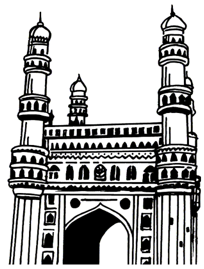
*Rahul's* ✔
*Topper's Voice*



# M.C.A.

## I Year II Sem

## *(Osmania University)*

Latest **2024** Edition

# OPERATIONS RESEARCH

- **Study Manual**
- **Important Questions**
- **Solved Model Papers**
- **Previous Question Papers**

Price ` . 199-00

- by -

WELL EXPERIENCED LECTURER

*Rahul Publications*™
**Hyderabad. Cell : 9391018098, 9505799122.**

# M.C.A.

## I Year II Sem

# OPERATIONS RESEARCH

*Inspite of many efforts taken to present this book without errors, some errors might have crept in. Therefore we do not take any legal responsibility for such errors and omissions. However, if they are brought to our notice, they will be corrected in the next edition.*



***Price `. 199/-***

***Sole Distributors :*** **Cell : 9391018098, 9505799122**

VASU BOOK CENTRE

**Shop No. 2, Beside Gokul Chat, Koti, Hyderabad.**

**Maternity Hospital Opp. Lane, Narayan Naik Complex, Koti, Hyderabad.**

**Near Andhra Bank, Subway, Sultan Bazar, Koti, Hyderabad -195.**

CONTENTS

# OPERATIONS RESEARCH

## STUDY MANUAL



## SOLVED MODEL PAPERS



## PREVIOUS QUESTION PAPERS

# FACULTY OF INFORMATICS

## M.C.A II - Semester (CBCS) Examinations,

## Noverber - 2020

# OPERATION RESEARCH

**Time : 3 Hours ]** **[Max. Marks : 70**

**Note : Answer any Four questions.**

1. Use simplex method to solve the LPP

Maximize $Z = 4x + 10y$

Subject to $2x + y \leq 50$

$2x + 5y \leq 100$

$2x + 3y \leq 90$

$x, y \geq 0$

2. Use dual simplex method to solve the following LPP

Maximize $Z = -2x - 2y - 4w$

Subject to $2x + 3y + 5w \geq 2$

$3x + y + 7w \leq 3$

$x + 4y + 6z \leq 5$

$x, y, z > 0$

3. Solve the following transportation problem for optimality

To

| | | D1 | D2 | D3 | D4 | D5 | Supply |
|---|---|---|---|---|---|---|---|
| From | F1 | 18 | 12 | 6 | 7 | 9 | 88 |
| | F2 | 6 | 4 | 10 | 16 | 12 | 22 |
| | F3 | 8 | 7 | 9 | 15 | 12 | 40 |
| | Demand | 20 | 30 | 40 | 25 | 35 | |

4. (a) Write about' north west corner rule method.

(b) Obtain initial feasible solution of following transportation problem.

| | D1 | D2 | D3 | D4 | Total |
|---|---|---|---|---|---|
| O1 | 1 | 2 | 1 | 4 | 30 |
| O2 | 3 | 3 | 2 | 1 | 50 |
| O3 | 4 | 2 | 5 | 9 | 20 |
| Total | 20 | 40 | 30 | 10 | 100 |

5. Find the optimal solution to the assignment problem.

| | A | B | C | D | E | |
|---|---|---|---|---|---|---|
| 1 | 52 | 58 | 58 | 53 | 54 | |
| 2 | 52 | 50 | 55 | 60 | 60 | |
| 3 | 55 | 56 | 57 | 58 | 59 | Operators |
| 4 | 56 | 51 | 51 | 56 | 59 | |
| 5 | 50 | 52 | 53 | 56 | 59 | |

6. Find the optimal integer solution to the following linear programming problem.

Maximize $z = x + 2y$

Subject to $2x \leq 7$

$x + y \leq 7$

$2x \leq 11$

$x, y \geq 0$ and are integers

7. Solve the LPP using dynamic programming

Maximize $z = 2x + 5y$

Subject to $2x + y \leq 430$

$2y \leq 440$

$x, y \geq 0$

8. Define :

(a) Principle of optimality

(b) Recursive function

(c) Stages

(d) State variable

(e) Forward recursion

9. Solve the following game using dominance property.

B

| | I | II | III | IV | |
|---|---|---|---|---|---|
| I | 3 | 2 | 4 | 0 | |
| II | 3 | 4 | 2 | 4 | A |
| III | 4 | 2 | 4 | 0 | |
| IV | 0 | 4 | 0 | 8 | |

10. Solve the following game graphically.

B

| | | | | |
|---|---|---|---|---|
| A | –6 | 0 | 6 | –3 / 2 |
| | 7 | 3 | –8 | 2 |

# FACULTY OF INFORMATICS

## M.C.A III - Semester (CBCS) Examination

## February - 2021

# OPERATIONS RESEARCH

**Time : 2 Hours]** **[Max. Marks : 70**

**Note : Answer any four Questions.**

1. (a) Define artificial variable.

   (b) Use big M-method to solve the following LPP.

   Maximum $Z = 5x_1 + 3x_2$

   S.T.C. $2x_1 + 4x_2 \leq 12$

   $2x_1 + 2x_2 = 10$

   $x_1, x_2 \geq 0.$

2. (a) State any two merits of duality.

   (b) Solve the following dual simplex method.

   Maximum $Z = 2x_1 + x_2$

   S.T.C $-3x_1 + x_2 \geq 3$

   $4x_1 + 3x_2 \geq 6$

   $x_1, x_2 \geq 0$

3. (a) What is degeneracy in transportation problem?

   (b) Solve by using North-West corner rule.

| | $W_1$ | $W_2$ | $W_3$ | $W_4$ | Capacity |
|---|---|---|---|---|---|
| $F_1$ | 19 | 30 | 50 | 10 | 7 |
| $F_2$ | 70 | 30 | 40 | 60 | 9 |
| $F_3$ | 40 | 8 | 70 | 20 | 18 |
| Requirement | 5 | 8 | 7 | 14 | |

4. (a) Write the steps for Vogel's approximation method.

   (b) Solve by VAM method.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Capacity |
|---|---|---|---|---|---|
| $O_1$ | 5 | 3 | 6 | 2 | 19 |
| $O_2$ | 4 | 7 | 9 | 1 | 37 |
| $O_3$ | 3 | 4 | 7 | 5 | 34 |
| Requirement | 16 | 18 | 31 | 25 | |

5. (a) Write steps for solving assignment problem.

   (b) Solve the following assignment problem

| | I | II | III | IV |
|---|---|---|---|---|
| A | 12 | 30 | 21 | 15 |
| B | 18 | 33 | 9 | 31 |
| C | 44 | 25 | 24 | 21 |
| D | 23 | 30 | 28 | 14 |

6. (a) Write Short notes on cutting plane algorithm.

   (b) Solve the following integer programming problem.

   Maximum $Z = x_1 + 2x_2$

   S.T.C. $2x_2 \leq 7$

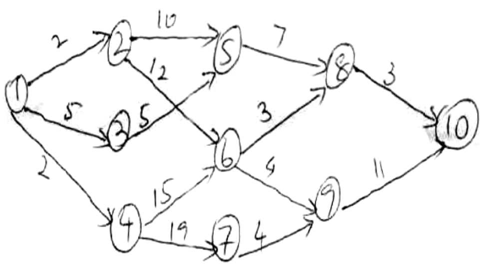
   $x_1 + x_2 \leq 7$

   $x_1, x_2 \geq 0$ and integers

7. (a) Write recursive equation for DPP.

   (b) Explain the applications of DPP.

8. Using dynamic programming find the shortest path from city 1 to city 10.



9. (a) Define saddle point and optimal strategy.

   (b) Solve the following game

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

10. (a) Define payoff matrix and value of the game.

    (b) Solve the following $2 \times n$ game by using graphical method.

B

A

| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 3 | 11 |
| 2 | 8 | 5 | 2 |

# FACULTY OF INFORMATICS

## M.C.A II-Semester (CBCS) Examination

## December - 2021

# OPERATIONS RESEARCH

**Time : 2 Hours]** **[Max. Marks : 70**

**PART - A - (4 × 17$^{1/2}$ = 70 Marks)**

**Note : Answer any four Questions.**

1. (a) Define slack and surplus variables.

   (b) Use two phase simplex methods to solve the following

   $\text{Min}z = 15 + 2x_1 - 3x_2$

   S.T.C $3x_1 - x_2 - x_3 \geq 3$

   $x_1 - x_2 + x_3 \geq 2$

   $x_1 x_2 x_3 \geq 0$

2. (a) Explain dual simplex method.

   (b) Solve the following LPP using Duality.

   $\text{Max } z = 2x_1 + 3x_2$

   S.T.C $x_1 + 2x_2 \leq 4$

   $x_1 + x_2 + \leq 6$

   $x_1 + 3x_2 \leq 9$

   $x_1 x_2 \geq 0$

3. (a) Write the steps for vogels approximation method.

   (b) Find IBFS by using VAM.

| | D1 | D2 | D3 | D4 | Supply |
|---|---|---|---|---|---|
| 01 | 10 | 11 | 11 | 15 | 110 |
| 02 | 8 | 16 | 19 | 20 | 90 |
| 03 | 6 | 10 | 18 | 10 | 48 |
| 04 | 16 | 4 | 8 | 9 | 152 |
| Demand | 73 | 89 | 95 | 143 | |

4. (a) What is Initial Basic feasible Solution.

   (b) Solve the following transportation problem.

| | D1 | D2 | D3 | D4 | Source |
|---|---|---|---|---|---|
| 01 | 1 | 2 | 1 | 4 | 30 |
| 02 | 3 | 3 | 2 | 1 | 50 |
| 03 | 4 | 2 | 5 | 9 | 20 |
| Destination | 20 | 40 | 30 | 10 | |

5. (a) Explain Hungarian method.

   (b) Solve the given assignment problem.

| Operators | Machines | | | | | |
|---|---|---|---|---|---|---|
| | | A | B | C | D | E |
| | P | 52 | 58 | 58 | 53 | 54 |
| | Q | 52 | 50 | 55 | 60 | 60 |
| | R | 55 | 56 | 57 | 58 | 59 |
| | S | 56 | 51 | 51 | 56 | 59 |
| | T | 50 | 52 | 52 | 56 | 59 |

6. (a) Define mixed integer Programming problem.

   (b) Solve the following problem by using branch and bound method

   max $z = 3x_1 + 5x_2$

   S.T.C. $2x_1 + 4x_2 \leq 25$

   $x_1 \leq 8$

   $2x_2 \leq 10$

   $x_1, x_2 \geq 0$ And Integers.

7. (a) Define principle of Optimality.

   (b) Explain Dynamic programming Algorithm.

8. (a) Explain forward and backward recursion process

   (b) Solve the following LPP by dynamic programming technique.

9. (a) Define Saddle point.

   (b) Solve the given m × 2 game.

| | B1 | B2 |
|---|---|---|
| A1 | –7 | 6 |
| A2 | 7 | –4 |
| A3 | –4 | –2 |
| A4 | 8 | –6 |

10. (a) Write about dominance property.

(b) Solve the given 2 × n matrix.

| B's | | | | | |
|---|---|---|---|---|---|
| A's | | I | II | III | IV |
| | P | 1 | 4 | 2 | -3 |
| | Q | 2 | 1 | 4 | 5 |

# FACULTY OF INFORMATICS

## M.C.A II - Semester (CBCS) Examinations,

## April / May - 2023

## OPERATION RESEARCH

**Time : 3 Hours ]** **[Max. Marks : 70**

**Note : Answer one question from each unit. All questions carry equal marks.**
**Missing data, if any, may be suitable assumed**
**All questions carry equal marks.**

**UNIT - I**

1. Use simplex method to solve the following problem.

Max. $Z = 3X_1 + 2X_2 + 5X_3$

Sub to $x_1 + 2x_2 + X_3 \leq 430$

$3X_1 + 2X_2 \leq 460$

$x_1 + 4x_2 \leq 420$

$x_1, x_2, x_3 \geq 0$

(OR)

2. Use two-phase simplex method to solve the problem.

Min. $Z = 15/2\ X_1 - 3X_2$

Sub to $3x_1 - x_2 \geq 3,$

$x_1 + x_2 + x_3 \geq 2,$

$x_1, x_2, x_3 \geq 0$

**UNIT - II**

3. (a) Obtain the basic feasible solution of the following transportation problem by VAM.

| | D1 | D2 | D3 | D4 | Supply |
|---|---|---|---|---|---|
| S1 | 5 | 2 | 4 | 3 | 60 |
| S2 | 6 | 4 | 9 | 5 | 60 |
| S3 | 2 | 3 | 8 | 1 | 90 |
| Demand | 50 | 65 | 65 | 30 | |

(b) How do you solve an unbalance transportatation problem?

OR

4. Five salesmen are to be assigned to five districts. Estimate of sales revenue (in thousands) for each salesman are given as follows :

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | 32 | 40 | 41 | 22 | 29 |
| 2 | 38 | 24 | 27 | 38 | 33 |
| 3 | 40 | 28 | 33 | 41 | 40 |
| 4 | 28 | 21 | 30 | 36 | 35 |
| 5 | 40 | 36 | 37 | 36 | 39 |

Find the assignment pattern that maximizes the sales revenue.

**UNIT - III**

5. A work shop contains four persons available for work on the four jobs. Only one person can work on any one job. The following table shows the cost of assigning each person to each job. The objective is to assign person to jobs such that the total assignment cost is a minimum.

Jobs

| Person | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 20 | 25 | 23 | 28 |
| B | 15 | 18 | 23 | 17 |
| C | 19 | 17 | 21 | 24 |
| D | 25 | 23 | 24 | 24 |

(OR)

6. (a) Write procedure for Hungarian Method.

(b) Discus the Gomory cutting plane Method.

**UNIT - IV**

7. Solve the following LPP by Dynamic programming.

Max $Z = 2X_1 + 5X_2$

Sub to $2X_1 + X_2 \leq 430$

$2X_2 \leq 460$

$X_1, X_2 \geq 0$

OR

8. (a) What are the applications of dynamic programming?

(b) What is a Queue? Explain the basic elements of Queues

**UNIT - V**

9. The following table represents the payoff matrix with respect to player A. Solve it optimally using dominance property.

Player B

| Player A: Strategies | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 4 | 6 | 5 | 10 | 6 |
| 2 | 7 | 8 | 5 | 9 | 10 |
| 3 | 8 | 9 | 11 | 10 | 9 |
| 4 | 3 | 4 | 10 | 6 | 4 |

10. Define the following terms in the given example

(a) Saddle point

(b) Two-person zero sum game

(c) Dominance Property

(d) Game with are strategies

# Contents

## UNIT - I



## UNIT - II



## UNIT - III

# *Important Questions*

## UNIT - I

**1. What is linear programming problem LPP? States the mathematical model of LPP.**

*Ans :*

Refer Unit-I, Q.No. 1

---

**2. Explain the major assumptions of LPP.**

*Ans :*

Refer Unit-I, Q.No. 3

---

**3. What do you mean by Graphical Method of LPP? State the characteristics of Graphical Method.**

*Ans :*

Refer Unit-I, Q.No. 4

---

**4. Write the computational procedure for simplex method.**

*Ans :*

Refer Unit-I, Q.No. 9

---

**5. Explain the various steps involved in Big M Method.**

*Ans :*

Refer Unit-I, Q.No. 11

---

**6. What is Two Phase Method ? Explain the steps involved in Two Phase Method to solve a LPP.**

*Ans :*

Refer Unit-I, Q.No. 12

---

**7. Discuss about special cases in finding solution to LPP by graphical method.**

*Ans :*

Refer Unit-I, Q.No. 16

## UNIT - II

**1. Explain the Mathematical formulation for Transportation Problem.**

*Ans :*

Refer Unit-II, Q.No. 2

**2. Explain the basic terminology are used in transportation problem.**

*Ans :*

Refer Unit-II, Q.No. 3

**3. Write about method to obtain initial basic feasible solution by North-West Corner Rule.**

*Ans :*

Refer Unit-II, Q.No. 6

**4. Explain Stepping Stone Method.**

*Ans :*

Refer Unit-II, Q.No. 11

**5. Explain the Modi Method for obtaining optimal solution.**

*Ans :*

Refer Unit-II, Q.No. 12

**6. Define Transshipment. State the characteristics of Transshipment Model.**

*Ans :*

Refer Unit-II, Q.No. 14

## UNIT - III

**1. What is an Assignment Problem ? How do you mathematically to formulate an assignment problem ?**

*Ans :*

Refer Unit-III, Q.No. 1

**2. Explain the concept of Zero-One Programming Model for Assignment Problem with an example.**

*Ans :*

Refer Unit-III, Q.No. 3

**3. Discuss the steps involved in the Hungarians Method used to find optimal solution to an Assignment Problem.**

*Ans :*

Refer Unit-III, Q.No. 7

**4. Explain the concept of Branch-and-Bound Technique for Assignment Problem.**

*Ans :*

Refer Unit-III, Q.No. 8

**5. Explain the concept of Cutting-Plane Algorithm.**

*Ans :*

Refer Unit-III, Q.No. 13

**6. Explain the concept of Zero-One Implicit Enumeration Algorithm.**

*Ans :*

Refer Unit-III, Q.No. 15

## UNIT - IV

**1. Define Dynamic Programming.**

*Ans :*

Refer Unit-IV, Q.No. 1

**2. What are the characteristics of dynamic programming problem?**

*Ans :*

Refer Unit-IV, Q.No. 2

**3. What are the various applications of dynamic programming ?**

*Ans :*

Refer Unit-IV, Q.No. 4

**4. Explain the formulation of LPP by dynamic programming.**

*Ans :*

Refer Unit-IV, Q.No. 5

**5. Explain the assumptions of linear programming.**

*Ans :*

Refer Unit-IV, Q.No. 6

## UNIT - V

**1. Define Game Theory. What are the characteristics of Game Theory ?**

*Ans :*

Refer Unit-V, Q.No. 1

**2. State the Assumptions of game Theory**

*Ans :*

Refer Unit-V, Q.No. 3

**3. Explain the advantages and disad-vantages of game theory**

*Ans :*

Refer Unit-V, Q.No. 4

**4. State the different types of games.**

*Ans :*

Refer Unit-V, Q.No. 6

**5. What is Saddle Point ? How do you determine Saddle Point ?**

*Ans :*

Refer Unit-IV, Q.No. 8

**6. Explain the Game without Saddle Point.**

*Ans :*

Refer Unit-IV, Q.No. 9

**7. Explain the concept of Dominance.**

*Ans :*

Refer Unit-IV, Q.No. 11

| UNIT I | **Linear Programming**: Introduction, Concept of Liner Programming Model, Development of LP models, Graphical Method, Linear Programming Methods, Special cases of Linear Programming, Duality, Sensitivity Analysis. |
|---|---|

## 1.1 Concept of Liner Programming Model

### 1.1.1 Introduction

**Q1. What is linear programming problem LPP? States the mathematical model of LPP.**

*Ans :* **(Imp.)**

- Linear programming is an optimization technique.
- It is a technique for specifying how to use limited resources or capacities of a business to obtain a particular objective.
- It is a mathematical modeling technique.
- The term 'programming' normally refers to determine a particular plan of action amongst several available alternatives.
- The term 'linear' indicates that all relations involved in the system are linear.

In 1947, George dantzig and his associates, while working in the U.S. department of Air Force, observed that a large of military programming and planning problems could be formulated as maximizing/minimizing a linear form of profit/cost function whose variables were restricted to values satisfying a system of linear constraints.

A linear form is meant a thematical expression of the type $a_1x_1 + a_2x_2 + ... + a_nx_n$, where $a_1, a_2, ..., a_n$ are constants and $x_1, x_2, ..., x_n$ are variables. The term 'Programming' refers to the process of determining a particular programme or plan of action. So Linear Programming (L.P.) is one of the most important optimization (maximization/minimization) techniques developed in the field of Operations Research.

The general LPP calls for optimizing (maximizing/ minimizing) a linear function of variables called the '*Objective Function*' subject to a set of linear equations and / or inequalities called the '*constraints*' or *restrictions*.

**Mathematical Model of LPP**

In order to find the values of n decision variables $x_1, x_2, ..., x_n$ to maximize or minimize the objective function

$$z = c_1x_1 + c_2x_2 + c_3x_3 + ... + c_nx_n$$

and also satisfy m-constants :

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1j}x_j + ... + a_{1n}x_n \; (\leq = \text{ or } \geq) \; b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2j}x_j + ... + a_{2n}x_n \; (\leq = \text{ or } \geq) \; b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n \; (\leq = \text{ or } \geq) \; b_i \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mj}x_j + ... + a_{mn}x_n \; (\leq = \text{ or } \geq) \; b_m \end{array}\right\}$$

where constraints may be in the form of any inequality ($\leq$ or $\geq$) or even in the form of an equation (=) and finally satisfy the non-negativity restrictions

$$x_1 \geq 0, \; x_2 \geq 0, \; ..., x_j \geq 0, \; ..., x_n \geq 0.$$

## 1.2 Development of LP Models

**Q2. State the Formulation of LPP**

*Ans :*

The formulation of linear programming problem as a mathematical model involves the following basic steps :

**Step 1**

Find the key-decision to be made from the study of the solution. (In this connection, looking for variables helps considerably).

**Step 2**

Identify the variables and assume symbols $x_1, x_2$ ... for variable quantities noticed in step 1.

**Step 3**

Express the possible alternatives mathematically in terms of variables. The set of feasible alternatives generally in the given situation is :

$$\{(x_1, x_2) \; ; \; x_1 > x_2 > 0\}$$

**Step 4**

Mention the objective quantitatively and express it as a linear function of variables.

**Step 5**

Express the constraints also as linear equalities / inequalities in terms of variables.

## PROBLEMS

**1. Sainath and Co. manufactures two brands of products namely Shivnath and Harinath. Both these models have to undergo the operations on three machines lathe, milling and grinding. Each unit of Shivnath gives a profit of Rs. 45 and requires 2 hours on lathe, 3 hours on milling and 1 hour on grinding. Each unit of Harinath can give a profit of Rs. 70 and requires 3, 5, and 4 hours on lathe, milling and grinding respectively. Due to prior commitment, the use of lathe hours are restricted to a maximum of 70 hours in a week. The operators to operate milling machines are hired for 110 hours / week. Due to scarce availability of skilled man power for grinding machine, the grinding hours are limited to 100 hours/week. Formulate the data into on LPP.**

*Sol :*

**Step 1: Selection of Variables**

In the above problem, we can observe that the decision is to be taken on how many products of each brand is to be manufactured. Hence the quantities of products to be produced per week are the decision variables.

Therefore we assume that the number of units of product Shivnath brand produced per week $= x_I$

The number of units of product of Harinath brand produced per week $= x_2$

**Step 2 : Setting Objective**

In the given problem the profits on the brands are given.

Therefore objective function is to maximize the profits.

Now, the profit on each unit of Shivnath brand = Rs. 45.

Number of units of Shivnath to be manufactured $= x_1$

$\therefore$ The profit on xj units of Shivnath brand $= 45\,x_1$

Similarly, the profit on each unit of Harinath brand = Rs. 70

Number of units of Harinath brand to be manufactured $= x_2$

$\therefore$ The profit on $x_2$ units of Harinath brand $= 70x_2$

The total profit on both brands $= 45x_i + 70x_2$

This total profit (say z) is to be maximised.

Hence, the objective function is to

$$\boxed{\text{Maximise } z = 45x_i + 70x_2}$$

**Step 3 : Identification of Constraint Set**

In the above problem, the constraints are the availability of machine hours.

**(i) Constraint on Lathe Machine :** Each unit of Shivnath brand requires 2 hours/ week

So $x_1$ units of Shivnath brand requires $2x_I$ hours / week.

Each unit of Harinath brand requires 3 hours / week

and So $x_2$ units of Harinath brand require $3x_2$ hours / week.

Total lathe hours utilized for both the brands is $2x_x + 3x_2$

and this cannot exceed 70 hours/ week.

$$\therefore \quad \boxed{2x_1 + 3x_2 \leq 70}$$

(Constraint on availability of lathe hours due to prior commitment)

**(ii) Constraint on Milling Machine:** Milling hours required for each unit of Shivnath brand = 3 hours/week.

$\therefore$ For $x_1$ units = $3x_1$ hours/week

Milling hours required for each unit of Harinath brand = 5 hours/week.

$\therefore$ For $x_2$ units = $5x_2$

Total milling hours = 3xj + $5x_2$

This can not be more than 110

$\therefore$ $\boxed{3x_1 + 5x_2 \leq 110}$

(Constraint on availability of milling machine hours due to hiring)

**(iii) Constraint on Grinding Machine :** One unit of Shivnath needs one hour/week and $x_1$ units need $x_1$ hours/week One unit of Harinath needs 4 hours/week and $x_2$ units need $4x_2$ hours/week. Total grinding hours = $x_1$ + $4x_2$ and this cannot be greater than 100 hours.

$\therefore$ $\boxed{x_1 + 4x_2 \leq 100}$

(Constraint on availability of grinding hours due to scarcity of skilled labour)

**Step 4 : Writing Conditions of Variables**

Both x i and $x_2$ are the number of products to be produced. There can not exist any negative production. Therefore $x_l$ and $x_2$ can not assume any negative values (i.e., non negative)

Mathematically

$\boxed{x_1 \geq 0 \text{ and } x_2 \geq 0}$



**Step 5 : Summary**

Maximise $Z = 45x_1 + 70x_2$

Subject to $2x_1 + 3x_2 \leq 70$

$3x_1 + 5x_2 \leq 110$

$x_1 + 4x_2 \leq 100$

$x_1 \geq 0$ and $x_2 \geq 0$

---

**2. Formulate the following problem as an LP Problem:**

**A firm engaged in producing 2 models $X_1$, $X_2$ performs 3 operations Painting. Assembly and Testing. The relevant data are as follows :**

| Model | Units Sales Price | Hours Required for each unit | | |
|---|---|---|---|---|
| | | Assembly | Paining | Testing |
| Model $X_1$ | Rs. 50 | 1.0 | 0.2 | 0.0 |
| Model $X_2$ | Rs. 80 | 1.5 | 0.2 | 0.1 |

**Total number of hours available are:**

**For Assembly 600**

**For Painting 100**

**For Testing 30**

**Determine weekly production schedule to maximise revenue.**

*Sol :*

**Step 1: Variables**

Let the weekly produced units of model $X_i = x_t$

i.e., Let the no. of units produced per week in Model Xj = $x_t$ and let the no. of units produced per week in model $X_2 = x_2$

**Step 2: Objective Function**

Unit sales price for model Xj = Rs. 50/-

For $x_1$ units, weekly sales revenue = 50 $x_1$

Similarly, unit sales price for model $X_2$ = Rs. 80/-

For $x_2$ units, weekly sales revenue = 80 $x_2$

Total sales revenue per week = $50\ x_1 + 80\ x_2$

Hence the objective function is to -

$$\boxed{\text{Maximise } Z = 50\ x_1 + 80\ x_2}$$

**Step 3: Constraint Set**

**For assembly,**

one unit of model $X_1$ requires 1.0 hrs.

$\therefore$ $x_1$ units require 1.0 $x_1$ hrs.

one unit of model $X_2$ requires 1.5 hrs.

$x_2$ units of model $X_2$ requires 1.5 $x_2$ hrs.

Total assembly hours per week = $1.0x_1 + 1.5\ x_2$.

This cannot exceed 600 hrs since only 600 hrs are available for assembly.

$$\therefore \quad \boxed{1.0x_1 + 1.5x_2 \leq 600}$$

(Constraint on availability of assembly hours)

Similarly,

**For painting**

$x_1$ units require 0.2 $x_1$ hrs

and $x_2$ units require 0.2 $x_2$ hrs

$$\therefore \quad \boxed{0.2x_1 + 0.2x_2 \leq 100}$$

(Constraint on availability of painting hours)

**For Testing**

$0.0\ x_1 + 0.1\ x_2 < 30$

or

$0.1x_2 \leq 30$

(Availability constraint on testing hours)

**Step 4: Conditions of Variables**

As the problem is to determine weekly production schedule and we cannot have a negative production.

Therefore,

$x_1 \geq 0$ and $x_2 \geq 0$

**Step 5: Summary of Formulation**

maximize $Z = 50\ x_1 + 80\ x_2$

subject to $1.0\ x_1 + 1.5\ x_2 \leq 600$

$0.2\ x_1 + 0.2\ x_2 \leq 100$

$0.1\ x_2 \leq 30$

$x_1, x_2 \geq 0$

---

**3. A firm manufactures two products in three departments. Product A contributes Rs. 5/- unit and requires 5 hrs. in dept. M, 5 hrs. in dept. N and one hour in dept. P. Product B contributes Rs. 10/- unit and requires 8 hrs. in dept. M, 3 hrs in dept. N and 8 hrs in dept. P. Capacities for departments M, N, P are 48 hours per week.**

*Sol :*



The problem is summarized below.

Unit contribution 5 10

**Step 1: Variables**

Let quantity produced of product A = $X_1$

Let quantity produced of product B = $X_2$

**Step 2: Objective Function**

Profit contribution of each unit of A = Rs. 5

Profit contribution of $X_1$ units of A = $5\ X_1$

Profit contribution of each unit of B = Rs. 10

Profit contribution of $X_2$ units of B = $10\ X_2$

Total profit = $5\ X_1 + 10\ X_2$

Therefore objective function is to maximise $Z = 5\ X_1 + 10\ X_2$

**Step 3: Constraint Set**

In Department M

Time required by each unit of product A = 5 hrs

Time required by $X_1$ units of product A = 5 $X_1$

Time required by each unit of product B = 8

Time required by $X_2$ units of product B = 8 $X_2$

Total hours available in dept. M = 48 hrs per week

$\therefore \quad \boxed{5X_1 + 8X_2 \leq 48}$

(Constraint on capacity in dept. M)

Similarly in department N,

$\boxed{5X_1 + 3X_2 \leq 48}$

(Constraint on capacity in dept. N.)

and in department P,

$\boxed{X_1 + 8X_2 \leq 48}$

(Constraint on capacity in department P)

**Step 4: Conditions**

$X_1$ and $X_2 \geq 0$

**Step 5: Summary**

Max $Z = 5X_1 + 10X_2$

Subject to $5X_1 + 8X_2 \leq 48$

$5X_1 + 3X_2 \leq 48$

$X_1 + 8X_2 \leq 48$

$X_1, X_2 \geq 0$

---

**4. Food X contains 6 units of Vitamins A per gram and 7 units of Vitamin B per gram and costs 12 paise per gram. Food Y contains 8 units of Vitamin A per gram and 12 units of Vitamin B and costs 20 paise per gram. The daily minimum requirements of Vitamin A and Vitamin B are 100 units and 120 units respectively. Find the minimum cost of product mix.**

*Sol :*

**Step 1: Selection of Variables**

Let the No. of grams produced in type X = $x_1$

Let the No. of grams produced in type Y = $x_2$

**Step 2: Objective Function**

Cost of each gram of type X= 12 paise

Cost of xj gms of type X = $12x_1$ ps.

Cost of each gm of type 7 = 20 ps.

Cost of $x_2$ gms of type 7=20 $x_2$ ps.

Total cost Z = $12x_1 + 20x_2$

$\therefore$ Objective function is to $\boxed{\text{minimize } Z = 12x_1 + 20x_2}$

**Step 3: Constraint Set**

**For Vitamin A**

Availability of vitamin A in each gm of food X = 6 units

Availability of vit-A in gms of food X = 6 $x_1$

Availability of vit-A in each gms of food Y = 8 units

Availability of vit-A in $x_2$ gms of food Y = 8 $x_2$

Total vitamin A = 6xj + $8x_2$

Daily minimum vitamin A required =100 units

$\boxed{6x_1 + 8x_2 \geq 100}$

(Requirement constraint on vitamin A)

**For Vitamin B**

Availability of vitamin B in each gm of food X = 7 units

Availability of vit-B in xj gms of food X= 7 $x_1$

Availability of vit-B in each gms of food Y = 12

units Availability of vit-B in $x_2$ gms of food Y = 12 $x_2$

Total vitamin B = 7 $x_1$ + 12 $x_2$

Daily minimum requirement of vitamin B = 120 units

Hence $\boxed{7x_1 + 12x_2 \geq 120}$

(Constraint on requirement of vitamin – B).

**Step 4: Conditions of Variables**

$x_1$ and $x_2$ cannot be negative

$\therefore$ $\boxed{x_1 \geq 0}$ & $\boxed{x_2 \geq 0}$

**Step 5: Summary**

minimize $= 12\,x_1 + 20\,x_2$

subject to $6x_1 + 8x_2 \geq 100$

$7x_1 + 12x_2 \geq 120$

$x_1, x_2 \geq 0$

---

**5. M/s. ABCL company manufactures two types of cassettes, a video and audio. Each video cassette takes twice as long to produce one audio cassette, and the company would have time to make a maximum of 2000 per day if it is produced only audio cassettes. The supply of plastic is sufficient to produce 1500 per day to both audio and video cassettes combined. The video cassette requires a special testing and processing of which there are only 6000 hrs. per day available. If the company makes a profit of Rs. 3/- and Rs. 5/- per audio and video cassette respectively, how many of each should be produced per day in order to maximize the profit ?**



*Sol :*

Let No of audio cassettes to be produced per day $= x_1$

No. of video casettes to be produced per day $= x_2$

**Objective Function**

Profit per one audio casette = Rs. 3

Profit on audio casettes = 3x

Profit on each video casette = 5

Profit on xj video casettes $= 5x_2$

Total profit $= 3\,x_1 + 5\,x_2$

It is to be maximized

$\therefore$ Objective function is to Maximize

$$Z = 3x_1 + 5x_2$$

**Constraint Set :**

**(i) Time Constraint**

As time of production of video is twice to that of audio cassettes, if $x_1$ audio are produced $\frac{x_2}{2}$ videos can be produced. Thus in terms of audio casette times, we get

$$x_1 + \frac{x_2}{2} \leq 2000 \qquad \text{(constraint on production time)}$$

$$\Rightarrow \quad 2x_1 + x_2 \leq 4000$$

**(ii) Plastic Constraint**

$$x_1 + x_2 \leq 1500$$

**(iii) Testing & Processing Time Constraint**

$$x_2 \leq 6000$$

Conditions Both $x_1 \geq 0$ and $x_2 \geq 0$

Summary Maximize $Z = 3x_1 + 5x_2$

Subject to $2x_1 + x_2 \leq 4000$

$x_1 + x_2 \leq 1500$

$x_2 \leq 6000$

$x_1, x_2 \geq 0$

**Q3. Explain the major assumptions of LPP.**

*Ans :* **(Imp.)**

1. **Proportionality :** A primary requirement of linear programming problem is that the objective function and every constraint function must be *linear*. Roughly speaking, it simply means that if 1 kg of a product costs Rs. 2, then 10 kg will cost Rs. 20. If a steel mill can produce 200 tons in 1 hour, it can produce 1000 tons in 5 hours.

   Intuitively, linearity implies that the product of variables such as $x_1 x_2$, powers of variables such as $x_3^2$, and combination of variables such as $a_1x_1 + a_2 \log x_2$, are not allowed.

2. **Additivity :** Additivity means if it takes $t_1$ hours on machine G to make product a and $t_2$ hours to make product B, then the time on machine G devoted to produce A and B both is $t_1 + t_2$, provided the time required to change the machine from product A to B is negligible.

   Then additivity may not hold, in general. If we mix several liquids of different chemical composition, then the total volume of the mixture may not be the sum of the volume of individual liquids.

3. **Multiplicativity :** It requires :

   (a) It takes one hour to make a single item on a given machine, it will take 10 hours to make 10 such items.

   (b) The total profit from selling a given number of units is the unit profit times the number of units sold.

4. **Divisibility :** It means that the fractional levels of variables must be permissible besides integral values.

5. **Deterministic :** All the parameters in the linear programming models are assumed to be known exactly. While in actual practice, production may depend upon change also.

6. **Sensitivity :** Further, the problem can be extended by sensitivity analysis to check the post optimal situations.

7. **Decision Making by Conditions :** The conditions on the answers stated at the beginning of the problem will aid the decision making. The assumptions such as non negativity (or) unrestricted variables can influence the answers.

## 1.3 Graphical Method

**Q4. What do you mean by Graphical Method of LPP? State the characteristics of Graphical Method.**

**(OR)**

**Explain the LPP by graphical method.**

**(OR)**

**Explain about the solution by graphical method.**

*Ans :* **(Imp.)**

Graphical method is a simple method to understand and also to use. This is effectively used in LPP's which involves only 2 variables. It gives the graphical representation of the solutions.

All types of solutions are highlighted in this method very clearly. The only drawback is that more the number of constraints, more will be the straight lines which makes the graph difficult to understand.

**Characteristics**

The following are the characteristics of graphical method of LPP :

1. Method is very simple and easy to understand.
2. Very sensitive analysis and can be illustrated very easily by drawing graphs.
3. Very easy to obtain optimal solution.
4. It consumes very less time.

**Q5. What are the merits and demerits of graphical method ?**

*Ans :*

**Merits**

1. Graphical solutions are easier to understand and reproduce.
2. It is a pictorial view is always a better representation.
3. A graphical solutions have gained prominence in Operations Research.

**Demerits**

However, graphical solutions have certain limitations such as :

1. Limited to the problems of two decision variables only.
2. Accuracy can not be obtained.
3. Some times it is difficult to represent certain expressions, particularly in the case of non linear expressions.

**Q6. Explain the procedure for generating solution to an LPP graphical method.**

*Ans :*

Simple linear programming problems of two decision variables can be easily solved by graphical method. The outlines of graphical procedure are as follows :

The steps involved in graphical method are as follows.

**Step 1**

Consider each inequality constraint as equation.

**Step 2**

Plot each equation on the graph as each will geometrically represent a straight line.

**Step 3**

Mark the region. If the inequality constraint corresponding to that line is $\leq$ then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraint $\geq$ sign, the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.

**Step 4**

Assign an arbitrary value say zero for the objective function.

**Step 5**

Draw the straight line to represent the objective function with the arbitrary value (i.e. a straight line through the origin).

**Step 6**

Stretch the objective function line till the extreme points of the region. In the maximization case this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

**Step 7**

Find the co-ordinates of the extreme points selected in step 6 and find the maximum (or) minimum value of Z.

## PROBLEMS

**6. Solve the following LPP by graphical method.**

**Minimize** $Z = 20X_1 + 10X_2$

**Subject to** $X_1 + 2X_2 \leq 40$

$3X_1 + X_2 \geq 30$

$4X_1 + 3X_2 \geq 60$

$X_1, X_2 \geq 0.$

*Sol :*

Replace all the inequalities of the constraints by equation

$X_1 + 2X_2 = 40$ If $X_1 = 0 \Rightarrow X_2 = 20$

If $X_2 = 0 \Rightarrow X_1 = 40$

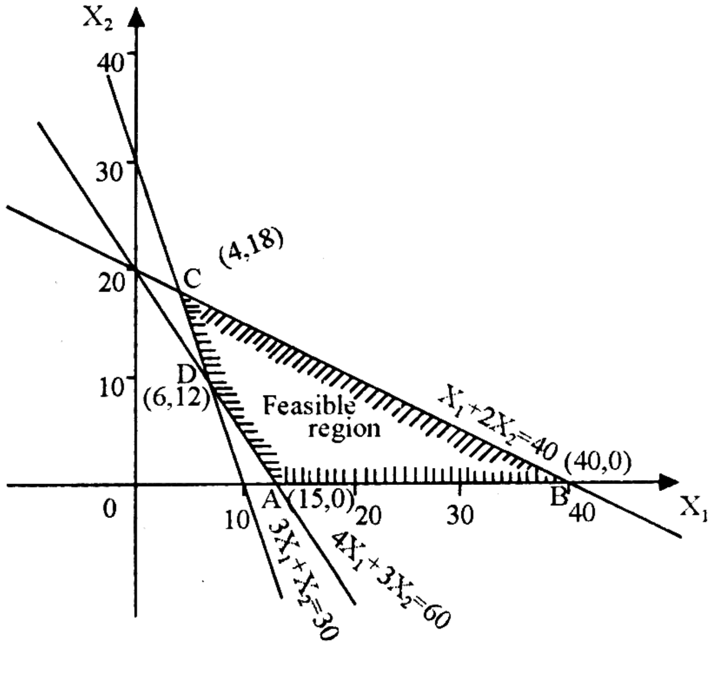
$\therefore X_1 + 2X_2 = 40$ passes through (0, 20) (40, 0)

$3X_1 + X_2 = 30$ passes through (0, 30) (10, 0)

$4X_1 + 3X_2 = 60$ passes through (0, 20) (15, 0)

Plot each equation on the graph.





The feasible region is ABCD

C and D are point of intersection of lines

$X_1 + 2X_2 = 40, 3X_1 + X_2 = 30$ and

$4X_1 + 3X_2 = 60, X_1 + X_2 = 30$

on solving web get C = (4, 18)

D = (6, 12)

| Corner points | value of $Z = 20x_1 + 10X_2$ |
|---|---|
| A(15, 0) | 300 |
| B(40, 0) | 800 |
| C(4,18) | 200 |
| D(6, 12) | 240 (Minimum value) |

∴ The minimum value of Z occurs at D(6, 12). Hence, the optimal solution is $X_1 = 6, X_2 = 12$

**7. Find the maximum value of $Z = 5X_1 + 7X_2$**

**Subject to the constraints**

$X_1 + X_2 \leq 4$

$3X_1 + 8X_2 \leq 24$

$10X_1 + 7X_2 \leq 35$

$X_1, X_2 > 0$

*Sol :*

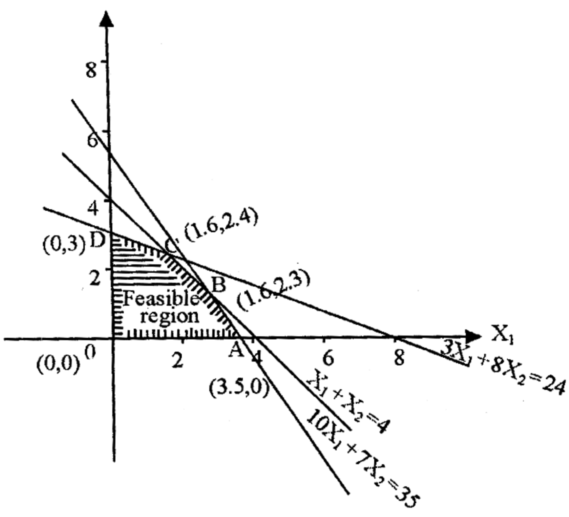
Replace all the inequalities of the constraints by forming by forming equations

$X_1 + X_2 = 4$ passes through (0,4) (4,0)

$3X_1 + 8X_2 = 24$ passes through (0, 3) (8,0)

$10X_1 + 7X_2 = 35$ passes through (0,5) (3.5,0)

Plot these lines in the graph and mark the region below the line as the inequality of the constraints is ≤ and is also lying in the firs, quadrant.



The feasible region is OABCD.

B and C are point of instruction of lines

$X_l + X_2 = 4$, $l0X_1 + 7X_2 = 35$ and

$3X_1 + 8X_2 = 24$, $X_l + X_2 = 4$.

On solving we get,

B = (1.6, 2.3)

C = (1.6, 2.4)

| Corner points | Value of $Z = 5X_1 + 7X_2$ |
|---|---|
| O (0, 0) | 0 |
| A (3.5, 0) | 17.5 |
| B (1.6, 2.3) | 25.1 |
| C (1.6, 2.4) | 24.8 (Maximum value) |
| D (0, 3) | 21 |

$\therefore$ The maximum value of Z occurs at C (1.6, 2.4) and the optimal solu-tion is $X_1 = 1.6$, $X_2 = 2.4$.

**8. Solve graphically the following LPP**

**Maximize (z) = $5x_1 + 3x_2$**

**Subject to Constraints :**

$3x_1 + 5x_2 \leq 15$; $5x_1 + 2x_2 \leq 10$; $x_1, x_2 \geq 0$

*Sol :*



In order to plot the constraints on the graph we convert inequalities into equations

i.e., $3x_1 + 5x_2 = 15 \Rightarrow$ (1)

$5x_1 + 2x_2 = 10 \Rightarrow$ (2)

To plot $3x_1 + 5x_2 = 15$

put $x_1 = 0$, $x_2 = 3 \Rightarrow$ (0, 3)

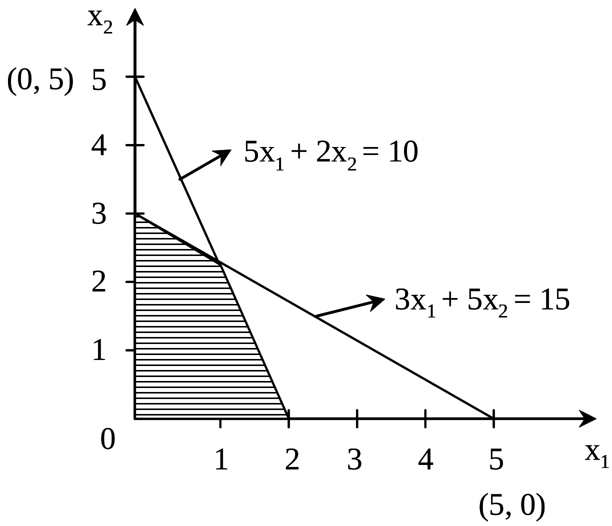
$x_2 = 0$, $x_1 = 5 \Rightarrow$ (5, 0)

To plot $5x_1 + 2x_2 = 10$

put $x_1 = 0$, $x_2 = 5 \Rightarrow$ (0, 5)

$x_2 = 0$, $x_1 = 2 \Rightarrow$ (2, 0)

Plotting these equations on the graph, we get




The area OABC is the figure satisfied by the constraints is shown by shaded area and is called the feasible solution region.

Hence Max z = 12.5, the solution to the given problem is

$\therefore \quad x_1 = 1,\ x_2 = 2.5$

**9. Solve graphically the following LPP**

**Maximize $Z = 3x_1 + 2z_2$**

**Subject to constraints**

$-2x_1 + x_2 \leq 1$

$x_1 \leq 2$

$x_1, x_2 \geq 0$



*Sol :*

Convert inequalities into equations

$-2x_1 + x_2 = 1$

$x_2 = 2$

$x_1 + x_2 = 3$

To plot $-2x_1 + x_2 = 1$,

put $x_1 = 0,\ x_2 = 1 \Rightarrow (0, 1)$

$x_2 = 0,\ x_1 = -\frac{1}{2} \Rightarrow \left(-\frac{1}{2}, 0\right)$

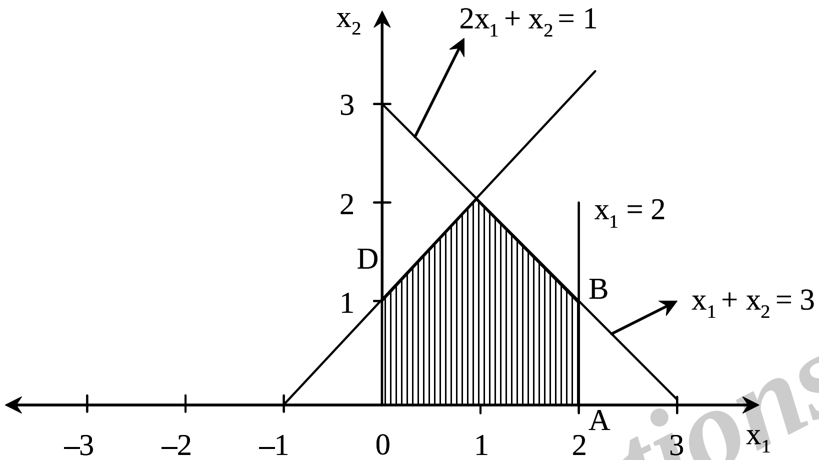
$x_1 = 2 \Rightarrow (2, 0)$

To plot $x_1 + x_2 = 3$,

put $x_1 = 0,\ x_2 = 3 \Rightarrow (0, 3)$

$x_2 = 0,\ x_1 = 3 \Rightarrow (3, 0)$

Plotting these equations on the graph, we get




| Corner Points | Coordinates | Max $Z = 3x_1 + 2x_2$ | Value |
|---|---|---|---|
| O | (0, 0) | 3(0) + 2(0) | 0 |
| A | (2, 0) | 3(2) + 2(0) | 6 |
| B | (2, 1) | 3(2) + 2(1) | 8 |
| C | (1, 2.3) | 3(1) + 2(2.3) | 7.6 |
| D | (0, 1) | 3(0) + 2(1) | 2 |

Hence Max Z = 8, $x_1 = 2$, $x_2 = 1$.

**10. Solve the following LP problem using graphical method .**

**Maximize : $Z = 6x_1 + 8x_2$**

**Subject to :**

**$5x_1 + 10x_2 \leq 60$**

**$4x_1 + 4x_2 \leq 40$**

**$x_1, x_2 \geq 0$.**

*Sol :*

In graphical method, the introduction of the non-negative constraints ($x_1 \geq 0$ and $x_2 \geq 0$) will eliminate the second, third and fourth quadrants of the $x_1$, $x_2$ plane as shown in the figure.

Now, we compute the coordinates on the $x_1$, $x_2$ plane. From the first constraint.

$$5x_1 + 10x_2 = 60$$

Let $x_1 = 0$ then $x_2 = 6$ similarly if $x_2 = 0$ then $x_1 = 12$. Now plot the points (0, 6) and (12, 0) on a graph as shown in figure.

$$4x_1 + 4x_2 = 40$$

Now in 2nd constraint, let $x_1 = 0$, $x_2 = 0$, similarly if $x_2 = 0$ then $x_1 = 0$ then $x_1 = 10$. Now plot the points (0, 10) and (10, 0) on a graph as shown in the figure.

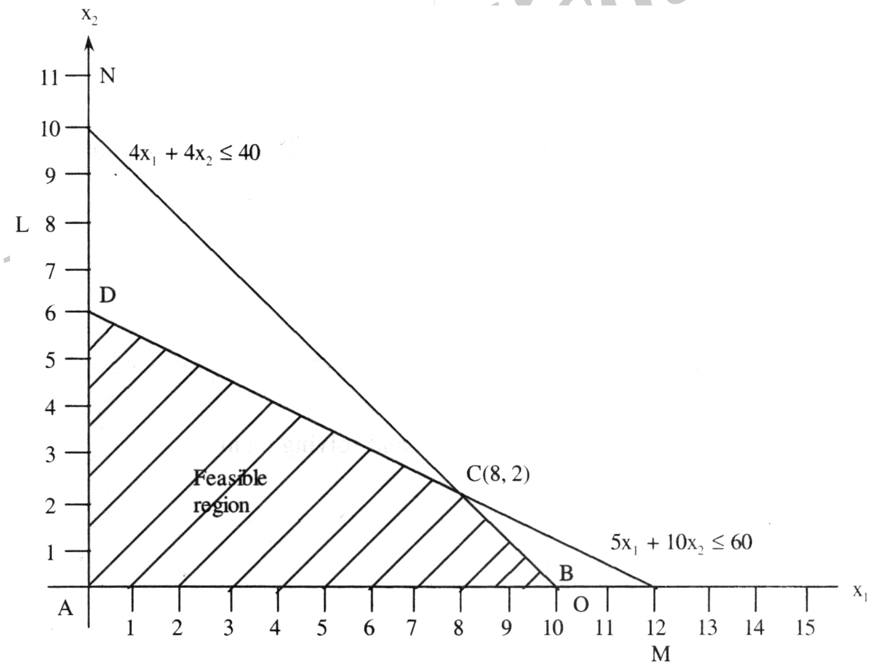
From the graph ABCD is found to be a feasible region. Points lying at the corner of the polygon must be substituted to obtain the value of an objective function as follows.

$Z(A) = 6 \times 0 + 8 \times 0 = 0$

$Z(B) = 6 \times 10 + 8 \times 10 = 60$

$Z(C) = 6 \times 8 + 8 \times 2 = 48 + 16 = 64$

$Z(D) = 6 \times 0 + 8 \times 6 = 48$




**Graph : Feasible Region**

Since, the type of the objective function here is maximization, the solution corresponding to the maximum Z value is to be selected as the optimum solution. The Z value is maximum for the corner point C. Hence, the corresponding solution is presented below.

$x_1^* = 8$, $x_2^* = 2$, max Z(optimum) = 64.

**11. Solve the following LPP graphically**

**Min Z = $4x_1 + 2x_2$**

**S.T.C. :**

$x_1 + 2x_2 \geq 2$

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1, x_2 \geq 0$

*Sol :*

Convert inequations into equations considering equations (1), (2) and (3)

$x_1 + 2x_2 = 2 \Rightarrow$ (1)

$3x_1 + x_2 = 3 \Rightarrow$ (2)

$4x_1 + 3x_2 = 6 \Rightarrow$ (3)

To plot $x_1 + 2x_2 = 2$

put $x_1 = 0,\ x_2 = 1 \Rightarrow (0,1)$

$x_2 = 0,\ x_1 = 2 \Rightarrow (2, 0)$

To plot $3x_1 + x_2 = 3$

put $x_1 = 0,\ x_2 = 3 \Rightarrow (0,3)$

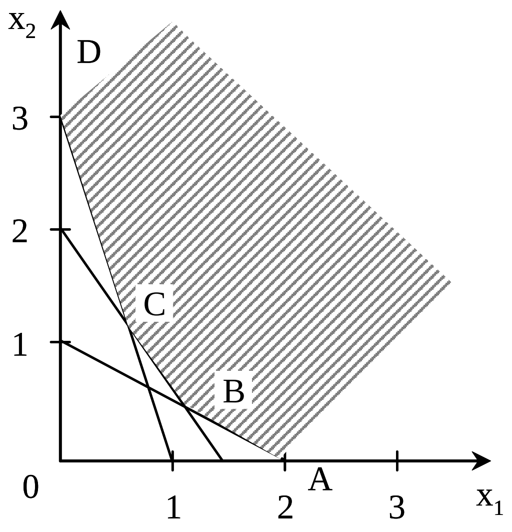
$x_2 = 0,\ x_1 = 1 \Rightarrow (1, 0)$

To plot $4x_1 + 3x_2 = 6$

put $x_1 = 0,\ x_2 = 2 \Rightarrow (0,2)$

$x_2 = 0,\ x_1 = 1.5 \Rightarrow (1.5, 0)$

Plotting these equations in the graphs, we get



| Corner Points | Coordinates | Max $Z = 4x_1 + 2x_2$ | Value |
|---|---|---|---|
| A | (2, 0) | 4(2) + 2(0) | 8 |
| B | (1.2, 0.4) | 4(1.2) + 2(0.4) | 5.6 |
| C | (0.6, 1.2) | 4(0.6) + 2(1.2) | 4.8 |
| D | (3, 0) | 4(3) + 2(0) | 6 |

The minimum value of z is 4.8 which occurs at C = (0.6, 1.2).

Hence, the solution to the above problem is $x_1 = 0.6$ ; $x_2 = 1.2$, min $z = 4.8$

**12. An auto company has three plants A, B and C and two major distribution centers in X and Y. The capacities of the three plants during the next quarter are 1000, 2300 and 1400 cars. The transportation costs (which depend on the mileage, transport company etc) between the plants and the distribution centers is as follows :**

| Plant Name | Distr. Center X | Distr. Center Y |
|---|---|---|
| Plant A | 80 | 215 |
| Plant B | 100 | 108 |
| Plant C | 102 | 68 |

**Which plant should supply how many cars to which outlet so that the total cost is minimum?**

*Sol :*



The problem can be formulated as a LP model:

Let $x_{ij}$ be the number of cars to be shipped from source i to destination j. Then our objective is to minimize the total cost which is $80x_{11} + 215x_{12} + I00x_{21} + I08x_{22} + 102x_{31} + 680x_{32}$. The constraints are the ones imposed by the number of cars to be transported from each plant and the number each center can absorb.

The whole model is:

Minimize $z = 80x_{11} + 215x_{12} + 100x_{21} + I08x_{22} + 102x_{31} + 68x_{32}$

subject to,

$x_{11} + x_{12} = 1000$

$x_{21} + x_{22} = 1500$

$x_{31} + x_{32} = 1200$

$x_{11} + x_{21} + x_{31} = 2300$

$x_{12} + x_{22} + x_{32} = 1400$

## 1.4 Linear Programming Methods

### 1.4.1 Simplex Method

**Q7. Define Simplex Method.**

*Ans :*

Simplex method is an iterative procedure for solving LPP in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be than at the previous vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps (or) indicates the existence of unbounded solution.

**Q8. Explain basic terminology is used in Simplex Method for solving LPP.**

*Ans :*

**Terminology of Simplex Method**

The following are the terminologies used in solving LPP through simplex method,

1. **Standard Form**

   A LPP in which all the constraints are written as equalities.

2. **Slack Variable**

   A variable added to LHS of ≤ constraints (maximizations LPP) to convert the constraints into equality. The value of this variable is equal to the amount of unused resource.

3. **Surplus Variable**

   A variable subtracted from the LHS of ≥ constraints (minimization LPP) to convert the constraints into equality. The value of this variable is equal to the amount over and above the required minimum level.

4. **Basic Solution**

   For a general LP with 'n' variables and 'm' constraints, a basic solution can be found by setting (n –m) variables equal to zeros and solving the constraint equations for the value of other m variables. If a unique solution exists, it is a basic solution.

5. Basic Feasible Solution

   It refers to the set of constraints corresponding to the extreme point of the feasible region.

6. **Simplex Table**

   A table used to keep track of the calculations made at each iteration when the simplex solution method is employed.

7. **Product Mix**

   A column in the simplex table that contains all of the variables in the solution.

8. **Basic**

   A set of variables which are not restricted or equal to zero in the current basic solution are listed in the product mix column. These variables are called basic variables.

9. **Iteration**

   The sequence of steps performed in moving from one basic feasible solution to another.

10. **$Z_j$ Row**

    The row containing the figures for gross profit or loss given up by adding one unit of a variable into the solution.

11. **$C_j – Z_j$ / Net Evaluation index Row**

    The-row- containing the net profit or loss that will result from introducing one unit of the variable indicated in that column in the solution. Number in the index row are also shadow prices or according prices.

12. **Pivot/Key Column**

    The column with the largest positive number in the $C_j – Z_j$ row of a maximization problem (largest negative for minimization problems). It indicates which variables will enter the solution next (i.e., entering variable/incoming variable).

**13. Pivot/Key Row**

The row' corresponding to the variable that will leave the basis in order to make room for the entering variable. The departing variable (outgoing variable) will correspond to the smaller positive ratio found by dividing the quantity column values by the key column values for each row.

**14. Pivot/Key Element**

The element of the intersection of key row and key column.

**Q9. Write the computational procedure for simplex method.**

**(OR)**

**Explain the procedure for simplex method.**

*Ans :* **(Imp.)**

At each step it projects the improvement in the objective function over its previous step. Thus, the solution becomes optimum when no further improvement is possible on the objective function.

**Simplex Algorithm**

The algorithm goes as follows :

**Step 1**

Formulation of LPP :

- Selection of decision variables
- Setting of objective function
- Identification of constraint set
- Writing the conditions of variables.

**Step 2**

Convert constraints into equality form.

- Add slack variable if constraints is $\leq$ type.
- Subtract surplus and add an artificial variable if the constraint is $\geq$ type.
- Add an artificial variable if constraint is exact (=) type.

**Step 3**

Find, Initial Basic Feasible Solution (IBFS)

- If m non identical equations have n variables (m < n) including all decision, slack / surplus and artificial variables, we get m number of variables basic and (n – m) variables non basic (i.e., equated to zero).
- First make all decision variables (and surplus) as non basic i.e., equate to zero to identity the IBFS.
- Find solution values for basic variables.

**Step 4**

Construct, Initial Simplex Tableau as given above with the following notations.

- $C_B$ : Coefficient of basic variable in the objective functions (or contribution of basic variables)
- BV : Basic variables (form IBFS)

- SV : Solution value (from IBFS)
- $C_j$ : Contribution of j[th] variables or coefficient of each variable (j[th]) in objective function.
- $Z_j - C_j$ : Net contribution.

| $C_B$ | BV | $C_j$ SV | $C_j$ / $x_i$, S & A | Min-Ratio | Remarks |
|---|---|---|---|---|---|
| Contribution of basic variable in objective function | Basic variables | Solution variables | $y_i$ Key Element (KE) | Most min ratio of SV/key co. vlaue | ← Key row |
| | | $Z_j$ | Sum of products of $C_B$ and $y_i$ | | |
| | | $Z_j - C_j$ | Most negative value ↑ Key column | | |

**Step 5**

Find 'out going' and 'incoming' variables.

- Find $Z_n$ by summation of products of $C_B$ and $y_i$ for each column
- Computed $Z_j - C_j$ value for each column
- To find key column use most negative value of $Z_j - C_j$
- Variable in key column is 'in coming variable' or 'entering' variable.
- The variable of key row is 'out going' or 'existing' variable.
- Find the minimum ratio of solution value to corresponding key column value to identify key row.
- The cross section of key column and key row is key element with which the next iteration is carried out.

**Step 6**

Re-write next tableau as per given set of rules.

- Replace the existing variable from the basis with the entering variable along with its coefficient (or contribution).
- You have to make key element as unity (i.e., 1) and other element in the key column as zeros.
- To make key element as unity, divide the whole key row by the key element. This is supposed as the new row in the place of key row in the next iteration table.
- To find other rows of next iteration table, use this new row. By appropriate adding or subtracting entire new row in the old rows, make other elements of the key column as zeros.

**Step 7**

Check whether all the values of $Z_j - C_j$ are positive. If all are positive, the optimal solution is reached. Write the solution values and find $Z_{opt}$. (i.e., $Z_{max}$ or $Z_{min}$ as the case may be).

If $Z_j - C_j$ values are still negative, again choose most negative among these and go to step 5 and repeat the iteration till all the values of $Z_j - C_j$ become positive.

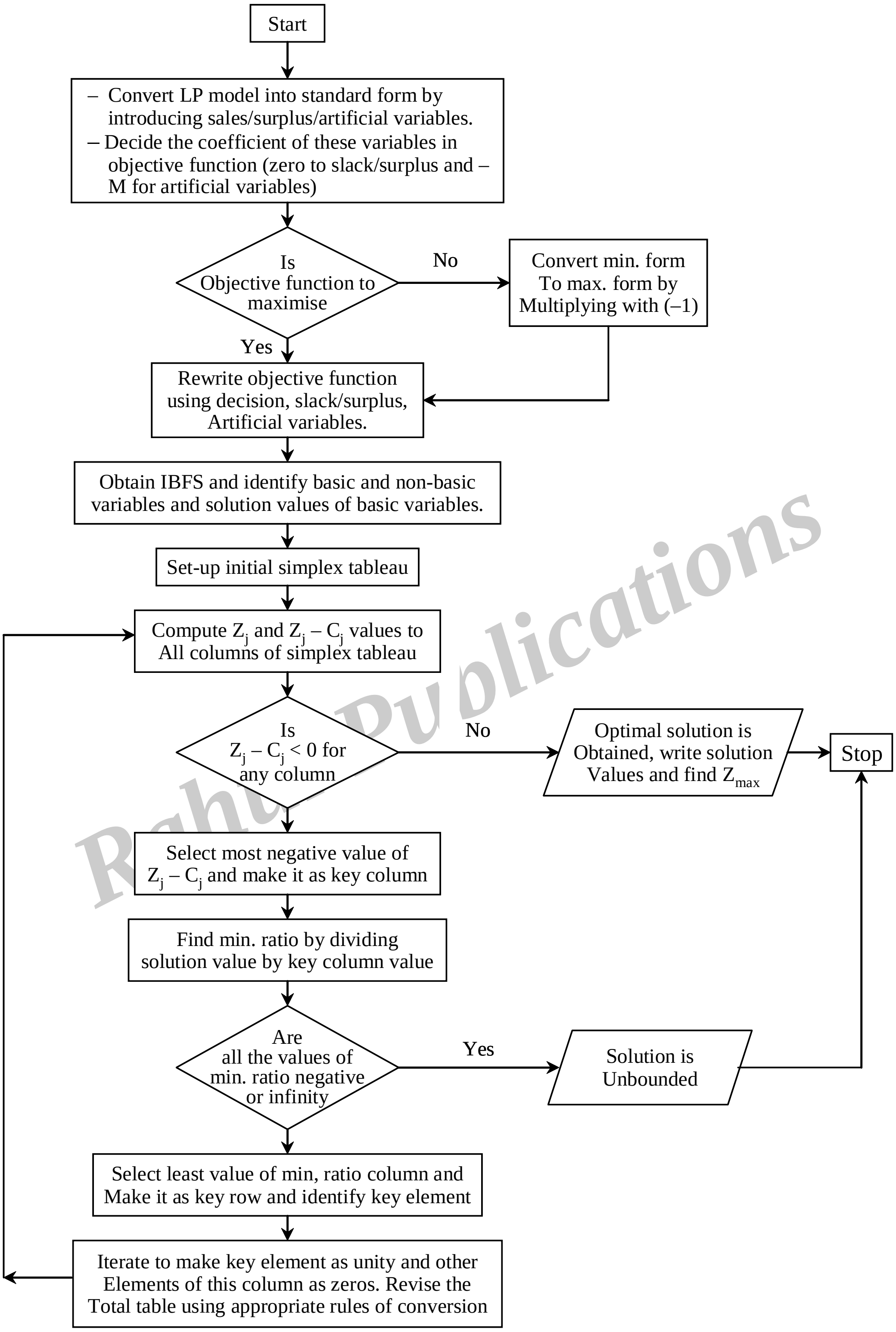



**Fig.: Flow Chart or Simplex Method**

## PROBLEMS

**13. Use simplex method to solve the LPP.**

**Max $Z = 3X_1 + 2X_2$**

**Subject to $X_1 + X_2 \leq 4$**

**$X_1 - X_2 \leq 2$**

**$X_1, X_2 \geq 0$**

*Sol :*

By introducing the slack variables $S_1$, $S_2$, convert the problem in standard form.

Max $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2$

Subject to $X_1 + X_2 + S_1 = 4$

$X_1 - X_2 + S_2 = 2$

$X_1, X_2, S_1, S_2 \geq 0$

$$\begin{bmatrix} X_1 & X_2 & S_1 & S_2 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

An initial basic feasible solution is given by

$X_B = B^{-1}$ b, where

$B = I_2$, $X_B = (S_1, S_2)$.

i.e., $(S_1\ S_2) = I_2(4, 2) = (4, 2)$.

**Initial Simplex Table**

$Z_j = C_B\, a_j$

$$Z_1 - c_1 = C_B\, a_1 - c_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3 = -3$$

$$Z_2 - c_2 = C_B\, a_2 - c_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1 - 1) - 2 = -2$$

$$Z_3 - c_3 = C_B\, a_3 - c_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1 - 0) - 0 = -0$$

$$Z_4 - c_4 = C_B\, a_4 - c_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (0 - 1) - 0 = -0$$

| | | $C_j$ | 3 | 2 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ | Min $\frac{X_B}{X_1}$ |
| 0 | $S_1$ | 4 | 1 | 1 | 1 | 0 | 4/1 = 4 |
| ←0 | $S_2$ | 2 | ① | –1 | 0 | 1 | 2/1 = 2 |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | |
| | $Z_j - C_j$ | | –3↑ | –2 | 0 | 0 | |

Since there are some $Z_j - C_j < 0$, the current basic feasible solution is not optimum.

Since $Z_1$ – C, = – 3 is the most negative, the corresponding non basic variable X, enters the basis.

The column corresponding to this X, is called the key column.

$$\text{To find the ratio} = \text{Min}\left\{\frac{X_{Bi}}{X_{ir}}, X_{ir} > 0\right\}$$

$$= \text{Min}\left\{\frac{4}{1}, \frac{2}{1}\right\} = 2 \text{ which corresponds to } S_2.$$

∴ The leaving variable is the basic variable $S_2$. This row is called the key row. Convert the leading element $X_{21}$ to units and all other elements in its column n i.e. ($X_1$) to zero by using the formula:

$$\text{New element} = \text{Old element} - \left[\frac{\text{Product of elements in key row and key column}}{\text{key element}}\right]$$

To apply this formula, first we find the ratio, namely

$$\frac{\text{the element to be zero}}{\text{key element}} = \frac{1}{1} = 1$$

Apply this ratio, for the number of elements that are converted in the key row. Multiply this ratio by key row element as shown below.

$1 \times 2 = 2$

$1 \times 1 = 1$

$1 \times -1 = -1$

$1 \times 0 = 0$

$1 \times 1 = 1$

Now subtract this element from the old element. The elements to be converted into zero, is called the old element row. Finally we have

4 –1 × 2 = 2

1 –1 × 1 = 0

1 – 1 × –1 = 2

1 – 1 × 0 = 1

0 – 1 × 1 = –1

∴ The improved basic feasible solution is given in the following simplex table.

**First Iteration**

| | | $C_j$ | 3 | 2 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ | Min $\frac{X_B}{X_2}$ |
| ← 0 | $S_1$ | 2 | 0 | (2) | 1 | – 1 | 2/1 = 1 |
| 3 | $X_1$ | 2 | 1 | – 1 | 0 | 1 | – |
| | $Z_j$ | 6 | 3 | – 3 | 0 | 0 | |
| | $Z_j – C_j$ | | 0 | –5↑ | 0 | 0 | |

Since $Z_2 – C_2$ is most negative, $X_2$ enters the basis.

To find Min $\left(\frac{X_B}{X_{i2}}, X_{i2} > 0\right)$

Min $\left(\frac{2}{2}\right)$ = 1.

This gives the out going variables. Convert the leading element into one. This is done by dividing all the elements in the key row by 2. The remaining element by zero using the formula as shown below –½ is the common ratio. Put this ratio 5 times and multiply each ratio by key row element.

$-\frac{1}{2} \times 2$

$-\frac{1}{2} \times 0$

$-\frac{1}{2} \times 2$

– 1/2 × 2

– 1/2 × – 1

Subtract this from the old element. All the row elements which are converted into zero, are called the old element.

$$2-\left(-\frac{1}{2}\times 2\right) = 3$$

$1-(-1/2 \times 0) = 1$

$-1-(-1/2 \times 2) = 0$

$0-(-1/2 \times 1) = 1/2$

$1-(-1/2 \times -1) = 1/2$

**Second Iteration**

| | | $C_j$ | 3 | 2 | 0 | 0 |
|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ |
| 2 | $X_2$ | 1 | 0 | 1 | ½ | – ½ |
| 3 | $X_1$ | 3 | 1 | 0 | ½ | ½ |
| | $Z_j$ | 11 | 3 | 2 | 5/2 | ½ |
| | $Z_j - C_j$ | | 0 | 0 | 5/2 | ½ |

Since all $Z_j - C_j \geq 0$, the solution is optimum, The optimal solution is Max Z = 11, $X_1 = 3$, and $X_2 = 1$.



**14. Solve the LPP**

**Max Z = $3X_1 + 2X_2$**

**Subject to $4X_1 + 3X_2 \leq 12$**

**$4X_1 + X_2 \leq 8$**

**$4X_1 - X_2 \leq 8$**

**$X_1, X_2 \geq 0$**

*Sol :*

Convert the inequality of the constraint into an equation by adding slack variables $S_1, S_2, S_3$,....

Max $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2 + 0S_3$

Subject to $4X_1 + 3X_1 + S_1 = 12$

$4X_1 + X_2 + S_2 = 8$

$4X_1 - X_2 + S_3 = 8$

$X_1, X_2, S_1, S_2, S_3 \geq 0$

$$\begin{bmatrix} X_1 & X_2 & S_1 & S_2 & S_3 \\ 4 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 \\ 4 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 8 \end{bmatrix}$$

**Initial Table**

| | | $C_j$ | 3 | 2 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | Min $\frac{X_B}{X_1}$ |
| 0 | $S_1$ | 12 | 4 | 3 | 1 | 0 | 0 | 12/4 = 3 |
| 0 | $S_2$ | 8 | 4 | 1 | 0 | 1 | 0 | 8/4 = 2 |
| ← 0 | $S_3$ | 8 | (4) | –1 | 0 | 0 | 1 | 8/4 = 2 |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | |
| | $Z_j - C_j$ | | –3↑ | –2 | 0 | 0 | 0 | |

$\because$ $Z_1 - C_1$ is most negative, $X_1$ enters the basis. And the min $\left(\frac{X_B}{x_{ij}}, x_{ij} > 0\right)$ = min (3, 2, 2) = 2 gives $S_3$ as the leaving variable.

Convert the leading element into, by dividing key row element by 4 and the remaining elements into 0.

