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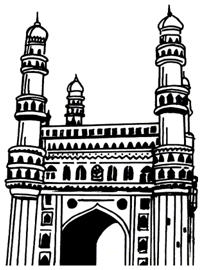

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## I Year I Sem

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## UNIT - II

**Theorem on Total Differentials** - Composite Functions - Differentiation of Composite Functions - Implicit Functions - Equality of fxy(a, b) and fyz(a, b) - Taylor's theorem for a function of two Variables - Maxima and Minima of functions of two variables – Lagrange's Method of undetermined multipliers.

## UNIT - III

**Curvature and Evolutes:** Introduction - Definition of Curvature - Radius of Curvature - Length of Arc as a Function, Derivative of arc - Radius of Curvature - Cartesian Equations - Newtonian Method - Centre of Curvature - Chord of Curvature.

**Evolutes:** Evolutes and Involutes - Properties of the evolute.

**Envelopes:** One Parameter Family of Curves - Consider the family of straight lines - Definition - Determination of Envelope.

## UNIT - IV

**Lengths of Plane Curves:** Introduction - Expression for the lengths of curves y = f(x) - Expressions for the length of arcs x = f(y); x = f(t), y = ?(t); r = f(?)

**Volumes and Surfaces of Revolution:** Introduction - Expression for the volume obtained by revolving about either axis - Expression for the volume obtained by revolving about any line - Area of the surface of the frustum of a cone - Expression for the surface of revolution - Pappus Theorems - Surface of revolution.

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|---|---|

## 1.1 Introduction - Functions of Two Variables

**1. Write a short notes on function of two variables.**

*Sol :*

A function in a two variables (x,y) is denoted by f(x,y) i.e if z is a function in two variables x,y we denote by z = f(x,y)

The region determine the point (x,y) is called domain of the point (x,y)

**Eg.** If $z = \sqrt{1-x^2-y^2}$ then we see that the point (x,y) for which $x^2+y^2 \leq 1$ lies on or within the circle 0 centre is origin & radius 1.

i.e. domain = $\left\{\frac{(x,y)}{x^2+y^2} = 1\right\}$.

## 1.2 Neighbourhood of a Point (a, b)

**2. Define a neighbourhood of a point (a,b)**

*Sol :*

Let $\delta$ be any positive number of the points (x,y) $\Rightarrow$ $a - \delta \leq x \leq a + \delta$, $b - \delta \leq n \leq b + \delta$.

Determine the square by the lines $x = a - \delta$, $x = a + \delta$, $y = a - \delta$, $y = b + \delta$ is the centre is at the point (a,b) this square is called a neighbourhood of point (a,b) for every value of the $\delta$ we will get neighbourhood.

Thus, the $\{(x,y) / a - \delta \leq x \leq a + \delta,\ b - \delta \leq x \leq b + \delta\}$ in a neighbourhood of the point (a,b)

## 1.3 Continuity of a Function of Two Variables

A function 'f' of two variables i.e. f(x,y) is said to be continuous at a point (a,b)if for every

$$\varepsilon > 0 \exists \delta > 0 \ni |f(x,y) - f(a,b)| < \varepsilon$$

$$\{(x,y)/a - \delta \leq x \leq a + \delta, b - \delta \leq y \leq b + \delta\}$$

**i.e.** $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

**Note :** A function is said to be continuous if it continous at every point of its domain.

## 1.4 Continuity at a point, Limit of a Function of two variables

➢ **Limit of a function of two variables :**

A function 'f' is said to be and to the limit '*l*' as (x,y) tends to (a,b) if for every

$\varepsilon > 0$, $\exists$ a positive number $\delta \ni |f(x,y) - l| < \varepsilon$, $x \in [a - \delta, a + \delta]$, $y \in [b - \delta, b + \delta]$

## 1.5 Partial Derivatives

➢ **Partial differentation**

A partial derivatives of a function of several variables is its derivatives with respect to one of those variables with the others held constant.

