values depend on the state of the observer.

Some of the common terms used in relativity are defined below :

(i) **Particle :** A particle is a small piece of matter, having practically no linear dimension, but only a position at a point. It is characterized by its mass and charge.

(ii) **Observer :** A person who locates, records, measures and interprets an event is called an observer.

(iii) **Event :** In relativity, an event implies anything that occurs suddenly or instantaneously at a point in space. It involves a position and a time of occurrence.

Frame of Reference

Even the basic physical quantities like displacement, velocity, time, mass, etc. are not absolute and the measured values are relative, depending upon the reference and the state of observer. To locate the coordinates of a point, we assign a specific x-, y- and Z-values of a coordinate system having its origin at O with $x = 0$, $y = 0$ and $z = 0$. If the origin of the coordinate system is changed, the coordinates of the point also change. We can state that the reference of the point has changed and the coordinate axes form a reference system.

A system of coordinate axes which defines the position of a particle or specifies the location of an event is called a frame of reference. The simplest frame of reference is the Cartesian system of coordinates in which the location of a point is specified by the three (x-, y- and z-) coordinates.

However, for complete specification of an event in a reference frame, i.e., for determination of its exact location as well as the exact time of its occurrence, in addition to the three space coordinates, another coordinate, time / of its occurrence should be specified. A frame of reference having four coordinates, x, y, z and t is referred to as a space-time frame the four axes defining a four-dimensional contirum is called space-time.

Q13. Describe Galilean Transformation.

Ans : **(June-17)**

Galilean or Classical Transformation

A point or particle at any instant has different coordinates in different reference systems. The equations which provide the relationship between the coordinates of two reference systems are called transformation equations.

The transformation of coordinates of a particle from one inertial frame to another is known as Galilean (or classical) transformation.

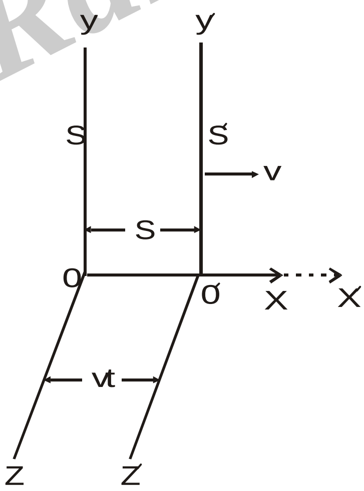


Fig.: Reference frame S' moves with velocity v (in the x direction) relative to reference frame S.

To detect the position of a particle at a certain time, we should represent it in both space as well as time. such a thing is called an event. The event may be conveniently represented by (x, y, z, t).

Consider a fixed frame of reference S with axes OX, OY and OZ. Let an event P be represented by (x, y, z, t). The same event is seen by an observer in the moving frame of reference S' with axes O'X'. O'Y' and O'Z'. The frame of reference S' is moving with a relative velocity ν. Let the origin O' at time t = t' = 0 be at O. Now, the observer in the moving frame S' sees the same event at time t'. Hence the coordinates be represented by (x', y', z', t'). It is now required to obtain the relation between x, y, z, t and x', y', z', t'. It is clear from fig. above that the measurements in S' with those in the frame S by the equations.

$$x' = x - vt \quad y' = y \quad z' = z \text{ and } t' = t \qquad ...(1)$$

The same can be represented in the frame S with those in frame S' by considering that S is moving a velocity –v with respect to S'. Thus

$$x = x' + vt \quad y = y' \quad z = z' \quad \text{and } t = t' \qquad ...(2)$$

These sets of equations are Galilean (or classical) transformations.

To convert velocity components measured in the frame S to their equivalents in the frame S'. we differentiate these equations and putting t' = t and dt' = dt, we get

$$v_x' = \frac{dx'}{dt'} = \frac{dx}{dt} - v = v_x - v$$

$$v'_y = \frac{dy'}{dt'} = \frac{dy}{dt} = v_y$$

$$v'_z = \frac{dz'}{dt'} = \frac{dz}{dt} = v_z$$

To convert acceleration components, we differentiate again

$$f_x' = \frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2} = f_x$$

$$f_y' = \frac{d^2y'}{dt'^2} = \frac{d^2y}{dt^2} = f_y$$

$$f_z' = \frac{d^2z'}{dt'^2} = \frac{d^2z}{dt^2} = f_z$$

Thus it is clear that acceleration of a body is the same in both the frames of reference. Hence Newton's second law of motion (F = mf) is equally valid in both the frames. So Newton's laws of motion are invarient under the Galilean (or classical) transformation.

3.10 Absolute Frame of Reference

Q14. Discuss about absolute frame of reference.

Ans : **(June-17)**

Absolute Frames of Reference

We know that the laws of mechanics are invariant under Galilean transformation and the laws of electro magnetism or electro dynamics and Maxwell's equations are not invariant under Galilean transformations. Hence, the velocity of light will have different values for different observers moving with different uniform velocities. From this, scientists thought that there exists a preferred inertial frame known as the absolute frame in which the velocity of light will be exactly equal to the value derived from

Maxwell's equations $c = \frac{1}{\sqrt{\mu_0 \in_0}} = 3 \times 10^8$

m/s and will be the same in all directions as the frame is at absolute rest.

Relative to such an absolute frame of reference, the velocity of light as determined in any inertial frame moving with a velocity v relative to the absolute frame will be c+v or c – v depending on the direction of motion. Scientists called such on absolute frame as ether, and tried to measure the change in velocity of light as measured from earth. Michelson and Morley experiment is aimed at this purpose. But it failed to observe any such change and the velocity of light is found to be independent of the state of motion of the observer.

To explain the null result of Michelson Morley experiment Einstein proposed his famous postulates of special theory of relativity that

1) The laws of physics, all laws, are the same in all inertial frames and

2) The speed of light in vacuum 'c' has the same value in all inertial frames.

Now, as Galilean transformations are insufficient to predict the invariance of all physical laws and the invariance of velocity of light, Lorentz transformations are to be used. When the laws of physics are expressed in Four Vector Formalism, they will be invariant.

The final conclusion is that there are no absolute frames of reference and all motion is relative.

3.11 Michelson - Morley Experiment

Q15. Describe Michelson Morely experiment. What is its significance?

Ans : **(June-19, Dec.-18, June-18, Dec.-16)**

In 1887, Michelson and Morley performed an experiment in order to determine the velocity of earth w.r.t. ether medium as the inertial frame of reference by using an interferometer. However, the result of the experiment ruled out the existence of this hypothetical medium.

Experimental arrangement

The experiments] arrangement is show fig. (3). Light from a monochromatic extended source S after being rendered parallel by

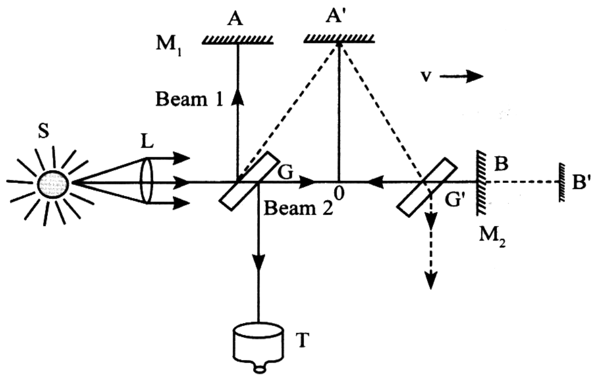


Fig.: Set-up for Michelson-Morley Experiment

a collimating lens L, falls on the semisilvered glass plate G inclined at an angle 45° to the beam. It is divided into two parts, one being reflected from the semisilvered surface G giving rise to ray 1 which travels towards mirror and the other being transmitted giving rise to ray 2 which travels towards mirror M_2. The two rays fall normally on mirrors M_1 and M_2 respectively and are reflected back along their original paths. The reflected rays again meet at the semisilvered surface of glass plate G and enter telescope where interference pattern is obtained. The optical distances at the mirror M_1 and M_2 from G are made equal with the help of a compensating plate not shown is the figure.

If the apparatus is at rest in ether, the two reflected rays would take equal time to return the glass plate G. But actually the whole apparatus is moving along with the earth. Let us suppose that the direction of motion of earth is in the direction of the initial beam. Due to the motion of the earth, the optical paths traversed by both the ray are not the same The reflections at mirrors. M_1 and M_2 do not take place at A and B but at A' and B' respectively as show n in fig. (3). Thus the times taken by the two rays to travel to mirrors and back to G will be different in this case.

Theory

Let the two mirrors M_1 and M_2 he art an equal distance / from the glass plate G. Further let c and v he the velocities of light and apparatus or earth respectively. It is obvious from fig. (3) than the reflected ray I from glass plate G strikes the mirror M_1 at A' and not at A due to the motion of the earth. The total path of the ray from G to A and back will he GA'G'. From ΔGA'D

or $$(GA')^2 = (AA')^2 + (A'D)^2 \qquad (\therefore GD = AA') \qquad ...(1)$$

If t be the time taken by the ray to move from G to A. Then from eq.(1), we have

$$(c\,t)^2 = (\upsilon\,t)^2 + (l)^2$$

or $$t^2(c^2 - \upsilon^2) = l^2 \quad \text{or} \quad t = \frac{l}{\sqrt{(c^2-\upsilon^2)}}$$

If t_1 be the time taken by the ray to travel the whole parth GAG'. then

$$t_1 = 2t = \frac{2l}{\sqrt{c^2-\upsilon^2)}} = \frac{2l}{c\left(1-\frac{\upsilon^2}{c^2}\right)^{1/2}}$$

$$= \frac{2l}{c}\left(1-\frac{\upsilon^2}{c^2}\right)^{-1/2} = \frac{2l}{c}\left[1+\frac{\upsilon^2}{2c^2}\right] \qquad ...(2)$$

Now consider the case of the transmitted ray 2 which is moving longitudinally towards minor M_2. It has a velocity $(c - \upsilon)$ relative to the apparatus when it is moving from G to B. During its return journey its velocity relative to apparatus is $(c + \upsilon)$. If t_2 be the total time taken by the longitudinal ray to reach G', then

$$t_2 = \frac{l}{(c-\upsilon)} + \frac{l}{(c+\upsilon)} \qquad (\because GB = G'B' = l)$$

$$\therefore \quad t_2 = \frac{l(c+\upsilon)+l(c-\upsilon)}{(c^2-\upsilon^2)} = \frac{2\,l\,c}{(c^2-\upsilon^2)}$$

$$= \frac{2\,l\,/\,c}{(c^2 - \upsilon^2\,/\,c^2)} = \frac{2\,l}{c}\left(1 - \frac{\upsilon^2}{c^2}\right)^{-1}$$

$$= \frac{2l}{c}\left[1 + \frac{\upsilon^2}{c^2}\right] \qquad ...(3)$$

Thus, difference in times of travel of longitudinal and transverse journeys is

$$\Delta t = t_2 - t_1$$

$$= \frac{2l}{c}\left(1 + \frac{\upsilon^2}{c^2}\right) - \frac{2l}{c}\,\frac{2l}{c}\left(1 + \frac{1}{2}\frac{\upsilon^2}{c^2}\right)$$

$$= \frac{2l}{c}\frac{\upsilon^2}{c^2} = \frac{l\upsilon^2}{c^3} \qquad ...(4)$$

∴ Optical path difference between two rays is given by

Optical path difference = velocity × Δt = c × Δt

$$= c \times \frac{l\upsilon^2}{c^3} = \frac{l\,\upsilon^2}{c^2}$$

If λ is the wavelength of light used, then path difference terms of wavelength = $\frac{l\,\upsilon^2}{c^2\lambda}$.

Michelson and Morley performed the experiment in two steps i.e., firstly by the setting shown in fig. and secondly by turning the apparatus through 90°. When the apparatus was turned through 90°, the positions of two mirrors are changed. Now the path difference is in opposite directions i.e., the path difference is $-l\,\upsilon^2/\lambda\,c^2$ wavelength. The resultant path difference now becomes $(l\,\upsilon^2/\lambda\,c^2) - (-l\,\upsilon^2/\lambda\,c^2) = 2l\,\upsilon^2/\lambda\,c^2 = 2\,l\,\upsilon^2/\lambda\,c^2)$ wavelength. We know that a change optical path difference by λ corresponds to a shift of one fringe and hence the path difference $(2\,l\,\upsilon^2/\lambda\,c^2)$ corresponds to a fringe shift $(2\,l\,\upsilon^2/\lambda\,c^2)$. Following were following data of Michelson and Morley experiment :

$l = 1.0 \times 10^3$ cm, $\lambda = 5.0 \times 10^{-5}$ cm,

$v = 3 \times 10^6$ cm/sec, and $c = 3 \times 10^{10}$ cm/sec

Change in fringe shift $n = \frac{2\,l\,\upsilon^2}{\lambda c^2}$

or $\quad n = \frac{2 \times 1.0 \times 10^3 \times (3 \times 10^6)}{5.0 \times 10^{-5}(3 \times 10^{10})^2} = 0.4$ fringe

Thus a shift of less than half a fringe was only expected. Michelson and Morley could observe a shift of about 0.01 of fringe. Of course, this shift is within the limits (or) the error of observations. They repeated the experiment at different points on the earth's surface and at different seasons of the year but they could not detect any measurable shift. So it was a null or negative result.

The negative result suggests that it is impossible to measure the speed of the earth relative to ether or the concept of a fixed frame of reference (like ether filling all space) cannot be checked by experiment. In this way the null result of experiment lead to the total rejection of ether hypothesis. This suggests that the speed of light in vacuum is the same in all frames of reference which are in uniform relative motion.

1. Ether drag: In order to explain the null result, it was argued that when the earth earth moves through ether, it drags ether along with it. As such, the velocity of the earth w.r.t. ether i.e., $\upsilon = O$.

In the equation (vii), setting $\upsilon = O$, we have

$$n = O$$

i.e., no fringe shift should be expected.

2. Loreutz-Fitzgerald Contraction: Lorentz and Fitzgerald suggested that a moving body contracts by a factor $\sqrt{1-\upsilon^2/c^2}$ along its direction of motion. However, no such contraction occurs in a direction perpendicular to the direction of motion.

According to contraction hypothesis, the distance for the transmitted part of the beam i.e., the distance of mirror M_2 from the glass plate contracts from to d to $d\sqrt{1-\upsilon^2/c^2}$. Using equation (ii), replacing d by $d\sqrt{1-\upsilon^2/c^2}$

$$t_2 = \frac{2d\sqrt{1-\upsilon^2/c^2}}{c}\left(1+\frac{\upsilon^2}{c^2}\right)$$

$$= \frac{2d}{c}\left(1-\frac{\upsilon^2}{c^2}\right)^{1/2}\left(1+\frac{\upsilon^2}{c^2}\right) = \frac{2d}{c}\left(1-\frac{\upsilon^2}{2c^2}\right)\left(1+\frac{\upsilon^2}{c^2}\right)$$

Neglecting higher power of υ^2 / c^2, we have

$$t^2 = \frac{2d}{c}\left(1+\frac{\upsilon^2}{2c^2}\right) = t_1$$

From above discussion contraction hypothesis allows shortening of the path of light parallel to the earth's motion just enough to equalise the transit times for the two paths and hence no fringe shift is observed.

Inferences from Michelson - Morley experiment

The null result of the Michelson - Morley experiment has some conclusions .

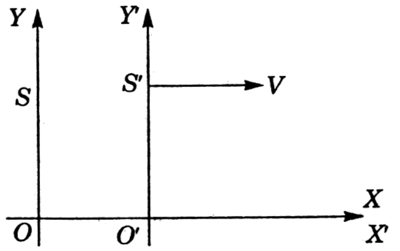
1. The negative result of the experiment implied that the motion of the earth through ether was undetectable. Hence, the concept of ether medium as a preferred inertial frame of reference must be discarded.
2. The null result indicated that the measured speed of light is same in all directions. It is not affected by the motion of the earth through the space. It is called the principle of consistency of the speed of light and is one of the two fundamental postulates of Einstein's special theory of relativity.

3.12 Lorentz Transformations

Q16. Explain and write Lorentz Transformations.

Ans : **(Dec.-16)**

Consider a reference frame S which is at Y U. rest. Consider another reference frame S' which is moving with a constant velocity V. The S position vector of an event in S' frame



$$r' = x - \upsilon t \quad ....(1)$$

The component from of equation (1) is written as

$$x' = x - \upsilon t \quad ....(2)$$

$$y' = y \quad ....(3)$$

$$z' = z \quad ....(4)$$

and $t' = t$ ....(5)

The equations (2) to (5) are called Galilean transformations. The relation between velocity components in S and S' frames is written as

$$\upsilon'_x = \upsilon_x - \upsilon \quad ....(6)$$

$$\upsilon'_y = \upsilon_y \quad ....(7)$$

$$\upsilon'_z = \upsilon_z \quad ....(8)$$

The eqs. (6), (7) and (8) disobey the postulates of special theory of relativity.

The relation between x and x' is written as

$$x' = k(x - \upsilon t) \quad ....(9)$$

where x is a constant of proportionality.

The inverse equation for equation (9) is written as

$$x = k'(x' + \upsilon t') \quad ....(10)$$

The other equations are written as

$$y' = y \quad ....(11)$$

$$z' = z \quad ....(12)$$

Times t and t' are not equal.

Putting the value of x' from equation (9) in equation (10), we have

$$x = kk'(x - \upsilon t) + k'\upsilon t'$$

$$t' = kt + \left(\frac{1 - kk'}{k'\upsilon}\right)x \quad ....(13)$$

For calculating k and k', the second postulate of special theory of relativity is used.

Consider a single of light is given out from the common origins of S and S' at t = t' = 0. The signal propagates in the two systems according to the equations

$$x = ct \quad ....(14)$$

$$x' = ct' \quad ....(15)$$

in system S and S' respectively.

Substituting the values of x' and t' into equation (15), one can has

$$k(x - \upsilon t) = ckt + \left(\frac{1 - kk'}{k'\upsilon}\right)cx$$

$$x = ct\left[\frac{1 + \frac{\upsilon}{c}}{1 - \left\{\frac{1}{kk'} - 1\right\}\frac{c}{\upsilon}}\right] \quad ....(16)$$

Comparing equation (16) with equation (14), we have

$$\frac{1 + \frac{\upsilon}{c}}{1 - \left\{\frac{1}{kk'} - 1\right\}\frac{c}{\upsilon}} = 1 \quad ....(17)$$

$$\sqrt{kk'} = \frac{1}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

We choose $k = k' = \dfrac{1}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$ ....(18)

Substituting these values in equations (9) and (13), we have

$$\left.\begin{aligned} x' &= \frac{x-\upsilon t}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}} \\ y' &= y \\ z' &= z \text{ and} \\ t' &= \frac{t-\dfrac{\upsilon x}{c^2}}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}} \end{aligned}\right\} \quad ....(19)$$

Equations (19) are called Lorentz transformations.

The inverse Lorentz transformation equations are written as

$$\left.\begin{aligned} x &= \frac{x'+\upsilon t'}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}} \\ y &= y' \\ z &= z' \\ t &= \frac{t'+\dfrac{\upsilon x'}{c^2}}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}} \end{aligned}\right\} \quad ....(20)$$

The measurements of position and time are found to depend upon the frame of reference of the observer. The Lorentz equations reduce to the Galilean transformations when the relative velocity is very small in comparison with the velocity of light.

3.13 Time Dilation

Q17. Explain the concept of Time Dilation.

Ans : **(Dec.-17)**

Time intervals are affected by relative motion. A clock moving with velocity υ with respect to an observer appears him to have slowed down by a factor $\sqrt{1-\dfrac{\upsilon^2}{c^2}}$, than when at rest with respect to him.

Suppose a clock is placed at the point x' in the moving frame S'. An observer in S' finds that the clock gives two ticks at times t'_1 and t'_2. The time interval between the ticks as judged from S' is

$$t_0 = t'_2 - t'_1$$

t_0 is the interval as measured in a frame in which the clock is at rest. Another observer measures the time interval between the same two ticks from a stationary frame of reference S, relative to which the clock is moving with velocity υ. If he records the ticks at times t_1 and t_2, the time interval appears to him as

$$t = t_2 - t_1$$

Using Lorentz transformations, we have

$$t_2 = \frac{t'_2 + \left(\dfrac{x'\upsilon}{c^2}\right)}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$$

and $$t_1 = \frac{t'_1 + \left(\dfrac{x'\upsilon}{c^2}\right)}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$$

$$\therefore \quad t = t_2 - t_1 = \frac{t'_2 - t'_1}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$$

$$\therefore \quad t = \frac{t_0}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$$

From this equation, it is evident that to the stationary observer in S, the time intervals appear to be lengthened by a factor $\dfrac{1}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$. A moving clock appears to be slowed down to a stationary observer. This concept is called time dilation.

If $\upsilon = c$, then $t = \infty$. It means that a clock moving with the speed of light appears to be completely stopped to a stationary observer.

Special Cases:

1. When υ is very small compared to c, $\frac{\upsilon^2}{c^2}$ is neglected

   $\therefore t = t_0$

2. If υ is comparable to c, $\sqrt{1-\frac{\upsilon^2}{c^2}}$ is less than unity. In this case $t > t_0$.

3.14 Length Contraction

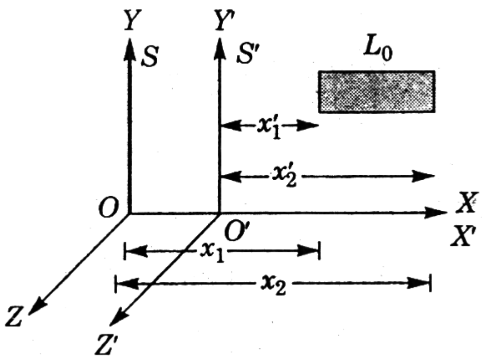
Q18. What is length contraction? Obtain expression for length contraction.

Ans : **(June-19, (Dec.-17)**

Consider two reference frames S and S' respectively.

S' is moving with a velocity υ with respect to S. An observer in frame S' is at rest with respect to S' and hence with respect to the rod. The rod is at rest with respect to this observer.

The length L_0 of the rod is written as



Where x'_2 and xj are the coordinates of the rod in S'. Let L frame is the length of the rod in S frame of reference.