**First Iteration**

| | | $C_j$ | 3 | 2 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | Min $\frac{X_B}{X_2}$ |
| 0 | $S_1$ | 4 | 0 | 4 | 1 | 0 | – 1 | 4/4 = 1 |
| ←0 | $S_2$ | 0 | 0 | (2) | 0 | 1 | – 1 | 0/2 = 1 |
| 3 | $X_1$ | 2 | 1 | –1/4 | 0 | 0 | 1/4 | – |
| | $Z_j$ | (6) | 3 | – 3/4 | 0 | 0 | 3/4 | |
| | $Z_j - C_j$ | | 0 | –11/4↑ | 0 | 0 | 3/4 | |

$8 - \frac{4}{4} \times 8 = 0$　　$12 - \frac{4}{4} \times 8 = 4$

$4 - \frac{4}{4} \times 4 = 0$　　$4 - \frac{4}{4} \times 4 = 0$

$1 - \frac{4}{4} \times -1 = 2$　　$3 - \frac{4}{4} \times -1 = 4$

$0 - \frac{4}{4} \times 0 = 0$　　$1 - \frac{4}{4} \times 0 = 1$

$1 - \frac{4}{4} \times 0 = 1$　　$0 - \frac{4}{4} \times 0 = 0$

$0 - \frac{4}{4} \times 1 = -1$　　$10 - \frac{4}{4} \times 1 = -1$

Since $Z_2 - C_2 = -3/4$ is the most negative $x_2$ enters the basis. To find the outgoing variable, find min $\left( {x_B}/{x_{i2}}, x_{i2} > 0 \right)$ min $\left( \frac{4}{4}, \frac{0}{2}, -1 \right) = 0$.



**First Iteration**

Therefore $S_2$ leaves the basis. Convert the leading element into 1 by dividing the key row elements by 2 and remaining element in that column as zero using the formula.

$$\text{New element} = \text{old element} - \left[ \frac{\text{Product of elements in key row and key column}}{\text{key element}} \right]$$

**Initial Table**

| | | $C_j$ | 3 | 2 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | Min $\frac{X_B}{S_3}$ |
| $\leftarrow 0$ | $S_1$ | 4 | 0 | 0 | 1 | – 2 | (1) | 4/1 = 1 |
| 2 | $X_2$ | 0 | 0 | 1 | 0 | 1/2 | –1/2 | – |
| 3 | $X_1$ | 2 | 1 | 0 | 0 | 1/8 | 1/8 | 2/1/8 = 16 |
| | $Z_j$ | 6 | 3 | 2 | 0 | 11/8 | –5/8 | |
| | $Z_j - C_j$ | | 0 | 0 | 0 | 11/8 | $-5/8 \uparrow$ | |

**Second Iteration**

Since $Z_5 - C_5 = -5/8$ is mot negative $S_3$ enters the basis and

$$\text{Min}\left(\frac{X_B}{S_{13}}, S_{i3}\right) = \text{Min}\left(\frac{4}{1}, -1\frac{2}{1/18}\right) = 4.$$

Therefore, $S_1$ leaves the basis, Convert the leading element into one and remaining elements as zero.

**Third iteration**

| | | $C_j$ | 3 | 2 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ |
| 0 | $S_3$ | 4 | 0 | 0 | 1 | – 2 | 1 |
| 2 | $X_2$ | 2 | 0 | 1 | 1/2 | –1/2 | 0 |
| 3 | $X_1$ | 3/2 | 1 | 0 | –1/8 | 3/8 | 0 |
| | $Z_j$ | 17/2 | 3 | 2 | 5/8 | 1/8 | 0 |
| | $Z_j - C_j$ | | 0 | 0 | 5/8 | 1/8 | 0 |

Since all $Z_j - C_j \geq 0$, the solutionis optimum and it is given by $X_1 = 3/2$, $X_2 = 2$ and. Max $Z = 17/2$.



**Q10. Define Artificial Variables Techniques.**

*Ans :*

LPP in which constraints may also have $\geq$ and = signs after ensuring that all $b_i \geq 0$ are considered in this section. In such cases basis matrix cannot be obtained as an identify matrix in the starting simplex table, therefore we introduce a new type of variable called the artificial variable. These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. To solve such LPP there are two methods.

(i) The Big M Method (or) Method of Penalties

(ii) The Two-phase Simplex Method.

**Q11. Explain the various steps involved in Big M Method.**

*Ans :* **(Imp.)**

The following steps are involved in solving an LPP using the Big M method.

**Step 1**

Express the problem in the standard form.

**Step 2**

Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type $\geq$ or $=$. However, addition of these artificial variable causes violation of the corresponding constraints. Therefore, we would like to get rid of these variables and would not allow them to appear in the final solution. This is achieved by assigning a very large penalty (-M for maximization and M for minimization) in the objective function.

**Step 3**

Solve the modified LPP by simplex method, until any one of the three cases may arise.

1. If no artificial variable appears in the basis and the optimality conditions are satisfied, then the current solution is an optimal basic feasible solution.
2. If at least one artificial variable in the basis at zero level and the optimality condition is satisfied then the current solution is an optimal basic feasible solution (though degenerated solution).
3. If at least one artificial variable appears in the basis at positive level and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function, since it contains a very large penalty M and is called pseudo optimal solution.

**Note**

While applying simplex method, whenever an artificial variable happens to leave the basis, we drop that artificial variable and omit all the entries corresponding to its column from the simplex table.



**PROBLEMS**

**15. Use penalty method to**

**Maximize $Z = 3x_1 + 2x_2$**

**Subject to the constraints**

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

*Sol :*

By introducing slack variable $S_1 \geq 0$, surplus variable $S_2 \geq 0$ and artificial variable $A_1 \geq 0$, the given LPP can be reformulated as:

Maximize $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1$

Subject to $2x_1 + x_2 + S_1 = 2$

$3x_1 + 4x_2 - S_2 + A_1 = 12$

The starting feasible solution is $S_1 = 2$, $A_1 = 12$.

**Initial Table**

| | | $C_j$ | 3 | 2 | 0 | 0 | – M | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $A_1$ | Min $x_B/x_2$ |
| ← 0 | $S_1$ | 2 | 2 | ① | 1 | 0 | 0 | 2/1 = 2 |
| – M | $A_1$ | 12 | 3 | 4 | 0 | –1 | 1 | 12/4 = 3 |
| | $Z_j$ | –12M | –3M | –4M | 0 | M | –M | |
| | $Z_j – C_j$ | – | – 3M –3 | – 4M – 2 | 0 | M | 0 | |
| | | | | ↑ | | | | |

Since some of $Z_j – C_j \leq 0$ the current feasible solution is not optimum. Choose the most negative $Z_j – C_j = – 4M –2$.

∴ $x_2$ variable enters the basis, and the basic variable $S_1$ leaves the basis.

**First iteration**

| | | $C_j$ | 3 | 2 | 0 | 0 | – M |
|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $A_1$ |
| 2 | $x_2$ | 2 | 2 | 1 | 1 | 0 | 0 |
| – M | $A_1$ | 4 | – 5 | 0 | – 4 | – 1 | 1 |
| | $Z_j$ | 4 – 4M | 4 + 5M | 2 | 2 + 4M | M | –M |
| | $Z_j – C_j$ | | 5M + 1 | 0 | 4M + 2 | M | 0 |

Since all $Z_j – C_j \geq 0$ and an artificial variable appears in the basis at positive level, the given LPP does not possess any feasible solution. But the LPP possesses a pseudo optimal solution.



**16. Solve the LPP**

**Minimize $Z = 4x_1 + x_2$**

**Subject to $3x_1 + x_2 = 3$**

**$4x_1 + 3x_2 \geq 6$**

**$x_1 + 2x_2 \leq 3$**

**$x_1, x_2 \geq 0.$**

*Sol :*

Since the objective function is minimization, we covert it into maximization using.

Min Z = – Max (z)

Maximize $Z = –4x_1 – x_2$

Subject to $3x_1 + x_2 = 3$

$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0.$$

Convert the given LPP into standard form by adding artificial variables $A_1$, $A_2$, surplus variable $S_1$ and slack variable $S_2$ to get the initial basic feasible solution.

Maximize $Z = -4x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject $3x_1 + x_2 + A_1 = 3$

$4x_1 + 3x_2 - S_1 + A_2 = 6$

$x_1 + 2x_2 + S_2 = 3$

The starting feasible solution is $A_1 = 3$, $A_2 = 6$, $S_2 = 3$.

**Initial Solution**

| | | $C_j$ | – 4 | – 1 | – M | 0 | –M | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $A_1$ | $S_2$ | $A_2$ | $S_2$ | Min $\frac{X_B}{S_2}$ |
| – M | $A_1$ | 3 | 3 | 1 | 1 | 0 | 0 | 0 | 3/3 = 1 |
| – M | $A_2$ | 6 | 4 | 3 | 0 | – 1 | 1 | 0 | 6/4 = 1.5 |
| ← 0 | $S_2$ | 3 | 1 | (2) | 0 | 0 | 0 | 1 | 3/1 = 3 |
| | $Z_j$ | – 9M | – 7M | – 4M | – M | M | – M | 0 | |
| | $Z_j - C_j$ | | –7M + 4 | –4 M+1↑ | 0 | M | 0 | 0 | |

Since some of the $Z_j - C_j \leq 0$, the current feasible solutio is not optimum. As $Z_1 - C_1$ is most negative, $x_1$ enters the basis and the basic variable $A_2$ leaves the basis.

**First Iteration**

| | | $C_j$ | – 4 | – 1 | – M | 0 | – M | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $A_1$ | $S_1$ | $A_2$ | $S_2$ | Min $\frac{X_B}{X_1}$ |
| – M | $A_1$ | 3/2 | 5/2 | 0 | 1 | 0 | 0 | – 1/2 | 3/5 |
| ← – M | $A_2$ | 3/2 | (5/2) | 0 | 0 | – 1 | 1 | – 3/2 | 3/5 |
| – 1 | $x_2$ | 3/2 | 1/2 | 1 | 0 | 0 | 0 | 1/2 | 3 |
| | $Z_j$ | – 3M – 3/2 | – 5M – 1/2 | – 1 | – M | + M | – M | 2M – 1/2 | |
| | $Z_j - C_j$ | | –5M + 7/2 ↑ | 0 | 0 | M | 0 | 2M – 1/2 | |

Since $Z_j - C_j$ is negative, the current feasible solution is not optimum. Therefore, $x_1$ variable enters the basis and the artificial variable $A_2$ leaves the basis.

**Second Iteration**

| | | $C_j$ | – 4 | – 1 | – M | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $A_1$ | $S_1$ | $S_2$ | Min $\frac{X_B}{X_1}$ |
| ← – M | $A_1$ | 0 | 0 | 0 | 1 | (1) | 1 | 0 |
| – 4 | $x_1$ | 3/5 | 1 | 0 | 0 | $-\frac{2}{5}$ | $-\frac{3}{5}$ | – |
| – 1 | $x_2$ | 6/5 | 0 | 1 | 0 | $-\frac{1}{5}$ | $\frac{4}{5}$ | – |
| | $Z_j$ | $-\frac{18}{5}$ | – 4 | – 1 | – M | $-M+\frac{9}{5}$ | $-M+\frac{8}{5}$ | |
| | $Z_j - C_j$ | | 0 | 0 | 0 | $-M+\frac{9}{5}$ ↑ | $-M+\frac{8}{5}$ | |

Since $Z_4 - C_4$ is most negative, S1 enters the basis and the artificial variable $A_1$ leaves the basis.

**Third Iteration**

| | | $C_j$ | – 4 | – 1 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ | Min $\frac{X_B}{S_2}$ |
| ← 0 | $S_1$ | 0 | 0 | 0 | 1 | (1) | 0 |
| – 4 | $x_1$ | 3/5 | 1 | 0 | 0 | $-\frac{1}{5}$ | – |
| – 1 | $x_2$ | 6/5 | 0 | 1 | 0 | 1 | 6/5 |
| | $Z_j$ | $-\frac{18}{5}$ | – 4 | – 1 | 0 | –1/5 | |
| | $Z_j - C_j$ | | 0 | 0 | 0 | – 1/5↑ | |

Since $Z_4 - C_4$ is most negative, $S_2$ enters the basis and $S_1$ leaves the basis,

**Fourth Iteration**

| | | $C_j$ | – 4 | – 1 | 0 | 0 |
|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ |
| 0 | $S_2$ | 0 | 0 | 0 | 1 | 1 |
| – 4 | x1 | 3/5 | 1 | 0 | 1/5 | 0 |
| – 1 | $x_2$ | 6/5 | 0 | 1 | – 1 | 1 |
| | $Z_j$ | – 18/5 | – 4 | – 1 | 1/5 | 0 |
| | $Z_j - C_j$ | | 0 | 0 | 1/5 | 0 |

Since all $Z_j - C_j \geq 0$ the solution is optimum and is given by $x_1 = 3/5$. $x_2 = 6/5$, and Max $Z = -18/5$.

$\therefore$ Min $Z = -$ Max $(-Z) = 18/5$.

**17. Solve by Big M method.**

**Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$**

**Subject to $x_1 + 2x_2 + 3x_3 = 15$**

**$2x_1 + x_2 + 5x_3 = 20$**

**$x_1 + 2x_2 + x_2 + x_4 = 10$**

*Sol :*

Since the constraints are equations, introduce artificial variables $A_1, A_2 \geq 0$. The reformulated problem is given as follows.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

Subject to $x_1 + 2x_2 + 3x_3 + A_1 = 15$

$2x_1 + x_2 + 5x_3 + A_2 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

Initial solution is given by $A_1 = 15$, $A_2 = 20$ and $x_4 = 10$.

**Initial table**

| | | $C_j$ | 1 | 2 | 3 | – 1 | – M | – M | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $x_4$ | $A_1$ | $A_2$ | Min $\frac{X_B}{X_3}$ |
| – M | $A_1$ | 15 | 1 | 2 | 3 | 0 | 1 | 0 | 15/3 = 5 |
| ← – M | $A_2$ | 20 | 2 | 1 | (5) | 0 | 0 | 1 | 20/5 = 4 |
| – 1 | $x_4$ | 10 | 1 | 2 | 1 | 1 | 0 | 0 | 10/1 = 10 |
| | $Z_j$ | – 35 M – 10 | – 3M – 1 | – 3M – 2 | – 8 M – 1 | – 1 | – M | – M | |
| | $Z_j - C_j$ | | – 3M – 2 | – 3M – 4 | – 8M – 4 ↑ | 0 | 0 | 0 | |

Since $Z_3 - C_3$ is most negative, $x_3$ enters the basis and the basic variable $A_2$ leaves the basis.

**First Iteration**

| | | $C_j$ | 1 | 2 | 3 | – 1 | – M | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $x_4$ | $A_1$ | Min $\frac{X_B}{X_2}$ |
| ← – M | $A_1$ | 3 | –1/5 | (7/5) | 0 | 0 | 1 | $\frac{3}{7/5}=\frac{15}{7}$ |
| 3 | $x_3$ | 4 | 2/5 | 1/5 | 1 | 0 | 0 | $\frac{4}{1/5}=\frac{20}{1}$ |
| – 1 | $x_4$ | 6 | 3/5 | 9/5 | 0 | 1 | 0 | $\frac{6}{9/5}=\frac{30}{9}$ |
| | $Z_j$ | – 3M + 6 | $\frac{1}{5}M+\frac{3}{5}$ | $-\frac{7}{5}M-\frac{6}{5}$ | 3 | –1 | – M | |
| | $Z_j - C_j$ | | $\frac{1}{5}M-\frac{2}{5}$ | $-\frac{7}{5}M-\frac{16}{5}$ | 0 | 0 | 0 | |
| | | | | ↑ | | | | |

Since $Z_2 - C_2$ is most negative $x_2$ enters the basis and the basic variable $A_1$ leaves the basis.

**Second Iteration**

| | | $C_j$ | 1 | 2 | 3 | – 1 | |
|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $x_4$ | Min $\frac{X_B}{X_1}$ |
| 2 | $x_2$ | 15/7 | $-\frac{1}{7}$ | 1 | 0 | 0 | – |
| 3 | $x_3$ | 25/7 | 3/7 | 0 | 1 | 0 | 25/3 |
| ← – 1 | $x_4$ | 15/7 | (6/7) | 0 | 0 | 1 | 15/6 |
| | $Z_j$ | $\frac{90}{7}$ | $\frac{1}{7}$ | 2 | 3 | –1 | |
| | $Z_j - C_j$ | | – 6/7 | 0 | 0 | 0 | |
| | | | ↑ | | | | |

Since $Z_1 - C_1 = -6/7$ in negative, the current feasible solution is not optimum. Therefore, $x_1$ enters the basis and the basic variable $x_4$ leaves the basis.

**Third Iteration**

| | | $C_j$ | 1 | 2 | 3 | – 1 |
|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $x_4$ |
| 2 | $x_2$ | 15/6 | 0 | 1 | 0 | 1/6 |
| 3 | $x_3$ | 15/6 | 0 | 0 | 1 | 3/6 |
| 1 | $x_1$ | 15/6 | 1 | 0 | 0 | 7/6 |
| | $Z_j$ | 15 | 1 | 2 | 3 | 3 |
| | $Z_j – C_j$ | | 0 | 0 | 0 | 4 |

Since all $Z_j – C_j \geq 0$ the solution is optimum and is given by $x_1 = x_2 = x_3 = 15/6 = 5/2$, and Max Z = 15.

**Q12. What is Two Phase Method ? Explain the steps involved in Two Phase Method to solve a LPP.**

*Ans :* **(Imp.)**

The two-phase simplex method is another method to solve a given LPP involving some artificial variable. The solution is obtained in two phases.

**Phase I**

In this phase, we construct an auxiliary LPP leading to a final simplex lable containing a basic feasible solution to the original problem.

**Step 1**

Assign a cost-1 to each artificial variable and a cost 0 to all other variables and get a new objective function $Z^* = – A_1 – A_2 – A_3$ ... where $A_1$ are artificial variable.

**Step 2**

Write down the auxiliary LPP in which the new objective function is to be maximized subject to the given set of constraints.

**Step 3**

Solve the auxiliary LPP by simplex method until either of the following three cases arise:

(i) Max $Z^* < 0$ and at least one artificial variable appears in the optimum basis at positive level.

(ii) Max $Z^* = 0$ and at least one artificial variable appears in the optimum basis at zero level.

(iii) Max $Z^* = 0$ and no artificial variable appears in the optimum basis.

In case (i), given LPP does not possess any feasible solution, where-as in case (ii) and (iii) we go to phase II.

**Phase II**

Use the optimum basic feasible solution of phase I as a starting solution for the original LPP. Assign the actual costs to the variable in the objective function and a zero cost to every artificial variable in the basis at zero level. Delete the artificial variable column from the table which is eliminated from the basis in phase I. Apply simplex method to the modified simplex table obtained at the end of phase I till an optimum basic feasible is obtained or till there is an indication of unbounded solution.

## PROBLEMS

**18. Using two phase simplex method to solve.**

**Maximize $Z = 5x_1 - 4x_2 + 3x_3$**

**Subject to $2x_1 + x_2 - 6x_3 = 20$**

**$6x_1 + 5x_2 + 10x_3$**

**$8x_1 - 3x_2 + 6x_3 \leq 50$**

**$x_1, x_2, x_3 \geq 0$**

*Sol :*

Introducing slack variables $S_1, S_2 \geq 0$ and an artificial variable $A_1 \geq 0$ in the constraints of the given LPP, the problem is reformulated in the standard form. Initial basic feasible solution is given by $A_1 = 20$, $S_1 = 76$ and $S_2 = 50$.

Assigning a cost -1 to the artificial variable $A_1$ and cost 0 to other variables, the objective function of the auxiliary LPP is

Maximize $Z^* = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - 1A_1$

Subject to $2x_1 + x_2 - 6x_3 + A_1 = 20$

$6x_1 + 5x_2 + 10x_3 + S_1 = 76$

$8x_1 - 3x_2 + 6x_3 + S_2 = 50$

$x_1, x_2, x_3, S_1, S_2, A_1 \geq 0$

**Initial table**

| | | $C_j$ | 0 | 0 | 0 | –1 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $A_1$ | $S_1$ | $S_2$ | Min $\frac{X_B}{X_1}$ |
| – 1 | $A_1$ | 20 | 2 | 1 | –6 | 1 | 0 | 0 | 20/2 = 10 |
| 0 | $S_1$ | 76 | 6 | 5 | 10 | 0 | 1 | 0 | 76/6 = 12.66 |
| ←0 | $S_2$ | 50 | (8) | – 3 | 6 | 0 | 0 | 1 | 50/8 = 6.25 |
| | $Z_j$ | – 20 | – 2 | – 1 | 6 | – 1 | 0 | 0 | |
| | $Z_j - C_j$ | | – 2↑ | – 1 | 6 | 0 | 0 | 0 | |

| | | $C_j$ | 0 | 0 | 0 | –1 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $A_1$ | $S_1$ | $S_2$ | Min $\frac{X_B}{X_2}$ |
| ←– 1 | $A_1$ | 15/2 | 0 | (7/4) | – 15/2 | 1 | 0 | –1/4 | 30/7 |
| 0 | $S_1$ | 77/2 | 0 | 29/4 | 11/2 | 0 | 1 | –3/4 | 154/29 |
| 0 | $x_1$ | 25/4 | 1 | –3/8 | 3/4 | 0 | 0 | 1/8 | – |
| | $Z_j$ | – 15/2 | 0 | – 7/4 | 15/2 | – 1 | 0 | 1/4 | |
| | $Z_j – C_j$ | | 0 | – 7/4↑ | 15/2 | 0 | 0 | 1/4 | |
| 0 | $x_2$ | 30/7 | 0 | 1 | –30/7 | 4/7 | 0 | –1/7 | |
| 0 | $S_1$ | 52/7 | 0 | (1) | 256/7 | –29/7 | 1 | 2/7 | |
| 0 | $x_1$ | 55/7 | 1 | 0 | –6/7 | 3/4 | 0 | 1/14 | |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| | $Z_j – C_j$ | | 0 | 0 | 0 | 0 | 0 | 0 | |

Since all $Z_j = C_j \geq 0$, an optimum solution to the auxiliary LPP has been obtained. Also Max Z* = 0 with no artificial variable in the basis. We go to phase II.

**Phase II**

Consider the final simplex table of phase I. Consider the actual cost associated with the original variables. Delete the artificial variable $A_1$ column from the table as it is eliminated in phase I.

| | | $C_j$ | 5 | –4 | 3 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $S_1$ | $S_2$ |
| – 4 | $x_2$ | 30/7 | 0 | 1 | –30/7 | 0 | –1/7 |
| 0 | $S_1$ | 52/7 | 0 | 0 | 256/7 | 1 | 2/7 |
| 5 | $x_1$ | 55/7 | 1 | 0 | –6/7 | 0 | 1/14 |
| | $Z_1$ | 155/7 | 5 | –4 | 90/7 | 0 | 13/14 |
| | $Z_j – C_j$ | 0 | 0 | 0 | 69/7 | 0 | 13/14 |

Since all $Z_j – C_j \geq 0$ an optimum basic feasible solution has been reached. Hence, an optimum feasible solution to the given LPP is $x_1$ = 55/7, $x_2$ = 30/7, $x_3$ = 0 and Max Z = 155/7.

**19. Solve using, two-phase simplex method**

**Solve the following LPP**

**Min $Z = 10X + 15Y$**

**S.T.C. ...,**

$Y \geq 3$

$X - Y \geq 0$

$Y \leq 12;$

$X + Y \leq 30$

$X \leq 20$ **and** $X, Y \geq 0$

*Sol :*

Rewritting the objective function and constraint set in standard form by introducing slack, surplus and artificial variables, we get,

Maxmize $Z = 10X + 15Y$

Subject to $Y - S_1 + A_1 = 3$

$X - Y - S_2 + A_2 = 0$

$Y + S_3 = 12$

$X + Y + S_4 = 30$

$X + S_5 = 20$

$X, Y \geq 0, S_1, S_2, S_3, S_4, S_5 \geq 0$ and $A_1, A_2 \geq 0$

(where X, Y are decision variables, $S_1$ and $S_2$ are surplus variables, $S_3$, $S_4$ and $S_5$ are variables and $A_1$ and $A_2$ are artificial variables).

**Phase - I**

For Phase - I, we consider the objective function as

Max. $Z_1 = -A_1 - A_2$

$Y - S_1 + A_1 = 3;\ X - Y - S_2 + A_2 = 0;\ Y + S_3 = 12,$

$X + Y + S_4 = 30;\ X + S_5 = 20$ and

$X, Y \geq 0,\ S_j \geq 0,\ A_1, A_2 \geq 0\ (j = 1, 2, 3, 4, 5)$

**IBFS**

Basic variables : $A_1 = 3,\ A_2 = 0,\ S_3 = 12,\ S_4 = 30,\ S_5 = 20$

Non basic variables : $X = Y = S_1 = S_2 = 0$

### Iteration Tableau I

Entering Variable (Y) — Delete column for next Iteration ($A_1$)

| $C_B$ | BV | $C_j$ / SV | 0 X | 0 (Y) | 0 $S_1$ | 0 $S_2$ | 0 $S_3$ | 0 $S_4$ | 0 $S_5$ | –1 $A_1$ | –1 $A_2$ | Min ratio |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| –1 | ($A_1$) | 3 | 0 | [1] KE | –1 | 0 | 0 | 0 | 0 | 1 | 0 | 3/1 ← KR |
| –1 | $A_2$ | 0 | 1 | –1 | 0 | –1 | 0 | 0 | 0 | 0 | 1 | – ve (Ignore) |
| 0 | $S_3$ | 12 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 12/1 |
| 0 | $S_4$ | 30 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 30/1 |
| 0 | $S_5$ | 20 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | ∞ (Ignore) |
| Leaving variable | | $Z_j$ | –1 | –1 | 1 | 1 | 0 | 0 | 0 | –1 | –1 | |
| | | $Z_j – C_j$ | –1 | –1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |

Key column (Y)

**Note**

We an choose X or Y as key column here, but better to choose Y and if X is taken, it may lead to confusions.

### Iteration Tableau II

Entering Variable (X) — Delete column ($A_2$)

| $C_B$ | BV | $C_j$ / SV | 0 (X) | 0 Y | 0 $S_1$ | 0 $S_2$ | 0 $S_3$ | 0 $S_4$ | 0 $S_5$ | –1 $A_2$ | Min ratio | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | Y | 3 | 0 | 1 | –1 | 0 | 0 | 0 | 0 | 0 | ∞ | $R_1^N \to R_1^0$ |
| –1 | ($A_2$) | 3 | [1] KE | 0 | –1 | –1 | 0 | 0 | 0 | 1 | 3/1 ← | $R_2^N \to R_2^0 + R_1^N$ |
| 0 | $S_3$ | 9 | 0 | 0 | –1 | 0 | 1 | 0 | 0 | 0 | ∞ | $R_3^N \to R_3^0 - R_1^N$ |
| 0 | $S_4$ | 27 | 1 | 0 | –1 | 0 | 0 | 1 | 0 | 0 | 27/1 | $R_4^N \to R_4^0 - R_1^N$ |
| 0 | $S_5$ | 20 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 20/1 | $R_5^N \to R_5^0$ |
| Leaving variable | | $Z_j$ | –1 | 0 | 1 | 1 | 0 | 0 | 0 | –1 | | |
| | | $Z_j – C_j$ | –1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | | |

Key column (X)

**Note :**

In the above iteration, we can observe that there is no need of changing $R_1$ and $R_5$ since the desired 1 and 0 are already available.

### Iteration Tableau III

| $C_B$ | BV | $C_j$ SV | 0 X | 0 Y | 0 $S_1$ | 0 $S_2$ | 0 $S_3$ | 0 $S_4$ | 0 $S_5$ | Min ratio | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | Y | 3 | 0 | 1 | –1 | 0 | 0 | 0 | 0 | | $R_1^N \to R_1^0$ |
| 0 | X | 3 | 1 | 0 | –1 | –1 | 0 | 0 | 0 | | $R_2^N \to R_2^0$ |
| 0 | $S_3$ | 9 | 0 | 0 | –1 | 0 | 1 | 0 | 0 | | $R_3^N \to R_3^0$ |
| 0 | $S_4$ | 24 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | | $R_4^N \to R_4^0 - R_2^N$ |
| 0 | $S_5$ | 17 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | | $R_5^N \to R_5^0 - R_2^N$ |
| | | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| | | $Z_j - C_j$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | This is now called auxiliary simplex tableau | |

Since $Z_j - C_j = 0$ for all the variable phase - I computation is complete at this state. Both artificial variables have been replaced from the basis. Therefore we proceed to phase - II now. Now for phase - II, we take above auxiliary simplex tableau with all numericals 'as is' except for the values of $C_j$. These are taken from the new objective function of second place.

**Phase - II :**

Objective Function : Max. $Z2 = 10X + 15Y$

Entering Variable: $S_1$

| $C_B$ | BV | $C_j$ SV | 0 X | 0 Y | 0 $S_1$ | 0 $S_2$ | 0 $S_3$ | 0 $S_4$ | 0 $S_5$ | Min ratio | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | Y | 3 | 0 | 1 | –1 | 0 | 0 | 0 | 0 | – ve | |
| 0 | X | 3 | 1 | 0 | –1 | –1 | 0 | 0 | 0 | – ve | |
| 0 | $S_3$ | 9 | 0 | 0 | –1 | 0 | 1 | 0 | 0 | – ve | |
| 0 | $S_4$ | 27 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | ∞ | |
| 0 | $S_5$ | 17 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 20/1 ← | |
| | | $Z_j$ | 10 | 15 | –25 | –10 | 0 | 0 | 0 | | |
| | | $Z_j - C_j$ | 0 | 0 | –25 | –10 | 0 | 0 | 0 | | |

Leaving variable: $S_5$ Key column: $S_1$

| $C_B$ | BV | $C_j$ SV | 10 X | 15 Y | 0 $S_1$ | 0 $S_2$ | 0 $S_3$ | 0 $S_4$ | 0 $S_5$ | Min ratio | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | Y | 20 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | | $R_1^N \to R_1^0 + R_5^N$ |
| 10 | X | 20 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | | $R_2^N \to R_2^0 + R_5^N$ |
| 0 | $S_3$ | 26 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | | $R_3^N \to R_3^0 + R_5^N$ |
| 0 | $S_4$ | 24 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | | |
| 0 | $S_5$ | 17 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | | $R_5^N \to R_5^0$ |
| | | $Z_j$ | 10 | 15 | 0 | 15 | 0 | 0 | 25 | | Zmax = 10×20 + 5×20 = 500 |
| | | $Z_j - C_j$ | 0 | 0 | 0 | 15 | 0 | 0 | 25 | | |

Since all the values of $Z_j - C_j$ are positive ($\geq$ 0), we arrived, at optimal solution.

The optimal solution is $X = 20$; $Y = 20$

$Z_{max} = 10 \times 20 + 15 \times 20 = 200 + 300 = 500.$

### 1.4.2 Special Cases in Simplex Method

**Q13. Define Unbounded Solution.**

*Ans :*

In some cases if the value of a variable is increased indefinitely, the constraints are not violated. This indicates that the feasible region is unbounded at least in one direction. Therefore, the objective function value can be increased indefinitely. This means that problem has been poorly formulated or conceived.

In simplex method, this can be noticed if $Z_j - Z_j$ value is negative to a variable (entering) which is notified as key column and the ratio of solution value to key column value is either negative or infinity (both are to be ignored) to all the variables. This indicates that no variable is ready to leave the basis, though a variable is ready to enter. We cannot proceed further and the solution is unbounded or not finite.

**PROBLEMS**

**20. Maximise $Z = 4x_1 + 3x_2$**

**Subject to $x_1 \leq 5$**

$x_1 - x_2 \leq 8$

$x_1, x_2 \geq 0$



*Sol :*

Let us introduce slack variables to express inequalities;

Maximise $Z = 4x_1 + 3x_2 + 0.S_1 + 0.S_2$

Subject to $x_1 + S_1 = 5$

$x_1 - x_2 + S_2 = 8$

$x_1, x_2 \geq 0,\ S_1, S_2 \geq 0$

IBFS $x_1 = 0,\quad x_2 = 0$ (Non basic)

$s_1 = 5,\quad s_2 = 8$ (basic)

With usual steps it is solved through the following iterations.

**Iteration Tableau 1**

Entering Variable

| $C_B$ | BV | $C_j$ SV | 4 $x_1$ | 3 $x_2$ | 0 $S_1$ | 0 $S_2$ | Min ratio | Remarks |
|---|---|---|---|---|---|---|---|---|
| 0 | $S_1$ | 5 | 1 | 0 | 1 | 0 | $\frac{5}{1} = 5$ | ← Key Row |
| 0 | $S_2$ | 8 | 1 | –1 | 0 | 1 | $\frac{8}{1} = 8$ | |
| Leaving variable | | $Z_j$ | 0 | 0 | 0 | 0 | | |
| | | $Z_j - C_j$ | –4 | –3 | 0 | 0 | | |

Key Row

**Iteration Tableau 2**

$R_1^N \rightarrow R_1^0; \quad R_2^N \rightarrow R_2^N - R_1^0$

Entering Variable

| $C_B$ | BV | $C_j$ SV | 4 $x_1$ | 3 $x_2$ | 0 $S_1$ | 0 $S_2$ | Min ratio | Remarks |
|---|---|---|---|---|---|---|---|---|
| 4 | $x_1$ | 5 | 1 | 0 | 1 | 0 | $\frac{5}{0} = \infty$ | to be ignored |
| 0 | $S_2$ | 3 | 0 | –1 | –1 | 1 | $\frac{3}{-1} = -ve$ | to be ignored |
| Leaving variable | | $Z_j$ | 0 | 0 | 0 | 0 | No variable is ready to leave the basis | |
| | | $Z_j$ - $C_j$ | 0 | –3 | 4 | 0 | Solution is UNBOUNDED | |

Key Row

From the above tableau - II, it is clear that $x_2$ is entering variable into the basis (key column variable whose $Z_j – C_j$ is negative) but no variable is read to leave since the ratio of solution value to key column value is infinity for $R_1$ and negative for $R_2$ both of which are to be ignored. Thus we cannot proceed further because key row (leaving variable) cannot be found. Thus, this problem yields *no finite solution* or in other words, as *unbounded solution*.

**Q14. Define Multiple Optimal Solution.**

*Ans :*

When the objective function is parallel to one of the constraints, the multiple optimal solutions may exist. As we have seen from graphical solutions, that the optimal solution exists at the extreme point on the feasible region, the multiple optimal solutions will be noticed on at least two points of the binding constraint parallel to that of objective function. Thus in simplex method also at least two solution can be found.

This alternate optima is identified in simplex method by using the following principle.

After reaching the optimality, if at least one of the non-basic (decision) variables possess a zero value in $Z_j – C_j$, the multiple optimal solutions exist.

**21. Maximise $Z = 3x_1 + 6x_2$**

**Subject to $x_1 + x_2 \leq 5$**

**$x_1 + 2x_2 \leq 6$**

**$x_1, x_2 \geq 0$**

*Sol :*

Converting inequalities into equations, we get

Maximise $Z = 3x_1 + 6x_2 + 0.S_1 + 0.S_2$

Subject to $x_1 + x_2 + S_1 = 5$

$x_1 + 2x_2 + S_2 = 6$

$x_1, x_2, S_1, S_2 \geq 0$

IBFS $S_1 = 5,\ S_2 = 6$ (Basic)

$x_1 = 0,\ x_2 = 0$ (Non basic)

**Iteration Tableau I**

Entering Variable: $x_2$

| $C_B$ | BV | $C_j$ / SV | 3 $x_1$ | 6 $x_2$ | 0 $S_1$ | 0 $S_2$ | Min ratio | Remarks |
|---|---|---|---|---|---|---|---|---|
| 0 | $S_1$ | 5 | 1 | 1 | 1 | 0 | 5 | |
| 0 | $S_2$ | 6 | 1 | 2 (Key element) | 0 | 1 | 3 | Key row |
| Leaving variable ($S_2$) | | Zj | 0 | 0 | 0 | 0 | | |
| | | $Z_j - C_j$ | –3 | –6 | 0 | 0 | | |

Key Row

**Iteration Tableau II**

Entering Variable: $x_1$

| $C_B$ | BV | $C_j$ / SV | 3 $x_1$ | 6 $x_2$ | 0 $S_1$ | 0 $S_2$ | Min ratio | Remarks |
|---|---|---|---|---|---|---|---|---|
| 0 | $S_1$ | 2 | $\frac{1}{2}$ (Key element) | 0 | 1 | $-\frac{1}{2}$ | 5 ← Key row | $R_1^N \to R_1^0 - R_2^N$ |
| 6 | $x_2$ | 3 | $\frac{1}{2}$ | 1 | 0 | $\frac{1}{2}$ | 3 | $R_2^N \to \frac{1}{2} R_2^0$ |
| Leaving variable ($S_1$) | | Zj | 3 | 6 | 0 | 3 | Since $Z_j - C_j$ □0 ∀ all variables, | |
| | | $Z_j - C_j$ | 0 | 0 | 0 | 3 | Optimal solution is $Z_{max} = 18$ | |

Key Row

From the above Iteration Tableau - II the optimality is already reached since, $Z_j - C_j \geq 0$ i.e., positive for all the variables. Therefore the solution is

$x_1 = 0,\ x_2 = 3,$

and $Z_{max} = 3 \times 0 + 6 \times 3 = 18$

However, the value of $Z_j - C_j = 0$ for the non basic decision variable $x_1$ in tableau - II. This indicates alternative optimal solution. Therefore choosing first column as key column, if we further iterate, we get the following tableau - III.

**Iteration Tableau III**

| $C_B$ | BV | $C_j$ SV | 3 $x_1$ | 6 $x_2$ | 0 $S_1$ | 0 $S_2$ | Min ratio | Remarks |
|---|---|---|---|---|---|---|---|---|
| 3 | $x_1$ | 4 | 1 | 0 | 2 | –1 | 5 | $R_1^N \to 2R_1^0$ |
| 6 | $x_2$ | 1 | 0 | 1 | –1 | 1 | 3 | $R_2^N \to R_2^0 - \frac{1}{2}R_1^N$ |
| | | Zj | 3 | 6 | 0 | 3 | $Z_j - C_j$ □ for all variable and hence | |
| | | $Z_j$ - $C_j$ | 0 | 0 | 0 | 3 | Optimal solution is $Z_{max}$ = 18 | |

This tableau - III yields another solution as,

$x_1 = 4, \; x_2 = 1,$

and $Z_{max} = 3 \times 4 + 6 \times 1 = 18$

We can observe that $Z_j - C_j$ value will be always zero for basic variables in any simplex tableau [observe for $S_1$ and $S_2$ in Tableau - I, for $S_1$, and $x_2$ in Tableau - II and for $x_1$ and $x_2$ in Tableau - III they have $Z_j - C_j$ values zero]. However, in Tableau - II, apart from basic variables $S_1$ and $x_2$, the other variable $x_1$ is also showing $Z_j - C_j = 0$. This indicates that $x_1$ is also worth coming into basis, as it is behaving similar to any basic variables. Thus it provides can alternate optimal solution.

However, if all the basic variables are not replaced by decision variables it does not mean any multiple solution unless this condition (i.e., $Z_j - C_j = 0$ is obeyed any non-basic decision variable.

**Q15. Define Infeasible Solution.**

*Ans :*

There may not exist any solution to certain LPP. This in LPP jargon is said to be infeasible solution. In this type of solution, there exists no feasible region. We do not get any infeasible solution with all constraints as 'less than or equal to type'.

**22. Maximise $Z = 2x_1 + 3x_2$**

**Subject to $x_1 \leq 5$**

**$x_1 - x_2 \geq 10$**

**$x_1 \geq 0, \; x_2 \geq 0$**

*Sol :*

Converting the problem into canonical form,

Maximise $Z = 2x_1 + 3x_2$

Subject to $x_1 \leq 5$

$-x_1 + x_2 \leq -10$

$x_1 \geq 0, \; x_2 \geq 0$

Now introduce slack variables, to get equations

$$x_1 + S_1 = 5$$
$$-x_1 + x_2 + S_2 = -10$$
$$x_1, x_2, S_1 \text{ and } S_2 \geq 0$$

IBFS : $\left.\begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}\right\}$ Non basic

$\left.\begin{array}{l} S_1 = 5 \\ S_2 = -10 \end{array}\right\}$ Basic

From the condition $S_2 \geq 0$, $S_2$ can not take value –10, therefore no IBFS can exist with such less than or equal to type constraints.

So also, this method of converting '$\geq$' type constraint to '$\leq$' type with a negative sign can not yield any result and not suitable in such cases.

## 1.5 Special Cases of Linear Programming

**Q16. Discuss about special cases in finding solution to LPP by graphical method.**

*Ans :* **(Imp.)**

All LP problems does not have unique optimal solutions, so these problems are required as special cases. Some of the special cases in graphical method of LPP are unbounded solution, infeasible solution and multiple optimal solution.

### 1. Unbounded Solution

A solution to LPP that has no limits on the constraints is called an unbounded solution i.e., the common feasible region, identified in graphical method is not bounded in any respect and the primary variable can take any value in the unbounded region.

In such situation, there may or may not be optimal solution to the LPP. If the value of the objective function goes on changing in the unbounded region, then there will be no optimal solution. If the value of the objective function in the region is less than the value of the vertex, the optimal solution will be existing.

### 2. Infeasible Solution

A solution that does not satisfy the constraints of an LP problem is called infeasible solution i.e., it is a condition that arises when there is no solution to a LPP that satisfies all the constraints. Graphically, it can be concluded as when there is no feasible solution region existing. This generally happens when the LPP was formulated with conflicting constraints.

### 3. Multiple Optimal Solution

Multiple optimal solution refers to a solution in which optimal objective function value can be generated from more than one solution. It is also termed as alternative optimal solution.

A solution can be have multiple solution if it satisfies the following two conditions,

(a) Constraint which creates the limits for feasible solutions region must be equal to slope of the objective function.

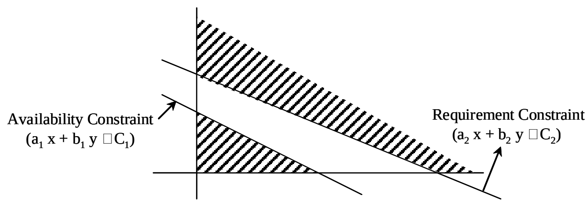
(b) Constraint should be active and create boundary on feasible region in optimal movement direction of objective function. Active constraint pass through one of the extreme points of feasible solution space.

**Q17. Define infeasible solution.**

*Ans :*

When feasible region does not exist, the solution we get is infeasible.

In graphical solution it is found when one constraint is availability ($\leq$) type and the other is requirement ($\geq$) type and these two can not produce any common area (non intersecting) in the specific quadrant (such as $x_1 \geq 0$, $x_2 \geq 0$ indicates first quadrant).



## PROBLEMS

**23. Solve Graphically**

**Maximise $Z = 50x_1 + 60x_2$**

**Subject to $x_1 + x_2 \leq 12$**

**$2x_1 + 3x_2 \geq 60$**

**$x_1, x_2 \geq 0$**

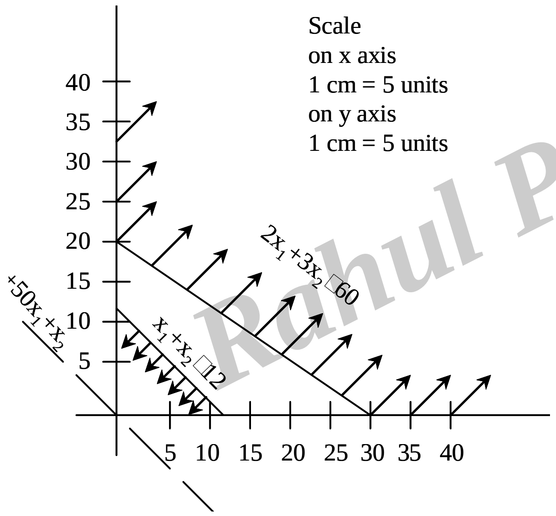
*Sol :*

$x_1 + x_2 = 12$

| | | |
|---|---|---|
| $x_1$ | 0 | 2 |
| $x_2$ | 12 | 0 |

$2x_1 + 3x_2 = 60$

| | | |
|---|---|---|
| $x_1$ | 0 | 30 |
| $x_2$ | 20 | 0 |




There is no feasible region and so the solution is infeasible.

**24. Solve graphically the following LPP.**

**$Z = 4x_1 + 5x_2$**

**Subject to constraints**

**$x_1 + x_2 \geq 1$**

**$-2x_1 + x_2 \leq 1$**

**$4x_1 + 2x_2 \leq 1$**

**and $x_1, x_2 \geq 0$**

*Sol :*

**Step - 1**

To identify the objective function and given constraints.

**Step - 2**

Considering inequality constraints as equality constraints and solve them.

$x_1 + x_2 = 1$

$-2x_1 + x_2 = 1$

$4x_1 + 2x_2 = 1$

Solving equation (1),

Put $x_1 = 0 \Rightarrow x_2 = 1$

Put $x_2 = 0 \Rightarrow x_1 = 1$

$\therefore$ P(0, 1) Q(1, 0)

Mark the points P, Q on graph and join them to obtain straight line PQ.

Solving equation (2),

Put $x_1 = 0 \Rightarrow x_2 = 1$

Put $x_2 = 0 \Rightarrow x_1 = -0.5$

$\therefore$ R(0, 1) S(–0.5, 0)

Mark points R and S on graph and join them to obtain straight line RS.

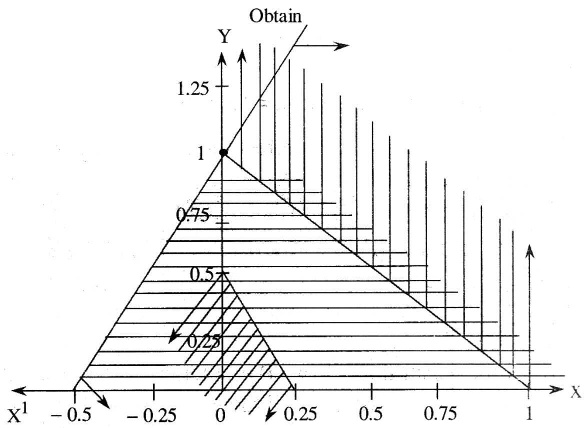
Solving equation (3),

Put $x_1 = 0 \Rightarrow x_2 = 0.5$

Put $x_2 = 0 \Rightarrow x_1 = 0.25$

$\therefore$ L(0, 0.5) M(0.25, 0)

Mark points L and M on graph and join them to obtain straight line LM.



**Step 3 : Identifying Feasible Region**

Since equation (1) is $\geq$ type, shading is done above the line and equations (2) and (3) are $\leq$ type and shading is done below the line.

There is no common feasible region obtained.

**Step 4 :**

Thus the solution is infeasible.

**Example 3 :**

**Minimizes** $Z = x_1 + 2x_2$

**Subject to** $x_1 - x_2 \geq 3$

$-x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$

**Solution :**

Convert the constraints into equation

$x_1 - x_2 = 3$

$-x_1 + x_2 = 4$

$\Rightarrow \quad x_1 - x_2 = -4$

To plot $x_1 - x_2 = 3$

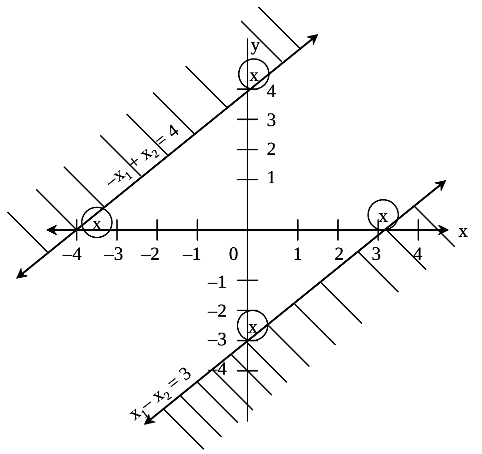
put $x_1 = 0 \Rightarrow x_2 = -3 \Rightarrow (0, -3)$

$x_2 = 0 \Rightarrow x_1 = 3 \Rightarrow (3, 0)$

To plot $x_1 - x_2 = -4$

put $x_1 = 0 \Rightarrow x_2 = 4 \Rightarrow (0, 4)$

$x_2 = 0 \Rightarrow x_1 = -4 \Rightarrow (-4, 0)$



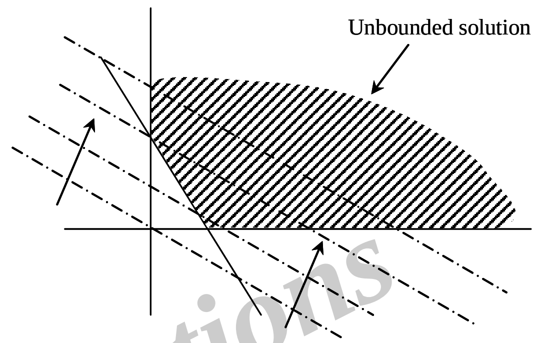

Since there is no feasible region, the solution is infeasible.

**Q18. Define Unbounded Solution.**

*Ans :*

If a distinct and finite solution can not be found or the solution exists at infinity, the solution is said to be unbounded.

In graphical solution, unbounded solutions are obtained if the feasible region is unbounded (formed by requirement constraints i.e., $\geq$ type) while the objective function is maximization.




(Since it has to be taken to infinity to locate maximum value we have no finite or unbounded solution).

**PROBLEMS**

**25. Solve the following LPP graphically**

**Maximize** $Z = 3x_1 + 2z_2$

**s.t.c.** $x_1 - x_2 \leq 1$

$x_1 + x_2 \geq 3 \ ; \ x_1, x_2 \geq 0$

*Sol :*

Convert inequations into equations

$x_1 - x_2 = 1$ and $x_1 + x_2 = 3$

To plot $x_1 - x_2 = 1$

put $x_1 = 0, \ x_2 = -1 \Rightarrow (0, -1)$

$x_2 = 0, \ x_1 = 1 \Rightarrow (1, 0)$

To plot $x_1 + x_2 = 3$

put $x_1 = 0, \ x_2 = 3 \Rightarrow (0, 3)$

$x_2 = 0, \ x_1 = 3 \Rightarrow (3, 0)$

Here the shaded region is unbounded. The two vertices of the region are B = (0, 3); A = (2, 1). The values of the objective function at these vertices are Z(A) = 6 and Z(B) = 8. But there exists points in the region for which the values of the objective function is more than 8. For example, the point (5, 5) lies in the region and the function value at this point is 25 which is more than 8. Hence, the maximum value of Z occurs at the point at infinity only and thus the problem has an unbounded solution.

## 1.6 Duality

**Q19. Define dual method and its properties ?**

*Ans :*

Corresponding to every given linear programming problem, there is another associated LPP, called dual. Primal for given problem and dual are replicas of each other. The difference is while primal is a maximisation problem, the dual would be a minimisation problem or vice versa. Thus, Duality plays an important role in business transaction analysis for the following reasons:

1. It has an important economic interpretation.
2. It can help in computer capacity limitations.
3. It can ease the calculations for the minisation or maximisation problems, whose number of variables is large.

**Properties**

1. Maximisation case can be turned into minimisation and vice versa for each of calculations, as for every primal, there exists a dual.
2. The optimal solution for the dual exists only where there exists an optimal solution to its primal.
3. The dual of the dual is the primal.
4. The dual variables may assume negative values.
5. The value of the objective function of the optimal solution in both the problems is the same.

**Q20. What is Economic Interpretations of Dual Variables ?**

*Ans :*

The dual variables are termed as shadow prices of the constraints. These can be named as marginal values or the opportunity costs of primal resources. Hence, there is one dual variable for every constraint in the primal and the change in the right hand side is proportional change in the objective function. Because of the same reason of one dual variable for every constraint, the dual being non-zero, indicates full utilization of that resource. The use of dual concept can be done in making decisions regarding change in resources such as addition, deletion or trade-off of resources. Hence a minisation problem can also be solved by solving its dual as a maximisation problem.

**Q21. State the mathematical formulation of primal to dual problem.**

*Ans :*

Consider the general LPP Maximize

$$z = c_1x_1 + c_2x_2 + ..... + c_nx_n$$

s.t.c $a_{11}x_1 + a_{12}x_2 + ..... + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + ..... + a_{2n}x_n \leq b_2$

. . . . . . . . . . . . . . . . . . . . . . . .

. . . . . . . . . . . . . . . . . . . . . . . .

$a_{m1}x_1 + a_{m2}x_2 + ..... +$

$a_{mn}x_n \leq b_n, \quad x_1, x_2, ..... x_n \geq 0$

If the above problem is referred to as primal, then its associated dual is,

Minimize $Z^* = b_1y_1 + b_2y_2 + ... + b_my_m$

s.t.c $a_{11}y_1 + a_{21}y_2 + a_{m1}y_m \geq c_1$

$a_{12}y_1 + a_{22}y_2 + a_{m2}y_m \geq c_2$

$a_{1n}y_1 + a_{2n}y_2 + a_{mn}y_m \geq c_n,$

$y_1, y_2, ... y_m \geq 0$

where $y_1, y_2 ... y_m$ are called dual variables.

The above problems are said to be in Symmetric form.

**Q22. Explain the procedure for contructing dual from primal.**

*Ans :*

1. If in the primal problem, the objective function is to be maximized, then in the dual problem, it is to be minimized and vice versa.
2. If the primal contains n variables and m constraints, then the dual will contain m variables and n constraints.
3. For a maximization primal with all $\leq$ type constraints there exists a minimization dual problem with all $\geq$ type constraints and vice versa.

4. If the primal constraint is an equality the corresponding dual variable is unrestricted in sign.
5. If the primal variable is Unrestricted, then the corresponding dual constraint is an equality.
6. The RHS of primal constraints become the coefficients of the dual variables in the dual objective function. Whereas the coefficients of primal variables in the primal objective function became the RHS of dual constraints.
7. The coefficients ($a_{ij}$) of the dual variables on the constraints are the same as the coefficient of the primal variables except that they are transposed.

**Q23. What are the Advantages of Duality ?**

*Ans :*

Duality will be more advantageous in the following cases :

1. Sometimes dual problem solution may be easier than primal solution particularly, when number of decision variables is considerably less than slack/surplus variables.
2. In the areas like economics, it is highly helpful in obtaining future decisions in the activities being programmed.
3. In physics, it is used in parallel circuit and series circuit theory.
4. In game theory, dual is employed by column player who wishes to minimise his maximum loss while his opponent while Row player applies primal to maximise his minimum gains. However, if one problem is solved, the solution for other also can be obtained from the simplex tableau.
5. When a problem does not yield any solution in primal, it can be verified with dual.
6. Economic interpretations can be made and shadow prices can be determined enabling the managers to take further decisions.

## PROBLEMS

**Case I : All constraints are " $\leq$ " type.**

**26. Write the dual of the following LPP.**

**Maximize** $Z = 3x_1 + 5x_2$

**s.t.c.** $2x_1 + 6x_2 \leq 50$

$3x_1 + 2x_2 \leq 35,\ 5x_1 - 3x_2 \leq 10,\ x_2 \leq 20$ **where** $x_1 \geq 0,\ x_2 \geq 0$

*Sol :*

Let $y_1, y_2, y_3$ and $y_4$ be the corresponding dual variables, then the dual problem is given by

Min $Z^* = 50y_1 + 35y_2 + 10y_3 + 20y_4$

s.t.c. $2y_1 + 3y_2 + 5y_3 \geq 3$ ;

$6y_1 + 2y_2 - 3y_3 + y_4 \geq 5$

where $y_1, y_2, y_3, y_4$ all $\geq 0$.

**27. Write the dual of the following LPP.**

**Max** $Z = x_1 - x_2 + 3x_3$

**s.t.c.** $x_1 + x_2 + x_3 \leq 10,$

$2x_1 + - 0.x_2 - x_3 \leq 2,\ 2x_1 - 2x_2 - 3x_3 \leq 6$

**where** $x_1, x_2, x_3 \geq 0.$

*Sol :*

Let $y_1, y_2$ and $y_3$ be the corresponding dual variables, then the dual problem is given by

Min $Z^* = 10y_1 + 2y_2 + 6y_3$

s.t.c. $y_1 + 2y_2 + 2y_3 \geq 1$

$y_1 + 0.y_2 - 2y_3 \geq -1,\ y_1 - y_2 - 3y_3 \geq 3$

where $y_1, y_2, y_3$ all $\geq 0$

**Case II : One or more constraints are " $\geq$ " type.**

**28. Write the dual of the following LPP.**

**Max** $Z = 3x_1 + 17x_2 + 9x_3$

**s.t.c.** $x_1 - x_2 + x_3 \geq 3,$

$-3x_1 + 2x_2 \leq 1 \quad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$

*Sol :*

We convert the $\geq$ constraint into $\leq$ constraint by multiplying both sides by –1.

i.e., $-x_1 + x_2 - x_3 \leq -3$

Let $y_1$ and $y_2$ be the corresponding dual variables, then the dual problem is given by

Min. $Z^* = -3y_1 + y_2$

s.t.c. $-y_1 - 3y_2 \geq 3,\quad y_1 \geq 17,$

$-y_1 + 2y_2 \geq 9$ where $y_1 \geq 0,\quad y_2 \geq 0$

**29. Write the dual of the following LPP.**

**Max $Z = 2x_1 + 5x_2 + 3x_3$**

**s.t.c. $2x_1 + 4x_2 - x_3 \leq 8$**

**$2x_1 + 2x_2 + 3x_3 \geq -7, \quad x_1 + 3x_2 - 5x_3 \geq -2$**

**$4x_1 + x_2 + 3x_3 \leq 4$ where $x_1, x_2, x_3 \geq 0$**

*Sol :*

We first convert "$\geq$" constraint into "$\leq$" constraints by multiplying both sides by –1 i.e., $-2x_1 + 2x_2 - 3x_3 \leq 7, \quad -x_1 - 3x_2 + 5x_3 \leq 2$

Let $y_1, y_2, y_3$ and $y_4$ be the corresponding dual variables, then the dual problem is given by

Min. $Z^* = 8y_1 + 7y_2 + 2y_3 + 4y_4$

s.t.c. $2y_1 - 2y_2 - y_3 + 4y_4 \geq 2$

$4y_1 - 2y_2 - 3y_3 + y_4 \geq 5, \quad -y_1 - 3y_2 + 5y_3 + 3y_4 \geq 3$ where $y_1, y_2, y_3, y_4 \geq 0$

**Case III : All constraints are "$\geq$" type.**

**30. Write the dual of the following LPP.**

**Max. $Z = 3x_1 + 4x_2$**

**s.t.c. $x_1 + x_2 \geq 4$**

**$-x_1 + 3x_2 \geq 4$ where $x_1 \geq 0, \quad x_2 \geq 0$**

*Sol :*



We first convert all "$\geq$" constraints into "$\leq$" constraint by multiplying both sides by –1.

i.e., $-x_1 - x_2 \leq -4, \quad x_1 - 3x_2 \leq 4$

Let $y_1$ and $y_2$ be the corresponding dual variables then the dual problem is given by

Min $Z^* = 4y_1 + 4y_2$

s.t.c. $-y_1 + y_2 \geq 3$

$-y_1 - 3y_2 \geq 4$ where $y_1 \geq 0, \quad y_2 \geq 0$

**31. Apply the principle of duality to solve the LPP**

**Minimize $Z = 4x_1 + 3x_2 + 3x_3$**

**s.t.c. $x_1 + 2x_2 \geq 2$ ; $3x_1 - x_2 + x_3 \geq 4$ ; $4x_3 \geq 1$**

**$x_1 + x_3 \geq 1$ ; $x_1, x_2, x_3 \geq 0$**

*Sol :*

The dual of the given problem is

Max $Z^* = 2y_1 + 4y_2 + y_3 + y_4$

s.t.c. $y_1 + 3y_2 + y_4 \leq 4$

$2y_1 - y_2 \leq 3, \quad y_2 + 4y_3 + y_4 \leq 3, y_1, y_2, y_3, y_4 \leq 0$

Introducing slack variables $y_5, y_6, y_7$, the problem becomes, Max $Z^* = 2y_1 + 4y_2 + y_3 + y_4 + 0y_5 + 0y_6 + 0y_7$

s.t.c. $y_1 + 3y_2 + y_4 + y_5 = 4$; $2y_1 - y_2 + y_6 = 3$

$y_2 + 4y_3 + y_4 + y_7 = 3$; $y_1, y_2, y_3, y_4, y_5, y_6, y_7 \geq 0$

**Staring Table**

| | | | $C_j \rightarrow$ | 2 | 4 | 1 | 1 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $C_B$ | BV | $y_B$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | $y_7$ | $y_b/y_2$ |
| ← | 0 | $y_5$ | 4 | 1 | (3) | 0 | 1 | 1 | 0 | 0 | $\frac{4}{3}$ (Min) |
| | 0 | $y_6$ | 3 | 2 | –1 | 0 | 0 | 0 | 1 | 0 | – |
| | 0 | $y_7$ | 3 | 0 | 1 | 4 | 1 | 0 | 0 | 1 | 3 |
| | | | $Z^* = 0$ | –2 | –4 | –1 | –1 | 0 | 0 | 0 | |

**First Iteration :** Introduce $y_2$ and drop $y_5$

| | | | | 2 | 4 | 1 | 1 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $C_B$ | BV | $y_B$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | $y_7$ | $y_b/y_2$ |
| | 4 | $y_2$ | $\frac{4}{3}$ | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 | $\infty$ |
| | 0 | $y_6$ | $\frac{13}{3}$ | $\frac{7}{3}$ | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | 1 | 0 | $\infty$ |
| ← | 0 | $y_7$ | $\frac{5}{3}$ | $-\frac{1}{3}$ | 0 | (4) | $\frac{2}{3}$ | $-\frac{1}{3}$ | 0 | 1 | $\frac{5}{12}$ (Min) |
| | | | $Z^* = \frac{16}{3}$ | $-\frac{2}{3}$ | 0 | –1 | $\frac{1}{3}$ | $\frac{4}{3}$ | 0 | 0 | |

**Second Iteration :** Introduce $y_3$ and drop $y_7$

| | | | | 2 | 4 | 1 | 1 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $C_B$ | BV | $y_B$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | $y_7$ | $y_b/y_2$ |
| | 4 | $y_2$ | $\frac{4}{3}$ | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 | 4 |
| ← | 0 | $y_6$ | $\frac{13}{3}$ | (7/3) | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | 1 | 0 | $\frac{13}{3}$ (Min) |
| | 0 | $y_3$ | $\frac{5}{12}$ | $\frac{1}{12}$ | 0 | 1 | $\frac{1}{6}$ | $-\frac{1}{12}$ | 0 | $\frac{1}{4}$ | 5 |
| | | | $Z^* = \frac{69}{12}$ | $-\frac{9}{12}$ | 0 | 0 | $\frac{1}{2}$ | $\frac{15}{12}$ | 0 | $\frac{1}{4}$ | |

**Third Iteration :** Introduce $y_1$ and drop $y_6$

| | | | 2 | 4 | 1 | 1 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $y_B$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | $y_7$ |
| 4 | $y_2$ | $5/7$ | 0 | 1 | 0 | $2/7$ | $2/7$ | $-1/7$ | 0 |
| 2 | $y_1$ | $13/7$ | 1 | 0 | 0 | $1/7$ | $1/7$ | $3/7$ | 0 |
| 1 | $y_3$ | $4/7$ | 0 | 0 | 1 | $5/28$ | $-1/214$ | $1/28$ | $1/4$ |
| | | $Z^* = 50/7$ | 0 | 0 | 0 | $48/28$ | $19/14$ | $9/28$ | $1/4$ |

Since all net evaluations are positive, the solution to the dual problem has been obtained. The solution is $y_1 = 13/7$, $y_2 = 5/7$, $y_3 = 4/7$.

**Primal Solution**

Since the prime variables $x_1, x_2, x_3$ correspond to slack variables $y_5, y_6, y_7$ respectively of the dual problem. From the above table, the net evaluations corresponding to the variables $y_5, y_6, y_7$ are $\frac{19}{14}, \frac{9}{28}, \frac{1}{4}$, respectively.

Hence optimum solution to the original primal will be $x_1 = \frac{19}{14}$, $x_2 = \frac{9}{28}$, $x_3 = \frac{1}{4}$, Max Z $= \frac{50}{8}$.

**32. Using duality solve the LPP.**

**Max** $\quad Z = 2x_1 + 3x_2$

**s.t.c.** $\quad x_1 + x_2 \geq 1;\ 2x_1 + 3x_2 \leq 9;\ x_1 - x_2 \leq 4;\ x_1, x_2 \geq 0$



*Sol :*

The given can be written as

Max $\quad Z = 2x_1 + 3x_2$

s.t.c. $\quad -x_1 - x_2 \leq -1;\ 2x_1 + 3x_2 \leq 9;\ x_1 - x_2 \leq 4;\ x_1, x_2 \geq 0.$

The dual of the problem is

Max $\quad Z^* = -y_1 + 9y_2 + 4y_3$

s.t.c. $\quad -y_1 + 2y_2 + y_3 \geq 2;\ -y_1 + 3y_2 - y_3 \geq 3;\ y_1, y_2, y_3 \geq 0.$

Expressing the dual in the standard form. Introducing surplus variables $y_4, y_5 \geq 0$ and artificial variables $a_1, a_2 \geq 0$.

Max $\quad Z^* = y_1 - 9y_2 - 4y_3 + 0y_4 + 0y_5 - Ma_1 - Ma_2$

s.t.c. $\quad -y_1 + 2y_2 + y_3 - y_4 + a_1 = 2;\ -y_1 + 3y_2 - y_3 - y_5 + a_2 = 3;$

$y_1, y_2, y_3, y_4, y_5 \geq 0.$

**Staring Table :**

| $C_B$ | BV | $y_B$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $a_1$ | $a_2$ | $y_B/y_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | –9 | –4 | 0 | 0 | –M | –M | |
| –M | $a_1$ | 2 | –1 | 2 | 1 | –1 | 0 | 1 | 0 | 1 |
| –M | $a_2$ | 3 | –1 | 3 | –1 | 0 | –1 | 0 | 1 | 1 |
| | | Z* = –5M | M-1 | –5M+9 | 4 | M | M | 0 | 0 | |
| | | | | ↑ | | | | | | |

Since the min ratio is same i.e., 1. Therefore, the Problem is degenerate one. To solve degeneracy, rearrange the column of the above tabnle so that Identify matrix comes first.

| | $C_B$ | BV | $y_B$ | $a_1$ | $a_2$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | First Column / Key Column |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | –M | –M | 1 | –9 | –4 | 0 | 0 | |
| | –M | $a_1$ | 2 | 1 | 0 | –1 | 2 | 1 | –1 | 0 | 1/2 |
| ← | –M | $a_2$ | 3 | 0 | 0 | –1 | (3) | –1 | 0 | –1 | 0 (Min) |
| | | | Z* = –5M | 0 | 0 | M–1 | –5M+9 | 4 | M | M | |
| | | | | | | | ↑ | | | | |

**First Iteration :** Introduce $y_2$ and drop $a_2$

| | $C_B$ | BV | $y_B$ | $a_1$ | $a_2$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_B/y_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | –M | –M | 1 | –9 | –4 | 0 | 0 | |
| ← | –M | $a_1$ | 0 | 1 | –2/3 | –1/3 | 0 | (5/3) | –1 | 2/3 | 0 |
| | –9 | $y_2$ | 1 | 0 | 1/3 | -1/3 | 1 | –1/3 | 0 | –1/3 | – |
| | | | Z* = –9 | 0 | $\frac{5M}{3}+\frac{9}{2}$ | $\frac{M}{3}+\frac{7}{2}$ | 0 | $\frac{-5M}{3}+\frac{17}{2}$ | M | $\frac{-2M}{3}+\frac{9}{2}$ | |
| | | | | | | | | ↑ | | | |

**Second Iteration :** Introduce $y_3$ and drop $a_1$

| $C_B$ | BV | $y_B$ | $a_1$ | $a_2$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|---|---|---|---|
| | | | –M | –M | 1 | –9 | –4 | 0 | 0 |
| –4 | $y_3$ | 0 | 3/5 | –2/5 | –1/5 | 0 | 1 | –3/5 | 2/5 |
| –9 | $y_2$ | 1 | 1/5 | 1/5 | –2/5 | 1 | 0 | –1/5 | –1/5 |
| | | Z* = –9 | $M-\frac{21}{5}$ | $M-\frac{1}{5}$ | 17/5 | 0 | 0 | 21/5 | 1/5 |

Since all net evaluations are $\geq$ 0 and the coefficient of M in the net evaluation is negative. Therefore the solution to the dual problem has been obtained.

The solution is $y_1 = 0$; $y_2 = 1$; Max $Z^* = -9 \Rightarrow$ Min $Z^* = 9$.

The primal solution is, $x_1 = \frac{21}{5}$; $x_2 = \frac{1}{5}$; Max $Z = 9$.

---

**33. One unit of product a contributes Rs. 7 and requires 3 units of raw material and 2 hrs. of labour. One unit of product B contributes Rs. 5 and requires one unit of raw material and one hour of labour. Availability of raw material at present is 48 units and n total 40 hours are available. Formulating the problem as LPP write its dual. Solve the dual by simplex and find the optimal product mix and the shadow prices of raw material and labour.**

*Sol :*

**Formulation :** (Primal)

Let $x_1$ and $x_2$ be the number of units of product a and B respectively to be produced.

Max $Z = 7x_1 + 5x_2$

s.t.c $3x_1 + x_2 \leq 48$

$2x_1 + x_2 \leq 40$

$x_1, \ x_2 \geq 0$

**Dual :** Let $y_1$ and $y_2$ are dual variables.

Max $Z = 48y_1 + 40y_2$

s.t.c $3y_1 + 2y_2 \geq 7$

$y_1 + y_2 \geq 5$

$y_1, \ y_2 \geq 0$

Solving the dual by Big M method.

Introducing surplus and artificial variables

$3y_1 + 2y_2 - y_3 + a_1 = 7$

$y_1 + y_2 - y_4 + a_2 = 5$

New objective function is,

Max $Z' = -48y_1 - 40y_2 + 0y_3 + 0y_4 - Ma_1 - Ma_2$

**Staring Table :**

| | | | –48 | –40 | 0 | 0 | –M | –M | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $y_B$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $a_1$ | $a_2$ | $y_B/y_1$ |
| –M | $a_1$ ← | 7 | (3) | 2 | –1 | 0 | 1 | 0 | 7/3 (Min) |
| –M | $a_2$ | 5 | 1 | 10 | –1 | 0 | 1 | 5/1 | |
| | | Z' = –12M | –4M + 48 ↑ | –3M + 40 | M | M | 0 | 0 | |

**Iteration 1 :** Introduce $y_1$ and drop $a_1$

| $C_B$ | BV | $y_B$ | $y_1$ (–48) | $y_2$ (–40) | $y_3$ (0) | $y_4$ (0) | $a_1$ (–M) | $a_2$ (–M) | $y_B/y_1$ |
|---|---|---|---|---|---|---|---|---|---|
| –48 | $y_1$ | 7/3 | 1 | 2/3 | –1/3 | 0 | 1/3 | 0 | – |
| ← –M | $a_2$ | 8/3 | 0 | 1/3 | (1/3) | –1 | –1/3 | 1 | 8 |
| | | | 0 | $\frac{-M}{3}+8$ | $\frac{-M}{3}+16$ ↑ | 0 | $16+\frac{4M}{3}$ | 0 | |

**Iteration 2 :** Introduce $y_3$ and drop $a_2$

| $C_B$ | BV | $y_B$ | $y_1$ (–48) | $y_2$ (–40) | $y_3$ (0) | $y_4$ (0) | $a_1$ (–M) | $a_2$ (–M) | $y_B/y_1$ |
|---|---|---|---|---|---|---|---|---|---|
| ← –48 | $y_1$ | 5 | 1 | (1) | 0 | –1 | 0 | 1 | 5 |
| 0 | $y_3$ | 8 | 0 | 1 | 1 | –3 | –1 | 3 | 8 |
| | | | 0 | –8 ↑ | 0 | 48 | M | M–48 | |

**Iteration 3 :** Introduce $y_2$ and drop $y_1$

| $C_B$ | BV | $y_B$ | $y_1$ (–48) | $y_2$ (–40) | $y_3$ (0) | $y_4$ (0) | $a_1$ (–M) | $a_2$ (–M) |
|---|---|---|---|---|---|---|---|---|
| –40 | $y_2$ | 5 | 1 | 1 | 0 | –1 | 0 | 1 |
| 0 | $y_3$ | 3 | –1 | 0 | 1 | –2 | –1 | 2 |
| | | Z' = –200 | 8 | 0 ↑ | 0 | 38 | 0 | M–40 |

Since all $Z_j - C_j \geq 0$, the optimum solution to the dual problem has been obtained.