**3. Define Partial derivatives.**

*Sol :*

Consider z = f(x,y) then

$$\lim \frac{f(a+h+b) - f(a,b)}{h}; h \to 0$$

If it exists, is said to be the partial derivatives of f w.r. to x at (a,b) and is denoted by $\left(\frac{\partial z}{\partial x}\right)_{(a,b)}$ or $f_x(a,b)$

$$\Rightarrow \lim \frac{f(a, b+k) - f(a, b)}{k} \text{ as } k \to 0$$

If it exists is called the partial derivative of f(x,y) w.r. to y at (a,b) & is denoted or $f_y$ (a,b) by $\left(\partial z/\partial y\right)_{(a,b)}$

- **Partial derivative of Higher order**

From the above first order partial derivatives, we form the partial derivative of higher order.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{yx}$$

$$\frac{\partial z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{xy}$$

## 1.6 Geometrical Representation of a Function of Two Variables

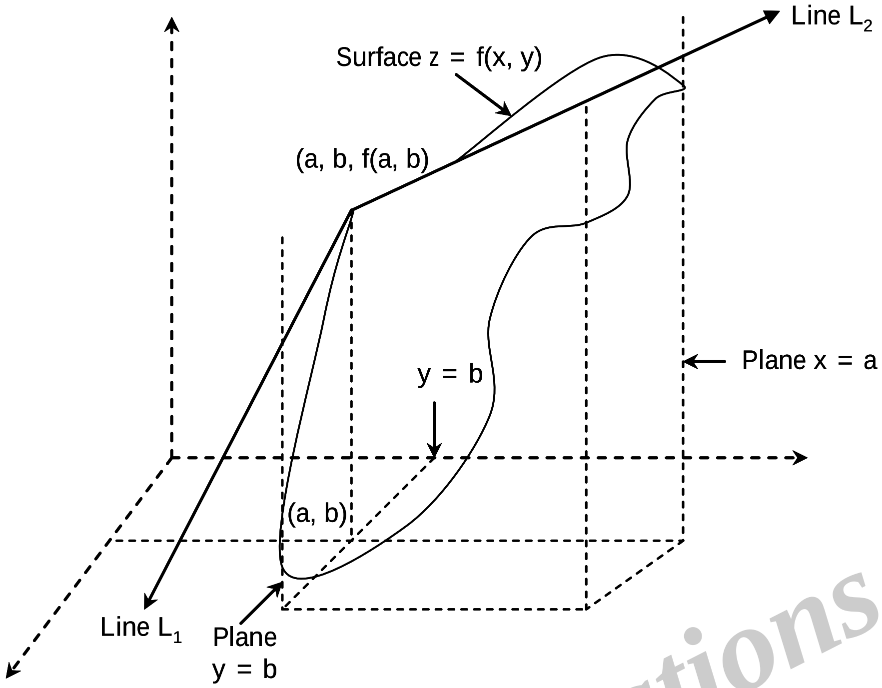
Let as we represent functions of one variable by means of curves, we seek to represent functions of two variables geometrically by means of surfaces. We achieve this representation by considering a rectangular co-ordinates x, y and u, and marking off above each point (x, y) of the range (R) of the function the point P with the third coordinate u = f(x, y). As the point point (x, y) ranges over the region R, the point P describes a surface in space. We take this surface as the geometrical representation of the function.

In analytical geometry, surfaces in space are represented by functions of two variables, so that there is between such surfaces and functions of two variables a reciprocal relationship.

### Geometrical Interpretation of Partial Derivatives

Assume that z = f(x, y) is a function of two variables which represents a surface in three-dimensional space. Compute the partial derivative $z_x$ and $z_y$. Evaluate these partial derivatives at the point (a, b) then

$z_x$ is the slope (measured along the x-axis) of line $L_1$, which is tangent to the surface at the point (a, b, f(a, b)), and $z_y$ is the slope (measured along the y-axis) of the $L_2$, which is a tangent to the tangent to the surface at the point (a, b, f(a, b)) line $L_1$ lies in the plane y = b, line $L_2$ lies in the plane x = a.