Using Lorentz transformation equations, we have

$$x'_1 = \frac{x_1 - \upsilon t_1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \quad .....(2)$$

$$x'_2 = \frac{x_2 - \upsilon t_2}{\sqrt{1-\frac{\upsilon^2}{\upsilon^2}}} \quad ..... (3)$$

$$L_0 = x'_2 - x'_1$$

$$= \frac{(x_2 - \upsilon t_2)}{\sqrt{1-\frac{\upsilon^2}{c^2}}} - \frac{(x_1 - \upsilon t_1)}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

$$L_0 = \frac{x_2 - x_1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \quad [\because t_1 = t_2]$$

$$L_0 = \frac{L}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \quad ..... (4)$$

This equation indicates that to the stationary observer in S, the rod placed in the moving frame S' appears to be contracted by a factor $\sqrt{1-\frac{\upsilon^2}{c^2}}$. This

contraction occurs in the direction of relative motion. If $\upsilon = c$, then $L = 0$. This means that the rod moving with the speed of light will appear as reduced to a point to a stationary observer. This type of contraction is called the Lorentz-Fitzgerald length contraction.

Special Cases:

1. When υ is very small compared c, $\frac{\upsilon^2}{c^2}$ is neglected. Thus $L = L_0$.

2. When υ is comparable to c, $\sqrt{1-\frac{\upsilon^2}{c^2}}$ is less than unity. Thus $L < L_0$. The length of the moving rod appears to be less than the length when it was at rest.

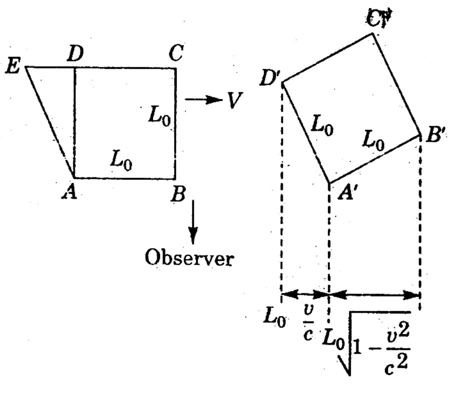
3. When υ is equal to c or greater than c, $\frac{\upsilon^2}{c^2}$ is equal to unity or greater than unity. Thus l = 0 or imaginary. This is impossible and no material body attains the speed of light.

4. The contraction takes place only along the direction of motion and remains unchanged in a perpendicular direction.

5. The contraction is not visualized as it really occurs.

6. The contraction is reciprocal

An interesting puzzle of the length contraction was proposed by Terrel in the year 1959. Consider a cube of side L_0 moves with uniform velocity v with respect to an observer situated at some distance. The direction of motion of the cube is perpendicular to the line of sight of the observer. The length of the cube is shortened to $L = L_0\sqrt{1-\frac{\upsilon^2}{c^2}}$ while the other dimensions were unaffected. The length DE must be equal to $L_0 \frac{\upsilon}{c}$.The observer sees not only the face AB but also AD. AB is perpendicular to the line of sight and AD is parallel to the line of sight. The length of the cube parallel to the direction of motion is shortened to $L = L_0\sqrt{1-\frac{\upsilon^2}{c^2}}$. The cube is appear to be rotated through an angle $\sin^{-1}\left(\frac{\upsilon}{c}\right)$. Thus the cube is subjected to an apparent rotation [Fig. 6]. Similarly a moving sphere appear as a sphere.



3.15 Addition of Velocities

Q19. Obtain the relativistic law for the addition of velocities.

Ans :

Consider two reference frames S and S'. The frame S' moves with a constant velocity v relative to S along the X-axis. Let a body moves a distance dx in a time interval dt in the frame S. The velocity of the body measured by an observer in S frame of reference is

$$u = \frac{dx}{dt} \quad ...(1)$$

The velocity of the body in S' frame of reference is

$$u' = \frac{dx'}{dt'} \quad ...(2)$$

Using Lorentz transformations, we have

$$x' = \frac{x - \upsilon t}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

and $$t' = \frac{t - \left(\frac{x\upsilon}{c^2}\right)}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

Differentiating, we have

$$dx' = \frac{dx - \upsilon dt}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

and $$dt' = \frac{dt - \left(\frac{\upsilon dx}{c^2}\right)}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

$$\therefore \quad u' = \frac{dx'}{dt'} = \frac{dx - \upsilon dt}{dt - \frac{\upsilon dx}{c^2}}$$

$$= \frac{\frac{dx}{dt} - \upsilon}{1 - \frac{\upsilon}{c^2}\frac{dx}{dt}}$$

$$\frac{dx}{dt} = u$$

$$\therefore \quad u' = \frac{u - \upsilon}{1 - \left(\frac{u\upsilon}{c^2}\right)} \qquad ...(3)$$

This equation gives the relativistic addition of velocities u and υ.

Special Cases :

1. If u' and υ are small when compared to c, $\frac{u'\upsilon}{c^2}$ is neglected.

   Thus u = u' + υ. T his is the classical formula.

2. If υ or u' = c then u = c. If one body moves with velocity c with respect to other, their relative velocity is c.

3. When u' = c = υ, then u = c. The addition of velocity of light reproduces the velocity of light.

3.16 Mass-Energy Relation

Q20. Derive Einstein mass-energy relation. Explain the verification of mass-energy relation.

Ans : **(June-19)**

As per Einsteins' mass - energy relation

$$m = \frac{m_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} \qquad ...(1)$$

m_0 = rest mass of the body

c = velocity of light

υ = velocity of body

m = mass of the body

According to Newtons' second law of motion,

$$F = \frac{d}{dt}(m\upsilon) = m\frac{d\upsilon}{dt} + \upsilon\frac{dm}{dt}$$

The change in kinetic energy of the body is equal to the workdone by the force F for a displacement dS of the body.

Thus dk = F. ds

$$= m\frac{d\upsilon}{dt}dS + \upsilon\frac{dm}{dt}dS$$

$$= m\frac{dS}{dt}d\upsilon + \upsilon\frac{ds}{dt}dm$$

$$= m\upsilon d\upsilon + \upsilon^2 dm \qquad ...(2)$$

Differentiating equation (1), we have

$$dm = m_0\left(-\frac{1}{2}\right)\left(1 - \frac{\upsilon^2}{c^2}\right)^{-3/2}\left(\frac{-2\upsilon d\upsilon}{c^2}\right)$$

$$= \frac{m_0}{c^2}\frac{\upsilon d\upsilon}{\left(1 - \frac{\upsilon^2}{c^2}\right)^{3/2}}$$

We know that $m = \frac{m_0}{\left(1 - \frac{\upsilon^2}{c^2}\right)^{1/2}}$

$$\therefore \quad dm = \frac{m\upsilon d\upsilon}{(c^2 - \upsilon^2)}$$

$$m\upsilon d\upsilon = \left(c^2 - \upsilon^2\right)dm$$

Putting this value in equation (2), one can has

$$dk = \left(c^2 - \upsilon^2\right)dm + \upsilon^2 dm = c^2 dm$$

The kinetic energy of a body is written as

$$k = \int dk = \int_{m_0}^{m} c^2 dm = c^2\left(m - m_0\right) \qquad ...(3)$$

m_0c^2 is the rest energy of the body. The total energy of a body is the sum of the kinetic energy and the rest energy.

$\therefore$ E = k + E_0

= $(m - m_0)c^2 + m_0c^2$

E = $mc^2 - m_0c^2 + m_0c^2$

E = mc^2 ...(4)

This equation represents the famous Einsteins' mass - energy relation.

3.17 Concept Four Vector Formalism

Q21. Explain four vector formalism?

Ans : **(Dec.-18)**

The four vector concept has its significance in theory of relativity. As per first postulate of special theory of relativity, the laws of physics are invariant in all inertial reference frames.

If any equation holds good from the point of special theory of relativity, it should be expressed in four vector form. As per Lorentz transformations, the value $x^2 + y^2 + z^2 - c^2 t^2$ must be same in all inertial reference frames. If S and S' are two inertial reference frames, then

$$x^2 + y^2 + z^2 - c^2t^2 = x_1^2 + y_1^2 + z_1^2 - c^2t'^2 \quad ... (1)$$

Let $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$

x_1, x_2, x_3 and x_4 are treated as components of a vector in 4D space. the vector length in 4 D space is written as

$$D^2 = x_1^2 + x_2^2 + x_2^3 + x_4^2$$

$$= \sum_{\mu=1}^{4} x_\mu^2 \quad ...(2)$$

If vector length is invariant in both frames S and S'. Using Lorentz transformation

$$\sum_{\mu=1}^{4} x_\mu^2 = \sum_{\mu=1}^{4} x_\mu'^2$$

$$x' = v(x - \upsilon t)$$

$$y' = y$$

$$z' = z$$

$$t' = v\left[t - \frac{\upsilon x}{c^2}\right]$$

where $v = \dfrac{1}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}} = \left(1-\beta^2\right)^{-1/2}, \quad \beta = \dfrac{\upsilon}{c}$

$$x_1' = v[x_1 - \upsilon t] = v[x_1 + i\beta x_4]$$

$$x_2' = x_2; x_3' = x_3$$

$$x_4' = ict' = v\left[x_4 - \frac{i\upsilon x_1}{c}\right] = v[x_4 - i\beta x_1] \quad ... (3)$$

Putting the above equations in matrix form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} v & 0 & 0 & i\beta v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta v & 0 & 0 & v \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad ....(4)$$

The above matrix form shown the position coordinates in four vector notation. The above matrix is valid for S and S' frames which have relative velocity along X-Matrix elements will be determined in any direction.

PROBLEMS

1. What will be the fringe - slift according to the ether theory in Michelson - Morley experiment. The effective length of each path is 5 m and light has 5000 Å wavelength?

Sol :

The fringe shift in Michelson - Morley experiment is

$$\Delta N = \frac{2l\upsilon^2}{c^2\lambda}$$

where $l = 5$m, $\upsilon = 3 \times 10^4$ m/s,

$c = 3 \times 10^8$ m/s and

$\lambda = 5000$ Å $= 5 \times 10^{-7}$ m

$$\therefore \quad \Delta N = \frac{2 \times 5 \times \left(3 \times 10^4\right)^2}{\left(3 \times 10^8\right)^2 \times 7 \times 10^{-7}}$$

$$= \frac{10 \times 9 \times 10^8}{9 \times 10^{16} \times 7 \times 10^{-7}}$$

$$= \frac{10 \times 9 \times 10^8}{9 \times 7 \times 10^{16-7}}$$

$$= \frac{90 \times 10^8}{63 \times 10^9} = 1.43 \times 10^{-1}$$

$$\Delta N = 0.14$$

2. A rod 0.1m long is moving along its length with a velocity of 0.5c. Calculate the length as it appears to a stationary observer?

Sol :

l_0 = 1m, υ = 0.5 c, l = ?

$$L = L_0\sqrt{1-\frac{\upsilon^2}{c^2}}$$

$$= \sqrt{1-\left(\frac{0.5c)^2}{2}\right)}$$

$$L = \sqrt{\frac{c^2-0.25c^2}{c^2}}$$

$$= r\sqrt{1-0.25} = 0.87 \text{ m}$$

L = 0.87 m

3. A space ship 30 m long passes the earth at a speed of 2.8 × 10⁸ m/s. What will be its apparent length ? (c = 3.0 × 10⁸ m/s)

Sol :

$$L = L_0\sqrt{1-\frac{\upsilon^2}{c^2}}$$

L_0 = rest length of space ship

L = Length of space ship

υ = Velocity of a body or space ship

= 2.8 × 10^8 m/s

c = velocity of light

$$L = 30\sqrt{1-\left(\frac{2.8}{3}\right)^2}$$

$$= 30\sqrt{\frac{9-7.84}{9}}$$

$$= \frac{30}{3}\sqrt{1.16}$$

= 10 × 1.07

L = 10.77 m

4. Calculate the length and the orientation of a meter rod in a frame of reference which is moving with a velocity equal to 0.8 c in a direction making an angle of 45° with the rod?

Sol :

υ = 0.8, θ = 45°, L = ?

The component of the length of the 1 – m rod along the direction of the motion of the frame is l cos 45°. The component of the length of the 1 – m rod along perpendicular direction is l sin 45°.

The apparent length along the moving frame is

$$L_x = l\cos 45^o\sqrt{1-\frac{\upsilon^2}{c^2}}$$

$$= 1 \cos 45^o \sqrt{1-\frac{(0.6)^2}{c^2}}$$

$$= 0.707 \sqrt{1-\frac{0.36c^2}{c^2}}$$

$$= 0.707 \sqrt{\frac{c^2-0.36c^2}{c^2}}$$

$$= 0.7070\sqrt{0.64}$$

= 0.707 × 0.8

L_x = 0.56 m

The length perpendicular to the direction of motion is

Ly = L sin 45° = sin 45°

Ly = 0.707 m

$$L = \sqrt{Lx^2 + Ly^2}$$

$$= \sqrt{(0.56)^2 + (0.707)^2}$$

$$= \sqrt{0.313 + 0.499} = \sqrt{0.81}$$

L = 0.9 m

If the rod makes an angle θ with the direction of motion

$$\tan\theta = \frac{Ly}{Lx} = \frac{0.707}{0.560} = 1.26$$

$$\theta = \tan^{-1}(1.26)$$

5. Calculate the percentage contraction of the rod moving with a velocity of 0.5 c in a direction inclined at 60° to its' own length?

Sol :

L_0 is the length of the rod at rest. Its component to the direction of motion is L_0 sin60°.

The apparent length along the direction of motion

$$= L_0 \cos 60^o \sqrt{1 - \frac{(0.6c)}{c^2}}$$

$$= L_0 \times 0.5\sqrt{0.63}$$

$$= L_0 \times 0.5 \times 0.79$$

$$= 0.40\ L_0$$

The apparent length perpendicular to the direction of motion

$$= L_0 \sin 60^o$$

$$= \frac{\sqrt{3}}{2} L_0$$

$\therefore$ The length of the moving rod

$$= \sqrt{(0.40L_0)^2 + \left(\frac{\sqrt{3}}{2}L_0\right)^2}$$

$$= L_0\sqrt{0.16 + 0.87}$$

$$= L_0\sqrt{1.03}$$

$$= 1.01\ L_0$$

Percentage contraction is $\frac{L_0 - L}{L_0} \times 100$

$$\frac{L_0 - 1.01L_0}{L_0} \times 100 = -0.01 \times 100$$

$$= -1.0\%$$

% contraction = 1.0%

6. A rocket ship is 99 m long on the ground. When it is in flight its length is 98 m to an observer on the ground. What is its' speed ? (c = 3 × 10⁸ m/s)

Sol :

L_0 = Length of the rocket ship on the ground

L = Length in the flight

We know that $L = L_0\sqrt{1 - \frac{\upsilon^2}{c^2}}$

$$1 - \frac{\upsilon^2}{c^2} = \left(\frac{L}{L_0}\right)^2$$

$$1 - \left(\frac{L}{L_0}\right)^2 = \frac{\upsilon^2}{c^2}$$

$$\upsilon = \sqrt{1 - \left(\frac{L}{L_0}\right)^2}.c$$

$$= \sqrt{1 - (0.98)^2}.c$$

$$= \sqrt{1 - 0.96}.c$$

$$\upsilon = 0.2c$$

7. Two particles are moving in opposite directions each with a speed of 0.8 c in laboratory frame of reference. Find the velocity of one particle relative to other.

Sol :

$$u = \frac{u' + \upsilon}{1 + \frac{u'\upsilon}{c^2}}$$

where u' = 0.8 c and θ = 0.8c

$$= \frac{0.8c + 0.8c}{1 + \frac{0.8c \times 0.8c}{c^2}}$$

$$= \frac{1.6c}{1 + 0.64}$$

u = 0.97 c

As per, Galilean transformations,

u = u' + υ = 0.8 c + 0.8 c = 1.6 c

This is greater than c and hence impossible

8. In the laboratory two particles are observed to travel in opposite directions each with a speed 2.8 × 10¹⁰ cm /sec. Deduce the relative speed of the particles? (c = 3.0 × 10¹⁰ cm/sec)

Sol :

$$u = \frac{u' + \upsilon}{1 + \frac{u'\upsilon}{c^2}}$$

$$= \frac{2.8 \times 10^{10} + 2.8 \times 10^{10}}{1 + \frac{2.8 \times 10^{10} \times 2.8 \times 10^{10}}{\left[3.0 \times 10^{10}\right]^2}}$$

$$= \frac{5.6 \times 10^{10}}{1 + \frac{2.8 \times 2.8 \times 10^{10}}{9.0}}$$

$$= \frac{5.6 \times 10^{10}}{1 + \frac{2.8 \times 2.8}{9.0}}$$

$$= \frac{5.6 \times 10^{10}}{1.871} = 2.99 \times 10^{10} \text{ cm/sec}$$

9. What is the velocity of π mesons whose proper mean life is 3 × 10⁻⁸ sec and observed mean life is 2 × 10⁻⁷ sec ?

Sol :

The observed mean life is

$$t = \frac{t_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

$t_0 = 3 \times 10^{-8}$ sec, $t = 2 \times 10^{-7}$ sec

where t_0 is the proper life

$$\sqrt{1 - \frac{\upsilon^2}{c^2}} = \frac{t_0}{t}$$

$$1 - \frac{\upsilon^2}{c^2} = \frac{t_0^2}{t^2}$$

$$\sqrt{1 - \frac{\upsilon^2}{c^2}} = \frac{3 \times 10^{-8}}{2 \times 10^{-7}}$$

$$= 1.5 \times 10^{-1} = 0.15$$

$$1 - \frac{\upsilon^2}{c^2} = (0.15)^2$$

$$= 0.0225$$

$$\frac{\upsilon^2}{c^2} = 1 - 0.0225 = 0.9775$$

$$\frac{\upsilon}{c} = 0.98$$

$$\therefore \quad \upsilon = 0.98\ c$$

10. On the surface of the earth the mass of the man is 95 kg. When he is in a rocket moving with a speed of 3 × 10⁷ m/s relative to the earth, what will be his mass as observed by

1) an observer on the earth

2) an observer in his rocket?

(c = 3 × 10⁸ m/s)

Sol :

m_0 = 95 kg

The mass as observed by a stationary observer when the man is moving in a rocket

$$m = \frac{m_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

$$= \frac{95}{\sqrt{1-\frac{(3\times10^7)^2}{c^2}}}$$

$$= \frac{95}{\sqrt{1-\left(\frac{3\times10^7}{3\times10^8}\right)^2}}$$

$$= \frac{95}{\sqrt{1-\frac{9\times10^{14}}{9\times10^{16}}}}$$

$$= \frac{95}{\sqrt{1-10^{14-16}}}$$

$$= \frac{95}{\sqrt{1-10^2}}$$

$$= \frac{95}{\sqrt{1-0.01}}$$

$$= \frac{95}{\sqrt{0.99}} = 95.95$$

m = 95.95

An observer moving in the rocket will find the mass of the man as the rest mass i.e., 95.95.

11. A charged particle shows an acceleration of 3 × 10^{12} cm/sec^2 under an electric field at low speed. Calculate the acceleration of the particle under the same field when the speed has reached a value 2.88 × 10^{10} cm/sec. The speed of light is 3.0 × 10^{10} cm/sec.

Sol :

The force acting on a particle is

$a_0 = 3 \times 10^{12}$ cm/sec^2, a = ?, F = qE

When $\upsilon = 2.88 \times 10^{10}$ cm/sec

acceleration $a_0 = \frac{F}{m_0} = \frac{qE}{m_0}$

$= 3 \times 10^{12}$ cm/sec^2

When the particle attains a speed 3.0 × 10^{10} cm/sec, its mass increases to m

$$m = \frac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}} = \frac{m_0}{\sqrt{1-\frac{(2.88\times10^{10})^2}{(3.0\times10^{10})^2}}}$$

$$m = \frac{m_0}{\sqrt{1-\frac{(2.88)^2}{(3.0)^2}}}$$

$$= \frac{m_0}{\sqrt{1-0.9216}}$$

$$= \frac{m_0}{0.0784}$$

$$\alpha = \frac{F}{m} = \frac{F}{\frac{m_0}{0,0784}}$$

$$= 0.0784 \times \frac{F}{m_0}$$

$\alpha = 0.0784 \times 3 \times 10^{12}$

$= 0.23 \times 10^{12}$ cm/sec^2

12. The rest mass of an electron is 9 × 10^{-31} kg. What will be its mass if it were moving with $\frac{2}{3}$rd of the speed of light ?

Sol :

We know that $m = \frac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$

$$\frac{\upsilon}{c} = \frac{2}{3}$$

$$\sqrt{1-\frac{\upsilon^2}{c^2}} = \sqrt{1-\frac{2^2}{3^2}}$$

$$= \sqrt{1-\frac{4}{9}}$$

$$= \sqrt{\frac{9-4}{9}}$$

$$= \sqrt{\frac{5}{9}}$$

$$\therefore\ m = \frac{9\times10^{-31}}{\sqrt{\frac{5}{9}}}$$

$$= \frac{9\times3\times10^{-31}}{\sqrt{5}}$$

$$= \frac{21\times10^{-31}}{\sqrt{5}}$$

$$= 9.39 \times 10^{-31}\text{kg}$$

13. Deduce the velocity at which the mass of a particle becomes 1.5 times its' rest mass (c = 3 × 10⁸ m/s)

Sol :

We know that $m = \dfrac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$

$$\frac{m}{m_0} = 1.5$$

$$\frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} = 1.5 = \frac{15}{10} = \frac{3}{2}$$

$$\sqrt{1-\frac{\upsilon^2}{c^2}} = \frac{1}{3/2} = 2/3$$

$$\left(1+\frac{\upsilon^2}{c^2}\right) = \frac{4}{9}$$

$$\frac{\upsilon^2}{c^2} = 1-\frac{4}{9}$$

$$= \frac{9-4}{9} = \frac{5}{9}$$

$$\frac{\upsilon}{c} = \sqrt{\frac{5}{9}}$$

$$\upsilon = \frac{\sqrt{5}}{3}c$$

$$= \frac{\sqrt{5}}{3} \times 3 \times 10^8 \text{ m/s}$$

$$\upsilon = 2.24 \times 10^8 \text{ m/s}$$

14. Deduce the rest energy of an electron in iozles and electron vists (m_0 = 9.1 × 10^{-31} kg, C = 3.0 × 10^8 m/s). Also deduce the speed at which the total relation static energy becomes 1.25 times the rest energy?