The solution is $y_1 = 0$; $y_2 = 5$;

max $z = -200 \Rightarrow$ min $z = 200$.

Here $y_1$ and $y_2$ are shadow prices of raw material and labour.

The primal solution : $x_1 = 0$, $x_2 = 5$; Max $z = 200$.

**Q24. What are the differences between primal and dual LPP.**

*Ans :*

| | Primal | Dual |
|---|---|---|
| **1.** | **With Reference to Formulation** | |
| 1. | If objective function is in Maximisation form | Objective function is in Minimisation form |
| 2. | If objective function is in Minimisation form | Objective function is in Maximisation form |
| 3. | Coefficient of decision variables in objective function | Constants of constraints set |
| 4. | Constants in constraint set | Coefficients of decision variables in objective function |
| 5. | Number of decision variables | Number of constraints |
| 6. | Number of constraints | Number of decision variables |
| 7. | Coefficients of variables in constraints row wise | Coefficients of variables in constraints column wise |
| 8. | Coefficients of variables in constraints column wise | Coefficients of variables row wise |
| 9. | $\leq$ type constraints | $\geq$ type constraints |
| 10. | $\geq$ type constraints | $\leq$ type constraints |
| 11. | Equality type (=) constraint | Corresponding variable unrestricted |
| 12. | Unrestricted variable | Corresponding constraints is equality (=) type |
| **2.** | **With Reference to Solution** | |
| 1. | Solution is taken from solution values column of final primal tableau (simplex) | Solution is taken Zj – Cj row of final simplex tableau of primal |
| 2. | Unbounded solution | Infeasible solution |
| 3. | Infeasible solution | Unbounded solutions |
| 4. | Solution variables | Shadow prices |
| 5. | Shadow prices | Solution values |



## 1.7 Sensitivity Analysis

**Q25. Explain the concept of sensitivity analysis.**

*Ans :*

Sensitivity analysis is a post-optimality analysis and is an integral part of the OR work. It is the control over the solution, when non-controllable parameters change, which may affect the controllable parameters and the optimality of the solution may change.

In fact, in a linear programming analysis, we presume that the parameters of cost decision variables and constraints are deterministic and constant. However, they may be subject to error and cost and resources may change with times slightly or substantially. This uncertainty of parameters may be a cause of doubt about the optimality of the solution. The sensitivity analysis is thus, a tool to assess the change in the optimal solution due to variations in the original parameter values in the discrete fashion.

There can be five types of discrete changes in the original LP model needing assessment or investigation during this post-optimality analysis.

(a) $C_j$, the unit cost of profit associated with both basic and non-basic decision variables.

(b) bi, the resources availability.

(c) aij, indicating consumption of resources per unit of product.

(d) any addition of a new variable to the problem (at a later date).

(e) Any addition of a new constraint to the original problem.

**(a) Change in $C_j$ :** Since $C_j$ represents the profit cost per unit, an increase or decrease in $C_j$ would indicate the corresponding change in the resource utilization i.e., diverting resources to or away from a more profitable activity. In the maximisation case, $C_j - Z_j \leq 0$ for all Ck is the coefficient of non-basic variable $x_k$, it does not affect any of the $C_j$ values associated with basic variables and such changes do not alter $Z_j$ values nor any of the $(C_j - Z_j)$ values except $C_k - Z_k$ due to change in $C_k$. If optimality is to be maintained within the variation limits, than $DC_k$ (= small variation in $C_k$) must satisfy the condition $(C_k + DC_k) - Z_k \leq 0$ for optimality of $Z_j - C_j \geq DC_k$. Then the objective function and the optimal solution will remain unchanged.

**(b) Change in $b_i$ :** Any change in the values of bi, i.e., the right hand side of the constraints does not affect the optimality condition. It only affects the values of the basic variables and the values of the objective function, because in the determination of the solution values. Values of resources involved and the range of variation in bi can be obtained by solving the system of inequalities.

**(c) Change in the input coefficients ($a_{ij}$) :** When the elements of the coefficient matrix are changed, here an be two cases worth considering.

(i) Change in aij for the columns in basic matrix.

(ii) Change in aij for the columns which do not belong to the basic matrix.

The changes in the values of aij associated with non-basic variables in the optimal simplex table can be analyzed by forming a corresponding dual constraint from the original set of constraints i.e.,

$$\sum_{i=1}^{m} aij \,.\, y_i \geq c_j \text{ for xj non-basic variables.}$$

**(d) Addition of new variables (Additional column) :** When we add another variable, there will be an extra column, the following situations arise

(i) $C_{n+1} - Z_{n+1} \leq 0$ the solution remains optimal.

(ii) $C_{n+1} - Z_{n+1} > 0$, then the solution can be improved by introducing a new column into the basic and complex method is continued for obtaining a new optimal solution.

**(e) Addition of a new constraint (Row) :** Addition of a new constraint would mean a simultaneous change in the objective function coefficient Cj as well as coefficients aij of the corresponding non-basic variable. Thus it will affect only the optimality of the problem. This means that the new variable will enter into the basis only if it improves the value of the objective function. If optimal solution of the original problem satisfies the new constraint, then the solution remains optimal and feasible. If the optimal solution of the original problem does not satisfy the new constraint, the optimal solution to the modified LP problem is re-obtained.

| UNIT II | **Transportation Problem:** Introduction, Mathematical Model for 'Transportation Problem, Types of Transportation Problem, Methods to solve Transportation Problem, Transshipment Model. |
|---|---|

## 2.1 Transportation Problem

### 2.1.1 Introduction

**Q1. What is Transportation Problem ?**

*Ans :*

Transportation problem is another case of application to linear programming problems, where some physical distribution (transportation) of resources is to be made from one place to another to meet certain set of requirements with in the given availability. The places from where the resources are to be transferred are referred to as sources or origins. These sources or origins will have the availability or capacity or supply of resources. The other side of this transportation i.e., to where the resources are transported are called sinks or destinations such as market entres, godowns etc. These will have certain requirements or demand.

### 2.1.2 Mathematical Model for Transportation Problem

**Q2. Explain the Mathematical formulation for Transportation Problem.**

*Ans :* **(Imp.)**

Transportation problem is applied to the situations in which a single product is transported from several origins (say $O_1, O_2, \ldots O_m$) to several destinations (say $D_1, D_2, \ldots D_m$). Let us assume the cost of trnasporting a unit product from $O_i$ to $D_j$ is $C_{ij}$ and the no. of units transported be $x_{ij}$. Let the capacity of $O_i$ be $a_i$ and requirement of $D_j$ be $b_j$. Then, the transportation problem can be mathematically written as

Minimise (Total transportation cost)

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij}\, x_{ij}$$

Subject to the constraints

$$\sum_{j=i}^{M} x_{ij} = a_{ij}, \quad i = 1, 2, \ldots m$$

(supply or availability constraints)

$$\sum_{i=1}^{M} x_{ij} = b_{ij}, \; j = 1, 2, \ldots n$$

(demanded or requirement constraints)

and $x_{ij} \geq$ for all i and j.

The above can be represented as table as follows :

| | | Destinations | | | | |
|---|---|---|---|---|---|---|
| | From \ To | $D_1$ | $D_2$ | ....... | $D_m$ | Supply or available |
| Origins | $O_1$ | $C_{11}$ $x_{11}$ | $C_{12}$ $x_{12}$ | ....... | $C_{1n}$ $x_{1n}$ | $a_1$ |
| | $O_2$ | $C_{21}$ $x_{21}$ | $C_{22}$ $x_{22}$ | ....... | $C_{2n}$ $x_{2n}$ | $a_2$ |
| | : | : | : | ....... | : | : |
| | $O_m$ | $C_{m1}$ $x_{m1}$ | $C_{m2}$ $x_{m2}$ | ....... | $C_{mn}$ $x_{mn}$ | $a_m$ |
| Demand Or Requirement | | $b_1$ | $b_2$ | ....... | $b_n$ | $\Sigma a_i$ / $\Sigma b_j$ |

**Note :**

In the above table, if total supply $\left(\sum_{i=1}^{m} a_i\right)$ = Total demand $\left(\sum_{j=1}^{n} b_j\right)$, the transportation problem is said to be balanced transportation, otherwise it is unbalnced transportation.

The above condition is said to be "balance condition" or "rim condition" of T.P.

In the standard interpretation of the model, there are m-supply points with the items available to be transported to n-demand points. Specifically, plant i can transport atmost $S_i$ items, and demand point j requires at least $D_j$ items. The cost of transporting each unit from plant i to demand points j is $C_{ij}$.

The mathematical description of the transportation problem is

Minimize : $\sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij}$

Subject to $\sum^{n} X_{ij} \le S_i$ for $i = 1, 2, 3, ..., m$ (Supply)

$\sum^{m} X_{ij} \le D_i$ for $j = 1, 2, 3, ..., n$ (Demand)

$X_{ij} \ge 0$ for all i and j.

The non-negative quantity $x_{ij}$ represents the amount of goods transported from plant i to all the demand points j.

**Q3. Explain the basic terminology are used in transportation problem.**

*Ans :* **(Imp.)**

**(i) Feasible Solution**

Any set of non negative allocations ($x_{ij}$ > Q) which satisfies the row and column sum is called a feasible solution.

**(ii) Basic Feasible Solution**

A feasible solution is called a basic feasible solution if the number of non negative allocations is equal to *m* + *n* – 1 where m is the number of rows, *n* the number of columns in a transportation table.

**(iii) Non-degenerate Basic Feasible Solution**

Any feasible solution to a transportation problem containing in origins and n destina-tions is said to be non-degenerate, if it contains m + n – 1 occupied cells and each allocation is in independent positions.

The allocations are said to be in independent positions, if it is impossible to form a closed path.

Closed path means by allowing horizontal and vertical lines and all the corner cells are occupied.

The allocations in the following tables are not in independent positions.

| | | |
|---|---|---|
| | * | * |
| | * | * |
| | | |

| | | |
|---|---|---|
| * | | * |
| | | |
| * | | * |

| | | | |
|---|---|---|---|
| | * | * | |
| | * | | |
| | * | * | |
| | | | |

The allocations in the following tables are in independent positions.

| | | |
|---|---|---|
| | * | |
| * | * | * |
| * | | |

| | | | |
|---|---|---|---|
| * | * | | |
| | | | |
| | * | | * |
| | | * | * |

**(iv) Degenerate Basic Feasible Solution**

If a basic feasible solution contains less than m + n – 1 non negative allocations, it is said to be degenerate.

**Q4. State the assumptions of transportation problem.**

*Ans :*

Transportation model is based on following assumptions,

1. It is assumed that quantity available at sources is equal to the quantity required at destinations.
2. Items are easily transported from source to destination.
3. Transportation cost per unit from source to destination is well known.
4. The main aim is to reduce total transportation cost for the whole organization.
5. There is direct relationship between transportation cost of specific route and number of units shipped for that specific route.

## 2.2 Methods to solve Transportation Problem

**Q5. What are the different methods of Finding Initial Feasible Solution ?**

*Ans :*

Optimal solution is a feasible solution (not necessarily basic) which minimizes the total cost.

The solution of a transportation problem can be obtained in two stages, namely initial solution and optimum solution.

Initial solution can be obtained by using any one of the three methods viz,

(i) North west corner rule (NWCR)

(ii) Least cost method or Matrix minima method

(iii) Vogel's approximation method (VAM).



### 2.2.1 North-West Corner Rule

**Q6. Write about method to obtain initial basic feasible solution by North-West Corner Rule.**

*Ans :* **(Imp.)**

The simplest of the procedures used to generate an initial feasible solution is NWCM. It is so called because we begin with the north west or upper left corner cell of our transportation table. Various steps of this method can be summarised as under :

**Step 1 :** Select the north west (upper left-hand) corner cell of the transportation table and allocate as many units as possible equal to the minimum between available supply and demand requirement i.e., min ($S_1$, $d_1$).

**Step 2 :** Adjust the supply and demand numbers in the respective rows and columns allocation.

**Step 3 :** (a) If the supply for the first row is exhausted, then move down to the first cell in the second row and first column and go to step 2.

(b) If the demand for the first column is satisfied, then move horizontally to the next cell in the second column and first row and go to step 2.

**Step 4 :** If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.

**Step 5 :** Continue the procedure until the total available quantity is fully allocated to the cells as required.

**Remarks :** The quantities so allocated are circled to indicate the value o the corresponding variable.

Empty cells indicate the value of the corresponding variables as zero, i.e., no unit is shipped to this cell.

**Advantages :** Simple and reliable method

Easy to compute, understand and interpret.

**Drawbacks :** This method of assignment does not take into considera-tion the shipping cost, consequently the initial solution obtained by this method usually requires several iterations before an optimum solution is obtained.

**Area of Application**

It is used in case of transportation within the campus of an organisation as costs are nt significant.

It is used for transportation to satisfy such obligations where cost is not the criteria. For example in case of Food Corporation of India Ltd.

## PROBLEMS

**1. Consider the following transportation problem**

| Source | Destination | | | | Total |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | |
| $O_1$ | 1 | 2 | 1 | 4 | 30 |
| $O_2$ | 3 | 3 | 2 | 1 | 50 |
| $O_3$ | 4 | 2 | 5 | 9 | 20 |
| Total | 20 | 40 | 30 | 10 | 100 |

**Determine the initial feasible solution.**

*Sol :*

The given problem is a balanced TP.

Allocation by NWCM

**ITERATION TABLEAU - I**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | 1 | 2 | 1 | 4 | 30 |
| $O_2$ | 3 | 3 | 2 | 1 | 50 |
| $O_3$ | 4 | 2 | 5 | 9 | 20 |
| Demand | (20) | 40 | 30 | 10 | 100 / 100 |



In the above TP table, the northwest corner cell is ($C_{11}$) we can transfer only 20 units from $O_1$ to $D_1$ (20 < 30). Since the $D_1$ requirement is exhausted we delete this column ($D_1$) for next iteration while we adjust the supply of $O_1$ from 30 to 10.

**ITERATION TABLEAU - II**

| | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|
| $O_1$ | 2 | 1 | 4 | (10) |
| $O_2$ | 3 | 2 | 1 | 50 |
| $O_3$ | 2 | 5 | 9 | 20 |
| Demand | ~~40~~ 30 | 30 | 10 | 80 / 80 |

Allocated cell ($C_{12}$) is ($O_1$, $D_2$) No. of units allocated is 10 Deletion for next allocated is 10 Deletion for next iteration is $O_1$ Readjustments for $D_2$ : 40 to 30 Cost of transportation from $O_1$ to $D_2$ = $2 \times 10 = 20$.

### ITERATION TABLEAU - III

Allocated cell ($C_{22}$) is ($O_2$, $D_2$) No. of units allocated is 30 Deletion for next iteration is $D_2$ Readjustments for $O_2$ is 50 to 20 Cost of transportation is $3 \times 30 = 90$.

| | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|
| $O_2$ | 3 [30] | 2 | 1 | ~~50~~ 20 |
| $O_3$ | 2 | 5 | 9 | 20 |
| Demand | (30) | 30 | 10 | 70 / 70 |

### ITERATION TABLEAU - IV

Allocated cell ($C_{23}$) is ($O_2$, $D_3$) No. of units allocated is 20 Deletion for next iteration is $O_2$ Readjustments for $D_3$ is 30 to 10 Cost of transportation is $2 \times 20 = 40$.

| | $D_3$ | $D_4$ | Supply |
|---|---|---|---|
| $O_2$ | 2 [20] | 1 | (20) |
| $O_3$ | 5 | 9 | 20 |
| Demand | ~~30~~ 10 | 10 | 40 / 40 |



### ITERATION TABLEAU - V

| | $D_3$ | $D_4$ | Supply |
|---|---|---|---|
| $O_3$ | 5 [10] | 9 [10] | 20 |
| Demand | 10 | 10 | 20 / 20 |

Allocate cells ($C_{33}$ and $C_{34}$) are ($O_3$, $D_3$) and ($O_3$, $D_4$)

No. of units allocated are 10 and 10 respectively.

Cost are $5 \times 10$ and $9 \times 10$

i.e., 50 and 90 respectively.

All the units are exhausted

$\therefore$ IBFS is obtained.

**Summary :**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | 1 [20] | 2 [10] | 1 | 4 | 30 |
| $O_2$ | 3 | 3 [30] | 2 [20] | 1 | 50 |
| $O_3$ | 4 | 2 | 5 [10] | 9 [10] | 20 |
| Demand | 20 | 40 | 30 | 10 | |

Cost Calculations of IBFS by NWCM

| Allocated Cell | From | To | Unit Cost | No. of Units | Total Cost |
|---|---|---|---|---|---|
| $C_{11}$ | $O_1$ | $D_1$ | 1 | 20 | 20 |
| $C_{12}$ | $O_1$ | $D_2$ | 2 | 10 | 20 |
| $C_{22}$ | $O_2$ | $D_2$ | 3 | 30 | 90 |
| $C_{23}$ | $O_2$ | $D_3$ | 2 | 20 | 40 |
| $C_{33}$ | $O_3$ | $D_3$ | 5 | 10 | 50 |
| $C_{34}$ | $O_3$ | $D_4$ | 9 | 10 | 90 |
| Grand Total Cost of Transportation | | | | | 310 |

Total cost of transportation = Rs. 310/-

**2. Obtain the initial basic feasible solution to the following transportation problem by North-West corner method**

| | $D_1$ | $D_2$ | $D_3$ | Available ($a_i$) |
|---|---|---|---|---|
| $O_1$ | 2 | 7 | 4 | 5 |
| $O_2$ | 3 | 3 | 1 | 8 |
| $O_3$ | 5 | 4 | 7 | 7 |
| $O_4$ | 1 | 6 | 2 | 14 |
| Requirement ($b_j$) | 7 | 9 | 18 | |



**Sol :**

| | $D_1$ | $D_2$ | $D_3$ | |
|---|---|---|---|---|
| $O_1$ | [5] 1 | 2 | 1 | 5 |
| $O_2$ | [2] 3 | [6] 3 | 2 | 8 |
| $O_3$ | 3 | [3] 3 | [4] 2 | 7 |
| $O_4$ | 4 | 2 | [14] 5 | 14 |
| | 7/2 | 9 | 18 | |

Start with the cell (1, 1) and allocate the min $(a_1, b_1)$ = min (5, 7) = 5. Therefore allocate 5 units to this cell satisfy the availability of origin $O_1$ and leaves a balance of 7 - 5 = 2 units of requirement at Destination $D_1$. Move vertically downward to the cell (2, 1) and allocate min $(a_2, b_1 - 5)$ = min (8, 2) = 2. This satisfy the requirement at $D_1$ completely. Move horizontally to the cell (2, 20 and allocate min $(a_2 - 2, b_2)$ = min (8 – 2, 9) = 6. This satisfy the availability at $0_2$ completely. Move again downwards to the cell (3, 2) and allocate min $(a_3, b_2 - 6)$ = min (7, 9 – 6) = 3. This satisfy the requirement at $D_2$. Move again horizontally to the cell (3, 3) and allocate min $(a_3 - 3, b_3)$ = min (7 – 3, 18) = 4. This satisfy the availability at $0_3$. Move downwards to the cell (4, 3) and allocate min $(a_4, b_3 - 4)$ = min (14, 18 – 4) = 14.

The total transportation cost is

$= 5 \times 2 + 2 \times 3 + 6 \times 3 + 3 \times 4 + 4 \times 7 + 14 \times 2$

$=$ Rs. 102

**3. Find the initial basic feasible solutin to the following transportation problem by North-West corner rule.**

| | | | | | Available |
|---|---|---|---|---|---|
| | **19** | **30** | **50** | **10** | **7** |
| | **70** | **30** | **40** | **60** | **9** |
| | **40** | **8** | **70** | **20** | **18** |
| **Requirements** | **5** | **8** | **7** | **14** | |

*Sol :*

| | | | | |
|---|---|---|---|---|
| [5] 19 | [2] 30 | 50 | 10 | 7 |
| [2] 70 | [6] 30 | [6] 40 | 60 | 9 |
| 40 | 8 | [14] 70 | [14] 20 | 18 |
| 5 | 8 | 7 | 14 | |

Total transportation Cost = Rs. 1,015.

**4. Obtain an intial basic feasible solution to the follwing trasportation problem :**

| | D | E | F | G | Available |
|---|---|---|---|---|---|
| A | 11 | 13 | 17 | 14 | 250 |
| B | 16 | 18 | 14 | 10 | 300 |
| C | 21 | 24 | 13 | 10 | 400 |
| Requirement | 200 | 225 | 275 | 250 | |

*Sol :*

| | | | | |
|---|---|---|---|---|
| [200] 11 | [50] 13 | 17 | 14 | 250 |
| 16 | [175] 18 | [125] 14 | 10 | 300 |
| 21 | 24 | [150] 13 | [250] 10 | 400 |
| 200 | 225 | 275 | 250 | |

The transportation cost according to the above route is given by,

$z = 200 \times 11 + 50 \times 13 + 175 \times 18 + 125 \times 14 + 150 \times 13 + 250 \times 10 = 12{,}200.$

### 2.2.2 Least Cost Method

**Q7. What is Least Cost Method ? Explain the steps to get an initial basic feasible solution by Least Cost Method.**

*Ans :*

Least cost method is also known as lowest cost entry method or matrix minima method. To achieve the objective of minimum transportation cost, this method consider those routes (or cells) with least unit transportation cost to transport the goods. The steps of LCM are as follows,

**Step 1**

Determine the smallest cost in the cost matrix of the transportation table. Let it be $C_{ij}$. Allocate $x_{ij}$ = min $(a_i , b_j)$ in the cell (i, j)

**Step 2**

If $X_{ij} = a_i$ cross off the $i^{th}$ row of the transportation table and decrease $b_j$ by $a_i$. Then go to step 3.

If $x_{ij} = b_j$ cross off the $j^{th}$ column of the transportation table and decrease $a_i$ by $b_j$. Go to step 3.

If $x_{ij} = a_i = b_j$ cross off either the $i^{th}$ row or the $j^{th}$ column but not both.

**Step 3**

Repeat steps 1 and 2 for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

**PROBLEMS**

**5. Obtain an initial basic feasible solution to the following transportation problem using matrix minima method.**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | 1 | 2 | 3 | 4 | 6 |
| $O_2$ | 4 | 3 | 2 | 0 | 8 |
| $O_3$ | 0 | 2 | 2 | 1 | 10 |
| Demand | 4 | 6 | 8 | 6 | 24 |

*Sol :*

Since $\Sigma a_i = \Sigma b_j = 24$, there exists a feasible solution to the TP using tre steps in the least cost method, the first allocation is made in the cell (3, 1) the magnitude being $x_{31} = 4$. Which satisfies the demand at the destination $D_1$ and we delete this column from the table as it is exhausted.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | 1 | 2 (6) | 3 | 4 | ~~6~~ 0 |
| $O_2$ | 4 | 3 | 2 (2) | 0 (6) | ~~8~~ ~~2~~ |
| $O_3$ | 0 (4) | 2 | 2 (6) | 1 | ~~10~~ ~~6~~ |
| Demand | ~~4~~ 0 | ~~6~~ 0 | ~~8~~ ~~2~~ 0 | ~~6~~ 0 | 24 |

The second allocation is made in the cell (2, 4). with magnitude $x_{24}$ = min (6, 8) = 6. Since it satisfies the demand at the destination $D_4$.

It is deleted from the table. From the reduced table the third allocation is made in the cell (3, 3) with magnitude $x_{33}$ = min (8, 6) = 6. The next allocation is made in the cell (2, 3) with magnitude $x_{23}$ of min (2, 2) = 2.

Finally the allocation is made in the cell (1, 2) with magnitude $x_{12}$ = min (6,6) = 6. Now all the rim requirements have been satisfied and hence, initial feasible solution is obtained.

The solution is given by

$x_{12} = 6, x_{23} = 2, X_{24} = 6, X_{31} = 4, X_{33} = 6$

Since the total number of occupied cell = 5 < m + n – 1.

We get a degenerate solution.

Total cost = 6 × 2 + 2 × 2 + 6 × 0 + 4 × 0 + 6 × 2

= 12 + 4 + 12 = Rs 28.

**6. Determine an initial basic feasible solution for the following TP, using least cost method.**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | 6 | 4 | 1 | 5 | 14 |
| $O_2$ | 8 | 9 | 2 | 7 | 16 |
| $O_3$ | 4 | 3 | 6 | 2 | 5 |
| Demand | 6 | 10 | 15 | 4 | 35 |

*Sol :*

Since $\Sigma a_i = \Sigma b_j$, there exists a basic feasible solution. Using the steps in least cost method we make the first allocation to the cell (1,3) with magnitude $X_{l3}$ = min (14, 15) = 14. (as it is the cell having the least cost).

This allocation exhaust the first row supply. Hence, the first row is deleted. From the reduced table, the next allocation is made in the next least cost cell (2, 3) which is chosen arbitrarily with magnitude $x_{23}$ = Min (1, 16) = 1. Which exhausts the 3rd column destination.

From the reduced table the next least cost cell is (3,4) which allocation > made with magnitude min (4, 5) = 4. Which exhausts the destination $D_4$ requirement. Delete this fourth column from the table. The next allocation is made in the cell (3,2) with magnitude $X_{32}$ = Min (1, 10) = 1 Which exhausts the 3rd origin capacity. Hence, the 3rd row is exhausted. From the reduced table the next allocation is given to the cell (2,1) with magnitude $X_{21}$ = min (6, 15) = 6. Which exhaust the first column require-ment. Hence it is deleted from the table.

Finally the allocation is made to the cell (2, 2) with magnitude $X_{22}$ = Min (9, 9) = 9 which satisfies the rim requirement. These allocation are shown in the transportation table as follows.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | 6 | 4 | 1 (14) | 5 | ~~14~~ |
| $O_2$ | 8 | 9 | 2 | 7 | 16 |
| $O_3$ | 4 | 3 | 6 | 2 | 5 |
| Demand | 6 | 10 | ~~15~~ 1 | 4 | |

(I allocation)

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_2$ | 8 | 9 | 2 (1) | 7 | ~~16~~ 15 |
| $O_3$ | 4 | 3 | 6 | 2 | 5 |
| Demand | 6 | 10 | ~~1~~ 0 | 4 | |

(II allocation)

| | $D_1$ | $D_2$ | $D_4$ | Supply |
|---|---|---|---|---|
| $O_2$ | [8] | [9] | [7] | 15 |
| $O_3$ | [4] | [3] | [2] ④ | ~~5~~ 1 |
| Demand | 6 | 10 | ~~4~~ 0 | |

(III allocation)

| | $D_1$ | $D_2$ | Supply |
|---|---|---|---|
| $O_2$ | [8] | [9] | 15 |
| $O_3$ | [4] | [3] ① | ~~1~~ 0 |
| Demand | 6 | ~~10~~ 9 | |

(IV allocation)

| | $D_1$ | $D_2$ | Supply |
|---|---|---|---|
| $O_2$ | [8] ⑥ | [9] ⑨ | 15 |
| Demand | 6 | 9 | |

(V, VI allocation)

The following table gives the initial basic feasible solution.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | [6] | [4] | [1] ⑭ | [5] | 14 |
| $O_2$ | [8] ⑥ | [9] ⑨ | [2] ① | [7] | 16 |
| $O_3$ | [4] | [3] ① | [6] | [2] ④ | 5 |
| Demand | 6 | 10 | 15 | 4 | |

Solution is given by

$X_{13} = 14; X_{21} = 6; X_{22} = 9; X_{23} = 1; X_{32} = 1; X_{34} = 4$

Transportation cost

$= 14 \times 1 + 6 \times 8 + 9 \times 9 + 1 \times 2 + 3 \times 1 + 4 \times 2$

= Rs 156.

### 2.2.3 Vogel's Approximation Method

**Q8. Write the steps for vogel's approximation method (VAM).**

*Ans :*

Vogel's method makes effective use of the cost information and the solution obtained is much more nearer to the optimal solution than the once obtained by the earlier methods. Various steps involved in this method can be summarized below :

**Step 1 :** Compute the difference between the costs of two cheapest routes for each origin and each destination. Each individual difference is considered as penalty cost for not choosing the cheapest route and is marked opposite each row and column.

**Step 2 :** Identify the row or column with largest difference (penalty cost) and assign as many units are possible to the lowest cost cell in that row or column so as to exhaust either the supply at a particular source or satisfy demand at a warehouse.

If a tie occurs in penalties, select that row / column which has minimum cost. If there is a tie in the minimum cost also, select that particular row / column which will have maximum possible assignments. Such an exercise will considerably reduce the computational work.

**Step 3 :** Reduce the row supply or the column demand by the amount assignment to the cell.

**Step 4 :** If the row supply is now zero, eliminate the row; if the column demand is zero, eliminate; if both row supply and column demand are zero, eliminate both the row and column.

**Step 5 :** Recompute the row and column difference for the reduce transportation table, omitting rows or columns crossed out in the preceding steps.

**Step 6 :** Repeat the above procedure until all the rim condition are satisfied.

## PROBLEMS

**7. Consider the transportation problem**

| Source | Destination | | | | Total |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | |
| $O_1$ | 1 | 2 | 1 | 4 | 30 |
| $O_2$ | 3 | 3 | 2 | 1 | 50 |
| $O_3$ | 4 | 2 | 5 | 9 | 20 |
| Total | 20 | 40 | 30 | 10 | 100 |

**Determine the initial feasible solution.**

*Sol :*

The given TP is already balanced since the total supply (30 + 50 + 20 = 100) is eaual total demand (20 + 40 + 30 + 10 = 100).

Calculation of penalities for rows and columns.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply | Penalty |
|---|---|---|---|---|---|---|
| $O_1$ | 1 | 2 | 1 | 4 | 30 | 0 (1–1) |
| $O_2$ | 3 | 3 | 2 | 1 | 50 | 1 (2–1) |
| $O_3$ | 4 | 2 | 5 | 9 | 20 | 2 (4–2) |
| Demand | 20 | 40 | 30 | 10 | 100 | |
| Penalty | 2 (3–1) | 0 (2–2) | 1 (2–1) | 3 (4–1) | | |

Penalty for $O_1$, row : least is 1, and next least is also 1, therefore difference is zero.

penalty for $O_2$ row : least is 1, next least is 2 therefore penalty is 1.

Similarly for $O_3$ : 4 – 2 = 2.

Also for $D_1$ : 3 – 1 = 2 for $D_2$ : 2 – 2 = 0 for $D_3$ : 2 – 1 = and $D_4$ : 4 – 1 = 3.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply | Penalty |
|---|---|---|---|---|---|---|
| $O_1$ | 1 | 2 | 1 | 4 | 30 | 0 |
| $O_2$ | 3 | 3 | 2 | [10] 1 | ~~50~~ 40 | 1 |
| $O_3$ | 4 | 2 | 5 | 9 | 20 | 2 |
| Demand | 20 | 40 | 30 | (10) | 100 | |
| Penalty | 2 | 0 | 1 | (3) | | |

Allocation at $C_{24}$ i.e., ($O_2$, $D_4$) No. of units 10, deletion $D_4$ columns Adjustment for $O_2$ from 50 to 40 Cost : 1 × 10 = 10.

| | $D_1$ | $D_2$ | $D_3$ | Supply | Penalty |
|---|---|---|---|---|---|
| $O_1$ | 1 | 2 | 1 | 30 | 0 |
| $O_2$ | 3 | 3 | 2 | 40 | 1 |
| $O_3$ | 4 | [20] 2 | 5 | (20) | (2) |
| Demand | 20 | ~~40~~ 20 | 30 | 90 | |
| Penalty | 2 | 0 | 1 | | |

Allocation at $C_{32}$ i.e., ($O_3$, $D_2$) No. of units 20, deletion $O_3$ Adjustnment for $O_2$ from 40 to 20 Cost : 2 × 20 = 40.

Here, highest penalties (2) are in tie, therefore we select arbitrarily as $O_3$.

| | $D_1$ | $D_2$ | $D_3$ | Supply | Penalty |
|---|---|---|---|---|---|
| $O_1$ | [20] 1 | 2 | 1 | ~~30~~ 10 | 0 |
| $O_2$ | 3 | 3 | 2 | 40 | 1 |
| Demand | 20 | 20 | 30 | 70 | |
| Penalty | (2) | 1 | 1 | | |

Allocation at $C_{11}$ i.e., ($O_1$, $D_1$) No. of unis 20, deletion $D_1$ Adjustment for $O_1$ from 30 to 10 Cost : 1 × 20 = 20.

| | $D_1$ | $D_3$ | Supply | Penalty |
|---|---|---|---|---|
| $O_1$ | 2 | [10] 1 | 10 | (1) |
| $O_2$ | 3 | 2 | 40 | 1 |
| Demand | 20 | ~~30~~ 20 | 50 | |
| Penalty | 1 | 1 | | |

Tie for penalty is resolved by arbitration and selected $O_1$ Allocation ($C_{13}$) i.e., ($O_1$, $D_3$) No. of units : 10, Deletion : $O_1$ Adjustment for $D_3$ from 30 to 20 Cost = 1 × 10 = 10

| | $D_2$ | $D_3$ | Supply | Penalty |
|---|---|---|---|---|
| $O_2$ | [20] 3 | [20] 2 | 40 | 1 |
| Demand | 20 | 20 | 40 | |

**Summary :**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | [20] 1 | 2 | [10] 1 | 4 | 30 |
| $O_2$ | 3 | [20] 2 | [20] 2 | [10] 1 | 50 |
| $O_3$ | 4 | [20] 2 | 5 | 9 | 20 |
| Demand | 20 | 40 | 30 | 10 | |

(Check whether sums of the allocated units are same as their respective demand/supply)

**Cost Calculations**

| Allocated Cell | From | To | No. of Units | Unit Cost | Total |
|---|---|---|---|---|---|
| $C_{11}$ | $O_1$ | $D_1$ | 20 | 1 | 20 |
| $C_{13}$ | $O_1$ | $D_3$ | 10 | 1 | 10 |
| $C_{22}$ | $O_2$ | $D_2$ | 20 | 3 | 60 |
| $C_{23}$ | $O_2$ | $D_3$ | 20 | 2 | 40 |
| $C_{24}$ | $O_2$ | $D_4$ | 10 | 1 | 10 |
| $C_{31}$ | $O_3$ | $D_2$ | 20 | 2 | 40 |
| | | | | Total | 180 |



**8. Find the initial basic feasible solution to the following transportation problem by VAM.**

| | | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Available |
|---|---|---|---|---|---|---|
| | $O_1$ | 10 | 2 | 20 | 11 | 15 |
| Origin | $O_2$ | 12 | 7 | 9 | 20 | 25 |
| | $O_3$ | 4 | 14 | 16 | 18 | 10 |
| | Required | 5 | 15 | 15 | 15 | |

*Sol :*

Identify the smallest and next to smallest cost in each row and column. Compute the differences between them and place them along side the transportation table by enclosing them in the parentheses against the respective rows (columns). These differences are called penalities.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | | |
|---|---|---|---|---|---|---|
| $O_1$ | 10 | 2 | 20 | 11 | 15 | (8) |
| $O_2$ | 12 | 7 | 9 | 20 | 25 | (2) |
| $O_3$ | [5] 4 | 14 | 16 | 18 | 10/5 | (1) ← |
| | 5 | 15 | 15 | 15 | | |
| | (6) | (5) | (7) | (7) | | |

Row 3 has the largest penalty i.e., 10 and cell (3, 1) has smallest cost in the row. Since the availability is 10 and requirement is 5, max possible amount is 5 i.e., min (10, 5) = 5. Allocate this to the cell (3, 1). $D_1$ is satisfied, so, column 1 is crossed out and reduce the corresponding availability by 5. The reduced matrix is

**Repeat the Procedure with the Reduced table :**

| | $D_1$ | $D_2$ | $D_4$ | | |
|---|---|---|---|---|---|
| $O_1$ | [15] 2 | 20 | 11 | 15 | (9) |
| $O_2$ | 7 | 9 | 20 | 25 | (2) |
| $O_3$ | 14 | 16 | 18 | 5 | (2) |
| | 15 | 15 | 15 | | |
| | (5) | (7) | (7) | | |

The largest penalty (9) is associated with row 1 and cell (1, 2) has smallest cost. Allocate min (15, 15) = 15 to the cell (1, 2) and satisfies row 1 and column 2 simultaneously. Arbitrarily cross out both row 1 and column 2.

| | | $D_3$ | $D_4$ | | |
|---|---|---|---|---|---|
| $O_1$ | [15] | 9 | 20 | 25/10 | (11) Largest penalty |
| $O_3$ | | 16 | 18 | 5 | (2) |
| | | 15 | 15 | | |
| | | (7) | (2) | | |

The reduced table is

| | $Q_4$ | |
|---|---|---|
| $Q_2$ | [10] 20 | 10 |
| $Q_3$ | [5] 18 | 5 |
| | 15 | |

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | |
|---|---|---|---|---|---|
| $O_1$ | 10 | [15] 2 | [10] 20 | 11 | 15 |
| $O_2$ | 12 | 7 | [15] 9 | [10] 20 | 25 |
| $O_3$ | [5] 4 | 14 | 16 | [5] 18 | 10 |
| | 5 | 15 | 15 | 15 | |

∴ The total transportation cost is,

$Z = 15 \times 2 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18$

$= \text{Rs. } 475.$

**9. Find an initial basic feasible solution to the transportation problem by VAM.**

| | I | II | III | IV | Supply |
|---|---|---|---|---|---|
| 1 | 18 | 15 | 6 | 15 | 15 |
| 2 | 5 | 7 | 5 | 6 | 21 |
| 3 | 21 | 23 | 10 | 25 | 9 |
| 4 | 12 | 16 | 2 | 18 | 13 |
| Demand | 9 | 15 | 20 | 14 | |

*Sol :*

| | I | II | III | IV | | Penalty |
|---|---|---|---|---|---|---|
| 1 | 18 | 15 | 6 | 15 | 15 | (9) |
| 2 | 5 | 7 | 5 | 6 | 21 | (0) |
| ~~3~~ | ~~21~~ | ~~23~~ | [5] ~~10~~ | ~~25~~ | ~~9~~ | ~~(11)~~ ← |
| 4 | 12 | 16 | 2 | 18 | 13 | (10) |
| | 9 | 15 | 20/11 | 14 | | |
| Penalty | (7) | (9) | (3) | (9) | | |

The largest of penalty (11) corresponds to the third row. The least cost cell in this row is (3, 3). Allocate min (9, 20) = 9 to this cell and cross out third row. Since third row is satisfied. Adjust the demand and compute penalties for the resulting table.

| | I | II | III | IV | | |
|---|---|---|---|---|---|---|
| 1 | 18 | 15 | 6 | 15 | 15 | (9) |
| 2 | 5 | 7 | 5 | 6 | 21 | (0) |
| 3 | 12 | 16 | [11] 2 | 18 | 13–11 =2 | (10) |
| | 9 | 15 | 11 | 14 | | |
| | (7) | (8) | (3) | (9) | | |

The largest of the penalties is 10, which corresponds to fourth row and least cost cell in this row is (4, 3). Allocate min (11, 13) = 11 to this cell and cross out the fourth row. The resulting table is

| | I | II | IV | | |
|---|---|---|---|---|---|
| 1 | 18 | 15 | 15 | 15 | (0) |
| 2 | 5 | 7 | [14] 6 | 21–4 =7 | (1) |
| 4 | 12 | 16 | 18 | 2 | (4) |
| | 9 | 15 | 14 | | |
| | (7) | (8) | (9) ↑ | | |

Proceed in the similar manner

| | | | | |
|---|---|---|---|---|
| 1 | 18 | 15 | 15 | (3) |
| 2 | 5 | [7] 7 | 7 | (2) |
| 4 | 12 | 16 | 2 | (4) |
| | 9 | 15–7=8 | | |
| | (7) | (8) ↑ | | |

| | | | |
|---|---|---|---|
| 18 | 15 | 15 | (3) |
| [2] 12 | 16 | 2 | (4) |
| 9–2=7 | 8 | | |
| (6) ↑ | (1) | | |

| | | |
|---|---|---|
| [7] 18 | [8] 15 | 15 |
| 7 | 8 | |

| | | | | |
|---|---|---|---|---|
| [7] 18 | [8] 15 | 6 | 15 | 15 |
| 5 | [7] 7 | 5 | [14] 6 | 21 |
| 21 | 23 | [9] 10 | 25 | 9 |
| [2] 12 | 16 | [11] 2 | 18 | 13 |
| 9 | 15 | 20 | 14 | |



The total transportation cost is

$$Z = 7 \times 18 + 18 \times 15 + 7 \times 7 + 9 \times 10 + 14 \times 6 + 12 \times 2 + 11 \times 2$$

$$Z = \text{Rs. } 515.$$

**10. Find the initial basic feasible solution for the following transportation problem by VAM.**

| | | *Destination* | | | | |
|---|---|---|---|---|---|---|
| | | $D_1$ | $D_2$ | $D_3$ | $D_4$ | *Supply* |
| Origin | $O_1$ | 11 | 13 | 17 | 14 | 250 |
| | $O_2$ | 16 | 18 | 14 | 10 | 300 |
| | $O_3$ | 21 | 24 | 13 | 10 | 400 |
| | *Demand* | 200 | 225 | 275 | 250 | 950 |

*Sol :*

Since $\Sigma a_i = \Sigma b_j = 950$ the problem is balanced and there exists a feasible solution to the problem.

First we find the row & column penalty $P_1$ as the difference between the least and next least cost. The maximum penalty is 5. Choose the first column arbitrarily. In this column choose the cell having the least cost name (1, 1). Allocate to this cell with minimum magnitude (i.e.(250,200) = 200.) This exhausts the first column. Delete this column. Since column is deleted, then there is a change in row penalty $P_{II}$ and column penalty $P_{II}$ remains the same. Continuing in this manner we get the remaining allocations as given in the table below.

I allocation

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply | $P_I$ |
|---|---|---|---|---|---|---|
| $O_1$ | 11 (200) | 13 | 17 | 14 | 50 ~~250~~ | 2 |
| $O_2$ | 16 | 18 | 14 | 10 | 300 | 4 |
| $O_3$ | 21 | 24 | 13 | 10 | 400 | 3 |
| Demand | ~~200~~ 0 | 225 | 275 | 250 | | |
| $P_I$ | 5↑ | 5 | 3 | 0 | | |

II allocation

| | $D_2$ | $D_3$ | $D_4$ | Supply | $P_{II}$ |
|---|---|---|---|---|---|
| $O_1$ | 13 (50) | 17 | 14 | ~~50~~ 0 | 3 |
| $O_2$ | 18 | 14 | 10 | 300 | 4 |
| $O_3$ | 24 | 13 | 10 | 400 | 3 |
| Demand | ~~225~~ 175 | 275 | 250 | | |
| $P_{II}$ | 5↑ | 3 | 0 | | |



III allocation

| | $D_2$ | $D_3$ | $D_4$ | Supply | $P_{III}$ |
|---|---|---|---|---|---|
| $O_2$ | 18 (175) | 14 | 10 | ~~300~~ 125 | 4 |
| $O_3$ | 24 | 13 | 10 | 400 | 3 |
| Demand | ~~175~~ 0 | 275 | 250 | | |
| $P_{III}$ | 6↑ | 1 | 0 | | |

IV allocation

| | $D_3$ | $D_4$ | Supply | $P_{IV}$ |
|---|---|---|---|---|
| $O_2$ | 14 | 10 (125) | ~~125~~ 0 | 4 ← |
| $O_3$ | 13 | 10 | 400 | 3 |
| Demand | 275 | ~~250~~ 125 | | |
| $P_{IV}$ | 1 | 0 | | |

V allocation

| | $D_3$ | $D_4$ | Supply | $P_V$ |
|---|---|---|---|---|
| $O_3$ | 13 (275) | 10 | ~~400~~ 125 | 3 |
| Demand | ~~275~~ 0 | 125 | | |
| $P_V$ | 13↑ | 10 | | |

VI allocation

| | $D_4$ | Supply | $P_{VI}$ |
|---|---|---|---|
| $O_3$ | 10 (125) | ~~125~~ 0 | 10 ← |
| Demand | ~~125~~ 0 | | |
| $P_{VI}$ | 10 | | |

Finally we arrive at the initial basic feasible solution which is shown in the following table.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | 11 (200) | 13 (50) | 17 | 14 | 250 |
| $O_2$ | 16 | 18 (175) | 14 | 10 (125) | 300 |
| $O_3$ | 21 | 24 | 13 (275) | 10 (125) | 400 |
| Demand | 200 | 225 | 275 | 250 | |

There are 6 positive independent allocations which equals to m + n –1 = 3 + 4 – 1. This ensures that the solution is a non- degenerate basic feasible solution.

∴ The transportation cost

= 11 × 200 + 13 × 50 + 18 × 175 + 10 × 125 + 13 × 275 + 10 × 125

= Rs 12,075.

**11. Find the initial solution to the following TP using VAM.**

| | | Destination | | | | |
|---|---|---|---|---|---|---|
| | | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
| Factory | $F_1$ | 3 | 3 | 4 | 1 | 100 |
| | $F_2$ | 4 | 2 | 4 | 2 | 125 |
| | $F_3$ | 1 | 5 | 3 | 2 | 75 |
| | Demand | 120 | 80 | 75 | 25 | 300 |

*Sol :*

Since $\Sigma a_i = \Sigma b_j$ the problem is a balance TP ∴ there exists a feasible solution.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply | $P_I$ | $P_{II}$ | $P_{III}$ | $P_{IV}$ | $P_V$ | $P_{VI}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_1$ | 3 (45) | 3 | 4 (30) | 1 (25) | 100 | 2 | 2 ← | 0 | 1 ← | 4 | 4 |
| $F_2$ | 4 | 2 (80) | 4 (45) | 2 | 125 | 0 | 0 | 2 ← | 0 | 4 ← | – |
| $F_3$ | 1 (75) | 5 | 3 | 2 | 75 | 1 | – | – ← | – | – | – |
| Demand | 120 | 80 | 75 | 25 | | | | | | | |
| $P_I$ | 2↑ | 1 | 1 | 1 | | | | | | | |
| $P_{II}$ | 1 | 1 | 0 | 1 | | | | | | | |
| $P_{III}$ | 1 | 1 | 0 | – | | | | | | | |
| $P_{IV}$ | 1 | – | 0 | – | | | | | | | |
| $P_V$ | – | – | 0 | – | | | | | | | |
| $P_{VI}$ | – | – | 4↑ | – | | | | | | | |

Finally we have the initial basic feasible solution as given in the following table.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $F_1$ | 3 (45) | 3 | 4 (30) | 1 (25) | 100 |
| $F_2$ | 4 | 2 (80) | 4 (45) | 2 | 125 |
| $F_3$ | 1 (75) | 5 | 3 | 2 | 75 |
| Demand | 120 | 80 | 75 | 25 | |

There are 6 independent non-negative allocations equal to m + n – 1 = 3 + 4 – 1 = 6. This ensures that the solution is non - degenerate basic feasible .

∴ The transportation cost

= 3 × 45 + 4 × 30 + 1 × 25 + 2 × 80 + 4 × 45 + 1 × 75

= 135 + 120 + 25 + 160 + 180 + 75 = Rs. 695

## 2.3 Types of Transportation Problem

**Q9. Define maximisation case of Balance transportation problem (TP)**

*Ans :*

In some transportation problems, we may come across transportation problems with maximisation objectives. In order to convert the maximisation into minimisation, we can we any of the following two methods :

(a) Multiplying with (–1) in all cells

(b) Subtracting each cell value from highest among the cell values.



**12. Obtain IBFS of transportation problem whose profit matrix is given below.**

| | | Markets | | | |
|---|---|---|---|---|---|
| | | $M_1$ | $M_2$ | $M_3$ | Stock |
| Godowns | $G_1$ | 4 | 4 | 9 | 25 |
| | $G_2$ | 3 | 5 | 8 | 20 |
| | Sales | 18 | 16 | 11 | 45 |

**Sol :**

As the given TP is in maximisation form, it is to be converted to equivalent cost matrix.

**Equivalent Cost Matrix :**

| | | Markets | | | |
|---|---|---|---|---|---|
| | | $M_1$ | $M_2$ | $M_3$ | Stock |
| Godowns | $G_1$ | 4 | 4 | 9 | 25 |
| | $G_2$ | 3 | 5 | 8 | 20 |
| | Sales | 18 | 16 | 11 | 45 |

Highest among all unit profits is 9. All other unit cost are calculated by subtracting from 9.

**Check Balancing :** As total stock (25 + 20 = 45) is equal to total sales (18 + 16 + 11 = 45), the TP is balanced.

**Calculation of Penalties**

| | $M_1$ | $M_2$ | $M_3$ | Stock | Penalty |
|---|---|---|---|---|---|
| $G_1$ | 4 | 4 | [11] 9 | 25 | (5) ← |
| $G_2$ | 3 | 5 | 8 | 20 | 3 |
| Sales | 18 | 16 | (11) | 45 | |
| Penalty | 1 | 1 | 1 | | |

Highest penalty is 5 among 5, 3, 1, 1, 1

∴ $G_1$ is chosen to allocated at $C_{13}$ i.e., $(G_1, M_3)$

No. of units : 11

Deletion : $M_3$

Adjustment to $G_1$ as 25 to 14

Cost = 0 × 11 = 9

Profit = 9 × 11 = 99

Repetition of steps 4 to 6

| | $M_1$ | $M_2$ | Stock | Penalty |
|---|---|---|---|---|
| $G_1$ | 5 | 5 | 14 | 0 |
| $G_2$ | 6 | 4 | ~~20~~ 4 | 2 ← |
| Sales | 18 | (16) | 34 | |
| Penalty | 1 | 1 | | |



Highest penalty is 2

∴ $G_2$ is selectedl; allocation to $C_{22}$

i.e., $(G_2, M_2)$

No. of units : 16

Deletion : $M_2$

Adjustment to $G_2$ from 20 to 4

Cost = 4 × 16 = 64

Profit = 5 × 16 = 80

**Summary :**

Equivalent Cost Matrix

| | $M_1$ | $M_2$ | $M_3$ | Stock |
|---|---|---|---|---|
| $G_1$ | 5 | [14] 5 | [11] 0 | 25 |
| $G_2$ | [4] 6 | [16] 4 | 1 | 20 |
| Sales | 18 | 16 | 11 | 45 |

Profit Matrix (Original)

| | $M_1$ | $M_2$ | $M_3$ | Stock |
|---|---|---|---|---|
| $G_1$ | [14] 4 | 4 | [11] 9 | 25 |
| $G_2$ | [4] 63 | [16] 5 | 18 | 20 |
| Sales | 18 | 16 | 11 | 45 |

**Profit Calculations**

| Allocated Cell | From | To | No. of Units | Unit Cost | Total |
|---|---|---|---|---|---|
| $C_{11}$ | $G_1$ | $M_1$ | 14 | 4 | 56 |
| $C_{13}$ | $G_1$ | $M_3$ | 11 | 9 | 99 |
| $C_{21}$ | $G_2$ | $M_1$ | 4 | 3 | 12 |
| $C_{22}$ | $G_2$ | $M_2$ | 16 | 5 | 80 |
| Grand total profit | | | | | 247 |

**13. Obtain IBFS for the profit matrix given below :**

| | $M_1$ | $M_2$ | $M_3$ | Stock |
|---|---|---|---|---|
| $G_1$ | 4 | 4 | 9 | 25 |
| $G_2$ | 3 | 5 | 8 | 20 |
| Sales | 18 | 16 | 11 | 45 |

*Sol :*

**Standard TP :** The given matrix is profit matrix, therefore it is to be maximised. As standard TP is to be minimised, an equivalent cost matrix is to obtained by one of the following two methods.

(i) Multiplying by (–1) to all cells.

(ii) Substracting all profits from highest value among all profit cells to make equivalent costs.

**Equivalent Cost Matrix :**

**Iteration Tableau 1(a)**

| | $M_1$ | $M_2$ | $M_3$ | |
|---|---|---|---|---|
| $G_1$ | 5 | 5 | 0 | 25 |
| $G_2$ | 6 | 4 | 1 | 20 |
| | 18 | 16 | 11 | |

(OR)

**Iteration Tableau 1(b)**

| | $M_1$ | $M_2$ | $M_3$ | |
|---|---|---|---|---|
| $G_1$ | – 4 | – 4 | – 9 | 25 |
| $G_2$ | – 3 | – 5 | – 8 | 20 |
| | 18 | 16 | 11 | |

**Balance :** It is already balances since total stock (25 + 20) = total sales (18 + 16 + 11)

**Allocation :** The least cost among the cells is 'zero' in 1(a) [or –9 in 1(b)] which appears in $C_{13}$ cells i.e., from $G_1$ to $M_3$. A maximum of 1l units can be allocated here. Subsequently, $M_3$ gets exhausted and deleted for further iteration while stock $G_1$ is adjusted to 14.

**Iteration Tableau 2(a)**

| | $M_1$ | $M_2$ | $M_3$ | |
|---|---|---|---|---|
| $G_1$ | 5 | 5 | [11] 0 | ~~25~~ 14 |
| $G_2$ | 6 | 4 | 1 | 20 |
| | 18 | 16 | (11) | |

(OR)

**Iteration Tableau 2(b)**

| | $M_1$ | $M_2$ | $M_3$ | |
|---|---|---|---|---|
| $G_1$ | – 4 | – 4 | [11] – 9 | ~~25~~ 14 |
| $G_2$ | – 3 | – 5 | – 8 | 20 |
| | 18 | 16 | (11) | |

**Repeat :** Till all the units are exhausted.

**Iteration Tableau 3(a)**

| | $M_1$ | $M_2$ | |
|---|---|---|---|
| $G_1$ | 5 | 5 | 14 |
| $G_2$ | 6 | [16] 4 | ~~20~~ 4 |
| | 18 | (16) | 34 |

(OR)

**Iteration Tableau 3(b)**

| | $M_1$ | $M_2$ | |
|---|---|---|---|
| $G_1$ | – 4 | – 4 | 14 |
| $G_2$ | – 3 | [16] – 5 | ~~20~~ 4 |
| | 18 | (16) | 34 |

Leat cost is 4 in iteration tableau – 3(a) or –5 in iteration 3(b), which appears in $C_{22}$ cell i.e., $G_2$ to $M_2$. So, 16 units can be allocated here since $M_2$ requires 16 only though $G_2$ can supply 20 units.

**Iteration Tableau 4(a)**

| | $M_1$ | |
|---|---|---|
| $G_1$ | [14] 5 | 14 |
| $G_2$ | [4] 6 | 4 |
| | 18 | 18 |

(OR)

**Iteration Tableau 4(b)**

| | $M_1$ | |
|---|---|---|
| $G_1$ | [14] – 4 | 14 |
| $G_2$ | [4] – 3 | 4 |
| | 18 | 18 |

IBFS : $G_1 – M_3$ : 11 units, $G_1 – M_1$ : 14 units; $G_2 – M_1$ : 4 ; $G_2 – M_2$ : 16

**Summary :**

The allocated units are now entered in the respective cells of original TP (profit matrix).

| | | | |
|---|---|---|---|
| [14] 4 | 4 | [11] 9 | 25 |
| [4] 3 | [16] 5 | 8 | 20 |
| 18 | 16 | 11 | |



**Profit Calculation**

| Allocated Cell | From | To | No. of Units | Unit Cost | Total |
|---|---|---|---|---|---|
| $C_{11}$ | $G_1$ | $M_1$ | 14 | 4 | 56 |
| $C_{13}$ | $G_1$ | $M_3$ | 11 | 9 | 99 |
| $C_{21}$ | $G_2$ | $M_1$ | 4 | 3 | 12 |
| $C_{22}$ | $G_2$ | $M_2$ | 16 | 5 | 80 |
| | | Total Units | 45 | Total Profit | 247 |

**Q10. Define Unbalanced TP.**

*Ans :*

The transportation problems with unequal supply and demand is said to be unbalanced transportation problems. To solve this problem, we make the unbalanced TP as balanced TP by adding a dummy row or column.

## PROBLEMS

**14. A dealer stocks and sells four types of Bicycles namely Atlas, Bharath, Champion, Duncan which he may procure from three different suppliers namely Priyanshu, Quershi and Raju. His anticipated sales for the bicycles for the coming seasons are 410, 680, 310 and 550 nos. respectively. He can obtain 900 bicycles from Priyanshu, 600 from Quershi and 560 from Raju at suitable prices. The profit per bicycle in rupees for each supplier is tabulated below :**

| Type / Supplier | Atlas (A) | Bharath (B) | Champion (C) | Duncan (D) |
|---|---|---|---|---|
| **Priyanshu (P)** | **21.50** | **26.00** | **19.50** | **21.00** |
| **Quershi (Q)** | **20.50** | **24.00** | **20.00** | **21.00** |
| **Raju (R)** | **18.00** | **19.50** | **19.00** | **19.50** |

**Formulate the above information as transportation model and obtain initial solution by North West Corner Rule.**

*Sol :*

Formulation

| Type / Supplier | Atlas (A) | Bharath (B) | Champion (C) | Duncan (D) | Availability |
|---|---|---|---|---|---|
| Priyanshu (P) | 21.50 | 26 | 19.5 | 21 | 900 |
| Qureshi (Q) | 20.50 | 24 | 20 | 21 | 600 |
| Raju (R) | 18 | 19.5 | 19 | 19.5 | 560 |
| Requirement | 410 | 680 | 310 | 550 | 2060 / 1950 |



The above problem contains profit matrix and therefore it is to be maximised. This is non-standard form. As standard T.P. is to be minimised an equivalent cost matrix is to be obtained by any of the two methods (i) Multiplying with (–1) in all cells (ii) Substractin each cell value from highest among the ell values. Here (i) is used.

| Type Supplier | Atlas (A) | Bharath (B) | Champion (C) | Duncan (D) | Dummy (E) | Availability |
|---|---|---|---|---|---|---|
| Priyanshu (P) | –21.5 | –26 | –19.5 | –21 | 0 | 900 |
| Qureshi (Q) | –20.5 | –24 | –20 | -21 | 0 | 600 |
| Raju (R) | –18 | –19.5 | –19 | –19.5 | 0 | 560 |
| Requirement | 410 | 680 | 310 | 550 | 110 | 2060 |

**Balancing**

The above problem is unbalanced since the requirement (1950) is not equal to availability (2060). As the availability is excess, we create a Dummy column (Say E) with no profit for any supplier and with a requirement of the deficit i.e., 110.

| Type Supplier | Atlas (A) | Bharath (B) | Champion (C) | Duncan (D) | Dummy (E) | Availability |
|---|---|---|---|---|---|---|
| Priyanshu (P) | 21.5 | 26 | 19.5 | 21 | 0 | 900 |
| Qureshi (Q) | 20.5 | 24 | 20 | 21 | 0 | 600 |
| Raju (R) | 18 | 19.5 | 19 | 19.5 | 0 | 560 |
| Requirement | 410 | 680 | 310 | 550 | 110 | 2060 / 2060 |

With usual rules we iterate as follows to get initial solution.

**Iteration Tableau 1**

| | A | B | C | D | E | Avl. |
|---|---|---|---|---|---|---|
| P | [410] – 21.5 | – 26 | – 19.5 | – 21 | 0 | ~~900~~ 490 |
| Q | – 21.5 | – 24 | – 19.5 | – 21 | 0 | 600 |
| R | – 18 | – 19.5 | – 19 | – 19.5 | 0 | 560 |
| Req. | (410) | 680 | 310 | 550 | 110 | ~~2060~~ 1650 |

Allocation to (P, A)
No. of units 410
Deletion column A
Adjustment to P 900 to 490
Cost is – 21.5 × 410 = – 8815
(Negative cost means profit here)

**Iteration Tableau 2**

| | A | B | C | D | E | Avl. |
|---|---|---|---|---|---|---|
| P | [410] – 21.5 | [490] – 26 | – 19.5 | – 21 | 0 | ~~900~~ (490) |
| Q | – 21.5 | – 24 | – 19.5 | – 21 | 0 | 600 |
| R | – 18 | – 19.5 | – 19 | – 19.5 | 0 | 560 |
| Req. | 410 | ~~680~~ 190 | 310 | 550 | 110 | ~~1650~~ 1160 |

Allocation to (P, B)
No. of units 490
Deletion row P
Adjustment to B 680 to 190
Cost– 26 × 490 = – 12740

**Iteration Tableau 3**

| | A | B | C | D | E | Avl. |
|---|---|---|---|---|---|---|
| P | [410] – 21.5 | [490] – 26 | – 19.5 | – 21 | 0 | 900 |
| Q | – 21.5 | [190] – 24 | – 19.5 | – 21 | 0 | ~~600~~ (410) |
| R | – 18 | – 19.5 | – 19 | – 19.5 | 0 | 560 |
| Req. | 410 | ~~680~~ 190 | 310 | 550 | 110 | ~~1160~~ 970 |

Allocation to (Q, B)
No. of u nits 190
Deletion column B
Adjustment to Q as 600 to 410
Cost– 24 × 190 = – 4560

### Iteration Tableau 4

| | A | B | C | D | E | Avl. |
|---|---|---|---|---|---|---|
| P | [410] – 21.5 | [490] – 26 | – 19.5 | – 21 | 0 | 900 |
| Q | – 20.5 | [190] – 24 | [310] – 20 | – 21 | 0 | ~~600~~ ~~410~~ 100 |
| R | – 18 | – 19.5 | – 19 | – 19.5 | 0 | 560 |
| Req. | 410 | 680 | (310) | 550 | 110 | ~~970~~ 660 |

Allocation to (Q, C)
No. of units 310
Deletion column C
Adjustment to Q as 410 to 100
Cost– 20 × 310 = – 6200

### Iteration Tableau 5 & 6

| | A | B | C | D | E | Avl. |
|---|---|---|---|---|---|---|
| P | [410] – 21.5 | [490] – 26 | – 19.5 | – 21 | 0 | 900 |
| Q | – 20.5 | [190] – 24 | [310] – 20 | [100] – 21 | 0 | ~~600~~ ~~410~~ (100) |
| R | – 18 | – 19.5 | – 19 | [450] – 19.5 | [110] 0 | ~~560~~ 110 |
| Req. | 410 | 680 | 310 | ~~550~~ (450) | (110) | ~~660~~ ~~560~~ 450 |

**Tableau – 5 :**
Allocation to (Q, D)
No. of units 100
Deletion row Q
Adjustment to D as 550 to 450
Cost – 21 × 100 = – 2100

**Tableau – 6 :**
Allocation to (R, D) and (R, E)
No. of units 450 and 110
(all units exhausted)
Cost : –19.5 × 450 and
0 × 100 = 8775 and 0

### Iteration Tableau 7

Final Tableau - 7 for IBFS by NWCM :

***Equivalent Cost Matrix***

| | A | B | C | D | E | Avl. |
|---|---|---|---|---|---|---|
| P | [410] – 21.5 | [490] – 26 | – 19.5 | – 21 | 0 | 900 |
| Q | – 20.5 | [190] – 24 | [310] – 20 | [100] – 21 | 0 | 600 |
| R | – 18 | – 19.5 | – 19 | [450] – 19.5 | [110] 0 | 560 |
| Req. | 410 | 680 | 310 | 550 | 110 | 2060 |

***Profit Matrix***

| | A | B | C | D | E | Avl. |
|---|---|---|---|---|---|---|
| P | [410] 21.5 | [490] 26 | 19.5 | 21 | 0 | 900 |
| Q | 20.5 | [190] 24 | [310] 20 | [100] 21 | 0 | 600 |
| R | 18 | 19.5 | 19 | [450] 19.5 | [110] 0 | 560 |
| Req. | 410 | 680 | 310 | 550 | 110 | 2060 |

### Profit Calculation

| Allocated Cell | From | To | Unit Cost | No. of Units | Total Cost |
|---|---|---|---|---|---|
| $C_{11}$ | P | A | 21.5 | 410 | 8815 |
| $C_{12}$ | P | B | 26 | 490 | 12740 |
| $C_{22}$ | Q | B | 24 | 190 | 4560 |
| $C_{23}$ | Q | C | 20 | 310 | 6200 |
| $C_{33}$ | Q | D | 21 | 100 | 2100 |
| $C_{34}$ | R | D | 19.5 | 450 | 8775 |
| $C_{35}$ | R | E | 0 | 110 | 00 |
| | | | | Total Profit | 43190 |

**15. Priyanshu Enterprise has three factories at locations A, B and C which supplies three warehouses located at D, E and F. Monthly factory capacities are 10, 80 and 50 units respectively. Monthly warehouse requirements are 75, 20 and 50 units respectively. Unit shipping cost (in Rs.) are given as**

Warehouse

| Factory | D | E | F |
|---|---|---|---|
| A | 5 | 1 | 7 |
| B | 6 | 4 | 6 |
| C | 3 | 2 | 5 |

**The penalty costs for not satisfying demand at warehouses D, E and F are Rs. 5, Rs. 3 and Rs. 2 per unit respectively. Determine the optimal distribution for Priyanshu using transportation technique.**

*Sol :*

**Formulation :** The given information is formulated as follows :

**Iteration Tableau 1**

Warehouse

| Factory | D | E | F | Capacity |
|---|---|---|---|---|
| A | 5 | 6 | 7 | 10 |
| B | 3 | 7 | 4 | 80 |
| C | 8 | 5 | 4 | 50 |
| Requirement | 75 | 20 | 50 | 140 / 145 |

**Check Balancing**

In the above TP, total capacity (140) is less than total requirement (145), Therefore, this TP is unbalanced. To balance this TP we create another dummy factory say K, with the deficit capacity, i.e., 145 – 140 = 5.

The transportation costs usually we assume zeros, but here these are taken as 5, 3 and 2 instead of zeros since these are penalty costs for not satisfying the demand.

Thus the reformulated TP tableau is

**Iteration Tableau 2 :**

Warehouse

| Factory | D | E | F | Capacity |
|---|---|---|---|---|
| A | 5 | 1 | 7 | 10 |
| B | 6 | 4 | 6 | 80 |
| C | 3 | 2 | 5 | 50 |
| K | 5 | 3 | 2 | 5 |
| Requirement | 75 | 20 | 50 | |

Penalty calculation and allocation and adjustments :

**Iteration Tableau 3 :**

Warehouse

| Factory | D | E | F | Capacity | Penalty |
|---|---|---|---|---|---|
| A | 5 | [10] 1 | 7 | (10) | (4) ← |
| B | 6 | 4 | 6 | 80 | 2 |
| C | 3 | 2 | 5 | 50 | 1 |
| K | 5 | 3 | 2 | 5 | 1 |
| Requirement | 75 | ~~20~~ 10 | 50 | 145 | |

Highest penalty for A i.e., 4
Allocation to $C_{12}$ i.e., A to E
No. of units 10
Deletion A (exhausted)
Adjustment to E as 20 to 10
Cost : 1 × 10 = 10

**Iteration Tableau 4 :**

Warehouse

| Factory | D | E | F | Capacity | Penalty |
|---|---|---|---|---|---|
| B | 6 | 4 | 6 | 80 | 2 |
| C | 3 | 2 | 5 | 5 | 1 |
| K | 5 | 3 | [10] 2 | (5) | 1 |
| Requirement | 75 | 10 | ~~50~~ 45 | 135 | |
| Penalty | 2 | 1 | 3 ↑ | | |

Highest penalty for F i.e., 3
Allocation to $C_{43}$ i.e., K to F
No. of units 5
Deletion K (exhausted)
Adjustment for F from 50 to 45
Cost : 2 × 5 = 10

**Iteration Tableau 5 :**

Warehouse

| Factory | D | E | F | Capacity | Penalty |
|---|---|---|---|---|---|
| B | 6 | 4 | 6 | 80 | 2 |
| C | [5] 3 | 2 | 5 | (50) | 1 |
| Requirement | ~~75~~ 25 | 10 | 45 | 130 | |
| Penalty | 3 ↑ | 2 | 1 | | |

Highest penalty is for i.e., 3
Allocation to $C_{43}$ i.e., K to F
No. of units 5
Deletion K (exhausted)
Adjustment for F from 50 to 45
Cost : 2 × 5 = 10

**Iteration Tableau 6 :**

Warehouse

| Factory | D | E | F | Capacity | Penalty |
|---|---|---|---|---|---|
| B | [10] 6 | [10] 4 | [45] 6 | 80 | 2 |
| Requirement | 25 | 10 | 45 | 80 | |
| Penalty | 3 ↑ | 2 | 1 | | |

Penalty calculation is not required
Allocations B to D : 25,
B to E : 10 and B to F : 45

**Summary :**

Warehouse

| Factory | D | E | F | Capacity |
|---|---|---|---|---|
| A | 5 | [10] 1 | 7 | 10 |
| B | [25] 6 | [10] 4 | [45] 6 | 80 |
| C | [50] 3 | 2 | 5 | 50 |
| K | 5 | 3 | 2 | 5 |
| Requirement | 75 | 20 | 50 | 145 |

**Cost Calculation**

| Allocated Cell | From | To | No. of Units | Unit Cost | Total Cost |
|---|---|---|---|---|---|
| $C_{12}$ | A | E | 10 | 1 | 10 |
| $C_{21}$ | B | D | 25 | 6 | 150 |
| $C_{22}$ | B | E | 10 | 4 | 40 |
| $C_{23}$ | B | F | 45 | 6 | 270 |
| $C_{31}$ | C | D | 50 | 3 | 150 |
| $C_{43}$ | K | F | 5 | 2 | *10 |
| | | Grand Total Transport Cost | | | 620 + 10 = 630 |

* Total transportation cost is Rs. 620 and penalty for not satisfying the demand of 5 units of F is 10. This total is Rs.630/-.

**16. A company has received a contract to supply gravel for three new construction projects located in towns A, B and C. Construction engineers have estimated the required amounts of gravel which will be needed at these construction projects**

| Project Location | Weekly requirement (Truck loads) |
|---|---|
| A | 144 |
| B | 204 |
| C | 82 |

**The company has three gravel pits locatd in towns, W, X and Y respectively. The gravel required by the construction projects can be supplied by thse three plants. The amount of gravel which can be supplied by each plant is as follows :**

| Plant | W | X | Y |
|---|---|---|---|
| Amount available (Truck loads) | 152 | 164 | 154 |

**The company has computed the deliver cost from each plant to each project site. These costs (in rupees) as showsn in the following table :**

**Cost per truck load**

| | | A | B | C |
|---|---|---|---|---|
| | **W** | **8** | **16** | **16** |
| **Plant** | **X** | **32** | **48** | **32** |
| | **Y** | **16** | **32** | **48** |

**Schedule the shipment from each plant to each project in such a manner so as to minimize the total transportation cost within the constraints imposed by plant capacities and project requirements. Find the minimum cost.**

*Sol :*

Since the total requirement (i.e., 430 truck loads) at project locations A, B and C is less than the total capacity (i.e., 470 truck loads) at plants W, X and Y by 470 – 430 = 40, the given problem is an unbalanced one. We, therefore, introduce a dummy project location D with its requirement of 40 truck loads and transportation cost from the dummy project location to all points as zero.

the initial solution is obtained by using Vogel's Approximation Method and it is given in following table 1.

**Table - I**

| From \ To | A | B | C | D | Avl. |
|---|---|---|---|---|---|
| W | [8] | [16] (70) | [16] (82) | [0] | 152 |
| X | [8] | [16] (124) | [16] | [0] 40 | 164 |
| Y | [8] (144) | [16] (10) | [16] | [0] | 154 |
| Req. | 144 | 204 | 82 | 40 | 470 |



In this case, allocation to the cell with zero transportation cost will be postponed until the capacities in all other plants are exhausted because the allocation from their project locations stands for the shortages.

**Table - II**

| From \ To | A | B | C | D | Avl. |
|---|---|---|---|---|---|
| W | [8] | [16] (152) | [16] | [0] | 152 |
| X | [32] | [48] (42) | [32] (82) | [0] 40 | 164 |
| Y | [16] (144) | [32] (10) | [48] | [0] | 154 |
| Req. | 144 | 204 | 82 | 40 | 470 |

The transportation cost associated with this solution is :

$= 152 \times 16 + 42 \times 58 + 82 \times 32 + 40 \times 0 + 144 \times 16 + 10 \times 32$

= Rs. 10,116.

## 2.4 Optimisation

### 2.4.1 Stepping Stone Method

**Q11. Explain Stepping Stone Method.**

*Ans :* **(Imp.)**

This method is developed by A. Charnes. In this method we use the improvement index. The algorithm is as follows.

**Algorithm :**

**Obtain IBFS**

**Step 1**

Obtain initial solution by any suitable method such as NWCM, VAM, LCEM, etc.