**4. If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 - \tan^{-1}\frac{x}{y}$ ; $xy \neq 0$. prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$**

*Sol :*

Given that $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

Partially Diff w.r.to y. Then we have

$$\frac{\partial u}{\partial y} = x^{\cancel{2}} \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{\cancel{x}} - 2y\tan^{-1}\frac{x}{y} + y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{x}{y^2}$$

$$= \frac{x.x^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} + \frac{y^4}{y^2+x^2}\frac{x}{y^2}$$

$$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} + \frac{xy^2}{y^2+x^2}$$

$$= \frac{x\left(x^2+y^2\right)}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$\frac{\partial u}{\partial y} = x - 2y\tan^{-1}\frac{x}{y}$$

Partially differentiate with respect to x.

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}\frac{x}{y}\right]$$

$$= 1 - 2y \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}$$

$$= 1 - \frac{2y^2}{y^2 + x^2}$$

$$= \frac{x^2 + y^2 - 2y^2}{x^2 + y^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$$

**5. If $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ : $x^2 + y^2 + z \neq 0$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$**

*Sol :*

$$u = \left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2} 2x$$

$$= -x\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left(-x\left(x^2 + y^2 + z^2\right)^{-3/2}\right)$$

$$= -\left(x^2 + y^2 + z^2\right)^{-1/3} + \frac{3}{\cancel{2}}x\left(x^2 + y^2 + z^2\right)^{-5/2}(\cancel{2}x)$$

$$= -\left(x^2 + y^2 + z^2\right)^{-1/3} + 3x^2\left(x^2 + y^2 + z^2\right)^{-5/2} \quad . \qquad ....(1)$$

$$\frac{\partial u}{\partial y} = \frac{-1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2} 2y$$

$$= -\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left[-y\left(x^2+y^2+z^2\right)^{-3/2}\right]$$

$$= -\left(x^2+y^2+z^2\right)^{-3/2} + \frac{3y}{2}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}-1} 2y$$

$$= -\left(x^2+y^2+z^2\right)^{-3/2} + 3y^2\left(x^2+y^2+z^2\right)^{-5/2} \quad ......(2)$$

$$\frac{\partial u}{\partial z} = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2} - 2z$$

$$= -z\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left[-z\left(x^2+y^2+z^2\right)^{-3/2}\right]$$

$$= \left(x^2+y^2+z^2\right)^{-3/2} + \frac{3}{2} z\left(x^2+z^2+y^2\right)^{-5/2} 2z$$

$$= -\left(x^2-y^2+z^2\right)^{-3/2} + 3z^2\left(x^2+y^2+z^2\right)^{-5/2} \quad .....(3)$$

Adding (1) & (2) & (3)

$$\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial y^2}+\frac{\partial^2 y}{\partial z^2} = -\left(x^2+y^2+z^2\right)^{-3/2} + 3x^2\left(x^2+y^2+z^2\right)^{-5/2} - \left(x^2+y^2+z^2\right)^{-3/2} + 3y^2\left(x^2+y^2+z^2\right)^{-5/2}$$

$$- \left(x^2+y^2+z^2\right)^{-3/2} + 3z^2\left(x^2+y^2+z^2\right)^{-5/2}$$

$$= -3\left(x^2+y^2+z^2\right)^{-3/2} + 3(x^2+y^2+z^2)\left(x^2+y^2+z^2\right)^{-5/2}$$

$$= -3\left(x^2+y^2+z^2\right)^{-3/2} + 3\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\therefore \frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial y^2}+\frac{\partial^2 y}{\partial z^2} = 0$$

**6. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$**

*Sol :*

We have $u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz} 3y^2-3xz$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}\ 3z^2 - 3zy$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{x^3+y^3+z^3-3xyz}$$

$$= 3(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= \frac{3}{x+y+z}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$\therefore \quad \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 = -\frac{9}{(x+y+z)^2}$$

---

**7. If $x^x y^y z^z = C$ Show that $x = y = z$ $\frac{\partial^2 z}{\partial x \partial y} = -(x \log_e x)^{-1.}$**

*Sol :*

Given that $x^x\ y^y\ z^z = C$

Taking Log on both sides

We have, $\log(x^x y^y z^z) - \log C$

$\log x^x + \log y^y + \log z^z = \log C$

$x \log x + y \log y + z \log z = \log C$

Differentiate partially with respect to x weget

$$x.\frac{1}{x} + 1.\log x + (z.\frac{1}{z} + 1.\log z)\frac{\partial z}{\partial x} = 0$$