Sol :

The rest energy is $E_0 = m_0c^2$

$$= 9.1 \times 10^{-31} \times (3 \times 10^8)^2$$

$$= 8.19 \times 10^{-14} \text{ J}$$

$$E_0 = \frac{8.19\times10^{-14}}{1.6\times10^{-19}}$$

$$= 0.51 \times 10^6 \text{ eV}$$

$$= 0.51 \text{ MeV}$$

$$\frac{E}{E_0} = \frac{mc^2}{m_0c^2} = \frac{m}{m_0} = 1.25$$

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

$$\frac{1}{1-\frac{\upsilon^2}{c^2}} = (1.25)^2 = 1.5625$$

$$1-\frac{\upsilon^2}{c^2} = \frac{1}{1.5625} = 0.64$$

$$\frac{\upsilon^2}{c^2} = 1 - 0.64 = 0.36$$

$$\frac{\upsilon}{c} = 0.6$$

υ = 0.6 C = 0.6 × 3 × 10^8 m/s

υ = 1.8 × 10^8 m/s

15. Show that the mass of an electron is equivalent to 0.51 MeV energy. State the minimum energy of γ -ray photon which can produce an electron positron pair.

Sol :

The rest mass m_0 of an electron is 9.1 × 10^{-31} kg.

E_0 = 0.51 MeV

To produce an electron position pair, the minimum energy of the γ -ray photon is the sum of the rest- mass energy of an electron and a position.

0.51 0.51 = 1.02 MeV

16. A particle of rest mass m_0 moves with a speed of $\frac{c}{\sqrt{3}}$. calculate its mass, momentum, total energy and kinetic energy?

Sol :

The relativistic mass of the particle is

$$m = \frac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}} = \frac{m_0}{\sqrt{1-\frac{c^2}{3c^2}}}$$

$$= \frac{m_0}{\sqrt{1-\frac{1}{3}}} = \frac{m_0}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}\, m_0$$

m = 1.22 m_0

The momentum is $p = m\upsilon = 1.22 m_0 \times \frac{c}{\sqrt{3}}$

= 0.70 c m_0

The total energy is E = mc^2 = 1.22 m_0c^2

The kinetic energy is K = E – m_0c^2

= 1.22 m_0c^2 – m_0c^2

K = 0.22 m_0c^2

17. An electron (rest mass 9.1 × 10^{-31}kg) is moving with speed 0.8c. What is its total energy? Find the ratio of Newtonian kinetic energy to the relativistic energy? (c = 3.0 × 10^8 m/s)

Sol :

We know that $m = \frac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$

$$= \frac{9.1\times10^{-31}}{\sqrt{1-(0.8)^2}}$$

$$= \frac{9.1\times10^{-31}}{\sqrt{1-0.64}}$$

$$= \frac{9.1\times10^{-31}}{\sqrt{0.36}}$$

$$= \frac{9.1\times10^{-31}}{0.6}$$

= 15.17 × 10^{-31} kg

The total energy of the electron is

E = mc^2

= 15.17 × 10^{-31} × (3.0 × 10^8)^2

= 136.53 × 10^{-31+16}

= 136.53 × 10^{-15}

= 1.36 × 10^{-31}J

The Newtonian kinetic energy is $\frac{1}{2}m_0\upsilon^2$

The ratio of Newtonian kinetic energy to the relativistic energy is

$$\frac{\frac{1}{2}m_0\upsilon^2}{mc^2 - m_0c^2} = \frac{1}{2}\frac{m_0}{(m-m_0)}\left(\frac{\upsilon}{c}\right)^2$$

$$= \frac{1}{2}\frac{9.1\times10^{-31}}{15.17\times10^{-31} - 9.1\times10^{-31}}$$

$$= \frac{1}{2}\frac{9.1\times10^{-31}}{6.07\times10^{-31}}$$

$$= \frac{1}{2}\times\frac{9.1}{6.07} = 0.75$$

18. How much energy will be obtained if 3.0 g of mass is completely converted into energy?

Sol :

Using mass - energy relation

$$\Delta E = (\Delta m)c^2$$

$$\Delta_m = 3g = 3\times10^{-2}\,kg$$

$$c = 3.0 \times 10^8 \text{ m/s}$$

$$\Delta E = (3 \times 10^{-2})(3.0 \times 10^8)^2 = 27 \times 10^{14}$$

$$1 \text{ MeV} = 1.60 \times 10^{-31} \text{ J}$$

$$\Delta E = \frac{27.0\times10^{14}}{1.60\times10^{-13}} \text{ MeV}$$

$$= 16.87 \times 10^{27} \text{ MeV}$$

19. What is the mass equivalent of the energy from an antenna radiating 20000 watts for 48 hours.

Sol :

The total energy radiated is

$$\Delta E = 20000 \times 48 \text{ watt-hours}$$

$$= 20000 \times 48 \times 3600 \text{ watt-sec}$$

$$= 3.45 \times 10^9 \text{ J}$$

The mass equivalent is

$$\Delta_m = \frac{\Delta E}{c^2}$$

$$= \frac{3.45\times10^9}{(3.0\times10^8)^2}$$

$$= \frac{3.45\times10^9}{9\times10^{16}}$$

$$= \frac{3.45}{9}\times10^{9-16}$$

$$= 0.38 \times 10^{-7} \text{ kg}$$

20. A clock keeps correct time. With what speed should it be moved relative to an observer so that it may appear to loose 5 minutes in 24 hours?

Sol :

$$t = \frac{t_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

$$t_0 = 24 \times 60 = 1440$$

$$t = 5 + 24 \times 60 = 1445$$

$$1445 = 1440\left(1+\frac{\upsilon^2}{2c^2}\right)$$

$$\frac{\upsilon^2}{2c^2} = \frac{1445}{1440} - 1 = \frac{5}{1440}$$

$$= 3.47 \times 10^{-3}$$

$$\upsilon^2 = 3.47\times10^{-3}\times2c^2$$

$$\upsilon = \sqrt{2\times2.47\times10^{-3}}\times c$$

$$= \sqrt{2\times0.00347}\times c$$

$$= \sqrt{0.00694}\times c$$

$$\upsilon = 0.0833\ c$$

Thus $\upsilon = 0.0833 \times 3 \times 10^8$

$$\upsilon = 0.25 \times 10^8 \text{ m/s}$$

21. Find the velocity with which a body should be moving such that it gets its rest mass doubled?

Sol :

We know that $m = \dfrac{m_0}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$

$$2m_0 = \frac{m_0}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$$

$$\left(1-\frac{\upsilon^2}{c^2}\right)^{1/2} = \frac{1}{2}$$

$$1-\frac{\upsilon^2}{c^2} = \frac{1}{4}$$

$$\upsilon^2 = \frac{3}{4}c^2 = \frac{\sqrt{3}}{2}c$$

υ = 0.86 c = 0.86 × 3 × 10^8

υ = 2.58 × 10^8 m/s

22. In Michelson- Morley experiment, the mirror is 10 m distance from the glass plate. Find fringe shift for 6000 Å radiation (υ = 3 × 10^4 m/s)

Sol :

Fringe shift $\delta = \dfrac{2lv^2}{c^2\lambda}$

$$= \frac{2\times 10\times 9\times 10^8}{9\times 10^{16}\times 6000\times 10^{-10}}$$

$$= \frac{180\times 10^8}{54\times 10^3\times 10^{16}\times 10^{-10}}$$

$$= \frac{180}{54}\times 10^{8-3-16+10}$$

= 3.33 × 10^{-1}

δ = 0.33

23. A meson particle decays in 2μs in a rest frame. When meson particle is moving with 0.6 c velocity. Find its decay time?

Sol :

υ = 0.6c, $t = \dfrac{t_0}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$

$$= \frac{2}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$$

$$= \frac{2}{\sqrt{1-\left(\dfrac{0.6c}{c}\right)^2}}$$

$$= \frac{2}{\sqrt{1-0.36}}$$

t_0 = 2 μs, t = ?

$$t = \frac{2}{\sqrt{0.64}}$$

t = 2.5 μs

24. Find the velocity with which a body should travel so that the length becomes half of the rest length.

Sol :

Given that,

For a body,

Length = $\dfrac{1}{2}$ (rest length)

$\Rightarrow$ $l = \dfrac{l'}{2}$

The variation oflength and velocity can be mathematically expressed by length contraction.

i.e., $l = l'\sqrt{1-\dfrac{v^2}{c^2}}$

$$\Rightarrow \quad \frac{l'}{2} = l'\sqrt{1-\frac{v^2}{c^2}}$$

$$\Rightarrow \quad \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2}$$

$$\Rightarrow \quad 1-\frac{v^2}{c^2} = \frac{1}{4}$$

$$\Rightarrow \quad \frac{-v^2}{c^2} = \frac{1}{4}-1$$

$$\Rightarrow \quad \frac{-v^2}{c^2} = \frac{-3}{4}$$

$$\Rightarrow \quad \frac{v^2}{c^2} = \frac{3}{4}$$

$$\Rightarrow \quad \frac{v}{c} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad v = \frac{\sqrt{3}}{2}\,c = 0.866\,c$$

$= 0.866 \times 3 \times 10^8$ [$\because c = 3 \times 10^8$ m/sec]

$= 2.598 \times 10^8$

$= 2.6 \times 10^8$ m/sec.

$\therefore$ At $v = 0.866\,c$ or $v = 2.6 \times 10^8$ m/sec, the lenght of body becomes half of the rest length.

Short Question and Answers

1. Central Force.

Ans :

A central force is defined as a force, which always acts on a particle or body towards or away from a fixed, point and whose magnitude depends upon only on the distance from the fixed point. This fixed point is known as the centre of the force.

2. What are inertial and non-inertial frames?

Ans :

(i) Inertial Frame of Reference

Inertial frame of reference can be defined as a frame in which the bodies obey Newton's law of inertia. In other words, a coordinate system wherein Newton's first law of motion holds good is known as Inertial frame of reference.

In this type of frame, a body moves with constant velocity i.e., zero acceleration. Hence, it is also known as unaccelerated frame.

Examples

(a) Any reference frame attached to earth

(b) A coordinate system fixed on earth having spinning motion

(c) A cart at rest or moving with constant velocity.

(ii) Non-inertial Frame of Reference

Non-inertial frame of reference can be defined as a frame in which the bodies do not obey Newton's first law of motion. In other words, a frame of reference having an acceleration with respect to an inertial frame is known as non-inertial frame of reference.

In this type of frame, a body moves with variable velocity i.e., certain amount of acceleration. Hence, it is also known as accelerated frame.

Examples

(i) Any reference frame with uniform linear acceleration

(ii) A cart moving with variable velocity

(iii) An aircraft making its take-off run.

3. What is velocity of the particle if its KE is equal for rest energy?

Ans :

Given that,

For a particle,

Total energy = E = m_oC^2

The variation of mass with velocity can be mathematically expressed as,

$$m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} \quad ...(1)$$

Where,

m = Relatively mass which includes both kinetic and rest energy

m_o = Rest mass

$\because \quad E = E_k + E_r$

$mc^2 - Ek + m_oc^2 \quad [\because E = mc^2]$

$\Rightarrow \quad E_k = mc^2 - m_oc^2$

$$\Rightarrow \quad E_k = \frac{m_oc^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_oc^2$$

$$\Rightarrow \quad m_oc^2 = \frac{m_oc^2}{1-\frac{v^2}{c^2}} - m_oc^2 \quad [\because \text{ Given}]$$

$$\Rightarrow \quad \frac{m_oc^2}{\sqrt{1-\frac{v^2}{c^2}}} = 2m_oc^2$$

$$\Rightarrow \quad \frac{m_oc^2}{\sqrt{1-\frac{v^2}{c^2}}} = 2$$

$$\Rightarrow \quad \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2}$$

$\Rightarrow \quad 1 - \frac{v^2}{c^2} = \left(\frac{1}{2}\right)^2$

$\Rightarrow \quad 1 - \frac{v^2}{c^2} = \frac{1}{4}$

$\Rightarrow \quad \frac{v^2}{c^2} = 1 - \frac{1}{4}$

$\Rightarrow \quad \frac{v^2}{c^2} = \frac{3}{4}$

$\Rightarrow \quad v^2 = c^2 \frac{3}{4}$

$\Rightarrow \quad v = c\sqrt{\frac{3}{4}}$

$\Rightarrow \quad v = \sqrt{0.75} \times 3 \times 10^{10} = 2.59 \times 10^{10}$

$\therefore$ Velocity of particle, $v = 2.59 \times 10^{10}$ cm/sec

4. Calculate the work done to keep two balls having a mass 500 gm each from infinite distance to 10cm apart.

Ans :

Given that,

Mass of balls,

$M_1 = m = 500$ gm $= 500 \times 10^{-3}$ kg

Distance, $r = 10$ cm $\times 10^{-2}$m

Work done in moving the balls from infinite distance to 10 cm apart is given as,

$$W = E = -\frac{GM_m}{r}$$

Where, G – Gravitational constant

$= 6.67 \times 0^{-11}$ Nm^2 kg^{-2}

Substituting the corresponding values in above equation,

$$W = -\frac{6.67 \times 10^{-11} \times 500 + 10^{-3} \times 500 \times 10^{-3}}{10 \times 10^{-2}}$$

$= -6.67 \times 25 \times 10^{-11+4} - 6 + 1$

$= -16.675 \times 10^{-12}$ J

$\therefore$ Work done, $W = -166.75 \times 10^{-12}$ J

5. Time Dilation.

Ans :

Time intervals are affected by relative motion. A clock moving with velocity υ with respect to an observer appears him to have slowed down by a factor $\sqrt{1-\frac{\upsilon^2}{c^2}}$, than when at rest with respect to him.

Suppose a clock is placed at the point x' in the moving frame S'. An observer in S' finds that the clock gives two ticks at times t'_1 and t'_2. The time interval between the ticks as judged from S' is

$$t_0 = t'_2 - t'_1$$

t_0 is the interval as measured in a frame in which the clock is at rest. Another observer measures the time interval between the same two ticks from a stationary frame of reference S, relative to which the clock is moving with velocity υ. If he records the ticks at times t_1 and t_2, the time interval appears to him as

$$t = t_2 - t_1$$

Using Lorentz transformations, we have

$$t_2 = \frac{t'_2 + \left(\frac{x'\upsilon}{c^2}\right)}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

and $$t_1 = \frac{t'_1 + \left(\frac{x'\upsilon}{c^2}\right)}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

$$\therefore \quad t = t_2 - t_1 = \frac{t'_2 - t'_1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

$$\therefore \quad t = \frac{t_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

From this equation, it is evident that to the stationary observer in S, the time intervals appear to be lengthened by a factor $\frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$. A moving clock appears to be slowed down to a stationary observer. This concept is called time dilation.

If $\upsilon = c$, then $t = \infty$. It means that a clock moving with the speed of light appears to be completely stopped to a stationary observer.

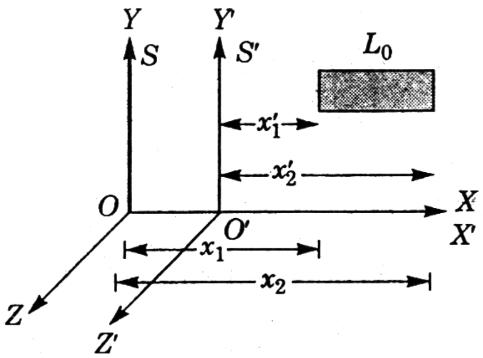
6. Length contraction.

Ans :

Consider two reference frames S and S' respectively.

S' is moving with a velocity υ with respect to S. An observer in frame S' is at rest with respect to S' and hence with respect to the rod. The rod is at rest with respect to this observer.

The length L_0 of the rod is written as



Where x'_2 and xj are the coordinates of the rod in S'. Let L frame is the length of the rod in S frame of reference.

Using Lorentz transformation equations, we have

$$x'_1 = \frac{x_1 - \upsilon t_1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \quad ...(2)$$

$$x'_2 = \frac{x_2 - \upsilon t_2}{\sqrt{1-\frac{\upsilon^2}{\upsilon^2}}} \quad ...(3)$$

$$L_0 = x'_2 - x'_1$$

$$= \frac{(x_2 - \upsilon t_2)}{\sqrt{1-\frac{\upsilon^2}{c^2}}} - \frac{(x_1 - \upsilon t_1)}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

$$L_0 = \frac{x_2 - x_1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \qquad [\because t_1 = t_2]$$

$$L_0 = \frac{L}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \quad ...(4)$$

This equation indicates that to the stationary observer in S, the rod placed in the moving frame S' appears to be contracted by a factor $\sqrt{1-\frac{\upsilon^2}{c^2}}$. This

contraction occurs in the direction of relative motion. If $\upsilon = c$, then L = 0. This means that the rod moving with the speed of light will appear as reduced to a point to a stationary observer. This type of contraction is called the Lorentz-Fitzgerald length contraction.

7. A meter scale length is recorded as 96 cm by an observer. Find it's velocity.

Ans :

Given that,

Length of a scale, l = 1m

Length observed, l = 96 cm = 96×10^{-2}m

Velocity of the observe, v is given as,

$$v = c\sqrt{1-\frac{l^2}{l^2}} \qquad \left[\because l = l'\sqrt{1-\frac{v^2}{c^2}}\right]$$

Where,

c – Velocity of light = 3×10^8 m/s

Substituting the corresponding values in above equation.

$$V = 3 \times 10^8 \sqrt{1 - \frac{(96 \times 10^{-2})}{(1)^2}}$$

$= 3 \times 10^8 \sqrt{1 - 0.9216}$

$= 3 \times 10^8 \times 0.28$

$= 0.84 \times 10^8$

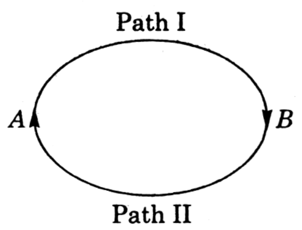
$= 84 \times 10^6$ m/s

$\therefore$ Velocity, $v = 84 \times 10^6$ m/s

8. Explain the conservative nature of central force?

Ans :

A force is said to be conservative if the work done in moving a particle from one point to another point is independent of the path followed. But depends on the end points. Consider a particle that moves from A to B. It can follow path I or path II or any other path. If the force is conservative, the work done along these paths is constant. The points A and B are fixed.



$$\int_A^B \vec{F}.\overline{dr} = \int_B^A \vec{F}.\overline{dr}$$

path I path II

The work done in moving a particle along a closed curve is zero.

W_{AB} = work done in moving from A to B

$\int_A^B \vec{F}.\overline{dr}$ ...(1)

W_{BA} = work done in moving from B to A = $+ W_{BA} = -W_{AB}$

$\therefore$ Net work done = $W_{AB} + W_{BA}$

$W_{AB} + W_{BA} = 0$ ...(2)

9. Kepler's laws of planetary motion.

Ans :

Kepler proposed three laws for planetary motion.

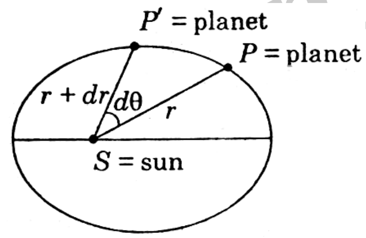
They are as follows:

1. First Law

Each planet revolves around the sun in an elliptical orbit around the sun. The sun is at the one of the foci of the ellipse [Fig. 7].

This law is called as the law of elliptical orbits.

This law gives us the shape of the orbit of a planet around the sun.



2. Second Law

The radius vector of any planet relative to the sun sweeps out equal areas in equal times. The areal velocity of the radius vector is constant.

This law is called as the law of areas. This law gives the relationship between the speed of the planet and its distance from the sun.

3. Third law

The square of the period of revolution of any planet around the sun is directly proportional to the cube of the length of semi-major axis of the elliptical orbit. This is called as harmonic law.

$$T^2 \propto a^3$$

T = period of revolution of planet around the sun

a = length of semi-major axis of the elliptical orbit.

10. Define postulates of special theory of relativity.

Ans :

Einstein published the special theory of relativity in 1905. This theory was based on the postulates.

Postulate 1

The laws of physics have the same form in all inertial frames of references moving with a constant velocity relative to one another.

This is called as principle of relativity. This postulate defines the absence of universal frame of reference.

As per this postulate, it is impossible by any means to demonstrate absolute motion. The absolute motion is meaningless. The motion of bodies relative to one another has physical meaning. There is no absolute motion according to Einstein. Undetection of absolute motion implies undetection of ether. If the laws of physics were different for observers in different frames in relative motion, it could be determined from these differences which objects are stationary in space and which are moving. There is no universal frame of reference, this distinction between objects cannot be made.

Postulate 2

The speed of light in free space is the same in all inertial frames of references. This is called as the principle of the constancy of the speed of light.

This postulate follows directly from the result of Michelson-Morley experiment. As per this postulate, the speed of light is same in all directions. It is the greatest velocity.

Q11. Concept Four Vector Formalism

Ans :

The four vector concept has its significance in theory of relativity. As per first postulate of special theory of relativity, the laws of physics are invariant in all inertial reference frames.

If any equation holds good from the point of special theory of relativity, it should be expressed in four vector form. As per Lorentz transformations, the value $x^2 + y^2 + z^2 - c^2 t^2$ must be same in all inertial reference frames. If S and S' are two inertial reference frames, then

$$x^2 + y^2 + z^2 - c^2t^2 = x_1^2 + y_1^2 + z_1^2 - c^2t'^2 \quad ...(1)$$

Let $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$

x_1, x_2, x_3 and x_4 are treated as components of a vector in 4D space. the vector length in 4 D space is written as

$$D^2 = x_1^2 + x_2^2 + x_2^3 + x_4^2$$

$$= \sum_{\mu=1}^{4} x_\mu^2$$

If vector length is invariant in both frames S and S'. Using Lorentz transformation

$$\sum_{\mu=1}^{4} x_\mu^2 = \sum_{\mu=1}^{4} x_\mu'^2$$

$$x' = v(x - \upsilon t)$$

$$y' = y$$

$$z' = z$$

$$t' = v\left[t - \frac{\upsilon x}{c^2}\right]$$

$$\text{where } v = \frac{1}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} = \left(1 - \beta^2\right)^{-1/2}, \quad \beta = \frac{\upsilon}{c}$$

$$x_1' = v[x_1 - \upsilon t] = v[x_1 + i\beta x_4]$$

$$x_2' = x_2; x_3' = x_3$$

$$x_4' = ict' = v\left[x_4 - \frac{i\upsilon x_1}{c}\right] = v[x_4 - i\beta x_1] \quad ...(3)$$

Putting the above equations in matrix form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} v & 0 & 0 & i\beta v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta v & 0 & 0 & v \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad ...(4)$$

The above matrix form shown the position coordinates in four vector notation. The above matrix is valid for S and S' frames which have relative velocity along X-Matrix elements will be determined in any direction.