**Step 2 :** Check degeneracy :

Check whether there is any degeneracy. This is checked by the degenecary condition that the number of allocated cells is less than sum of number of rows and number of columns minus one.

Non-degenerate : $n(C_{ij}) \geq n(r) + n(c) - 1$ [or simply $n(C_{ij}) \geq r + c - 1$]

Degenerate : $n(C_{ij}) < n(r) + n(c) - 1$ [or simply $n(C_{ij}) < r + c - 1$]

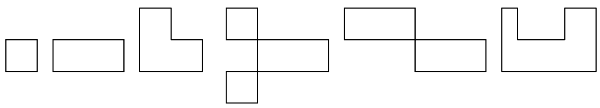
Where $n(C_{ij})$ is no. of allocated cells in IBFS, n(r) is no. of rows and n(c) is no. o columns of TP. If there is degeneracy, resolve it, else move to step-3. (Degeneracy case is explained later).

**Step 3 : Calculate Improvement Index**

IBFS may or may not be optimum. To check whether the IBFS is optimum or not, we calculate Improvement Index, which ndicates possibility of cost reduction by assigning one unit to unallocated cell.

This is done as follows :

Connect one unallocated cell at a time with all other allocated cells as corners of a loop. The loop need not always be a square. It may be any shape as given below.



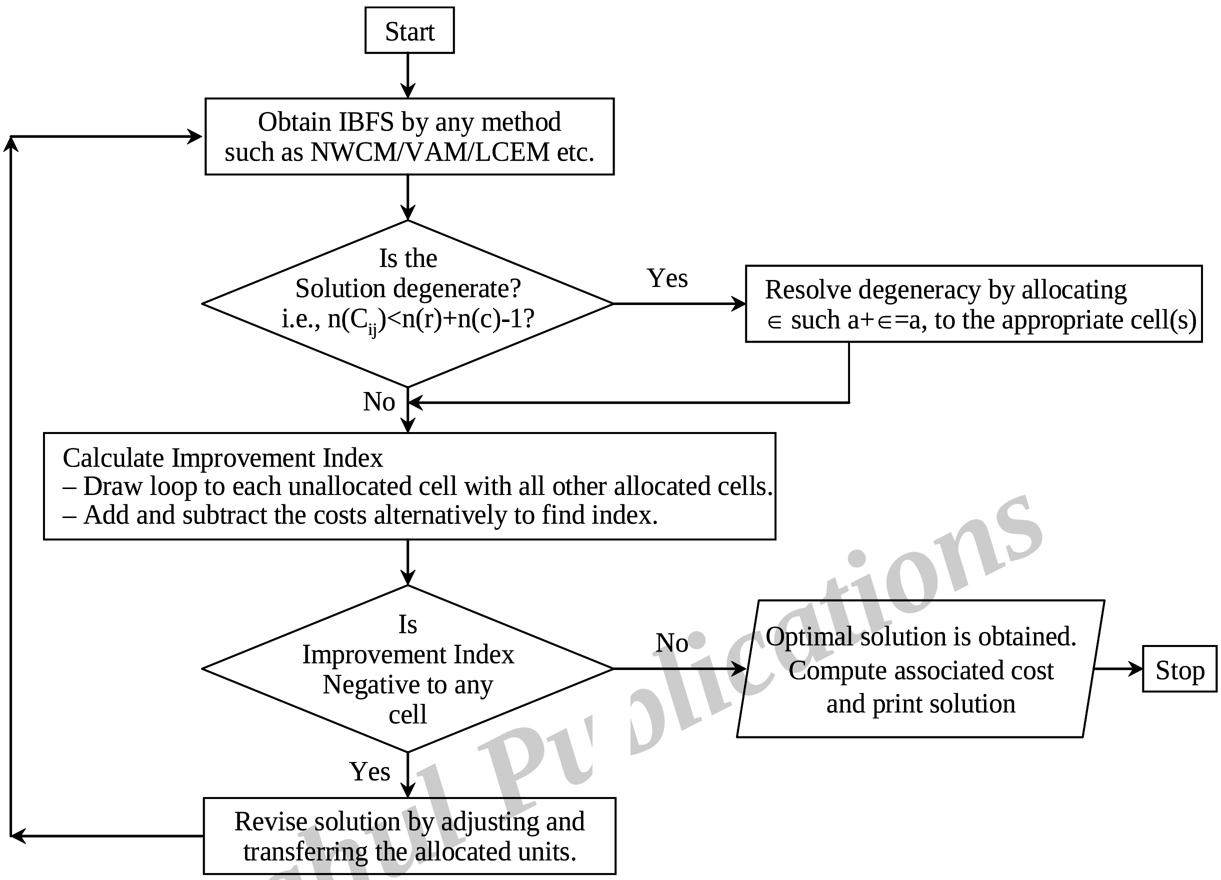
Further, the loop should always be closed and should contain either horizontal or vertical lines only, provided all corners are allocated cells except one whose index is to be found. Also the loop lines may intersect, but the point of intersection is not a corner and has no importance. The loop line may go over an unallocated or allocated cell and such cells have no interference in calculation of index.

Thus after drawing the loop, give positive sign to the unallocated cell cost and alternatively negative and positive to the corners of the cells. Sum up these costs of cells to find the index.

**Step 4 :** **Revision to Find Improved Matrix :** If all the indices are positive, the optimal solution is reached (BFS itself is OBFS). But, if any value is negative, select unallocated cell with most negative then transfer possible number of units to the selected cell along the loop and readjust the supply/demand of allocated cells.

**Step 5 :** Repeat step 3 and 4 untill all the indices are positive.

**Flow Chart for Stepping Stone Method**






**PROBLEMS**

**17. Obtain optimal solution for the following TP by stepping stone method**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | 1 | 2 | 1 | 4 | 30 |
| $O_2$ | 3 | 3 | 2 | 1 | 50 |
| $O_3$ | 4 | 2 | 5 | 9 | 20 |
| Demand | 20 | 40 | 30 | 10 | |

*Sol :*

**Step 1 : Obtain IBFS :** The IBFS by North west corner rule is,

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | [20] 1 | [10] 2 | 1 | 4 | 30 |
| $O_2$ | 3 | [30] 3 | [20] 2 | 1 | 50 |
| $O_3$ | 4 | 2 | [10] 5 | [10] 9 | 20 |
| Demand | 20 | 40 | 30 | 10 | 100 |

**Step 2 :** Degeneracy test by $n(C_{ij}) \geq r + c - 1$

No. of allocated cells = 6

$[(O_1, D_1), (O_1, D_2), (O_2, D_2), (O_2, D_3), (O_3, D_3), (O_3, D_4)]$

No. of rows = 3

No. of columns = 4

As no. of allocated cells i.e., $n(C_{ij})$ is (6) equal to $n(r) + n(c) - 1$

i.e., $3 + 4 - 1 = 6$, there is no degeneracy in the TP.

**Step 3 : Optimality Test by Improvement Index :** Loops are drawn for unallocated cells as given below

Loop for $(O_1, D_3)$ Cell.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|
| $O_1$ | [20] 1 | [10] 2 − | 1 + | 4 |
| $O_2$ | 3 | [30] 3 + | [20] 2 − | 1 |
| $O_3$ | 4 | 2 | [10] 5 | [10] 9 |

**Index :** $+1 - 2 + 3 - 2 = 0$

Loop for $(O_1, D_4)$

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|
| $O_1$ | [20] 1 | [10] 2 − | 1 | 4 + |
| $O_2$ | 3 | [30] 3 + | [20] 2 − | 1 |
| $O_3$ | 4 | 2 | [10] 5 + | [10] 9 − |

**Index :** $+4 - 0 + 5 - 2 + 3 - 2 = -1$

Loop for $(O_2, D_1)$ Cell.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|
| $O_1$ | [20] 1 − | [10] 2 + | 1 | 4 |
| $O_2$ | 3 + | [30] 3 − | [20] 2 | 1 |
| $O_3$ | 4 | 2 | [10] 5 | [10] 9 |

**Index :** $+3 - 1 + 2 - 3 = +1$

Loop for $(O_3, D_1)$

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|
| $O_1$ | − [20] 1 | + [10] 2 | 1 | 4 |
| $O_2$ | 3 | − [30] 3 | − [20] 2 | 1 |
| $O_3$ | + 4 | 2 | − [10] 5 | [10] 9 |

**Index :** $+4 - 5 + 2 - 3 + 2 - 1 = -1$

Loop for $(O_2, D_4)$ Cell.

Loop for $(O_3, D_2)$

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|
| $O_1$ | [20] 1 | [10] 2 | 1 | 4 |
| $O_2$ | 3 | [30] 3 | [20] −2 | +1 |
| $O_3$ | 4 | 2 | [10] +5 | [10] −9 |

**Index :** + 1 – 2 + 5 – 9 = – 1

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|
| $O_1$ | [20] 1 | [10] 2 | 1 | 4 |
| $O_2$ | 3 | [30] −3 | [20] +2 | 1 |
| $O_3$ | 4 | +2 | [10] −5 | [10] 9 |

**Index :** + 2 – 5 + 2 – 3 = – 4

Of all the above the most negative index is for ($O_2$, $D_4$) cell i.e., –5. Therefore we transfer max. possible number of units (i.e., 10) to this cell from ($O_3$, $D_4$).

**Step 4 : Revision :** When 10 units are transferred from ($O_3$, $D_4$) to ($O_2$, $D_4$), we have to reduce to units from ($O_2$, $D_3$) and add at ($O_3$, $D_3$). This addition or subtraction can be easily understood by sign on teh corner of the loop in the cell. Thus the revised matrix is

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | |
|---|---|---|---|---|---|
| $O_1$ | [20] 1 | [10] 2 | 1 | 4 | 30 |
| $O_2$ | 3 | [30] −3 | [10] +2 | [10] 1 | 50 |
| $O_3$ | 4 | +2 | [20] −5 | 9 | 20 |
| | 20 | 40 | 30 | 10 | |

**Step 5 : Repeating Step 2 to 4 :** No degeneracy is found and the indices for unallocated cells are as follows.

| Unallocated | Loop | Index |
|---|---|---|
| ($O_1$, $D_3$) | ($O_1$, $D_3$) → ($O_2$, $D_3$) → ($O_2$, $D_2$) → ($O_1$, $D_2$) | + 1 – 2 + 3 – 2 = 0 |
| ($O_1$, $D_4$) | ($O_1$, $D_4$) → ($O_2$, $D_4$) → ($O_2$, $D_2$) → ($O_1$, $D_2$) | + 4 – 1 + 3 – 2 = + 4 |
| ($O_2$, $D_1$) | ($O_2$, $D_1$) → ($O_1$, $D_1$) → ($O_1$, $D_2$) → ($O_2$, $D_2$) | + 3 – 1 + 2 – 3 = + 1 |
| ($O_3$, $D_1$) | ($O_3$, $D_1$) → ($O_1$, $D_1$) → ($O_1$, $D_2$) → ($O_2$, $D_2$) → ($O_2$, $D_3$) → ($O_3$, $D_3$) | + 4 – 1 + 2 – 3 + 2 – 5 = –1 |
| ($O_3$, $D_2$) | ($O_3$, $D_2$) → ($O_2$, $D_2$) → ($O_2$, $D_3$) → ($O_3$, $D_3$) | + 2 – 3 + 2 – 5 = (–4) ← |
| ($O_3$, $D_4$) | ($O_3$, $D_4$) → ($O_2$, $D_4$) → ($O_2$, $D_3$) → ($O_3$, $D_3$) | + 9 – 1 + 2 – 5 = + 5 |

As ($O_3$, $D_2$) shows most negative index we hae to transfer 20 units, (this max possible number of units to be transferred can be found by least allocated units among the negative cornered cells of the loop). The revised solution is

| | | | | |
|---|---|---|---|---|
| [20] 1 | [10] 2 | 1 | 4 | 30 |
| 3 | [10] 3 | [30] 2 | [10] 1 | 50 |
| 4 | [20] 2 | 5 | 9 | 20 |
| 20 | 40 | 30 | 10 | |

Again the indices in the revised matrix are :

for $(O_1, D_3)$ : 0 ; $(O_1, D_4)$ : +4 ; $(O_2, D_1)$ : + 1 remain same as above tableau.

for $(O_3, D_1)$ : Loop is $(O_3, D_1) \to (O_3, D_2) \to (O_1, D_2) \to (O_1, D_1)$

and index is + 4 – 2 + 2 – 1 = + 3

for $(O_3, D_3)$ : Loop is $(O_3, D_3) \to (O_2, D_3) \to (O_2, D_2) \to (O_3, D_2)$

and index is + 5 – 2 + 3 – 2 = + 4

for $(O_3, D_4)$ : Loop is $(O_3, D_4) \to (O_2, D_4) \to (O_2, D_2) \to (O_3, D_2)$

index is + 9 – 1 + 3 – 2 = + 9

Thus all the indices are positive, therefore optimal solution is obtained. Solution is

| Allocated | From | To | No. of Units | Unit Cost | Total Cost |
|---|---|---|---|---|---|
| $C_{11}$ or $(O_1, D_1)$ | $O_1$ | $D_1$ | 20 | 1 | 20 |
| $C_{12}$ or $(O_1, D_2)$ | $O_1$ | $D_2$ | 10 | 2 | 20 |
| $C_{22}$ or $(O_2, D_2)$ | $O_2$ | $D_2$ | 10 | 3 | 30 |
| $C_{23}$ or $(O_2, D_3)$ | $O_2$ | $D_3$ | 30 | 2 | 60 |
| $C_{24}$ or $(O_2, D_4)$ | $O_2$ | $D_4$ | 10 | 1 | 10 |
| $C_{32}$ or $(O_3, D_2)$ | $O_3$ | $D_2$ | 20 | 2 | 40 |
| | | | | Grand Total Cost | 180 |

**18. Maximise the TP**

| | $M_1$ | $M_2$ | $M_3$ | |
|---|---|---|---|---|
| $G_1$ | 4 | 4 | 9 | 25 |
| $G_2$ | 3 | 5 | 8 | 20 |
| | 18 | 16 | 11 | 45 |

*Sol :*

**Step 1 :** The IBFS by VAM or LCEM is

**Equivalent Cost Matrix**

| | | | |
|---|---|---|---|
| 5 [14] | 5 | 0 [11] | 25 |
| 6 [4] | 4 [16] | 1 | 20 |
| 18 | 16 | 11 | |

**Profit Matrix (Original)**

| | | | |
|---|---|---|---|
| 4 [14] | 4 | 9 [11] | 25 |
| 3 [4] | 5 [16] | 8 | 20 |
| 18 | 16 | 11 | |

**Step 2 : Degeneracy test :**

No. of rows (2) + No. of column (3) – 1 = 4

No. of allocated cells = 4; i.e., $n(C_{ij}) = n(r) + n(c) - 1$

$\therefore$ No degeneracy

**Step 3 : Optimality Test :**

For $(G_1, M_2)$

| | | | |
|---|---|---|---|
| [14] 5 | 5 | [11] 0 | 25 |
| [4] 6 | [16] 4 | 1 | 20 |
| 18 | 16 | 11 | |

**Index :** + 5 – 4 + 6 – 5 = + 2

For $(G_2 – M_3)$

| | | | |
|---|---|---|---|
| [14] 5 | 5 | [11] 0 | 25 |
| [4] 6 | [16] 4 | 1 | 20 |
| 18 | 16 | 11 | |

**Index :** + 1 – 0 + 5 – 6 = 0

Since indices for all unallocated cells is positive (or zero), the optimal solution is attained. And the solution is

Max profit = 4 × 14 + 9 × 11 + 3 × 4 + 5 × 16 = 247.

**Remark :** The above problem yields multiple optimal soluition. This can be identified by zero index value for non-allocated cells.

Thus solution is

| | $M_1$ | $M_2$ | $M_3$ | |
|---|---|---|---|---|
| $G_1$ | [18] 4 | 4 | [7] 0 | 25 |
| $G_2$ | 3 | [16] 5 | [4] 8 | 20 |
| | 18 | 16 | 11 | |

Profit = 4 × 18 + 9 × 7 + 5 × 16 + 8 × 4

= 72 + 63 + 80 + 32 = 247.



### 2.4.2 MODI Method

**Q12. Explain the Modi Method for obtaining optimal solution.**

*Ans :* **(Imp.)**

Once the initial basic feasible solution has been computed, the next step in the problem is to determine whether the solution obtained is optimum or not.

Optimality test can be conducted to any initial basic feasible solution of a TP provided such allocations has exactly m + n – 1 non-negative allocations. Where *m* is the number of origins and *n* is the number of destinations. Also these allocations must be in independent positions.

To perform this optimality test, we shall discuss the modified distribution method (MODI). The various steps involved in MODI method for performing optimality test are given below.

**MODI Method**

**Step 1**

Find the initial basic feasible solution of a TP by using any one of the three methods.

**Step 2**

Find out a set of numbers $u_i$ and $v_j$ for each row and column satisfying $u_i + v_j = c_{ij}$ for each occupied cell. To start with we assign a number '0' to any row of column having maximum number of allocations. If this maximum number of allocations is more than one, choose any one arbitrarily.

**Step 3**

For each empty (unoccupied) cell, we find the sum $u_i$ and $v_j$ written in the bottom left comer of that cell.

**Step 4**

Find out for each empty cell the net evaluation value $\Delta_{ij} = c_{ij} - (u_i + v_j)$ and which is written at the bottom right corner of that cell. This step gives the optimality conclusion,

(i) If all $\Delta_{ij} > 0$ (ie. all the net evaluation value) the solution is optimum and a unique solution exists.

(ii) If $\Delta_{ij} \geq 0$ then the solution is optimum, but an alternate solution exists.

(iii) If atleast one $\Delta_{ij} < 0$, the solution is not optimum. In this case we go to the next step, to improve the total transportation cost.

**Step 5**

Select the empty cell having the most negative value of $\Delta_{ij}$. From this cell we draw a closed path by drawing horizontal and vertical lines with the corner cells occupied. Assign sign + and – alternately and find the minimum allocation from the cell having negative sign. This allocation should be added to the allocation having positive sign and subtracted from the allocation having negative sign.

**Step 6**

The above step yield a better solution by making one (or more) occupied cell as empty and one empty cell as occupied. For this new set of basic feasible allocations repeat from the step(2) till an optimum basic feasible solution is obtained.



**<u>PROBLEMS</u>**

**19. Obtain optimal solution by using MODI method.**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | 1 | 2 | 1 | 4 | 30 |
| $O_2$ | 3 | 3 | 2 | 1 | 50 |
| $O_3$ | 4 | 2 | 5 | 9 | 20 |
| Demand | 20 | 40 | 30 | 10 | 100 |

*Sol :*

**Step 1 :** Obtain Initial Solution :

The IBFS for the above TP by North West Corner Method is as follows :

**Iteration Tableau 1 :**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | [20] 1 | [10] 2 | 1 | 4 | 30 |
| $O_2$ | 3 | [30] 3 | [20] 2 | 1 | 50 |
| $O_3$ | 4 | 2 | [10] 5 | [10] 9 | 20 |
| Demand | 20 | 40 | 30 | 10 | 100 |

**Step 2 :** Degeneracy Test :

No. of occupied cells $n(C_{ij}) = 6$

No. of rows i.e., $n(r) = 3$

No. of columns i.e., $n(c) = 4$

$n(C_{ij}) = n(r) + n(c) - 1$.

There is no degeneracy.

**Iteration Tableau 2 :**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply | Shadow price ($u_i$) |
|---|---|---|---|---|---|---|
| $O_1$ | [20] 1 | [10] 2 | 1 | 4 | 30 | $u_1 = 0$ (assumed) |
| $O_2$ | 3 | [30] 3 | [20] 2 | 1 | 50 | $u_2 = 1$ |
| $O_3$ | 4 | 2 | [10] 5 | [10] 9 | 20 | $u_3 = 4$ |
| Demand | 20 | 40 | 30 | 10 | | |
| Shadow Price ($v_j$) | $v_1 = 1$ | $v_2 = 2$ | $v_3 = 1$ | $v_4 = 5$ | | |

(Assumed) For Occupied Cells Assume $C_{ij} = u_i + v_j$

$u_1 + v_1 = 1,\ u_1 + v_2 = 2,\ u_2 + u_2 = 3,$

$u_2 + v_3 = 2,\ u_3 + v_3 = 5,\ u_3 + v_4 = 9$

We get above 6 equations with seven variables $u_1$, $u_2$, $u_3$ and $v_1$, $v_2$, $v_3$ and $v_4$. Thus one of the variables (say $u_1$) is assumed to be zero.

Then all other values can be found as mentioned in the above tableau.

**Step 4 :** Optimality test : To check $C'_{ij} - (u_i + v_j) \geq 0$



**Iteration Tableau 3 :**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply | |
|---|---|---|---|---|---|---|
| $O_1$ | [20] 1 | [10] 2 | 0 / 1 | – 1 / 4 | 30 | $u_1 = 0$ |
| $O_2$ | +1 / 3 | [30] 3 | – [20] 2 | – 5 / + 1 | 50 | $u_2 = 1$ |
| $O_3$ | – 1 / 4 | – 4 / 2 | + [10] 5 | – [10] 9 | 20 | $u_3 = 4$ |
| Demand | 20 | 40 | 30 | 10 | | |
| $v_j$ | $v_1 = 1$ | $v_2 = 2$ | $v_3 = 1$ | $v_4 = 5$ | | |

**Sample Calculations :**

for $(O_1, D_3)$ cell $C'_{ij} = 1$, $u_i = 0$, $v_j = 1$

$\therefore\ C'_{ij} - (u_i + v_j) = 1 - (0 + 1) = 0$

for $(O_2, D_4)$ cell $C_{24} = 1$, $u_2 = 1$, $v_4 = 5$

$\therefore\ C_{24} - (u_2 + v_4) = 1 - (1 + 5) = -5$

for $(O_3, D_1)$ $4 - (4 + 1) = -1$

and similarly for other unoccupied cells

**Step 5 :** Revising the Solution : From the above calculations, we find most negative value of $C'_{ij}$ – ($u_i$ – $v_j$) for ($O_2$, $D_4$) cell as –5. We start constructing closed loop from this cell with other occupied cells i.e., ($O_2$, $D_4$) $\rightarrow$ ($O_2$, $D_3$) $\rightarrow$ ($O_3$, $D_3$) $\rightarrow$ ($O_3$, $D_4$) and on the corners of this loop we give +, – alternatively from ($O_2$, $D_4$) onwards. Thus ($O_2$, $D_4$) and ($O_3$, $D_3$) get addition (+) while ($O_2$, $D_3$) and ($O_3$, $D_4$) get subtraction (–). Among thse negative marked cells, ($O_3$, $D_4$) has least allocation as 10 and thus direction is form this cell. (10 units are transferred upwards). Revising the allocations in this way, we get.

**Iteration Tableau 4 :**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $O_1$ | [20] 1 | [10] 2 | 1 | 4 | 30 |
| $O_2$ | 3 | [30] 3 | [10] 2 | [10] 1 | 50 |
| $O_3$ | 4 | 2 | [20] 5 | 9 | 20 |
| Demand | 20 | 40 | 30 | 10 | 100 |

**Step 6 :** Repeat steps 2 to 5 :

**Iteration Tableau 5 :**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply | $u_i$ |
|---|---|---|---|---|---|---|
| $O_1$ | [20] 1 | [10] 2 | 0 / 1 | + 4 / 4 | 30 | 0 |
| $O_2$ | +1 / 3 | [30] – 3 | [20] + 2 | [10] 1 | 50 | 1 |
| $O_3$ | – 1 / 4 | – 4 / + 2 | [20] – 5 | + 5 / 9 | 20 | 4 |
| Demand | 20 | 40 | 30 | 10 | | |
| $v_j$ | 1 | 2 | 1 | 0 | | |



**Iteration Tableau 6 :**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply | $u_i$ |
|---|---|---|---|---|---|---|
| $O_1$ | [20] 1 | [20] 2 | 0 / 1 | + 4 / 4 | 30 | 0 |
| $O_2$ | +1 / 3 | [10] 3 | [30] 2 | [10] 1 | 50 | 1 |
| $O_3$ | + 3 / 4 | [20] 2 | + 4 / 5 | + 0 / 9 | 20 | 0 |
| Demand | 20 | 40 | 30 | 10 | | |
| $v_j$ | 1 | 2 | 1 | 0 | | |

In the above solution, the values of $C'_{ij}$ – ($u_i$ – $v_j$) $\geq$ 0 for all unoccupied cells. Therefore the optimality is reached.

**Step 7 :** Cost calculations :

The solution is tabulated as follows.

| Allocated | From | To | Unit Cost | No. of Units | Total Cost |
|---|---|---|---|---|---|
| $C_{11}$ or $(O_1, D_1)$ | $O_1$ | $D_1$ | 1 | 20 | 20 |
| $C_{12}$ or $(O_1, D_2)$ | $O_1$ | $D_2$ | 2 | 10 | 20 |
| $C_{22}$ or $(O_2, D_2)$ | $O_2$ | $D_2$ | 3 | 10 | 30 |
| $C_{23}$ or $(O_2, D_3)$ | $O_2$ | $D_3$ | 2 | 30 | 60 |
| $C_{24}$ or $(O_2, D_4)$ | $O_2$ | $D_4$ | 1 | 10 | 10 |
| $C_{32}$ or $(O_3, D_2)$ | $O_3$ | $D_2$ | 2 | 20 | 40 |
| | | | | Grand Total Cost | 180 |

**20. Obtain an optimum solution to the following transportation problem by MODI method.**

| | | | | |
|---|---|---|---|---|
| 11 | 22 | 6 | 5 | 75 |
| 16 | 31 | 14 | 15 | 60 |
| 5 | 21 | 4 | 9 | 40 |
| 30 | 65 | 55 | 25 | |

*Sol :*

(i) Find initial basic feasible solution using V-AM. The solution is

| | | | | |
|---|---|---|---|---|
| 11 | 22 [50] | 6 | 5 [25] | 75 |
| 16 | 31 [15] | 14 [45] | 15 | 60 |
| 5 [30] | 21 | 4 [10] | 9 | 40 |
| 30 | 65 | 55 | 25 | |

Since the number of basic cells are 6 i.e., 3 + 4 – 1, the initial solution is non-degenerate. Thus an optimum solution can be obtained.

(ii) To calculate the values of $u_i$ and $v_j$ for each basic cels, assume $u_1 = 0$ and use relation $u_i + v_j = c_{ij}$

$c_{12} = u_1 + v_2 \Rightarrow 22 = 0 + v_2 \Rightarrow v_2 = 22$

$c_{14} = u_1 + v_4 \Rightarrow 5 = 0 + v_4 \Rightarrow v_4 = 5$

$c_{22} = u_2 + v_2 \Rightarrow 31 = u_2 + 22 \Rightarrow u_2 = 9$

$c_{23} = u_2 + v_3 \Rightarrow 14 = 9 + v_3 \Rightarrow v_3 = 5$

$c_{33} = u_2 + v_3 \Rightarrow 4 = u_3 + 5 \Rightarrow u_3 = -1$

$c_{31} = u_3 + v_1 \Rightarrow 5 = -1 + v_1 \Rightarrow v_1 = 6$

| | | | | | $u_i$ |
|---|---|---|---|---|---|
| | (– 5) 11 | [50] 22 | (– 1) 6 | [25] 5 | $u_1 = 0$ |
| | (– 1) 16 | [15] 31 | [45] 14 | (– 1) 15 | $u_2 = 9$ |
| | [30] 5 | (0) 21 | [10] 4 | (– 5) 9 | $u_3 = -1$ |
| $v_j$ | $v_1 = 6$ | $v_1 = 22$ | $v_3 = 5$ | $v_4 = 5$ | |

(iii) The opportunity cost for each of the non-basic cell is determined by using the relation $d_{ij} = u_i + v_j - c_{ij}$

$d_{11} = u_1 + v_1 - c_{11} = 0 + 6 - 11 = -5$

$d_{13} = u_1 + v_3 - c_{13} = 0 + 5 - 6 = -1$

$d_{21} = u_2 + v_1 - c_{21} = 9 + 6 - 15 = -1$

$d_{24} = u_2 + v_4 - c_{24} = 9 + 5 - 15 = -1$

$d_{32} = u_3 + v_2 - c_{32} = -1 + 22 - 21 = 0$

$d_{34} = u_3 + v_4 - c_{34} = -1 + 5 - 9 = -5$

Since all $d_{ij} \leq 0$, the optimum soliution has been obtained. The solution is $x_{12} = 50$; $x_{14} = 25$; $x_{22} = 15$; $x_{23} = 45$; $x_{31} = 30$; $x_{33} = 10$;

Minimum transportation cost

$= 50 \times 22 + 25 \times 5 + 15 \times 5 + 15 \times 31$

$+ 45 \times 14 + 30 \times 5 + 10 \times 4$

= Rs. 2510.

**21. A cement company has three factories which manufacture cement which is then transported to four distribution centres. The quantity of monthly production of each factory, the demand of each distribution centre and the associated transportation cost per quintal are given below :**

| Factory | Distribution Centres | | | | Monthly Production |
|---|---|---|---|---|---|
| | W | X | Y | Z | (quintals) |
| A | 10 | 8 | 5 | 4 | 7,000 |
| B | 7 | 9 | 15 | 8 | 8,000 |
| C | 6 | 10 | 14 | 8 | 10,000 |
| Monthly demand (in quintals) | 6,000 | 6,000 | 8,000 | 5,000 | 25,000 |

**(a) Suggest the optimum transportation schedule.**

**(b) Is there any other transportation schedule which is equally attractive ? If so, write that.**

**(c) If the company wants that atleast 5,000 quintals of cement are transported from factory C to distribution centre Y, will the transportation schedule be any different ? If so, what will be the new optimum schedule and the effect on cost?**

*Sol :*

(a) The initial feasible schedule using VAM method is as follows,

| | Distribution Centres | | | | Monthly Production (quintals) | Penalties | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Factory | W | X | Y | Z | | $RP_1$ | $RP_2$ | $RP_3$ | $RP_4$ | $RP_5$ | |
| A | 10 | 8 | [7000] 5 | 4 | ~~7000~~ | 1 | - | - | - | - | - |
| B | 7 | [6000] 9 | [1000] 15 | [1000] 8 | ~~1000~~ ~~7000~~ 8000 | 1 | 1 | 1 | 1 | 1 | 8 |
| C | [6000] 6 | 10 | 14 | [4000] 8 | 4000 ~~10,000~~ | 2 | 2 ← | 2 ← | - | - | - |
| Monthly demand (in quintals) | ~~6000~~ | ~~6000~~ | 1000 ~~8000~~ | 1000 ~~5000~~ | | | | | | | |
| $CP_1$ | 1 | 1 | [9] ↑ | 4 | | | | | | | |
| $CP_2$ | 1 | 1 | 1 | 0 | | | | | | | |
| $CP_3$ | - | 1 | 1 | 0 | | | | | | | |
| $CP_4$ | - | 9 | 15 ↑ | 8 | | | | | | | |
| $CP_5$ | - | 9 ↑ | - | 8 | | | | | | | |
| $CP_6$ | - | - | - | 8 | | | | | | | |

Number of allocation = 6

M + n –1 = 4 + 3 – 1

= 7 –1 = 6

∴ Total transportation cost

= (5 × 7000) + (9 × 6000) + (15 × 1000) + (8 × 1000) + (8 × 1000) + (6 × 6000) + (8 × 4000)

= 35000 + 54000 + 15000 + 8000 + 36000 + 32000

= 1,80,000

**Optimality Check through Modi Method Using Occupied Cells**

$C_{ij} = u_i + v_j$ ; Assume $u_1 = 0$

$C_{13} = u_1 + v_3 = 5$ ; $0 + v_3 = 5$; $v_3 = 5$

$C_{23} = u_2 + v_3 = 15$; $u_2 + 5 = 15$; $u_2 = 10$

$C_{22} = u_2 + v_2 = 9$; $10 + u_2 = 9$; $v_2 = -1$

$C_{24} = u_2 + v_4 = 8$; $10 + v_4 = 8$; $v_4 = -2$

$C_{34} = u_3 + v_4 = 8$; $u_3 - 2 = 8$; $u_3 = 10$

$C_{31} = u_3 + v_1 = 6$; $10 + v_1 = 6$; $v_1 = -4$

Determining opportunity cost for unoccupied cells using the equations as,

$\Delta_{ij} = C_{ij} - (u_j + v_j)$

$\Delta_{11} = 10 - (0 - 4) = 10 + 4 = 14$

$\Delta_{12} = 8 - (0 - 1) = 8 - 1 = 9$

$\Delta_{14} = 4 - (0 - 2) = 4 + 2 = 6$

$\Delta_{21} = 7 - (10 - 4) = 7 - 6 = 1$

$\Delta_{32} = 10 - (10 - 1) = 10 - 9 = 1$

$\Delta_{33} = 14 - (10 + 5) = 14 - 15 = -1$

Since, we have a negative value in cell (3,3), the solution is not optimum. Construct a loop from cell (3,3) with '+'sign.

| | | | | |
|---|---|---|---|---|
| [14] 10 | [9] 8 | (7000) 5 | [6] 4 | $u_1 = 0$ |
| [1] 7 | (6000) 9 | (1000) 15 $-\theta$ | (1000) 8 $+\theta$ | $u_2 = 10$ |
| (6000) 6 | [1] 10 | [1] 14 $+\theta$ | (4000) 8 $-\theta$ | $u_3 = 10$ |
| $V_1 = -4$ | $V_2 = -1$ | $V_3 = 5$ | $V_4 = -2$ | |

$\theta$ = Min (allocations with $-\theta$ assignments)

$\theta$ = Min (4,000, 1000)

$\theta$ = 1000

The transportation table with modified allocations are as follows,

The minimum negative allocation is 1000. Add this to cells with '+' sign and subtract from cells with '–' sign as follows,

(2,3) : 1000 – 1000 = 0

(2,4) : 1000 + 1000 = 2000

(3,3) : 0 + 1000 = 1000

(3,4) : 4000 – 1000 = 3000

| | | | | |
|---|---|---|---|---|
| 10 | 8 | (7000) 5 | 4 | $u_1 = 0$ |
| 7 | (6000) 9 | 15 | (2000) 8 | $u_2 = 9$ |
| (6000) 6 | 10 | (1000) 14 | (3000) 8 | $u_3 = 9$ |
| $v_1 = -3$ | $v_2 = 0$ | $v_3 = 5$ | $v_4 = -1$ | |

**Occupied Cells**

Assume $u_1 = 0$

$u_1 + v_3 = 5;\ 0 + v_3 = 5;\ v_3 = 5$

$u_3 + v_3 = 14;\ u_3 + 5 = 14;\ u_3 = 9$

$u_3 + v_4 = 8; 9 + v_4 = 8; v_4 = -1$

$u_2 + v_2 = 9; 9 + v_2 = 9; v_2 = 0$

$u_3 + v_1 = 6; 9 + v_1 = 6; v_1 = -3$

**Unoccupied Cells**

$\Delta_{ij} = C_{ij} - (u_i + v_j)$

$\Delta_{11} = 10 - (0 - 3) = 10 + 3 = 13$

$\Delta_{12} = 8 - (0 + 0) = 8$

$\Delta_{14} = 4 - (0 - 1) = 4 + 1 = 5$

$\Delta_{21} = 7 - (9 - 3) = 7 - 6 = 1$

$\Delta_{23} = 15 - (9 + 5) = 15 - 14 = 1$

$\Delta_{32} = 10 - (9 - 0) = 10 - 9 = 1$

Since there are no negative values in unoccupied cells, the solution is optimal,

Total transportation cost,

$= (5 \times 7000) + (9 \times 6000) + (8 \times 2000) + (6 \times 6000) + (14 \times 1000) + (8 \times 3000)$

= ` 1,79,000/-

(b) The above solution is unique there is no other transportation schedule which is equality attractive.

(c) 5000 quintals of cement are to be transported from c to y. The balance supply from factory c will be 5, 000 quintals and balance demand at y will be 3000 quintals as shown below.

| Factory | Distribution Centres | | | | Monthly Production (Quintals) |
|---|---|---|---|---|---|
| | W | X | Y | Z | |
| A | 10 | 8 | [3000] 5 | [4000] 4 | ~~4000~~ ~~7000~~ |
| B | [1000] 7 | [6000] 9 | 15 | [1000] 8 | ~~1000~~ ~~2000~~ ~~8000~~ |
| C | [5000] 6 | 10 | 14 | 8 | ~~5000~~ |
| | ~~6000~~ ~~1000~~ | ~~6000~~ | ~~3000~~ | ~~5000~~ 1000 | 20000 |

**Penalties (RP)**

| | $RP_1$ | $RP_2$ | $RP_3$ | $RP_4$ | $RP_5$ | $RP_6$ |
|---|---|---|---|---|---|---|
| A | 1 | 4 ← | – | – | – | – |
| B | 1 | 1 | 1 | 1 | 1 | 7 ← |
| C | 2 | 2 | 2 ← | – | – | – |

| Penalty (CP) | W | X | Y | Z |
|---|---|---|---|---|
| $CP_1$ | 1 | 1 | 9↑ | 4 |
| $CP_2$ | 1 | 1 | – | 4 |
| $CP_3$ | 1 | 1 | – | 0 |
| $CP_4$ | 7 | 1 | – | 8 |
| $CP_5$ | 7 | 9↑ | – | 8↑ |
| $CP_6$ | 7↑ | – | – | – |

**Check for Optimality Using MODI Method**

$= m + n - 1$

$= 3 + 4 - 1$

$= 7 - 1$

$= 6$

**Determining Opportunity Cost for Occupied Cells Using $C_{ij} = u_i + v_j$ , Assume $u_1 = 0$**

$C_{13} = u_1 + v_3 = 0 + 5 \Rightarrow v_3 = 5$

$C_{14} = u_1 + v_4 = 0 + 4 \Rightarrow v_4 = 4$

$C_{24} = u_2 + v_4 = u_2 + 4 \Rightarrow u_2 = 4$

$C_{21} = u_2 + v_1 = 4 + v_1 = 7 \Rightarrow v_1 = 3(7 - 4)$

$C_{22} = u_2 + v_2 = 4 + v_2 = 9 \Rightarrow v_2 = 5(9 - 4)$

$C_{31} = u_3 + v_1 = u_3 + 3 \Rightarrow u_3 = 3$

**For Unoccupied Cells Using the Equations as $\Delta_{ij} = C_{ij} - (u_i + v_j)$**

$\Delta_{11} = C_{11} - (u_1 + v_1) \Rightarrow 10 - (0 + 3) = 7$

$\Delta_{12} = C_{12} - (u_1 + v_2) \Rightarrow 8 - (0 + 5) = 3$

$\Delta_{23} = C_{23} - (u_2 - v_3) \Rightarrow 15 - (4 + 5) = 6$

$\Delta_{32} = C_{32} - (u_3 + v_2) \Rightarrow 10 - (3 + 5) = 2$

$\Delta_{33} = C_{33} - (u_3 + v_3) \Rightarrow 14 - (3 + 5) = 6$

$\Delta_{34} = C_{34} - (u_3 - v_4) \Rightarrow 8 - (3 + 4) = 1$

| | | | | |
|---|---|---|---|---|
| 10 | 8 | [3000] 5 | [4000] 4 | $u_1 = 0$ |
| [1000] 7 | [6000] 9 | 15 | [1000] 8 | $u_2 = 4$ |
| [5000] 6 | 10 | [5000] 14 | 8 | $u_3 = 3$ |
| $v_1 = 3$ | $v_2 = 5$ | $v_3 = 5$ | $v_4 = 4$ | |

Since there are no negative values in unoccupied cells, the solution is optimal,

Total transportation cost,

= (5 × 3000) + (4 × 4000) + (7 ×1000) + (9 ×6000) + (8 × 1000) + (6 × 5000) + (14 × 5000) (Route C – Y)

= 15,000 + 16,000 + 7,000 + 54,000 +8,000 + 30,000 + 70,000

= 2,00,000.

## 2.5 Degeneracy in TP

**Q13. What is degeneracy in transportation problem ? How it is resolved ?**

*Ans :*

**Degeneracy in TP**

Degeneracy in TP is said to occur when the number of allocated cells is less than m + n – 1.

Where,

m = Number of rows

n = Number of columns

If degeneracy is occurred in a TP, then it is not possible to draw a closed loop for every occupied cell while solving the problem by MODI method. This degeneracy may occur during:

(a) Initial stage

(b) Testing of optimal solution.

**Resolving Degeneracy TP**

In order to resolve degeneracy, an artificial quantity $\varepsilon$ with a 'o' cost is added to one or more of the unoccupied cells such that total allocated cells will be equal to $m + n - 1$. While placing $\varepsilon$ in unoccupied cells, it should be considered that it has been placed in the cell containing the lowest transportation cost. After placing $\varepsilon$ in the unoccupied cell, the problem can be solved as usual by MODI or stepping stone method. The $\varepsilon$ will remain in the problem until degeneracy is removed or a final solution is obtained.

## PROBLEMS ON DEGENERACY

**22. A company has three plants at locations A, B and C which supply to warehouses located at D, E, F, G and H. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in `) are given below,**

| | | To | | | | |
|---|---|---|---|---|---|---|
| | | D | E | F | G | H |
| From | A | 5 | 8 | 6 | 6 | 3 |
| | B | 4 | 7 | 7 | 6 | 5 |
| | C | 8 | 4 | 6 | 6 | 4 |

**Determine an optimum distribution for the company in order to minimize the total transportation cost.**

**OR**

**Solve the following transportation problem for optimal solution,**

| | P | Q | R | S | T | Supply |
|---|---|---|---|---|---|---|
| A | 5 | 8 | 6 | 6 | 3 | 8 |
| B | 4 | 7 | 7 | 6 | 5 | 5 |
| C | 8 | 4 | 6 | 6 | 4 | 9 |
| Demand | 4 | 4 | 5 | 4 | 8 | 22 / 25 |

*Sol :*

| | | Stores D | E | F | G | H | Production capacity |
|---|---|---|---|---|---|---|---|
| Factories | A | 5 | 8 | 6 | 6 | 3 | 800 |
| | B | 4 | 7 | 7 | 6 | 5 | 500 |
| | C | 8 | 4 | 6 | 6 | 4 | 900 |
| Requirement | | 400 | 400 | 500 | 400 | 800 | 2200 / 2500 |

As total production capacity (2200 units) $\neq$ total requirement (2500 units).

Hence, the given transportation problem is unbalanced. To convert into a balanced transportation problem, add a dummy factory located at *D* with production capacity 3 units and unit cost of supplying from various factories is equal to zero.

This balanced transportation problem is then solved using VAM to obtain IBFS.

### Vogel's Approximation Method

| | D | E | F | G | H | Supply | Penalty | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 8 | [5] 6 | 6 | [3] 3 | ~~8~~ ~~3~~ | 2 | 2 | 2 | 3 | ←3 | – | |
| B | [4] 4 | 7 | 7 | [1] 6 | 5 | ~~5~~ ~~1~~ | 1 | 1 | 1 | 1 | 1 | 1 | ←1 |
| C | 8 | [4] 4 | 6 | 6 | [5] 4 | ~~9~~ ~~5~~ | 0 | 0 | 0 | 0 | 2 | ←2 | |
| D (Dummy) | 0 | 0 | 0 | [3] 0 | 0 | ~~3~~ | 0 | – | – | – | – | – | |
| Demand | ~~4~~ | ~~4~~ | ~~5~~ | ~~4~~ ~~1~~ | ~~8~~ ~~5~~ | | | | | | | | |
| Penalty | 4 | 4 | 6 | 6↑ | 3 | | | | | | | | |
| | 4 | 4 | 6↑ | 0 | 1 | | | | | | | | |
| | 4↑ | 4 | – | 0 | 1 | | | | | | | | |
| | – | 4↑ | – | 0 | 1 | | | | | | | | |
| | – | – | – | 0 | 1 | | | | | | | | |
| | – | – | – | 0 | 1 | | | | | | | | |

Total transportation cost $= \Sigma C_{ij} . x_{ij}$

$= (6 \times 5) + (3 \times 3) + (4 \times 4) + (6 \times 1) + (4 \times 4) + (4 \times 5) + (0 \times 3)$

= ` 97



### Optimal Solution using MODI Method

Number of occupied cells = 7

$m + n - 1 = 4 + 5 - 1 = 9 - 1 = 8$

Since the number of occupied cells $< m + n - 1$ there is degeneracy. To resolve degeneracy, '$\in$' is added to unoccupied cells with least cost [i.e., cell (AS 1)].

### Note

$\in$ is very small value but not equal to zero and

$a + \in = a - \in = a$

Compute $u_i$, $v_j$ $(u_i + v_j)$ and $\Delta_{ij}$ values and check for optimality.

| | D | E | F | G | H | Supply |
|---|---|---|---|---|---|---|
| A | (ε) 5 | 8 | (5) 6 | 6 | (5) 3 | 8 |
| B | (4) 4 | 7 | 7 | (1) 6 | 5 | 5 |
| C | 3 | (4) 4 | 6 | 6 | (5) 4 | 9 |
| D | 0 | 0 | 0 | (3) 0 | 0 | 4 |
| Demand | 4 | 4 | 5 | 4 | 8 | 25 / 25 |

Determining $u_i$ and $v_j$ values for occupied cells using,

$C_{ij} = u_i + v_j$

$u_1 + v_1 = 5$

$u_1 + v_3 = 6$

$u_1 + v_5 = 3$

$u_2 + v_1 = 4$

$u_2 + v_4 = 6$

$u_3 + v_2 = 4$

$u_3 + v_5 = 4$

$u_4 + v_4 = 0$

There are 9 shadow values, but 8 equations available. Assume $u_i = 0$ and solve remaining values.

| | |
|---|---|
| $u_1 = 0$ | $v_1 = 5$ |
| $u_2 = -1$ | $v_2 = 3$ |
| $u_3 = 1$ | $v_3 = 6$ |
| $u_4 = -7$ | $v_4 = 7, v_5 = 3$ |

Consider unoccupied cells and find their evaluations.

$\Delta_{ij} = C_{ij} - (u_i + v_j)$

or

$\Delta_{ij} = C_{ij} - u_j - v_j$

$C_{12} - u_1 - v_2 = 8 - 0 - 3 = +5$

$C_{14} - u_1 - v_4 = 6 - 0 - 7 = -1$

$C_{22} - u_2 - v_2 = 7 + 1 - 3 = +5$

$C_{23} - u_2 - v_3 = 7 + 1 - 6 = +2$

$C_{25} - u_2 - v_5 = 5 + 1 - 3 = +3$

$C_{31} - u_3 - v_1 = 8 - 1 - 5 = +2$

$C_{33} - u_3 - v_3 = 6 - 1 - 6 = -1$

$C_{34} - u_3 - v_4 = 6 - 1 - 7 = -2$

$C_{41} - u_4 - v_1 = 0 + 7 - 5 = +2$

$C_{42} - u_4 - v_2 = 0 + 7 - 3 = +4$

$C_{43} - u_4 - v_3 = 0 + 7 - 6 = +1$

$C_{45} - u_4 - v_5 = 0 + 7 - 3 = +4$

| | D | E | F | G | H |
|---|---|---|---|---|---|
| A | [ε] −θ 5 | [5] 8 | 6 | 6 | [3] +θ 3 |
| B | [4] 4 +θ | 7 | 7 | [1] −θ 6 | 5 |
| C | 8 | [4] 4 | 6 | +θ 6 | [5] −θ 4 |
| D | 0 | 0 | 0 | [3] 0 | 0 |

Consider the most negative all $C_{34}(\Delta_{ij} = -2)$ and use loop method to find optimum solution.

$[(\theta - \in, 500 - \theta, 100 - \theta)]$ select minimum positive value $\theta = \in$

Modified allocation table,

| | | | | |
|---|---|---|---|---|
| 5 | 8 | [5] −θ 6 | 6 | [3] +θ 3 |
| [4] 4 | 7 | 7 | [1] 6 | 5 |
| 8 | [4] 4 | +θ 6 | [∈] 6 | [5] −θ 4 |
| 0 | 0 | 0 | [3] 0 | 0 |

**Optimality Test for Occupied Cells**

$C_{ij} = u_i + v_j$ for allocated cells

Assume $u_3 = 0$

$u_i + v_3 = 6$ $\quad \therefore u_1 = -1$ $\quad v_1 = 4$

$u_1 + v_3 = 3$ $\quad u_2 = 0$ $\quad v_2 = 4$

$u_2 + v_1 = 4$ $\quad u_3 = 0$ $\quad v_3 = 7$

$u_2 + v_4 = 6$ $\quad u_4 = -6$ $\quad v_4 = 6$

$u_3 + v_2 = 4$ $\quad v_5 = 4$

$u_3 + v_4 = 6$

$u_3 + v_5 = 4$

$u_4 + v_4 = 0$

**For Unoccupied Cells**

$\Delta_{11} = 5 + 1 - 4 = 2$

$\Delta_{12} = 8 + 1 - 4 = 5$

$\Delta_{14} = 6 + 1 - 6 = 1$

$\Delta_{22} = 7 + 0 - 4 = 3$

$\Delta_{23} = 7 + 0 - 7 = 0$

$\Delta_{25} = 5 + 0 - 4 = 1$

$\Delta_{31} = 8 + 0 - 4 = 4$

$\Delta_{33} = 6 + 0 - 7 = -1$

$\Delta_{41} = 0 + 6 - 4 = 2$

$\Delta_{45} = 0 + 6 - 4 = 2$

Consider – ve cells & from loop from cell (3,3)

$[500 - \theta, 500 - \theta]$ minimum

Modified allocation table,

| | | | | |
|---|---|---|---|---|
| 5 | 8 | [$\in$] 6 | 6 | [8] 3 |
| [4] 4 | 7 | 7 | [1] 6 | 5 |
| 8 | [4] 4 | [5] 6 | [$\in$] 6 | 4 |
| 0 | 0 | 0 | [3] 0 | 0 |

As there occurs degeneracy at this stage, include one more '$\in$' at (3,1) cell to resolve it.

**Check optimality for Occupied Cells**

$u_1 + v_3 = 6 \therefore \quad u_1 = 0 \quad v_1 = 4$

$u_1 + v_5 = 3 \quad u_2 = 0 \quad v_2 = 4$

$u_2 + v_1 = 4 \quad u_3 = 0 \quad v_3 = 6$

$u_2 + v_4 = 6 \quad u_4 = -6 \;\; v_4 = 6$

$u_3 + v_2 = 4 \quad v_5 = 3$

$u_3 + v_3 = 6$

$u_3 + v_4 = 6$

$u_4 + v_4 = 0$

**For Unoccupied Cells**

$\Delta_{11} = 5 - 0 - 4 = 1$

$\Delta_{12} = 8 - 0 - 4 = 4$

$\Delta_{14} = 6 - 0 - 6 = 0$

$\Delta_{22} = 7 - 0 - 4 = 3$

$\Delta_{23} = 7 - 0 - 6 = 1$

$\Delta_{25} = 5 - 0 - 3 = 2$

$\Delta_{31} = 8 - 0 - 4 = 4$

$\Delta_{33} = 4 - 0 - 3 = 1$

$\Delta_{41} = 0 \;\; 6 - 4 = 2$

$\Delta_{45} = 0 + 6 - 3 = 3$

Since , all $\Delta_{ij} \geq 0$ the solution obtained here is an optimal one.

$= [(3 \times 8) + (4 \times 4) + (6 \times 1) + (4 \times 4)$

$+ (6 \times 3)] \times 10$

$= [24 + 16 + 6 + 16 + 12 + 18] \times 10$

$\therefore$ Minimum total transportation cost = 9,20

| Cell | Allocation | Cost |
|---|---|---|
| $A \rightarrow F$ | $6 \times \in$ | $\in$ |
| $A \rightarrow H$ | $3 \times 8$ | 24 |
| $B \rightarrow D$ | $4 \times 4$ | 16 |
| $B \rightarrow G$ | $6 \times 1$ | 6 |
| $C \rightarrow E$ | $4 \times 4$ | 16 |
| $C \rightarrow F$ | $6 \times 5$ | 30 |
| $C \rightarrow G$ | $6 \times \in$ | $\in$ |
| $D \rightarrow G$ | $0 \times 3$ | 0 |

## 2.6 TRANSSHIPMENT MODEL

**Q14. Define Transshipment. State the characteristics of Transshipment Model.**

*Ans :* **(Imp.)**

**Definition :**

A transportation problem in which available commodity frequently moves from one source to another source or destination before reaching its actual destination is called a transhipment problem.

**Main Characteristics of Transhipment Problems**

Following are the main characteristics of transhipment problems :

1. The number of sources and destinations in the transportation problem are ma dn n respectively. But in transhipment problems, we have m + n sources and destinations.
2. If $S_i$ denotes the ith source and $D_j$ denotes the jth destination, then commodity can move along the route $S_i \rightarrow D_i \rightarrow D_j$, $S_i \rightarrow S_j \rightarrow D_i \rightarrow D_j$, $S_i \rightarrow D_i \rightarrow S_j \rightarrow D_j$, or in various other ways. Clearly, transportation cost from $S_i$ to $S_i$ is zero and the transportation costs from $S_i$ to $S_j$ or $S_i$ to $D_i$ do not have to be symmetrical, i.e., in general, $S_i \rightarrow S_j \neq S_j \rightarrow S_i$.
3. While solving the transhipment problem, we first obtan the optimum solution to the transportation problem, and then proceed in the same manner as in solving the transportation problems.
4. The basic feasible solution contains 2m + 2n – 1 basic variables. If we omit the variables appearing in the (m + n) diagonal cells, we are left with m + n – 1 basic variables.

**PROBLEMS**

**23. Consider the following transhipment problem with two sources and two destinations, the costs for shipment in rupees are given below. Determine the shipping schedule**

| | $S_1$ | $S_2$ | $D_1$ | $D_2$ | |
|---|---|---|---|---|---|
| $S_1$ | 0 | 1 | 3 | 4 | 5 |
| $S_2$ | 1 | 0 | 2 | 4 | 25 |
| $D_1$ | 3 | 2 | 0 | 1 | |
| $D_2$ | 4 | 4 | 1 | 0 | |
| | | | 20 | 10 | 30 |

*Sol :*

**Step 1 : (To get modified transportation problem)**

In the transhipment problem, each given source and destination can be considered a source or a destination. If we now take the quantity available at each of the sources $D_1$ and $D_2$ to be zero and also at each of the destinations $S_1$ and $S_2$ the requirement to be zero, then to have a supply and demand from all the points (sources or destinations) a fictitious supply and demand quantity termed as 'buffer stock' is assumed and is added to both supply and demand of all the points. Generally, this buffer stock is chosen equal to $\sum_i a_i$ or $\sum_j b_j$. In our problem, the buffer-stock comes-out to be 30 units.

| | $S_1$ | $S_2$ | $D_1$ | $D_2$ | Available |
|---|---|---|---|---|---|
| $S_1$ | 0 | 1 | 3 | 4 | 35 |
| $S_2$ | 1 | 0 | 2 | 4 | 55 |
| $D_1$ | 3 | 2 | 0 | 1 | 30 |
| $D_2$ | 4 | 4 | 1 | 0 | 30 |
| Required | 30 | 30 | 50 | 40 | |

**Step 2 : (To find initial solution of modified problem)**

By adding 30 units of commodity to each point of supply and demand, an initial basic feasible solution is obtained in Table 2 by using Vogel's Approximation method.



**Starting Table 2**

| | | | | | $a_i$ ↓ |
|---|---|---|---|---|---|
| | 30 (0) • | (1) | (3) | 5 (4) • | 35 |
| | (1) | 30 (0) • | 20 (2) • | 5 (4) • | 55 |
| | (3) | (2) | 30 (0) • | (1) | 30 |
| | (4) | (4) | (1) | 30 (0) • | 30 |
| $b_j$ → | 30 | 20 | 50 | 40 | |

**Step 3 : (To apply optimality test)**

The variable $u_i$ (i = 1, 2, 3, 4) and $v_j$ (j = 1, 2, 3, 4) have been determined by using successively the relations $u_i + v_j = c_{ij}$ for all the basic (occupied) cells. These values are then used to compute the net-evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for all the non-basic (empty) cells. Clearly $d_{34}$ (= –1) is the only negative quantity. Hence an unknown quantity $\theta$ is assigned to this cell (3, 4). After identifying the loop, we find that $\theta = 5$ and that the cell (2, 4) leaves the basis (i.e., becomes empty).

**Table 3**

| | | | | $u_i$ |
|---|---|---|---|---|
| 30 • (0) | 1 (1) 0 + 0 | 1 (3) 0 + 2 | 5 • (4) | 0 |
| 1 (1) 0 + 0 | 30 • (0) | 20 + θ • (2) | 5 − θ • (4) | 0 |
| 5 (3) −2 + 0 | 4 (2) − 2 + 0 | 30 − θ • (0) | − 1 + θ √ (1) − 2 + 4 | − 2 |
| 8 (4) − 4 + 0 | 8 (4) − 4 + 0 | 3 (1) − 4 + 2 | 30 • (0) | − 4 |
| $v_j$ 0 | 0 | 2 | 4 | |

**Step 4 :** Introduce the cell (3, 4) into the basis and drop the cell (2, 4) from the basis. Then, again test the optimality of the revised solution :

Since all the current net evaluations are non-negative, the current solution is an optimum one. It is shown in table 48. The minimum transportation cost is :

$$z^* = 5 \times 4 + 25 \times 2 + 5 \times 1 = 7$$

and the optimum transportation route is as shown below.

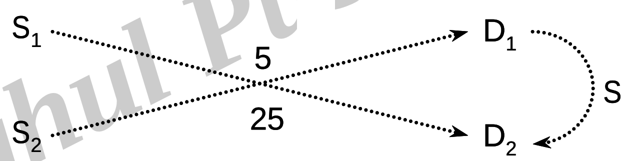





**Table 4**

| | | | | $u_i$ |
|---|---|---|---|---|
| 30 • (0) | 0 (1) − 1 | 0 (3) 3 | 5 • (4) | 0 |
| 2 (1) − 1 | 30 • (0) | 25 • (2) | 1 (4) 3 | − 1 |
| 6 (3) − 3 | 4 (2) − 2 | 25 • (0) | 5 • (1) | − 3 |
| 8 (4) − 4 | 7 (4) − 3 | 2 (1) − 1 | 30 • (0) | − 4 |
| $v_j$ 0 | 1 | 3 | 4 | |

**24. Consider the following transshipment problem involving 4 sources and 2 destinations. The supply values of the sources $S_1$, $S_2$, $S_3$ and $S_4$ are 100 units, 200 units, 150 units and 350 units, respectively. The demand values of destinations $D_1$ and $D_2$ are 350 units and 450 units, respectively. The transportation cost per unit between different sources and destinations are summarized as in Table below. Solve the transhipment problem.**

$C_{ij}$ Values for Example

| | | Destination | | | | | |
|---|---|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $D_1$ | $D_2$ |
| Source | $S_1$ | 0 | 4 | 20 | 5 | 25 | 12 |
| | $S_2$ | 10 | 0 | 6 | 10 | 5 | 20 |
| | $S_3$ | 15 | 20 | 0 | 8 | 45 | 7 |
| | $S_4$ | 20 | 25 | 10 | 0 | 30 | 6 |
| | $D_1$ | 20 | 18 | 60 | 15 | 0 | 10 |
| | $D_2$ | 10 | 25 | 30 | 23 | 4 | 0 |

*Sol :*

Here, the number of sources is 4, and the number of destinations is 2. Therefore, the total number of starting nodes as well as the total number of ending nodes of the transshipment problem is equal to 6 (i.e. 4 + 2 = 6). We also have

$$B = \sum_{i=1}^{4} a_i = \sum_{j=1}^{2} b_j = 800$$

A detailed format of the transshipment problem after including the sources and the destinations as transient nodes is shown in Table below. where the value of *B* is added to all the rows and all the columns.

Table : (b) Detailed Format of Transshipment Problem

| | | Destination | | | | | | Supply |
|---|---|---|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $D_1$ | $D_2$ | |
| Source | $S_1$ | 0 | 4 | 20 | 5 | 25 | 12 | 100 + 800 = 900 |
| | $S_2$ | 10 | 0 | 6 | 10 | 5 | 20 | 200 + 800 = 1000 |
| | $S_3$ | 15 | 20 | 0 | 8 | 45 | 7 | 150 + 800 = 950 |
| | $S_4$ | 20 | 25 | 10 | 0 | 30 | 6 | 350 + 800 = 1150 |
| | $D_1$ | 20 | 18 | 60 | 15 | 0 | 10 | 800 |
| | $D_2$ | 10 | 25 | 30 | 23 | 4 | 0 | 800 |
| Demand | | 800 | 800 | 800 | 800 | 350 + 800 = 1150 | 450 + 800 = 1250 | |

The solution to the problem in Table (b) below is shown in Table (c) and the corresponding total cost of transportation is Rs. 5,250. The allocations in the main diagonal cells are to be ignored. The shipping pattern is diagrammatically presented in below table which shows the shipments related to the off-diagonal cells alone.

**Table : (C)** Solution

| | | Destination | | | | | | Supply |
|---|---|---|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $D_1$ | $D_2$ | |
| Source | $S_1$ | 800 | 100 | – | – | – | – | 900 |
| | $S_2$ | – | 700 | – | – | 300 | – | 1000 |
| | $S_3$ | – | – | 800 | – | – | 150 | 950 |
| | $S_4$ | – | – | – | 800 | – | 350 | 1150 |
| | $D_1$ | – | – | – | – | 800 | – | 800 |
| | $D_2$ | – | – | – | – | 50 | 750 | 800 |
| Demand | | 800 | 800 | 800 | 800 | 1150 | 1250 | |

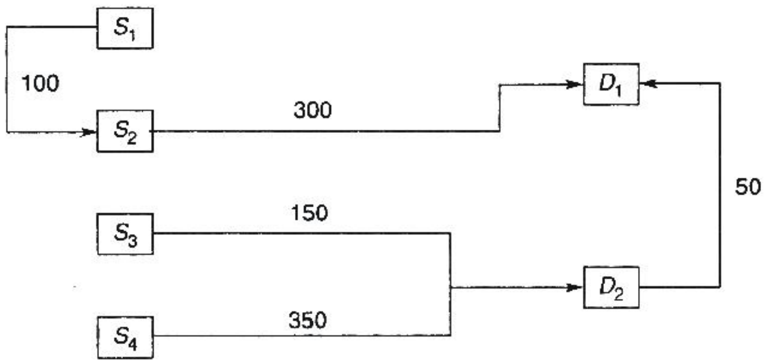



Fig. (a): Optimal shipping pattern

**25. The supply values of the sources $S_2$ and $S_3$ are 300 units, 450 units and 250 units, respectively. The demands of the destinations $D_1$ $D_2$ and Z), are 150 units, 200 units and 400 units, respectively. The cost of transportation (in rupees) per unit between different source and destination combinations are shown in Table (a). Find the optimal shipping plan for this transshipment problem.**

| | | Destination | | | | | |
|---|---|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_3$ | $D_1$ | $D_2$ | $D_3$ |
| Source | $S_1$ | 0 | 1 | 12 | 9 | 20 | 6 |
| | $S_2$ | 4 | 0 | 15 | 8 | 5 | 4 |
| | $S_3$ | 6 | 10 | 0 | 3 | 12 | 11 |
| | $D_1$ | 15 | 15 | 6 | 0 | 3 | 20 |
| | $D_2$ | 18 | 11 | 12 | 17 | 0 | 15 |
| | $D_3$ | 17 | 13 | 4 | 15 | 16 | 0 |

**Table (a) : Data for Example**

*Sol :*

The given problem is an unbalanced problem, because the sum of the supply values is not equal to the sum of the demand values. The maximum of these two sums is 1000 units.

i.e. $\sum_{i=1}^{3} a_i = 1000$ and $\sum_{j=1}^{3} b_j = 750$

The sum of the supplies is more than the sum of the demands by 250 units. So, a dummy column ($D_4$) is to be introduced with a demand of 250 units to absorb the excess supply. The value of B which is to be added to all the supply values as well to all the demand values is 1000 units. The balanced problem is shown in Table (b). In Table (b), the cell values in the row corresponding to $D_4$ except the last cell in that row are made as $\infty$ mainly to avoid allocations to those cells. The total number of starting nodes (sources) as well as the total number of ending nodes (destinations) of this transshipment problem is 7 (3 + 4).

| | | Destination | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_3$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
| | $S_1$ | 0 | 1 | 12 | 9 | 20 | 6 | 0 | 300 + 1000 = 1300 |
| | $S_2$ | 4 | 0 | 15 | 8 | 5 | 4 | 0 | 450 + 1000 = 1450 |
| | $S_3$ | 6 | 10 | 0 | 3 | 12 | 11 | 0 | 250 + 1000 = 1250 |
| Source | $D_1$ | 15 | 15 | 6 | 0 | 3 | 20 | 0 | 1000 |
| | $D_2$ | 18 | 11 | 12 | 17 | 0 | 15 | 0 | 1000 |
| | $D_3$ | 17 | 13 | 4 | 15 | 16 | 0 | 0 | 1000 |
| | $D_4$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 0 | 1000 |
| Demand | | 1000 | 1000 | 1000 | 150+ 1000 =1150 | 200+ 1000 =1200 | 400+ 1000 =1400 | 250+ 1000 =1250 | |

**Table (b) : Balanced Problem of Example**

The solution of the problem shown in Table (b) is presented in Table (c). The optimal shipment plan as per the solution given in Table (c) is shown in Figure (a). The corresponding total cost of shipment is Rs. 3,200.

| | | Destination | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_3$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
| | $S_1$ | 1000 | 150 | – | – | – | – | 150 | 1300 |
| | $S_2$ | – | 850 | – | – | 200 | 400 | – | 1450 |
| | $S_3$ | – | – | 1000 | 150 | – | – | 100 | 1250 |
| Source | $D_1$ | – | – | – | 1000 | – | – | – | 1000 |
| | $D_2$ | – | – | – | – | 1000 | – | – | 1000 |
| | $D_3$ | – | – | – | – | – | 1000 | – | 1000 |
| | $D_4$ | – | – | – | – | – | – | 1000 | 1000 |
| Demand | | 1000 | 1000 | 1000 | 1150 | 1200 | 1400 | 1250 | |

**Table (c) : Balanced Problem of Example**

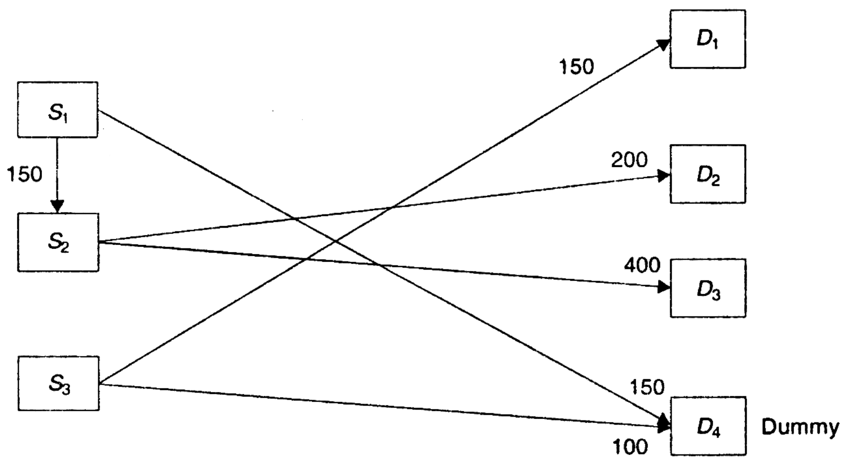



**Table (a) : Balanced Problem of Example**

# UNIT III

**Assignment Problem:** Introduction, Zero-One Programming Model, Types of Assignment Problem, Hungarian Method, Branch-and-Bound Technique for Assignment Problem.

**Integer Programming**: Introduction, Integer Programming Formulations, The Cutting-Plane Algorithm, Branch-and-Bound Technique, Zero-One Implicit Enumeration Algorithm.

## 3.1 Assignment Problem

### 3.1.1 Introduction

**Q1. What is an Assignment Problem ? How do you mathematically to formulate an assignment problem ?**

*Ans :* **(Imp.)**

An assignment problem is a particular case of transportation problem where the objective is to assign a number of origins to the equal number of destinations at a minimum cost or maximum profit.

Suppose there are m jobs (activities) to be performed and m persons are available for doing these jobs. Assume that each person can do each job at a time. Let $c_{ij}$ be the cost incurred in assigning ith person to jth job. The problem is to assign each person to one and only job so that the cost of performing all jobs is minimum. The assignment problem can be stated in the form of m $\times$ m cost matrix shown below.

| | 1 | 2 | j ..... | m | Availability |
|---|---|---|---|---|---|
| 1 | $c_{11}$ ..... | $c_{12}$ ..... | $c_{1j}$ ..... | $c_{1m}$ | 1 |
| : | | | | | : |
| 2 | $c_{21}$ ..... | $c_{22}$ ..... | $c_{2j}$ ..... | $c_{2m}$ | 1 |
| : | | | | | : |
| i | $c_{i1}$ ..... | $c_{i2}$ ..... | $c_{ij}$ ..... | $c_{im}$ | 1 |
| : | | | | | : |
| m | $c_{m1}$ ..... | $c_{m2}$ ..... | $c_{mj}$ ..... | $c_{mm}$ | 1 |
| Requirement | 1 ...... | 1 ...... | 1 ...... | 1 | |

**Mathematical Formulation of AP**

Mathematically, the assignment problem can be stated as :

Minimize the total cost : $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$, i = 1, 2, ..., n ; j = 1, 2, ..., n

Subject to restrictions of the form :

$$x_{ij} = \begin{cases} 1 \text{ if ith person is assigned jth job} \\ 0 \text{ if not} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{(one job is done by the ith person, } i = 1, 2, ..., n)$$

and $$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{(only one person should be assigned the jth job, } j = 1, 2, ..., n)$$ where $x_{ij}$ denotes that jth job is to be assigned to the ith person.

**Q2. State the assumptions of an Assignment Problem.**

*Ans :*

**Assumptions Made in Assignment Models**

(i) The number of assignees (workers or machines) and the number of jobs or tasks are the same (say the number is 'n').

(ii) Each assignee is to be assigned to exactly one job or task.

(iii) Each job or task is to be performed by exactly one assignee.

(iv) There is a cost $c_i$ associated with assignee 'i' (i = 1, 2....,n) performing task 'j' (j = 1, 2, n).

(v) The objective is to determine how all 'n' assignments should be made to minimise the total cost.

Any problem satisfying all these assumptions can be solved very efficiently by algorithms designed specifically for assignment problems.

The first three assumptions are fairly restrictive. Many applications do not quite satisfy these assumptions. When these assumptions are not satisfied, it is possible to reformulate the problem to make it fit.

For example, dummy assignees or dummy jobs or tasks can be used for this purpose.

Many solution algorithms have been proposed to assignment problems. They are:

(i) Total enumeration of all possibilities

(ii) Linear programming

(iii) A transportation approach

(iv) Dynamic programming

(v) A binary branch and bound approach and

(vi) An efficient approach developed specifically for the assignment problem known as the Hungarian method or algorithm, which is generally used.

## 3.2 Zero-One Programming Model

**Q3. Explain the concept of Zero-One Programming Model for Assignment Problem with an example.**

*Ans :* **(Imp.)**

A zero-one programming model for the assignment problem is presented below :

Minimize

$$Z = C_{11}X_{11} + C_{12}X_{12} + .... + C_{1m}X_{1m} + C_{21}X_{21} + C_{22}X_{22} + ... + C_{2m}X_{2m} + ...$$
$$+ C_{i1}X_{i1} + C_{i2}X_{i2} + ... + C_{im}X_{im} + ... + C_{m1}X_{m1} + C_{m2}X_{m2} + ... + C_{mm}X_{mm}$$

Subject to

$$\begin{aligned}
X_{11} + X_{12} + \cdots + X_{1j} + \cdots + X_{1m} &= 1\\
X_{21} + X_{22} + \cdots + X_{2j} + \cdots + X_{2m} &= 1\\
\vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
X_{i1} + X_{i2} + \cdots + X_{ij} + \cdots + X_{im} &= 1\\
\vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
X_{m1} + X_{m2} + \cdots + X_{mj} + \cdots + X_{mm} &= 1\\
X_{11} + X_{21} + \cdots + X_{i1} + \cdots + X_{m1} &= 1\\
X_{12} + X_{22} + \cdots + X_{i2} + \cdots + X_{m2} &= 1\\
\vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
X_{1j} + X_{2j} + \cdots + X_{ij} + \cdots + X_{mj} &= 1\\
\vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
X_{1m} + X_{2m} + \cdots + X_{im} + \cdots + X_{mm} &= 1
\end{aligned}$$

$$X_{ij} = 0 \text{ or } 1, \text{ for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, m$$

The above model is presented in a short form as :

Minimize $\quad Z = \sum_{i=1}^{m}\sum_{j=1}^{m} C_{ij} X_{ij}$

Subject to $\quad \sum_{j=1}^{m} X_{ij} = 1, 2, 3, \ldots. m$

and $\quad \sum_{i=1}^{m} X_{ij} = 1, j, = 1, 2, 3, \ldots. m$



where $X_{ij} = 0$ or 1 for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, m$, m being the number of rows (jobs) as well as the number of columns (operators) and $C_{ij}$ the time/cost (processing time, travel time, etc.) of assigning the row *i* to the column j. Thus,

$X_{ij}$ = 1, if the row i is assigned to the column j

= 0, otherwise.