$$1 + \log x + (1 + \log z)\frac{\partial z}{\partial x} = 0$$

$$(1 + \log z)\frac{\partial z}{\partial x} = -[1 + \log x]$$

$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}$$

Differentiate partially with respect to 'y' we get

$$y\cdot\frac{1}{y} + 1.\log y + z.\frac{1}{z} + 1.\log z\frac{\partial z}{\partial y} = 0$$

$$1 + \log y + (1 + \log z)\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log z}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{1+\log x}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial y}$$

$$= -\frac{(1+\log x)}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\frac{(1+\log y)}{(1+\log z)}$$

$$= -\frac{(1+\log x)^2}{(1+\log x)^3}\cdot\frac{1}{x} \qquad \text{Since } x = y = z$$

$$= -\frac{1}{x(1+\log x)}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x\log ex} = -(x\log ex)^{-1}$$

**8. If $u = 3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$ & $l^2 + m^2 + n^2 = 1$**

**Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$**

*Sol :*

$$u = 3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$$

$$\frac{\partial u}{\partial x} = 6(lx + my + nz)\,l - 2x$$

$$\frac{\partial^2 u}{\partial x^2} = 6\,l\,.\,l - 2$$
$$= 6\,l^2 - 2$$
$$\frac{\partial u}{\partial y} = 6(lx + my + nz)\,m - 2y$$
$$= 6m\,(lx + my + nz) - 2y$$
$$\frac{\partial u^2}{\partial y^2} = 6m.m - 2$$
$$= 6m^2 - 2$$
$$\frac{\partial u}{\partial z} = 6(lx + my + nz)\,n - 2z$$
$$= 6n\,(lx + my + nz) - 2z$$
$$\frac{\partial^2 u}{\partial z^2} = 6n^2 - 2$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6l^2 - 2 + 6m^2 - 2 + 6n^2 - 2$$
$$= 6l^2 + 6m^2 + 6n^2 - 6$$
$$= 6\,[l^2 + m^2 + n^2] - 6$$
$$= 6(1) - 6$$
$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

**9. If $u = \log(\tan x + \tan y + \tan z)$ prove that $(\text{Sin } 2x)\frac{\partial u}{\partial x} + (\text{Sin } 2y)\frac{\partial u}{\partial y} + (\text{Sin } z)\frac{\partial u}{\partial z} = 2$**

*Sol :*

$$u = \log(\tan x + \tan y + \tan z)$$
$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z}\sec^2 x$$

Simplify Sin 2x on both side

$$(\text{Sin } 2x)\frac{\partial u}{\partial x} = \frac{\text{Sin}2x\ \text{Sec}^2 x}{\tan x + \tan y + \tan z}$$
$$= \frac{2\tan x}{\tan x + \tan y + \tan z}$$
$$(\text{Sin } 2y)\frac{\partial u}{\partial y} = \text{Sin}2y\ \frac{1}{\tan x + \tan y + \tan z}\sec^2 y$$
$$= \frac{2\tan y}{\tan x + \tan y + \tan z}$$

$$(\text{Sin } 2z)\frac{\partial u}{\partial z} = \text{Sin } 2z \frac{1}{\tan x + \tan y + \tan z} \sec^2$$

$$= \frac{2\tan z}{\tan x + \tan y + \tan z}$$

$$(\text{Sin } 2x)\frac{\partial u}{\partial x} + (\text{Sin } 2x)\frac{\partial u}{\partial y} + (\text{Sin } 2z)\frac{\partial u}{\partial z} = \frac{2\tan x + 2\tan y + 2\tan z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)}$$

$$\therefore (\text{Sin } 2x)\frac{\partial u}{\partial x} + (\text{Sin } 2x)\frac{\partial u}{\partial y} + (\text{Sin } 2z)\frac{\partial u}{\partial z} = 2$$