12. Define gravitational field.

Ans :

The region around a body where the gravitational force of attraction is present is called the gravitational field.

Gravitational attraction or the intensity of the gravitational field at a point in the field is the force experienced by a unit is placed at that point.

The intensity of the gravitational field is

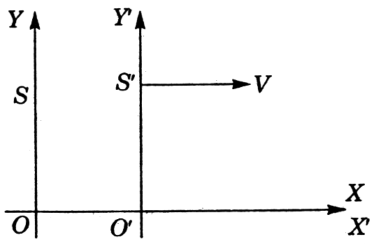
$$F = G\frac{M \times 1}{r^2}$$

$$F = \frac{GM}{r^2}$$

13. Lorentz Transformations.

Ans :

Consider a reference frame S which is at Y U. rest. Consider another reference frame S' which is moving with a constant velocity V. The S position vector of an event in S' frame



$$r' = x - \upsilon t \quad ...(1)$$

The component from of equation (1) is written as

$$x' = x - \upsilon t \quad ...(2)$$

$$y' = y \quad ...(3)$$

$$z' = z \quad ...(4)$$

and $t' = t$ ...(5)

The equations (2) to (5) are called Galilean transformations. The relation between velocity components in S and S' frames is written as

$$\upsilon'_x = \upsilon_x - \upsilon \quad ...(6)$$

$$\upsilon'_y = \upsilon_y \quad ...(7)$$

$$\upsilon'_z = \upsilon_z \quad ...(8)$$

The eqs. (6), (7) and (8) disobey the postulates of special theory of relativity.

The relation between x and x' is written as

$$x' = k(x - \upsilon t) \quad ...(9)$$

where x is a constant of proportionality.

The inverse equation for equation (9) is written as

$$x = k'(x' + \upsilon t') \quad ...(10)$$

The other equations are written as

$$y' = y \quad ...(11)$$

$$z' = z \quad ...(12)$$

Times t and t' are not equal.

Putting the value of x' from equation (9) in equation (10), we have

$$x = kk'(x - \upsilon t) + k'\upsilon t'$$

$$t' = kt + \left(\frac{1 - kk'}{k'\upsilon}\right)x \quad ...(13)$$

For calculating k and k', the second postulate of special theory of relativity is used.

Consider a single of light is given out from the common origins of S and S' at t = t' = 0. The signal propagates in the two systems according to the equations

$$x = ct \quad ...(14)$$

$$x' = ct' \quad ...(15)$$

in system S and S' respectively.

Substituting the values of x' and t' into equation (15), one can has

$$k(x - \upsilon t) = ckt + \left(\frac{1 - kk'}{k'\upsilon}\right)cx$$

$$x = ct\left[\frac{1 + \frac{\upsilon}{c}}{1 - \left\{\frac{1}{kk'} - 1\right\}\frac{c}{\upsilon}}\right] \quad ...(16)$$

Comparing equation (16) with equation (14), we have

$$\frac{1+\frac{\upsilon}{c}}{1-\left\{\frac{1}{kk'}-1\right\}\frac{c}{\upsilon}}=1 \quad ...(17)$$

$$\sqrt{kk'}=\frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

We choose $k=k'=\frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$ ...(18)

Substituting these values in equations (9) and (13), we have

$$\left.\begin{aligned} x' &= \frac{x-\upsilon t}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \\ y' &= y \\ z' &= z \text{ and} \\ t' &= \frac{t-\frac{\upsilon x}{c^2}}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \end{aligned}\right\} \quad ...(19)$$

Equations (19) are called Lorentz transformations.

The inverse Lorentz transformation equations are written as

$$\left.\begin{aligned} x &= \frac{x'+\upsilon t'}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \\ y &= y' \\ z &= z' \\ t &= \frac{t'+\frac{\upsilon x'}{c^2}}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \end{aligned}\right\} \quad ...(20)$$

14. Mass-energy relation.

Ans :

As per Einsteins' mass - energy relation

$$m = \frac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

m_0 = rest mass of the body

c = velocity of light

υ = velocity of body

m = mass of the body

According to Newtons' second law of motion,

$$F = \frac{d}{dt}(m\upsilon)=m\frac{d\upsilon}{dt}+\upsilon\frac{dm}{dt}$$

The change in kinetic energy of the body is equal to the workdone by the force F for a displacement dS of the body.

Thus dk = F. ds

$$= m\frac{d\upsilon}{dt}dS+\upsilon\frac{dm}{dt}dS$$

$$= m\frac{dS}{dt}d\upsilon+\upsilon\frac{ds}{dt}dm$$

$$= m\upsilon d\upsilon+\upsilon^2 dm \quad ...(2)$$

Differentiating equation (1), we have

$$dm = m_0\left(-\frac{1}{2}\right)\left(1-\frac{\upsilon^2}{c^2}\right)^{-3/2}\left(\frac{-2\upsilon d\upsilon}{c^2}\right)$$

$$= \frac{m_0}{c^2}\frac{\upsilon d\upsilon}{\left(1-\frac{\upsilon^2}{c^2}\right)^{3/2}}$$

We know that $m = \frac{m_0}{\left(1-\frac{\upsilon^2}{c^2}\right)^{1/2}}$

$$\therefore \quad dm = \frac{m\upsilon d\upsilon}{(c^2-\upsilon^2)}$$

$$m\upsilon d\upsilon = \left(c^2-\upsilon^2\right)dm$$

Putting this value in equation (2), one can has

$$dk = \left(c^2 - \upsilon^2\right)dm + \upsilon^2 dm = c^2 dm$$

The kinetic energy of a body is written as

$$k = \int dk = \int_{m_0}^{m} c^2 dm = c^2\left(m - m_0\right) \qquad ...(3)$$

m_0c^2 is the rest energy of the body. The total energy of a body is the sum of the kinetic energy and the rest energy.

$\therefore \quad E = k + E_0$

$= (m - m_0)c^2 + m_0c^2$

$E = mc^2 - m_0c^2 + m_0c^2$

$E = mc^2 \qquad ...(4)$

This equation represents the famous Einsteins' mass - energy relation.

15. Addition of velocities.

Ans :

Consider two reference frames S and S'. The frame S' moves with a constant velocity v relative to S along the X-axis. Let a body moves a distance dx in a time interval dt in the frame S. The velocity of the body measured by an observer in S frame of reference is

$$u = \frac{dx}{dt} \qquad ...(1)$$

The velocity of the body in S' frame of reference is

$$u' = \frac{dx'}{dt'} \qquad ...(2)$$

Using Lorentz transformations, we have

$$x' = \frac{x - \upsilon t}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

and
$$t' = \frac{t - \left(\frac{x\upsilon}{c^2}\right)}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

Differentiating, we have

$$dx' = \frac{dx - \upsilon dt}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

and
$$dt' = \frac{dt - \left(\frac{\upsilon dx}{c^2}\right)}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

$$\therefore \quad u' = \frac{dx'}{dt'} = \frac{dx - \upsilon dt}{dt - \frac{\upsilon dx}{c^2}}$$

$$= \frac{\frac{dx}{dt} - \upsilon}{1 - \frac{\upsilon}{c^2}\frac{dx}{dt}}$$

$$\frac{dx}{dt} = u$$

$$\therefore \quad u' = \frac{u - \upsilon}{1 - \left(\frac{u\upsilon}{c^2}\right)} \qquad ...(3)$$

This equation gives the relativistic addition of velocities u and υ .

16. If the earth be one-half of its present distance from the sun what will be the number of days in a year?

Ans :

Let x be the distance between the sun and the earth. When the earth was at half of its present distance from the sun,. then the distance becomes $\frac{x}{2}$.

According to Kepler's third law, the square of the time period of revolution of the planet is proportional to cube of semi-major axis i.e.,

$T^2 \alpha\, a^2$

$$\Rightarrow \quad \left(\frac{T_1}{T_2}\right)^2 = \left(\frac{a_1}{a_2}\right)^3$$

$$\Rightarrow \quad \frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

$$\Rightarrow \quad \frac{T_1^2}{a_1^3} = \frac{T_2^2}{a_2^3}$$

$$\Rightarrow \quad T_2^2 = \left(\frac{a_2}{a_1}\right)^3 \times T_1^2 \qquad ...(1)$$

Here, $T_1 = 1$ year, $a_1 = x$, $a_2 = \frac{1}{2}x$

Substituting the corresponding values in quation (1),

$$T_2^2 = \left(\frac{\frac{x}{2}}{x}\right) \times 1 \text{ year}$$

$$= \left(\frac{1}{2}\right)^3 \times 1 \text{ year}$$

$$\Rightarrow \quad T_2^2 = \frac{1}{8} \text{ years}$$

$$\Rightarrow \quad T_2 = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}} \times 365 \text{ days}$$

[$\because$ 1 year = 365 days]

$$\Rightarrow \quad T_2 = 125 \text{ days}$$

$$\therefore \quad T_2 = 129 \text{ days}$$

17. Galilean Transformation

Ans :

Galilean or Classical Transformation

A point or particle at any instant has different coordinates in different reference systems. The equations which provide the relationship between the coordinates of two reference systems are called transformation equations.

The transformation of coordinates of a particle from one inertial frame to another is known as Galilean (or classical) transformation.

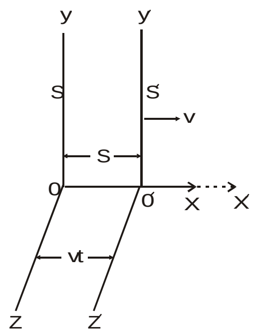


Fig.: Reference frame S' moves with velocity v (in the x direction) relative to reference frame S.

To detect the position of a particle at a certain time, we should represent it in both space as well as time. such a thing is called an event. The event may be conveniently represented by (x, y, z, t).

18. The total electrical energy generated in a station in a particular year was 7.5 × 10^{11} KWH. Find the mass equivalent of this energy.

Ans :

Given that,

In a station,

Total electrical energy generated

$= 7.5 \times 10^{11}$ KWH

$= 7.5 \times 10^{11} \times 3.60 \times 10^6$ J

[$\therefore$ 1 KWH = 3.6×10^6 J]

$= 2.7 \times 10^{18}$ J

The expression for mass equivalent of electrical energy is given as,

$$E = mc^2$$

$$\Rightarrow \quad m = \frac{E}{c^2}$$

Substituting the corresponding values in above equation.

$$m = \frac{2.7 \times 10^{18}}{(3 \times 10^8)^2}$$

$$= \frac{2.7 \times 10^{18}}{9 \times 10^{16}}$$

$$= 0.3 \times 10^2$$

$\therefore$ Mass equivalent of the 7.5×10^{11} kWH = 30kg

Choose the Correct Answers

1. Gravitational potential inside a spherical shell is ________. [a]
   (a) equal to that on the surface (b) greater than that on the surface
   (c) less than that on the surface (d) zero

2. The velocity of the earth around the sun is ________. [c]
   (a) 3 km/s (b) 30 km/s
   (c) 0.3 km/s (d) 300 km/s

3. A body under the inverse square force will move along a circular path if total energy is ________. [c]
   (a) zero
   (b) positive
   (c) negative but equal to minimum potential energy
   (d) negative but greater than minimum potential energy.

4. A satellite revolves round a plant of radius R in time T. What will be the period of revolution around another planet of radius 3R ? [a]
   (a) 3T (b) $3\sqrt{3}T$
   (c) $\sqrt{T}$ (d) 9 T

5. The angular velocity of rotation of a star (of mass m and radius R) at which the matter will start escaping from its equator is ________. [b]
   (a) $\sqrt{\frac{2GR}{m}}$ (b) $\sqrt{\frac{2Gm}{R^3}}$
   (c) $\sqrt{\frac{2Gm}{R}}$ (d) $\sqrt{\frac{2Gm^2}{R}}$

6. An infinite number of identical point masses each equal to m are placed at point x = 1, x = 2, x = 4, x = 8, ... The total gravitational potential at point x = 0 is ________. [b]
   (a) – Gm (b) – 2Gm
   (c) + 2 Gm (d) infinite

7. If the radius of earth were to decrease 1%, its mass remaining the same, acceleration due to gravita on the surface of the earth. [d]
   (a) will increase by 1% (b) will decrease by 1%
   (c) will decrease by 2% (d) will increase by 2%

8. If the K.E. of a staellite revolving in an orbit near the earth surface is doubled then: [d]
   (a) its orbital velocity is doubled (b) its period of revolution is doubled
   (c) it will be broken into pieces (d) it escapes out of earth's field

9. Michelson - Morley experiment was performed to ________. [c]
   (a) measure speed of light
   (b) prove existence of ether
   (c) measure speed of earth relative to ether
   (d) test the isotropy of space

10. Rest volume L^3_0 is connected to relativistic volume as ________. [c]
   (a) $L_0^3(1-\beta^2)^{3/2}$ (b) $L_0^3(1-\beta^2)$
   (c) $L_0^3\sqrt{1-\beta^2}$ (d) $\frac{L_0^3}{\sqrt{1-\beta^2}}$

11. Relativistic transformations were suggested by ________. [d]
   (a) Newton (b) Einstein
   (c) Huygen (d) H.A.Lorentz

12. A body moves with 0.2c velocity. The ratio of the moving mass to rest mass is ________. [b]
   (a) 1.2 (b) 1.02
   (c) 0.2 (d) 1.0

13. The rest mass of an electron is m_0. When it moves will velocity v = 0.6c then its mass is ________. [b]
   (a) m_0 (b) 5/4 m_0
   (c) 4/5m_0 (d) 2m_0

14. In accordance with the special theory of relativity if n and u' be the velocities of a particle in the laboratory and in the moving frames respectively, then which of the following relations is correct. [b]
   (a) $u'_y = u_y$ (b) $u'_x = u_x$
   (c) $u'_z = u_z$ (d) none of the above

15. In a perfectly elastic, relativistic collision between two masses, which one of the following quantities is NOT conserved ? [b]
   (a) momenturn (b) energy
   (c) rest mass (d) angular momentum

Fill in the Blanks

1. The condition for conservative force is ________.
2. If the total energy of a particle is negative but not minimum, the path is ________.
3. Gravitational forces are ________.
4. The force required to keep the satellite in the orbit is provided by ________.
5. If the eccentricity of a trajectory is zero, the trajectory is________.
6. A body under the action of inverse square force will follow an elliptic path if eccentricity is ________
7. A body under the inverse square force will move along a circular path if total energy is ________
8. An inetial frame of reference must ________.
9. In a perfectly elastic, relativistic collision between two masses, quantities is NOT conserved________
10. Michelson-Morley experiment to detect the presence of ether is based in the phenomen of ________.
11. Two photons recede from each other. Their relative velocity will be ________.
12. The rect mass of an electron is M_0. When it moves with a velocity v = 0.6 c then its mass is ________.
13. At what velocity the kinetic energy of a particle is equal to the rest mass energy? ________.
14. Einstein's mass-energy relation ($E = mc^2$) shows that mass and energy disappers to reappear as ________ and ________.

ANSWERS

1. $\vec{F} = \vec{\nabla} - V$
2. elliptical
3. weak
4. gravitation
5. circle
6. e < 1
7. negative but equal to minimum potential energy.
8. not accelerate
9. energy
10. interference
11. c
12. 5/4 m_0
13. $\sqrt{3}/2c$
14. energy and mass

UNIT IV	**OSCILLATIONS :** Simple harmonic oscillator, and solution of the differential equation– Physical characteristics of SHM, torsion pendulum measurements of rigidity modulus, compound pendulum, measurement of g, combination of two mutually perpendicular simple harmonic vibrations of same frequency and different frequencies, Lissajous figures. Damped harmonic oscillator, solution of the differential equation of damped oscillator. Energy considerations, logarithmic decrement, relaxation time, quality factor, differential equation offorced oscillator and its solution, amplitude resonance, velocity resonance.

4.1 Simple Haromonic Oscillator

Q1. Discuss the basic terms involved in oscillator, motion?

Ans :

Introduction

A motion which repeats itself after equal intervals of time is called periodic motion or harmonic motion.

A body or a particle is said possess oscillatory or vibratory motion if it moves back and forth repeatedly about the mean position.

Few terms regarding the oscillatory motion:

(i) Perodic time

The periodic time 'T' of an oscillatory motion is defined as the time taken for one oscillation.

(ii) Frequency

The 'frequency' n or v is defined as the number of oscillations in one second. It is reciprocal of periodic time. i.e., $n = \frac{1}{T}$ cycles per second.

(iii) Displacement

The Distance of the particle in any direction from the equilibrium position at any instant is called the displacement of the particle at that instant.

(iv) Amplitude

The maximum displacement or the distance between the equilibrium position at and the extremeposition is known as amplitude 'a' of the oscillation.

(v) Phase

The phase of an oscillatory particle at any instant defines the state of the particle as regards its position and direction of motion at that instant.

(vi) Restoring force

In the equilibrium position of the oscillationg particle, no net force acts on it, when the particle is displaced from its equilibrium position, a periodic force acts on it in such a direction as to bring the particle to its equilibrium position. This is called the restoring force F.

Q2. Define simple harmonic motion? Write the Equation for simple harmonic oscillator?

Ans : **(Imp.)**

Simple Harmonic Motion

It is defined as the motion of an oscillatory particle which is acted upon by a restoring force which is directly proportional to the displacement but opposite to it in direction.

Following are the characteristics of simple harmonic motion :

(a) The motion is periodic.

(b) The motion is along a straight line about the mean or equilibrium position.

(c) The Acceleration is proportional to the displacement.

(d) Acceleration is directed towards the mean or equilibrium position.

Simple Harmonic Oscillator

When a particle or body moves around such that is acceleration is always directed towards a fixed point and varies directly as its distance from the point, the particle or body is said to executer S.H.M. the particles or body executing simple harmonic motion is called a simple oscillator

Equation of Motion of a Simple Oscillator

Consider a particle P of mass m executing S.H.M about equilibrium position O along X-axis as shown in fig (2). By defination, the force under which the particle is oscillating is proportional to its displacement directed towards the mean position. Let x be the displacement of P from O at any instant. The instantaneous force Facting upon P is given by

$$F \;\alpha - x \text{ or } F = -k\,x, \;....... (1)$$

Where k is proportionality factor which represents the force per unit displacement. The negative sign is used to show that the force F is opposite to the displacement. The negative sign is used to show that the force F is opposite to the displacement.

According to Newton's second law of motion the restoring force on mass m produces as acceleration $\frac{d^2x}{dt^2}$ in the mass, so that

Force = mass × acceleration

$$\text{i.e., } F = m\frac{d^2x}{dt^2} \quad .... (1)$$

From eq (1) & (2)

$$m\frac{d^2x}{dt^2} = -kx \text{ or } \frac{d^2x}{dt^2} = \frac{-k}{m}x$$

Let us put $\frac{k}{m} = \omega^2$. Thus,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \;....... (3)$$

This is known as the differential equation of simple harmonic oscillator.

4.1.1 Solution of Differential Equation

Q3. Derive the differential Equation of simple harmonic oscillator ?

Ans :

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left[\frac{dx}{dt}\right] = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = v\frac{dv}{dx}$$

$$\left[\because \frac{dx}{dt} = v\right]$$

Substituting the value of $\frac{d^2x}{dt^2}$ in (3)

$$\text{We get } V = \frac{dV}{dx} = -\omega^2 x$$

$$Vdv = -\omega^2 x\,dx........ (4)$$

(OR)

Integrating eq (4) we get

$$\frac{V^2}{2} = -\omega^2\frac{x^2}{2} + C_1$$

{where C_1 = Constant of Integration}

The value of C_1 can be obtained by applying the condition that at x = a (Amplitude of vibration the velocity of particle is zero.

$$O = \frac{-\omega^2 a^2}{2} + C_1$$

$$C_1 = -\frac{\omega^2 a^2}{2}$$

$$\frac{V^2}{2} = \frac{-\omega^2 x^2}{2} + \frac{\omega^2 a^2}{2}$$

$$V^2 = \omega^2 (a^2 - x^2)$$

$$V = \omega\sqrt{(a^2 - x^2)} \qquad .... (5)$$

(OR)

As $V = \frac{dx}{dt}$, eq (5) can be written as

$$\frac{dx}{\sqrt{(a^2 - x^2)}} = \omega dt \qquad .... (6)$$

So Integrate eq (6) we put x= a sin θ, Hence,

$dx = a\cos\theta\, d\theta$

$\frac{a\cos\theta d\theta}{a\cos\theta}$ or $d\theta = \omega dt$ ...... (7)

Integrating eq (7), we get $\theta = (\omega t + \phi)$, where ϕ is constant. Now the displacement

$x = a\sin(\omega t + \phi)$ ...... (8)

This gives the displacement of particle at any instant.

If the motion takes place along y-axis, then

$y = a\sin(\omega t + \phi)$ .... (9)

Second treatment

The simple harmonic oscillator equation is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Let us assume a trial solution of the form

$$x = Ce^{\alpha t}$$

Where C and α are arbitary constants. Differentiating it, we get

$$\frac{dx}{dt} = C\alpha e^{\alpha t} \text{ and } \frac{d^2x}{dt^2} = C\alpha^2 e^{\alpha t}$$

Substituting these values in the equation of simple oscillator, we have

$$C\alpha^2 e^{\alpha t} + \omega^2 C e^{\alpha t} = 0$$

(OR)

$$Ce^{\alpha t}(\alpha^2 + \omega^2) = 0$$

$$(\alpha^2 = \omega^2) = 0 \quad [\because C \neq 0 \text{ and } e^{\alpha t} \neq 0]$$

$$\therefore \alpha = \pm\sqrt{(-\omega^2)} = \pm j\omega, \text{ Where } j = \sqrt{(-1)}$$

Now, $x = Ce^{+j\omega t}$ and $x = Ce^{-j\omega t}$

So the general solution can be written as

$$x = C_1 e^{+j\omega t} + C_2 e^{-j\omega t}$$

Where C_1 and C_2 are arbitary constants

Further,

$$x = C_1(\cos\omega t + j\sin\omega t) + C_2(\cos\omega t - j\sin\omega t)$$

$$x = (C_1 + C_2)\cos\omega t + j(C_1 - C_2)\sin\omega t$$

Let us put $C_1 + C_2 = a\sin\phi$ and $j(C_1 - C_2) = a\cos\phi$

Where a and ϕ are new constants

$$\therefore x = a\sin\phi\cos\omega t + a\cos\phi\sin\omega t$$

(or)

$$x = a\sin(\omega t + \phi)$$

This is the solution of the equation of simple harmonic oscillator.