In this model, the objective function minimizes the total cost of assigning the rows to the columns. The first set of constraints ensures that each row (job) is assigned to only one column (operator). The second set of constraints ensures that each column (operator) is assigned to only one row (job).

**Example**

**Consider the assignment problem as shown in Table below. In this problem, 5 different jobs are to be assigned to 5 different operators such that the total processing time is minimized. The matrix entries represent processing times in hours.**

| | | Operator 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| | 1 | 10 | 12 | 15 | 12 | 8 |
| | 2 | 7 | 16 | 14 | 14 | 11 |
| Job | 3 | 13 | 14 | 7 | 9 | 9 |
| | 4 | 12 | 10 | 11 | 13 | 10 |
| | 5 | 8 | 13 | 15 | 11 | 15 |

Develop a zero-one programming model.

*Sol :*

Let

$X_{ij}$ = 1, if the job *i* is assigned to the operator j

= 0, otherwise.

A zero-one programming model for the assignment problem to minimize the total processing time presented below :

$$\text{Minimize } Z = 10X_{11} + 12X_{12} + 15X_{13} + 12X_{14} + 8X_{15}$$
$$+ 7X_{21} + 16X_{22} + 14X_{23} + 14X_{24} + 11X_{25}$$
$$+ 13X_{31} + 14X_{32} + 7X_{33} + 9X_{34} + 9X_{35}$$
$$+ 12X_{41} + 10X_{42} + 11X_{43} + 13X_{44} + 10X_{45}$$
$$+ 8X_{51} + 13X_{52} + 15X_{53} + 11X_{54} + 15X_{55}$$

Subject to

$$X_{11} + X_{12} + X_{13} + X_{14} + X_{15} = 1$$
$$X_{21} + X_{22} + X_{23} + X_{24} + X_{25} = 1$$
$$X_{31} + X_{32} + X_{33} + X_{34} + X_{35} = 1$$
$$X_{41} + X_{42} + X_{43} + X_{44} + X_{45} = 1$$
$$X_{51} + X_{52} + X_{53} + X_{54} + X_{55} = 1$$
$$X_{11} + X_{21} + X_{31} + X_{41} + X_{51} = 1$$
$$X_{12} + X_{22} + X_{32} + X_{42} + X_{52} = 1$$
$$X_{13} + X_{23} + X_{33} + X_{43} + X_{53} = 1$$
$$X_{14} + X_{24} + X_{34} + X_{44} + X_{54} = 1$$
$$X_{15} + X_{25} + X_{35} + X_{45} + X_{55} = 1$$

$X_{ij} = 0$ or 1, $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3, 4, 5$

The optimal solution of the above model is presented in table below.

| | | Operator 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Job | 1 | 10 | 12 | 15 | 12 | $X_{15} = 1$ 8 |
| | 2 | $X_{21} = 1$ 7 | 16 | 14 | 14 | 11 |
| | 3 | 13 | 14 | $X_{33} = 1$ 7 | 9 | 9 |
| | 4 | 12 | $X_{42} = 1$ 10 | 11 | 13 | 10 |
| | 5 | 8 | 13 | 15 | $X_{54} = 1$ 11 | 15 |

## 3.3 Types of Assignment Problem

### 3.3.1 Minimisation in an AP

**PROBLEMS**

**1. A company is faced with the problem of assigning five jobs to five machines, each job must be done on only one machine the cost of processing each job on each machine is given below :**

| | | A | B | C | D | E |
|---|---|---|---|---|---|---|
| Jobs | 1 | 7 | 5 | 9 | 8 | 11 |
| | 2 | 9 | 12 | 7 | 11 | 10 |
| | 2 | 8 | 5 | 4 | 6 | 9 |
| | 4 | 7 | 3 | 6 | 9 | 5 |
| | 5 | 4 | 6 | 7 | 5 | 11 |

**Solve the problem assuming that the objective is to minimize the total cost.**

*Sol :*

**Step 1 :**

Select the minimum element in each row and subtract this element from every element in that row. Minimum element is 5 in first row, 7 in second row, 4 in row 3, 3 in row 4 and 4 in row 5. The resulting matrix is in table 1.

| | | | | |
|---|---|---|---|---|
| 2 | 0 | 4 | 3 | 6 |
| 2 | 5 | 0 | 4 | 3 |
| 4 | 1 | 0 | 2 | 5 |
| 4 | 0 | 3 | 6 | 2 |
| 0 | 2 | 3 | 1 | 7 |

**Tabel (1)**

| | | | | |
|---|---|---|---|---|
| 2 | 0 | 4 | 2 | 4 |
| 2 | 5 | 0 | 3 | 1 |
| 4 | 1 | 0 | 1 | 3 |
| 4 | 0 | 3 | 5 | 0 |
| 0 | 2 | 3 | 0 | 5 |

**Table (2)**

**Step 2 :**

Select the minimum element in each column and subtract this element from every element in that column. The column having zero element will not change, the minimum element in column 4 is 1, in column 5 is 2 the resulting matrix is in table 2.

**Step 3 :**

Examine the rows starting from row 1, one by one until a row with exactly single zero element is found make an assignment indicated by '☐' to that cell and cross all other zeros in the column in which the assignment was made.

When the all the rows has been completely examined, the procedure is applied the columns. Starting with column examine columns until a column containing exactly one remaining zero is found. Make an assignment in that position and corss other seros in the row in which the assignments was made.

| | A | B | C | D | E | |
|---|---|---|---|---|---|---|
| 1 | 2 | [0] | 4 | 2 | 4 | |
| 2 | 2 | 5 | [0] | 3 | 1 | 3 |
| 3 | 4 | 1 | ~~0~~ | 1 | 3 | 3 |
| 4 | 4 | ~~0~~ | 3 | 5 | [0] | |
| 5 | [0] | 2 | 3 | ~~0~~ | 5 | |
| | | | 3 | | | |

The solution is not optimal since only four assignments are made. Since row 3 and column 3 donot have any assignment.

**Step 4 :**

(i) Mark (3) row 3 since it has no assignment

(i) Mark (3) column 3, since row 3 has zero elements in that column

(iii) Mark (3) row 2, since column 3 have an assignment in row 2.

(iv) Since o other row or columns can be marked draw the straight lines through the unmarked row and marked columns.

| | A | B | C | D | E | |
|---|---|---|---|---|---|---|
| 1 | 2 | [0] | 4 | 2 | 4 | |
| 2 | 2 | 5 | [0] | 3 | 1 | 3 |
| 3 | 4 | 1 | 0̸ | 1 | 3 | 3 |
| 4 | 4 | 0̸ | 3 | 5 | [0] | |
| 5 | [0] | 2 | 3 | 0̸ | 5 | |
| | | | 3 | | | |

The minimum number of lines drawn is 4, which is less than the order of the matrix i.e., 4 < 5, indicating that the current assignment is not optimum.

**Step 5 :**

To create one more line, examine the elements not covered by these lines and select the smallest element i.e., 1 is the smallest element not covered by these lines. Subtract this smallest element from all the uncovered elements and add it to the element lying at the intersection of the two lines. The reduced matrix obtained is shown below :

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | 2 | 0 | 5 | 2 | 4 |
| 2 | 1 | 4 | 0 | 2 | 0 |
| 3 | 3 | 0 | 0 | 0 | 2 |
| 4 | 4 | 0 | 4 | 5 | 0 |
| 5 | 0 | 2 | 4 | 0 | 5 |

Repeat step (3)

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | 2 | [0] | 5 | 2 | 4 |
| 2 | 1 | 4 | [0] | 2 | 0̸ |
| 3 | 3 | 0̸ | 0̸ | [0] | 2 |
| 4 | 4 | 0̸ | 4 | 5 | [0] |
| 5 | [0] | 2 | 4 | 0̸ | 5 |

Since each row and each column has one and only one assignment, an optimum solution is reached.

The optimum assignment is

$1 \rightarrow B,\ 2 \rightarrow C,\ 3 \rightarrow D,\ 4 \rightarrow E,\ 5 \rightarrow A$

The total cost = 5+7+6+5+4 = Rs. 27.

**2. Find the minimum cost assignment for the following problem, explaining each step.**

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 6 | 5 | 8 | 11 | 16 |
| B | 1 | 13 | 16 | 1 | 10 |
| C | 16 | 11 | 8 | 8 | 8 |
| D | 9 | 14 | 12 | 10 | 16 |
| E | 10 | 13 | 11 | 8 | 16 |



*Sol :*

**Step 1 & 2 :**

As the given A.P has cost matrix i.e., to minimise, it is the standard form. Also as no. of row = no. of columns, given A.P. is balanced.

| | Jobs | | | | |
|---|---|---|---|---|---|
| Workers | I | II | III | IV | V |
| A | 6 | 5 | 8 | 11 | 16 |
| B | 1 | 13 | 16 | 1 | 10 |
| C | 16 | 11 | 8 | 8 | 8 |
| D | 9 | 14 | 12 | 10 | 16 |
| E | 10 | 13 | 11 | 8 | 16 |

**Step 3 :**

***(a) Row Iteration***

Select least member of each row and subtract in its corresponding row memebrs.

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 1 | 0 | 3 | 6 | 11 |
| B | 0 | 12 | 15 | 0 | 19 |
| C | 8 | 3 | 0 | 0 | 0 |
| D | 0 | 5 | 3 | 1 | 7 |
| E | 2 | 5 | 3 | 0 | 8 |

***(b) Column Iteration***

Select least no. of each column and subtract from members of corresponding column.

Since every column is having a zero, this step results in the same matrix as that step 3(a).

**Step 4 :**

**Allocation :** Allocate by entering ( □ ) on zeros such that each row/column will have one and only one allocation. When a zero is allocated, the other zeros in its row and column will be crossed out. ( × ).

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 1 | [0] | 3 | 6 | 11 |
| B | ×0 | 12 | 15 | ×0 | 19 |
| C | 8 | 3 | [0] | ×0 | ×0 |
| D | [0] | 5 | 3 | 1 | 7 |
| E | 2 | 5 | 3 | [0] | 8 |

**Step 5 :**

**Meaning :** As workers B is not assigned & Job V is unassigned, we move towards optimization by marking by following rules.

(a) Mark (3) the unassigned row i.e., Row 'B'

(b) Mark the columns in which unassigned marked row has zero, i.e., column I & IV.

(c) Mark the row in which marked columns have the assignment i.e., row D & E and strike off unmarked rows and marked column by a line.

| | I | II | III | IV | V | |
|---|---|---|---|---|---|---|
| A | 1 | [0] | 3 | 6 | 11 | 3 |
| B | ×0 | 12 | 15 | ×0 | 19 | 3 |
| C | 8 | 3 | [0] | ×0 | ×0 | |
| D | [0] | 5 | 3 | 1 | 7 | 3 |
| E | 2 | 5 | 3 | [0] | 8 | 3 |
| | 3 | | | 3 | | |

All the zeros are connected with zeros and the number of lines (4) is less than the no. of rows/ columns. (5) Hence move towards step 6.

**Step 6 :**

**Optimisation**

For finding optimal solution we iterate the above table as follows.

(a) Identify least digit of unlined numbers. i.e., 3

(b) Subtract this number in all unlined numbers

(c) Add this member at the intersected numbers (junctions of two lines).

(d) Keep all others (lined) unaltered.

**Step 7**

Repeat step 4 i.e., allocation

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 4 | [0] | 3 | 9 | 11 |
| B | [0] | 9 | 12 | ×0 | 6 |
| C | 11 | 3 | ×0 | 3 | [0] |
| D | ×0 | 2 | [0] | 1 | 4 |
| E | 2 | 2 | ×0 | [0] | 5 |

The optimal assignment is

| Worker | Job | Cost |
|---|---|---|
| A | II | 5 |
| B | I | 1 |
| C | V | 8 |
| D | III | 12 |
| E | IV | 8 |
| Total | 34 | |

Total cost (min.) = 34 units.

### 3.3.2 Maximization in an AP

**Q4. Define the term maximization in AP.**

*Ans :*

There are problems where certain facilities have to be assigned to a number of jobs so an to maximize the overall performance of the assignment. Such problems may be solved by covering the given maximization problem into a minimization problem in either of the following ways.

(i) Multiply each element of given matrix by –1 so as to convert the profit values into cost values.

(ii) Subtract all element of the matrix from the highest element of the matrix.

**PROBLEMS**

**3. Find the maximum profit possible through optimum assignments.**

*Sol :*

The given maximization problem can be converted into a maximization problem by subtracting all the elements from the largest element i.e., 62 of the table. The revised assignment table is given below

| | | | | |
|---|---|---|---|---|
| 32 | 25 | 22 | 34 | 22 |
| 22 | 38 | 35 | 41 | 26 |
| 22 | 30 | 29 | 32 | 27 |
| 27 | 24 | 22 | 26 | 26 |
| 33 | 0 | 21 | 28 | 23 |

Row reduced matrix

| | | | | |
|---|---|---|---|---|
| 10 | 3 | 0 | 12 | 0 |
| 0 | 16 | 13 | 17 | 4 |
| 0 | 8 | 7 | 10 | 5 |
| 15 | 2 | 0 | 4 | 4 |
| 12 | 0 | 0 | 7 | 2 |

Column reduced matrix

| | | | | |
|---|---|---|---|---|
| 10 | 3 | 0 | 8 | 0 |
| 0 | 16 | 13 | 13 | 4 |
| 0 | 8 | 7 | 6 | 5 |
| 15 | 2 | 0 | 0 | 4 |
| 12 | 0 | 0 | 3 | 2 |

| | | | | | |
|---|---|---|---|---|---|
| 10 | 3 | ~~0~~ | 8 | [0] | |
| [0] | 16 | 13 | 13 | 4 | 3 |
| ~~0~~ | 8 | 7 | 6 | 5 | 3 |
| 15 | 2 | ~~0~~ | [0] | 4 | |
| 12 | [0] | ~~0~~ | 3 | 2 | |
| 3 | | | | | |

The minimum number of lines drawn is 4, which is less than order of matrix i.e., 4 < 5, indicating the current assignment is not optimum.

Select the smallest element which is not covered by these lines i.e., 4. Subtract this element from all the uncovered elements and add it to the element lying at the interaction of two lines. The reduced matrix is

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | 14 | 3 | [0] | 8 | ~~0~~ |
| 2 | ~~0~~ | 12 | 9 | 0 | [0] |
| 3 | [0] | 4 | 3 | 2 | 1 |
| 4 | 19 | 2 | ~~0~~ | [0] | 4 |
| 5 | 16 | [0] | ~~0~~ | 3 | 2 |

Since each row and each column has one and only one assignment, an optimum solution is reached.

The optimum assignment is

$1 \rightarrow C, \quad 2 \rightarrow E, \quad 3 \rightarrow A, \quad 4 \rightarrow D, \quad 5 \rightarrow B$

Maximum Profit is 40 + 36 + 40 + 36 + 62 = Rs. 214.

### 3.3.3 Unbalanced AP

**Q5. Write a brief note on unbalanced assignment problem.**

*Ans :*

An assignment problem is said to be unbalanced if the number of rows is not equal to the number of columns. Number of rows n(r) = Number of columns n(c) balanced AP. Number of rows n(4) ≠ number of column n(c) unbalanced AP.

The AP is balanced if the AP matrix is a square and unbalanced if it is not a square matrix.

In unbalanced AP, we notice that either number of jobs are greater or lesser than number of men or machine. In such case, a dummy row/column will be created with zero costs to each cell. Observe the following example to convert unbalanced AP to balanced AP.

| | $M_1$ | $M_2$ | $M_3$ |
|---|---|---|---|
| $J_1$ | $x_{11}$ | $x_{12}$ | $x_{13}$ |
| $J_2$ | $x_{21}$ | $x_{22}$ | $x_{23}$ |
| $J_4$ | $x_{31}$ | $x_{32}$ | $x_{33}$ |
| $J_5$ | $x_{41}$ | $x_{42}$ | $x_{43}$ |

**Unbalanced :**
Machines (3) ≠ jobs (4)

| | $M_1$ | $M_2$ | $M_3$ | Dummy |
|---|---|---|---|---|
| $J_1$ | $x_{11}$ | $x_{12}$ | $x_{13}$ | 0 |
| $J_2$ | $x_{21}$ | $x_{22}$ | $x_{23}$ | 0 |
| $J_4$ | $x_{31}$ | $x_{32}$ | $x_{33}$ | 0 |
| $J_5$ | $x_{41}$ | $x_{42}$ | $x_{43}$ | 0 |

**Balanced :**
Machines (4) = jobs (4) here D is Dummy machine.

Similarly, if jobs are less, we create a dummy row with zero costs.

## PROBLEMS

**4. Solve the following assignment problem of minimizing total time for doing all the jobs.**

Job

| Operator | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 6 | 2 | 5 | 2 | 6 |
| 2 | 2 | 5 | 8 | 7 | 7 |
| 3 | 7 | 8 | 6 | 9 | 8 |
| 4 | 6 | 2 | 3 | 4 | 5 |
| 5 | 9 | 3 | 8 | 9 | 7 |
| 6 | 4 | 7 | 4 | 6 | 8 |

*Sol :*

Since the given matrix is not a square matrix. The problem is unbalanced. Introduce a dummy job with duration zero. The revised assignment problem is given below.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 6 | 2 | 5 | 2 | 6 | 0 |
| 2 | 2 | 5 | 8 | 7 | 7 | 0 |
| 3 | 7 | 8 | 6 | 9 | 8 | 0 |
| 4 | 6 | 2 | 3 | 4 | 5 | 0 |
| 5 | 9 | 3 | 8 | 9 | 7 | 0 |
| 6 | 4 | 7 | 4 | 6 | 8 | 0 |

Since each row contains zero, the matrix will be the same after row reduction. Subtract the smallest value in each column from all the values of that column and making assignments.

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | ~~0~~ | 2 | [0] | 1 | ~~0~~ | |
| [0] | 3 | 5 | 5 | 2 | ~~0~~ | |
| 5 | 6 | 3 | 7 | 3 | [0] | 3 |
| 4 | ~~0~~ | [0] | 2 | ~~0~~ | ~~0~~ | |
| 7 | 1 | 5 | 7 | 2 | ~~0~~ | 3 |
| 2 | 5 | 1 | 4 | 3 | ~~0~~ | 3 |
| | | | | | 3 | |

Since the minimum number of lines are four, which is not equal to the number of assignments. Select the smallest element which is not covered by these lines i.e., 1. Subtract this element from all the uncovered elements and add it to the element lying at the intersection of two lines.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | [0] | 1 | 1 |
| 2 | [0] | 3 | 5 | 5 | 2 | 1 |
| 3 | 4 | 5 | 2 | 6 | 2 | [0] |
| 4 | 4 | 0 | 0 | 2 | [0] | 1 |
| 5 | 6 | [0] | 4 | 6 | 1 | 0 |
| 6 | 1 | 4 | [0] | 3 | 2 | 0 |

Since each row and each column has one and only one assignment, an optimum solution is reached. The optimum assignment is $1 \to 4$, $2 \to 1$, $3 \to 6$, $4 \to 5$, $5 \to 2$, $6 \to 3$.

Minimum Time is $2 + 2 + 0 + 5 + 3 + 4 = 16$.

### 3.3.4 Restricted / Prohibited AP

**Q6. Explain prohibited AP (or) Restricted AP.**

*Ans :*

Some times due to certain reason an assignment cannot be made in a particular cell. For example, a particular machine cannot be installed at a particular place or a worker cannot be given a particular job to perform. To resolve this, we put either a very large cross or dash (–) to avoid assignments in those cells where there is a restriction of assignment. The procedure involved in case of prohibited assignment is as follows :

**Step 1 :** Set up a matrix with the m objects (rows) to the assignment tasks (columns).

**Step 2 :** Enter the cost, profit, or other measure of performance in the matrix cell corresponding to each object-task combination. Use a–m profit or a–m cost (or $\infty$) if the specific object task combination is unacceptable or prohibited.

**Step 3 :** If the number of rows is not equal to the number of columns, add dummy rows or dummy columns until the number of rows and number of columns are equal. Use 0 values for the new elements in the assignment matrix.

**Step 4 :** If the problem involves maximization, convert the matrix to an opportunity loss matrix by subtracting each element from the highest element of the matrix.

## PROBLEMS

**5. In the modification of a plant layout of a factory four new machines $M_1$, $M_2$, $M_3$ and $M_4$ are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine $M_2$ cannot be placed at C and M cannot be placed at A. The cost of locating of machine i to place j in rupees is shown below. Find optimum assignment schedule.**

| | A | B | C | D | E |
|---|---|---|---|---|---|
| $M_1$ | 9 | 11 | 15 | 10 | 11 |
| $M_2$ | 12 | 9 | – | 10 | 9 |
| $M_3$ | – | 11 | 14 | 11 | 7 |
| $M_4$ | 14 | 8 | 12 | 7 | 8 |

*Sol :*

**Step 1 :**

As there are five vacant places and four machines we take one dummy machine $M_D$ with zero cost of installation. Further the cost for cells $M_3A$ and $M_2C$ which are not to be filled is taken as $\infty$. Accordingly the cost matix will be :

| | A | B | C | D | E |
|---|---|---|---|---|---|
| $M_1$ | 9 | 11 | 15 | 10 | 11 |
| $M_2$ | 12 | 9 | $\infty$ | 10 | 9 |
| $M_3$ | $\infty$ | 11 | 14 | 11 | 7 |
| $M_4$ | 14 | 8 | 12 | 7 | 8 |
| $M_D$ | 0 | 0 | 0 | 0 | 0 |



**Step 2 :**

Subtracting the minimum element from each row and each column and allocating assignment, we get

| | A | B | C | D | E |
|---|---|---|---|---|---|
| $M_1$ | [0] | 2 | 6 | 1 | 2 |
| $M_2$ | 3 | [0] | $\infty$ | 1 | 0 |
| $M_3$ | $\infty$ | 4 | 7 | 4 | [0] |
| $M_4$ | 7 | 1 | 5 | [0] | 1 |
| $M_D$ | 0 | 0 | [0] | 0 | 0 |

Since each row and column is having only one assignment, the optimum assignment is reached and is given by

$M_1$ – A, $M_2$ – B, $M_3$ – E, $M_4$ – D (Space C will be kept vacant)

with that cost = 9 + 9 + 7 + 7 = 32.

**6. An airline company has drawn up a new flight schedule involving five flights. To assist in allocating five pilots to the flights, it has asked them to state their preference scores by giving each flight a number out of 10. The higher the number, the greater is the preference. Certain of these flights are unsuitable to some pilots owing to domestic reasons. These have been marked with a '*' what should be the allocating of the pilots to flights in order to meet as many preferences as possible.**

Flight Number

| | | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| | A | 8 | 2 | * | 5 | 4 |
| | B | 10 | 9 | 2 | 8 | 4 |
| Pilot | C | 5 | 4 | 9 | 6 | * |
| | D | 3 | 6 | 2 | 8 | 7 |
| | E | 5 | 6 | 10 | 4 | 3 |

*Sol :*

(Whenever a particular source cannot be assigned to perform a particular activity, in such cases, the cost of performing that activity is assume to the M or $\infty$, which is very large).

Since the given problem is to maximize the total preference, converting the problem into minimization problem by subtracting all the elements from the highest element of matrix.

| | | | | |
|---|---|---|---|---|
| 2 | 8 | M | 5 | 6 |
| 0 | 1 | 8 | 2 | 6 |
| 4 | 6 | 1 | 4 | M |
| 5 | 4 | 8 | 2 | 3 |
| 5 | 4 | 0 | 6 | 7 |

Selecting minimum element from each row and subtracting from other elements of that row, we get

| | | | | |
|---|---|---|---|---|
| 0 | 6 | M | 3 | 4 |
| 0 | 1 | 8 | 2 | 6 |
| 4 | 5 | 0 | 3 | M |
| 5 | 2 | 6 | 0 | 1 |
| 5 | 4 | 0 | 6 | 7 |

Selecting minimum element from each columns and subtracting from other elements of that columns and making assignments.

| | | | | | |
|---|---|---|---|---|---|
| [0] | 5 | M | 3 | 3 | |
| X | [0] | 8 | 2 | 5 | 3 |
| 4 | 4 | [0] | 3 | M | |
| 5 | 1 | 6 | [0] | X | |
| 5 | 3 | X | 6 | 6 | 3 |
| | | 3 | | | |

The minimum number of lines drawn is less than the order of the matrix i.e., 4 < 5. Select the smallest element which it not covered by these lines. Subtract this lement from all the uncoverd elements and add it to the element lying at the intersection of two lines.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | [0] | 5 | M | 3 | 3 |
| B | X | [0] | 11 | 2 | 5 |
| C | 1 | 1 | X | [0] | M |
| D | 5 | 1 | 9 | X | [0] |
| E | 2 | X | [0] | 3 | 3 |

Since each row and each column has one and only one assignment, an optimum solution is reached.

The optimum assignments is A $\rightarrow$ 1, B $\rightarrow$ 2, C $\rightarrow$ 4, D $\rightarrow$ 5, E $\rightarrow$ 3

Maximum preferences,

8 + 9 + 6 + 7 + 10 = 40.

## 3.4 Hungarian Method

**Q7. Discuss the steps involved in the Hungarians Method used to find optimal solution to an Assignment Problem.**

*Ans :* **(Imp.)**

**Steps Involved in the Hungarian Method (in Solving Minimisation Problems)**

**Step 1**

Check whether the numbers of rows are equal to the number of column. If the number of rows equals the number of columns, the problem is a balanced one and Hungarian method can be used. If not, then the assignment problem is unbalanced and application of Hungarian method to an unbalanced problem yields an incorrect solution. Hence any assignment problem should be balanced by the additions of one or more dummy points (i.e., rows and columns). For dummy rows and columns, the value at the point of intersection of row and column is of zero value (i.e., zero cost in a cost matrix)

**Step 2**

Find the minimum element (or cost) in each row of the (m x m) cost matrix. Construct a new matrix by subtracting from each cost the minimum cost in its row. For this new matrix, find the minimum cost in each column. Construct a new matrix (called the reduced cost matrix) by subtracting from each cost, the minimum cost in its column.

This step may also be stated as below:

(a) **Row Subtraction : Subtract the minimum** element (say cost) of each row from all elements in that row. (Note : If there is zero in each row, there is no need for row subtraction).

(b) **Column Subtraction :** Subtract the minimum element of each column (of the new matrix obtained after row subtraction) from all elements of that column (Note : if there is zero in each column, there is no need for column subtraction).

**Step 3**

Draw the minimum number of lines (horizontal, vertical or both) that are needed to cover ail the zeros in the reduced cost matrix. If 'm' lines are required then an optimal solution is available among the covered zeros in the matrix. If fewer than 'm' lines are needed, then proceed to Step 4.

[**Note :** To draw the minimum number of lines the following procedure may be followed:

(a) Select a row containing exactly one uncovered zero and draw a vertical line through the column containing this zero and repeat the process till no such row is left.

(b) Select a column containing exactly one uncovered zero and draw a horizontal line through the row containing the zero and repeat the process till no such column is left].

**Step 4 :**

Find the smallest non-zero element (call its value K) in the reduced cost matrix that is uncovered by the lines drawn in Step 3. Now subtract K from each uncovered element of the reduced cost matrix and add K to each element that is covered by two lines.

**Step 5 :**

Repeat steps 3 and 4 till minimum number of lines covering ail zeros is equal to the size of the matrix (i.e., 'm' lines in a 'm x m' matrix)

**Step 6** :

**Assignment :** Use the matrix obtained in Step 5 (without horizontal or vertical lines) select a row containing exactly one unmarked zero and surround it by a ☐ and draw a vertical line through the column containing this zero. Repeat the process till no such row is left, then select a column containing exactly one covered zero and surround it by a ☐ and draw a horizontal line through the row containing this zero and repeat this process till no such column is left.

[**Note :** if there are more than one unmarked zero in any row or column, it indicates that an alternative solution exists. In this case, select any one arbitrarily and pass two lines horizontally and vertically.]

**Step 7 :**

Add up the value attributable to the allocation which shall be the minimum value.

**Step 8 :**

**Alternate solution :** If there are more than one covered zero in any row or column, select the other one (i.e., other than the one selected in step 6) and pass two lines horizontally and vertically. Add up the value attributable to the allocation, which shall be the minimum value.

## PROBLEMS

**7. Solve the following assignment problem by Hungarian assignment method.**

| Worker | Time (in minutes) | | |
|---|---|---|---|
| | Job 1 | Job 2 | Job 3 |
| A | 4 | 2 | 7 |
| B | 8 | 5 | 3 |
| C | 4 | 5 | 6 |

*Sol :*

Given,

| Worker | Time (in minutes) | | |
|---|---|---|---|
| | Job 1 | Job 2 | Job 3 |
| A | 4 | 2 | 7 |
| B | 8 | 5 | 3 |
| C | 4 | 5 | 6 |



**Row Reduced Matrix (Row Reduction)**

Select the minimum number from each row and subtract it from each element of row.

| Worker | Job 1 | Job 2 | Job 3 |
|---|---|---|---|
| A | 2 | 0 | 5 |
| B | 5 | 2 | 0 |
| C | 0 | 1 | 2 |

**Column Reduced Matrix (Column Reduction)**

No need to do column reductions as each column is having a 'zero'.

Now, cover maximum zeros from the reduced matrix.

| Worker | Job 1 | Job 2 | Job 3 |
|---|---|---|---|
| A | 2 | $\boxed{0}$ | 5 |
| B | 5 | 2 | $\boxed{0}$ |
| C | $\boxed{0}$ | 1 | 2 |

Since, the number of assignments is equal to the order of matrix, the current solution is said to be optional.

Allocations are,

Worker A $\rightarrow$ Job 2 = 2

Worker B $\rightarrow$ Job 3 = 3

Worker C $\rightarrow$ Job 1 = 4

Minimum Cost 9

**8. A company has 4 machines available for assignment to 4 tasks. Any machine can be assigned to any task, and each task requires processing by one machine. The time required to set up each machine for the processing of each task is given in the table below.**

**What kind of assignment will allow the company to minimize the total setup time needed for the processing of all four tasks?**

| | **TIME (Hours)** | | | |
|---|---|---|---|---|
| | **Task 1** | **Task 2** | **Task 3** | **Task 4** |
| **Machine 1** | **13** | **4** | **7** | **6** |
| **Machine 2** | **1** | **11** | **5** | **4** |
| **Machine 3** | **6** | **7** | **2** | **8** |
| **Machine 4** | **1** | **3** | **5** | **9** |

*Sol :*

| | **TIME (Hours)** | | | |
|---|---|---|---|---|
| | **Task 1** | **Task 2** | **Task 3** | **Task 4** |
| **Machine 1** | 13 | 4 | 7 | 6 |
| **Machine 2** | 1 | 11 | 5 | 4 |
| **Machine 3** | 6 | 7 | 2 | 8 |
| **Machine 4** | 1 | 3 | 5 | 9 |

**Step 1**

**Row Reduced Matrix (Row Reduction)**

Select the minimum number from each row and subtract it from each element of row.

| | **Task 1** | **Task 2** | **Task 3** | **Task 4** |
|---|---|---|---|---|
| **Machine 1** | 9 | 0 | 3 | 2 |
| **Machine 2** | 0 | 10 | 4 | 3 |
| **Machine 3** | 4 | 5 | 0 | 6 |
| **Machine 4** | 0 | 2 | 4 | 8 |

**Step 2**

**Column Reduced Matrix (Column Reduction)**

Select the minimum number from each column and subtract it from each element of column.

| | Task 1 | Task 2 | Task 3 | Task 4 |
|---|---|---|---|---|
| Machine 1 | 9 | [0] | 3 | 0 |
| Machine 2 | 0 | 10 | 4 | 1 |
| Machine 3 | 4 | 5 | [0] | 4 |
| Machine 4 | [0] | 2 | 4 | 6 |

**Step 3**

**Assignment**

As number of assignment $\neq$ order of matrix. Thus applying Hungarian rule. The minimum cost uncovered element is '1'. So add '1' where the lines are intersecting and subtract '1' from all uncovered elements.

**New Cost Matrix**

| | Task 1 | Task 2 | Task 3 | Task 4 |
|---|---|---|---|---|
| Machine 1 | 9 | [0] | 3 | 0 |
| Machine 2 | 0 | 10 | 4 | 1 |
| Machine 3 | 4 | 5 | [0] | 4 |
| Machine 4 | [0] | 2 | 4 | 6 |

**Step 4: Optimality Test**

Number of assignments = 4

Number of row/columns = 4

Since, the number of assignments = Number of row/ columns

Therefore, the current solution is optimal.

**Optimal Solution**

Machine 1 $\rightarrow$ Task 2 = 4

Machine 2 $\rightarrow$ Task 4 = 4

Machine 3 $\rightarrow$ Task 3 = 2

Machine 4 $\rightarrow$ Task 1 = 1

Min. time = 11

$\therefore$ The minimum total setup time is 11 hours.

**9. Solve the minimal assignment problem for the cost matrix given below :**

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| **A** | 12 | 13 | 14 | 15 |
| **B** | 14 | 15 | 16 | 17 |
| **C** | 17 | 18 | 19 | 18 |
| **D** | 13 | 15 | 18 | 14 |

*Sol :*

**Step 1 : Row Subtraction :**

Subtract the smallest element in the row from each element in that row. The resulting cost matrix is shown:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| **A** | 0 | 1 | 2 | 3 |
| **B** | 0 | 1 | 2 | 3 |
| **C** | 0 | 1 | 2 | 1 |
| **D** | 0 | 2 | 5 | 1 |

**Step 2 : Column subtraction :**

In the cost matrix obtained in Step 1 : subtract the smallest element in the column from each element in that column.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| **A** | 0 | 1 | 2 | 3 |
| **B** | 0 | 1 | 2 | 3 |
| **C** | 0 | 1 | 2 | 1 |
| **D** | 0 | 2 | 5 | 1 |

The reduced cost matrix is as shown below :

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| **A** | 0 | 0 | 0 | 2 |
| **B** | 0 | 0 | 0 | 2 |
| **C** | 0 | 0 | 0 | 0 |
| **D** | 0 | 1 | 3 | 0 |

Since no signal zero exists in any row column we have the following alterative columns

(i)

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | [0] | ~~0~~ | ~~0~~ | 2 |
| B | ~~0~~ | [0] | ~~0~~ | 2 |
| C | ~~0~~ | ~~0~~ | [0] | ~~0~~ |
| D | ~~0~~ | 1 | 3 | [0] |

(ii)

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | X | [0] | X | 2 |
| B | X | X | [0] | 2 |
| C | X | X | X | [0] |
| D | [0] | 1 | 3 | 0 |

(iii)

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | X | X | [0] | 2 |
| B | X | [0] | X | 2 |
| C | X | X | X | [0] |
| D | [0] | 1 | 3 | X |

The possible optimal solutions are

| | | Cost (Rs) |
|---|---|---|
| (i) | A → 1 → | 12 |
| | B → 2 → | 15 |
| | C → 3 → | 19 |
| | D → 4 → | 14 |
| | **Total** | **60** |

| | | Cost (Rs) |
|---|---|---|
| (ii) | A → 2 → | 13 |
| | B → 3 → | 16 |
| | C → 4 → | 18 |
| | D → 1 → | 13 |
| | **Total** | **60** |

| | | Cost (Rs) |
|---|---|---|
| (iii) | A → 3 → | 14 |
| | B → 2 → | 15 |
| | C → 4 → | 18 |
| | D → 1 → | 13 |
| | **Total** | **60** |

## 3.5 Branch-and-Bound Technique for Assignment Problem

**Q8. Explain the concept of Branch-and-Bound Technique for Assignment Problem.**

*Ans :* **(Imp.)**

The assignment problem can also be solved using a branch-and-bound algorithm. It is a curtailed enumeration technique. The terminologies of the branch and bound technique applied to the assignment problem are presented below.

Let $k$ be the level number in the branching tree (for root node, it is 0), $\sigma$ be an assignment made in the current node of a branching tree. $P_\sigma^k$ be an assignment at level $k$ of the branching tree, A be the set of assigned cells (partial assignment) up to the node $P_\sigma^k$ from the root node (set of $i$ and $j$ values with respect to the assigned cells up to the node $P_\sigma^k$ from the root node), and $V_\sigma$ be the lower bound of the partial assignment, A up to $P_\sigma^k$, such that,

$$V_\sigma = \sum_{i,j\in A} C_{ij} + \sum_{i\in X}\left(\sum_{j\in Y} \min C_{ij}\right)$$

where $C_{ij}$ is the cell entry of the cost matrix with respect to the $i^{th}$ row and $j^{th}$ column, X be the set of rows which are not deleted up to the node $P_\sigma^k$ from the root node in the branching tree, and Y be the set of columns which are not deleted up to the node $P_\sigma^k$ from the root node in the branching tree.

### Branching Guidelines

1. At Level k, the row marked as k of the assignment problem, will be assigned with the best column of the assignment problem.

2. If there is a tie on the lower bound, then the terminal node at the lower-most level is to be considered for further branching.

3. *Stopping rule* : If the minimum lower bound happens to be at any one of the terminal nodes at the $(n-1)^{th}$ level, the optimality is reached. Then the assignments on the path from the root node to that node along with the missing pair of row-column combination will form the optimum solution.

## PROBLEMS

**10. Solve the assignment problem using the branch-and-bound algorithm. The cell entries represent the processing time in hours ($C_{ij}$) of the job i if it is assigned to the operator j.**

| | | Operator j | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Job i | 1 | 23 | 20 | 21 | 24 |
| | 2 | 19 | 21 | 20 | 20 |
| | 3 | 20 | 18 | 24 | 22 |
| | 4 | 22 | 18 | 21 | 23 |

*Sol :*

Initially, no job is assigned to any operator. So, the assignment ($\sigma$) at the root node (level 0) of the branching tree is a null set and the corresponding lower bound $V_\sigma$ is also 0, as shown in Figure 4.1.

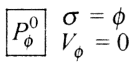



**Figure 4.1 : Branching tree at the root node.**

**Further Branching**

The four different sub-problems under the root node are shown as in Figure 4.2. The lower bound for each of the sub-problems is shown on its right-hand side.

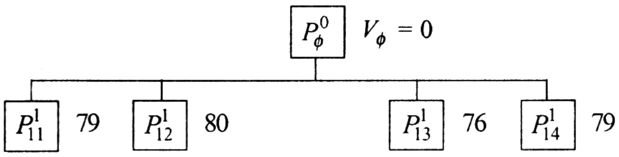



**Figure 4.2** Tree with lower bounds after branching from $P_\phi^0$.

Sample calculations to compute the lower bound for the first and the third sub-problems are shown below.

**Lower bound for $P_{11}^1$**

$$V_\sigma = \sum_{i \text{ and } j \in A} C_{ij} + \sum_{i \in N} \left( \sum_{j \in Y} \min C_{ij} \right)$$

where $\sigma = \{(11)\}$, $A = \{(11)\}$, $X = \{2, 3, 4\}$, $Y = \{2, 3, 4\}$

Then

$$V_{(11)} = C_{11} + \sum_{i \in (2,3,4)} \left( \sum_{i \in (2,3,4)} \min C_{ij} \right) = 23 + (20 + 18 + 18) = 79$$

**Lower bound for $P^1_{13}$**

$\sigma = \{(13)\}, A = \{(13)\},$

$X = \{2, 3, 4\}, Y = \{1, 2, 4\},$

Then

$$V_{(13)} = C_{13} + \sum_{i \in (2,3,4)} \left( \sum_{j \in (1,2,4)} \min C_{ij} \right)$$

$$= 21 + (19 + 18 + 18) = 76$$

**Further branching**

Further branching is done from the terminal node which has the least lower bound. At this stage, the nodes $P^1_{11}, P^1_{12}, P^1_{13}$, and $P^1_{14}$ are the terminal nodes. Among these nodes, the node $P^1_{13}$ has the least-lower bound. Hence, further branching from this node is shown as in Figure 4.3. The lower bound of each of the newly created nodes is shown by the side of it. As an example, the calculation pertaining to the lower bound of the node $P_{22}$ is presented below.

$\sigma = \{(22)\}, \ A = \{(13), (22)\},$

$X = \{3, 4\}, \ Y = \{1, 4\}$

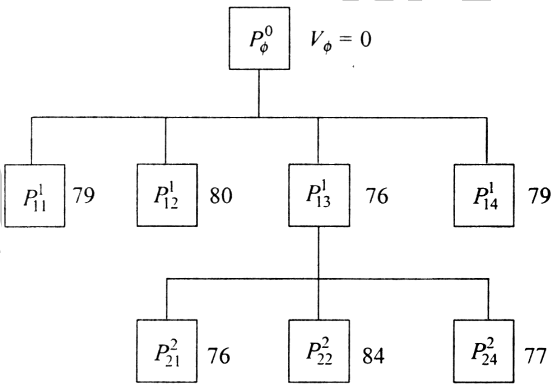



**Fig. : Branching tree after branching from $P^1_{13}$**

Then

$$V_{(22)} = C_{13} + C_{22} + \sum_{i \in (3,4)} \left( \sum_{j \in (1,4)} \min C_{ij} \right)$$

$$= 21 + 21 + (20 + 22) = 84$$

**Further branching**

At this stage, the nodes $P^1_{11}, P^1_{12}, P^2_{21}, P^2_{22}, P^2_{24}$ and $P^1_{14}$ are the terminal nodes. Among these nodes, the node $P^2_{21}$ has the least lower bound. Hence further branching from this node is shown as in Figure 4.4. The lower bound of each of the newly created nodes is shown by the side of it.

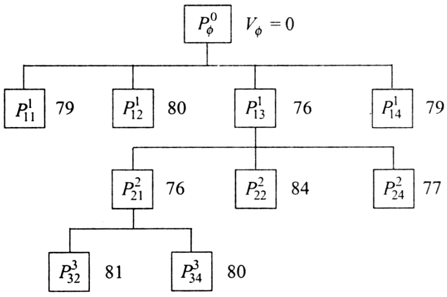



**Fig. : Tree with lower bounds after branching from $P_{21}^2$..**

### Further branching

At this stage, the nodes $P_{11}^1$, $P_{12}^1$, $P_{32}^3$, $P_{34}^3$, $P_{34}^3$, $P_{22}^2$, $P_{24}^2$ and $P_{14}^1$ are the terminal nodes. Among these nodes, the node $P_{24}$ has the least lower bound. Hence, further branching from this node is shown as in Figure 4.5. The lower bound of each of the newly created nodes is shown by the side of it.

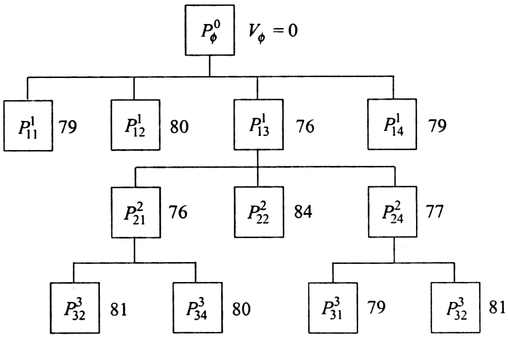



**Figure 4.5** Tree with lower bounds after branching from $P_{24}^2$.

### Further branching

At this stage, the nodes $P_{11}^1$, $P_{12}^1$, $P_{34}^3$, $P_{22}^2$, $P_{32}^3$, and $P_{14}^1$ are the terminal nodes. Among these nodes, there are three nodes with the least lower bound of 79. So, the node $P_{31}^3$ which is at the bottom-most level is considered for further branching. Since this node lies at $(n - l)^{th}$ level (k = 3) of the branching tree, where n is the size of the assignment problem, optimality is reached. The corresponding solution is traced from the root node to the node $P_{31}^3$ along with the missing pair of job and operator combination, (4, 2) as shown in Table 4.52.

**Table 4.52** Optimal Solution of Example 4.8

| Job | Operator | Time (in hours) |
|---|---|---|
| 1 | 3 | 21 |
| 2 | 4 | 20 |
| 3 | 1 | 20 |
| 4 | 2 | 18 |

Hence, total time = 79 hours.

## 3.6 Integer Programming

### 3.6.1 Introduction

**Q9. What is integers programming problem?**

*Ans :*

'Integer Linear Programming Problems' are the special class of linear programming problems where all or some of the variables in the optimal solution are restricted to non-negative integer values. Such problems are called as 'all integer' or 'mixed integer' problems depending, respectively, on whether all or some of the variables are restricted to integer values.

There was a need to develop a systematic procedure in order to identify the optimal integer solution to such problems.

In 1956, R.E. Gomory suggested first of all the systematic method to obtain an optimum integer solution to an 'all integer programming problem'. Later, he extended the method to deal with the more complicated case of 'mixed integer programming problems' when only some of the variables are required to the integer. These algorithms are proved to converge to the optimal integer solution in a finite number of iterations making use of familiar dual simplex method. This is called the "cutting plane algorithm" because it mainly introduces the clever idea of constructing "secondary" constraints which, when added to the optimum (non-integer) solution, will effectively cut the solution space towards the required result. Successive application of these constraints should gradually force the non-integer optimum solution toward the desired "all-integer" or "mixed integer" solution.

**Q10. Explain the importance of integer programming problems.**

*Ans :*

In LPP all the decision variables were Allowed to take any non-negative real values as it is quite possible and appropriate to have fractional values in many situation. These are several frequently occurring circumstances in business and industry that lead to planning models involving integer valued variables. For example in production, manufacturing is frequently scheduled in terms of batches, lots or runs. In allocation of goods, a shipment must involves a discrete number of tracks, aircrafts. In such cases the fractional values of variables like 13/3 may be meaningless in the context of the actual decision problem.

This is the main reason why integer programming is so important for marginal decision.

**Q11. State the applications of integer programming problems.**

*Ans :*

Integer programming is applied in business and industry. All assignment and transportation problems are integer programming problems, as in assignment and travelling salesmen problem all the decision variables are either zero or one.

i.e., $x_{ij} = 0$ or

Other examples are capital budgeting and production scheduling problems. In fact any situation involving decisions of the types "either to day a job or not to do" can be viewed as an LPP. In all such situations

$x_{ij} = 1$ if the $j^{th}$ activity is performed,

0 if the $j^{th}$ activity is not performed.

In addition, alternation problems involving the allocation of men, and machines give rise to IPP, since such communities can be assigned only in integers and not in functions.

**Note :** If the non-integer variable is rounded off, it violates the feasibility and also there is no guarantee that the rounded off solution will be optimal. Due to these difficulties there is a need for developing a systematic and efficient procedure for obtaining the exact optimal integer solution to such problems.

### 3.6.2 Integer Programming Formulations

**Q12. State the formulation of Integer Programming.**

*Ans :*

In a linear programming problem if all variables are required to take integral values then it is called the pure (all) integer programming problem.

If only some of the variables in the optimal solution of a LPP are restricted to assume non-negative integer values while the remaining variables are free to take any non-negative values then it is called a mixed integer programming problem (Mixed I.P.P).

Further if all the variables in the optimal solution are allowed to take values 0 or 1, then the problem is called the 0–1 programming problem or Standard discrete programming problem.

The general integer programming problem is given by Max Z = CX

Subject to the constraints

$A \ x \leq b$

$x \geq 0$ and some or all variables are integers.

## 3.7 The Cutting-Plane Algorithm

**Q13. Explain the concept of Cutting-Plane Algorithm.**

*Ans :* **(Imp.)**

**Gomory's All Integer Programming Method**

In this technique, we first find the optimum solution of the given I.P.P. by regular simplex method as discussed earlier, disregarding the integer condition of variables. Then, we observe the following :

(i) If all the variables in the optimum solution thus obtained have integer values, the current solution will be the desired optimum integer solution.

(ii) If not, the considered L.P.P. requires modification by introducing a secondary constraint (also called Gomory's constraint) that reduces some of the non-integer values of variables to integer one, but does not eliminate any feasible integer.

(iii) Then, an optimum solution to this modified L.P.P. is obtained by using any standard algorithm. If all the variables in this solution are integers, then the optimum integer solution is obtained. Observe, another secondary constraint is added to the L.P.P. and the entire procedure is repeated.

In this way, the optimum integer solution will be obtained eventually after introducing the sufficient number of new constraints. Thus, it becomes specially important to discuss below – how the additional constraints (Gomory's constraints) are constructed.

**How to Construct Gomory's Constraint**

The secondary constraints which will force the solution toward an all-integer point are constructed as follows :

Let the optimum non-integer solution to the maximization L.P.P. has been obtained. In our usual notations, this solution can be shown by the following optimal simplex table.

**Table 1**

| BASIC VAR. | $C_B$ | $X_B$ | BASIC | | | | | | NON-BASIC | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $X_1$ $(\beta_1)$ | $X_2$ $(\beta_2)$ | ... | $X_i$ $(\beta_i)$ | ... | $X_m$ $(\beta_m)$ | $X_{m+1}$ | ... | $X_n$ | |
| $x_1$ | $c_{B1}$ | $x_{B1}$ | 1 | 0 | ... | 0 | ... | 0 | $x_{1,m+1}$ | ... | $x_{1n}$ | |
| $x_2$ | $c_{B2}$ | $x_{B2}$ | 0 | 1 | ... | 0 | ... | 0 | $x_{2,m+1}$ | ... | $x_{2n}$ | |
| : | : | : | : | : | | : | | : | : | | : | |
| $\rightarrow x_i$ | $c_{Bi}$ | $x_{Bi}$ | 0 | 0 | ... | 1 | ... | 0 | $x_{i,m+1}$ | ... | $x_{in}$ | |
| : | : | : | : | : | : | : | : | : | : | : | : | |
| $x_m$ | $c_{Bm}$ | $x_{Bm}$ | 0 | 0 | ... | 0 | ... | 1 | $x_{m,m+1}$ | .. | $x_{mn}$ | |
| · | $z = C_B X_B$ | | 0 | 0 | ... | 0 | ... | 0 | $\Delta_{m+1}$ | ... | $\Delta_n$ | $\leftarrow \Delta_j$ |

In this table, the variable ($x_{Bi}$, i = 1, 2, ..., m) represent the basic variables and the remaining (n – m) variables $x_{m+1}, x_{m+2}, ..., x_n$ are the non-basic variables. However, these variables have been arranged in this order, for our convenience.

Let the ith basic variable $x_{Bi}$ possesses a non-integer value which is given by the constraint equation.

$$x_{Bi} = 0x_1 + 0x_2 + ... 1x_1 + ... + 0x_m + x_{i,m+1}\, x_{m+1} + ... + x_{in}\, x_n$$

or $$x_{Bi} = x_i + \sum_{j=m+1}^{n} x_{ij}\, x_j$$

or $$x_i = x_{Bi} + \sum_{j=m+1}^{n} x_{ij}\, x_j \quad ... (1)$$

Now, let $\boxed{x_{Bi} = I_{Bi} + f_{Bi}}$ and $x_{ij} = I_{ij} = f_{ij}$, where $I_{Bi}$ and $I_{ij}$ are the largest integer parts of $x_{Bi}$ and $x_{ij}$, respectively, such that $I_{Bi} \le x_{Bi}$ and $I_{ij} \le x_{ij}$. It follows that $0 < f_{Bi} < 1$ and $0 \le f_{ij} < 1$; that is, $f_{Bi}$ is strictly positive fraction while $f_{ij}$ is a non-negative fraction. For example,

| $x_a$ | $I_a$ | $f_a = x_a - I_a$ |
|---|---|---|
| $2\frac{1}{2}$ | 2 | $\frac{1}{2}$ |
| $-1\frac{1}{3}$ | –2 | $\frac{2}{3}$ |
| –2 | –2 | 0 |
| $-\frac{3}{5}$ | –1 | $\frac{2}{5}$ |



Now substituting above values in eqn. (1) for $x_i$, we get

$$x_i = (I_{Bi} + f_{Bi}) - \sum_{j=m+1}^{n} (I_{Bi} + f_{Bi})\, x_j \quad ... (2)$$

or $$f_{Bi} = \sum_{j=m+1}^{n} f_{ij}\, x_j = x_i - I_{Bi} + \sum_{j=m+1}^{n} I_{ij}\, x_j \quad ... (3)$$

Now for all the variables $x_i$ (i = 1, 2, ..., m) and $x_j$ (j = m + 1, ..., n) to be integer valued, the right hand side of the above equation must be an integer. This implies that left-hand side of (3), i.e.,

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j$$

must also be an integer. Since $0 < f_{Bi} < 1$ and $\sum_{j=m+1}^{n} f_{ij}\, x_j \ge 0$, it follows that the equality condition is satisfied if

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j \le 0. \quad ... (4)$$

This is true because $f_{Bi} = \sum_{j=m+1}^{n} f_{ij}\, x_j \le f_{Bi} < 1$.

But, since $f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j$ is an integer, then it can be either a zero or a negative integer.

Now the constraint (4) can be put in the form

$$f_{Bi} = \sum_{j=m+1}^{n} f_{ij}\, x_j + g_i = 0$$

or

$$\boxed{-f_{Bi} = -\sum_{j-m+1}^{n} f_{ij}\, x_j + g_i} \qquad \ldots (5)$$

Where $g_i$ is a non-negative Gomorian slack variable which by definition must also be an integer. The constraints equation (5) defines the so-called Gomory's cutting plane. From Table 1, the non-basic variables $x_j = 0$ (j = m + 1, ..., n) and thus by virtue of (5) it is clearly infeasible. Thus in order to clear this infeasibility, we have no alternative except to use the dual simplex method (as described earlier). Practically, that is equivalent to cutting of the solution space towards the optimal integer solution.

Now, after adding the Gomory's constraint (5) the optimum simplex Table 1 take the form :

| BASIC VAR. | $X_B$ | $X_1$ $(\beta_1)$ | $X_2$ $(\beta_2)$ | ... | $X_i$ $(\beta_i)$ | ... | $X_m$ $(\beta_m)$ | $X_{m+1}$ | ... | $X_n$ | $G_1$ $(\beta_{m+1})$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | $x_{B1}$ | 1 | 0 | ... | 0 | ... | 0 | $x_{1,m+1'}$ | ... | $x_{1n}$ | 0 |
| $x_2$ | $x_{B2}$ | 0 | 1 | ... | 0 | ... | 0 | $x_{2,m`+1}$ | ... | $x_{2n}$ | 0 |
| : | : | : | : | : | : | : | : | : | : | : | : |
| $\rightarrow x_i$ | $x_{Bi}$ | 0 | 0 | .. | 1 | ... | 0 | $x_{i,m+1}$ | ... | $x_{in}$ | 0 |
| : | : | : | : | | | | | : | | : | : |
| $x_m$ | $x_{Bm}$ | 0 | 0 | ... | 0 | ... | 1 | $x_{m,m+1}$ | ... | $x_{mn}$ | 0 |
| $g_i$ | $-f_{Bi}$ | **0** | **0** | **...** | **0** | **...** | **1** | $-f_{B,m+1}$ | ... | $-f_{Bm}$ | **1** |
| | $z$ | 0 | 0 | ... | 0 | ... | 0 | $\Delta_{m+1}$ | ... | $\Delta_n$ | ...0 |

$\leftarrow \Delta_j$

If the new solution (after applying the dual simplex method) is all-integer one, the process ends. Otherwise, second Gomory's constraint is again constructed from the resulting optimal table and then the dual simplex method is again used to clear the infeasibility. This process is repeated so long as an all integer solution is obtained. However, if at any iteration, the dual simplex algorithm indicates that no feasible solution exists then the problem has no feasible integer solution.

**Gomory's Cutting-plane (All I.P.P.) Algorithm**

The step-by-step procedure for the solution of all-integer programming problem is as follows :

**Step 1 :** If the I.P.P. is in the minimization form, convert it to maximization form.

**Step 2 :** Then convert the inequalities into equations by introducing slack and/or surplus variables (if necessary) and obtain the optimum solution of the L.P.P. (after ignoring the integer condition) by usual simplex method.

**Step 3 :** Now, test the integerality of the optimum solution obtained in Step 2.

(i) If the optimum solution contains all integer values, then an optimum integer basic feasible solution has been achieved.

(ii) If not, go to next step.

**Step 4 :** Examine the constraint equations corresponding to the current optimal solution. Let these constraints be expressed by

$$x_{Bi} = x_i + \sum_{j-m+1}^{n} x_{ij} x_j \quad (i = 1, 2, ..., m).$$

Select the largest fraction of $x_{Bi}$'s i. e. find max $[f_{Bi}]$. Let it be $f_{Bk}$ for $i = k$.

**Step 5 :** Express the negative fraction, if any in the &th row of the optimum simplex table, as the sum of a negative integer and a non-negative fraction.

**Step 6 :** At this stage, construct the Gomorian constraint:

$$f_{Bi} - \sum_{j-m+1}^{n} f_{ij} x_j \leq 0.$$

as described in the preceding section, and then introduce the Gomorian equation

$$-f_{Bi} = -\sum_{j-m+1}^{n} f_{ij} x_j + g_i$$

to the current set of equality constraints.

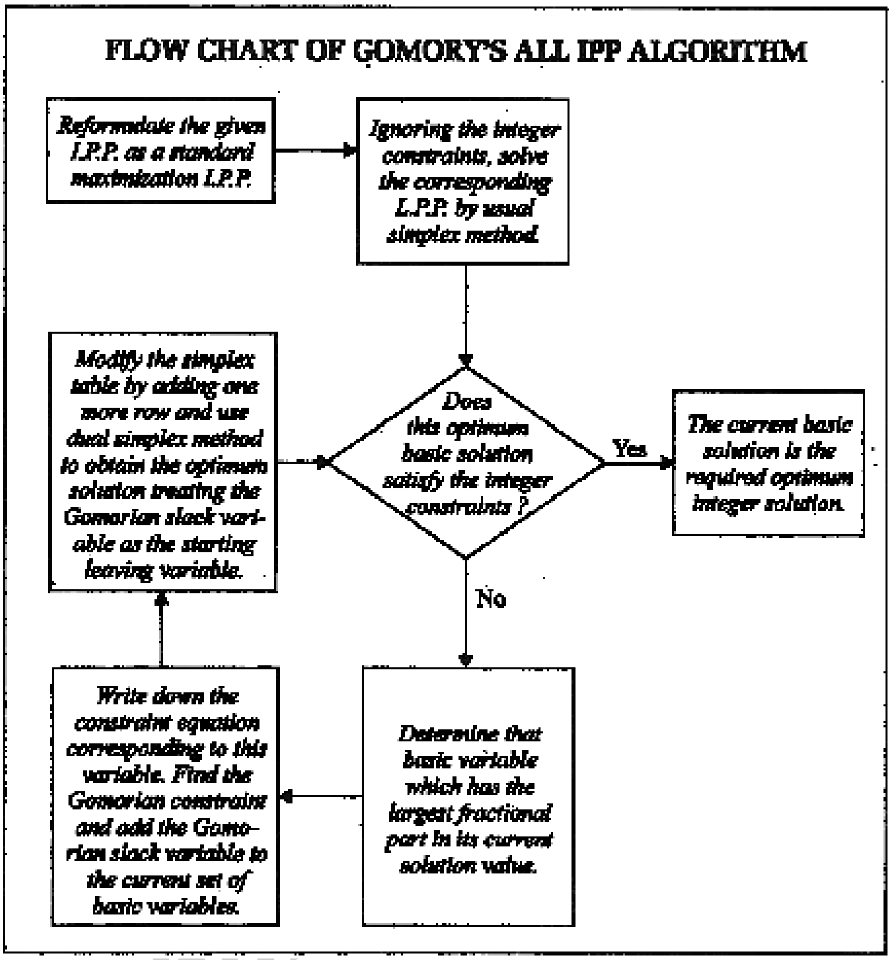
**Step 7 :** Starting with this new set of constraint equations, obtain the new optimum solution by using dual simplex method in order to clear infeasibility. The slack variable $g_i$ will be the initial leaving basic variable.

**Step 8 :** Now two possibilities may arise :

(i) If this new optimum solution for the Modified LPP is an all-integer solution, it is also feasible and optimum for the given LPP.

(ii) Otherwise, we return to step-4 and repeat the entire process until an optimum feasible integer solution is obtained.

All above steps of Gomory's algorithm can be more precisely demonstrated by the following flow chart :




**PROBLEMS**

**11. Solve the integer programming problem :**

**Maximize $z = 7x_1 + 9x_2$**

**Subject to Constraints :**

**$-x_1 + 3x_2 \leq 6,\ 7x_1 + x_2 \leq 35,\ x_1 \geq 0, x_2 \geq 0$, and integers**

*Sol :*

**Step 1 :** Since the problem is already given in standard maximiation form, we go to the next step.

**Step 2 :** Introducing the slack variables, we get the constant equations

$$-x_1 + 3x_2 + x_3 = 6$$

$$7x_1 + x_2 + x_4 = 35$$

Now ignoring the integer conditions and then using the regular simplex method we get the following set of tables. The optimal table to the LPP is given below.

**Optimal Table**

| BASIC VARIABLES | $C_B$ | $X_B$ | $X_1$ (7) | $X_2$ (9) | $X_3$ (0) | $X_4$ (0) | MIN. $(X_B / X_k)$, $X_k > 0$ |
|---|---|---|---|---|---|---|---|
| $\leftarrow x_3$ | 0 | 6 | −1 | [3] | 1 | 0 | $6/3 \leftarrow$ |
| $x_4$ | 0 | 35 | 7 | 1 | 0 | 1 | 35/1 |
| | $z = C_B X_B = 0$ | | −7 | −9 ↑ | 0 ↓ | 0 | $\leftarrow \Delta_j$ |
| $x_2$ | 9 | 2 | −1/3 | 1 | 1/3 | 0 | — |
| $\leftarrow x_4$ | 0 | 33 | [22/3] | 0 | −1/3 | 1 | $33/(22/3) \leftarrow$ |
| | $z = 18$ | | −10 ↑ | 0 | 3 | 0 ↓ | $\leftarrow \Delta_j$ |
| $x_2$ | 9 | $3\frac{1}{2}$ | 0 | 1 | 7/22 | 1/22 | |
| $x_1$ | 7 | $4\frac{1}{2}$ | 1 | 0 | −1/22 | 3/22 | |
| | $z = C_B X_B = 63$ | | 0 | 0 | 28/11 | 15/11 | $\leftarrow \Delta_j$ |

The optimum solution thus obtained is :

$$x_1 = 4\frac{1}{2},\ x_2 = 3\frac{1}{2},\ z = 63.$$

**Step 3 :** Since the optimum solution obtained as above is not an integer solution because of $x_1 = 4\frac{1}{2}$, we go to the next step.

**Step 4 :** We now select the constraint corresponding to max $(f_{Bi})$ = max. $(f_{B1}, f_{B2})$.

Since $x_{B1} = 1_{B1} + f_{B1} = 3 + \frac{1}{2}$, and $x_{B2} = 1_{B2} + f_{B2} = 4 + \frac{1}{2}$,

we have $f_{B1} = f_{B2} = \frac{1}{2}$.

Hence, max $(f_{B1}, f_{B2})$ = max. $\left[\frac{1}{2}, \frac{1}{2}\right] = \frac{1}{2}$.

Thus, in the problem, since both the equation have the same value of $f_{Bi}$, that is, $f_{Bi} = f_{B2} = \frac{1}{2}$, either one of the two equations can eb used. Let us consider the $x_2$ - equation, i.e., first-row of optimum table.

**Step 5 :** Negative fraction does not exist.

**Step 6 :** To construct the Gomorian Constraint

The Gomorian constraint is given by,

$$-f_{Bi} = -\sum_{j=m+1}^{n} f_{ij} x_j + g_i$$

Here $m = 2$, $n = 4$, $i = 1$, $f_{B1} = \frac{1}{2}$.

Thus above constraint becomes :

$$-f_{B1} = -\sum_{j=3}^{4} f_{ij}\, x_j + g_1$$

or $$-f_{B1} = -\, f_{13}x_3 - f_{14}x_4 + g_1$$

(since $x_3$, $x_4$ are slack variables)

Substituting the values :

$f_{13} = 7/22$, $f_{14} = 1/22$, $f_{B1} = 1/2$,

we get the required Gomorian Constraint as

$$-\frac{1}{2} = -\frac{7}{22}x_3 - \frac{1}{22}x_4 + g_1$$

($x_3 = x_4 = 0$, being non-basic)

Obviously, the coefficients of remaining variables $x_1$ and $x_2$ in the above Gomorian constraint will be taken 0. Thus complete Gomorian Constraint can be written as

$$-\frac{1}{2} = 0x_1 + 0x_2 - \frac{7}{22}x_3 - \frac{1}{22}x_4 + g_1,$$

Adding this new constraint to the above Optimal Table , we get the new table as below :

| | | $c_j \rightarrow$ | 7 | 9 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| **BASIC VARIABLES** | $C_B$ | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $G_1$ | |
| $x_2$ | 9 | $3\frac{1}{2}$ | 0 | 1 | 7/22 | 1/22 | 0 | |
| $x_1$ | 7 | $4\frac{1}{2}$ | 1 | 0 | –1/22 | 3/22 | 0 | |
| $g_1$ | 0 | → – 1/2 | 0 | 0 | [–7/22] | –1/22 | 1 | |
| | | $z = C_B X_B = 63$ | 0 | 0 | 28/11 ↑ | 15/11 | 0 ↓ | $\leftarrow \Delta_j$ |

**Step 7 :** To apply dual simplex method.

i) leaving vector is $G_1$, i.e., $\beta_3$. Therefore $r = 3$.

ii) Entering vector is obtained by

$$\frac{\Delta_k}{x_{rk}} = \max.\left[\frac{\Delta_3}{x_{33}}, \frac{\Delta_4}{x_{34}}\right] = \max\left[\frac{28/11}{-7/22}, \frac{15/11}{-1/22}\right]$$

$$= \max.\ [-8, -30] = 8 = \frac{\Delta_3}{x_{33}}.$$

Therefore, k = 3, Hence we must enter the vector a3 corresponding to which $X_3$ is given in the above table.

Thus, we get the following transformed table.

| BASIC VARIABLES | $C_B$ | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $G_1$ |
|---|---|---|---|---|---|---|---|
| | | $c_j \rightarrow$ | 7 | 9 | 0 | 0 | 0 |
| $x_2$ | 9 | 3 | 0 | 1 | 0 | 0 | 1 |
| $x_1$ | 7 | $4\frac{4}{7}$ | 1 | 0 | 0 | 1/7 | −1/7 |
| $x_3$ | 0 | $1\frac{4}{7}$ | 0 | 0 | 1 | 1/7 | −22/7 |
| | $z = C_{X_B} = 59$ | | 0 | 0 | 0 | 1 | 8 ← $\Delta_j$ |

$$\Delta_4 = C_B X_4 - c_4 = (9,7,0)\left(0, \frac{1}{7}, \frac{1}{7}\right) - 0 = (0 + 1 + 0) = 1$$

$$\Delta_5 = C_B G_1 - c_5 = (9, 7, 0)\left(1, -\frac{1}{7}, -\frac{22}{7}\right) - 0 = (9 - 1 + 0) = 8$$

The non-integer optimum solution given by above table is :

$$x_1 = 4\frac{4}{7},\ x_2 = 3,\ x_3 = 1\frac{4}{7},\ z = 59.$$

**Step 8 :** The optimal solution as obtained above by dual simplex method is still non-integer. Thus, a new Gomory's constraint is to be constructed again. Selecting $x_1$ -equation (i.e., IInd row of above table) to generate the cutting plant (becuase largest fractional part can be $f_{B2}$ = $\left. f_{B3} = \frac{4}{7}\right)$, we get the Gomory's constraint as

$$-\frac{4}{7} = -\frac{1}{7}x_4 - \frac{6}{7}g_1 + g_2$$

or $$-\frac{4}{7} = 0x_1 + 0x_2 + 0x_3 - \frac{1}{7}x_4 - \frac{6}{7}g_1 + g_2$$

Adding this constraint to the above table we get the new table as below :

| BASIC VARIABLES | $C_B$ | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $G_1$ | $G_2$ |
|---|---|---|---|---|---|---|---|---|
| | | $c_j \rightarrow$ | 7 | 9 | 0 | 0 | 0 | 0 |
| $x_2$ | 9 | 3 | 0 | 1 | 0 | 0 | 1 | 0 |
| $x_1$ | 7 | $4\frac{4}{7}$ | 1 | 0 | 0 | 1/7 | −1/7 | 0 |
| $x_3$ | 0 | $1\frac{4}{7}$ | 0 | 0 | 1 | 1/7 | −22/7 | 0 |
| $g_2$ | 0 | −4/7 | 0 | 0 | 0 | [−1/7] | −6/7 | 1 |
| | $z = C_B X_B = 59$ | | 0 | 0 | 0 | 1 ↑ | 8 | 0 ↓ ← $\Delta_j$ |

We again apply dual simplex method.

i) Leaving vector in $G_2$ (i.e. $\beta_4$). Therefore, r = 4.

ii) Entering vector will be obtained by

$$\frac{\Delta_k}{x_{4k}} = \max.\left[\frac{\Delta_4}{x_{44}}, \frac{\Delta_5}{x_{45}}\right] = \max.\left[\frac{1}{-1/7}, \frac{8}{-6/7}\right]$$

$$= \max.\left[-7, -9\frac{1}{3}\right] = -7 = \frac{\Delta_4}{x_{44}}.$$

Therefore, $k$ = 4. Hence we must enter $a_4$ corresponding to which $X_4$ given in the above table. Thus we get the transformed table as below :

| | $c_j \rightarrow$ | | 7 | 9 | 0 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| **BASIC VAR.** | $C_B$ | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $G_1$ | $G_2$ | |
| $x_2$ | 9 | 3 | 0 | 1 | 0 | 0 | 1 | 0 | |
| $x_1$ | 7 | 4 | 1 | 0 | 0 | 0 | –1 | 1 | |
| $x_3$ | 0 | 1 | 0 | 0 | 1 | 0 | –4 | 1 | |
| $x_4$ | 0 | 4 | 0 | 0 | 0 | 1 | 6 | –7 | |
| | $z = C_B X_B = 55$ | | 0 | 0 | 0 | 0 | 2 | 7 | $\leftarrow \Delta_j$ |

$\Delta_5$ = $C_B G_1 - c_5$ = (9, 7, 0, 0) (1, –1, –4, 6) – 0

= (9 – 7 + 0 + 0) = 2

$\Delta_6$ = $C_B G_2 - c_6$ = (9, 7, 0, 0) (0, 1, 1, – 7) – 0

= (0 + 7+ 0 + 0) = 7

Thus, finally we get the optimal integer solution :

$x_1 = 4, x_2 = 3$, max z = 55

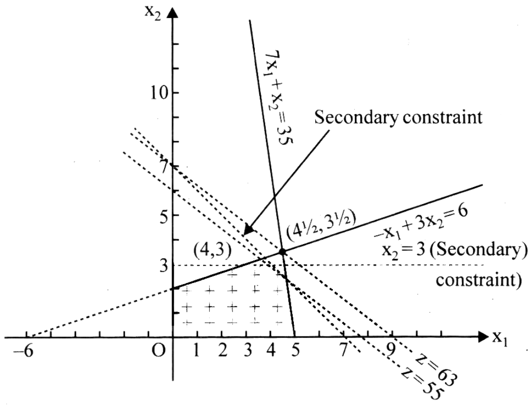
**Verification by Graphical Method.** It can be easily verified that the addition of the above Gomory's constraints effectively 'cut' the solution space as desired. Thus the Gomory's first constraint :

$$-\frac{7}{22}x_3 - \frac{1}{22}x_4 + g_1 = -\frac{1}{2},$$

can be expressed in terms of $x_1$ and $x_2$ only by substituting :

$x_3 = 6 + x_1 - 3x_2$ and $x_4 = 35 - 7x_1 - x_2$

from the original constraint equations treating as a slack variable in step 2.




This gives $g_1 + x_2 = 3$ or $x_2 \le 3$, treating $g_1$ as a slack variable.

Similarly, for the Gomory's second constraint, $-\frac{1}{7}x_3 - \frac{6}{7}g_1 + g_2 = -\frac{4}{7}$, the equivalent constraint in terms of $x_1$ and $x_2$ is obtained as $x_1 + x_2 < 7$.