**10. If $V = At^{-1/2} e^{-x^2/4a^2t}$ prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$**

*Sol :*

$$V = At^{-1/2} e^{-x^2/4a^2t}\left(\frac{-2x}{4a^2t}\right)$$

$$\frac{\partial v}{\partial x} = At^{-1/2} e^{-x^2/4a^2t}\left(\frac{-2x}{4a^2t}\right)$$

$$= -v\frac{2x}{4a^2t}$$

$$= \frac{-x}{2a^2t}V$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{-1}{2a^2t}v - \frac{x}{2a^2t}\frac{\partial v}{\partial x}$$

$$= -\frac{1}{2a^2t}\left[v + x\frac{\partial v}{\partial x}\right]$$

$$= -\frac{1}{2a^2t}\left[v + x\left(\frac{-xv}{2a^2t}\right)\right]$$

$$= -\frac{2a^2tv}{4a^4t^2} + \frac{x^2v}{4a^4t^2}$$

$$= \frac{v}{4a^4t^2}\left[-2a^2t + x^2\right]$$

$$\frac{\partial v}{\partial t} = At^{-1/2} e^{-x^2/4a^2t}\left(\frac{-x^2}{-4a^2t^2}\right) - A\frac{1}{2}t^{-1/2-1} e^{-x^2/4a^2t}$$

$$= At^{-1/2} e^{-x^2/4a^2t}\left(\frac{x^2}{4a^2t^2}\right) - A\frac{1}{2}t^{-3/2} e^{x^2/4a^2t}$$

$$= At^{-1/2} e^{-x^2/4a^2t}\left[\frac{x^2}{4a^2t^2} - \frac{1}{2}t^{-1}\right]$$

$$= At^{-1/2}e^{-x^2/4a^2t}\left[\frac{x^2}{4a^2t^2} - \frac{1}{2t}\right]$$

$$= \frac{v}{4a^2t}\left[x^2 - \frac{4a^2t^2}{2t}\right]$$

$$\frac{\partial v}{\partial t} = \frac{v}{4a^2t}[x^2 - 2a^2 t] \qquad .....(1)$$

$$a^2\frac{\partial^2 v}{\partial x^2} = a^2 \frac{v}{4a^4t^2}(-2a^2 t + x^2)$$

$$= \frac{v}{4a^2t^2}(-2a^2t + x^2) \qquad .....(2)$$

$$\therefore \quad \frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2}$$

---

**11. If $z = \tan^{-1}\left(\frac{y}{x}\right)$ verify that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.**

*Sol :*

$$z = \tan^{-1}(y/n)$$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(y/x\right)^2}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$= -\frac{x^2}{x^2+y^2}\frac{y}{x^2}$$

$$= -\frac{y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{\left(x^2+y^2\right)0 + y(2x)}{\left(x^2+y^2\right)^2} = \frac{2xy}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}$$

$$= \frac{x^{2}}{x^2+y^2}\frac{1}{x} \Rightarrow \frac{x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \Rightarrow \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) \Rightarrow \frac{\left(x^2+y^2\right)0-x(2y)}{\left(x^2+y^2\right)^2}$$

$$= \frac{-2xy}{\left(x^2+y^2\right)}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = 0$$

---

**12. If $z(x + y) = x^2 + y^2$ show that $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2 = 4\left[1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]$.**

*Sol :*

$z(x + y) = x^2 + y^2$

$$z = \frac{x^2+y^2}{x+y}$$

$$\frac{\partial z}{\partial x} = \frac{(x+y)(2x)-\left(x^2+y^2\right)1}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)2y-\left(x^2+y^2\right)1}{(x+y)^2} = \frac{2xy+2y^2-x^2-y^2}{(x+y)^2}$$

$$= \frac{2xy-x^2+y^2}{(x+y)^2}$$

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2 \Rightarrow \left[\left(\frac{x^2+2xy-y^2}{(x+y)^2}\right)-\left(\frac{y^2-x^2+2xy}{(x+y)^2}\right)\right]^2$$

$$= \left[\frac{x^2+2xy-y^2-x^2+x^2-2xy}{(x+y)^2}\right]^2$$

$$= \left[\frac{2x^2-2y^2}{(x+y)^2}\right]^2 \Rightarrow \frac{4\left(x^2-y^2\right)^2}{(x+y)^4}$$

$$4\frac{(x-y)^2}{(x+y)^2} = \frac{4(x+y)^2(x-y)^2}{(x+y)^{4\,2}}$$

$$\Rightarrow 4\left[1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right] = 4\left[1-\left(\frac{x^2+2xy-y^2}{(x+y)^2}\right)\right]-\left[\left(\frac{2xy-x^2+y^2}{(x+y)^2}\right)\right]$$

$$= \frac{4\left[(x+y)^2-x^2-2xy+y^2-2xy+x^2-y^2\right]}{(x+y)^2}$$

$$= 4\left[\frac{x^2+y^2+2xy-x^2-2xy+y^2-2xy+x^2-y^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2(x-y)^2}{(x+y)^2}\right]$$

$$\therefore \quad \left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2 = 4\left(\frac{(x-y)^2}{(x+y)^2}\right)$$