4.1.2 Physical characteristics of simple Harmonic motion

Q4. Write Physical Characteristics of simple Harmonic motion?

Ans : **(Imp.)**

1. Displacement

The displacement of any yarticle ay any instant executing S.H.M. is given by

$x = a\sin(wt + \phi)$

The maximum displacement from the mean position is called amplitude. Here the amplitude is a

2. Velocity

The velocity v od the oscillating particle can be obtained by differentiating eq (8).

Thus $V = \frac{dx}{dt}$

$$= \omega a\cos(\omega t + \phi) = \omega\sqrt{(a^2 - x^2)} \quad ..... (1)$$

At the mean position i.e, at x = 0 , the velocity is maximum (ωa). So $V_{max} = \omega a$. The Velocity is zero at the extreme positions.

3. Periodic time

Time taken for one complete oscillation is called as periodic time and is denoted by T.

Let t be increased by $\frac{2\pi}{\omega}$ in eq (8) then

$$x = a\sin\left[\omega\left[t + \frac{2\pi}{\omega}\right] + \phi\right]$$

$$= a\sin(\omega t + 2\pi + \phi) = a\sin(\omega t + \phi)$$

This shows that the displacement repeats itself after a time $\left[\frac{2\pi}{\omega}\right]$. Therefore, $\left[\frac{2\pi}{\omega}\right]$ is known as periodic time.

$$\therefore \; T = \left[\frac{2\pi}{\omega}\right]$$

$$\left\{\because \omega = \left[\left[\frac{d^2x}{dt^2}\right]\Big/ x\right]^{1/2}\right\}$$

$$\text{Now, } T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left[\frac{d^2x/dt^2}{x}\right]}}$$

$$= 2\pi\sqrt{\left[\frac{x}{(d^2x/dt^2)}\right]} = 2\pi\sqrt{\left[\frac{\text{displacement}}{\text{acceleration}}\right]}$$

4. Frequency

The number of oscillations made in one second is called as frequency and is denoted by n or V. Hence.

$$n \text{ or } V = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)} \quad ..... (3)$$

5. Phase

The angle $(\omega t + \phi)$ is called the phase of vibrations. Phase of a body executing S.H.M at any instant represent its state as regards its position and direction at that instant.

6. Epoch

The value of phase when t = 0 is called the phase or epoch. In our case ϕ is the epoch.

It is, therfore, termed as the length of equivalent of simple pendulum.

1. Centre of suspension

The point s of intersection of horizontal axis with the vertical plane passing through the centre of gravity is called the centre of suspension.

2. Centre of oscillation

The point O at a distance $\{l + (k^2/l)$ equal to the length of the equivalent simple pendulum from the point of suspension S, is called the centre of oscillation corresponding to centre of suspension S. It lies on the line joining S to the centre of gravity of the body and produced.

3. Conditions for maximum and minimum time period

The time period of a compound pendulum is

$$T = 2\pi\sqrt{\left(\frac{(k^2/\ell) + \ell}{g}\right)}$$

or

$$T^2 = \frac{4\pi^2(k^2 + \ell^2)}{lg} = \frac{4\pi^2}{g}\left(\frac{k^2}{\ell} + \ell\right)$$

Differentiating this expression with respect to *l*, we have

$$2T.\frac{dT}{d\ell} = \frac{4\pi^2}{g}\left(\frac{k^2}{\ell^2} + 1\right)$$

For T to be maximum or minimum we put $\frac{dT}{d\ell} = 0$ which gives $\ell = \pm k$. Further $\frac{d^2T}{d\ell^2}$ comes out to be positive for this value of *l*. This means that time period T is a minimum when $\ell = \pm k$ or we can state that when the distance between the centre of suspension and centre of mass is equal to the radius of gyration of the pendulum about an axis passing through its centre of mass and perpendicular to the plane of oscillation, the time period will be minimum.

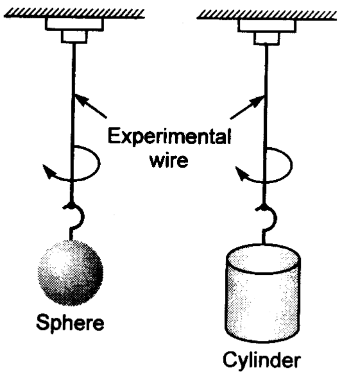
4.2 Torsional Pendulum

4.2.1 Measurements of Rigidity Modulus

Q5. Define torsion pendulum? How do you determine modulus of rigidity using torsion pendulum?

Ans : **(July-21)**

The torsional pendulam is shown in fig. It consists of heavy metal or cylinder suspended from a rigid support by means at experiment wire. When the sphere or cylinder is signty twisted in the horizontal plane and then released the pendulum starts torsional oscillations about the axis of suspension.



Theory

Let a sphere or cylinder at mass M be suspended at one end of a wire of length *l* and radius r keeping its other end fixed at a rigid support. this behaves like a torsional pendulum.

Let the pendulum slightly twisted in horizontal plane through an angle θ radian and then released. the pendulum starts executing torsional oscillations. Let *I* be the moment of inertia of cylinder or sphere about the axis of suspension within elastic limits. The couple or torque acting on the wire is proportional to angular displacement.

Therefore, T = Iα

Where angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$ and

inertial couple acting, T = $I\frac{d^2\theta}{dt^2}$

If C be the torsional rigidity of suspension wire (i.e., required to produce unit radian twist in the wire), then restoring couple (τ) required to produced θ radian is $-C\theta$.

In equilibrium $I\frac{d^2\theta}{dt^2} = -C\theta$

Therefore, the equation of motion of the pendulum will be $\frac{Id^2\theta}{dt^2} + C\theta = 0$ or $\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$

$\frac{d^2\theta}{dt^2} + \omega^2 = 0$ Where $\omega^2 = \frac{C}{I}$ ..... (2)

(or)

This is differential eq of simple harmonic motion whose time period T is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(\frac{C}{I}\right)}} = 2\pi\sqrt{\left(\frac{I}{C}\right)}$$

$$T = 2\pi\sqrt{\frac{I}{C}} \quad \text{....... (3)}$$

We know that torsional rigidity C of a wire is given by

$$C = \frac{\pi n r^4}{2\ell} \quad \text{....... (4)}$$

Where n is the modulus of rigidity material of wire and I is moment of inertia

In case of sphere I = $\frac{2}{4}MR^2$

Where M = mass of sphere and

R = radius of sphere

In case of cylinder I = $\frac{1}{2}MR^2$

Where M = mass of cylinder and

R = Radius of cylinder substituting the volume of C from eq (4) in eq (4) we get

$$T = 2\pi\sqrt{\left(\frac{1}{\pi n r^3 / 2\ell}\right)} = 2\pi\sqrt{\frac{2I\ell}{\pi n r^4}}$$

or $T^2 = \frac{8\pi^2 I\ell}{\pi n r^4} = \frac{8\pi I}{\pi r^4}$

Measurement of Rigidity modulus by Torsional Pendulum

The following procedure is adopted

(i) The sphere or the cylinder is suspended a rigid support with the help of experimental wire as shown in fit.

(ii) The sphere of cylinder is sightly roated about the wire and released so that it begins to execute tortional oscillations of small amplitude about wire as axis.

(iii) Start stop watch and simultaneously count the number of oscillations in the way find time period T.

$$T = \frac{\text{total number of oscillations}}{\text{total time taken}}$$

(iv) Measure the length *l* and radius of the wire. The radius of the wire is measured with the help of screw gauge and length *l* with help of meter scale.

(v) With the help of verneir calliper measure the radius R of sphere or cylinder.

(vi) Measure the mass M (in kg) of (cylinder or sphere) with help of physical balance calculate

$$I = \frac{2}{5} MR^2$$

$$I = \frac{1}{2} MR^2$$

Using the formula, $\eta = \frac{8\pi I\ell}{T^2 r^4}$ calculate the value of rigidity of the wire

Therefore $\eta = \frac{8\pi\ell}{T^2 r^4}\left(\frac{1}{2}MR^2\right) = \frac{4\pi MR^2 I}{T^2 r^4}$ for cylinder and $\eta = \frac{8\pi\ell}{T^2 r^4}\left(\frac{2}{5}MR^2\right) = \frac{16\pi MR^2 I}{5T^2 r^4}$ for sphere

4.3 Compound Pendulum

Q6. Explain Compound Pendulum?

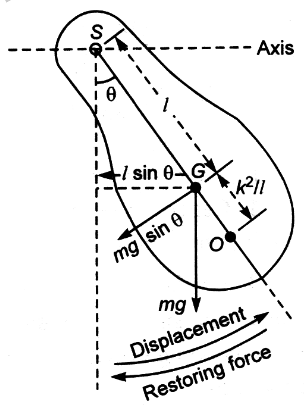
Ans :

Compound Pendulum

A compound pendulum is a rigid body capable of oscillating about a horizantal axis passing through it (not through its centre of gravity) in a vertical plane.

Let fig represent the vertical section of any irregular rigid body pivoted at a point S. In the equillibrium position of the body, the centre of mass lies vertically below S. Let m be the mass of the body and *l* the distance between the point of Suspension s and centre of gravituy G.

Let , at any instant t, the body be dispached through on angle θ. Now a restoring couple acts on the body to bring it in its mean position of rest. Due to intertia, it does not stop in the position of rest but swings to opposite side, i.e., the body executes simple hormonic motion.



Theory

The time period may be calculated as follows Restoring couple = weight × perpendicular distance of G from S.

τ = mg × $l\sin\theta$

τ = mg$l$ ($\because$ $\sin\theta = \theta$, when θ is small)

If I is the moment of inertia of the body about an axis through s perpendicular to the plane of oscillation and $\frac{d^2\theta}{dt^2}$ angular acceleration, then the torque acting will be

$$\tau = I\frac{d^2\theta}{dt^2}$$

$$I\frac{d^2\theta}{dt^2} = -mg\ell\theta$$

and thus

negative sign indicating that angular acceleration is always towards the position of rest. Then

$$\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\theta = -P^2\theta,$$

where,

$$\frac{mg\ell}{I} = P^2$$

This is the equation of simple harmonic motion whose time period T is given

$$\boxed{T = \frac{2\pi}{P} = 2\pi\sqrt{\frac{I}{mgl}}} \qquad ......(a)$$

If I_g be the moment of inertia of the body about centre of gravity, then from theorem of parallel axis

$I = I_g + ml^2$

$I = - mk^2 + ml^2$ ......(b)

or

where k is the radius of gyration about an axis through the centre of gravity

putting the value of I from eq(b) into (a)

$$T = 2\pi\sqrt{\left(\frac{mk^2 + m\ell^2}{mg\ell}\right)} = 2\pi\sqrt{\left(\frac{\frac{k^2}{\ell}+1}{g}\right)}$$

comparing the above time period with the periodic time of the simple pendulum $T = 2\pi\sqrt{\left(\frac{L}{g}\right)}$.

We note that

$$\boxed{L = \ell + \frac{k^2}{\ell}}$$

4.4 Combination of Two Mutually Perpendicular Simple Harmonic Vibrations of Same Frequency and Different Frequencies

Q7. Discuss the combination of two mutually simple harmonic vibrations of same frequencies with neat diagrams?

Ans : **(Imp.)**

Equal frequenties Perpendicular

Let us consider the case when two simple harmonic motion have the same frequency (or time period) one acting along the x-axis and the y-axis. Let the two vibrations be represented by

$x = a\sin(\omega t + \phi)$ ...... (1)

$y = b\sin\omega t$ ...... (2)

When a and b are the amplitudes of x and y vibrations respectively. The x motion is ahead of the y motion by angle ϕ i.e. the phase difference between the two vibrations is ϕ

The equation of resultant vibration can be obtained by climinating t between eqs (1) and (2).

From eq (2), we have $\sin\omega t = \left(\frac{y}{b}\right)$

$$\therefore \cos\omega t = \sqrt{(1-\sin^2\omega t)} = \sqrt{\left[1-\left(\frac{y^2}{b^2}\right)\right]}$$

Expanding eq (1) and substituting the values of $\sin\omega t$ and $\cos\omega t$, we get

$$\frac{x}{a} = \sin\omega t\cos\phi + \cos\omega t\sin\phi$$

(or) $$\frac{x}{a} = \frac{y}{b}\cos\phi + \sqrt{\left[1-\frac{y^2}{b^2}\right]}\sin\phi$$

(or) $$\frac{x}{a} = \frac{y}{b}\cos\phi = \sqrt{\left[1-\frac{y^2}{b^2}\right]}\sin\phi$$

Squaring both sides, we have

$$\left(\frac{x}{a}-\frac{y}{b}\cos\phi\right)^2 = \left(1-\frac{y^2}{b^2}\right)\sin^2\phi$$

(or) $$\frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\phi - \frac{2xy}{ab}\cos\phi$$

$$= \sin^2\phi - \frac{y^2}{b^2}\sin^2\phi$$

(or) $\frac{x^2}{a^2} + \frac{y^2}{b^2}(\cos^2\phi + \sin^2\phi) - \frac{2xy}{ab}\cos\phi$

$= \sin^2\phi$

(or) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi = \sin^2\phi$ ...(3)

The equation represents on oblique ellipse, which is the resultant path of the particle, Here we consider the following important cases:

(i) When $\phi = 0$ (two vibrations are in phase)

In this case, $\sin\phi = 0$ and $\cos\phi = 1$

The eq(3) becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$

(or) $\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$ or $\pm\left(\frac{x}{a} - \frac{y}{b}\right) = 0$

(or) $\pm y = \pm \frac{b}{a}x$ ...... (4)

This eq. represents two coincident straight lines passing through the origin and inclined to x-axis at the angle θ, given by

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

This is the resultant path of the particle as shown in fig [14(a)]

(ii) When $\phi = \frac{\pi}{4}$, we have

$\sin\phi = \frac{1}{\sqrt{2}}$ and $\cos\phi = \frac{1}{\sqrt{2}}$

Now eq (3) becomes

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\frac{1}{\sqrt{2}} = \frac{1}{2}$ ..... (5)

This represents an oblique ellipse, as shown in fig [14(b)]

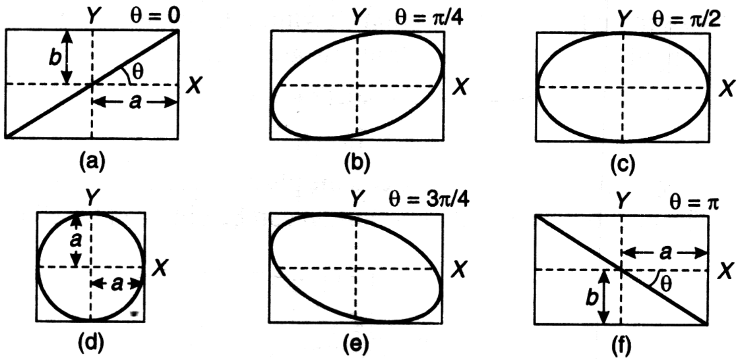
(iii) When $\phi = \frac{\pi}{2}$, we have

$\sin\phi = 1$ and $\cos\phi = 0$

the eq (3) reduces to

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...... (6)

The resultant path is an ellipse whose major axis coincides with the coordinate axis as shown in fig. If a = b, then $x^2 + y^2 = a^2$, so the resultant path of the particle is a circle of radius a as shown in fig.



(iv) When $\phi = \frac{3\pi}{4}$, we have

$$\sin\phi = \frac{1}{\sqrt{2}} \text{ and } \cos\phi = -\frac{1}{\sqrt{2}}$$

The eq (3) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \text{ ...... (7)}$$

This represents an oblique ellipse as shown in fig .

(v) When $\phi = \pi$, we have

$\sin\phi = 0$ and $\cos\phi = -1$

Now eq (3) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$$

(or) $$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$

(or) $$\pm\left(\frac{x}{a} + \frac{y}{b}\right) = 0$$

(or) $$\pm y = \pm \frac{b}{a}x \quad \text{...... (8)}$$

This again represents a pair of coincident straight lines passing through the origin and inclined to x-axis at an angle θ given by

$$\theta = \tan^{-1}\left(-\frac{b}{a}\right)$$

4.5 Lissajous Figures and Their Graphical Representations

Q8. Explain lissajous figures.

Ans :

The resultant path traced out by a particle when it is acted upon simultaneously by two simple harmonic motions at right angles to each other is known as Lissajous figure.

The nature of resultant path depends upon:

(1) The amplitude of vibrations

(2) The frequencies of two vibrations

(3) Phase difference between them

Graphical representation of Lissajous figures

Here we consider the method of combination of two rectangular simple harmonic motions of amplituded a and b graphically in the following cases :

(1) Same frequency and having a phase difference zero

(2) Same frequency but having a phase difference $\frac{\pi}{4}$

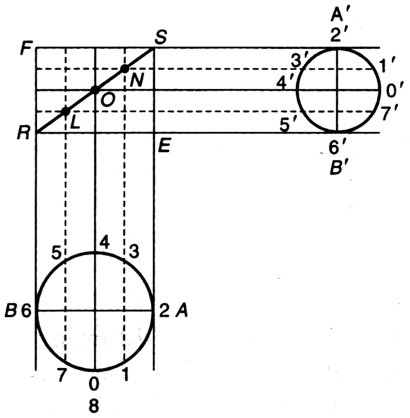
(3) Frequencies in the ratio 1 : 2 and phase difference zero

(4) Frequencies in the ratio 1 : 2 and phase difference $\frac{\pi}{2}$

(1) Same frequency and having a phase difference zero

(i) Draw two circles of reference of radii a and b equal to the amplitudes of the corresponding simple harmonic motions taking place along AB and A' B' axes respectively.

(ii) Divide these circles into equal parts, say for example in 8 parts, so each part corresponds to an angle $\frac{\pi}{4}$. Start numbering on the circle diameter AB from the position shown in fig. (16) while on the circle diameter A' B' from the position shown in fig. (16) because the two vibrations are perpendicular to each other and there is no phase difference between them.



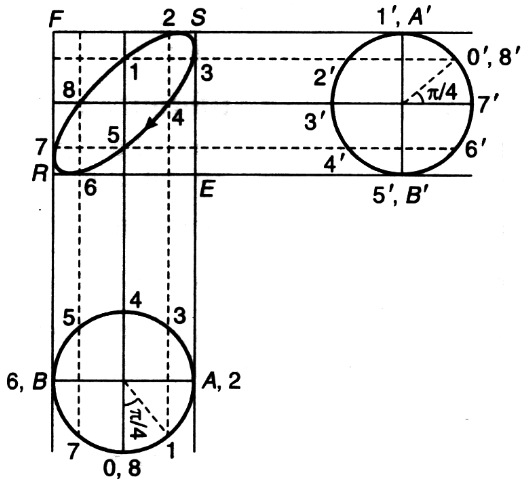
(iii) Draw lines through these points perpendicular to the line AB and A' B' respectively, so as to enclose a rectangle FRES.

(iv) At the start both perpendiculars intersect at O, the position initially taken up by the vibration particles. In the positions (1,1'), (2,2'), (3,3') they are found to intersect respectively at N, S, N and so on till after a complete time period the starting point O is reached again.

(v) The straight line RS then represents the resultant of the two simple harmonic vibrations.

(2) Same frequency but having a phase difference $\frac{\pi}{4}$

Consider two rectangular S.H.M.s taking place parallel to AB and A' B', the two lines mutually re-pendicular to each other. Draw two circles of reference of radii a and b equal to the amplitudes of the corresponding simple harmonic motions taking place along AB and A' B' axes respectively. Divide the circumference of each circle into 8 equal parts (As the periods are equal), and produce it as shown in the fig. (17) so as to enclose a rectangle FRES. Start numbering on the circle of diameter AB from the position P while on the circle of diameter A' B' from the position of angle $\frac{\pi}{4}$ because there is a phase difference of $\frac{\pi}{4}$ between them.



When the particle first starts from P, the second particle starts from 0'i.e., the position of angle $\frac{\pi}{4}$. The resultant position is marked by point 1 in the rectangle FRES.

Following are come uses of Lissajous figures :

(i) The ratio of the frequencies of two vibrating Systems can be obtained from their Lissajous figures provided the ratio is in whole number i.e., 1 : 1, 1 : 2, 1 : 3 and so on.

(ii) The Lissajous figures provide a good method for ajusting the frequencies of two forks to a given ratio.

(iii) Lissajous figures may be used to determine the frequency of a tuning fork provided the frequency of other tuning fork producing the figure is known are Commensurable i.e., in a whole number ratio.

(iv) These figures are useful in testing the accuracy of tuning of some simple intervals between two forks. This is possible because a slight mistuning causes the figure to vary in form.

(v) The figures may be employed to investigate how the period of a rod, fixed at one end, varies with the length of the rod.

(vi) Helmholtz used these figures to investigate the vibration of a violin string.

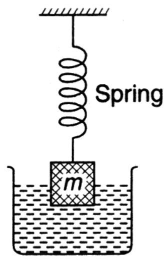
4.6 Damped Harmonic Oscillator, Solution of the Differential Equation of Mamped Oscillator

Q9. What are damped osccillations? Solve the differential Equation of damped harmonic oscillator ?

Ans : **(July-21)**

For an ideal hamonic oscillator, the amplitude of vibration remains constant for an infinite time. When a body vibrates in air or in any other medium which offers resistance to its motion, the amplitude of vibration decreases gradually and ultimately the body comes to rest. This is due to the fact that the body is subjected to frictional forces arising from air resistance. The motion of the body is known as damped simple harmonic motion. As an example, if we displace a pendulum from its equilibrium position it will oscillate with a decreasing amplitude and finalles come to rest in equilibrium position.

Let us consider another example of damped vibrations as shown in fig(1). Here a mass m is suspended from the spring and set to vibrate. It is observed that mass vibrates for a longer time in air as compared to the mass which vibrates partially in air and partially in liquid kept below the mass as shown in fig(1). The damping force is more when the mass moves in liquid and hence the vibrations die out more quickly in liquid than in aid.