Now plotting the Gomory's constraints $x_2 \le 3$ and $x_1 + x_2 \le 7$ in addition to the constraints of the given problem, we find that it will result in the new (optimal) extreme point (4, 3) as shown in the figure.

**12. Find the optimum integer solution to the following all LPP**

Max. $z = x_1 + 2x_2$ **subject to the constraints :**

$2x_2 \le 7$, $x_1 + x_2 \le 7$, $2x_1 \le 11$, $x_1 \ge 0, x_2 \ge 0$, **and $x_1$, $x_2$ are integers.**

*Sol :*

**Step 1.** Introducing the slack variables, we get

$$2x_2 + x_3 = 7$$
$$x_1 + x_2 + x_4 = 7$$
$$2x_1 + x_5 = 11$$
$$x_1, x_2, x_3, x_4, x_5 \ge 0.$$

**Step 2.** Ignoring the integer condition, we get the initial simplex table as follows :

**Initial Simplex Table**

| | | $c_j \rightarrow$ | 1 | 2 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| **BASIC VAR.** | $C_B$ | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | **MIN. RATIO** $(X_B / X_2)$, $X_2 > 0$ |
| $\leftarrow x_3$ | 0 | 7 | 0 | 2 | 1 | 0 | 0 | 7 / 2 ← |
| $x_4$ | 0 | 7 | 1 | 1 | 0 | 1 | 0 | 7 / 1 |
| $x_5$ | 0 | 11 | 2 | 0 | 0 | 0 | 1 | — |
| | $z = 0$ | | –1 | –2 ↑ | 0 ↓ | 0 | 0 | $\leftarrow \Delta_j$ |

Introducing $X_2$ and leaving $X_3$ from the basis, we get the following table:

| | | $c_j \rightarrow$ | 1 | 2 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| **BASIC VAR.** | $\mathbf{C_B}$ | $\mathbf{X_B}$ | $\mathbf{X_1}$ | $\mathbf{X_2}$ | $\mathbf{X_3}$ | $\mathbf{X_4}$ | $\mathbf{X_5}$ | **MIN. $(X_B/X_k)$, $X_k > 0$** |
| $x_2$ | 2 | $3\frac{1}{2}$ | 0 | 1 | 1/2 | 0 | 0 | — |
| $\leftarrow x_4$ | 0 | $3\frac{1}{2}$ | [1] | 0 | – 1/2 | 1 | 0 | $3\frac{1}{2}/1 \leftarrow$ |
| $x_5$ | 0 | 11 | 2 | 0 | 0 | 0 | 1 | 11/2 |
| | $z = \mathbf{C_B X_B} = 7$ | | $-1$ ↑ | 0 | 1 | 0 ↓ | 0 | $\leftarrow \Delta_j$ |

$\Delta_1 = C_B X_1 - c, = (2, 0, 0)\ (0,1, 2) - 1 = -1,$

$\Delta_3 = C_B X_3 - c_3 = (2, 0, 0) \left(\frac{1}{2}, \frac{1}{2}, 0\right) - 0 = 1.$

Introducing $X_1$ and leaving $X_4$ , we get the following optimal table.

**Optimal Table**

| | | $c_j \rightarrow$ | 1 | 2 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| **BASIC VAR.** | $\mathbf{C_B}$ | $\mathbf{X_B}$ | $\mathbf{X_1}$ | $\mathbf{X_2}$ | $\mathbf{X_3}$ | $\mathbf{X_4}$ | $\mathbf{X_5}$ | |
| $x_2$ | 2 | $3\frac{1}{2}$ | 0 | 1 | 1/2 | 0 | 0 | |
| $x_1$ | 1 | $3\frac{1}{2}$ | 1 | 0 | – 1/2 | 1 | 0 | |
| $x_5$ | 0 | 4 | 0 | 0 | 1 | – 2 | 1 | |
| | $z = 10\frac{1}{2}$ | | 0 | 0 | 1/2 | 1 | 0 | $\leftarrow \Delta_j$ |

$$\Delta_3 = C_B X_3 - c_3 = (2,\ 1,\ 0)\left(\frac{1}{2}, -\frac{1}{2}, 1\right) - 0 = \left(1 - \frac{1}{2} + 0\right) = \frac{1}{2}$$

$$\Delta_4 = C_B X_4 - c_4 = (2,\ 1,\ 0)\ (0,\ 1,\ -2) - 0 = (0 + 1 + 0) = 1$$

The optimum solution thus obtained is :

$$x_1 = 3\frac{1}{2}, x_2 = 3\frac{1}{2}, z = 10\frac{1}{2}.$$

**Step 3.** Since the optimum solution obtained above is not an integer solution, we must go to next step.

**Step 4.** We now select the constraint corresponding to the criterion

$$\max_i\ (f_{Bi}) = \max\ (f_{B1}, f_{B2}, f_{B3}) = \max\left(\frac{1}{2}, \frac{1}{2}, 0\right) = \frac{1}{2}$$

Since in this problem, the $x_2$ - equation and $x_1$ -equation both have the same value of $f_{Bi}$, i. e. $\frac{1}{2}$, either one of the two equations can be used. Let us consider the first-row of the above optimum table.

The Gomory's constraint to be added is therefore

$$-\sum_{j=3,4} f_{1j}x_j + g_1 = -f_{B1}$$

or $\quad -f_{13}x_3 - f_{14}x_4 + g_1 = -f_{B1}$

or $\quad -\frac{1}{2}x_3 + g_1 = -\frac{1}{2} (x_3 = x_4 = 0)$

Adding this new constraint to above optimal *table,* we get the new table as below :

**New table after adding Gomory constraint**

| | | $c_j \rightarrow$ | 1 | 2 | 0 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| **BASIC VAR.** | $C_B$ | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $G_1$ | |
| $x_2$ | 2 | $3\frac{1}{2}$ | 0 | 1 | 1/2 | 0 | 0 | 0 | |
| $x_1$ | 1 | $3\frac{1}{2}$ | 1 | 0 | −1/2 | 1 | 0 | 0 | |
| $x_5$ | 0 | 4 | 0 | 0 | 1 | −2 | 1 | 0 | |
| $g_1$ | 0 | →−1/2 | 0 | 0 | [−1/2] | 0 | 0 | 1 | |
| | | $z = C_B X_B = 10\frac{1}{2}$ | 0 | 0 | 1/2 ↑ | 1 | 0 | 0 ↓ | $\leftarrow \Delta_j$ |

**Step 5. To apply dual simplex method.** Now, in order to remove the infeasibility of the optimum solution :

$x_1 = 3\frac{1}{2}, x_2 = 3\frac{1}{2}, x_5 = 4, g_1 = -\frac{1}{2},$

we use the dual simplex method.

i) Leaving vector is $G_1$ (i.e., $\beta_4$). Therefore, r = 4.

ii) Entering vector is given by

$$\frac{\Delta_k}{x_{4k}} = \max.\left[\frac{\Delta_j}{x_{4j}}, x_{4j} < 0\right]$$

$$= \max.\left[\frac{\Delta_3}{x_{43}}\right] = \max.\left[\frac{\frac{1}{2}}{-\frac{1}{2}}\right] = \frac{\Delta_3}{x_{43}}.$$

Therefore, k = 3. So we must enter $a_3$ corresponding to which $X_3$ is given in the above table. Thus, dropping $G_1$ and introducing $X_3$ we get the following dual simplex table :

| | | $c_j \rightarrow$ | 1 | 2 | 0 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| **BASIC VAR.** | $\mathbf{C}_B$ | $\mathbf{X}_B$ | $\mathbf{X}_1$ | $\mathbf{X}_2$ | $\mathbf{X}_3$ | $\mathbf{X}_4$ | $\mathbf{X}_5$ | $\mathbf{G}_1$ | |
| $x_2$ | 2 | 3 | 0 | 1 | 0 | 0 | 0 | 1 | |
| $x_1$ | 1 | 4 | 1 | 0 | 0 | 1 | 0 | −1 | |
| $x_5$ | 0 | 3 | 0 | 0 | 0 | −2 | 1 | 2 | |
| $x_3$ | 0 | 1 | 0 | 0 | 1 | 0 | 0 | −2 | |
| | $z = \mathbf{C_B X_B} = 10$ | | 0 | 0 | 0 | 1 | 0 | 1 | $\leftarrow \Delta_j$ |

$\Delta_4 \;=\; C_B X_4 - c_4 = (2, 1, 0, 0)\,(0, 1, -2, 0) - 0 = 1$

$\Delta_6 \;=\; C_B G_1 - c_6 = (2, 1, 0, 0)\,(1, -1, 2, -2) - 0 = 1.$

This shows that the optimum feasible solution has been obtained in integers. Thus, finally, we get the integer optimuin solution to the given I.P.P. as : $x_x = 4$, $x_2 = 3$, and max $z = 10$.

**13. Solve the following integer programming problem:**

**Max. $z = 2x_1 + 20x_2 - 10x_3$ subject to the constraints:**

$$2x_1 + 20x_2 + 4x_3 < 15,$$

$$6x_l + 20x_2 + 4x_3 = 20,$$

**and $x_1, x_2, x_3 > 0$; and are integers.**

**Solve the problem as a (continuous) linear program, then show that it is impossible to obtain feasible integer solution by using simple rounding. Solve the problem using any integer program algorithm.**

*Sol :*

Introducing the slack variable $x_4 \geq 0$ and an artificial variable $a_t \geq 0$, an initial basic feasible solution is $x_4 = 15$ and $a_1 = 20$.

Now computing the net-evaluations ($\Delta_j$) and then using simplex method, the following optimal simplex table is obtained.

| | | $c_j \rightarrow$ | 2 | 20 | −10 | 0 | |
|---|---|---|---|---|---|---|---|
| **BASIC VARIABLES** | $\mathbf{C}_B$ | $\mathbf{X}_B$ | $\mathbf{X}_1$ | $\mathbf{X}_2$ | $\mathbf{X}_3$ | $\mathbf{X}_4$ | |
| $x_2$ | 20 | 5/8 | 0 | 1 | 1/5 | 3/40 | |
| $x_1$ | 2 | 5/4 | 1 | 0 | 0 | −1/4 | |
| | $z = 15$ | | 0 | 0 | 14 | 1 | $\leftarrow \Delta_j$ |

Thus the following non-integer optimum solution is obtained:

$$x_x = 5/4,\ x_2 = 5/8,\ x_3 = 0,\ \max z = 15.$$

The rounded solution will be $x_1 = 1$, $x_2 = 0$, $x_3 = 0$.

Since this solution satisfies the first constraint only, it is not possible to obtain a feasible solution by using simple rounding. So to obtain the integer-valued solution, we proceed as follows :

$$\text{Max.}(f_{B1}, f_{B2}) = \text{Max.}\left(\frac{5}{8}, \frac{1}{4}\right) = \frac{5}{8}$$

Therefore, from the first row of optimal table, we have

$$\frac{5}{8} = 0x_1 + x_2 + \frac{1}{5}x_3 + \left(\frac{5}{8}, \frac{1}{4}\right) = \frac{3}{40}\ x_4$$

$$\text{or } \left(0 + \frac{5}{8}\right) = (0 + 0)\ x_1 + (1 + 0)\ x_2 + \left(0 + \frac{1}{5}\right) x_3 + \left(0 + \frac{3}{40}\right) x_4$$

The corresponding fractional cut will be

$$-\frac{5}{8} = 0x_1 + 0x_2 - \frac{1}{5}x_3 - \frac{3}{40}x_4 + g_1$$

Now inserting the additional constraint in the optimum simplex table, the following modified table is obtained.

| | | $c_j \rightarrow$ | 2 | 20 | – 10 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| **BASIC VARIABLES** | $\mathbf{C}_B$ | $\mathbf{X}_B$ | $\mathbf{X}_1$ | $\mathbf{X}_2$ | $\mathbf{X}_3$ | $\mathbf{X}_4$ | $\mathbf{G}_1$ | |
| $x_2$ | 20 | 5/8 | 0 | 1 | 1/5 | 3/40 | 0 | |
| $x_1$ | 2 | 5/4 | 1 | 0 | 0 | – 1/4 | 0 | |
| $g_1$ | 0 ↓ | – 5/8 | 0 | 0 | – 1/5 | [– 3/40] | 1 | |
| | $z = 15$ | | 0 | 0 | 14 | 1 ↑ | 0 ↓ | $\leftarrow \Delta_j$ |

First Iteration. Remove $G_1$ and insert $X_4$ by dual simplex method to get the following table :

| | | $c_j \rightarrow$ | 2 | 20 | – 10 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| **BASIC VARIABLES** | $\mathbf{C}_B$ | $\mathbf{X}_B$ | $\mathbf{X}_1$ | $\mathbf{X}_2$ | $\mathbf{X}_3$ | $\mathbf{X}_4$ | $\mathbf{G}_1$ | |
| $x_2$ | 20 | 0 | 0 | 1 | 0 | 0 | 1 | |
| | 2 | 10/3 | 1 | 0 | 2/3 | 0 | – 10/3 | |
| $x_4$ | 0 | 25/3 | 0 | 0 | 8/3 | 1 | – 40/3 | |
| | $z = 20/3$ | | 0 | 0 | 34/3 | 0 | 40/3 | $\leftarrow \Delta_j$ |

Again, since the solution is non-integer one, insert one more fractional cut. From the third row of above table

$$25/3 = 8/3\,x_3 + x_4 - 40/3\,g_1$$

or

$$(8 + 1/3) = (2 + 2/3)\,x_3 + (1 + 0)\,x_4 + (-14 + 2/3)\,g_1$$

The corresponding fractional cut will be $-1/3 = 0x_1 + 0x_2 - 2/3\,x_3 + 0x_4 - 2/3\,g_1 + g_2$.

Inserting this constraint in above table, the following modified table is obtained

| BASIC VAR. | $C_B$ | $X_B$ ($c_j \rightarrow$) | $X_1$ (2) | $X_2$ (20) | $X_3$ (−10) | $X_4$ (0) | $G_1$ (0) | $G_2$ (0) |
|---|---|---|---|---|---|---|---|---|
| $x_2$ | 20 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| $x_1$ | 2 | 10/3 | 1 | 0 | 2/3 | 0 | −10/3 | 0 |
| $x_4$ | 0 | 25/3 | 0 | 0 | 8/3 | 1 | −40/3 | 0 |
| $g_2$ | 0 | → −1/3 | 0 | 0 | [−2/3] | 0 | −2/3 | 1 |
| | $z = 20/3$ | | 0 | 0 | 34/3 ↑ | 0 | 40/3 | 0 ↓ ← $\Delta_j$ |

Second Iteratin, using dual simplex method remove $G_2$ and introduce $X_3$ as given in the table below :

| BASIC VAR. | $C_B$ | $X_B$ ($c_j \rightarrow$) | $X_1$ (2) | $X_2$ (20) | $X_3$ (−10) | $X_4$ (0) | $G_1$ (0) | $G_2$ (0) |
|---|---|---|---|---|---|---|---|---|
| $x_2$ | 20 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| $x_1$ | 2 | 3 | 1 | 0 | 0 | 0 | −4 | 1 |
| $x_4$ | 0 | 7 | 0 | 0 | 0 | 1 | −16 | 4 |
| $x_3$ | −10 | 1/2 | 0 | 0 | 1 | 0 | 1 | −3/2 |
| | $z = 1$ | | 0 | 0 | 0 | 0 | 2 | 17 ← $\Delta_j$ |

Since the solution is still non-integer, a third fractional cut is required. From the last row of above table, we can construct the Gomarian constraint $-1/2 = -1/2\,g_2 + g_3$.

Inserting this additional constraint in the above table, the modified simplex table becomes as below

| BASIC VAR. | $C_B$ | $X_B$ ($c_j \rightarrow$) | $X_1$ (2) | $X_2$ (20) | $X_3$ (−10) | $X_4$ (0) | $G_1$ (0) | $G_2$ (0) | $G_3$ (0) |
|---|---|---|---|---|---|---|---|---|---|
| $x_2$ | 20 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| $x_1$ | 2 | 3 | 1 | 0 | 0 | 0 | −4 | 1 | 0 |
| $x_4$ | 0 | 7 | 0 | 0 | 0 | 1 | −16 | 4 | 0 |
| $x_3$ | −10 | 1/2 | 0 | 0 | 1 | 0 | 1 | [−3/2] | 0 |
| $g_3$ | 0→ | −1/2 | 0 | 0 | 0 | 0 | 0 | −1/2 | 1 |
| | $z = 1$ | | 0 | 0 | 0 | 0 | 2 | 17 ↑ | 0 ↓ ← $\Delta_j$ |

Third iteration. Using dual simplexd method, remove $G_3$ and introduce $G_2$ to obtain the following table :

| | | $c_j \rightarrow$ | 2 | 20 | −10 | 0 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|
| **BASIC VAR.** | $C_B$ | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $G_1$ | $G_2$ | $G_3$ | |
| $x_2$ | 20 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | |
| $x_1$ | 2 | 2 | 1 | 0 | 0 | 0 | −4 | 0 | 2 | |
| $x_4$ | 0 | 3 | 0 | 0 | 0 | 1 | −16 | 0 | 8 | |
| $x_3$ | −10 | 2 | 0 | 0 | 1 | 0 | 1 | 0 | −3 | |
| $g_2$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | −2 | |
| | $z = -16$ | | 0 | 0 | 0 | 0 | 2 | 0 | 34 | $\leftarrow \Delta_j$ |

Thus an optimum integer solution is obtained as $x_1 = 2, x_2 = 0, x_3 = 2$, max.z = −16.

**14. Find the optimum integer solution to the following LPP**

**Max $Z = x_2 + x_2$**

**Subject to constraints**

$$3x_1 + 2x_2 \le 5$$

$$x_2 \le 2$$

**$x_1, x_2 \ge 0$, and are integers.**

*Sol :* **(Imp.)**

Introducing the non-negative slack variables $S_1, S_2 \ge 0$, the standard form of the LPP becomes

Max $Z = x_1 + x_2\ 0S_1 + 0S_2$

Subject to $3x_1 + 2x_2 + S_1 = 5$

$0x_1 + x_2 + S_2 = 2$

$x_1, x_2, S_1, S_2 \ge 0$

Ignoring the integrality condition, solve the problem by simplex method. The initial basic feasible solution is given by

$S_1 = S$ and $S_2 = 2$

Since all $Z_j - C_j \ge 0$ an optimum solution is obtained, given by

Max $Z = 7/3$, $x_1 = 1/3$, $x_2 = 2$

To obtain an optimum integer solution, we have to add a fractional cut constraint in the optimum simplex table.

Since $x_B = 1/3$, the source row is the first row,

Expressing the negative fraction – 2/3 as a sum of negative integer and positive fraction, we get

$-2/3 = -1 + 1/3$

| | | $C_j$ | 1 | 1 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $Min\ \frac{x_B}{x_1}$ |
| ←0 | $S_1$ | 5 | ③ | 2 | 1 | 0 | 5/3 |
| 0 | $S_2$ | 2 | 0 | 1 | 0 | 1 | – |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | $Min\ \frac{x_B}{x_2}$ |
| | $Z_j–C_j$ | | –1↑ | –1 | 0 | 0 | |
| 1 | $x_1$ | 5/3 | 1 | 2/3 | 1/3 | 0 | 5/2=2–5 |
| ←0 | $S_2$ | 2 | 0 | ① | 0 | 1 | 2/1=2 |
| | $Z_j$ | 5/3 | 1 | 2/3 | 1/3 | 0 | |
| | $Z_j–C_j$ | | 0 | –1/3↑ | 1/3 | 0 | |
| 1 | $x_1$ | 1/3 | 1 | 0 | 1/3 | –2/3 | |
| 1 | $x_2$ | 2 | 0 | 1 | 0 | 1 | |
| | $Z_j$ | 7/3 | 1 | 1 | 1/3 | 1/3 | |
| | $Z_j – C_j$ | | 0 | 0 | 1/3 | 1/3 | |

Since $x_1$ is the source row, we have

$1/3 = x_1 + 1/3\ S_1 – 2/3\ S_2$

i.e., $1/3 = x_1 + 1/3\ S_1 + (–1 + 1/3)\ S_2$

The fractional cut (Gomorian) constraint is given by

$1/3\ S_1 + 1/3\ S_2 \geq 1/3 \Rightarrow –1/3\ S_1 – 1/3\ S_2 \leq – 1/3$

$\Rightarrow\ –1/3\ S_1 – 1/3\ S_2 + G_1 = –1/3$

where $G_1$ is the Gomorian slack. Add this fractional cut constraint at the bottom of the above optimal simplex table.



| | | $C_j$ | 1 | 1 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $G_1$ |
| 1 | $x_1$ | 1/3 | 1 | 0 | 1/3 | –2/3 | 0 |
| 1 | $x_2$ | 2 | 0 | 1 | 0 | 1 | 0 |
| ←0 | $G_1$ | –1/3 | 0 | 0 | (–1/3) | –1/3 | 1 |
| | $Z_j$ | 7/3 | 1 | 1 | 1/3 | 1/3 | 0 |
| | $Z_j–C_j$ | | 0 | 0 | 1/3↑ | 1/3 | 0 |

We apply dual simplex method. Since $G_1$ = –1/3, $G_1$ leaves the basis. To find the entering variable we find.

$$\text{Max}\left\{\frac{Z_j - C_j}{-a_{ij}},\ a_{ij} < 0\right\} = \text{Max}\left\{\frac{1/3}{-1/3}, \frac{1/3}{-1/3}\right\}$$

Max {–1, –1} = –1

We choose $S_1$ as the entering variable arbitrarily.

| | | $C_j$ | 1 | 1 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $G_1$ |
| 1 | $x_1$ | 0 | 1 | 0 | 0 | –1 | 1 |
| 1 | $x_2$ | 2 | 0 | 1 | 0 | 1 | 1 |
| 0 | $S_1$ | 1 | 0 | 0 | 1 | 1 | –3 |
| | $Z_j$ | 2 | 1 | 1 | 0 | 0 | 1 |
| | $Z_j$–$C_j$ | | 0 | 0 | 0 | 0 | 1 |

Since all $Z_j - C_j \geq 0$ and all $x_{Bi} \geq 0$ we obtain a optimal feasible integer solution.

∴ The optimum integer solution is

Max $z = 2, x_1 = 0, x_2 = 2$.

## 3.8 Branch-and-Bound Technique

**Q14. Explain the concept of Branch-and-Bound Technique.**

*Ans :*

If the number of decision variables in an integer programming problem is only two, a branch-and-bound technique can be used to find its solution graphically. Various terminologies of branch-and- bound technique are explained as under:



**Branching**

If the solution to the linear programming problem contains non-integer values for some or all decision variables, then the solution space is reduced by introducing constraints with respect to any one of those decision variables. If the value of a decision variable $X_1$ is 2.5, then two more problems will be created by using each of the following constraints.

$X_1 \leq 2$ and $X_1 \geq 3$

**Lower bound**

This is a limit to define a lower value for the objective function at each and every node. The *lower bound* at a node is the value of the objective function corresponding to the truncated values (integer parts) of the decision variables of the problem in that node.

**Upper bound**

This is a limit to define an upper value for the objective function at each and every node. The *upper bound* at a node is the value of the objective function corresponding to the linear programming solution in that node.

**Fathomed subproblem/node**

A problem is said to be *fathomed* if any one of the following three conditions is true:

1. The values of the decision variables of the problem are integer.
2. The upper bound of the problem which has non-integer values for its decision variables is not greater than the current best lower bound.
3. The problem has infeasible solution.

This means that further branching from this type of fathomed nodes is not necessary.

**Current Best Lower Bound**

This is the best lower bound (highest in the case of maximization problem and lowest in the case of minimization problem) among the lower bounds of all the fathomed lodes. Initially, it is assumed as infinity for the root node.

**Branch-and-bound algorithm applied to maximization problem**

**Step 1:** Solve the given linear programming problem graphically. Set, the current best lower bound, $Z_B$ as $\infty$ .

**Step 2:** Check, whether the problem has integer solution. If yes, print the current solution as the optimal solution and stop; otherwise go to step 3.

**Step 3:** Identify the variable $X_k$ which has the maximum fractional part as the branching variable. (In case of tie, select the variable which has the highest objective function coefficient.)

**Step 4:** Create two more problems by including each of the following constraints to the current problem and solve them.

$X_k$ < Integer part of $X_k$

$X_k$ > Next integer of $X_k$

**Step 5:** If any one of the new subproblems has infeasible solution or fully integer values for the decision variables, the corresponding node is fathomed. If a new node has integer values for the decision variables, update the current best lower bound as the lower bound of that node if its lower bound is greater than the'previous current best lower bound.

**Step 6:** Are all terminal nodes fathomed? If the answer is yes, go to step 7; otherwise, identify the node with the highest lower bound and go to step 3.

**Step 7:** Select the solution of the problem with respect to the fathomed node whose lower bound is equal to the current best lower bound as the optimal solution.

**PROBLEMS**

**15. Solve the following integer programming problem using branch-and-bound technique.**

**Maximize** $Z = 10X_1 + 20X_2$

**Subject to**

$6X_1 + 8X_2 \leq 48$

$X_1 + 3X_2 \leq 12$

$X_1, X_2 \geq 0$ **and integers**

*Sol :*

The introduction of the non-negative constraints $X_1 \geq 0$ and $X_2 \geq 0$ will eliminate the second, third and fourth quadrants of the $X_1 X_2$ plane as shown in Figure.

Now, from the first constraint in equation form

$6X_1 + 8X_2 = 48$

we get $X_2 = 6$, when $X_1 = 0$; and $X_1 = 8$, when $X_2 = 0$. Similarly from the second constraint in equation form

$X_1 + 3X_2 = 12$

we have $X_2 = 4$, when $X_1 = 0$; and $X_1$, = 12, when $X_2 = 0$.

Now, plot the constraints 1 and 2 as shown in Figure.

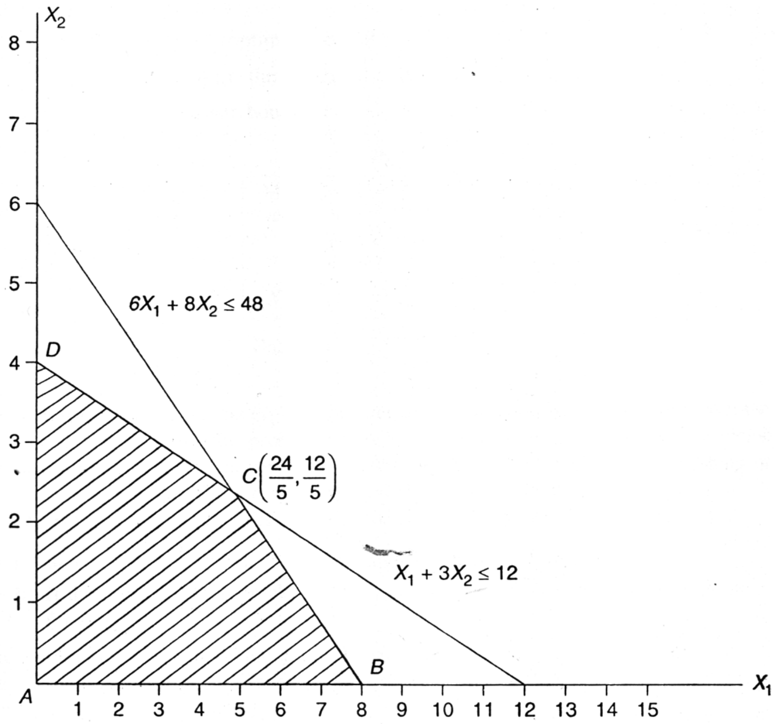



**Fig.: Feasible region of**

The closed polygon ABCD is the feasible region. The objective function value at each of the comer points of the closed polygon is computed as follows by substituting its coordinates in the objective function:

$Z(A) = 10 \times 0 + 20 \times 0 = 0$

$Z(B) = 10 \times 8 + 20 \times 0 = 80$

$Z(C) = 10 \times \frac{24}{5} + 20 \times \frac{12}{5} = 96$

$Z(D) = 10 \times 0 + 20 \times 4 = 80$

Since, the type of the objective function is maximization, the solution corresponding to the maximum Z value is to be selected as the optimum solution. The Z value is maximum for the corner point C. Hence, the corresponding solution of the continuous linear programming problem is presented below.

$$X_1 = \frac{24}{5}, X_2 = \frac{12}{5}, Z(\text{optimum}) = 96$$

These are jointly shown as problem P, in Figure. The notations for different types of lower bound are defined as follows:

$Z_U$ = Upper bound = Z (optimum) of LP problem

$Z_L$ = Lower bound w.r.t. the truncated values of the decision variables

$Z_B$ = Current best lower bound

$P_1$

Maximize $Z = 10X_1 + 20X_2$
subject to

$$6X_1 + 8X_2 \le 48$$
$$X_1 + 3X_2 \le 12$$

$X_1$ and $X_2 \ge 0$ and integers

$X_1 = 24/5$, $X_2 = 12/5$, $Z_U = 96$, $Z_L = 80$, $Z_B = \infty$

**Fig.: Solution of given linear programming problem.**

Since both the values of $X_1$ and $X_2$ are not integers, the solution is not optimum from the view point of the given problem. So, the problem is to be modified into two problems by including integer constraints one by one. The lower bound of the solution of P, is 80. This is nothing but the value of the objective function for the truncated values of the decision variables.

The rule for selecting of the variable for branching is explained as follows:

1. Select the variable which has the highest fractional part.
2. If there is a tie, then break the tie by choosing the variable which has the highest objective function coefficient.

In the continuous solution of the given linear programming problem $P_1$ the variable $X_1$ has the highest fractional part (4/5). Hence, this variable is selected for further branching as shown in Figure.

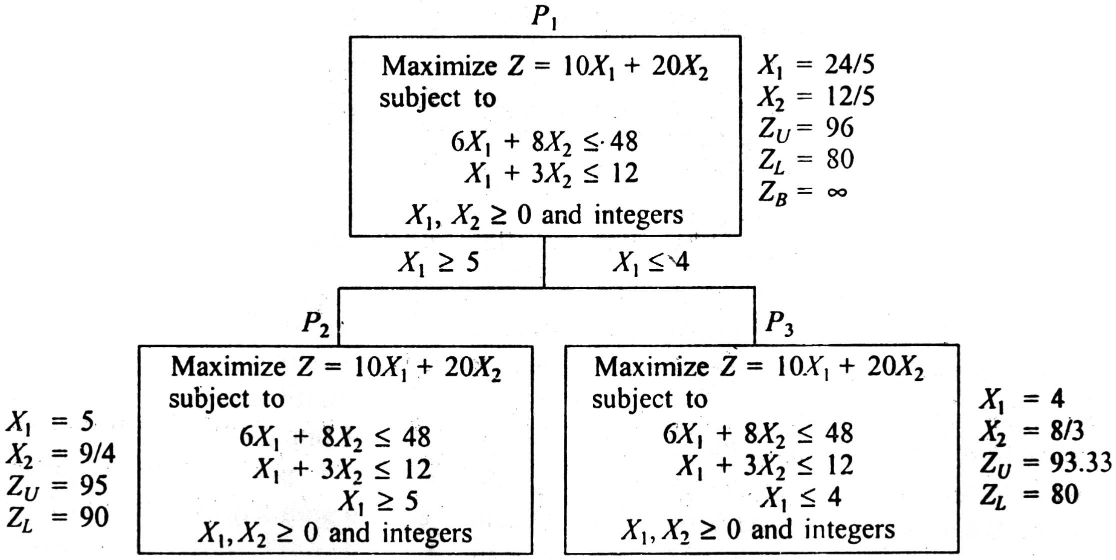



**Fig.: Branching from $P_1$.**

In Figure, the problems, $P_2$ and $P_3$ are generated by adding an additional constraint. The subproblem, $P_2$ is created by introducing '$X_1 \ge 5$' in problem $P_1$ and the problem $P_3$ is created by introducing '$X_1 \le 4$' in problem $P_1$. The corresponding effects in slicing the non-integer feasible region ire shown in Figures, respectively. The solution for each of the subproblems, $P_2$ and $P_3$

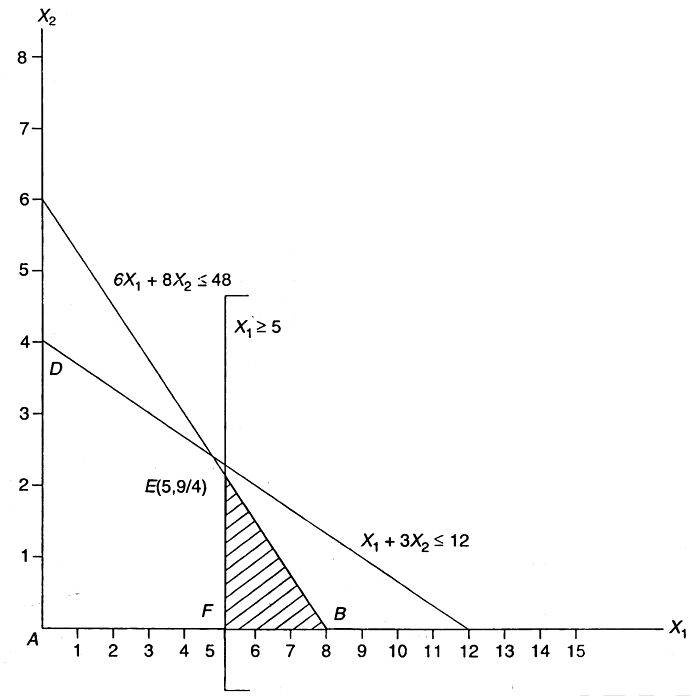



**Fig.: Feasible region of P2 after introducing $X_1 \geq 5$ to $P_1$**

is obtained from Figures, respectively. These are summarized in Figure. The problem has the highest lower bound of 90 among the unfathomed terminal nodes. So, the further branching done from this node as shown in Figure.

In Figure, the problems, $P_4$ and $P_5$ are generated by adding an additional constraint to $P_2$. The problem, $P_4$ is created by including '$X_2 \geq 3$' in problem $P_2$, and problem $P_5$ is created by including '$X_2 \leq 2$' in problem $P_2$. The corresponding effects in slicing the non-integer feasible region are shown Figures, respectively. The solution for each of the problems $P_4$ and $P_5$ is obtained from uses 6.7 and 6.8, respectively. The problem $P_4$ has infeasible solution. So, this node is fathomed. The lower bound of the node $P_5$ is 90. But, the solution of the node $P_5$ is still non-integer.

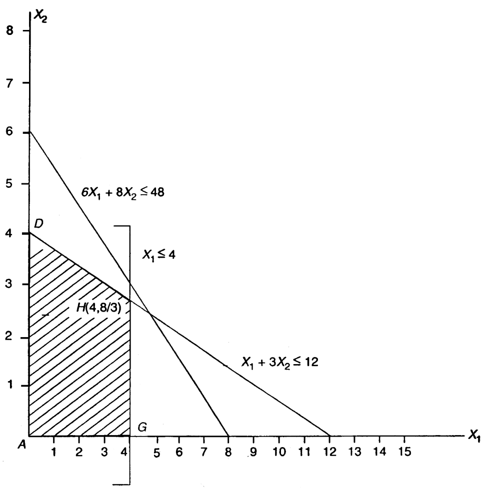



**Fig.: Feasible region of $P_3$ after introducing $X_1 \leq 4$ to $P_1$**

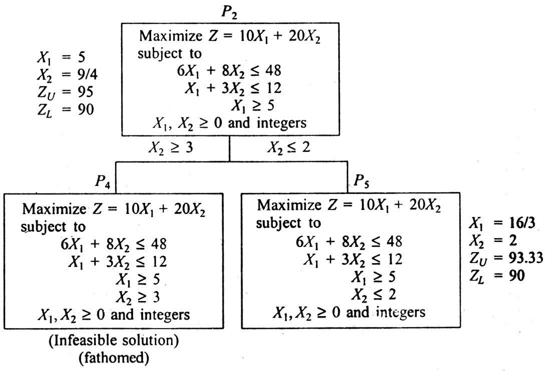



**Fig.: Branching from P2**

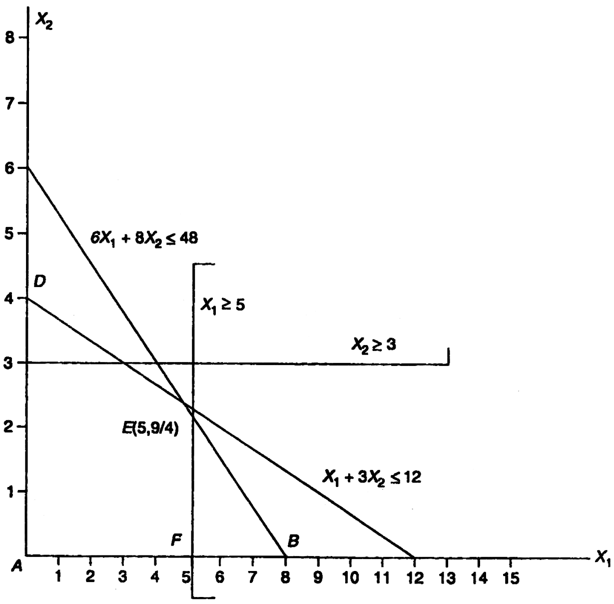



**Fig.: Infeasible region of $P_4$ after introducing $X_2 \geq 3$ to $P_2$**

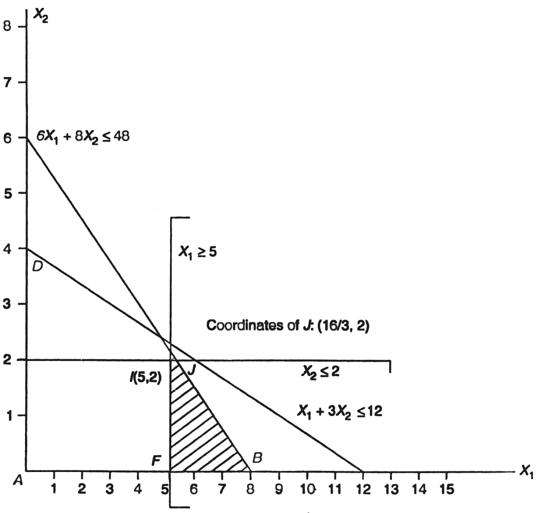



**Fig.: Feasible region of $P_5$ after introducing $X_2 \leq 2$ to $P_2$**

Now, the lower bound of the node $P_5$ is the maximum when compared to that of all other unfathomed terminal nodes (only $P_3$) at this stage. So, the further branching should be done from the node, $P_5$ as shown in Figure.

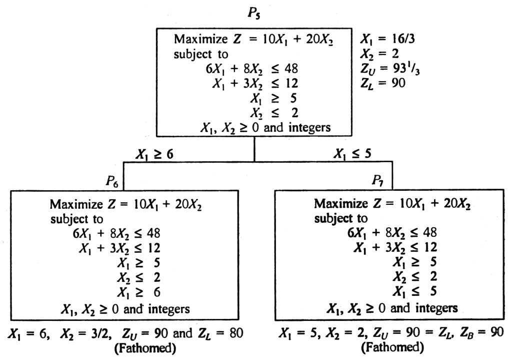



**Fig.: Branching from $P_5$**

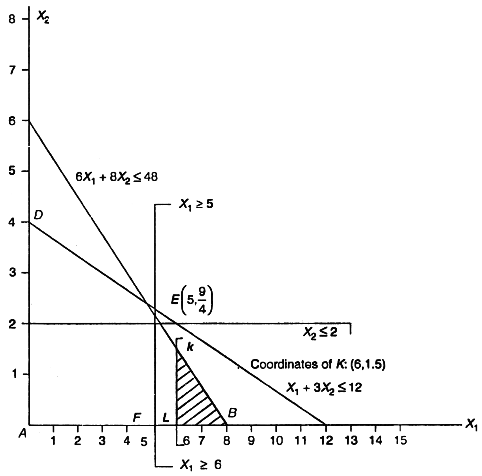



**Fig.: Feasible region of $P_6$ after introducing $X_1 \geq 6$ to $P_5$.**

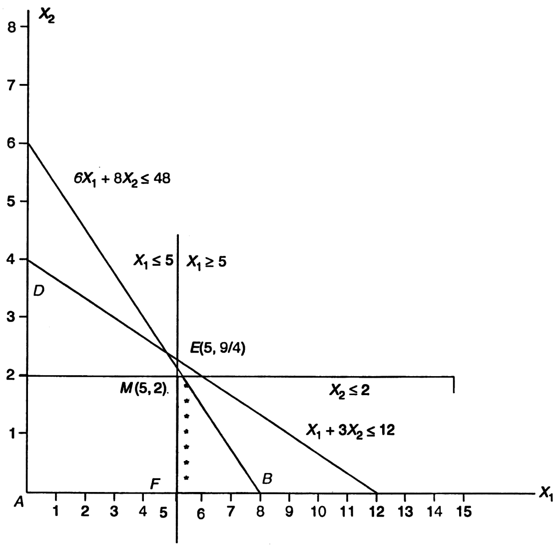



**Fig.: Feasible region of $P_7$ after introducing $X_1 \leq 5$ to $P_5$.**

**[Feasible region is the vertical line from M(5, 2) to f(5, 0) indicated by *s.]**

In Figure, the problems, $P_6$ and $P_7$ are generated by adding an additional constraint to $P_5$. The problem $P_6$ is created by including '$X_1 \geq 6$' in the problem $P_5$ and problem $P_7$ is created by including '$X_1 \leq 5$' in problem $P_5$. The corresponding effects in slicing the non-integer feasible region are shown in Figures, respectively. The solution for each of the problems $P_6$ and $P_7$ are also obtained from these figures, respectively. The problem $P_7$ has integer solution. So, it is a fathomed node. Hence, the current best lower bound ($Z_B$) is updated to its objective function value, 90.

The solution of the node $P_6$ is non-integer and its lower bound and upper bound are 80 and 90, respectively. Since, the upper bound of the node $P_6$ is not greater than the current best lower bound of 90, the node $P_6$ is also fathomed and it has infeasible solution in terms of not fulfilling integer constraints for the decision variables.

Now, the only unfathomed terminal node is $P_3$. The further branching from this node is shown in Figure.

In Figure, the problems $P_9$ and $P_9$ are generated by adding an additional constraint to $P_3$. The problem $P_8$ is created by including '$X_2 \geq 3$' in problem $P_3$ and problem $P_9$ is created by including

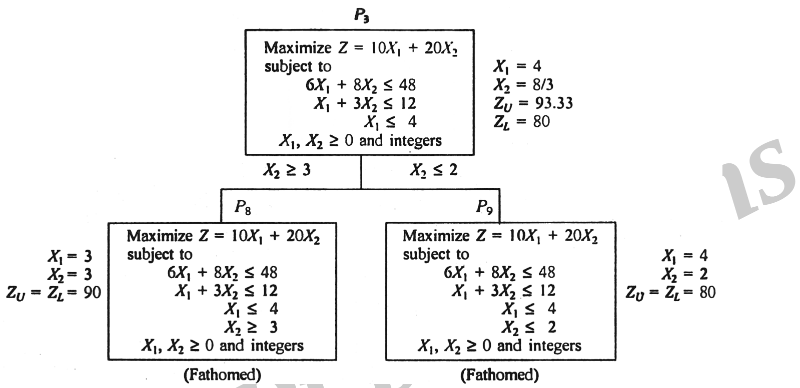



**Fig.: Branching from $P_3$.**

'$X_2 < 2$' in problem $P_3$. The corresponding effects in slicing the non-integer infeasible region are shown in Figures, respectively. The solution for each of the problems $P_8$ and $P_9$ are obtained from the Figures, respectively. The problems $P_8$ and $P_9$ have integer solution. So, these two nodes are fathomed. But the objective function value of these nodes are not greater than the current best lower bound of 90. Hence, the current best lower bound is not updated.

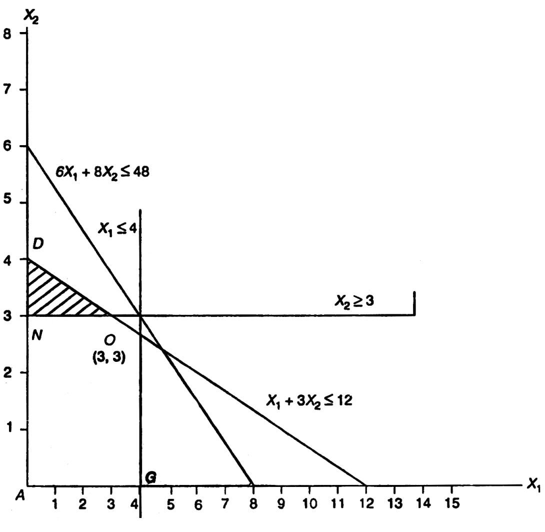



**Fig.: Feasible region of P8 after introducing $X_2 \geq 3$ to $P_3$.**

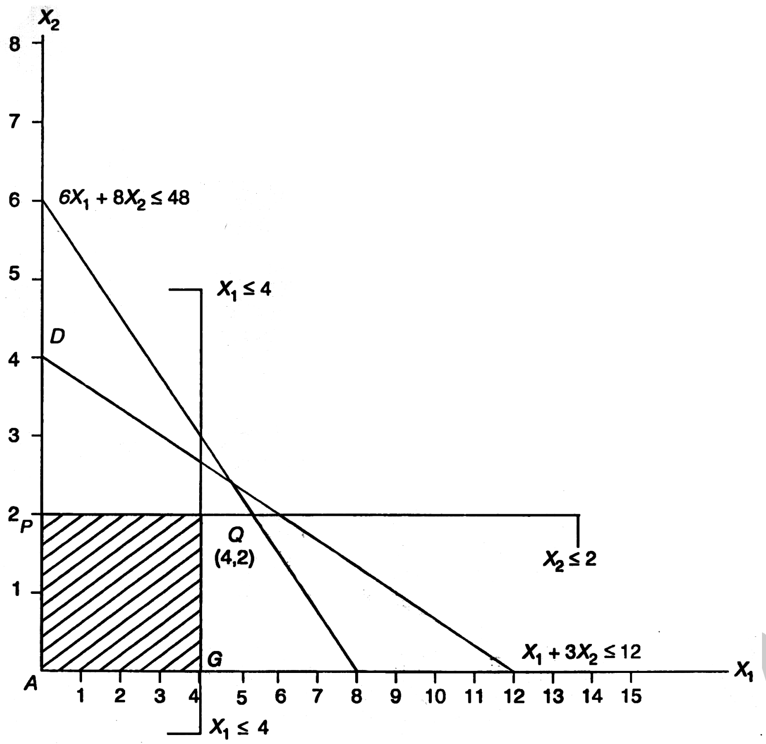



**Fig.: Feasible region of P9 after introducing $X_2 \leq 2$ to $P_3$.**

Now, all the terminal nodes are fathomed. The feasible fathomed node with the current best lower bound is $P_7$. Hence, its solution is treated as the optimal solution as listed below. A complete branching tree is shown in Figure.

$X_1 = 5, X_2 = 2$, Z(optimum) = 90

**Note:** This problem has alternate optimum solution at $P_8$ with $X_1 = 3, X_2 = 3$, Z(optimum) = 90.

**16. Use Branch and Bound technique to solve the following :**

**Max $Z = x_1 + 4x_2$**

**Subject to $2x_1 + 4x_2 \leq 7$**

**$5x_1 + 3x_2 \leq 15$**

**$x_1, x_2 \geq 0$ and are integers**

*Sol :*

Ignoring the integrality condition we solve the LPP

Max $Z = x_1 + 4x_2$

Subject to $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_1, x_2 \geq 0$

Introducing slack variables $S_1, S_2 \geq 0$ the standard form of LPP becomes

Max $Z = x_1 + 4x_2 + 0S_1 + 0S_2$

Subject to $2x_1 + 4x_2 + S_2 = 7$

$5x_1 + 3x_2 + S_2 = 15$

| | | $C_J$ | 1 | 4 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | Min$x_B/x_2$ |
| ←0 | $S_1$ | 7 | 2 | (4) | 1 | 0 | 7/4 |
| 0 | $S_2$ | 15 | 5 | 3 | 0 | 1 | 15/3=5 |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | |
| | $Z_j-C_j$ | | –1 | – 4↑ | 0 | 0 | |
| 4 | $x_2$ | 7/4 | $\frac{1}{2}$ | 1 | 1/4 | 0 | |
| 0 | $S_2$ | $\frac{39}{4}$ | $\frac{7}{2}$ | 0 | –3/4 | 1 | |
| | $Z_j$ | 7 | 2 | 4 | 1 | 0 | |
| | $Z_j-C_j$ | | 1 | 0 | 1 | 0 | |

Since all $Z_j - C_j \geq 0$ an optimum solution is obtained

$x_1 = 0,\ x_2 = 7/4$ and Max $Z = 7$

Since $x_2 = \frac{7}{4}$, this problem should be branched into wo rub-problems

For $x_2 = \frac{7}{4}$ $1 < x_2 < 2$

$= x_2 \leq 1,\ x_2 \geq 2$

Applying these two conditions separately in the given LPP we get wo sub-problems.

Sub - problem (1)

Max $Z = x_1 + 4x_2$

Subject to $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \leq 1$

$x_1, x_2 \geq 0$

Sub -problem(2)

Max $Z = x_1 + x_2$

Subject to $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 1.5$

$\geq$

$x_1, x_2 \geq$

**Sub-problem (1)**

| | | $C_j$ | 1 | 4 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | Min $\frac{x_B}{x_2}$ |
| 0 | $S_1$ | 7 | 2 | 4 | 1 | 0 | 0 | 7/4 |
| 0 | $S_2$ | 15 | 5 | 3 | 0 | 1 | 0 | 15/3 |
| –0 | $S_3$ | 1 | 0 | (1) | 0 | 0 | 1 | 1/1 |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | |
| | $Z_j – C_j$ | | –1 | – 4↑ | 0 | 0 | 0 | Min $\frac{x_B}{x_1}$ |
| –0 | $S_1$ | 3 | (2) | 0 | 1 | 0 | –4↑ | 3/2 |
| 0 | $S_2$ | 12 | 5 | 0 | 0 | 1 | –3 | 12/5 |
| 0 | $x_2$ | 1 | 0 | 1 | 0 | 0 | 1 | |
| | $Z_j$ | 4 | 0 | 4 | 0 | 0 | 4 | |
| | $Z_j – C_j$ | | –1↑ | 0 | 0 | 0 | 4 | |

| | | $C_j$ | 1 | 4 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ |
| 1 | $x_1$ | 3/2 | 1 | 0 | 1/2 | 0 | –2 |
| 0 | $S_2$ | 9/2 | 0 | 0 | –5/2 | 1 | 7 |
| 4 | $x_2$ | 1 | 0 | 1 | 0 | 0 | 1 |
| | $Z_j$ | 11/2 | 1 | 4 | 1/2 | 0 | 2 |
| | $Z_j – C_j$ | | 0 | 0 | 1/2 | 0 | 2 |

Since all $Z_j — C_j \geq 0$ the solution is optimum, given by $x_1 = 3/2$ $x_2 = 1$, and Max Z = 11/2

Since $x_1 = 3/2$ is not an integer, this sub-problem is branched again.

**Sub-problem (2)**

Max $Z = x_1 + 4x_2$

Subject to $2x_x + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \geq 2$

$x_1, x_2 \geq 0.$

| | | $C_j$ | 1 | 4 | 0 | 0 | 0 | $-M$ | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $A_1$ | $\text{Min}\,x_B/x_2$ |
| $\leftarrow 0$ | $S_1$ | 7 | 2 | (4) | 1 | 0 | 0 | 0 | 7/4 |
| 0 | $S_2$ | 15 | 5 | 3 | 0 | 1 | 0 | 0 | 15/3 |
| $-M$ | $A_1$ | 2 | 0 | 1 | 0 | 0 | $-1$ | 1 | 2/1 |
| | $Z_j$ | $-2M$ | 0 | $-M$ | 0 | 0 | $M$ | $-M$ | |
| | $Z_j-C_j$ | | $-1$ | $-M-4$ | 0 | 0 | $M$ | 0 | |
| 4 | $x_2$ | 7/4 | 1/2 | 1 | 1/4 | 0 | 0 | 0 | |
| 0 | $S_2$ | 39/4 | 7/2 | 0 | $-3/4$ | 1 | 0 | 0 | |
| $-M$ | $A_1$ | 1/4 | $-1/2$ | 0 | $-1/4$ | 0 | $-1$ | 1 | |
| | $Z_j$ | $7-\frac{5M}{4}$ | $2+\frac{M}{2}$ | 4 | $1+\frac{M}{4}$ | 0 | $M$ | $-M$ | |
| | $Z_j-C_j$ | | $\frac{M}{2}+1$ | 0 | $\frac{M}{4}+1$ | 0 | $M$ | 0 | |

Since all $Z_j - C_j \geq 0$, but an artificial variable $A_1$ is in the basis at pos-itive level there exists no feasible solution. Hence, this sub-problem is dropped.

In Sub-problem (1) Since $x_1 = 3/2$

We have $1 \leq x_1 \leq 2$

$= x_1 \leq 1, x_1 \geq 2$

Applying these two conditions separately in the sub-problem (1) we get two sub-problems.

Sub-problem (3)

Max $Z = x_1 + 4x_2$

Subject to $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \leq I$

$x_1 \leq 1$

$x_1\, x_2 \geq 0$

Sub-problem (4)

Max $Z = x_1 + x_2$

Subject to $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$\leq$

$\geq$

$x_1, x_2 \geq$

**Sub-problem (3)**

Since all $Z_j - C_j > 0$, an optimum solution is obtained. It is given b> $x_1 = 1, x_2 = 1$, and Max Z=5

Since this solution is integer valued this sub-problem cannot be branched further. The lower bound of the objective function is 5.

| | | $C_j$ | 1 | 4 | 0 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| $C_{B_l}$ | B | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | Min $\frac{x_B}{x_2}$ |
| 0 | $S_1$ | 7 | 2 | 4 | 1 | 0 | 0 | 0 | 7/4 |
| 0 | $S_2$ | 15 | 5 | 3 | 0 | 1 | 0 | 0 | 15/3 |
| ←0 | $S_3$ | 1 | 0 | (1) | 0 | 0 | 1 | 0 | 1/1 |
| 0 | $S_4$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | – |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| | $Z_j$–$C_j$ | | –1 | –4↑ | 0 | 0 | 0 | 0 | Min $\frac{x_B}{x_1}$ |
| 0 | $S_1$ | 3 | 2 | 0 | 1 | 0 | –4 | 0 | 3/2 |
| 0 | $S_2$ | 12 | 5 | 0 | 0 | 1 | –3 | 0 | 12/5 |
| 4 | $x_2$ | 1 | 0 | 1 | 0 | 0 | 1 | 0 | – |
| ←0 | $S_4$ | 1 | (1) | 0 | 0 | 0 | 0 | 1 | 1/1 |
| | $Z_j$ | 4 | 0 | 4 | 0 | 0 | 4 | 0 | |
| | $Z_j$–$C_j$ | | –1↑ | 0 | 0 | 0 | 4 | 0 | |

| | | $C_j$ | 1 | 4 | 0 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | B | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
| 0 | $S_1$ | 1 | 0 | 0 | 1 | 0 | –4 | –2 |
| 0 | $S_2$ | 7 | 0 | 0 | 0 | 1 | –3 | –5 |
| 4 | $x_2$ | 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| 1 | $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| | $Z_j$ | 5 | 1 | 4 | 0 | 0 | 4 | 1 |
| | $Z_j$–$C_j$ | | 0 | 0 | 0 | 0 | 4 | 1 |

**Sub-problem (4)**

Max $Z = x_1 + 4x_2$

Subject to $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \leq 1$

$x_2 \geq 2$

$x_1, x_2 \geq 0$

Since all $Z_j - C_j \geq 0$ the solution is optimum and is given by

$x_1 = 2$, $x_2 = 3/4$

| | $C_B$ | $x_B$ | $B$ | $x_1$ (1) | $x_2$ (4) | $S_1$ (0) | $S_2$ (0) | $S_3$ (0) | $S_4$ (0) | $A_1$ (–M) | Min $x_B/x_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 7 | $S_1$ | 2 | 4 | 1 | 0 | 0 | 0 | 0 | 7/2 |
| | 0 | 15 | $S_2$ | 5 | 3 | 0 | 1 | 0 | 0 | 0 | 15/5 |
| | 0 | 1 | $S_3$ | 0 | 1 | 0 | 0 | 1 | 0 | 0 | – |
| ← | –M | 2 | $A_1$ | (1) | 0 | 0 | 0 | 0 | –1 | 1 | 2/1 |
| | | –2M | $Z_j$ | –M | 0 | 0 | 0 | 0 | M | –M | |
| | | – | $Z_j–C_j$ | –M–1 | –4 | 0 | 0 | 0 | M | 0 | |

| $C_B$ | $B$ | $x_B$ | $x_1$ (1) | $x_2$ (4) | $S_1$ (0) | $S_2$ (0) | $S_3$ (0) | $S_4$ (0) | $A_1$ (–M) | Min $x_B/x_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| ←0 | $S_1$ | 3 | 0 | (4) | 1 | 0 | 0 | 2 | –2 | 3/4 |
| 0 | $S_2$ | 5 | 0 | 3 | 0 | 1 | 0 | 5 | –5 | 5/3 |
| 0 | $S_3$ | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1/1 |
| 1 | $x_1$ | 2 | 1 | 0 | 0 | 0 | 0 | –1 | 1 | – |
| | $Z_j$ | 2 | 1 | 0 | 0 | 0 | 0 | –2 | 1 | |
| | $Z_j–C_j$ | | 0 | –4 | 0 | 0 | 0 | –2 | 1+M | |
| 4 | $x_2$ | 3/4 | 0 | 1 | 1/4 | 0 | 0 | 1/2 | – | |
| 0 | $S_2$ | 11/4 | 0 | 0 | –3/4 | 1 | 0 | 7/2 | – | |
| 0 | $S_3$ | 1/4 | 0 | 0 | –1/4 | 0 | 1 | –1/2 | – | |
| 1 | $x_1$ | 2 | 1 | 0 | 0 | 0 | 0 | –1 | – | |
| | $Z_j$ | 5 | 1 | 4 | 1 | 0 | 0 | 1 | – | |
| | $Z_j–C_j$ | | 0 | 0 | 1 | 0 | 0 | 1 | – | |

Since $x_2 = 3/4$ is not an integer, this sub-problem is branched further.

In sub-problem (4) since $x_1 = 3/4 \;\; 0 \leq x_2 \leq 1$

$= x_2 \leq 0$, or $x_2 \geq 1$

Applying these two conditions in the sub-problem (4)

We get two sub-problems.

Sub - problem (5)

Max $Z = x_1 + 4x_2$

Subject to $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \leq 1$

$x_1 \geq 2$

$x_2 \leq 0$

$x_1, x_2 \geq 0$

Sub - problem (6)

Max $Z = x_1 + x_2$

Subject to $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \leq 1$

$x_1 \geq 2$

$x_2 \geq 1$

$x_1, x_2 \geq 0.$

## Sub-problem (5)

| | $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $A_1$ | $S_5$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $C_j$ | 1 | 4 | 0 | 0 | 0 | 0 | $-M$ | 0 | Min $x_B/x_1$ |
| | 0 | $S_1$ | 7 | 2 | 4 | 1 | 0 | 0 | 0 | 0 | 0 | 7/2 |
| | 0 | $S_2$ | 15 | 5 | 3 | 0 | 1 | 0 | 0 | 0 | 0 | 15/5 |
| | 0 | $S_3$ | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | – |
| ← | $-M$ | $A_1$ | 2 | (1) | 0 | 0 | 0 | 0 | –1 | 1 | 0 | 2/1 |
| | 0 | $S_5$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | – |
| | | $Z_j$ | –2M | –M | 0 | 0 | 0 | 0 | M | –M | 0 | |
| | | $Z_j - C_j$ | | –M–1 | –4 | 0 | 0 | 0 | M | 0 | 0 | Min $x_B/x_2$ |
| | 0 | $S_1$ | 3 | 0 | 4 | 1 | 0 | 0 | 2 | – | 0 | 3/4 |
| | 0 | $S_2$ | 5 | 0 | 3 | 0 | 1 | 0 | 5 | – | 0 | 5/3 |
| | 0 | $S_3$ | 1 | 0 | 1 | 0 | 0 | 1 | 0 | – | 0 | 1/1 |
| | 1 | $x_1$ | 2 | 1 | 0 | 0 | 0 | 0 | –1 | – | 0 | – |
| ← | 0 | $S_5$ | 0 | 0 | (1) | 0 | 0 | 0 | 0 | – | 1 | 0/1 |
| | | $Z_j$ | 2 | 1 | 0 | 0 | 0 | 0 | –2 | – | 0 | |
| | | $Z_j - C_j$ | | 0 | –4↑ | 0 | 0 | 0 | –2 | – | 0 | Min $x_B/S_4$ |
| | 0 | $S_1$ | 3 | 0 | 0 | 1 | 0 | 0 | 2 | – | 0 | 3/2 |
| ← | 0 | $S_2$ | 5 | 0 | 0 | 0 | 1 | 0 | (5) | – | 0 | 1 |
| | 0 | $S_3$ | 1 | 0 | 0 | 0 | 0 | 1 | 0 | – | –1 | – |
| | 1 | $x_1$ | 2 | 1 | 0 | 0 | 0 | 0 | –1 | – | 0 | – |
| | 4 | $x_2$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | – | 1 | – |
| | | $Z_j$ | 2 | 1 | 4 | 0 | 0 | 0 | –1 | – | 0 | |
| | | $Z_j - C_j$ | | 0 | 0 | 0 | 0 | 0 | –1↑ | – | 0 | |
| | 0 | $S_1$ | 1 | 0 | 0 | 1 | –2/5 | 0 | 0 | – | 0 | |
| | 0 | $S_4$ | 1 | 0 | 0 | 0 | $\frac{1}{5}$ | 0 | 1 | – | 0 | |
| | 0 | $S_3$ | 1 | 0 | 0 | 0 | 0 | 1 | 0 | – | 0 | |
| | 1 | $x_1$ | 3 | 1 | 0 | 0 | 1/5 | (1) | 0 | – | 0 | |
| | 4 | $x_2$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | – | 1 | |
| | | $Z_j$ | 3 | 1 | 4 | 0 | 3/5 | 0 | 0 | | 4 | |
| | | $Z_j - C_j$ | | 0 | 0 | 0 | 3/5 | 0 | 0 | – | 4 | |

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by $x_1 = 3$, $x_2 = 0$ and Max $Z = 3$. This sub-problem yields an optimum integer solution. Hence, this sub-problem is dropped.

## Sub-problem (6)

Max $Z = x_1 + 4x_2$

Subject to $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \leq 1$

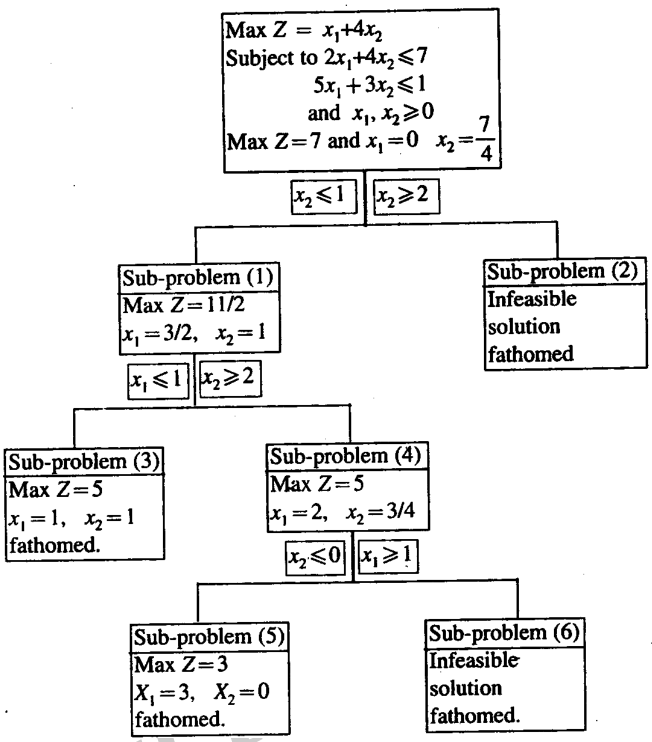
$x_2 \geq 2$

$x_2 > 1$

$x_1, x_2 \geq 0$

This sub-problem has no feasible solution. Hence, this sub-problem is also fathoned.

**Original problem**




Among the available integer valued solution, the best integer solution is given by sub-problem (3).

$\therefore$ The optimum integer solution is

Max Z = 5 and $x_1 = 1, x_2 = 1$

The best available integer optimal solution is

Max Z = 5, and $x_1 = 1, x_2 = 1$

**17. Use branch and bound technique to Maximise**

$$Z = 3x_1 + 2x_2$$

**Subject to** $2x_1 + 2x_2 \leq 7$

$x_1 \leq 2$

$x_2 \leq 2$ **where** $x_1\ x_2 \geq 0$ **and are integers.**

*Sol :*

Ignoring the integrality condition, we solve the given LPP by introducing slack variables $S_1, S_2, S_3, \geq 0$.

The standard form of LPP is given by

Max Z = $3x_1 + 2x_2 + 0S_1 + 0S_2 + 0S_3$

Subject to $2x_1 + 2x_2 + S_1 = 7$

$x_1 + S_2 = 2$

$x_2 + S_3 = 2$

The initial basic feasible solution is given by $S_1 = 7$, $S_2 = 2$, and $S_3 = 2$.

| | | $C_j$ | | 3 | 2 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| | $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | Min $x_B/x_1$ |
| | 0 | $S_1$ | 7 | 2 | 2 | 1 | 0 | 0 | 7/2 |
| ← | 0 | $S_2$ | 2 | (1) | 0 | 0 | 1 | 0 | 2/1 |
| | 0 | $S_3$ | 2 | 0 | 1 | 0 | 0 | 1 | – |
| | | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | |
| | | $Z_j–C_j$ | | –3↑ | –2 | 0 | 0 | 0 | Min $x_B/x_2$ |
| ← | 0 | $S_1$ | 3 | 0 | (2) | 1 | –2 | 0 | 3/2 |
| | 3 | $x_1$ | 2 | 1 | 0 | 0 | 1 | 0 | – |
| | 0 | $S_3$ | 2 | 0 | 1 | 0 | 0 | 1 | 2/1 |
| | | $Z_j$ | 6 | 3 | 0 | 0 | 3 | 0 | |
| | | $Z_j–C_j$ | | 0 | –2↑ | 0 | 3 | 0 | |

| | $C_j$ | | 3 | 2 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ |
| 2 | $x_2$ | 3/2 | 0 | 1 | 1/2 | –1 | 0 |
| 3 | $x_1$ | 2 | 1 | 0 | 0 | 1 | 0 |
| 0 | $S_3$ | 1/2 | 0 | 0 | –1/2 | 1 | 1 |
| | $Z_j$ | 9 | 3 | 2 | 1 | 1 | 0 |
| | $Z_j–C_j$ | | 0 | 0 | 1 | 1 | 0 |

Since all $Z_j$ – 0, solution is optimal. Since $x_2 = 3/2$ is a non integer, this problem should be branched into two sub-problems.

For $x_2 = 3/2, 1 < x_2 < 2$

$\Rightarrow x_2 \geq 2, x_2 \leq 1.$

Applying these two conditions separately in the given LPP.

We have two sub-problems.

Sub - problem (1)

Max Z = $3x_1 + 2x_2$

Subject to $2x_1 + 2x_2 \leq 7$

$x_1 \leq 2$

$x_2 \leq 2$

$x_2 \leq 1$

Sub - problem (2)

Max Z = $3x_1 + 2x_2$

Subject to $2x_1 + 2x_2 \leq 7$

$x_1 \leq 2$

$x_2 \leq 2$

$x_2 \geq 2$

**Sub-oroblem (1)**

| | $C_j$ | | 3 | 2 | 0 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $\text{Min}\, x_B/x_1$ |
| 0 | $S_1$ | 7 | 2 | 2 | 1 | 0 | 0 | 0 | 7/2 |
| ← 0 | $S_2$ | 2 | (1) | 0 | 0 | 1 | 0 | 0 | 2/1 |
| 0 | $S_3$ | 2 | 0 | 1 | 0 | 0 | 1 | 0 | – |
| 0 | $S_4$ | 1 | 0 | 1 | 0 | 0 | 0 | 1 | – |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| | $Z_j - C_j$ | | –3↑ | –2 | 0 | 0 | 0 | 0 | |

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by $x_1 = 2$, $x_2 = $ I, and Max Z = 8. Since $x_1$ and $x_2$ are integers, this sub-proble can't be branched further.

**Sub-problem (2)**

Max $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4 - MA_1$

Subject to $2x_1 + 2x_2 + S_1 = 7$

$x_1 + S_2 = 2$

$x_2 + S_3 = 2$

$x_2 - S_4 + A_1 = 2$

$x_1, x_2, S_1, S_2, S_3, S_4, A_1 \geq 0$



| | $C_j$ | | 3 | 2 | 0 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $\text{Min}\, x_B/x_2$ |
| 0 | $S_1$ | 3 | 0 | 2 | 1 | –2 | 0 | 0 | 3/2 = 1.5 |
| 3 | $x_1$ | 2 | 1 | 0 | 0 | 1 | 0 | 0 | – |
| 0 | $S_3$ | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 2/1 = 2 |
| — 0 | $S_4$ | 1 | 0 | (1) | 0 | 0 | 0 | 1 | 1/1 = 1 |
| | $Z_j$ | 6 | 3 | 0 | 0 | 3 | 0 | 0 | |
| | $Z_j - C_j$ | | 0 | –2↑ | 0 | 3 | 0 | 0 | |
| 0 | $S_1$ | 1 | 0 | 0 | 1 | –2 | 0 | –2 | |
| 3 | $x_1$ | 2 | 1 | 0 | 0 | 1 | 0 | 0 | |
| 0 | $S_3$ | 1 | 0 | 0 | 0 | 0 | 1 | –1 | |
| 2 | $x_2$ | 1 | 0 | 1 | 0 | 0 | 0 | 1 | |
| | $Z_j$ | 8 | 3 | 2 | 0 | 3 | 0 | 2 | |
| | $Z_j - C_j$ | | 0 | 0 | 0 | 3 | 0 | 2 | |

| | | $C_j$ | 3 | 2 | 0 | 0 | 0 | 0 | $-M$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $A_1$ | Min $x_B/x_2$ |
| 0 | $S_1$ | 7 | 2 | 2 | 1 | 0 | 0 | 0 | 0 | 7/2 |
| 0 | $S_2$ | 2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | – |
| 0 | $S_3$ | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 2/1 = 2 |
| ← $-M$ | $A_1$ | 2 | 0 | (1) | 0 | 0 | 0 | –1 | 1 | 2/1 = 2 |
| | $Z_j$ | $-2M$ | 0 | $-M$ | 0 | 0 | 0 | $M$ | $-M$ | |
| | $Z_j-C_j$ | | –3 | $-M-2$ | 0 | 0 | 0 | $M$ | 0 | Min $x_B/x_1$ |
| ← 0 | $S_1$ | 3 | (2) | 0 | 1 | 0 | 0 | 2 | – | 3/2 |
| 0 | $S_2$ | 2 | 1 | 0 | 0 | 1 | 0 | 0 | – | 2/1 |
| 0 | $S_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | – | – |
| 2 | $x_2$ | 2 | 0 | 1 | 0 | 0 | 0 | –1 | – | – |
| | $Z_j$ | 4 | 0 | 2 | 0 | 0 | 0 | –2 | – | |
| | $Z_j-C_j$ | | –3↑ | 0 | 0 | 0 | 0 | –2 | | |
| 3 | $x_1$ | 3/2 | 1 | 0 | 1/2 | 0 | 0 | 1 | – | |
| 0 | $S_2$ | 1/2 | 0 | 0 | –1/2 | 1 | 0 | –1 | – | |
| 0 | $S_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | – | |
| 2 | $x_2$ | 2 | 0 | 1 | 0 | 0 | 0 | –1 | – | |
| | $Z_j$ | 17/2 | 3 | 2 | 3/2 | 0 | 0 | 1 | – | |
| | $Z_j-C_j$ | | 0 | 0 | 3/2 | 0 | 0 | 1 | | |

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by $x_1 = 3/2$, $x_2 = 2$. Since $x_1 = 3/2$ is not an integer, this problem should be branched into two sub-problems.

For $x_1 = 3/2$, $1 < x_1 < 2 \Rightarrow x_1 \leq 1$, $x_1 \geq 1$.

Applying these two conditions separately in the sub-problem (2), we obtain another two sub-problems.