**13. If $\theta = t^n.e^{-r^2/4t}$ find the value of n which, will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$**

*Sol :*

$$\theta = t^n e^{-r^2/4t}$$

partial differentiate with respect to ‘r’

$$\frac{\partial\theta}{\partial r} = t^n e^{\frac{-r^2}{4t}}\left[\frac{-2r}{4t}\right]$$

$$= t^n e^{-r^2/4t}\left[\frac{-r}{2t}\right]$$

$$r^2\frac{\partial\theta}{\partial r} = r^2\left[t^n e^{-r^2/4t}\left[\frac{-r}{2t}\right]\right]$$

$$= \frac{-r^3t^{n-1}}{2}e^{-r^2/4t}$$

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial\theta}{\partial r}\right] = \frac{\partial}{\partial r}\left[\frac{-r^3t^{n-1}}{2}e^{-r^2/4t}\right]$$

$$= \frac{-t^{n-1}}{2}\left[3r^2e^{-r^2/4t} + r^3e^{-r^2/4t}.\left[\frac{-2r}{4t}\right]\right]$$

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)\right] = \frac{1}{r^2}\left[\frac{-t^{n-1}}{2}\left(3r^2e^{-r^2/4t} - \frac{r^4}{2t}e^{-r^2/4t}\right)\right]$$

$$= \frac{-3t^{n-1}}{2}e^{-r^2/4t} + \frac{r^2t^{n-2}}{4}e^{-r^2/4t}$$

Consider $\theta = t^n e^{-r^2/4t}$

$$\frac{\partial\theta}{\partial t} = nt^{n-1}e^{-r^2/4t} + t^n e^{-r^2/4t}\left(\frac{-r^2}{-4t^2}\right)$$

$$= n\,t^{n-1}e^{-r^2/4t} + \frac{t^{n-2}}{4}r^2e^{-r^2/4t}$$

Since $\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$

$$\Rightarrow \frac{-3}{2}t^{n-1}e^{-r^2/4t} + \cancel{\frac{r^2}{4}t^{n-2}}e^{-r^2/4t} = nt^{n-1}e^{-r^2/4t} + \cancel{\frac{t^{n-2}}{4}r^2}e^{-r^2/4t}$$

$$\Rightarrow \frac{-3}{2}t^{n-1}\cancel{e^{-r^2/4t}} = nt^{n-1}\cancel{e^{-r^2/4t}}$$

$$\boxed{n = \frac{-3}{2}}$$

**14. If u = log ($x^2$ + $y^2$ + $z^2$) prove that $x\dfrac{\partial^2 u}{\partial y \partial z} = y\dfrac{\partial^2 u}{\partial z \partial x} = z\dfrac{\partial^2 u}{\partial x \partial y}$**

*Sol :*

$$\frac{\partial u}{\partial z} = \frac{1}{x^2+y^2+z^2}\,2z$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left(\frac{2z}{x^2+y^2+2z}\right) = \frac{\left(x^2+y^2+z^2\right)0-2z(2y)}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{-4zy}{\left(x^2+y^2+z^2\right)^2}$$

$$x\frac{\partial^2 u}{\partial y \partial z} = \frac{-4xyz}{\left(x^2+y^2+z^2\right)^2} \qquad .....(1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2+z^2}\,2x \Rightarrow \frac{2x}{x^2+y^2+z^2}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial z}\left(\frac{2x}{x^2+y^2+z^2}\right)$$

$$= \frac{\left(x^2+y^2+z^2\right)0-2x(2z)}{\left(x^2+y^2+z^2\right)^2} = \frac{-4xz}{\left(x^2+y^2+z^2\right)^2}$$

$$y\left(\frac{\partial^2 u}{\partial z \partial x}\right) = \frac{-4xyz}{\left(x^2+y^2+z^2\right)^2} \qquad .....(2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2+z^2}\cdot 2y$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2+z^2}\right) = \frac{\left(x^2+y^2+z^2\right)(0)-2y(2x)}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{-4xy}{\left(x^2+y^2+z^2\right)^2}$$

$$z\frac{\partial^2 u}{\partial x \partial y} = \frac{-4xyz}{\left(x^2+y^2+z^2\right)^2} \quad .....(3)$$