Equation of damped hamonic oscillator

The damped system is subjected to :

(i) A restoring force which is proportional to displacement but oppositely directed. This is written as $-\mu x$ where μ is a constant of proportionality or force constant.

(ii) A frictional force proportional to velocity but oppositely directed. This may be written as

$-r\left[\frac{dx}{dt}\right]$. where r is the frictional force per unit velocity.

Since, force = Mass × Acceleration = $m\frac{d^2x}{dt^2}$

Therefore, the equation of motion of the particle is given by

$$m\frac{d^2x}{dt^2} = -\mu x - r\frac{dx}{dt}$$

$$\text{(or)} \quad \frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{\mu}{m}x = 0$$

$$\text{(or)} \quad \frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0 \text{ ...... (1)}$$

Where $\frac{r}{m} = 2b$ and $\frac{\mu}{m} = \omega^2$

Here b is damping constant or $\left[\frac{1}{b}\right]$ as decay modulus.

Equation (1) is a differential equation of damped harmonic motion.

Solution of the equation for various boundary conditions

Solution of the equation

Equation (1) is a differential equation of second degree. Let its solution be $x = Ae^{\alpha t}$ ....(2)

Where A and α are arbitrary constants.

Differentiating eq (2) with respect to t, we get

$$\frac{dx}{dy} = A\alpha e^{\alpha t} \text{ and } \frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$$

Substituting these values in eq(1), we have

$$A\alpha^2 e^{\alpha t} + 2bA\alpha e^{\alpha t} + w^2 Ae^{\alpha t} = 0$$

$$\text{or} \quad Ae^{\alpha t}(\alpha^2 + 2b\alpha + w^2) = 0$$

$$\text{As} \quad Ae^{\alpha t} \neq 0 \quad \therefore \alpha^2 + 2b\alpha + w^2 = 0$$

This give $\alpha = -b \pm \sqrt{(b^2 - w^2)}$

The general solution of eq(1) is given by

$$x = A_1 e^{\left[-b+\sqrt{(b^2-\omega^2)}\right]t} + A_2 e^{\left[-b-\sqrt{(b^2-\omega^2)}\right]t} \text{ ...... (3)}$$

Where A_1 and A_2 are arbitary constants

Dpending upon the relative values of b and ω, three cases are possible :

4.7 Energy Considerations, Comparison with UnDamped Haromic Oscillator

Q10. Discuss Energy consideration in damped harmonic motion?

Ans : **(July-21)**

Energy of damped harmonic oscillator

Whenever a system is set into oscillations, its motion is opposed by frictional (damping) force due to air resistance. The work done against these force is dissipated out in the form of heat. So the mechanical energy of the system continuously decreases with time and amplitude of oscillation gradually decays to zero. Here we shall obtain an expression for the energy dissipation from the oscillation.

The displacement of a damped harmonic oscillator at any time is given by

$$x = ae^{-bt}\sin\left[\sqrt{(\omega^2 - b^2)}t + \phi\right]$$

$$= ae^{-bt}\sin(\omega^t t + \phi).$$

Where $\omega^t = \sqrt{(\omega^2 - b^2)}$ .... (1)

The instantaneous velocity is given by

$$\left[\frac{dx}{dt}\right] = -abe^{-bt}\sin(\omega^t t + \phi)$$

$$+ \ a\,\omega^t e^{-bt}\cos(\omega^t t + \phi) \text{ .....(2)}$$

In practice, the damping is very small i.e. < < ω. Hence the term $-abe^{-bt}\sin(\omega^t t + \phi)$ can be neglected in comparison with the term $a\omega^t e^{-bt}\cos(\omega^t t + \phi)$. Now, we have

$$\left[\frac{dx}{dt}\right] = a\omega^t e^{-bt}\cos\left(\omega^t t+\phi\right) \quad ......\ (3)$$

The mechanical energy E of the oscillator is given by

E = Kinetic energy + potential energy

$$= \frac{1}{2}m\left[\frac{dx}{dt}\right]^2 + \frac{1}{2}\mu x^2 \quad ......\ (4)$$

Substituting the values of $\left[\frac{dx}{dt}\right]$ and x from (3) and (1) in (4)

We get

$$E = \frac{1}{2}ma^2\omega^{t2}e^{-2bt}\cos^2\left(\omega^t t+\phi\right)$$

$$+\frac{1}{2}\mu a^2 e^{-2bt}\sin^2\left(\omega^t t+\phi\right)$$

$$= \frac{1}{2}ma^2\left[\frac{\mu}{m}\right]e^{-2bt}\cos^2\left(\omega^t t+\phi\right)$$

$$+\frac{1}{2}\mu a^2 e^{-bt}\sin^2\left(\omega^t t+\phi\right)$$

$$\left[\because \omega^t = \sqrt{\left(\omega^2-b^2\right)} = \omega = \sqrt{\left(\frac{\mu}{m}\right)}\right]$$

$$= \frac{1}{2}a^2\mu e^{-2bt}\left[\cos^2\left(\omega^t t+\phi\right)+\sin^2\left(\omega^t t+\phi\right)\right]$$

$$= \frac{1}{2}a^2\omega e^{-2bt} = \frac{1}{2}a^2 m\omega^2 e^{-2bt}$$

$$E = \frac{1}{2}a^2\mu e^{-2bt} \quad ....\ (5)$$

This shows that the energy of oscillator decreases with time.

4.8 Logrithm Decrement, Relaxation Time and Quality Factor

Q11. Define & Explain Logarithmic decrement relaxation time & quality factor?

Ans :

Following are the three different methods of describing the damping of an oscillator.

1. Logarithmic decrement

Logartithmic decrement measures the rate at which the amplitude disc away. The amplitude of damped harmonic oscillator is given by amplitude = a.e^{-bt} at t = 0, amplitude a_0 = a.

Let a_1, a_2, a_3 .... be the amplitudes at time t = T, 2T, 3T ..... respectively where T = period of oscillation. Then

$a_1 = a.e^{-bT}$

$a_2 = a.\ e^{-b\ (2T)}$

$a_3 = a.e^{-b(3T)}$

From these equations , we get

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = ... = e^{bT} = e^{\lambda} \text{ (where } bT = \lambda)$$

λ is known as logarithmic decrement.

Taking the natural logarithm, we get,

$$\lambda = \log_e\frac{a_0}{a_1} = \log_e\frac{a_1}{a_2} = \log_e\frac{a_2}{a_3} = ...$$

Thus logarithm decrement is defined as the natural logarithm of the ratio between two successive maximum amplitudes which are separated by one period.

2. Relaxation Time

The relaxation time is defined as the time taken for the total mechanical energy to decay (1/c) of its original value.

The mechanical energy of damped harmonic oscillator is given by

$$E = \frac{1}{2}a^2.\mu.e^{-2bt}$$

Let $E = E_0$ when $t = 0$, $\therefore E_0 = \frac{1}{2}a^2.\mu$

Now, $E = E_0.e^{-2bt}$ ....(2)

Let T be the relaxation time i.e., $t = T$. $E = \frac{E_0}{e}$.

Making this substitution in eqn (2), we get.

$$\left(\frac{E_0}{e}\right) = E_0.e^{-2b\tau} \text{ or } e^{-1} = e^{-b\tau}$$

(or) $-1 = -2b\tau$

$$\boxed{\tau = \left(\frac{1}{2}b\right)}$$

From eqns (2) and (3), we get

$$\boxed{E = E_0.e^{-t/\tau}}$$

The expression of power dissipation, can be written as.

$$\boxed{P = \frac{E}{\tau}}$$

3. Quality factor

The quality factor is defined as 2π times the ratio of the energy stored in the system to the energy lost per period.

$$Q = 2\pi.\frac{\text{energy stored in system}}{\text{energy lost per period}}\ 2.\pi.\frac{E}{PT}$$

Where P is power dissipated and t is periodic time. We know that $P = \frac{E}{\tau}$, where τ is relaxation time. So,

$$Q = 2\pi\frac{E}{(E/\tau).T} = \frac{2\pi\tau}{T} = \omega T$$

$$\boxed{Q = \omega T}$$

Where $\omega = \left(\frac{2\pi}{T}\right)$ = angular frequency

It is clear form eq.(6) that the higher the value of Q, the higher would be the value of relaxation time. τ i.e., lower damping.

4.9 Forced Differential Equation of Forced Oscillator and its Solution

Q12. Explain forced vibrations?Obtain differential equation of forced oscillator & its solution?

Ans : **(Imp.)**

Forced vibrations can be defined as the vibrations in which the body vibrates with a frequency other than its natural frequency under the action of an external periodic force.

Theory of Forced Vibrations

1. Equation of forced vibrations

The force acted upon the particle are :

(i) A restoring force proportional to the displacement but oppositely directed, given by.

$$f_r \alpha - x \text{ (or) } f_r = -\mu x$$

where μ is known as the force constant.

(ii) A frictional force proportional to velocity but oppositely directed, given by

$$f_f \alpha - \frac{dx}{dt} \text{ (or) } f_f = -r.\frac{dx}{dt}$$

where r is frictional force per unit velocity, and

(iii) The external periodic force, represented by.

$f_e = F \sin Pt$

where F is the maximum value of this force and $\frac{P}{2\pi}$ is its frequency. So, the total force acting on the particle is given by

$$f_t = -\mu x - r.\frac{dx}{dt} + F\sin Pt$$

impressed periodic force is called driver and the body executing forced vibrations is called driven oscillation.

By Newton's second law of motion this must be equal to the product of mass m of the particle and its instantaneous acceleration.

i.e., $m.\frac{d^2x}{dt^2}$, hence

$$m.\frac{d^2x}{dt^2} = -\mu x - r.\frac{dx}{dt} + F \sin pt$$

$$m.\frac{d^2x}{dt^2} + r.\frac{dx}{dt} + \mu x = F \sin pt$$

$$\frac{d^2x}{dt^2} + \frac{r}{m}.\frac{dx}{dt} + \frac{\mu}{m}x = \frac{F}{m}\sin pt$$

$$\boxed{\frac{d^2x}{dt^2} + 2b.\frac{dx}{dt} + \omega^2.x = f \sin pt}$$

where $\frac{r}{m} = 2b$, $\frac{\mu}{m} = \omega^2$ and $\frac{F}{m} = f$

Eqn(1) is the differential eqn of the motion of the particle.

2. Solution of equation of forced oscillations

(Amplitude and phase of forced vibrations)

In this case, when the steady state is set up, the particle vibrates with the frequency of applied force, and not with its own natural frequency. The solution of differential eqn(2) must be of type.

$$x = A \sin(\rho t - \theta) \text{ .... (2)}$$

Where A is the steady amplitude of vibrations and θ is the angle which the displacement x logs behind the applied force F sinpt A and θ being orbitrary constant.

Differentiating eqn (2), we have

$$\frac{dx}{dt} = A.\rho \cos(\rho t - \theta)$$

and $$\frac{d^2x}{dt^2} = -A\rho^2 \sin(\rho t - \theta)$$

Substituting these values in eqn (1), we get

$$-A.P^2 \sin(Pt - \theta) + 3b\, A.\, P \cos(Pt - \theta)$$

$$+ \omega^2 A \sin(Pt - \theta)$$

$$= f \sin Pt = + \sin\{(Pt - \theta) + \theta\}$$

If this relation holds good for all values of t, the coefficients of sin (Pt –θ) and cos (Pt – θ) terms on both sides of this equation must be equal the coefficients of sin (Pt – θ) and cos (Pt – θ) on both sides, we have

$$A(\omega^2 - \rho^2) = f \cos\theta$$

and $$2b.\, A\, P = f.\sin\theta$$

Squaring eqn (3) and (4) and then adding, we get

$$A^2(\omega^2 - P^2)^2 + 4b^2.A^2P^2 = f^2$$

$$A^2\left[\left(\omega^2 - P^2\right)\right)^2 + 4b^2P^2\Big]$$

$$A = \frac{f}{\sqrt{\left(\omega^2 - P^2\right)^2 + 4b^2P^2]}}$$

While on dividing eqn (4) by eqn (3) we have

$$\tan\theta = \frac{2b.A.\rho}{A\left(\omega^2 - P^2\right)} = \frac{2.b.\rho}{\left(\omega^2 - P^2\right)}$$

(or) $$\boxed{\theta = \tan^{-1}\left(\frac{2b\rho}{\omega^2 - P^2}\right)}$$

Substituting the value of A from eqn(5) in eqn (2), we get.

$$x = \frac{f}{\sqrt{\left[\left(\omega^2 - P^2\right)^2 + 4b^2.P^2\right]}}\sin(\rho t - \theta)$$

Eqn(5) give the amplitude of forced vibration while eqn (6) its phase. Depending upon the relative values of P and ω, there cases are possible.

4.10 Amplitude Resonance & Velocity Resonance

Q13. Explain the terms amplitude resonance & velocity resonance?

Ans : **(Imp.)**

Amplitude Resonance

The amplitude of forced oscillations varies with the frequency of applied force and become maximum at a paritcular frequency. This phenomenon is known as amplitude resonance.

Condition of amplitude resonance

In case of forced vibrations, we have

$$A = \frac{f}{\sqrt{\left[\left(\omega^2-\rho^2\right)^2+4b^2P^2\right]}}$$

and $$\theta = \tan^{-1}\left[\frac{2b\rho}{\left(\omega^2-P^2\right)}\right]$$

The expression (1) shows that the amplitude varies with the frequency of the P. for a particular value of P. the amplitude becomes maximum. This phenomenon is known as amplitude resonance. The amplitude is maximum when

$$\sqrt{\left[\left(\omega^2-P^2\right)^2+4b^2P^2\right]} \text{ is minimum}$$

$$\frac{d}{d\rho}\left[\left(\omega^2-P^2\right)^2+4b^2P^2\right] = 0$$

$$2\left(\omega^2-P\right)\left(-2P\right)+4b^2\left(2P\right) = 0$$

$$\omega^2-P^2 = 2b^2$$

$$P = \sqrt{\left(\omega^2-2b^2\right)}$$

Thus the amplitude is maximum when the frequency $\frac{P}{2\pi}$ of the impressed force becomes $\frac{\sqrt{\left(\omega^2-2b^2\right)}}{2\pi}$. This is resonant frequency. This gives frequencies of the system both in presence of damping i.e., $\frac{\sqrt{\left\{\omega^2-2b^2\right\}}}{2\pi}$ and in the absence of damping i.e., $\frac{\omega}{2\pi}$. If the damping is small, then it can be neluted and the condition of maximum is reduced to P = w

Putting condition (3) in eq (1), we get

$$A_{max} = \frac{f}{\sqrt{\left\{\left(\omega^2-\omega^2+2b^2\right)^2+4b^2\left(\omega^2-2b^2\right)\right\}}}$$

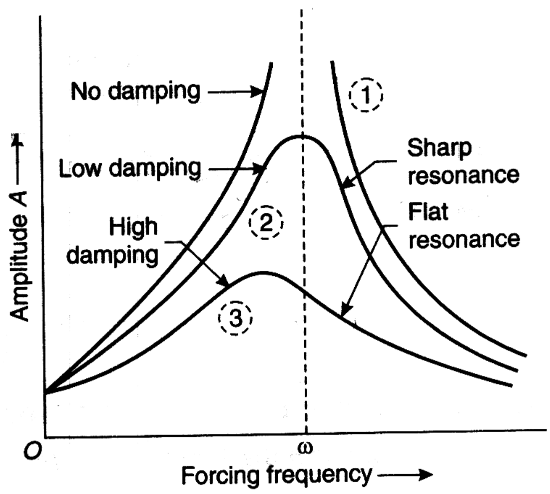
$$= \frac{f}{\left(4b^2\omega^2-4b^4\right)} = \frac{f}{2b\sqrt{\left(\omega^2-b^2\right)}}$$

$$= \frac{f}{2b\sqrt{\left(P^2+b^2\right)}} \quad \left[\because P^2 = \omega^2-2b^4\right]$$

and for law damping it reduce to

$$A_{max} \simeq \frac{f}{2bP}$$

$$A_{max} \to \infty \text{ as } b \to 0$$



Showing that

Shows the variation of amplitude with forcing frequency at different amounts of damping curve (1) shows the amplitude when there is no damping i.e., b = 0. In this case the amplitude in practise due to frictional resistance, as slight damping is always present, curve (2) and (3) show the effect of damping on the amplitude.

(ii) Velocity resonance [Energy resonance]

At very low during frequencies ($\rho << \omega$), the velocity amplitude is given by

$$u_{max} \simeq \frac{(F/m)P}{\omega^2} = \frac{(F/m)\rho}{(\mu/m)} \simeq \frac{FP}{\mu}$$

Thus, the amplitude is mainly governed by the force constant μ, when ($\rho << \omega$), then

$$\mu_{max} = \frac{f}{P} \simeq \frac{F}{mP}$$

In this case, the amplitude is mainly governed by the mass m, i.e., inertia factor.

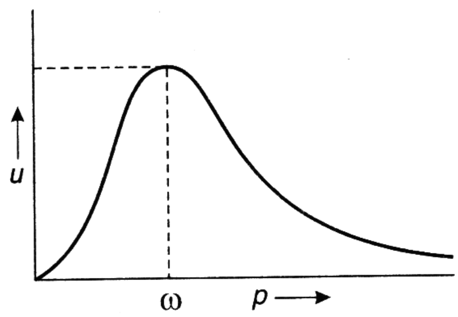
At frequecies comparable with natural frequency, the amplitude is maximum for a particular frequency. For the frequency

$$\mu_{max} = \frac{f}{\sqrt{\left[\left[\frac{\omega^2 - P^2}{P^2}\right]^2 + 4b^2\right]}}$$

This is maximum when denominator is minimum, i.e.,

$$\left(\frac{\omega^2 - P^2}{P}\right)^2 = 0 \text{ or } (\omega^2 - P^2) = 0 \text{ or } P = \omega$$

At this frequency of applied force, the velocity (or kinetic energy) of the oscillator is maximum. The phenomenon is known as velocity resonance. So when the druing frequency is equal to the natural undamped frequency of the oscillator, the velocity amplitude is maximum. Fig. shows the variation of velocity amplitude of forced oscillation with frequency of applied force.



PROBLEMS

1. When the body is in S.H.M. the time period is 0.001 sec, amplitude of vibration is 0.5 cm. Find the acceleration when it displaces to a distance of 0.2 cm from rest.

Sol :

We known that in S.H.M.,

$$a = \omega^2 y = \frac{4\pi^2}{T^2} y \text{ m/sec}^2$$

$$a = \frac{4 \times (3.14)^2}{(0.001)^2} \times 0.2. \ \frac{0.8 \times (3.14)^2}{(0.001)^2}$$

$$= 7.889 \times 10^6 \text{ m/sec}^2$$

2. A particle performing S.H.M. has a maximum velocity of 0.4 m/s and a maximum acceleration of 0.8 m/sec^2. Calculate the amplitude and the period of the oscillator.

Sol :

Given that (V) max = 0.4 m/s

and $(acc)_{max}$ = 0.8 m/sec^2

We know that (V) max = a ω

and (acc) max = aω^2

$$\therefore \frac{(acc)_{max}}{(V)_{max}} = \frac{a\omega^2}{a\omega} = \omega = \frac{2}{T}\pi$$

$$\text{or } T = 2\pi \times \frac{(V)_{max}}{(acc)_{max}} = 2\pi \times \frac{0.4}{0.8} = \pi \text{ sec}$$

or T = 3. 14 sec

From eq(1) $\omega = \frac{0.8}{0.4} = 2$

So, $(v)_{mass} = a\omega$ or 0.4 = a(2)

$\therefore$ a = 0.2 meter

3. A particle under S.H.M. has displacement of 0.4 at the velocity 0.3 m/s and a displacement 0.3 m at the velocity 0.4 m/s. Calculate amplitude and frequency of the oscillation.

Sol :

We know that, $V = \omega\sqrt{(a^2 - y^2)}$

Considering displacement of oscillation along y-axis.

Now, $0.3\omega\sqrt{\{a^2 - (0.3)^2\}} = \omega\sqrt{(a^2 - 0.016)}$

$0.4 = \omega\sqrt{\{a^2 - (0.3)^2\}} = \omega\sqrt{(a^2 - 0.09)}$

Dividing eq. (1) by eq. (2) we get

$\frac{3}{4} = \frac{\sqrt{(a^2 - 0.16)}}{\sqrt{(a^2 - 0.09)}}$ or $\frac{9}{16} = \frac{a^2 - 0.16}{a^2 - 0.09}$

or $16a^2 - 2.56 = 9a^2 - 0.81$ or $7a^2 = 1.75$

or $a^2 = \left[\frac{1.75}{7}\right] = 0.25$

$\therefore$ a = ± 0.5 m

Substituting the value of a in eq(2) we get

$0.4 = \omega\sqrt{(0.25 - 0.09)} = 0.4\ \omega$ or $\omega = 1$ and rad/sec or $2\pi n = 1$

$\therefore n = \frac{1}{2\pi}$ Hz

4. The displacement equation of a particle simple harmonic motion is x = 0.01 sin 50π(t + 0.007) metre. Calculate the amplitude time period, maximum velocity and initial phase for the particle

Sol :

Given x = 0.001 sin 50π (t + 0.007)

Standard equation of S.H.M. is

$x = a \sin(\omega t + \phi)$

Comparing the two equation we get

a = 0.01 metre, $\omega = 50\pi$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{50\pi} = \frac{1}{25}$ or T = 0.04 sec

Maximum velocity $V_{max} = \omega w$

$\therefore V_{max} = 0.01 \times 50\pi = \frac{1}{100} \times 50 \times 3.14$

= 1.57 m/sec

Initial plane $\phi = 50\pi \times 0.007$

$= 50 \times 3.14 \times 0.007$

$\phi = 1.099$ or 0.35π radian

5. A particle excutes S.H.M with a period of 0.002 sec and amplitude 10 cm. Find its acceleration when it is 4 cm away from it mean position and also obtain its maximum velocity.