Sub - problem - (3)

Max $Z = 3x_1 + 2x_2$

Subject to $2x_1 + 2x_2 \leq 7$

Sub - problem - (4)

Max $Z = 3x_1 + 2x_2$

Subject to $2x_1 + 2x_2 \leq 7$

| | |
|---|---|
| $x_1 \le 2$ | $x_1 \le 2$ |
| $x_2 \le 2$ | $x_2 \le 2$ |
| $x_2 \ge 2$ | $x_2 \ge 2$ |
| $x_1 \le 1$ | $x_1 \ge 1$ |

**Sub-problem (3)**

| | | $C_j$ | 3 | 2 | 0 | 0 | 0 | 0 | $-M$ | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $A_1$ | $S_5$ | Min $x_B / x_2$ |
| 0 | $S_1$ | 7 | 2 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 7/2 |
| 0 | $S_2$ | 2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | – |
| 0 | $S_3$ | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 2/1 = 2 |
| ← $-M$ | $A_1$ | 2 | 0 | (1) | 0 | 0 | 0 | –1 | 1 | 0 | 2/1 = 2 |
| 0 | $S_5$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | – |
| | $Z_j$ | $-2M$ | 0 | $-M$ | 0 | 0 | 0 | $M$ | $-M$ | 0 | |
| | $Z_j-C_j$ | | –3 | $-M-2$ | 0 | 0 | 0 | $M$ | 0 | 0 | |
| | | | | ↑ | | | | | | | Min$x_B/x_1$ |
| 0 | $S_1$ | 3 | 2 | 0 | 1 | 0 | 0 | 2 | – | 0 | 3/2 |
| 0 | $S_2$ | 2 | 1 | 0 | 0 | 1 | 0 | 0 | – | 0 | 2/1– |
| 0 | $S_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | – | 0 | – |
| 2 | $x_2$ | 2 | 0 | 1 | 0 | 0 | 0 | –1 | – | 0 | – |
| ← 0 | $S_5$ | 1 | (1) | 0 | 0 | 0 | 0 | 0 | – | 1 | 1/1 |
| | $Z_j$ | 4 | 0 | 2 | 0 | 0 | 0 | –2 | – | 0 | |
| | $Z_j-C_j$ | | –3↑ | 0 | 0 | 0 | 0 | –2 | – | 0 | Min$x_B/S_1$ |
| 0 | $S_1$ | 1 | 0 | 0 | 1 | 0 | 0 | 2 | | –2 | 1/2 |
| 0 | $S_2$ | 1 | 0 | 0 | 0 | 1 | 0 | (1) | – | 0 | – |
| ← 0 | $S_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | – | 0 | 0 |
| 2 | $x_2$ | 2 | 0 | 1 | 0 | 0 | 0 | –1 | – | 0 | – |
| 3 | $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | – | 1 | – |
| | $Z_j$ | 7 | 3 | 2 | 0 | 0 | 0 | –2↑ | – | 3 | |

| | $C_j$ | 3 | 2 | 0 | 0 | 0 | 0 | –M | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | B | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $A_1$ | $S_5$ |
| 0 | $S_1$ | 1 | 0 | 0 | 1 | 0 | –2 | 0 | – | –2 |
| 0 | $S_2$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | – | 0 |
| 0 | $S_4$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | – | 0 |
| 2 | $x_2$ | 2 | 0 | 1 | 0 | 0 | 1 | 0 | – | 0 |
| 3 | $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | – | 1 |
| | $Z_j$ | 7 | 3 | 2 | 0 | 0 | 2 | 0 | – | 3 |
| | $Z_j$–$C_j$ | | 0 | 0 | 0 | 0 | 2 | 0 | – | 3 |

Since all $Z_j – C_j \geq 0$, the solution is optimum and also since $x_1 = 1$, $x_2 = 2$ are integers, Max Z = 7 this sub-problem can't be branched further.

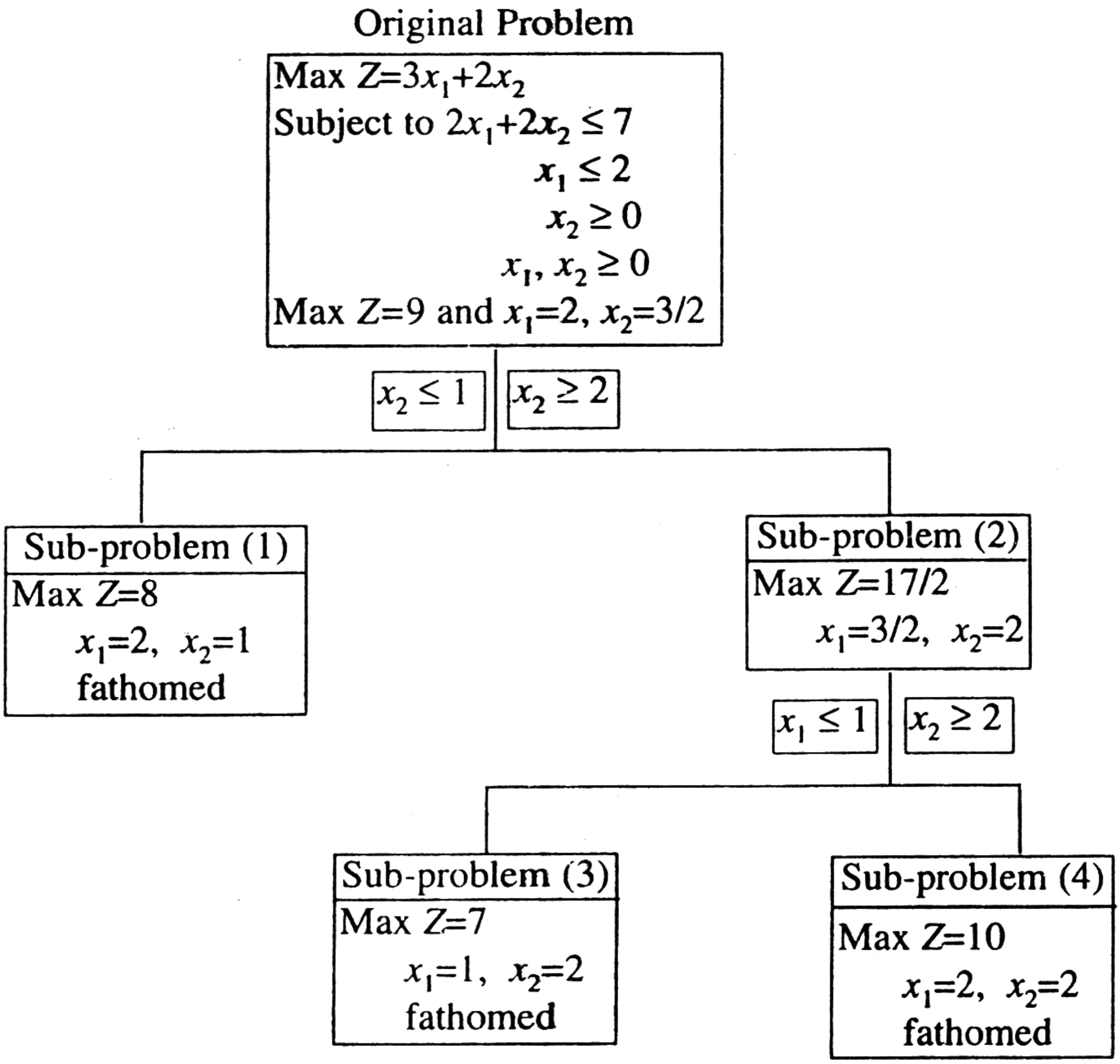
**Sub-problem (4)**

| | $C_j$ | 3 | 2 | 0 | 0 | 0 | 0 | M | 0 | –M | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | B | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $A_1$ | $S_5$ | $A_2$ | Min $\frac{x_B}{x_1}$ |
| 0 | $S_1$ | 7 | 2 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 7/2 |
| 0 | $S_2$ | 2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 2/1 |
| 0 | $S_3$ | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | – |
| –M | $A_1$ | 2 | 0 | 1 | 0 | 0 | 0 | –1 | 1 | 0 | 0 | – |
| –M | $A_2$ | 1 | (1) | 0 | 0 | 0 | 0 | 0 | 0 | –1 | 1 | 1/1 |
| | $Z_j$ | –3M | –M | –M | 0 | 0 | 0 | M | –M | M | –M | |
| | $Z_j$–$C_j$ | | –M–3↑ | –M–2 | 0 | 0 | 0 | M | 0 | M | 0 | |
| 0 | $S_1$ | 5 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 2 | – | Min $x_B/x_2$ |
| 0 | $S_2$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | – | 5/2 |
| 0 | $S_3$ | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | – | 2/1 = 2 |
| ← –M | $A_1$ | 2 | 0 | (1) | 0 | 0 | 0 | –1 | 1 | 0 | – | 2/1 = 2 |
| 3 | $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | –1 | – | – |
| | $Z_j$ | –2M+3 | 3 | –M | 0 | 0 | 0 | M | –M | –3 | | |
| | $Z_j$–$C_j$ | | 0 | –M–2 ↑ | 0 | 0 | 0 | M | 0 | –3 | | |

Since all $Z_j — C_j \geq 0$, and also an optimum integer solution is obtained, this sub-problem cannot be branched further. The solution is given by $x_1 = 2$, $x_2 = 2$, and Max Z = 10. Among the available integer solution the best integer solution is given by sub-problem (4).

The optimum integer solution is given by Max Z = 10, and $x_1 – 2$, $x_2 – 2$.

| | | $C_j$ | 3 | 2 | 0 | 0 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $C_B$ | $B$ | $x_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | Min $x_B/S_5$ |
| ← | 0 | $S_1$ | 1 | 0 | 0 | 1 | 0 | 0 | 2 | 2 | 1/2 |
| | 0 | $S_2$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1/1 |
| | 0 | $S_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | – |
| | 2 | $x_2$ | 2 | 0 | 1 | 0 | 0 | 0 | –1 | 0 | – |
| | 3 | $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | –1 | – |
| | | $Z_j$ | 7 | 3 | 2 | 0 | 0 | 0 | –2 | –3 | Min $x_B/S_4$ |
| | | $Z_j-C_j$ | | 0 | 0 | 0 | 0 | 0 | –2 | –3↑ | |
| | 0 | $S_5$ | 1/2 | 0 | 0 | 1/2 | 0 | 0 | 1 | 1 | $\frac{1/2}{1}$ |
| | 0 | $S_2$ | 1/2 | 0 | 0 | –1/2 | 1 | 0 | –1 | 0 | – |
| ← | 0 | $S_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| | 2 | $x_2$ | 2 | 0 | 1 | 0 | 0 | 0 | (–1) | 0 | – |
| | 3 | $x_1$ | 2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | – |
| | | $Z_j$ | 10 | 3 | 2 | 0 | 3 | 0 | (–2) | 0 | |
| | | $Z_j-C_j$ | | 0 | 0 | 0 | 3 | 0 | –2↑ | 0 | |
| | 0 | $S_5$ | 1/2 | 0 | 0 | 1/2 | 0 | –1 | 0 | 1 | |
| | 0 | $S_2$ | 1/2 | 0 | 0 | –1/2 | 1 | 1 | 0 | 0 | |
| | 0 | $S_4$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | |
| | 2 | $x_2$ | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | |
| | 3 | $x_1$ | 2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | |
| | | $Z_j$ | 10 | 3 | 2 | 0 | 3 | 2 | 0 | 0 | |
| | | $Z_j-C_j$ | | 0 | 0 | 0 | 3 | 2 | 0 | 0 | |




The best available solution is Max Z = 10, and $x_1 = 2$, $x_2 = 2$.

## 3.9 Zero-One Implicit Enumeration Algorithm

**Q15. Explain the concept of Zero-One Implicit Enumeration Algorithm.**

*Ans :* **(Imp.)**

Zero-one (0-1) programming is a special kind of linear programming problem. In this type of problem, all the variables are restricted to either 0 or 1. This type of problem exists in many realistic situations like, capital budgeting problem, assignment problem, scheduling problem, portfolio problem, etc.

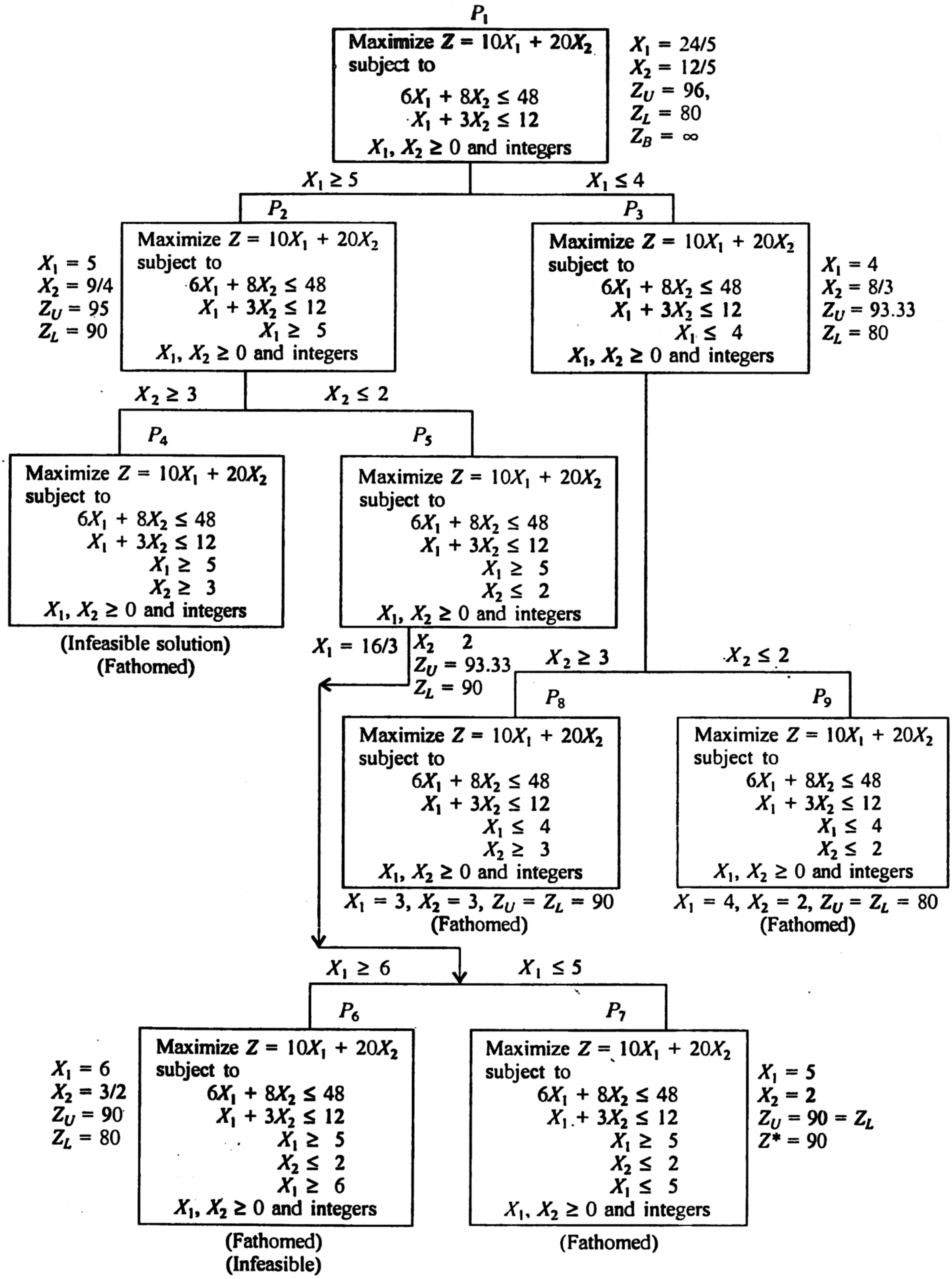



**Fig.: Complete Tree of Example**

## 1. Generalized 0-1 Programming Problem

A generalized model of the 0 - 1 programming problem is presented below.

$$\text{Minimize } Z = \sum_{j=1}^{n} C_j X_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} X_j \leq b_i, \; i = 1, 2, 3..., m$$

$$X_j = 0 \text{ or } 1, \quad j = 1, 2, 3..., m$$

In this model, the variable $X_j$ is restricted to either zero or one. Hence, a specialized procedure which is known as zero-one implicit enumeration algorithm is required to solve this problem, and the same is demonstrated in the next section using a numerical problem. This algorithm is also known as additive algorithm, since it employs only additions and subtractions.

## 2. Zero-One Implicit Enumeration Technique

The technique for data preparation is discussed as under:

1. Convert the problem into the minimization form with all $\leq$ type constraints.
2. If some of the coefficients in the revised objective function are negative, then using the following transformation, convert them into positive coefficients. Simultaneously, substitute the following equation in all the constraints and obtain the modified constraint set.

   $Y_j = 1 - X_j$, for all the variables with negative coefficient in the objective function

   $= X_j$, for the remaining variables
3. Present the problem in a convenient format as shown in Table 6.18.

**Table : Format of Presenting Problem**

| | $X_1$ | $X_2$ | ... | $X_n$ | $S_1$ | ... | $S_m$ | RHS |
|---|---|---|---|---|---|---|---|---|
| | . | . | | . | . | | . | Z |
| $S_1$ Row | . | . | | . | . | | . | . |
| $S_2$ Row | . | . | | . | . | | . | . |
| . | | | | | | | | |
| . | | | | | | | | |
| . | | | | | | | | |
| $S_m$ Row | . | . | | . | . | | . | . |

A branch and bound procedure is used to solve the 0-1 programming problem. The related terminologies are presented below.

### Free Variable

At any node of the tree, a binary variable is said to be a free variable if it is not elected by any of the branches leading to that node.

### Partial Solution

A partial solution provides a specific binary assignment (0 or 1) for some of the variables. Let, $J_t$ be the partial solution at the $t^{th}$ node (or iteration), where, + j means $X_j = 1$ and – j means $X_j = 0$.

The set $J_t$ contains these elements. If the subscript of a binary variable is +j, then it denotes that variable is fixed (value = 1); if it is – j, then it denotes that variable is un fixed (value = 0). The set $J_t$ is an ordered set which means that a new element is always augmented on the right of the partial solution.

### Rules for Fathomed Partial Solution

A partial solution is said to be fathomed if any one of the following conditions is true.

1. Further branching from the partial solution cannot lead to a better value of the objective function.
2. Further branching from the partial solution cannot lead to a feasible solution.

The general rule for generating next partial solution from a fathomed node is that if all the elements of a fathomed partial solution $J_t$ are negative, then the enumeration is complete (stop the algorithm), otherwise, select the right-most positive element and delete all the negative elements to its right. Then, complement the selected right-most positive element with minus sign.

### Tests for Further Branching

To branch further from a partial node (that is to select a free variable for elevating to level one), we need to do the following tests.

Let $J_t$ be the partial solution at node t. Initially $J_0$ is equal to null set which means that all the variables are free and $Z_t$ is the associated objective function value, $\bar{Z}$ is the current best upper bound of the objective function (initially $\bar{Z}$ is equal to infinity).

**Test 1**

For a free variable $X_p$, if all the technological coefficients ($a_{ip}$) with respect to $S_i^t < 0$ are greater than or equal to 0, then $X_p$ cannot improve the infeasibility of the problem and must be discarded as non-promising.

**Test 2**

For any variable $X_p$ if $C_P + Z_t \geq \bar{Z}$ then $X_p$ cannot lead to an improved solution and hence must be discarded.

**Test 3**

Consider the ith constraint for which $S_i^t < 0$ as shown below:

$$a_{i1}X_1 + a_{i2}X_2 + ... + a_{in}X_n + S_i = b_i$$

Let $N_t$ defines the set of free variables not discarded by Test 1 and Test 2. If for at least one $S_i^t < 0$, the following condition is satisfied, then all the free variables in $N_t$ are not promising. Under such situation, the set $J_t$ is fathomed.

$$\sum_{j \in N_t} \min(0, a_{ij}) > S_i^t$$

**Test 4**

After Test 3, if the set $N_t$ is not an empty set, find $V_j^t$ using the following formula for all j belonging to $N_t$. Then identify the value of j for which $V_j^t$ is maximum. Let it be k. The corresponding $X_k$ is the branching variable.

$$V_j^t = \sum_{i=1}^{m} \min(0, S_i^t - a_{ij})$$

## PROBLEMS

**18. Consider the capital budgeting problem where five projects are being considered for execution over the next 3 years. The expected returns for each project and the yearly expenditures (in thousands of rupees) are shown in Table 6.19. Assume that each approved project will be executed over the 3-year period. The objective is to select a combination of projects that will maximize the total returns.**

**Table : Example**

| | Expenditure for | | | |
|---|---|---|---|---|
| Project | Year 1 | Year 2 | Year 3 | Returns |
| 1 | 5 | 1 | 8 | 20 |
| 2 | 4 | 7 | 10 | 40 |
| 3 | 3 | 9 | 2 | 20 |
| 4 | 7 | 4 | 1 | 15 |
| 5 | 8 | 6 | 10 | 30 |
| Maximum available funds | 25 | 25 | 25 | – |

Formulate the problem as a zero-one integer programming problem and solve it by the additive algorithm.

*Sol :*

Let,

$Y_j$ = 1, if the jth project is selected.

= 0, otherwise.

Therefore, a zero-one programming model is:

Maximize

$Y_0 = 20Y_1, + 40Y_2 + 20Y_3 + 15Y_4 + 30Y_5$

Subject to

$5Y_1 + 4Y_2 + 3Y3 + 7Y_4 + 8Y_5 \leq 25$

$Y_1 + 7Y_2 + 9Y_3 + 4Y_4 + 6Y_5 \leq 25$

$8Y_1 + 10Y_2 + 2Y_3 + Y_4 + 10Y_5 \leq 25$

$Y_j$ = 0 or 1, j = 1, 2, 3, 4 and 5

Convert the objective function into minimization type as under

Minimize

$Z_0 = -20Y_1 - 40Y_2 - 20Y_3 - 15Y_4 - 30Y_5$

Since all the coefficients in the objective are negative, substitute the following formula in the objective function and the constraints of the model and rearrange the terms

$Y_j = I - X_j$, j = 1, 2, ..., 5

The resulting model is presented below:

Minimize

$Z = 20X_1 + 40X_2 + 20X_3 + 15X_4 + 30X_5$

Subject to

$-5X_1 - 4X_2 - 3X_3 - 7X_4 - 8X_5 \le -2$

$-X_1 - 7X_2 - 9X_3 - 4X_4 - 6X_5 \le -2$

$-8X_1 - 10X_2 - 2X_3 - X_4 - 10X_5 \le -6$

$X_j = 0$ or 1, j = 1, 2, 3, 4 and 5

[**Note:** The sum of the constants in the objective function is omitted.]

The above model is represented in a tabular form as in Table 6.20 in which $S_1$, $S_2$ and $S_3$ are the slack variables of the constraints.

**Table : Initial Table of Example**

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $S_1$ | $S_2$ | $S_3$ | RHS |
|---|---|---|---|---|---|---|---|---|
| 20 | 40 | 20 | 15 | 30 | 0 | 0 | 0 | Z |
| –5 | –4 | –3 | –7 | –8 | 1 | 0 | 0 | –2 |
| –1 | –7 | –9 | –4 | –6 | 0 | 1 | 0 | –2 |
| –8 | –10 | –2 | –1 | –10 | 0 | 0 | 1 | – 6 |

**Iteration 0:**

For $J_0$ = [null set], $\bar{Z}$ = infinite. The corresponding tree is shown in Figure 6.16.

$(S_1^0, S_2^0, S_3^0) = (-2, -2, -6), Z_0 = 0$

List of free variables, $N_0$ = (1, 2, 3, 4, 5)

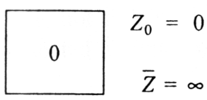



**Figure : Root node**

**Test 1**

This test does not exclude any free variable.

**Test 2**

At this stage, the Test 2 is not applicable, since $\bar{Z}$ is infinity.

**Test 3**

$S_1$: – 5 – 4 – 3 – 7 – 8 = – 27 < – 2

$S_2$: –1 –7 – 9 – 4 – 6 = – 27 < – 2

$S_3$: – 8 – 10 – 2 – 1 – 10 = – 31 < – 6

So, the set $N_0$ cannot be discarded.

**Test 4**

$V_1^0 = 0 - 1 + 0 = -1$

$V_2^0 = 0 + 0 + 0 = 0$

$V_3^0 = 0 + 0 - 4 = -4$

$V_4^0 = 0 + 0 - 5 = -5$

$V_5^0 = 0 + 0 + 0 = 0$

The maximum value of $V_j^0$ is 0 when j is equal to 2 and 5. Out of these two values, 5 is selected randomly as the value of k. Therefore, $X_5$ is the branching variable in the next iteration.

**Iteration 1**

For $J_1$ = [5], the corresponding tree is shown in Figure 6.17.

$(S_1^1, S_2^1, S_3^1) = (-2 + 8, -2 + 6, -6 + 10)$

$= (6, 4, 4), Z_1 = 30$

Since it is a feasible solution, $\bar{Z} = Z_1 = 30$. Thus $J_1$ is fathomed.

Hence, for $J_1$ = [5], $\bar{Z}$ = 30. These are indicated in Figure 6.17.

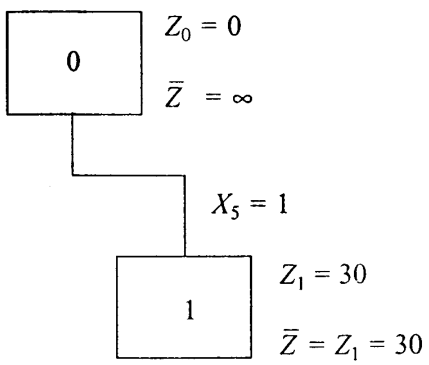



**Figure : Branching tree for $J_1$ = [5]**

**Iteration 2**

For $J_2$ = [– 5], the corresponding tree is shown in Figure 6.18.

$(S_1^2, S_2^2, S_3^2) = (-2, -2, -6), Z_2 = 0$

Since the above solution is infeasible, node 2 is not fathomed. Hence, $\bar{Z}$ is not updated.

Hence for $J_2$ = [– 5], $\bar{Z}$ = 30. These are indicated in Figure 6.18. Also, we have, $N_2$ = (1, 2, 3, 4).

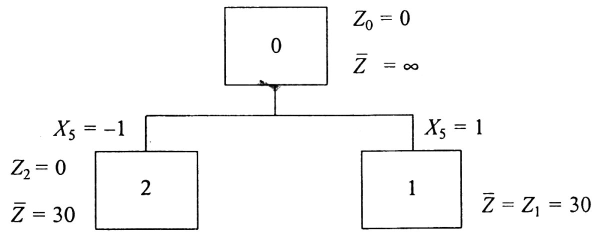



**Figure 6.18 Branching tree for $J_x$ = [– 5].**

**Test 1**

This test does not exclude any of the variables in $N_2$.

**Test 2**

Here $C_2$ = 40. Then $Z_2 + C_2 = 0 + 40 = 40$, which is more than $\bar{Z}$ (30). So, $X_2$ is excluded. Therefore, we have $N_2$ = (1, 3, 4).

**Test 3**

$S_1: -5 - 3 - 7 = -15 < -2$

$S_2: -1 - 9 - 4 = -14 < -2$

$S_3: -8 - 2 - 1 = -11 < -6$

So, the free variables in $N_2$ cannot be discarded.

**Test 4**

$V_1^2 = 0 - l + 0 = - l$

$V_3^2 = 0 + 0 - 4 = -4$

$V_4^2 = 0 + 0 - 5 = -5$

Hence, k is equal to 1 and the corresponding branching variable is $X_1$.

**Iteration 3**

For $J_3$ = [– 5, 1], the corresponding tree is shown in Figure 6.19.

$(S_1^3, S_2^3, S_3^3)$ = (– 2 + 5, – 2 + 1, – 6 + 8)

= (3, – 1, 2)

$Z_3$ = 20

Since the above solution is infeasible, $\bar{Z}$ is not updated.

For, $J_3$ = [– 5, 1], $\bar{Z}$ = 30. These are indicated in Figure 6.19.

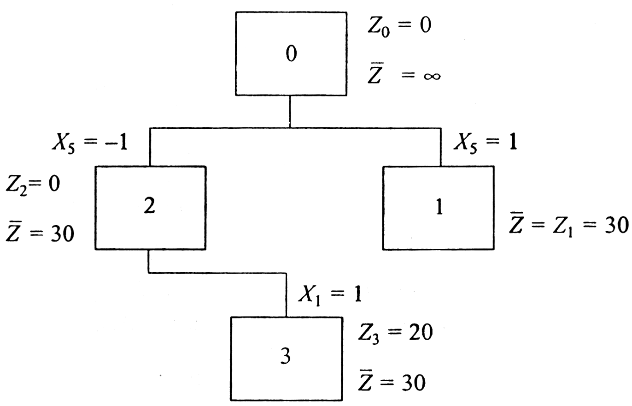



**Figure : Branching tree for $J_1$ = [– 5, 1]**

Also, $N_3$ = (2, 3, 4)

**Test 1**

This test does not exclude any free variable.

**Test 2**

$C_2 + Z_3$ (40 + 20 = 60) > $\bar{Z}$ (30). So, exclude $X_2$

$C_3 + Z_3$ (20 + 20 = 40) > $\bar{Z}$ (30). So, exclude $X_3$

$C_4 + Z_3$ (15 + 20 = 35) > $\bar{Z}$ (30). So, exclude $X_4$

All the free variables at this stage are excluded.

Since $N_0$ = null set, $J_3$ is fathomed.

**Iteration**

$J_4$ = [– 5, – 1]. The corresponding tree is shown in Figure 6.20

$(S_1^1, S_2^4, S_3^4)$ = (– 2, – 2, – 6), $Z_4$ = 0

Since the above solution is infeasible, $\bar{Z}$ is not updated.

For $J_4$ = [– 5, – 1], $\bar{Z}$ = 30

These are indicated in Figure 6.20. Also, we have $N_4$ = (2, 3, 4).

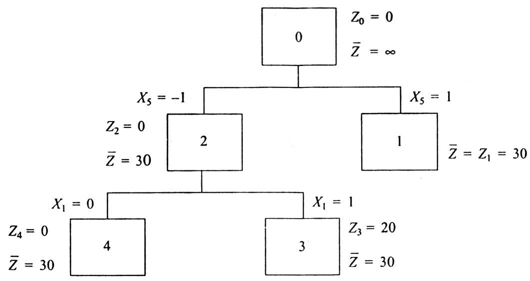



**Figure : Branching tree for $J_1$ = [– 5, –1]**

**Test 1**

This test does not excludeany free variable.

**Test 2**

$C_2 + Z_4$ (40 + 0 = 40) > $\bar{Z}$ (30)

$C_3 + Z_4$ (20 + 0 = 20) < $\bar{Z}$ (30)

$C_4 + Z_4$ (15 + 0 = 15) < $\bar{Z}$ (30)

So, discard only $X_2$. Hence, $N_4$ = (3, 4).

**Test 3**

$S_1$: – 10 < – 2

$S_2$: – 13 < – 2

$S_3$: – 3 > – 6

Since for $S_3$ row, $\Sigma_{j \in N_4} \min(0, a_{ij}) > S_3^4$, discard $N_4$. Thus $J_4$ is fathomed. Since all the elements of $J_4$ are negative, the enumeration is complete and $J_1$ is the optimum fathomed node, i.e. $J_1$ = [5]. So, $X_5$ = 1 and all other variables are zero. The values for $Y_1$, $Y_2$, $Y_3$, $Y_4$ and $Y_5$ are derived as shown in Table 6.21.

**Table : Summary of Results**

| $j$ | $X_j$ | $Y_j = 1 - X_j$ | Objective function coefficient ($C_j$) | $C_j * Y_j$ |
|---|---|---|---|---|
| 1 | 0 | 1–0 = 1 | 20 | 20 |
| 2 | 0 | 1–0 = 1 | 40 | 40 |
| 3 | 0 | 1–0 = 1 | 20 | 20 |
| 4 | 0 | 1–0 = 1 | 15 | 15 |
| 5 | 1 | 1–1 = 0 | 30 | 0 |
| | | | Total | 95 |

The final results are given below :

$Y_1 = 1, Y_2 = 1, Y_3 = 1, Y_4 = 1, Y_5 = 0$

Interpretation : All the projects except the project 5 are to be selected which will result with a total maximum return of 95 (in thousands of rupees).

# UNIT IV

**Dynamic Programming:** Introduction, Applications of Dynamic Programming, Solution of Linear Programming Problem through Dynamic Programming.

## 4.1 Dynamic Programming

### 4.1.1 Introduction

**Q1. Define Dynamic Programming.**

*Ans :* **(Imp.)**

Dynamic programming is a mathematical technique of optimization using multistage decision process. It is a systematic procedure for determining the combination of decisions which maximize the overall objective. Decisions regarding a certain problem are optimized in stages rather than simultaneously. The original problem is split into sub-problems such that the outcome of each depends on the result of the previous one. The solution is obtained in an orderly manner by going from one stage to the next and the complete solution is obtained at the final stage.

Decision-making process consists of selecting a combination of plans from a large number of alternatives. All the combinations must be specifically known beforehand. Then only we can select the optimal policy. If we deal with the problem as a whole it involves lot of computation work and time. Certain combinations may not be feasible. In such a situation dynamic programming becomes very useful. We break the problem into sub-problems (stages). Only one stage is considered at a time and the infeasible combinations are eliminated. The solution is obtained by moving from one stage to the next and is completed at the final stage. At each stage there are many alternatives and the selection of one feasible alternative is called stage decision. The variables which specify the condition of the process and summarize the current status (state) are called state variables. At any stage there could be a finite or infinite number of states.

Bellman's principle of optimality states that "an optimal policy has the property that whatever the initial stages and decisions are, the remaining decisions must constitute an optimal policy with regards to the state resulting from the first decisions". This implies that a wrong decision taken at one stage does not prevent from taking optimum decisions for the remaining stages.

**Q2. What are the characteristics of dynamic programming problem?**

*Ans :* **(Imp.)**

The basic features which characterize the dynamic programming problem are as follows.

1. The problem can be divided into stages with a policy decision required at each stage.
2. Every stage consists of a number of states associated with it. The states are the different possible conditions in which the system may find itself at that stage of the problem.
3. Decision at each stage converts the current stage into state associated with the next stage.
4. The state of the system at a stage is described by a set of variables called state variables.
5. When the current state is known, an optimal policy for the remaining stages is independent of the policy of the previous ones.
6. The solution procedure begins by finding the optimal policy for each state to the last stage.
7. A recursive relationship which identifies the optimal policy for each state with *n* stages remaining, given the optimal policy for each state with (n–1) stages left.
8. Using recursive equation approach, each time the solution procedure moves backward stage by stage for obtaining the optimum policy of each state for the particular stage, till it attains the optimum policy beginning at the initial stag.

**Q3. What are the Advantages of Dynamic Programming ?**

*Ans :*

i) The decision making process consists of selecting a combination of plans from a large number of alternative combinations, which also need a lot of computational work, where too much time is involved. Also, the number of combinations is very large.

ii) These drawbacks can be avoided by using DPP as it divides the given problem into sub-problems or stages. Only one stage is considered at a time and various infeasible combinations an eliminated by reducing the volume of computations.

## 4.2 Applications of Dynamic Programming

**Q4. What are the various applications of dynamic programming ?**

*Ans :* **(Imp.)**

The dynamic programming can be applied to many real-life situations. A sample list of applications of the dynamic programming is given below. The details of these problems are explained while solving them.

1. Capital budgeting problem
2. Reliability improvement problem
3. Stage-coach problem (shortest-path problem)
4. Cargo loading problem
5. Minimizing total tardiness in single machine scheduling problem
6. Linear programming problem

They are discussed in the following sections:

**1. Capital Budgeting Problem**

A capital budgeting problem is a problem in which a given amount of capital is allocated to a set of plants by selecting the most promising alternative for each selected plant such that the total revenue of the organization is maximized. This is demonstrated using a numerical problem.

**2. Reliability Improvement Problem**

Generally, electronic equipments are made up of several components in series or parallel. Assuming that the components are connected in series, if there is a failure of a component in the series, it will make the equipment inoperative. The reliability of the equipment can be increased by providing optimal number of standby units to each of the components in the series such that the total reliability of the equipment is maximized subject to a cost constraint. Application of dynamic programming technique to this problem is illustrated in Example below.

**3. Stage-coach Problem (Shortest-path Problem)**

Stage-coach problem is a shortest-path problem in which the objective is to find the shortest distance and the corresponding path from a given source node to a given destination node in a given distance network. Application of dynamic program-ming technique to this problem is illustrated using Example 8.3.

**4. Cargo Loading Problem**

Cargo loading problem is an optimization problem in which a logistic company is left with the option of loading a desirable combination of items in a cargo subject to its weight or volume or both constraints. In this process, the return to the company is to be maximized. The application of dynamic programming to the cargo loading problem which has only the weight constraint is illustrated.

**5. Minimizing Total Tardiness in Single Machine Scheduling Problem**

Single machine scheduling problem consists of n independent jobs which require processing in the same single machine.

Let,

n be the total number of independent jobs

$t_j$ be the processing time of the job j

$d_j$ be the due date of the job j

$C_j$ be the completion time of the job j

$T_j$ be the tardiness of the job j

$$T_j = C_j - d_j, \text{ if } C_j > d_j$$
$$= 0, \text{ otherwise.}$$

There are many measures of performance which are to be optimized in single machine scheduling problem. The dominant one is the minimization of total tardiness. There will be n\ sequences. The computation of the total tardiness for a sample sequence is demonstrated using the data shown in Table below.

**Table : Data for Sample Single Machine Scheduling Problem**

| Job j | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $t_j$ | 6 | 8 | 3 | 9 |
| $d_j$ | 10 | 16 | 8 | 18 |

Consider a sample sequence: 3-2-1-4. The calculations to determine the total tardiness for this sequence are summarized in Table below.

**Table : Summary of Calculations for Total Tardiness**

| Job j | 3 | 2 | 1 | 4 |
|---|---|---|---|---|
| $t_j$ | 3 | 8 | 6 | 9 |
| $C_j$ | 3 | 11 | 17 | 26 |
| $d_j$ | 8 | 16 | 10 | 18 |
| $T_j$ | 0 | 0 | 7 | 8 |

$$\text{Total tardiness } \sum_{j=1}^{4} T_j = 0 + 0 + 7 + 8 = 15$$

The application of dynamic programming to determine the sequence(s) which minimizes/ minimize the total tardiness in the single machine scheduling problem is illustrated in this section.

**6. Linear Programming Problem**

Technically, linear programme may be formally defined as "a method of optimizing (that is maximizing or minimizing) a linear function with a number of constraints (limitations) expressed in the form of inequa-lities".

(or) Mathematically, the problem of linear programming may be stated as ' 'one of optimizing a linear objective function, under the given constraints and non-negativity of some variable also".

"Linear programming methods are a technique for choosing the best alternative from a set of feasible alternatives in situations where the objective functions as well as the constraints are expressed as linear mathematical functions."



## 4.3 Solution of Linear Programming Problem through Dynamic Programming

**Q5. Explain the formulation of LPP by dynamic programming.**

*Ans :* **(Imp.)**

Consider the general LPP.

$$\text{Max } Z = C_1 x_1 + C_2 + C_2 + x_2 + ... + C_n x_n$$

Subject to the constraints

$$a_{i1} x_1 + a_{i2} x_2 + ... + a_{in} x_n \leq b_i \text{ where } i = 1, 2 ... m$$

$$\text{and } x_j \geq 0 \text{ where } j = 1, 2 ... n.$$

The problem can be formulated as a dynamic programming problem as follows.

Let the general LPP be considered as a multistage problem with each activity j(j = 1, 2 ... m) as a multistage problem with each activity j (j = 1, 2 ... m) as an individual stage. Then this is a n stage problem and the decision variables are the level of activities $x_j (\geq 0)$ at stage j. As $x_j$ is continuous, each activity has an infinite number of alternatives within the feasible region.

We know that allocation problem are the particular type of LPP. These problem required the allocation of available resources to the activities. Each constraint represents the limitation of different resources and $b_1, b_2 ... b_m$ are the amounts of available resource. Since there are m resources, states must be represented by an m-dimensional vector, given by $(\beta_1, \beta_2, ... \beta_m)$.

Let $f_n(\beta_1, \beta_2, ... \beta_m)$ be the maximum value of the general LPP defined for stages $x_1, x_2 ... x_n$ for states $(b_1, b_2 ... b_m)$ using forward recursive equations is

$f_j(\beta_1, \beta_2, ... \beta_m) = \max (C_j x_j + f_{j-1} (\beta_1 - a_{1j} x_2, \beta_2 - a_{2j} x_j ... \beta_m - a_{mj} x_j)$ $0 \le x_j \le \beta$

The maximum value of b that xj can assume is

$$\beta = \min \left\{ \frac{b_1}{a_{1j}}, \frac{b_2}{a_{2j}} ... \frac{b_m}{a_{mj}} \right\}$$

because the minimum value satisfies the set of constraints simultaneously.

**Q6. Explain the assumptions of linear programming.**

*Ans :* **(Imp.)**

Practically speaking, objectives and constraints are linear. Hence, linear programming problems are based on the following assumptions:

**1. Proportionality**

Proportionally is an assumption about both the objective function and functional constraints. The contribution of each activity to the value of the objective function Z is proportional to the level of the Xj activities represented by CjXj the term in the objective function.

Similarly, the contribution of each activity to the left- hand side of each functional constraint is proportional to the level of the activity $x_j$, as represented by the $a_j X_j$ term in the constraint. This means that the constraints increase or decrease proportionately to the level of each activity. This condition represents constant returns to scales rather than economies or dis-economies of scale.

**2. Additively**

Every function in a linear programming model, whether the objective functions or the function on the left- hand side of a functional constraint is the sum of the individual contributions of the respective activities.

It implies that the total contribution of all activities to the constraints is equal to the sum of the contributions for each activity individually. In other words, the whole is equal to the sum of its part.

**3. Divisibility**

This assumption concerns itself with the values allowed for the decision variables. Decision variables in a linear programming model are allowed to have any values, including non-integer values that satisfy the functional and non-negativity constraints.

**4. Certainty**

Certainty is concerned with the parameters of the model namely the coefficient in the objective function Cj, the coefficients in the functional constraints $a_{ij}$ and the right- hand sides of the functional constraints $b_i$.

The value assigned to each parameter of a linear programming model is assumed to be a known constant. This means that each coefficient ($c_i$, $a_{ij}$, $b_i$) is fixed and known with certainty.



## PROBLEMS

**1. Solve the following L.P.P by dynamic programming method, maximize $z = 2x_1 + 5x_2$ subject to $2x_1 + x_2 \le 430$, $2x_2 \le 460$ and $x_1, x_2 \ge 0$.**

*Sol :*

Given data,

Maximize $Z = 2x_1 + 5x_2$

$2x_1 + x_2 \le 430$, $2x_2 \le 460$ and $x_1, x2 \ge 0$

The number of decision variables = 2

Number of stage in the problem = 2

Stage - 1 is assigned to decision variable $x_1$

Stage - 2 is assigned to decision variable $x_2$

**Backward Recursion**

| Stage j | Decision variable | Set of states |
|---|---|---|
| 2 | $x_2$ | $\{b_{12}, b_{22}\}$ |
| 1 | $x_1$ | $\{b_{11}, b_{21}\}$ |

**Stage - 2**

$f_2(b_{12}, b_{22})$ = Maximum $5x_2$

To maintain feasibility, $x_2$ should be minimum of $\frac{b_{12}}{1}$ and $\frac{b_{22}}{2}$, the above objective is modified as follows,

$$f_2(b_{12}, b_{22}) = 5\text{ Min}\left(\frac{b_{12}}{1}, \frac{b_{22}}{2}\right)$$

$$x_2 = \text{Min}\left(\frac{b_{12}}{1}, \frac{b_{22}}{2}\right)$$

**Stage – 1**

$\text{Max} f_1(b_{11}, b_{21})$

$$= \underset{0 \le 2x_1 \le b_{11}}{\text{Max}}\left[2x_1 + f_2(b_{11} - 2x_1), b_{11}\right]$$

$$= \underset{0 \le 2x_1 \le b_{11}}{\text{Max}}\left[2x_1 + 5\text{Min}\left(\frac{b_{11} - 2x_1}{1}\right), \frac{b_{21}}{2}\right]$$

Stage 1 is the last in backward recursion

$b_{11} = 430, b_{21} = 460$

$$x_2 = \frac{460}{2} = 230$$

x1 can range from $0 \le x_1 \le 230$

$$\frac{430 - 2x_1}{1} = 230 \Rightarrow x_1 = 100$$

$$\frac{430 - 2x_1}{1} = 0 \Rightarrow x_1 = 215$$

Range of x1 can be given as

$0 \le x_1 \le 100$ and $100 \le x_1 \le 215$

$\text{Max } f_1(b_{11}, b_{21})$

$$= \text{Max}\begin{bmatrix} 2x_1 + 5\left(\frac{460}{2}\right) \\ 2x_1 + 5\left(\frac{430 - 2x_1}{1}\right) \end{bmatrix}$$

Substituting

$x_1 = 100$, as it is common for both ranges

$$= \text{Max}\begin{bmatrix} 2(100) + 5(230) \\ 2(100) + 5\left(\frac{430 - 2(100)}{1}\right) \end{bmatrix}$$

$$= \text{Max}\begin{bmatrix} 200 + 1150 \\ 200 + 1150 \end{bmatrix} = \text{Max } [1350, 1350]$$

$f_2(b_{11}, b_{21}) = \text{Max}[1350, 1350] = 1350$

$x_1 = 100 \; f_1 = 1350$

To find $x_2$ by substituting the above values in equation below

$b_{12} = b_{11} - 2x_1 \Rightarrow 430 - 2(100) = 230$

$b_{22} = b_{21} - 0 \Rightarrow 460 - 0 = 460$

$$x_2 = \text{Min}\left[\frac{230}{1}, \frac{460}{2}\right] = \text{Min } (230, 230) = 230$$

Optimal solution is,

$x_1 = 100, x_2 = 230$ and Max Z = 1350.

**2. Solve the following problem, Min z = $x_2 + y_2 + z_2$ Subject to x + y + z $\ge$ 10, x,y,z $\ge$ 0**

*Sol :*

Division variables x,y,z and stage $S_1, S_2, S_3$

$$\begin{Bmatrix} S_3 = x + y + z \ge 10 \\ S2 = x + y = S_3 - z \\ s_1 = y_1 = S2 - y \end{Bmatrix} \text{ and}$$

$$\begin{Bmatrix} F_3(S_3) = \min\left(z^2 + F_2(S_2)\right) \\ F_2(S_2) = \min\left(y^2 + F_1(S_1)\right) \\ F_1(S_1) = x^2\left(S^2 - y\right)^2 \end{Bmatrix}$$

Thus,

$F_2(S_2) = [y_2 + (S_2 - y)^2]$

**Calcus Method**

$y^2 + (S_2 - y)^2$ is minimum if its derivative with respect to y is zero

$$\frac{d}{dy}\left[y^2 + (S_2 - y)^2\right] = 0$$

$\Rightarrow$ $2y + 2(S_2 - y)^2 (-1) = 0 \Rightarrow 2y - 2(S_2\text{-}y) = 0$

$\Rightarrow$ $2y - 2S_2 + 2y = 0 \Rightarrow 4y - 2S_2 = 0$

$\Rightarrow$ $2y - S_2 = 0 \Rightarrow y = \dfrac{S_2}{2}$

$$F_2(S_2) = \frac{S_2^2}{2}$$

Now, $F_3(s_3) = \text{Min}\,[z^2 + F_2(S_2)]$

$= \text{Min}\,[z^2 + (S_3 - z)^2/2$ [By suing bellman's principle

Again using calculus method for minimum of the function of single variable z.

$2z - (S3 - Z) = 0 \Rightarrow z = \dfrac{S_3}{3}$

Hence,

$$F_3(S_3) = \frac{S_3^2}{3},\ S_3 \geq 10$$

Since, $F_3(S_3)$ is minimum for $S_3 = 10$, the minimum value of $x_2 + y_2 + z_2$ becomes 33.33, where x = y = z = 3.333.

**3. Solve the following problem, maximize Z = $u_1^2 + u_2^2 + u_3^2$ subject to $u_1^2, u_2^2, u_3^2$ = 6, $u_1$, $u_2$, $u_3 \geq 0$.**

*Sol :*

Let the stage variables be x1, $x_2$ and $x_3$ such that

$x_3 = u_1^2, u_2^2, u_3^2$

$x_2 = u_1^2, u_2^2 = \dfrac{X_3}{u_3^2}$, $x_1 = u_1^2 = \dfrac{X_3}{u_2^2.u_3^2}$

Applying the recursive formula

$f_1(x_1) = \text{Maxi}\,[u_1^2]$

$f_2(x_2) = \underset{u_2}{\text{Maxi}}\left\{u_1^2.u_2^2\right\} = \underset{u_2}{\text{Maxi}}\left\{\dfrac{X_3}{u_3^2}\right\}$

$= \underset{u3}{\text{Maxi}}\left\{\dfrac{X_2}{u_2^2}\right\}$

$$\frac{\partial f_2(x_2)}{\partial(u_2)} = \frac{-2x_2}{u_3^3}$$

$$f_3(x_3) = \underset{u3}{\text{Maxi}}\left\{u_3^2.f_2(x_2)\right\}$$

$$\underset{u3}{\text{Maxi}}\left\{u_3^2.\frac{X_3}{u_2^2}\right\}$$

Which after differentiating $f_3(x_3)$ with respect to $u_3$ and equating to zero, gives $u_3 = \frac{x_3^2}{4}$

Hence,

$$f_3(x_2) = \frac{x_3^2}{4} = \frac{36}{4}$$

Thus the maximum value of $u_1^2 + u_2^2 + u_3^2 = 36$

with $u_1 = 1$, $u_2 = 2$ and $u_3 = 3$

If the variables $u_1$, $u_2$ and $u_3$ are positive integers only, the problem can be solved by other methods also like tabular method (or) enumeration method.

**4. Solve the following Linear Programming Problem using dynamic programming technique, Maximize**

**$Z = 30x_1 + 15x_2 + 6x_3$ Subject ot**

**$6x_1 + 8x_2 + 9x_3 \leq 210;$**

**$12x_2 + 6x_3 \leq 180; x_1, x_2$ and $x_3 \geq 0.$**

*Sol :*

The number of decision variables in the given problem is equal to 3. So, there will be three stages (i.e., stage-1) is assigned to the decision variable $x_1$, stage-2 is assigned to the decision variable $x_2$ and stage-3 is assigned to the decision variable x3).

Since backward recursion is used to solve the given problem, stage-3 is to be considered first. The sets of states of different stages are presented in below table.

| Stage j | Decision variable | Set of states |
|---|---|---|
| 3 | $x_3$ | $\{b_{13}, b_{23}, b_{33}\}$ |
| 2 | $x_2$ | $\{b_{12}, b_{22}, b_{32}\}$ |
| 1 | $x_1$ | $\{b_{11}, b_{21}, b_{31}\}$ |

**Table : Sets of Stages of Different Stages**

Recursive function for stage-3 with respect to $x_3$ is based on the backward recursion. Therefore,

$f_3\ \{b_{13}, b_{23}, b_{33}\}$

$$\text{Max}\ (6x_3) = \begin{matrix} 0 & 9x_3 & b_{13} \\ 0 & 6x_3 & b_{23} \end{matrix}$$

To maintain feasibility, $x_3$ should be the minimum of $\frac{b_{13}}{9}$ and $\frac{b_{23}}{6}$, the above objective function is modified as follows,

$$f_3\{b_{13}, b_{23}, b_{33}\} = 60 \min\left(\frac{b_{13}}{9} \frac{b_{23}}{6}\right)$$

$$x_3^* = \min\left(\frac{b_{13}}{9} \frac{b_{23}}{6}\right)$$

Recursive function for stage 2 with respect to $x_2$ is $f_1\{b_{12}, b_{22}, b_{32}\}$

$$= \underset{0 \le 6x_1 \le b_{12}}{\text{Max}}\left[15x_2 + f_3\left(\frac{b_{12} - 6x_1}{8}\right)\frac{b_2^2}{12}, \frac{b_3^2}{b}\right]$$

The stage -1 is the last in the series of backward recursion. Therefore,

$b_{12} = 210$, $b_{21} = 180$

To determine the upper limit $x_1$, we have,

$x_1 = 35$, $x_2 = 0$ and $x_3 = 0$

Maximum,

$Z = 30x_1 + 15x_2 + 6x_3 = 30 \times 35 + 1 + 0 = 1050.$

---

**5. Solve the folloiwng LPP using dynamic programming technique**

**Maximize $Z = 10X_1 + 30X_2$**

**Subject to**

**$3x_1 + 6x_2 \le 168$**

**$12x_2 \le 240$**

**$X_1$ and $X_2 \ge 0$**

*Sol :*

The number of decision variables in the given problem is equal to 2. So, there will be two stages (i.e. stage 1 is assigned to the decision variable $X_x$ and stage 2 is assigned to the decision variable $X_2$). Since backward recursion is used to solve the problem, stage 2 is to be considered first. The sets of states of different stages are summarized in Table below.

**Table : Sets of States of different Stages**

| Stage j | Decision variable | Set of states |
|---|---|---|
| 2 | $X_2$ | $\{b_{12}, b_{22}\}$ |
| 1 | $X_1$ | $\{b_{11}, b_{21}\}$ |

Recursive function for stage 2 with respect to $X_2$ is based on the backward recursion. therefore,

$$f_2(b_{12}, b_{22}) = \max_{\substack{0 \le 6X_2 \le b_{12} \\ 0 \le 12X_2 \le b_{22}}} 30X_2$$

To maintain feasibility, $X_2$ should be the minimum of $b_{12}/6$ and $b_{22}/12$, the above objective function is modified as follows :

$$f_2(b_{12}, b_{22}) = 30 \min\left(\frac{b_{12}}{6}, \frac{b_{22}}{12}\right) \text{ and } X_2^* = \min\left(\frac{b_{12}}{6}, \frac{b_{22}}{12}\right)$$

Recursive function for stage 1 with respect to $X_1$ is

$$f_1(b_{11}, b_{21}) = \max_{0 \le 3X_1 \le b_{11}} [10X_1 + f_2(b_{11} - 3x_1, b_{21})]$$

$$= \max_{0 \le 3X_1 \le b_{11}} \left[10X_1 + 30 \min\left(\frac{b_{11} - 3X_1}{6}, \frac{b_{21}}{12}\right)\right]$$

The stage 1 is the last in the seris of backward recursion, Therefore, $b_{11} = 168$ and $b_{21} = 240$

Determine the upper limit $X_1^*$, we have

$$f_1\left(\frac{X_1}{b_{11}}, b_{21}\right) = \max\left[f_1\left(\frac{X_1}{168}, 240\right)\right]$$

$$= \max\left[10X_1 + 30 \min\left(\frac{168 - 3X_1}{6}, \frac{240}{12}\right)\right]$$

Now, to which in the ranges of $X_1$ is defined, $(168 - 3X_1)/6$ can be as high as 20 or as low as 0. So, equate it to 20 as well as to 0 and solve for $X_1$, as explained below

$$\text{or } X \frac{168 - 3X_1}{6} = 20 \text{ or } X_1 = 16$$

$$\text{or } X \frac{168 - 3X_1}{6} = 0 \text{ or } X_1 = 56$$

the ranges for $X_1$ are follows :

$0 \le X_1 \le 16$ and $16 \le X_1 \le 56$

Now, $f_1(x_1/b_{11}, b_{21})$ is rewritten as :

$$f_l\left(\frac{X_1}{b_{11}}, b_{21}\right) = \max\begin{bmatrix} 10X_1 + 30\min\left(\frac{168 - 3X_1}{6}\right), & 0 \le X_1 \le 16 \\ 10X_1 + 30\min\left(\frac{168 - 3X_1}{6}, 20\right) & 16 \le X_1 \le 56 \end{bmatrix}$$

$$f_l\left(\frac{X_1}{168}, 240\right) = \max\begin{bmatrix} 10X_1 + 30 \times 20, & 0 \le X_1 \le 16 \\ 10X_1 + 30 \times \dfrac{168 - 3X_1}{6} & 16 \le X_1 \le 56 \end{bmatrix}$$

$$= \max\begin{bmatrix} 10X_1 + 600, & 0 \le X_1 \le 16 \\ 10X_1 + 840 - 15X_1 & 16 \le X_1 \le 56 \end{bmatrix}$$

$$= \max\begin{bmatrix} 10X_1 + 600, & 0 \le X_1 \le 16 \\ 10X_1 + 840 - 15X_1 & 16 \le X_1 \le 56 \end{bmatrix}$$

The minimize each of the above cases, substitute 16 for X1 Now, we get

$$f_1\left(\frac{X_1}{168}, 240\right) = \max\,(760m\ 760) = 760$$

Therefore

$$X_1^* = 16 \text{ and } f_1\left(\frac{X_1}{168}, 240\right) = 760$$

For tracing the value of $X_1^*$, we have

$$b_{12} = b_{11} - 3X_1 = 168 - 3 \times 16 = 120$$

$$b_{22} = b_{21} - 0 = 240 - 0 = 240$$

therefore,

$$X_1^* = \min\left(\frac{b_{12}}{6}, \frac{b_{22}}{12}\right) = \min\left(\frac{120}{6}, \frac{240}{12}\right) = \min\,(20, 20) = 20$$

The optimal results are :

$$X_1^* = 16,\ X_1^* = 20,\ Z\,(\text{optimum}) = 760$$

**6. Use dynamic programming to solve the following LPP.**

**Max $Z = 3x_1 + 5x_2$**

**Subject to the constraints**

$$x_1 \le 4$$

$$x_2 \le 6$$

$$3x_1 + 2x_2 \le 18$$

$$x_1, x_2 \le 0$$

*Sol :*

The problem consists of three resources and two decision variables. Hence the problem has two stages and 3 state variables.

Let $B_{1j}$, $B_{2j}$, $B_{3j}$ be the stte of the system at stage j and $f_j$ ($B_{1j}$, $B_{2j}$, $B_{3j}$) be the optimal (maximum) value of the objective function for state j = 1, 2 given the state ($B_{1j}$, $B_{2j}$, $B_{3j}$). Using backward computation procedure, we have

$$f_2(B_{1j}, B_{2j}, B_{3j}) = \text{Max } [5x_2]$$

$$0 \le x_2 \le P_{22}$$

$$0 \le 2x_2 \le P_{32}$$

$$= 5 \text{ Max } (x_2)$$

$$0 \le x_2 \le \beta_{22}$$

$$0 \le x_2 \le \frac{\beta_{32}}{2}$$

Since Max $x_2$ which satisfies $0 \le x_2 \le \beta_{22}$,

$0 \le x_2 \le \frac{\beta_{32}}{2}$ is the minimum of $\beta_{22}, \frac{\beta_{32}}{2}$

$$\text{i.e., Max } (x_2) = x_2^* = \text{Min}\left(\beta_{22}, \frac{\beta_{32}}{2}\right) \quad \text{... (1)}$$

$$\therefore f_2(\beta_{12}, \beta_{22}, \beta_{32}) = 5 \text{ Min}\left(\beta_{22}, \frac{\beta_{32}}{2}\right) \quad \text{... (2)}$$

Also

$$f_1(\beta_{11}, \beta_{21}, \beta_{31}) = \text{Max}$$

$$0 \le x_1 \le \beta_{11}[3x_1 + f_1(\beta_{11} - x_1; \beta_{21} - 0; \beta_{31} - 3x_1]$$

$$0 \le 3x_1 \le \beta_{31}$$

From $f_2(\beta_{11} - x_1, (\beta_{21} - 0, \beta_{11} - 3x_1) = 5 \text{ min}\left(\beta_{21}, \frac{\beta_{31} - 3x_1}{2}\right)$

$$\therefore f_1(\beta_{11}, \beta_{21}, \beta_{31}) = \text{Max}$$

$$0 \le x_1 \le \beta_{11}\left[3x_1 + 5 \text{ min}\left(\beta_{21}, \frac{\beta_{31} - 3x_1}{2}\right)\right]$$

$$0 \le 3x_1 \le \frac{\beta_{31}}{3}$$

Since it is a two stage problem, at the first stage

$$\beta_{11} = 4, \beta_{21} = 6, \beta_{31} = 18$$

$\therefore$ $f_1(\beta_{11}, \beta_{21}, \beta_{31})$ = Max

$$0 \le x_1 \le 4 \quad \left\{3x_1 + 5 \min\left(6, \frac{18 - 3x_1}{2}\right)\right\}$$

$$0 \le x_1 \le 6 = \text{Max}$$

$$0 \le x_1 \le 4 \left\{3x_1 + 5 \min\left(6, \frac{18 - 3x_1}{2}\right)\right\} \quad \text{... (3)}$$

Now Min

$$0 \le x_1 \le 4 \left(6, \frac{18 - 3x_1}{2}\right) = \begin{cases} 6 & \text{if } 0 \le x_1 \le 2 \\ \dfrac{18 - 3x_1}{2} & \text{if } 2 \le x_1 \le 4 \end{cases}$$

From (3)

$$f_1(\beta_{11}, \beta_{21}, \beta_{31}) = \text{Max} \begin{Bmatrix} 3x1 + 5(6) \text{ if } 0\ x_1 \le 2 \\ 3x1 + 5\left(\dfrac{18 - 3_{x1}}{2}\right) \text{if } 2 \le x_1 \le 4 \end{Bmatrix}$$



Since Max of $3x_1 + 30$, $0 \le x \le 2$ occurs at $x_1 = 2$ and

Max of $\frac{90 - 9x_1}{2}$, $2 \le x_1 \le 4$ also occurs at $x_1 = 2$

$$\therefore \quad f_1(\beta_{11}, \beta_{21}, \beta_{31}) = 3 \times 2 + 30 = 36$$

$$\text{Now } x_2 = \text{Min } \left\{\beta_{21}, \frac{\beta_{31} - 3x_1}{2}\right\}$$

$$= \text{Min } \left\{6, \frac{18 - 3x_1}{2}\right\} = \text{Min } (6, 6) = 6$$

The optimal solution is Max Z = 36, $x_1 = 2$, $x_2 = 6$

**7. Use dynamic programming to solve the LPP max $Z = x_1 + 9x_2$**

**Subject to the constraints**

$$2x_1 + x_2 \leq 25$$

$$x_2 \leq 11$$

$$x_1\ x_2 \geq 0$$

*Sol :*

The problem has two resources and two decision variables. The states of the equivalent dynamic programming are $\beta_{1j}, \beta_{2j}$, j = 1, 2.

$$f_2(b_{12}, b_{22}) = \text{Max}\ (9x_2)$$

$$0 < x_2 \leq 25$$

$$0 < x_2 \leq 11$$

i.e., $f_2(\beta_{12}, \beta_{22}) = 9\ \text{Max}\ (x_2)$

$= 9\ \text{Max}\ (25, 11)$

Since the Max of $x_2$ satisfying the conditions of $x_2 \leq 25$, $x_2 \leq 11$, is the Min of (25, 11)

$$\therefore\ x_2^* = 11$$

Now $f_1\ (\beta_{11}, \beta_{21}) = \text{Max}\ [x_1 + f_2(\beta_{11} - 2x_1, \beta_{21} - 0)]$

$$0 \leq x_1 \leq \frac{25}{2}$$

At this last stage, substitute $\beta_{11} = 25$, $\beta_{21} = 11$

$$f_1(25, 11) = \text{Max}\ [x_1 + 9\ \text{min}\ (25 - 2x_1, 11)]$$

Min $(25 - 2x_1, 11) = \{11, 0 \leq x_1 \leq 7\}$

$$25 - 2x_1,\ 7 \leq x_1 \leq \frac{25}{2}$$

$$\therefore\ x_1 + 9\ \text{min}\ (25 - 2x_1, 11) = x_1 + 99,\ 0 \leq x_1 \leq 7$$

$$225 - 17x_1,\ 7 \le x_1 \le \frac{25}{2}$$

Since the maximum of both $x_1 + 99$, $225 - 17x_1$ occur at $x_1 = 7$

$$f_1(25, 11) = 7 + 9 \min(11, 11)$$

$$= 106 \text{ at } x_1^* = 7$$

$$x_1^* = \min(25 - 2x_1^*, 71) = \min(11, 11) = 11$$

Hence the optimum solution is

$$x_1^* = 7,\ x_2^* = 11 \text{ and Max } Z = 106$$

# UNIT V

**Game Theory**: Introduction, Game with Pure Strategies, Game with Mixed Strategies, Dominance Property, Graphical Method for 2 x n or m x 2 Games, Linear Programming Approach for Game Theory.

## 5.1 Game Theory

### 5.1.1 Introduction

**Q1. Define Game Theory. What are the characteristics of Game Theory ?**

*Ans :* **(Imp.)**

In the competitive world, it is essential for an executive to study or at least guess the activities or actions of his competitor. Moreover, he has to plan his course of actions or reactions or counter actions when his competitor uses certain technique. Such war or game is a regular feature in the market and the competitors have to make their decisions in choosing their alternatives among the predicted outcomes so as to maximize the profits or minimizing the loss.

**Characteristics**

There can be various types of games. They can be classified on the basis of the following characteristics.

i) **Chance of Strategy :** If in a game activities are determined by skill, it is said to be a game of strategy; if they are determined by change, it is a game of chance. In general, a game may involve game of strategy as well as a game of chance.

ii) **Number of Persons :** A game is called an n-person game if the number of persons playing in n. The person means an individual or a group aiming at a particular objective.

iii) **Number of Activities :** These may be finite or infinite.

(iv) **Number of Alternatives (choices) Available to Each Person** in a particular activity may also be finite or infinite. A finite game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is said to be infinite.

(v) **Information to the players about the past activities of other players** is completely available, partly available, or not available at all.

(vi) **Payoff :** A quantitative measure of satisfaction a person gets at the end of each play is called a payoff. It is a real-valued function of variables in the game. Let $v_i$ be the payoff to the player $P_i$, $1 \leq i \leq n$, in an n-person game. If $\Sigma_i^n v = 0$ then the game is said to be a zero-sum game.

**Q2. State the basic terminology are used in game theory.**

*Ans :*

1. **Game Theory :** A mathematical theory based on which strategy steps are employed to win a game played in a conflicting situation to maximize the benefits (or profit) or to minimize the damage (or loss).

2. **Game :** A competitive situation having the following characteristics:

   (i) The situation is competitive.

   (ii) There are a finite number of competitors (or players).

   (iii) Each player has a finite number of strategies available to him or her.

(iv) The game is said to be played when both competitors initiate actions based on their chosen strategies.

(v) Every game results in an outcome.

(vi) Every outcome has stakes, i.e., payment given or taken.

3. **Number of players :** A game involving only two players (competitors) is called a two-person game. If the number of players exceeds two, then the game is known as "'n'-person game" where 'n' denotes the number of players.

4. **Sum of gains and losses :** If in a game the gains of one player are exactly the same as the losses to another player, such that the sum of the gains and losses equals zero, then the game is said to be a "zero-sum game". Otherwise it is said to be "non-zero sum game".

5. **Strategy :** It is a "plan of action" conceived and carefully executed by each party to the game. It involves a list of all possible actions (or moves or courses of action) that a player will take for every outcome (pay-off) that might arise. The rules governing the choices are known in advance to the players. Also the outcome resulting from a particular choice is also known to the player in advance and is expressed in terms of numerical values. The players need not have a definite information about each other's strategies.

6. **Optimal strategy :** A particular strategy by which a player optimizes his gains or losses without knowing the competitor's strategies is called "optimal strategy".

7. **Pure strategy :** This is a predetermined plan of action based on which the games are played and which does not change during the game. It is a decision rule which is always used by the player to select the particular course of action. Each player knows in advance of all strategies available to each and out of which he or she always selects only one particular strategy irrespective of the strategy the opponent may choose. Pure strategy is used by a player to achieve the objective of maximizing the gains or minimizing the losses.

8. **Mixed strategy :** It is a plan of action which is changed while the game is in progress, when both players are guessing as to which course of action to be selected on a particular occasion with some fixed probability. Thus, there is a probabilistic situation and objective of the players is to maximize expected gains or to minimize expected losses by making a choice among pure strategies with fixed probabilities.

9. **Pay-off :** The outcome of the game is known as "pay-off'.

10. **Pay-off Matrix :** A table (in the form of a matrix) showing the outcome of the game (in terms of gains or losses) when different strategies are adopted by the players.

11. **Fair game :** A game is said to be fair when the value of the game is zero.

12. **Value of the game :** The maximum guaranteed expected outcome per play when players follow their optimal strategy is called the "value of the game".

13. **Solution of a game :** When the best strategies of both players are determined and the value of the game is determined, we say, the game is solved or the solution of the game is obtained.

14. **Maximin :** The maximum value of the minimum pay-offs in each row.

15. **Minimax :** The minimum value of the maximum pay-offs in each column.

16. **Saddle point :** The game value is called the saddle point in which each player has a pure strategy. The saddle point is the lowest numerical value in a row and the largest numerical value in e column, which are equal to each other.

17. **Strictly determinable game :** A game is said to be strictly determinable if the maximin value is equal to the minimax value. In other words for the optimal strategy for both players, the pay-off for both players will be the same, i.e., the gain of one player equals the loss of another."

**Q3. State the Assumptions of game Theory**

*Ans :* **(Imp.)**

**Assumptions**

The underlying assumptions, the rules of the game as given as follows :

1. The player act rationally and intelligently.
2. Each player has available to him a finite set of possible courses of action.
3. The player attempt to maximize gains and minimize losses.
4. All relevant information is known to each players.
5. The players make individual decisions without direct communication.
6. The players simultaneously select their respective courses of action.
7. The pay off is fixed and determined in advance.

**Q4. Explain the advantages and disadvantages of game theory**

*Ans :* **(Imp.)**

**Advantages**

1. Game theory keeps deep insight to few less known aspects, which arise in situations of conflicting interests.
2. Game theory creates a structure for analysis of decision-making in various situations like interdependence of firms etc.
3. For arriving at optimal strategy, game theory develops a scientific quantitative technique for two person zero-sum games.

**Disadvantages**

1. The highly unrealistic assumption of game theory is that the firm has prior knowledge about its competitor's strategy and is able to construct the payoff matrix for possible solutions, which is not correct. The main fact is that any firm is not exactly aware of its competitor's strategy. He can only make guesses about its strategy.
2. The hypothesis of maximin and minimax clearly shows that players are not risk lover and have whole knowledge about the strategies but the fact is that it is not possible.
3. It is totally impractical to understand that the several strategies followed by the rival player against others lead to an endless chain.
4. Most economic problems occur in the game if many players are involved in comparison to two-person constant sum game, which is not easy to understand. For example, the number of sellers and buyers is quite large in monopolistic competition and the game theory does not provide any solution to it.
5. In real market situations, it is doubtful to find the use of mixed strategies for making non zero-sum games.

**Q5. Explain briefly about pay off matrix.**

*Ans :*

Payoff is the outcome of playing the game. A payoff matrix is a table 'showing the amount received by the player named at the left hand side after all possible plays of the game. The payment is made by the player named at the top of the table.

If a player A has m-courses of action and player B has n-courses, then a payoff matrix may be constructed using the following steps.

(i) Row designations for each matrix are the course of action available to A

(ii) Column designations for each matrix are the course of action avail-able to B

(iii) With a two person zero sum game, the cell entries in B's payoff matrix will be the negative of the corresponding entries in A's pay-off matrix and the matrices will be as shown below.

$$\text{Player A}\overset{\text{Player B}}{\begin{bmatrix} 1 & 2 & 3 & \cdots & j & \cdots & n \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & & & \cdots & & \cdots & \\ m & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}}$$

A's payoff matrix

$$\text{Player A}\begin{matrix} \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ m \end{matrix}\overset{\text{Player B}}{\begin{matrix} \begin{matrix} 1 & 2 & 3 & \cdots & j & \cdots & n \end{matrix} \\ \begin{bmatrix} -a_{11} & -a_{12} & a_{13} & \cdots & -a_{1j} & \cdots & -a_{1} \\ -a_{21} & -a_{22} & a_{23} & \cdots & -a_{2j} & \cdots & a_{2} \\ \vdots & & & & & & \\ -a_{i1} & -a_{i2} & -a_{i3} & \cdots & -a_{1j} & \cdots & a_{in} \\ \vdots & & & \cdots & & \cdots & \\ -a_{m1} & -a_{m2} & -a_{m3} & \cdots & -a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{matrix}}$$

**Q6. State the different types of games.**

*Ans :* **(Imp.)**

1. **Two-person games and n-person games:** In two person games the players may have many possible choices open to them for each play of the game but the number of players remain only two. Hence it is called a two person game. In case of more than two persons, the game is generally called n-person game.