$\therefore$ from (1), (2) & (3)

$$x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x} = z\frac{\partial^2 u}{\partial x \partial y}$$

---

**15. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$ show that $u_x + u_y + u_z = 0$**

*Sol :*

$u = x^2(y-z) - y^2(x-z) + z^2(x-y)$

$\frac{\partial u}{\partial x} = 2x(y–z) + x^2(0) - [y^2(1-0) + 0(x-z)] + z^2(1-0) + 0(x-y)]$

$= 2x(y-z) - y^2 + z^2$

$= 2xy - 2xz - y^2 + z^2$

$\frac{\partial u}{\partial y} = x^2(1-0) - [2y(x-z) + y^2(0-0)] + z^2(0-1) + 0(x-y)]$

$= x^2 - 2xy + 2yz - z^2$

$\frac{\partial u}{\partial z} = x^2(0-1) + 0(y-z) - [y^2(0-1) + 0(x-z)] + 2z(x-y) + z^2(0-0)]$

$= -x^2 + y^2 + 2xz - 2zy$

$= y^2 + 2zx - 2zy - x^2$

$\therefore$ $2xy - 2xz - y^2 + z^2 + x^2 - 2xy + 2yz - z^2 + y^2 + 2zx - 2zy - x^2$

$= 0$

---

**16. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ when $u = \sin^{-1}\frac{x}{y}$**

*Sol :*

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] \ \& \ \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

Consider

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{1}{y}\right)$$

$$= \frac{1}{\frac{\sqrt{y^2-x^2}}{\sqrt{y^2}}}\left(\frac{1}{y}\right) = \frac{y}{\sqrt{y^2-x^2}}\frac{1}{y}$$

$$= \frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{-x}{y^2}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}}\frac{-x}{y^2} \Rightarrow \frac{-x}{y\sqrt{y^2-x^2}}$$

Consider

$$\frac{\partial^2 u}{\partial x \partial y} \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{y\sqrt{y^2-x^2}}\right)$$

$$= \frac{-1}{y}\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{y^2-x^2}}\right]$$

$$= \frac{-1}{y}\frac{\partial}{\partial x}\left[x\left(y^2-x^2\right)^{-1/2}\right]$$

$$= \frac{-1}{y}\left[x.\frac{-1}{2}\left(y^2-x^2\right)^{-3/2}(-2x) + 1\left(y^2-x^2\right)^{-1/2}\right]$$

$$= \frac{-1}{y}\left[\frac{x^2}{\sqrt{\left(y^2-x^2\right)^{3/2}}} + \frac{1}{\sqrt{y^2-x^2}}\right]$$

$$= \frac{-1}{y}\left[\frac{1}{\sqrt{y^2-x^2}}\left[\frac{x^2}{y^2-x^2}+1\right]\right]$$

$$= \frac{-1}{y}\left[\frac{1}{\sqrt{y^2-x^2}}\left(\frac{x^2+y^2-x^2}{y^2-x^2}\right)\right] = \frac{-1}{y}\left[\frac{y^2}{\left(y^2-x^2\right)^{3/2}}\right]$$

$$= \frac{-y}{\left(y^2-x^2\right)^{3/2}}$$

Consider

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{y^2 - x^2}}\right)$$

$$= \frac{-1}{\not{2}\left(y^2 - x^2\right)^{3/2}}(\not{2}y)$$

$$= \frac{-y}{\left(y^2 - x^2\right)^{3/2}}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