Sol :

We know that $x = a\sin(\omega t + \phi)$

The velocity and acceleration are given by

$v = \frac{dx}{dt} = \omega a \cos(\omega t + \phi)$ and $V_{max} = \omega a$

acceleration $= \frac{d^2x}{dt^2} = -\omega^2 a \sin(\omega t + f)$

$= -\omega^2 x$

Here, $\omega = \frac{2\pi}{T} = \frac{2x3.14}{0.002} = \frac{2 \times 3.14}{0.002}$

$= 3.14 \times 10^3\ sec^{-1}$,

$x = 4$ cm and a = 10 cm

$\therefore$ acceleration $= -(3.14 \times 10^3)^2 \times 4$

$= -3.9 \times 10^7\ cm/sec^2$

$V_{max} = \omega a = 3.14 \times 10^3 \times 10$

$= 3.14 \times 10^4$ cm/sec

6. A particle of mass 5×10^{-13} kg executes S.H.M and amplitude of 0.08 m. Its frequency is 16 Hz. Find its maximum velocity and energy at mean position.

Sol :

In case of S.H.M $x = a \sin(\omega t + \phi)$

The maximum velocity $V_{max} = \omega a = 2\pi na$

$\therefore V_{max} = 2 \times 3.14 \times 16 \times 0.08 = 0.038$ m/sec

The energy at mean position is entirely kinetic

$$\therefore E = K_{max} = \frac{1}{2} M V_{max}^2$$

$$= \frac{1}{2} \times (5 \times 10^{-5} kg) \times (8.038)^2 = 0.16 \text{ joule}$$

7. The displacement of the particle executing S.H.M is given by x = 10 cos (4πt + π/3) metre. Find out the frequency and displacement after time 1 second.

Sol :

Give $x = 10 \cos(4\pi t + \pi/3)$

The equation of S.H.M. is given by

$x = a \cos(\omega t + \phi)$

Comparing eqs (1) and (2), we get

$\omega = 4\pi$ but $\omega = 2\pi n$

$\therefore 2\pi n = 4\pi$ or $n = 2$ Hz

$$\text{Displacement at 1 sec} = 10 \cos\left[4\pi \times 1 + \frac{\pi}{3}\right]$$

$$= 10 \cos\left[4\pi + \frac{\pi}{3}\right] = 10 \cos\left[\frac{13\pi}{3}\right]$$

8. A body of 0.5 kg mass is hung to spring and made to oscillate. For time t = 0, displacement is 0.44 m, acceleration is – 0.0176 m/sec. Find the force constant of spning

Sol :

$$\text{Time period of spning } T = 2\pi\sqrt{\left[\frac{m}{k}\right]}$$

$$\therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi\sqrt{\left[\frac{m}{k}\right]}} = \sqrt{\left[\frac{k}{m}\right]}$$

$$\text{or } \omega^2 = \frac{k}{m}$$

We know that acceleration, $a = \omega^2$

$$x = \frac{k}{m} \times x$$

$$k = \frac{ma}{x} = \frac{0.5 \times 0.0176}{0.44} = 0.2 \text{ N/m}$$

9. The length of a weightness spring increases by 2cm when a weight of 1.0 kg is suspended from it. The weight is pulled down by 10cm and released. Determine

(i) Period of oscillation of spring

(ii) P.E of oscillation of spring

Sol :

$$\text{We know that } K = \frac{F}{x} = \frac{1.0 \times 9.8}{0.02} = 490 \text{ N/m}$$

Now,

$$T = 2\pi\sqrt{\left(\frac{m}{k}\right)} = 2\pi\sqrt{\left(\frac{1.0}{490}\right)}$$

$$= \frac{2\pi}{7 \times \sqrt{(10)}} = \text{sec} = 0.29 \text{ sec}$$

$$P.E = \frac{1}{2}kx^2 = \frac{1}{2} \times 490 \times (0.1)^2 = 2.45 \text{ Joule}$$

10. A spring of force constant 20 N/m is hung vertically and loaded with a mass 0.1kg and allowed to oscillate calculated the time period and frequency of oscillation of the spring.

Sol :

The angular frequency

$$w = \sqrt{\left(\frac{k}{m}\right)} = \sqrt{\left(\frac{20}{0.1}\right)} = \sqrt{200}$$

$$= 14.14 \text{ red/sec}$$

We know that $w = \frac{2\pi}{T}$ or Time period

$$T = \frac{2\pi}{w}$$

$$\therefore \quad T = \frac{2 \times 3.14}{14.14} = 04414 \text{ sec}$$

and frequency,

$$n = \frac{1}{T} = \frac{1}{0.4414} = 2.265\text{Hz}$$

11. A spring is strenched by 8 cm by a factor of 10N. Find its force constant. What will be the frequency of a 4kg mass suspended from it?

Sol :

The frequency is given by $n = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)}$

where k = spring constant

$$F = kx \text{ (or) } k = \frac{F}{x} = \frac{10}{0.08} = 125 \text{ N/m}$$

We know that

x = 8cm = 0.08m and m = 4kg

$$\therefore \quad n = \frac{1}{2 \times 3.14} \times \sqrt{\frac{125}{4}} = 0.9\text{Hz}$$

12. Lissajous figures are produced with two tuning forks whose frequencies are approximately 2:1 complete cycle of changes of form takes 5 second. Fork of higher pitch is slightly loaded such that the period of cycle is raised to 10 second. If the frequency of lower fork is 500, find the frequency of the other fork before and after loading.

Sol :

Let n_1 and n_2 be the frequencies of higher and lower forks respectively. Before loading, the figures are repeated after 5 second. i.e., the higher fork makes one vibration more or less than double number made by lower fork.

Thus,

$$n_1 \times 5 \sim z(n_2 \times 5) \text{ (or) } n_1 \sim 2n_2 = \frac{1}{5}$$

$$\text{or } n_1 = 2n_2 \pm \left[\frac{1}{5}\right] = 2 \times 500 \pm \frac{1}{5}$$

$$= 1000.2 \text{ or } 999.8$$

$$\text{After loading } n_1 = 2n_2 \pm \frac{1}{10} = 5 \times 500 = \frac{1}{10}$$

$$= 1000.1 \text{ or } 999.9$$

As on loading, the frequency should decrease and hence frequency before loading = 1000.2 Hz and after loading 1000.1 Hz.

Short Question & Answers

1. Discuss the basic terms involved in oscillator, motion.

Ans :

Introduction

A motion which repeats itself after equal intervals of time is called periodic motion or harmonic motion.

A body or a particle is said possess oscillatory or vibratory motion if it moves back and forth repeatedly about the mean position.

Few terms regarding the oscillatory motion:

(i) Perodic time

The periodic time 'T' of an oscillatory motion is defined as the time taken for one oscillation.

(ii) Frequency

The 'frequency' n or v is defined as the number of oscillations in one second. It is reciprocal of periodic time. i.e., $n = 1/T$ cycles per second.

(iii) Displacement

The Distance of the particle in any direction from the equilibrium position at any instant is called the displacement of the particle at that instant.

(iv) Amplitude

The maximum displacement or the distance between the equilibrium position at and the extremeposition is known as amplitude 'a' of the oscillation.

(v) Phase

The phase of an oscillatory particle at any instant defines the state of the particle as regards its position and direction of motion at that instant.

(vi) Restoring force

In the equilibrium position of the oscillationg particle, no net force acts on it, when the particle is displaced from its equilibrium position, a periodic force acts on it in such a direction as to bring the particle to its equilibrium position. This is called the restoring force F.

2. Write Physical Characteristics of simple Harmonic motion?

Ans :

1. Displacement

The displacement of any yarticle ay any instant executing S.H.M. is given by

$$x = a \sin(wt + \phi)$$

The maximum displacement from the mean position is called amplitude. Here the amplitude is a

2. Velocity

The velocity v od the oscillating particle can be obtained by differentiating eq (8).

$$\text{Thus } V = \frac{dx}{dt}$$

$$= \omega a \cos(\omega t + \phi) = \omega\sqrt{(a^2 - x^2)} \quad ..... (1)$$

At the mean position i.e, at x = 0 , the velocity is maximum (ωa). So $V_{max} = \omega a$. The Velocity is zero at the extreme positions.

3. Periodic time

Time taken for one complete oscillation is called as periodic time and is denoted by T.

Let t be increased by $\frac{2\pi}{\omega}$ in eq (8) then

$$x = a \sin\left[\omega\left[t + \frac{2\pi}{\omega}\right] + \phi\right]$$

$$= a \sin(\omega t + 2\pi + \phi) = a \sin(\omega t + \phi)$$

This shows that the displacement repeats itself after a time $\left[\frac{2\pi}{\omega}\right]$. Therefore, $\left[\frac{2\pi}{\omega}\right]$ is known as periodic time.

$$\therefore \quad T = \left[\frac{2\pi}{\omega}\right]$$

$$\left\{\because \omega = \left[\left[\frac{d^2x}{dt^2}\right]\Big/x\right]^{1/2}\right\}$$

$$\text{Now, } T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left[\frac{d^2x/dt^2}{x}\right]}}$$

$$= 2\pi\sqrt{\left[\frac{x}{(d^2x/dt^2)}\right]} = 2\pi\sqrt{\left[\frac{\text{displacement}}{\text{acceleration}}\right]}$$

4. Frequency

The number of oscillations made in one second is called as frequency and is denoted by n or V. Hence.

$$n \text{ or } V = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)} \quad ..... (3)$$

5. Phase

The angle $(\omega t + \phi)$ is called the phase of vibrations. Phase of a body executing S.H.M at any instant represent its state as regards its position and direction at that instant.

6. Epoch

The value of phase when t = 0 is called the phase or epoch. In our case ϕ is the epoch.

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3. Explain lissajous figures.

Ans :

The resultant path traced out by a particle when it is acted upon simultaneously by two simple harmonic motions at right angles to each other is known as Lissajous figure.

The nature of resultant path depends upon:

(1) The amplitude of vibrations

(2) The frequencies of two vibrations

(3) Phase difference between them

Graphical representation of Lissajous figures

Here we consider the method of combination of two rectangular simple harmonic motions of amplituded a and b graphically in the following cases :

(1) Same frequency and having a phase difference zero

(2) Same frequency but having a phase difference $\frac{\pi}{4}$

(3) Frequencies in the ratio 1 : 2 and phase difference zero

(4) Frequencies in the ratio 1 : 2 and phase difference $\frac{\pi}{2}$

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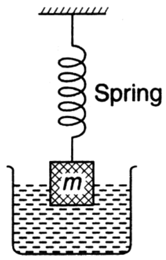
4. What are damped osccillations?

Ans :

For an ideal hamonic oscillator, the amplitude of vibration remains constant for an infinite time. When a body vibrates in air or in any other medium which offers resistance to its motion, the amplitude of vibration decreases gradually and ultimately the body comes to rest. This is due to the fact that the body is subjected to frictional forces arising from air resistance. The motion of the body is known as damped simple harmonic motion. As an example, if we displace a pendulum from its equilibrium position it will oscillate with a decreasing amplitude and finalles come to rest in equilibrium position.

Let us consider another example of damped vibrations as shown in fig(1). Here a mass m is suspended from the spring and set to vibrate. It is observed that mass vibrates for a longer time in air as compared to the mass which vibrates partially in air and partially in liquid kept below the mass as shown in fig(1). The damping force is more when the mass moves in liquid and hence the vibrations die out more quickly in liquid than in aid.



5. Explain the terms amplitude resonance & velocity resonance?

Ans :

Amplitude Resonance

The amplitude of forced oscillations varies with the frequency of applied force and become maximum at a paritcular frequency. This phenomenon is known as amplitude resonance.

Condition of amplitude resonance

In case of forced vibrations, we have

$$A = \frac{f}{\sqrt{\left[\left(\omega^2 - \rho^2\right)^2 + 4b^2P^2\right]}}$$

and $$\theta = \tan^{-1}\left[\frac{2b\rho}{\left(\omega^2 - P^2\right)}\right]$$

The expression (1) shows that the amplitude varies with the frequency of the P. for a particular value of P. the amplitude becomes maximum. This phenomenon is known as amplitude resonance.

6. Expain the term Velocity resonance [Energy resonance].

Ans :

At very low during frequencies ($\rho << \omega$), the velocity amplitude is given by

$$u_{max} \simeq \frac{(F/m)P}{\omega^2} = \frac{(F/m)\rho}{(\mu/m)} \simeq \frac{FP}{\mu}$$

Thus, the amplitude is mainly governed by the force constant μ, when ($\rho << \omega$), then

$$\mu_{max} = \frac{f}{P} \simeq \frac{F}{mP}$$

In this case, the amplitude is mainly governed by the mass m, i.e., inertia factor.

At frequecies comparable with natural frequency, the amplitude is maximum for a particular frequency. For the frequency

$$\mu_{max} = \frac{f}{\sqrt{\left[\left[\frac{\omega^2 - P^2}{P^2}\right]^2 + 4b^2\right]}}$$

This is maximum when denominator is minimum, i.e.,

$$\left(\frac{\omega^2 - P^2}{P}\right)^2 = 0 \text{ or } \left(\omega^2 - P^2\right) = 0 \text{ or } P = \omega$$

At this frequency of applied force, the velocity (or kinetic energy) of the oscillator is maximum.

7. Define Logarithmic decrement.

Ans :

Logartithmic decrement measures the rate at which the amplitude disc away. The amplitude of damped harmonic oscillator is given by amplitude $= a.e^{-bt}$ at $t = 0$, amplitude $a_0 = a$.

Let a_1, a_2, a_3 .... be the amplitudes at time t = T, 2T, 3T ..... respectively where T = period of oscillation. Then

$a_1 = a.e^{-bT}$

$a_2 = a.\ e^{-b\ (2T)}$

$a_3 = a.e^{-b(3T)}$

From these equations , we get

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = ... = e^{bT} = e^{\lambda}$$

(where $bT = \lambda$)

λ is known as logarithmic decrement.

Taking the natural logarithm, we get,

$$\lambda = \log_e \frac{a_0}{a_1} = \log_e \frac{a_1}{a_2} = \log_e \frac{a_2}{a_3} = ...$$

Thus logarithm decrement is defined as the natural logarithm of the ratio between two successive maximum amplitudes which are separated by one period.

8. Define term Relaxation Time.

Ans :

The relaxation time is defined as the time taken for the total mechanical energy to decay (1/c) of its original value.

The mechanical energy of damped harmonic oscillator is given by

$$E = \frac{1}{2} a^2.\mu.e^{-2bt}$$

Let $E = E_0$ when $t = 0$, $\therefore E_0 = \frac{1}{2}a^2.\mu$

Now, $E = E_0.e^{-2bt}$ ....(2)

Let T be the relaxation time i.e., $t = T$. $E = \frac{E_0}{e}$.

Making this substitution in eqn (2), we get.

$$\left(\frac{E_0}{e}\right) = E_0.e^{-2b\tau} \text{ or } e^{-1} = e^{-b\tau}$$

(or) $-1 = -2b\tau$

$$\boxed{\tau = \left(\frac{1}{2}b\right)}$$

From eqns (2) and (3), we get

$$\boxed{E = E_0.e^{-t/\tau}}$$

The expression of power dissipation, can be written as.

$$\boxed{P = \frac{E}{\tau}}$$

9. Quality factor.

Ans :

The quality factor is defined as 2π times the ratio of the energy stored in the system to the energy lost per period.

$$Q = 2\pi . \frac{\text{energy stored in system}}{\text{energy lost per period}} \quad 2.\pi.\frac{E}{PT}$$

Where P is power dissipated and t is periodic time. We know that $P = \frac{E}{\tau}$, where τ is relaxation time. So,

$$Q = 2\pi\frac{E}{(E/\tau).T} = \frac{2\pi\tau}{T} = \omega T$$

$$\boxed{Q = \omega T}$$

Where $\omega = \left(\frac{2\pi}{T}\right)$ = angular frequency

It is clear form eq.(6) that the higher the value of Q, the higher would be the value of relaxation time. τ i.e., lower damping.

10. Equation of Motion of a Simple Oscillator.

Ans :

Consider a particle P of mass m executing S.H.M about equilibrium position O along X-axis as shown in fig (2). By defination, the force under which the particle is oscillating is proportional to its displacement directed towards the mean position. Let x be the displacement of P from O at any instant. The instantaneous force Facting upon P is given by

$F \; \alpha -x$ or $F = -k\,x$, ... (1)

Where k is proportionality factor which represents the force per unit displacement. The negative sign is used to show that the force F is opposite to the displacement. The negative sign is used to show that the force F is opposite to the displacement.

According to Newton's second law of motion the restoring force on mass m produces as acceleration $\frac{d^2x}{dt^2}$ in the mass, so that

Force = mass × acceleration

i.e., $F = m\frac{d^2x}{dt^2}$ .... (1)

From eq (1) & (2)

$$m\frac{d^2x}{dt^2} = -kx \text{ or } \frac{d^2x}{dt^2} = \frac{-k}{m}x$$

Let us put $\frac{k}{m} = \omega^2$. Thus,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \text{ .... (3)}$$

This is known as the differential equation of simple harmonic oscillator.

Choose the Correct Answers

1. The potential energy of a particle executing SHM is given by [c]

   (a) $\frac{1}{4}kx^2$ (b) $\frac{1}{3}kx^2$

   (c) $\frac{1}{2}kx^2$ (d) $-kx$

2. For a particle executes simple harmonic motion, the phase difference is ______. [a]

   (a) $\frac{\pi}{2}$ (b) π

   (c) $\frac{\pi}{6}$ (d) $-\frac{\pi}{2}$

3. A particle executes SHM with a frequency 'f'. The frequency with which kinetic energy oscillates is ______. [b]

   (a) f (b) 2f

   (c) 4f (d) $\frac{f}{2}$

4. At what phase P.E and K.E are equal in case of SHM. [d]

   (a) 30^o (b) 60^o

   (c) 90^o (d) 45^o

5. If the length of seconds pendulum is increased by 2%. How many seconds it will lose per day. [a]

   (a) 864 sec (b) 3927 sec

   (c) 3429 sec (d) None of the above

6. For small amplitude oscillations the potential energy curve. [b]

   (a) circular (b) parabolic

   (c) elliptical (d) hyperbolic

7. In a simple harmonic motion the amplitude is 5cm and time period is 31.4 sec. The maximum velocity is (in cm/sec) [d]

   (a) 1.4 (b) 2

   (c) 2.4 (d) 1

8. In the expression $x = A\sin(wt + \theta)$; is x is ______. [b]

   (a) amplitude (b) displacement

   (c) phase (d) velocity

9. The length of pendulum which has a period 2.4 sec is [a]

   (a) 1.43 cm (b) 2.46 cm

   (c) 1.62 cm (d) 2.17 m

10. When the amplitude of a particle executing SHM decreases, its time period. [b]

   (a) decreases

   (b) remains unchanged

   (c) increases

   (d) may increase or decrease depending upon the phase

Fill in the Blanks

1. The phase differencebetween displacement and acceleration of SHO is ______.
2. MKS unit of spring constant is ______.
3. The energy possessed by a body a virtue of its position is known as ______.
4. A vibrating body is having maximum kinetic energy at ______.
5. The frequency of SHO is decreased by ______ force constant.
6. The time period of a second pendulum is ______.
7. If the point of suspension and point of oscillation are interchanged in a compound pendulum, then the time period ______.
8. The time period of Torsion pendulum is ______.
9. The period of SHO is ______.
10. The time period of a pendulum of infinite lengths ______.