2. **Zero Sum Game:** A zero sum game is one in which the sum of the payment to all the competitors is zero for every possible outcome of the game is in a game if the sum of the points won equals the sum of the points lost i.e.

3. **Two person zero sum game:** A game with two players, where the gain of one player equals the loss to the other is known as a two person Zero sum game. It is also called a rectangular game because their payoff matrix is in the rectangular form. The characteristics of such a game are

   (i) Only two players participate in the game

   (ii) Each player has a finite number of strategies to use

   (iii) Each specific strategy results in a payoff

   (iv) Total payoff to the two players at the end of each play is zero.

**Q7. State the assumptions of a two person zero sum game**

*Ans :*

1. Each player has available with him a finite number of possible courses of action. The list may not be same for each player.
2. Player A attempts to maximize gains and player B minimize losses.
3. The decisions of both players are made individually prior to the play with no communication between them.
4. The decisions are made simultaneously and also announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
5. Both the players know not only possible payoffs to themselves but also that of each other.

## 5.2 GAME WITH PURE STRATEGIES

**Q8. What is Saddle Point ? How do you determine Saddle Point ?**

*Ans :* **(Imp.)**

The saddle point in a pay-off matrix is one which is the smallest value in its row and the largest value in its column.

It is also known as equilibrium point in the game theory. An element of a matrix that is simultaneously minimum of the row in which it occurs and the maximum of the column in which it occurs is a saddle point of the matrix. In a game having a saddle point, optimum strategy for player A is always to play the row containing a saddle point and for the player B to play the column that contains a saddle point. Saddle point also gives the value of a game. If there is saddle point, we say the corresponding strategies as optimum strategies. A game with no saddle point is solved by adopting mixed strategies.

**Steps**

**Step 1 :** Write pay-off matrix.

**Step 2 :** Select the minimum value in each row of the pay-off matrix under the head row minimum written at right end of each row put a circle (◯) over these elements in the pay-off matrix.

**Step 3 :** Select large among the row minima found from step-2 and write beneath are row minima.

**Step 4 :** Select largest value (maximum) of each column and put a box or rectangle ( □ ) over these values in the pay-off matrix. Write these values of column maximum beneath the pay-off matrix.

**Step 5 :** Select minimum value of column maxima (found in step-4) and write this at the right end.

**Step 6 :** If Max ($R_{min}$) = Min ($C_{max}$) i.e., value in step-3 equals value in step-5 then saddle point exists. Thus saddle point is found at the element on which both circle and box are enclosed. Note this pay-off elements as the value of the game.

**PROBLEMS**

**1. For the following pay-off matrix for firm A, determine the optimal strategies for both the firms and the value of the game (using maximin-minimax principle).**

**Firm B**

**Firm A** $\begin{bmatrix} 3 & -1 & 4 & 6 & 7 \\ -1 & 8 & 2 & 4 & 12 \\ 16 & 8 & 6 & 14 & 12 \\ 1 & 11 & -4 & 2 & 1 \end{bmatrix}$

*Sol :*

It is evident from the given matrix, that

minimum of first row = min (3, –1, 4, 6, 7)

= –1

minimum of second row = min (–1, 8, 2, 4, 12)

= –1

minimum of third row = min (16, 8, 6, 14, 12)

= 6

minimum of fourth row = min (–1, –1, –4, 2, 1)

= – 4

∴ Maximin value = max (–1, –1, 6, –4, 6)

= 6

Again,

maximum of first column = max (3, –1, 16, 1)

= 16

maximum of second column

= max (–1, 8, 8, 11) = 11

maximum of third column

= max (4, 2, 6, –4) = 6

maximum of fourth column

= max (6, 4, 14, 2) = 14

maximum of fifth column

= max (7, 12, 12, 1) = 12

∴ Minimax value

= max (16, 11, 6, 14, 12) = 6

The saddle point exists at the position (3, 3) and thus the optimal strategy for firm A is its 3rd strategy and the optimal strategy for firm B is its 3rd strategy. The value of the game of A is 6.

**2. The pay-off matrix of a game is given below. Find the solution of game to A and B.**

**Player B**

| | | I | II | III | IV | V |
|---|---|---|---|---|---|---|
| **Player A** | **I** | –2 | 0 | 0 | 5 | 3 |
| | **II** | 4 | 2 | 1 | 2 | 5 |
| | **III** | –4 | –3 | 0 | –3 | 6 |
| | **IV** | 5 | 1 | –5 | –2 | –6 |

*Sol :*

We first find out the saddle point by encircling each row minima and putting squares around each column maxima. Thus, we obtain the saddle point which is enclosed by a circle and a square both, as shown below :

B ↑ Optimum Strategy for B

| | I | II | III | IV | V | Row Minimum |
|---|---|---|---|---|---|---|
| 1 | 2 | 0 | 0 | 5 | 3 | |
| Optimum Strategy For A 2 | 4 | 2 | 1 | 2 | 5 | 1 (Minimax Value) |
| 3 | –4 | –3 | 0 | –3 | 6 | –4 |
| 4 | 5 | 1 | –5 | –2 | –6 | –6 |
| Column-Maximum | 5 | 3 | 1 | 5 | 5 | |

Maximin-Value

Hence, the solution to this game is given by :

i) the best strategy for player A is 2nd

ii) the best strategy for player B is 3rd

iii) the value of the game is 1 to A and –1 to B.

**3. Solve the following game**

| | | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|---|
| **A** | $A_1$ | –5 | –2 | 0 | 7 |
| | $A_2$ | 5 | –6 | 0 | 7 |
| | $A_3$ | 4 | 0 | 2 | 3 |

*Sol :*

Finding $R_{min}$ and then maximum among $R_{min}$

| | | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $R_{min}$ |
|---|---|---|---|---|---|---|
| A | $A_1$ | –5 | –2 | 0 | 7 | –5 |
| | $A_2$ | 5 | –6 | –4 | 8 | –6 |
| | $A_3$ | 4 | 0 | 2 | 3 | 0 |
| | | | | | | 0 |

Frin $C_{max}$ and then min of $C_{max}$

| | | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $R_{min}$ |
|---|---|---|---|---|---|---|
| A | $A_1$ | –5 | –2 | 0 | 7 | –5 |
| | $A_2$ | 5 | –6 | –4 | 8 | –6 |
| | $A_3$ | 4 | 0 | 2 | 3 | 0 ← Max ($R_{min}$) |
| | $C_{max}$ | 5 | 0 | 2 | 8 | 0 ↑ Min ($C_{min}$) |

As Max ($R_{min}$) = Min ($C_{max}$)

Saddle point exists at $A_3$, $B_2$

$\therefore$ A uses pure (deterministic) strategy $A_3$

B uses pure (deterministic) strategy $B_2$

and value of the game (v) = 0

i.e., Game is drawn (i.e., neither A nor B wins)

(This is a fair game as v = 0).

**4. Solve the game whose pay off matrix is given by.**

$$\text{Player A}\begin{matrix} & B_1 & B_2 & B_3 \\ A_1 & 1 & 3 & 1 \\ A_2 & 0 & -4 & -3 \\ A_3 & 1 & 5 & -1 \end{matrix}$$

(Player B across the columns)

*Sol :*

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{Row minima} \\ \hline A_1 & 1 & 3 & 1 & 1 \\ A_2 & 0 & -4 & -3 & -4 \\ A_3 & 1 & 5 & -1 & -1 \\ \hline \text{Column maxima} & 1 & 5 & 1 & \end{array}$$

Maxi (minimum) = Max (1, –4, –1) = 1

Mini (maximum) = Min (1, 5, 1) = 1.

i.e., Maximin value $\underline{\gamma}$ = 1 = Minimax value $\overline{\gamma}$

$\therefore$ Saddle point exists. The value of the game is the saddle point which is 1. The optimal strategy is the position of the saddle point and is given by ($A_1$ $B_1$).

**5. For what value of 1, the game with the following matrix is strictly determinable?**

$$\text{Player A}\begin{matrix} & B_1 & B_2 & B_3 \\ A_1 & \lambda & 6 & 2 \\ A_2 & -1 & \lambda & -7 \\ A_3 & -2 & 4 & \lambda \end{matrix}$$

(Player B across the columns)

*Sol :*

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{Row minima} \\ \hline A_1 & \lambda & 6 & 2 & 2 \\ A_2 & -1 & \lambda & -7 & -7 \\ A_3 & -2 & 4 & \lambda & -2 \\ \hline \text{Column maxima} & -1 & 6 & 2 & \end{array}$$

The game is strictly determinable, if

$\underline{\gamma} = \gamma = \overline{\gamma}$. Hence $\overline{\gamma} = 2$, $\overline{\gamma} = -1$

$\Rightarrow -1 \le \lambda 2.$

**6. Determine which of the following two person zero sum games are strictly determinable and fair. Given the optimum strategy for each player in the case of strictly determinable games.**

(a) $$\text{Player A}\begin{matrix} & B_1 & B_2 \\ A_1 & -5 & 2 \\ A_2 & -7 & -4 \end{matrix}$$

(b) $$\text{Player A}\begin{matrix} & B_1 & B_2 \\ A_1 & 1 & 1 \\ A_2 & 4 & -3 \end{matrix}$$

(Player B across the columns)

*Sol :*

(a)

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{Row minima} \\ \hline A_1 & -5 & 2 & -5 \\ A_2 & -7 & -4 & -7 \\ \hline \text{Column maxima} & -5 & 2 & \end{array}$$

Since $\underline{\gamma} = \overline{\gamma} - 5 = 0$, the game is strictly determinable. There exists as saddle point = –5. Hence the value of the game is –5. The optimal strategy is the position of the saddle point given y ($A_i$, $B_i$).

(b)

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{ccc} B_1 & B_2 & \text{Row minima} \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \end{array}\begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}\begin{array}{c} 1 \\ -3 \end{array} \\ \text{Column maxima} & 4 \quad 2 \end{array}$$

Maxi (minimum) = $\underline{\gamma}$ = Max (1, –3) = 1.

Mini (maximum) = $\overline{\gamma}$ = Min (4, 1) = 1.

Since $\underline{\gamma} = \overline{\gamma} = 1 \neq 0$, the game is strictly determinable. The value of game is 1. The optimal strategy is ($A_2$, $B_2$).

**7. Solve the game whose payoff matrix is given below.**

$$\begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix}$$

*Sol :*

|  |  | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | Row minima |
|---|---|---|---|---|---|---|---|
| Player A | $A_1$ | –2 | 0 | 0 | 5 | 3 | –2 |
|  | $A_2$ | 3 | 2 | 1 | 2 | 2 | 1 |
|  | $A_3$ | –4 | –3 | 0 | –2 | 6 | –4 |
|  | $A_4$ | 5 | 3 | –4 | 2 | –6 | –6 |
| Column maxima |  | 5 | 3 | 1 | 5 | 6 |  |

Maxi (minimum) = $\underline{\gamma}$ = Max (–2, 1, –4, –6)

= 1.

Mini (maximum) = $\gamma$ = Min (5, 3, 1, 5, 6)

= 1.

Since $\underline{\gamma}$ = $\overline{\gamma}$ $\neq$ 1, there exists a saddle point. The value of the game is 1. The position of the saddle point is the optimal strategy and is given [$A_2$. $B_3$].

## 5.3 Game with Mixed Strategies

**Q9. Explain the Game without Saddle Point.**

*Ans :* **(Imp.)**

The problems of game where the maximin is not equal to minmax i.e., the saddle point does not exist are taken as mixed strategies. It means both players will mix different strategies with certain probabilities to optimize.

A mixed strategy can be solved by following methods :

(i) 2 × 2 game – Arithmetic Method

(ii) 2 × m or n × 2 game – Graphical Method

(iii) m × n game when both m and n > 2 - Linear programming method

**Q10. What is arithmetic method of solving game theory ?**

*Ans :*

**Solution of 2 × 2 Games :**

Consider a 2 × 2 game with the pay-off matrix

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \end{array}$$

Let $p_i$ be the probability player A plays row i with i = 1, 2, and $q_j$ be the probability player B plays column j with j = 1, 2,. Since $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$

We can write $p_2 = 1 - p_1$ ; $q_2 = 1 - q_1$

**Algorithm :**

1. Examine the pay-off matrix for a saddle point.

   If exists, the optimal minimax strategies are pure strategies. The saddle point is the value of a game. If a saddle point does not exist, go to step 2.

Let $p_1^* = \dfrac{p_{22} - p_{21}}{p_{11} + p_{22} - p_{12} - p_{21}}$ ; $p_2 = 1 - p_1$

$q_1^* = \dfrac{p_{22} - p_{12}}{p_{11} + p_{22} - p_{12} - p_{21}}$ ; $p_2 = 1 - q_1$

will be optimal strategies for players A and B

2. The value of the game is

$$V = \frac{p_{11}p_{22} - p_{21}p_{12}}{p_{11} + p_{22} - (p_{12} + p_{21})}$$

**PROBLEMS**

**8. Solve the following 2 × 2 game.**

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} B_1 \quad B_2 \\ \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

*Sol :*

Since the given matrix has no saddle point, calculate optimal strategies as follows :

For player A; $p_1 = \dfrac{p_{22} - p_{21}}{p_{11} + p_{22} - p_{12} - p_{21}}$

$$= \frac{4-3}{5+4+1-3} = \frac{1}{5}$$

$$p_1 = 1 - p_1 = 1 - \frac{1}{5} = 4/5$$

$$q_1 = \frac{p_{22} - p_{12}}{p_{11} + p_{22} - p_{12} - p_{21}}$$

$$= \frac{4-3}{5+4-1-3} = \frac{3}{5}$$

$$q_2 = 1 - q_1 = 1 - \frac{3}{5} = 2/5$$

The value of the game is

$$V = \frac{5 \times 4 - 1 \times 3}{5 + 4 - (1+3)} = \frac{20-3}{9-4} = \frac{17}{5}$$

Optimal strategies for player A and B are

$$\left(\frac{1}{5}, \frac{4}{5}\right) \text{ and } \left(\frac{3}{5}, \frac{2}{5}\right)$$

The value of the game is $17/5$

**9. Find the optimal strategies and value of the game of following 2 × 2 pay off matrix.**

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} B_1 \quad B_2 \\ \begin{bmatrix} 3 & -5 \\ -1 & 1 \end{bmatrix} \end{array}$$

*Sol :*

Since given matrix has no saddle points, the optimal strategies are calculate as follows.

For Player A : $p_1^* = \dfrac{1-(-1)}{3+1-(-1)-(-5)} = \dfrac{2}{10}$;

$$p_2^* = 1 - p_1^* = \frac{8}{10}$$

For Player B : $q_1^* = \dfrac{1-(-5)}{10} = \dfrac{6}{10}$;

$$q_2^* = 1 - q_1^* = \frac{4}{10}$$

The value of the game is

$$V = \frac{3 \times 1 - (-1) \times (-5)}{3 + 1 - (-5-1)} = \frac{-2}{10}$$

Optimal strategies for player A and B are

$$\left(\frac{2}{10}, \frac{8}{10}\right) \text{ and } \left(\frac{6}{10}, \frac{4}{10}\right)$$

The value of the game is $= \dfrac{2}{10}$.

**10. Solve the give game matrix.**

**B's strategy**

$$\textbf{A's strategy} \quad \begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} b_1 \quad b_2 \\ \begin{bmatrix} 12 & -8 \\ -6 & 4 \end{bmatrix} \end{array}$$

*Sol :*

Test whether saddle point exists using the Maximin-Minimax principle.

| | $b_1$ | $b_2$ | Row minima |
|---|---|---|---|
| $a_1$ | 12 | –8 | –8 |
| $a_2$ | [– 6] | (4) | [– 6] ← Maximin value |
| Column maxima | 12 | (4) ← Minimax value | |

No saddle point exists. Proceed with algebraic method.

Let A plays a strategy $a_1$ with a probability x and strategy $a_2$ with a probability (1 – x).

1. Then, (1 – x) will be probability that player B will play strategy $a_2$.
2. Let B play strategy $b_1$ with probability y and strategy $b_2$ with probability (1 – y).

   Given that,

$$a_{11} = 12, a_{12} = -8, a_{21} = -6, a_{22} = 4$$

**Value of x ($a_1$)**

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{4 - (-6)}{(12 + 4) - (-8 - 6)}$$

$$= \frac{4 + 6}{16 - (-14)}$$

$$= \frac{10}{16 + 14}$$

$$= \frac{10}{30} = \frac{1}{3}$$

$$a_2 = 1 - x = 1 - \frac{1}{3} = \frac{2}{3}$$

**Value of y ($b_1$)**

$$y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{4 - (-8)}{(12 + 4) - (-8 - 6)}$$

$$= \frac{12}{30} = \frac{2}{5}$$

$$b_2 = 1 - y$$

$$= 1 - \frac{2}{5} = \frac{3}{5}$$

**Value of Game v**

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(12 \times 4) - (-8 \times -6)}{(12 + 4) - (-8 - 6)}$$

$$= \frac{48 - 48}{30} = 0$$

**Optimal Strategy**

| | Strategy | Probability |
|---|---|---|
| For player A | $a_1$ | $\frac{1}{3}$ |
| | $a_2$ | $1 - \frac{1}{3} = \frac{2}{3}$ |
| For player B | $b_1$ | $\frac{2}{5}$ |
| | $b_2$ | $1 - \frac{2}{5} = \frac{3}{5}$ |
| Value of game | | v = 0 |

i.e., it is a fair game.

**11. In a game of machine coins with two players, suppose A wins one unit value, when there are two heads, wins nothing when there are two tails, and losses $\frac{1}{2}$ unit of value when there is one head and one tail. Determine the payoff matrix, the best strategy for each player, and the value of the game to A.**

*Sol :*

For the given problem matrix formulated is,

$$\begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix} \end{array}$$

Test whether saddle point exists using Maximin-Minimax principle.

| | H | T | Row minima |
|---|---|---|---|
| | 1 | $-1/2$ | $-1/2$ |
| | $\boxed{-1/2}$ | $\textcircled{0}$ | $\boxed{-1/2}$ ← Maximin value |
| Column maxima | 1 | $\textcircled{0}$ ← Minimax value | |

Since, Maximin value $\neq$ Minimax value

Saddle point does not exist.

1. Let A play strategy H with probability x and strategy T with probability (1 – x).
2. Let B play strategy H with probability y and strategy with (1 – y).

$$\therefore \; a_{11} = 1,\; a_{12} = -\frac{1}{2},\; a_{21} = -\frac{1}{2},\; a_{22} = 0$$

**Value of x(H)**

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{0 - \left(\frac{-1}{2}\right)}{(1+0) - \left(\frac{-1}{2} - \frac{1}{2}\right)}$$

$$\Rightarrow \quad x = \frac{\frac{1}{2}}{(1+1)} = \frac{1}{4},$$

$$T = 1 - x = 1 - \frac{1}{4} = \frac{3}{4}$$

**Value of y (H)**

$$y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{0 - \left(\frac{-1}{2}\right)}{(1+0) - \left(\frac{-1}{2} - \frac{1}{2}\right)}$$

$$\Rightarrow \quad y = \frac{1}{4},$$

$$T = 1 - y = 1 - \frac{1}{4} = \frac{3}{4}$$

**Value of Game v**

$$v = \frac{(a_{11} \times a_{22}) - (a_{12} \times a_{21})}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(1 \times 0) - \left(\frac{1}{4}\right)}{(1+0) - \left(-\frac{1}{2} - \frac{1}{2}\right)}$$

$$= \frac{-\frac{1}{4}}{\frac{3}{4}} = -\frac{1}{4} \times \frac{4}{3} = -\frac{1}{3}$$

**Optimal Strategy**

| | Strategy | Probability |
|---|---|---|
| For player A | H | $\frac{1}{4}$ |
| | T | $\frac{3}{4}\left(1 - \frac{1}{4} = \frac{3}{4}\right)$ |
| For player B | H | $\frac{1}{4}$ |
| | T | $\frac{3}{4}\left(1 - \frac{1}{4} = \frac{3}{4}\right)$ |

$\therefore$ Value of game, $v = \frac{-1}{3}$

## 5.4 Dominance Property

**Q11. Explain the concept of Dominance.**

*Ans :* **(Imp.)**

Principle of dominance is also applicable to pure strategy and mixed strategy problems. The principle of dominance states that if the strategy of a player dominates over the other strategy in all conditions than the later strategy can be ignored because it will not affect the solution in any way. A strategy dominates over the other only if it is preferable over other in all conditions. For instance, from the gainer point of view, if a strategy gives more gain than another strategy for all conditions (for all strategies of loser), then first strategy dominates over the other and second strategy can be ignored altogether. Similarly from lower point of view, if a strategy involves lesser loss then other in all conditions, then second can be omitted without affecting decision.

**Rules for Dominance :**

1. If all the element in a row are less than or equal to the corresponding elements in another row, then that row is dominated.
2. If all the elements in a column are greater than or equal to the corresponding elements in another column, then that column is dominated.
3. Dominated rows columns are deleted which reduces the size of the game.

**PROBLEMS**

**12. Following is the pay-off matrix for Player A**

$$\begin{bmatrix} 3 & 5 & 4 & 2 \\ 5 & 6 & 2 & 4 \\ 2 & 1 & 4 & 0 \\ 3 & 3 & 5 & 2 \end{bmatrix}$$

**Using the dominance properly, obtain the optimum strategies for both the players and determine the value of the game.**

*Sol :*

Since all elements of row 1 dominates row 3, i.e., 3 > 2, 5 > 1, 4 = 4, 2 > 0

Therefore row 3 can be eliminated. The matrix reduces to

$$\begin{bmatrix} 3 & 5 & 4 & 2 \\ 5 & 6 & 2 & 4 \\ 3 & 3 & 5 & 2 \end{bmatrix}$$

All elements in column 2 dominates column 1. Therefore column 2 can be eliminated. The matrix reduces to

$$\begin{bmatrix} 3 & 4 & 2 \\ 5 & 2 & 4 \\ 3 & 5 & 2 \end{bmatrix}$$

Column 1 dominates column 3, so column 1 can be eliminated to give

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \\ 5 & 2 \end{bmatrix}$$

Row 3 dominates row 1, so row 1 can be eliminated. The game reduces to 2 × 2 sub game

$$\begin{bmatrix} 2 & 4 \\ 5 & 2 \end{bmatrix}$$

The optimum strategies for Player A and Player B are

$$\left(0, \frac{3}{5}, 0, \frac{2}{5}\right) \text{ and } \left(0, 0, \frac{2}{5}, \frac{3}{5}\right)$$

The volume of the game is = $\frac{16}{5}$.

**13. Use dominance to reduce the size of the following game to 2 × 2 game and hence find the optimal strategies and value of the game.**

| | | Player B | | | |
|---|---|---|---|---|---|
| | | **I** | **II** | **III** | **IV** |
| | **I** | **3** | **2** | **4** | **0** |
| **Player A** | **II** | **3** | **4** | **2** | **4** |
| | **III** | **4** | **2** | **4** | **0** |
| | **IV** | **0** | **4** | **0** | **8** |

*Sol :*

**Step 1 :**

Since all elements of the third row are greater than or equal to the corresponding entries in the first row, the first row is dominated by third row and row 1 is deleted.

| | I | II | III | IV |
|---|---|---|---|---|
| I | 3 | 4 | 2 | 4 |
| II | 4 | 2 | 4 | 0 |
| IV | 0 | 4 | 0 | 8 |

**Step 2 :**

Since all elements of column 1 are greater than or equal to corresponding elements in the column II. So delete column I.

| | II | III | IV |
|---|---|---|---|
| II | 4 | 2 | 4 |
| III | 2 | 4 | 0 |
| IV | 4 | 0 | 8 |

**Step 3 :**

The first column is dominated by average of column III and IV i.e.,

$$4 > \frac{3+4}{2};\ 2 = \frac{4+0}{2};\ 4 = \frac{0+8}{2}.$$

So column II is deleted. We get

$$\begin{array}{c} \\ II \\ III \\ IV \end{array}\begin{array}{c} \begin{array}{cc} II & IV \end{array} \\ \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ 0 & 8 \end{bmatrix} \end{array}$$

**Step 4 :**

Row II is equal to the average of rows III and IV. Delete Row II. We get

$$\begin{array}{c} \\ III \\ IV \end{array}\begin{array}{c} \begin{array}{cc} III & IV \end{array} \\ \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \end{array}$$

$$p_1^* = \frac{8-0}{(4+8)-(0+0)}$$

$$= \frac{2}{3};\ p_2 = 1 - p_1$$

$$= \frac{1}{3}$$

and value of the game is given by

$$V = \frac{4\times8-0\times0}{(4+8)-(0+0)} = \frac{8}{3}$$

Hence, the optimal strategy for A is $\left(0\ 0\ \frac{2}{3}\ \frac{1}{3}\right)$, for B $\left(0\ 0\ \frac{2}{3}\ \frac{1}{3}\right)$ and value of game is

$= \frac{8}{3}$.

**14. Solve the following game using dominance rule:**

**Player B**

| **Player A** | | **$B_1$** | **$B_2$** |
|---|---|---|---|
| | **$A_1$** | **1** | **3** |
| | **$A_2$** | **4** | **5** |
| | **$A_3$** | **9** | **–7** |
| | **$A_4$** | **–3** | **–4** |
| | **$A_5$** | **2** | **1** |

*Sol :*

The second row dominates on the first row, i.e., elements of second row are greater than first row. So, eliminating the first row, we get the following matrix:

| | $B_1$ | $B_2$ |
|---|---|---|
| $A_2$ | 4 | 5 |
| $A_3$ | 9 | –7 |
| $A_4$ | –3 | –4 |
| $A_4$ | 2 | 1 |

Now first row dominates on the fourth row. Hence eliminating fourth row, we get the following matrix:

| | $B_1$ | $B_2$ |
|---|---|---|
| $A_2$ | 4 | 5 |
| $A_3$ | 9 | –7 |
| $A_4$ | –3 | –4 |

The first dominates on the third row, so reduced matrix will be as follows :

| | $B_1$ | $B_2$ |
|---|---|---|
| $A_2$ | 4 | 5 |
| $A_3$ | 9 | –7 |

Hence the reduced payoff matrix of order $2\times2$ is as follows :

Player B

$$\text{Player A} \begin{bmatrix} 4 & 5 \\ 9 & -7 \end{bmatrix}$$

Let consider the optimum mixed strategies for player A and B is in the following form :

$$S_A = \begin{bmatrix} A_2 & A_3 \\ p_1 & p_2 \end{bmatrix}, p_1 + p_2 = 1$$

and $$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}, q_1 + q_2 = 1$$

Where,

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$$

$$p_1 = \frac{-7-9}{(4-7)-(5+9)} = \frac{-16}{-17} = \frac{16}{17}$$

$$p_2 = 1 - \frac{16}{17} = \frac{1}{17}$$

$$q_1 = \frac{-7-5}{(4-7)-(5+9)} = \frac{-12}{-17} = \frac{12}{17}$$

$$p_2 = 1 - \frac{16}{17} = \frac{1}{17}$$

$$q_1 = \frac{-7-5}{(4-7)-(5+9)} = \frac{-12}{-17} = \frac{12}{17}$$

$$q_2 = 1 - \frac{12}{17} = \frac{5}{17}$$

The optimum strategy of the given payoff matrix is given by the following:

$$S_A = \begin{bmatrix} A_2 & A_3 \\ \frac{16}{17} & \frac{1}{17} \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{12}{17} & \frac{5}{17} \end{bmatrix}$$

The values of the game

$$(v) = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} - a_{21})}$$

$$= \frac{(4 \times -7) - (5 \times 9)}{(4 - 7) - (5 + 9)} = \frac{-28 - 45}{-17} = \frac{73}{17}$$

**15. Reduce the following game with the help of dominance rule and then determine the value of the game:**

**Player B**

| Player A | | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|---|
| | $A_1$ | 3 | 1 | 4 | 0 |
| | $A_2$ | 3 | 4 | 2 | 4 |
| | $A_3$ | 4 | 2 | 4 | 0 |
| | $A_4$ | 0 | 4 | 0 | 8 |

*Sol :*

Since this matrix does not contain any saddle point, hence we have to reduce the size of pay-off matrix using dominance rule. The third row dominates on the first row, so after eliminating first row, we get the following pay-off matrix:

Player B

| Player A | | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|---|
| | $A_2$ | 3 | 4 | 2 | 4 |
| | $A_3$ | 4 | 2 | 4 | 0 |
| | $A_4$ | 0 | 4 | 0 | 8 |

Again, it is clear from the above matrix that the first column dominates on the third column, hence eliminating the first column, we get the following:

Player B

| Player A | | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| | $A_2$ | 4 | 2 | 4 |
| | $A_3$ | 2 | 4 | 0 |
| | $A_4$ | 4 | 0 | 8 |

There is no row or column is dominated on others. On the other hand, the first column is dominated by the average of second and third columns:

$$\begin{bmatrix} \frac{2+4}{2} \\ \frac{4+0}{2} \\ \frac{0+8}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Now after eliminating the first column, we get the following pay-off matrix:

Player B

| Player A | | $B_3$ | $B_4$ |
|---|---|---|---|
| | $A_2$ | 2 | 4 |
| | $A_3$ | 4 | 0 |
| | $A_4$ | 0 | 8 |

Similarly, average of second and third rows $\left(\frac{4+0}{2}, \frac{0+8}{2}\right) = (2,4)$ dominated by the first row, hence eliminating the first row, we get the 2 x 2 game matrix as follows:

Player B

| Player A | | $B_3$ | $B_4$ |
|---|---|---|---|
| | $A_3$ | 4 | 0 |
| | $A_4$ | 0 | 8 |



Let this matrix be of the following form :

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

The optimum mixed strategies for player A and B is given by following matrices:

$$S_A = \begin{bmatrix} A_3 & A_4 \\ p_1 & p_2 \end{bmatrix}, \; p_1 + p_2 = 1$$

and $S_B = \begin{bmatrix} B_3 & B_4 \\ q_1 & q_2 \end{bmatrix}, \; q_1 + q_2 = 1$

Where,

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, \; p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, \; q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$$

$$p_1 = \frac{8 - 0}{(4 + 8) - (0 + 0)} = \frac{8}{12} = \frac{2}{3}$$

$$p_2 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$q_1 = \frac{8 - 0}{(4 + 8) - (0 + 0)} = \frac{8}{12} = \frac{2}{3}$$

$$q_2 = 1 - \frac{2}{3} = \frac{1}{3}$$

The optimum strategy of the game is given by the following matrices:

$$S_A = \begin{bmatrix} A_3 & A_4 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_3 & B_4 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

The values of the game

$$(v) = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(4 \times 8) - (0 \times 0)}{(4 + 8) - (0 + 0)} = \frac{32}{12} = \frac{8}{3}$$

## 5.5 Graphical Method for 2 x n OR m x 2 Games

**Q12. Explain the graphical method for 2 × n or m × 2 games.**

*Ans :*

Principle of dominance is applied to rectangular games without saddle point so that the size of the matrix can be reduced and appropriate solution method can be employed. After applying the dominance principle, if the size of the matrix is reduced to 2 × 2, then the algebraic method can be used to solve the game similar to Type-II rectangular games without saddle point.

But if the size reduces to 2 × n or m × 2, then a new solution methodology called graphical method is used.

By using graphical approach, it is aimed to reduce the game to the size of 2 × 2 by identifying and eliminating dominated strategies and then solve it by the analytical method i.e., (algebraic method).

**Graphical Method**

The steps of the graphical method are summarized as follows,

Ensure the game has no saddle point.

**Step 1**

Using dominance rule, reduce the size of the m × n matrix to 2 × n or m × 2 matrix.

**Step 2**

Draw two vertical parallel axis of distance one unit each axis representing the two strategies of player A, in case of 2 × n matrix and player B, in case of m × 2 matrix.

**Step 3**

Plot the strategies on the graph, one line for each strategy. Thus, there will be n lines for 2 × n matrix and m lines form m × 2 matrix.

**Step 4**

Identify the,

(i) Maximum ordinate of the convex set bounded above (lower envelope) for 2 × n matrix and

(ii) Minimum ordinate of the convex set bounded below (upper envelope) for m × 2 matrix.

**Step 5**

This ordinate gives the value of the game.

**Step 6**

Identify the 2 strategies, out of the *m* or *n* strategies which, bounds the convex set. The 2 strategies lead to reduction of the size of the payoff matrix to 2 × 2.

**Step 7**

Solve the 2 × 2 matrix by algebraic method.



**PROBLEMS**

**16. Solve the following game graphically.**

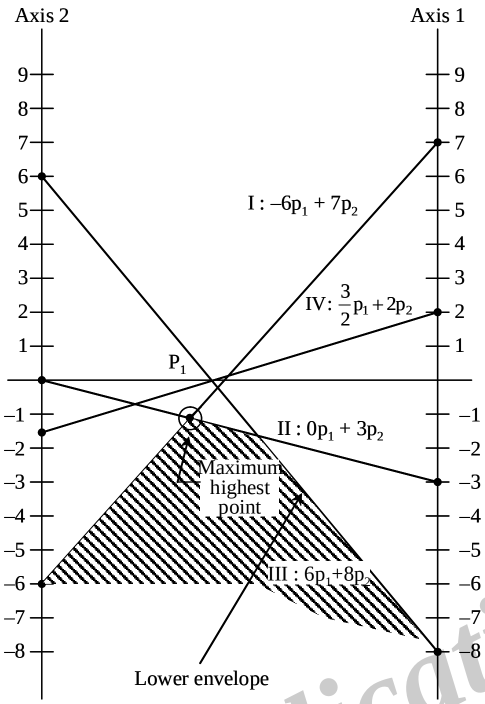
$$\begin{bmatrix} -6 & 0 & 6 & -3/2 \\ 7 & -3 & -8 & 2 \end{bmatrix}$$

*Sol :*

Let row player's (say A) strategies are $A_1$ and $A_2$ are used with the probabilities $p_1$ and $p_2$, then his expected pa-offs when his opponent (column player) uses his pure strategies are shown below :

| Column players pure strategies | Row player (A's) expected pay-off |
|---|---|
| I | $-6p_1 + 7p_2$ |
| II | $0p_1 - 3p_2$ |
| III | $0p_1 - 8p_2$ |
| IV | $-(3/2)\ p_1 + 2p_2$ |

These are graphically represented as follows as two parallel axes of unit distance apart.




The highest point on lower envelop appear at the intersection of the lines represented by column players I and II strategies.

$-6p_1 + 7p_2$ and $0p_1 - 3p_2$

$\therefore$ The required $2 \times 2$ matrix is

$$\begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{bmatrix} -6 & 0 \\ 7 & -3 \end{bmatrix} \end{array}$$

We have $-6p_1 + 7p_2 = 0p_1 - 3p_2 \Rightarrow 6p_1 = 0p_2$

$p_1 + p_2 = 1$

$p_1 = \frac{10}{16} = \frac{5}{8}, \; p_2 = \frac{6}{16} = \frac{3}{8}$

and $-6q_1 + 0q_2 = 7q_1 - 3q_2$

$\Rightarrow 13q_1 = 3q_2$

$\frac{q_1}{q_2} = \frac{3}{13}$ and $q_1 + q_2 = 1$

$\therefore q_1 = \frac{3}{16}, \qquad q_2 = \frac{13}{16}$

∴ Row player will use his strategies with the probabilities are (5/8, 3/8) respectively where the column player uses the first two strategies mix at the probabilities of (3/16, 13/16).

The value of the game

$$-6 \times \frac{5}{8} + 7 \times \frac{3}{8} = -\frac{9}{8}$$

or $$0 \times \frac{5}{8} - 3 \times \frac{3}{8} = -\frac{9}{8}$$

The column player wins the game with a gain of 9/8 units or row player loses the game with a loss of 9/8 units.

**17. Solve the following game using graphical method:**

**Player B**

| Player A | | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|---|
| | $A_1$ | –5 | 5 | 0 | –1 | 8 |
| | $A_2$ | 8 | –4 | –1 | 6 | –5 |

*Sol :*

After calculating the saddle point, it is clear that this does not contain any saddle point. Next, we have to reduce the size of the pay-off matrix using dominance rule. As the dominance rule cannot reduce the matrix, so let apply the graphical method. The player A's expected pay-offs along with the player B's pure strategies are shown in table below:

| Player B's Pure Strategies | Player A's Expected Payoffs |
|---|---|
| 1 | $-5x_1 + 8(1-x_1) = -13x_1 + 8$ |
| 2 | $5x_1 - 4(1-x_1) = 9x_1 - 4$ |
| 3 | $0x_1 - 1(1-x_1) = x_1 - 1$ |
| 4 | $-lx_1 + 6(1-x_1) = -7x_1 + 6$ |
| 5 | $8x_1 - 5(1-x_1) = 13x_1 - 5$ |

Figure shows the graph for B's strategies where five lines are plotted as the functions of X]. The two lines ($A_1$ and $A_2$) parallel to each other are at a distance of one unit. Join the mark –5 located on line $A_1$ with mark 8 located on the line $A_2$ for representing B's first strategy. Similarly, joint mark 5 located on line A] with mark – 4 located on the line $A_2$ for representing B's second strategy and so on for other three strategies and bound the graph as shown in figure.

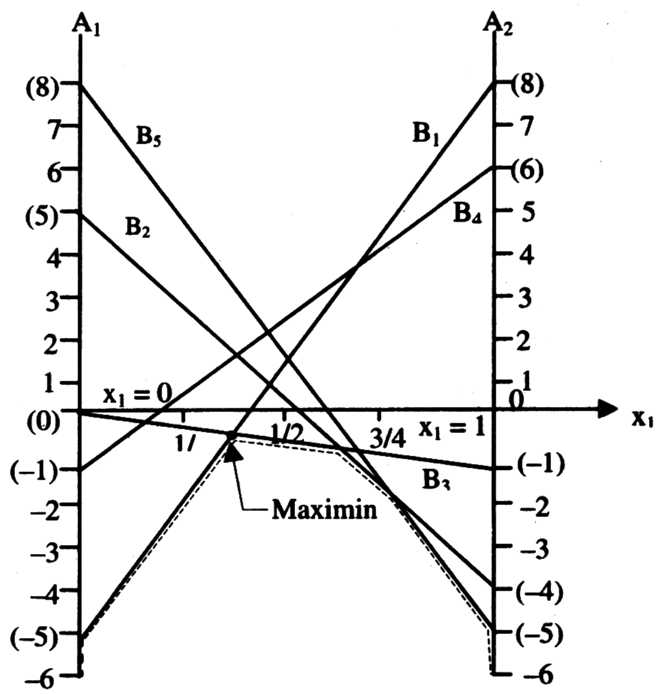



**Fig.:**

As the player A wants to maximize the minimum expected payoff, the two lines which are cutting at the upper point of the lower bound represent the two course of action Bi and $B_3$ that player B select in his best strategy.

Thus one can reduce the $2 \times 5$ game matrix into $2 \times 2$ matrix that can be easily solved with the help of arithmetic method.

Now, the pay-off matrix of $2 \times 2$ game will be as follows:

Player B

| Player A | | $B_1$ | $B_3$ |
|---|---|---|---|
| | $A_1$ | –5 | 0 |
| | $A_2$ | 8 | –1 |

This matrix will be of the following form:

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

The optimum mixed strategies for player A and B illustrated by following matrices:

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}, \; p_1 + p_2 = 1$$

and $$S_B = \begin{bmatrix} B_1 & B_3 \\ q_1 & q_2 \end{bmatrix}, \; q_1 + q_2 = 1$$

Where,

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})},\ p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})},\ q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$$

$$p_1 = \frac{-1-8}{(-5-1)-(0+8)} = \frac{-9}{-14} = \frac{9}{14}$$

$$p_2 = 1 - \frac{9}{14} = \frac{5}{14}$$

$$q_1 = \frac{-1-0}{(-5-1)-(0+8)} = \frac{-1}{-14} = \frac{1}{14}$$

$$q_2 = 1 - \frac{1}{14} = \frac{13}{14}$$

The optimum strategy of payoff matrix is now as follows :

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{9}{14} & \frac{5}{14} \end{bmatrix},\ S_B = \begin{bmatrix} B_1 & B_3 \\ \frac{1}{14} & \frac{13}{14} \end{bmatrix}$$

The values of the game

$$(v) = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(-5 \times -1) - (0 \times 8)}{(-5-1) - (0+8)} = -\frac{5}{14}$$

**18. Solve the following game using graphical method:**

**Player B**

**Player A**

| | | |
|---|---|---|
| **Player A** | **–3** | **1** |
| | **5** | **3** |
| | **6** | **–1** |
| | **1** | **4** |
| | **2** | **2** |
| | **0** | **–5** |

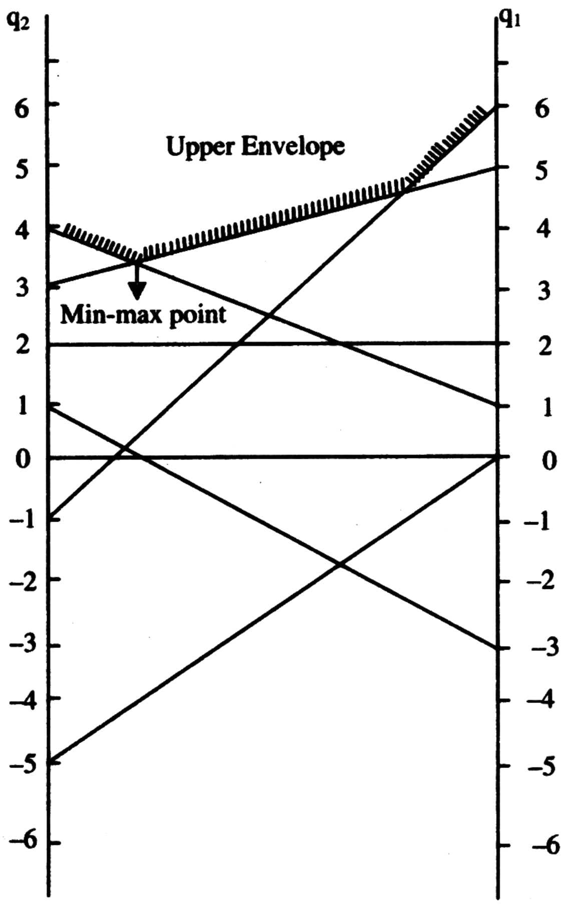
*Sol :*

As this problem does not contain any saddle point, we have to apply the mixed strategy. Let assume that player B's mix strategy is given by $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_2 = 1 - qi$ against player A.

The player B's mix strategy is illustrated in table below:

| A's Pure Strategy | B's Expected Pay-Off |
|---|---|
| $A_1$ | $-3q_1 + q_2$ |
| $A_2$ | $5q_1 + 3q_2$ |
| $A_3$ | $6q_1 - q_2$ |
| $A_4$ | $q_1 + 4q_2$ |
| $A_5$ | $2q_1 + 2q_2$ |
| $A_6$ | $0q_1 - 5q_2$ |

Let take the min-max point located on the upper envelope of B's expected pay-offs as the player B wants to minimize his maximum expected pay-off.




**Figure**

Now the reduced form of pay-off matrix will be as follows:

Player B

$$\text{Player A}\quad \begin{array}{c} \\ A_2 \\ A_4 \end{array}\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix} \end{array}$$

The optimum strategies are is given by the following matrices:

$$S_A = \begin{bmatrix} A_2 & A_4 \\ p_1 & p_2 \end{bmatrix},\ S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{4 - 1}{(5 + 4) - (3 + 1)} = \frac{3}{5}$$

$p_1 = 1 – p_1 = 1 – 3/5 = 2/5$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4 - 3}{(5 + 4) - (3 + 1)} = \frac{1}{5}$$

$q_2 = 1 – q_1 = 1\ 1/5 = 4/5$

$$\text{Value of the game } v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{5 \times 4 - 3 \times 1}{(5 + 4) - (3 + 1)} = \frac{17}{5}$$

**19. Solve the following 2 × 2 game graphically**

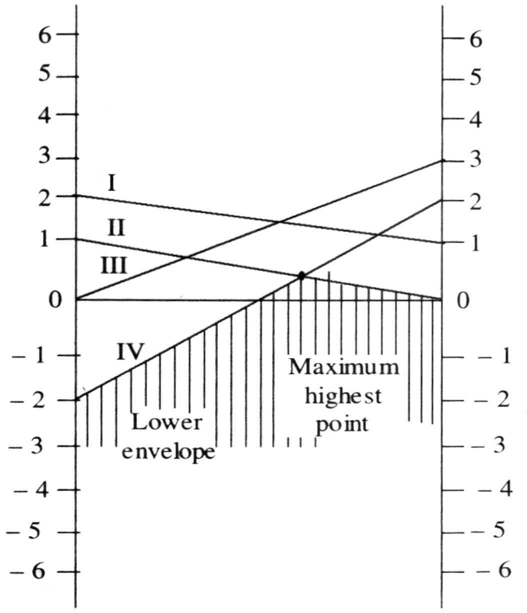
**Player B**

$$\textbf{Player A}\ \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{-2} \\ \mathbf{1} & \mathbf{0} & \mathbf{3} & \mathbf{2} \end{bmatrix}$$

*Sol :*

Column player pure strategies Row players (A/s) expected payoff

$$\begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array}\begin{bmatrix} 2x_1 & + & 1x_2 \\ 1x_1 & + & 0x_2 \\ 0x_1 & + & 3x_2 \\ -2x_1 & + & 2x_2 \end{bmatrix}$$




The highest point on lower envelope appears at the intersection of the lines represented by column II and IV strategies.

$1x_1 + 0x_2$ and $-2x_1 + 2x_2$

The required 2 × 2 matrix is,

Player B

$$\text{Player A}\quad \begin{matrix} & q_1 & q_2 \\ P_1 & 1 & 0 \\ P_2 & -2 & 2 \end{matrix}$$

1. Let A play strategy $P_1$ with 'x' as probability and $P_2$ with (1 – x) as probability.
2. Let B play strategy $q_1$ with 'y' as probability and $q_2$ with (1 – y) as probability.

**Value of x** $= \dfrac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

$$= \frac{2-(-2)}{(1+2)-(0+(-2))} = \frac{4}{(3)-(-2)} = \frac{4}{5}$$

$$1 - x = 1 - \frac{4}{5} = \frac{1}{5}$$

**Value of y** $= \dfrac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

$$= \frac{2-(0)}{(1+2)-(0+(-2))} = \frac{2}{(3)-(-2)} = \frac{2}{5}$$

$$1 - y = 1 - \frac{2}{5} = \frac{3}{5}$$

Value of the game

$$\nu = \frac{(a_{11} \times a_{22}) - (a_{12} \times a_{21})}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(1 \times 2) - (0 \times (-2))}{(1+2)-(0+(-2))} = \frac{2-(-0)}{3-(-2)} = \frac{2}{5}$$

**Optimal Strategy**

Row player $\left[\frac{4}{5}, \frac{1}{5}\right]$

Column player $\left[\frac{2}{5}, \frac{3}{5}, 0, 0\right]$

Value of the game, $\nu = \frac{2}{5}$

**20. Solving the game whose payoff matrix is,**

$$\begin{matrix} & & B & & \\ & \text{I} & \text{II} & \text{III} & \text{IV} \\ \text{I} & 1 & 4 & -2 & -3 \\ \text{II} & 2 & 1 & 4 & 5 \end{matrix}$$

*Sol :*

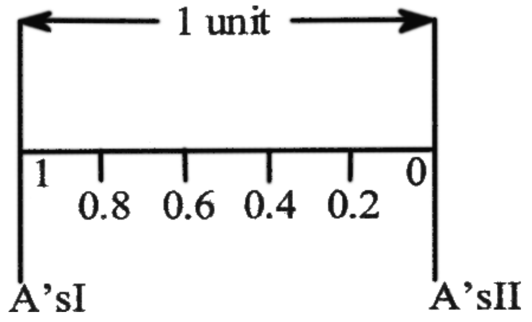
The game has no saddle point

**Step 1**

The size of the matrix is already of the required size 2 × n to use graphical method.

**Step 2**

Draw two parallel lines to represent the 2 strategies of A with a distance of one unit.




Figure

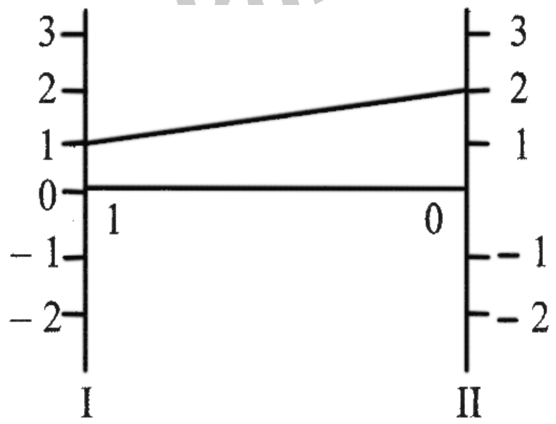
**Step 3**

Plot the strategies of B on the graph.

For example,

| | | B: I | B: II |
|---|---|---|---|
| A | I | 1 | |
| | II | 2 | |

B's 1 strategy is plotted as (1, 2) i.e, 1 on As'1 and 2 on A's II and join the two points to form a straight line.

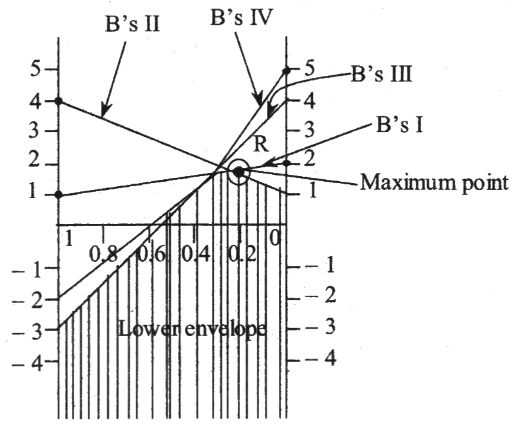



Figure

Similarly, plot the other strategies

**Step 4**

Identify the maximum value of the lower envelope as it is a 2 × n matrix.




**Step 5**

Maximum point is R i.e., value of game $\cong$ 2.

**Step 6**

The two strategies of B passing through the maximum point are B's I and B's II. Thus the reduced 2 × 2 matrix is,



| | | B: I | B: II |
|---|---|---|---|
| A | I | 1 | 4 |
| | II | 2 | 1 |

**Step 7**

Using algebraic method, the given matrix is solved.

$a_{11}$ = 1, $a_{12}$ = 4, $a_{21}$ = 2 and $a_{22}$ = 1

(i) Let the player of select strategy I with probability x and strategy II with probability 1 – x.

(ii) Let the player B of select strategy I with probability y and strategy II with probability 1 – y.

**Value of x**

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{1 - 2}{(1 + 1) - (4 + 2)}$$

$$= \frac{-1}{2-6} = \frac{-1}{-4} = \frac{1}{4}$$

$$1 - x = 1 - \frac{1}{4} = \frac{3}{4}$$

**Value of y**

$$y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{1-4}{(1+1)-(4+2)} = \frac{-3}{-4} = \frac{3}{4}$$

$$1 - y = 1 - \frac{3}{4} = \frac{1}{4}$$

**Value of Game (v)**

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(1 \times 1) - (4 \times 2)}{(1+1)-(4+2)}$$

$$= \frac{-7}{-4} - \frac{7}{4}$$

**Optimal Strategy**

| Players | Strategy | Probability |
|---|---|---|
| For player A | I | $\frac{1}{4}$ |
| | II | $\frac{3}{4}$ |
| For player B | I | $\frac{3}{4}$ |
| | II | $\frac{1}{4}$ |
| | III | 0 |
| | IV | 0 |

Value of game, $v = \frac{7}{4}$

**21. Solving the following game using graphical method.**

B's Strategy

$$\text{A's Strategy} \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{cc} b_1 & b_1 \\ \hline \end{array}$$

| | $b_1$ | $b_1$ |
|---|---|---|
| $a_1$ | –7 | 6 |
| $a_2$ | 7 | –4 |
| $a_3$ | –4 | –2 |
| $a_4$ | 8 | –6 |

*Sol :*

**Step 1**

The given matrix is of required size m × 2 to use graphical method.

**Step 2**

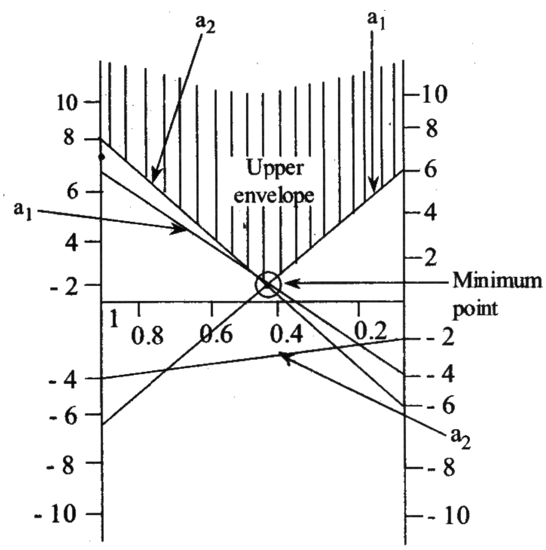
Draw two parallel lines to represent $b_1$ and $b_2$ strategies.

**Step 3**

Plot the strategies of A.

**Step 4**

Identify the minimum point of upper envelope.




**Figure**

The two strategies passing through the minimum point are $a_x$ and $a_r$

Thus, the reduced payoff matrix is,

$$\begin{array}{c} \\ a_a \\ a_2 \end{array}\begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} -7 & 6 \\ 7 & -4 \end{bmatrix} \end{array}$$

Where,

$a_{11} = -7$, $a_{12} = 6$, $a_{21} = 7$ and $a_{22} = -4$

**Value of x**

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{-4 - 7}{(-7 - 4) - (6 + 7)}$$

$$= \frac{11}{24}$$

$$1 - x = 1 - \frac{11}{24} = \frac{13}{24}$$

**Value of y**

$$y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{-4 - 6}{(-7 - 4) - (6 + 7)} = \frac{5}{12}$$

$$1 - y = 1 - \frac{5}{12} = \frac{7}{12}$$

**Value of Game**

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(-7 \times -4) - (6 \times 7)}{(-7 - 4) - (6 + 7)} = \frac{7}{12}$$

**Optimal Strategy**

| Players | Strategy | Probability |
|---|---|---|
| **For player A** | $a_1$ | $\frac{11}{24}$ |
| | $a_2$ | $\frac{13}{24}$ |
| | $a_3$ | 0 |
| | $a_4$ | 0 |
| **For player B** | $b_1$ | $\frac{5}{12}$ |
| | $b_2$ | $\frac{5}{12}$ |

Value of game, $v = \frac{7}{4}$

## 5.6 Linear Programming Approach for Game Theory

**Q13. Write briefly about linear programming approach for game theory.**

*Ans :*

Games which are of size m × n, where m > 2, n > 2 even after reducing the size using dominance principle, can be solved using linear programming approach. Every game matrix can be formulated from the point of view of the player in the row or from the point of view of the player in the column.

**General Formulation of Game Theory**

B's strategy

| A's strategy | | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|---|
| | $B_1$ | $a_{11}$ | $a_{12}$ | $a_{13}$ |
| | $B_2$ | $a_{21}$ | $a_{22}$ | $a_{23}$ |
| | $B_3$ | $a_{31}$ | $a_{32}$ | $a_{33}$ |

**From A's Point of View**

The player A is interested in maximizing the minimum gain.

Let this value of game be called 'U'.

Let $x_1$, $x_2$, and $x_3$ be the probabilities with which A plays the strategies $A_1$, $A_2$ and $A_3$.

Let $E_1$, $E_2$ and $E_3$ be the expected payoffs of A when B decides to play $B_1$, $B_2$ and $B_3$ respectively.

$$E_1 = a_{11}x_1 + a_{21}x_2 + a_{31}x_3$$

$$E_2 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3$$

$$E_3 = a_{13}x_1 + a_{23}x_2 + a_{33}x_3$$

Maximize $U = x_1 + x_2 + x_3$

Subject to $a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \geq U$ (i.e., $E_1$)

$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \geq U$ (i.e., $E_2$)

$a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \geq U$ (i.e., $E_3$)

Divide all the constraint by U and set $X_i = \frac{x_i}{U}$

Minimize $\frac{1}{U} = X_1 + X_2 + X_3$

Subject to, $a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \geq 1$

$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \geq 1$

$a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \geq 1$

and $x_1, x_2, x_3 \geq 0$

Similarly develop the LPP from B's point of view.

Maximize $\frac{1}{V} = Y_1 + Y_2 + Y_3$

Subject to, $3Y_1 - 4Y_2 + 2Y_3 \leq 1$

$Y_1 - 3Y_2 + 7Y_3 \leq 1$

$3Y_1 - 4Y_2 + 2Y_3 \leq 1$

and $Y_1, Y_2, Y_3 \geq 0$

This, is the dual LPP from A's point of view, Hence, solving any one LPP, we can get the solution can be obtained using simplex method.

**22. Two companies A and B are competing for the same product. Their different strategies are given in the following payoff matrix.**

$$A\overset{B}{\begin{bmatrix} 3 & -4 & 2 \\ 1 & -3 & -7 \\ -2 & 4 & 7 \end{bmatrix}}$$

**Solve the game using linear programming approach.**

*Sol :*

Let the LPP be formed from player B's point of view.

Let V be the value of the game.

Let $y_1$, $y_2$ and $y_3$ be the probabilities in respect to B choosing $b_1$, $b_2$ and $b_3$.

Set, $Y_i = \frac{y_i}{V}$

Maximize, $\frac{1}{V} = Y_1 + Y_2 + Y_3$

Subject to, $3Y_1 - 4Y_2 + 2Y_3 \leq 1$

$Y_1 - 3Y_2 - 7Y_3 \leq 1$

$-2Y_1 + 4Y_2 + 7Y_3 \leq 1$

and $Y_1, Y_2, Y_3 \geq 0$



**Standard Form**

Maximize, $\frac{1}{V} = Y_1 + Y_2 + Y_3 + 0s_1 + 0s_2 + 0s_3$

Subject to, $3Y_1 - 4Y_2 + 2Y_3 + s_1 = 1$

$Y_1 - 3Y_2 - 7Y_3 + s_2 = 1$

$-2Y_1 + 4Y_2 + 7Y_3 + s_2 = 1$

$Y_1, Y_2, Y_3 \geq 0$, and $s_1, s_2, s_3 \geq 0$

**IBFS (Initial Basic Feasible Solution)**

Let $Y_1 = 0, Y_2 = 0, Y_3 = 0$

Then $s_1 = 1, s_2 = 1, s_3 = 1$

| $c_B$ | B | $c_j$ / $x_B$ | 1 / $Y_1$ | 1 / $Y_2$ | 1 / $Y_3$ | 0 / $s_1$ | 0 / $s_2$ | 0 / $s_3$ | Min ratio | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $s_1$ | 1 | 3 | –4 | 2 | 1 | 0 | 0 | 1 / 3 → | Leaving variable |
| 0 | $s_2$ | 1 | 1 | –3 | –7 | 0 | 1 | 0 | 1 | |
| 0 | $s_3$ | 1 | –2 | 4 | 7 | 0 | 0 | 1 | – | |
| | $z_j$ | | 0 | 0 | 0 | 0 | 0 | 0 | | |
| | $c_j - z_j$ | | 1 | 1 | 1 | 0 | 0 | 0 | | |

Entering variable (chosen arbitrarily)

| $c_B$ | B | $c_j$ / $x_B$ | 1 / $Y_1$ | 1 / $Y_2$ | 1 / $Y_3$ | 0 / $s_1$ | 0 / $s_2$ | 0 / $s_3$ | Min ratio | Formulae | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $Y_2$ | 1 / 3 | 1 | –4 / 3 | 2 / 3 | 1 / 3 | 0 | 0 | – | $R_1^* = \frac{R_1}{3}$ | |
| 0 | $s_2$ | 2 / 3 | 0 | –5 / 3 | –23 / 3 | –1 / 3 | 1 | 0 | – | $R_2^* = R_2 - (1 \times R_1^*)$ | |
| 0 | $s_3$ | 5 / 3 | 0 | 4 / 3 | 25 / 3 | 2 / 3 | 0 | 1 | 5 / 4 | $R_3^* = R_3 - (-2 \times R_1^*)$ | Leaving variable |
| | $z_j$ | | 1 | –4 / 3 | 2 / 3 | 1 / 3 | 0 | 0 | | | |
| | $c_j - z_j$ | | 0 | 7 / 3 | 1 / 3 | –1 / 3 | 0 | 0 | | | |

Entering variable

| $c_B$ | B | $c_j$ / $x_B$ | 1 / $Y_1$ | 1 / $Y_2$ | 1 / $Y_3$ | 0 / $s_1$ | 0 / $s_2$ | 0 / $s_3$ | Min ratio | Formulae | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $Y_1$ | 2 | 1 | 0 | 27 / 3 | 1 | 0 | 1 | | $R_1^* = R_1 - \left(\frac{-4}{3} \times R_3^*\right)$ | No entering variable could be identified |
| 0 | $s_2$ | 11 / 4 | 0 | 0 | 11 / 4 | 1 / 3 | 1 | 5 / 4 | | $R_2^* = R_2 - \left(\frac{-5}{3} \times R_3^*\right)$ | |
| 1 | $Y_2$ | 5 / 4 | 0 | 1 | 25 / 4 | 1 / 2 | 0 | 3 / 4 | | $R_3^* = \frac{R_3}{\frac{4}{3}}$ | |
| | $z_j$ | | 1 | 1 | 61 / 4 | 3 / 2 | 0 | 7 / 4 | | | |
| | $c_j - z_j$ | | 0 | 0 | –57 / 4 | –3 / 2 | 0 | –7 / 4 | | | |

From the optimal simplex table,

$$Y_1 = 2, Y_2 = \frac{5}{4} \text{ and } Y_3 = 0.$$

$$\frac{1}{V} = Y_1 + Y_2 + Y_3$$

$$\frac{1}{V} = 2 + \frac{5}{4} + 0 = \frac{13}{4}$$

$\therefore$ Value of game, $V = \frac{4}{13}$

$$Y_i = \frac{y_i}{V} \Rightarrow y_i = Y_i \times V$$

$$y_1 = Y_1 \times V = 2 \times \frac{4}{13} = \frac{8}{13}$$

$$y_2 = Y_2 \times V = \frac{5}{4} \times \frac{4}{13} = \frac{5}{13}$$

$$y_3 = Y_3 \times V = 0 \times \frac{4}{13} = 0$$

We can read the values of the dual variable $x_1$, $x_2$ and $x_3$ from $c_j - z_j$ row of optimal simplex table.

$$X_1 = \frac{3}{2}$$

$$X_2 = 0$$

$$X_3 = \frac{7}{4}$$

$$\frac{1}{U} = X_1 + X_2 + X_3$$

$$\frac{1}{U} = \frac{3}{2} + 0 + \frac{7}{4} = \frac{13}{4}$$

$$\therefore \quad U = \frac{4}{13}$$

$$X_i = \frac{x_i}{U} \Rightarrow x_i = X_i \times U$$

$$x_1 = X_1 + U = \frac{3}{2} \times \frac{4}{13} = \frac{6}{13}$$

$$x_2 = X_2 + U = 0 \times \frac{4}{13} = 0$$

$$x_3 = X_3 + U = \frac{7}{4} \times \frac{4}{13} = \frac{7}{13}$$

**Optimal Strategy**

For player A : $\left(\frac{6}{13}, 0, \frac{7}{13}\right)$

For player B : $\left(\frac{8}{13}, \frac{5}{13}, 0\right)$

Value of the game (U or V) = $\frac{4}{13}$.

# FACULTY OF INFORMATICS

## M.C.A. I Year II Semester Examination

## Model Paper - I

# OPERATIONS RESEARCH

Time : 3 Hours] Max. Marks : 70

**(5 × 14 = 70 Marks)**

**Note :** Answer all the question according to the internal choice

**ANSWERS**

1. Solve the LPP

   Max $Z = 3X_1 + 2X_2$

   Subject to $4X_1 + 3X_2 \leq 12$

   $4X_1 + X_2 \leq 8$

   $4X_1 - X_2 \leq 8$

   $X_1, X_2 \geq 0$ **(Unit-I, Prob.14)**

OR

2. What do you mean by Graphical Method of LPP? State the characteristics of Graphical Method. **(Unit-I, Q.No.4)**

3. A company has three plants at locations A, B and C which supply to warehouses located at D, E, F, G and H. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in `) are given below,

| | | To | | | | |
|---|---|---|---|---|---|---|
| | | D | E | F | G | H |
| From | A | 5 | 8 | 6 | 6 | 3 |
| | B | 4 | 7 | 7 | 6 | 5 |
| | C | 8 | 4 | 6 | 6 | 4 |

Determine an optimum distribution for the company in order to minimize the total transportation cost.

OR

Solve the following transportation problem for optimal solution,

| | P | Q | R | S | T | Supply |
|---|---|---|---|---|---|---|
| A | 5 | 8 | 6 | 6 | 3 | 8 |
| B | 4 | 7 | 7 | 6 | 5 | 5 |
| C | 8 | 4 | 6 | 6 | 4 | 9 |
| Demand | 4 | 4 | 5 | 4 | 8 | 22 / 25 |

**(Unit-II, Prob.22)**

OR

4. Write about method to obtain initial basic feasible solution by North-West Corner Rule. **(Unit-II, Q.No.6)**

5. Explain the concept of Zero-One Programming Model for Assignment Problem with an example. **(Unit-III, Q.No.3)**

OR

6. Find the optimum integer solution to the following LPP

   Max $Z = x_2 + x_2$

   Subject to constraints

   $3x_1 + 2x_2 \leq 5$

   $x_2 \leq 2$

   $x_1, x_2 \geq 0$, and are integers. **(Unit-III, Prob.14)**

7. Explain the formulation of LPP by dynamic programming. **(Unit-IV, Q.No.5)**

OR

8. Solve the following problem,

   Min $z = x_2 + y_2 + z_2$ Subject to $x + y + z \geq 10$, $x, y, z \geq 0$ **(Unit-IV, Prob.2)**

9. Solve the following game using graphical method:

Player B

Player A

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | –5 | 5 | 0 | –1 | 8 |
| $A_2$ | 8 | –4 | –1 | 6 | –5 |

**(Unit-V, Prob.17)**

OR

10. What is Saddle Point ? How do you determine Saddle Point ? **(Unit-V, Q.No.8)**

# FACULTY OF INFORMATICS

## M.C.A. I Year II Semester Examination

## Model Paper - II

# OPERATIONS RESEARCH

Time : 3 Hours] Max. Marks : 70

**(5 × 14 = 70 Marks )**

**Note :** Answer all the question according to the internal choice

**ANSWERS**

1. Solve the following LP problem using graphical method .

 Maximize : $Z = 6x_1 + 8x_2$

 Subject to :

 $5x_1 + 10x_2 \leq 60$

 $4x_1 + 4x_2 \leq 40$

 $x_1, x_2 \geq 0.$ **(Unit-I, Prob.10)**

OR

2. Solve using, two-phase simplex method

 Solve the following LPP

 Min $Z = 10X + 15Y$

 S.T.C. ...,

 $Y \geq 3$

 $X - Y \geq 0$

 $Y \leq 12;$

 $X + Y \leq 30$

 $X \leq 20$ and $X, Y \geq 0$ **(Unit-I, Prob.19)**

3. A company has three plants at locations A, B and C which supply to warehouses located at D, E, F, G and H. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in `) are given below,

| | | To | | | | |
|---|---|---|---|---|---|---|
| | | D | E | F | G | H |
| From | A | 5 | 8 | 6 | 6 | 3 |
| | B | 4 | 7 | 7 | 6 | 5 |
| | C | 8 | 4 | 6 | 6 | 4 |

 Determine an optimum distribution for the company in order to minimize the total transportation cost. **(Unit-II, Prob.22)**

OR

4. Define Transshipment. State the characteristics of Transshipment Model. **(Unit-II, Q.No.14)**

5. Solve the following integer programming problem using branch-and-bound technique.

Maximize $Z = 10X_1 + 20X_2$

Subject to

$6X_1 + 8X_2 \leq 48$

$X_1 + 3X_2 \leq 12$

$X_1, X_2 \geq 0$ and integers **(Unit-III, Prob.15)**

OR

6. Solve the following assignment problem by Hungarian assignment method.

| Worker | Time (in minutes) | | |
|---|---|---|---|
| | Job 1 | Job 2 | Job 3 |
| A | 4 | 2 | 7 |
| B | 8 | 5 | 3 |
| C | 4 | 5 | 6 |

**(Unit-III, Prob.7)**

7. Solve the folloiwng LPP using dynamic programming technique

Maximize $Z = 10X_1 + 30X_2$

Subject to

$3x_1 + 6x_2 \leq 168$

$12x_2 \leq 240$

$X_1$ and $X_2 \geq 0$ **(Unit-IV, Prob.5)**

OR

8. What are the characteristics of dynamic programming problem? **(Unit-IV, Q.No.2)**

9. Reduce the following game with the help of dominance rule and then determine the value of the game:

Player B

| Player A | | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|---|
| | $A_1$ | 3 | 1 | 4 | 0 |
| | $A_2$ | 3 | 4 | 2 | 4 |
| | $A_3$ | 4 | 2 | 4 | 0 |
| | $A_4$ | 0 | 4 | 0 | 8 |

**(Unit-V, Q.No.1)**

OR

10. State the basic terminology are used in game theory. **(Unit-V, Q.No.2)**

# FACULTY OF INFORMATICS

## M.C.A. I Year II Semester Examination

## Model Paper - III

# OPERATIONS RESEARCH

Time : 3 Hours] Max. Marks : 70

**(5 × 14 = 70 Marks)**

**Note :** Answer all the question according to the internal choice

**ANSWERS**

1. M/s. ABCL company manufactures two types of cassettes, a video and audio. Each video cassette takes twice as long to produce one audio cassette, and the company would have time to make a maximum of 2000 per day if it is produced only audio cassettes. The supply of plastic is sufficient to produce 1500 per day to both audio and video cassettes combined. The video cassette requires a special testing and processing of which there are only 6000 hrs. per day available. If the company makes a profit of Rs. 3/- and Rs. 5/- per audio and video cassette respectively, how many of each should be produced per day in order to maximize the profit ? **(Unit-I, Prob.5)**

OR

2. Discuss about special cases in finding solution to LPP by graphical method. **(Unit-I, Q.No.16)**

3. A dealer stocks and sells four types of Bicycles namely Atlas, Bharath, Champion, Duncan which he may procure from three different suppliers namely Priyanshu, Quershi and Raju. His anticipated sales for the bicycles for the coming seasons are 410, 680, 310 and 550 nos. respectively. He can obtain 900 bicycles from Priyanshu, 600 from Quershi and 560 from Raju at suitable prices. The profit per bicycle in rupees for each supplier is tabulated below :

| Supplier \ Type | Atlas (A) | Bharath (B) | Champion (C) | Duncan (D) |
|---|---|---|---|---|
| Priyanshu (P) | 21.50 | 26.00 | 19.50 | 21.00 |
| Quershi (Q) | 20.50 | 24.00 | 20.00 | 21.00 |
| Raju (R) | 18.00 | 19.50 | 19.00 | 19.50 |

Formulate the above information as transportation model and obtain initial solution by North West Corner Rule. **(Unit-II, Prob.14)**

OR

4. Explain the Modi Method for obtaining optimal solution. **(Unit-II, Q.No.12)**

5. Explain the concept of Branch-and-Bound Technique for Assignment Problem. **(Unit-III, Q.No.8)**

OR

6. Use Branch and Bound technique to solve the following :

Max $Z = x_1 + 4x_2$

Subject to $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_1, x_2 \geq 0$ and are integers **(Unit-III, Prob.16)**

7. Explain the assumptions of linear programming. **(Unit-IV, Q.No.6)**

OR

8. Use dynamic programming to solve the LPP max $Z = x_1 + 9x_2$

Subject to the constraints

$2x_1 + x_2 \leq 25$

$x_2 \leq 11$

$x_1\ x_2 \geq 0$ **(Unit-IV, Prob.7)**

9. Define Game Theory. What are the characteristics of Game Theory ? **(Unit-V, Q.No.1)**

OR

10. Two companies A and B are competing for the same product. Their different strategies are given in the following payoff matrix.

$$\begin{array}{c} \quad B \\ A\begin{bmatrix} 3 & -4 & 2 \\ 1 & -3 & -7 \\ -2 & 4 & 7 \end{bmatrix} \end{array}$$

Solve the game using linear programming approach. **(Unit-V, Prob.22)**