Answers

1. π radian (or) 180^o
2. newton / meter
3. potential energy
4. mean position
5. mean position
6. 2 seconds
7. remains same
8. $T = 2\pi\sqrt{\frac{I}{m}}$
9. independent of amplitude
10. $2\pi\sqrt{\frac{R}{g}}$

FACULTY OF SCIENCE

B.Sc. I - Semester(CBCS) Examination

Subject : Physics

Paper-I : Mechanics and Oscillations

MODEL PAPER - I

Time : 3 Hours] [Max. Marks : 80

Part - A (8 × 4 = 32 Marks)

Note : Answer any Eight questions

ANSWERS

1. Show that $\vec{F} = (y^2 - x^2)\hat{i} + 2xy\hat{j}$ conservative. **(Unit-I, SQA-2)**
2. Curl of a Vector Field. **(Unit-I, SQA-4)**
3. Gauss's divergence theorem. **(Unit-I, SQA-6)**
4. Calculate the thrust on a rocket. **(Unit-II, SQA-7)**
5. Define Torques. prove that the rate of change of angular momentum is equal to torque. **(Unit-II, SQA-1)**
6. Inelastic collision. **(Unit-II, SQA-11)**
7. Length contraction. **(Unit-III, SQA-6)**
8. Calculate the work done to keep two balls having a mass 500 gm each from infinite distance to 10cm apart. **(Unit-III, SQA-4)**
9. What are inertial and non-inertial frames? **(Unit-III, SQA-2)**
10. What are damped osccillations? **(Unit-IV, SQA-4)**
11. Expain the term Velocity resonance (Energy resonance). **(Unit-IV, SQA-6)**
12. Quality factor. **(Unit-IV, Prob.9)**

Part - B (4 × 12 = 48 Marks)

Note : Answer all the questions

13. (a) Define gradient of a scalar field funct-ion. Explain the physical significance for the gradient of a scalar field. **(Unit-I, Q.No.15)**

OR

(b) What are line, surface and volume integrals? Explain. **(Unit-I, Q.No.19)**

14. (a) Derive the equation of motion of variable mass system. **(Unit-II, Q.No.6)**

OR

(b) Explain in detail about collisions in two and three dimensions. **(Unit-II, Q.No.15)**

15. (a) Describe Galilean Transformation. **(Unit-III, Q.No.13)**

OR

(b) Explain and write Lorentz Transformations. **(Unit-III, Q.No.16)**

16. (a) Define torsion pendulum? How do you determine modulus of rigidity using torsion pendulum? **(Unit-IV, Q.No.5)**

OR

(b) Explain the terms amplitude resonance & velocity resonance? **(Unit-IV, Q.No.13)**

FACULTY OF SCIENCE

B.Sc. I - Semester(CBCS) Examination

Subject : Physics

Paper-I : Mechanics and Oscillations

MODEL PAPER - II

Time : 3 Hours] [Max. Marks : 80

Part - A (8 × 4 = 32 Marks)

Note : Answer any Eight questions

		ANSWERS
1.	Prove that $\nabla.(A\times r) = r.(\nabla \times A)$	**(Unit-I, SQA-8)**
2.	What are different kinds of vectors?	**(Unit-I, SQA-12)**
3.	What are scalar and vector fileds?	**(Unit-I, SQA-7)**
4.	Distinguish between elastic and inelastic collisions.	**(Unit-II, SQA-9)**
5.	Derive Euler's equation for a rigid body.	**(Unit-II, SQA-8)**
6.	Moment of Inertia.	**(Unit-II, SQA-15)**
7.	Central Force.	**(Unit-III, SQA-1)**
8.	What is velocity of the particle if its KE is equal for rest energy?	**(Unit-III, SQA-3)**
9.	Define postulates of special theory of relativity.	**(Unit-III, SQA-10)**
10.	Explain lissajous figures.	**(Unit-IV, SQA-3)**
11.	Define Logarithmic decrement.	**(Unit-IV, SQA-7)**
12.	Equation of Motion of a Simple Oscillator.	**(Unit-IV, SQA-10)**

Part - B (4 × 12 = 48 Marks)

Note : Answer all the questions

13. (a) What is called divergence? Derive expression for divergence of a vector field. **(Unit-I, Q.No.16)**

OR

(b) State and prove Gauss's divergence theorem. **(Unit-I, Q.No.21)**

14. (a) Describe the principle of motion of a rocket as system of variable mass. **(Unit-II, Q.No.7)**

OR

(b) What is a symmetric top ? Derive an expression for the angular velocity of precession of a symmetric top. **(Unit-II, Q.No.21)**

15. (a) State and obtain kepler's law motion planetary. **(Unit-III, Q.No.10)**

OR

(b) What is length contraction? Obtain expression for length contraction. **(Unit-III, Q.No.18)**

16. (a) What are damped osccillations? Solve the differential Equation of damped harmonic oscillator ? **(Unit-IV, Q.No.9)**

OR

(b) Write Physical Characteristics of simple Harmonic motion? **(Unit-I, Q.No.4)**

FACULTY OF SCIENCE

B.Sc. I - Semester(CBCS) Examination

Subject : Physics

Paper-I : Mechanics and Oscillations

MODEL PAPER - III

Time : 3 Hours] [Max. Marks : 80

Part - A (8 × 4 = 32 Marks)

Note : Answer any Eight questions

ANSWERS

1. Define gradient of a scalars filed obtain an expression for it. **(Unit-I, SQA-1)**
2. Explain divergence of a vector field and its physical significance. **(Unit-I, SQA-3)**
3. Prove that Curl of a gradient is zero. **(Unit-I, SQA-5)**
4. Explain the working of multistage rocket. **(Unit-II, SQA-14)**
5. State and explain Newton's Law of Motion. **(Unit-II, SQA-2)**
6. The kinetic energy of a metal disc rotaing at a constant speed of 5 revolutions per second is 100 joule. Find the angular momentum of the disc. **(Unit-II, Prob.27)**
7. Time Dilation. **(Unit-III, SQA-5)**
8. A meter scale length is recorded as 96 cm by an observer. Find it's velocity. **(Unit-III, SQA-7)**
9. Kepler's laws of planetary motion. **(Unit-III, SQA-9)**
10. Write Physical Characteristics of simple Harmonic motion? **(Unit-IV, SQA-2)**
11. Explain the terms amplitude resonance & velocity resonance? **(Unit-IV, SQA-5)**
12. Discuss the basic terms involved in oscillator, motion. **(Unit-IV, SQA-1)**

Part - B (4 × 12 = 48 Marks)

Note : Answer all the questions

13. (a) What is curl of a vector field ? Obtain expression for curl of a vector field. **(Unit-I, Q.No.17)**

OR

(b) State and explain Green's Theorem applications. **(Unit-I, Q.No.22)**

14. (a) What are the various stages of the rocket(multistage rocket) in motion? **(Unit-II, Q.No.8)**

OR

(b) Derive Euler's equations of rotation of a rigid body about a fixed point. **(Unit-II, Q.No.32)**

15. (a) Define gravitational field and gravitational potential. Obtain Expression for gravitational potential due to a point mass. **(Unit-III, Q.No.7)**

OR

(b) Describe Michelson Morely experiment. What is its significance? **(Unit-III, Q.No.15)**

16. (a) Define simple harmonic motion? Write the Equation for simple harmonic oscillator? **(Unit-IV, Q.No.2)**

OR

(b) Discuss Energy consideration in damped harmonic motion? **(Unit-IV, Q.No.10)**

FACULTY OF SCIENCE

B.Sc. I - Semester Regular Examination

July / August - 2021

Subject : Physics

Paper-I : Mechanics and Oscillations

Time : 2 Hours] [Max. Marks : 80

Part - A (5 × 4 = 20 Marks)

Note : Answer any five questions.

ANSWERS

1. What are scalar and vector fields? Give examples. **(Unit-I, SQA-9)**
2. Define the curl of a vector field. **(Unit-I, SQA-4)**
3. If $A = 2 \times z^2 \hat{i} - y z \hat{j} + 3xz^2 \hat{k}$. Find $\nabla . A$ at (1, 1, 1). **(Unit-I, Prob.46)**
4. Distinguish between elastic and inelastic collisions. **(Unit-II, SQA-9)**
5. Explain working of Gyroscope. **(Unit-II, SQA-10)**
6. A 500 gm stone is revolved at the end of a 0.4 m long string at the rate of 12.5 rad/s. Find its angular momentum. **(Unit-II, Prob.34)**
7. State Keplers laws of planetary motion. **(Unit-III, SQA-9)**
8. State postulates of special theory of relativity. **(Unit-III, SQA-10)**
9. Find the velocity with which a body should travel so that the length becomes half of the rest length. **(Unit-III, Prob.24)**
10. What are the physical characteristics of simple harmonic motion? **(Unit-IV, SQA-2)**
11. What are Lissajous figures? Mention its applications. **(Unit-IV, SQA-3)**
12. A particle executing SHM has an acceleration of 0.5 m/s^2, when the displacement is 2m. Find its time period. **(Unit-IV, Prob.1)**

Part - B (3 × 20 = 60 Marks)

Note : Answer any three questions.

13. Explain the gradient of a scalar field. Derive the equation for it and write the significance. **(Unit-I, Q.No.15)**
14. Define divergence of a vector field. State and prove Gauss divergence theorem. **(Unit-I, Q.No.16,21)**
15. State Newton's laws of motion. Derive the equation of motion of a system of variable mass. **(Unit-II, Q.No.2,6)**

16. Explain the precessional motion of a symmetric Top. Obtain an expression for precisional velocity. **(Unit-II, Q.No.33)**

17. What are central forces? Mention its features. Show that conservative force is equivalent to negative gradient of potential energy. **(Unit-III, Q.No.1,3)**

18. Describe Michelson Moreley experiment and discuss its disadvantages. **(Unit-III, Q.No.15)**

19. Explain with necessary theory, how do you determine the rigidity modulus of a given wire using torsional pendulum. **(Unit-IV, Q.No.5)**

20. Derive the equation of motion of a damped harmonic oscillator. Deduce the solution and discuss the under damped and critically damped conditions. **(Unit-IV, Q.No.9,10)**

FACULTY OF SCIENCE

B.Sc. I - Semester(CBCS) Examination

November / December - 2019

Subject : Physics

Paper-I : Mechanics and Oscillations

Time : 3 Hours] [Max. Marks : 80

Part - A (8 × 4 = 32 Marks)

Note : Answer any Eight of the following questions

ANSWERS

1. What are scalar and vector fields? Give examples. **(Unit-I, SQA-9)**
2. Define curl of a vector and explain its significance. **(Unit-I, SQA-4)**
3. What are line, surface and volume integrals? **(Unit-I, SQA-10)**
4. Explain the working of multistage rocket. **(Unit-II, SQA-14)**
5. Prove the law of conservation of energy using Euler's equations. **(Unit-II, SQA-8)**
6. An α-particle with kinetic energy 6×10^{-14} J is scattered at an angle of 60° by coulomb field of stationary nucleus. Find the impact parameters. **(Unit-II, Prob.24)**
7. Distinguish between inertial and non-inertial frames. **(Unit-III, SQA-2)**
8. State Kepler's laws of motion. **(Unit-III, SQA-9)**
9. Assuming that earth is revolving around the Sun in circular orbit of radius 1.5×10^{11} m, estimate the mass of the sun. ($G = 6.67 \times 10^{-11}$ N-m^3/kg^2). **(Unit-III, SQA-16)**
10. Write the expression of Lorentz and Galilean transformation. **(Unit-III, SQA-13,17)**
11. Explain significance of four vector formalism. **(Unit-III, SQA-11)**
12. The total electrical energy generated in a station in a particular year was 7.5×10^{11} KWH. Find the mass equivalent of this energy. **(Unit-III, SQA-18)**

Part - B (4 × 12 = 48 Marks)

Note : Answer All the questions

13. (a) Explain line, surface and volume integrals in vector fields and explain their significance. **(Unit-I, Q.No.19)**

OR

(b) Explain the divergence of a vector field. State and prove Gauss-divergence theorem. **(Unit-I, Q.No.16,21)**

14. (a) Derive the equation of motion of system of variable mass. **(Unit-II, Q.No.6)**

OR

(b) Calculate the processional velocity of a symmetric top and show that $\tau = \omega_p \times L$ where τ is torque, ω_p is precessional velocity and L is angular momentum. **(Unit-II, Q.No.33)**

15. (a) What are central forces? Mention its characteristics. Show that the central force is equal to negative gradient of potential energy. **(Unit-III, Q.No.1,3)**

OR

(b) Define gravitational field and gravitational potential. Derive the equation of motion of a planet under inverse square law. **(Unit-III, Q.No.7)**

16. (a) State postulates of special theory of relativity. Explain length contraction and time dilation. **(Unit-III, Q.No.11,17,18)**

OR

(b) Describe Michelson-Morley experiment and explain the significance of negative result. **(Unit-III, Q.No.15)**

FACULTY OF SCIENCE

B.Sc. I - Semester(CBCS) Examination

May / June - 2019

MECHANICS

Time: 3 Hours] [Max. Marks: 80

PART – A (5 x 4 = 20 Marks)

Note: Answer any FIVE of the following questions

ANSWERS

1. If $\phi = (x^2 + y^2 + z^2)^{1/2}$ then find grad ϕ. **(Unit-I, SQA-11)**
2. Explain the terms gradient and divergence. **(Unit-I, SQA-1,3)**
3. What are the elastic and inelastic collisions? Give examples. **(Unit-II, SQA-11,12)**
4. Show that $\tau = I\alpha$ for a rotating rigid body. **(Unit-II, SQA-13)**
5. Prove that central forces are conservative forces. **(Unit-III, SQA-8)**
6. Prove that areal velocity is constant. **(Unit-III, SQA-14)**
7. Explain about addition of velocities. **(Unit-III, SQA-15)**
8. Discuss Lorentz transformations. **(Unit-III, SQA-13)**

PART – B (4 x 15 = 60 Marks)

Note: Answer ALL the questions

9. (a) What are the scalar and vector fields? State and prove Stoke's theorem. **(Unit-I, Q.No.14,21)**

OR

(b) State and prove Gauss divergence theorem. **(Unit-I, Q.No.21)**

10. (a) Derive the equation for final velocities of two particles in an elastic collision in two dimension. **(Unit-II, Q.No.15)**

OR

(b) Explain the precessional motion of a symmetric top. Obtain an expression for its precessional velocity. **(Unit-II, Q.No.33)**

11. (a) State and explain coriolis force. What are the consequences of coriolis force? **(Out of Syllabus)**

OR

(b) State Kepler's law. Prove Kepler's first law. **(Unit-III, Q.No.10)**

12. (a) Describe Michelson-Morley experiment and discuss the significance of negative result. **(Unit-III, Q.No.15)**

OR

(b) Derive Einstein mass-energy relation. Explain the verification of mass-energy relation. **(Unit-III, Q.No.20)**

FACULTY OF SCIENCE

B.Sc. I - Semester(CBCS) Examination

November / December - 2018

MECHANICS

Time: 3 Hours] [Max. Marks: 80

PART – A (5 x 4 = 20 Marks)

Note: Answer any FIVE of the following questions

ANSWERS

1. Define Scalar field. What is gradient of a scalar field? Explain its significance. **(Unit-I, SQA-9)**
2. Define line, surface and volume integrals. Give one example for each. **(Unit-I, SQA-10)**
3. Distinguish between elastic and inelastic collision. **(Unit-II, SQA-9)**
4. Explain the concept of impact parameter. **(Unit-II, SQA-3)**
5. Explain the working of Gyroscope. Mention few fields where Gyroscope use in mandatory. **(Unit-II, SQA-10)**
6. Define Kepler's laws of planetary motion. Give the significance of each law. **(Unit-III, SQA-9)**
7. Mention the postulates of special theory of relativity. **(Unit-III, SQA-10)**
8. Obtain an expression for mass-energy relation. **(Unit-III, SQA-14)**

PART – A (4 × 15 = 60 Marks)

Note: Answer any FIVE of the following questions

9. (a) Define divergence of a vector field. State and prove Gauss theorem. Mention one application of Gauss theorem. **(Unit-I, Q.No.16,24)**

OR

(b) (i) State and prove Stoke's theorem. **(Unit-I, Q.No.20)**

(ii) If r is a position vector, prove that $\vec{\nabla}.\vec{r} = 3$ **(Unit-I, Q.No.28)**

10. (a) Derive Euler's equations of a rigid rotating body. **(Unit-II, Q.No.32)**

OR

(b) Derive an expression for Rutherford scattering cross section. **(Unit-II, Q.No.18)**

11. (a) What is coriolis force? Obtain an expression for freely falling body due to coriolis force. **(Out of Syllabus)**

OR

(b) Derive Kepler's First law of planetary motion. **(Unit-III, Q.No.9)**

12. (a) Describe Michelson-Morley experiment and discuss the negative result. **(Unit-III, Q.No.15)**

OR

(b) Explain the concept of four vectors. Explain length contraction. **(Unit-III, Q.No.21,18)**

FACULTY OF SCIENCE

B.Sc. I - Semester(CBCS) Examination

May / June - 2018

Subject : Physics

Paper-I : (Mechanics)

Time : 3 Hours] [Max. Marks : 80

Part - A (5 × 4 = 20 Marks)

Note : Answer any Five of the following questions

ANSWERS

1. State Green's and Gauss's Theorems. **(Unit-I, SQA-6,7)**
2. Prove that $\nabla.(A \times r) = r.(\nabla \times A)$. **(Unit-I, SQA-8)**
3. Calculate the Thrust on a rocket. **(Unit-II, SQA-7)**
4. Obtain Euler's equations of a rigid body rotating about a fixed point. **(Unit-II, SQA-8)**
5. What are central forces? Write their characteristics. **(Unit-III, SQA-1)**
6. Define gravitational field and give two examples. **(Unit-III, SQA-12)**
7. State Einstein's postulates and obtain Lorentz transformation equations for position and time. **(Unit-III, SQA-13)**
8. Explain the concept of four vector. **(Unit-III, SQA-11)**

Part - B (4 × 15 = 60 Marks)

Note : Attempt All the questions

9. (a) Prove that Curl Curl A = Grad div A- ∇^2A. **(Unit-I, Q.No.38)**

OR

(b) Explain the curl of a vector field. State and prove Stoke's theorem. **(Unit-I, Q.No.17,20)**

10. (a) Obtain an expression for precessional velocity of the symmetric top. Explain the principle and working of Gyroscope. **(Unit-II, Q.No.33,34)**

OR

(b) Define impact parameter. Derive an expression for Rutherford scattering cross-section. **(Unit-II, Q.No.16,18)**

11. (a) Show that the central forces are conservative. State Kepler's third law from inverse square law of gravitation. **(Unit-III, Q.No.2,9)**

OR

(b) What is Coriolis force and obtain its expression? Derive Kepler's third law from inverse-square law of gravitation. **(Out of Syllabus)**

12. (a) Define inertial and non-inertial frames. Explain the concept of time dilation with example. **(Unit-III, Q.No.17)**

OR

(b) Describe Michelson-Morley experiment. Discuss the importance of negative result. **(Unit-III, Q.No.15)**

FACULTY OF SCIENCE

B.Sc. I - Semester(CBCS) Examination

November / December - 2017

Subject : Physics

Paper-I : (Mechanics)

Time : 3 Hours] [Max. Marks : 80

Part - A (5 × 4 = 20 Marks)

Note : Answer any Five of the following questions

ANSWERS

1. Define Gradient, Divergence and Curl. Give examples to each. What are their physical significance. **(Unit-I, SQA-1,3,4)**
2. Prove that Curl of a gradient is zero. **(Unit-I, SQA-5)**
3. Describe the principle of motion of a rocket as a system of variable mass. **(Unit-II, SQA-6)**
4. Define impact parameter and scattering cross section. **(Unit-II, SQA-3,4)**
5. Are central forces are conservative? Give two examples of central forces. **(Unit-III, SQA-8)**
6. State and explain Kepler's laws of planetary motion. **(Unit-III, SQA-9)**
7. Mention the postulates of special theory of relativity. **(Unit-III, SQA-10)**
8. Explain the concept of four vector formalism. **(Unit-III, SQA-11)**

Part - B (4 × 15 = 60 Marks)

Note : Attempt All the questions

9. (a) Define surface and volume integral. State and prove Gauss's divergence theorem. **(Unit-I, Q.No.19,21)**

OR

(b) Define Green's theorem. Give the proof of Green's theorem. **(Unit-I, Q.No.22)**

10. (a) Define elastic and inelastic collisions. Give the theory of elastic collisions in two dimensions. **(Unit-II, Q.No.13,15)**

OR

(b) What is a symmetric top? Explain the precession of top and obtain an expression for precession velocity, of symmetric top. **(Unit-II, Q.No.33)**

11. (a) Show that conservative force as a negative gradient of potential energy. **(Unit-II, Q.No.3)**

(b) What is Coriolis force and obtain its expression? **(Out of Syllabus)**

OR

(c) Derive Kepler's second law and third law of planetary motion. **(Unit-III, Q.No.9,10)**

12. (a) Describe the working of Michelson-Morley experiment and derive the expression for the fringe shift. **(Unit-III, Q.No.15)**

OR

(b) What is length contraction? Obtain expression for length contraction. **(Unit-III, Q.No.18)**

(c) Explain the concept of time dilation. **(Unit-III, Q.No.17)**

FACULTY OF SCIENCE

B.Sc. I - Semester(CBCS) Examination

May / June - 2017

MECHANICS

Time: 3 Hours] **[Max. Marks: 80**

PART – A (5 x 4 = 20 Marks)

Note: Answer any FIVE of the following questions

ANSWERS

1. Explain the Divergence of vector field and its physical significance. **(Unit-I, SQA-3)**
2. Show that $\vec{F} = \left(y^2 - x^2\right)\hat{i} + 2xy\,\hat{j}$ conservative **(Unit-I, SQA-2)**
3. Define : (i) Impact parameter (ii) Scattering cross section **(Unit-II, SQA-3,4)**
4. Describe Gyroscope. **(Unit-II, SQA-5)**
5. Calculate the work done to keep two balls having a mass 500 gm each from infinite distance to 10cm apart. **(Unit-III, SQA-4)**
6. Explain the coriolis force. **(Out of Syllabus)**
7. Explain : (i) Time dilation (ii) Length contraction **(Unit-III, SQA-5,6)**
8. A meter scale length is recorded as 96 cm by an observer. Find his velocity. **(Unit-III, SQA-7)**

PART – B (4 x 15 = 60 Marks)

Note: Attempt ALL the questions.

9. (a) State and prove Gauss divergence theorem. **(Unit-I, Q.No.21)**

OR

(b) State and prove Green's theorem. **(Unit-I, Q.No.22)**

10. (a) Discuss elastic collision in two dimensions **(Unit-II, Q.No.15)**

OR

(b) Obtain Euler's equation. Prove the law of conservation of energy from them. **(Unit-II, Q.No.32,10)**

11. (a) What are central forces? Show that the central force is equal to negative gradient of potential energy. **(Unit-III, Q.No.1,3)**

OR

(b) State and prove Kepler's law of planetary motion **(Unit-III, Q.No.10)**

12\. (a) Describe Michelson Morley experiment and explain Physical significance of negative result. **(Unit-III, Q.No.15)**

OR

(b) What are the Frames of reference? Explain the Galilean transformation. **(Unit-III, Q.No.14,13)**

FACULTY OF SCIENCE

B.Sc. I - Semester(CBCS) Examination

December - 2016

MECHANICS

Time: 3 Hours] **[Max. Marks: 80**

PART – A (5 x 4 = 20 Marks)

Note: Answer any FIVE of the following questions

ANSWERS

1. Define gradient of a scalar field. Obtain an expression for it. **(Unit-I, SQA-1)**
2. Prove that $\vec{F} = a\left(x\hat{i} + y\hat{j}\right)$ is a conservative force. **(Unit-I, SQA-2)**
3. Define Torque. Prove that the rate of change of angular momentum is equal to Torque. **(Unit-II, SQA-1)**
4. State Newton's Laws of motions. **(Unit-II, SQA-2)**
5. What are the central forces? **(Unit-III, SQA-1)**
6. Explain the coriolis force. **(Out of Syllabus)**
7. What are intertial and non-inertial frames. Give examples. **(Unit-III, SQA-2)**
8. What is velocity of the particle if its KE is equal to rest energy **(Unit-III, SQA-3)**

PART – B (4 x 15 = 60 Marks)

Note: Attempt ALL the questions.

9. (a) State and prove stokes theorem. **(Unit-I, Q.No.20)**

OR

(b) State and prove Green's theorem. **(Unit-I, Q.No.22)**

10. (a) Derive the equations of motion of system of variable mass, **(Unit-II, Q.No.6)**

OR

(b) Obtain an expression for angular momentum of a rigid body rotating about a fixed axis. A wheel is rotating with 500 revolutions per minuteb about an axis. Another similar wheel which is at rest is added to the axis of first wheel and if the both wheels rotate with uniform velocity find their uniform velocity **(Unit-II, Q.No.29)**

11. (a) Obtain the equation of motion of a particle moving under the influence of central force. Find the central force due to potential energy function $U = -Kr^{2.}$ **(Unit-III, Q.No.4,6)**

OR

(b) State and obtain Kepler's laws of planetary motion. **(Unit-III, Q.No.10)**

12. (a) Describe Michelson Morley experiment. What is its significance? **(Unit-III, Q.No.15)**

OR

(b) Explain the Lorentz transformations. **(Unit-III, Q.No.16)**