---

**17. Find the value of $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$ when $a^2x^2 + b^2 y^2 - c^2 z^2 = 0$.**

*Sol :*

Given that $a^2x^2 + b^2 y^2 - c^2 z^2 = 0$

$c^2 z^2 = a^2 x^2 + b^2 y^2$

Partially differentiate with respect to x

$$c^2 . 2z \frac{\partial z}{\partial x} = 2xa^2$$

$$z \frac{\partial z}{\partial x} = \frac{a^2}{c^2} . x \qquad .....(c)$$

Again, differentiate with respect to x

$$z.\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial x} = \frac{a^2}{c^2}$$

$$z.\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2 = \frac{a^2}{c^2}$$

$$z. \frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2} - \left(\frac{\partial z}{\partial x}\right)^2$$

$$z \frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2} - \left(\frac{a^2 x}{c^2 z}\right)^2$$

$$\frac{\partial^2 z}{\partial z^2} = \frac{1}{z}\left[\frac{a^2}{c^2} - \frac{a^4 x^2}{c^4 z^2}\right]$$

$\Rightarrow$ Consider $c^2 z^2 = a^2 x^2 + b^2 y^2$

Partiallty differentiate with respect to y

$$c^2 . 2z \frac{\partial z}{\partial y} = 2y\, b^2$$

$$z \frac{\partial z}{\partial y} = \frac{b^2}{c^2} y.$$

Again differentiate with respect to 'y'

$$z\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial y} = \frac{b^2}{c^2}$$

$$z\frac{\partial^2 z}{\partial y^2} = \frac{b^2}{c^2} - \left(\frac{\partial z}{\partial y}\right)^2$$

$$z\frac{\partial z}{\partial y} = \frac{b^2}{c^2} - \left(\frac{\partial z}{\partial y}\right)^2$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{z}\left[\frac{b^2}{c^2} - \left(\frac{b^2 y}{c^2 z}\right)^2\right]$$

$$= \frac{1}{z}\left[\frac{b^2}{c^2} - \frac{64y^2}{c^4 z^2}\right]$$



Given that $\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2} + \dfrac{1}{b^2}\dfrac{\partial^2 z}{\partial y^2}$

$$= \frac{1}{a^2}\left[\frac{1}{z}\left(\frac{a^2}{c^2} - \frac{a^4 x^2}{c^4 z^2}\right)\right] + \frac{1}{b^2}\left[\frac{1}{z}\left(\frac{b^2}{c^2} - \frac{b^4 y^2}{c^4 z^2}\right)\right]$$

$$= \frac{1}{a^2}\left[\frac{a^2}{c^2 z} - \frac{a^4 x^2}{c^4 z^3}\right] + \frac{1}{b^2}\left[\frac{b^2}{zc^2} - \frac{b^4 y^2}{c^4 z^3}\right]$$

$$= \frac{1}{a^2}\frac{a^2}{c^2 z} - \frac{a^4 x^2}{a^2 c^4 z^3} + \frac{b^2}{b^2 zc^2} - \frac{b^4 y^2}{b^2 c^4 z^3}$$

$$= \frac{1}{c^2 z} - \frac{a^2 x^2}{c^4 z^3} + \frac{1}{zc^2} - \frac{b^2 y^2}{c^4 z^3}$$

$$= \frac{1}{c^2 z} - \left[\frac{a^2x^2 - c^2z^2 + b^2y^2}{c^4 z^3}\right]$$

$[\therefore a^2 x^2 + b^2 y^2 - c^2 z^2 = 0]$

$$= \frac{1}{c^2 z} - \frac{0}{c^4 z^3}$$

$$= \frac{1}{c^2 z}$$

$$\therefore \quad \frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2 z^2}$$

---

**18. If u = $\sin^{-1}\left(\frac{x}{y}\right)$ + $\tan^{-1}\left(\frac{y}{x}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ .**

*Sol :*

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

Partially differentiate with respect to x.

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{1}{y}\right) + \frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)}\left(\frac{-y}{x^2}\right)$$

$$= \frac{1}{y}\left[\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\right] - \frac{y}{x^2}\left[\frac{1}{1+\frac{y^2}{x^2}}\right]$$

Partially differentiate with respect to y

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{-x}{y^2}\right) + \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(\frac{1}{x}\right)$$

$$= \frac{-x}{y^2}\left[\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\right] + \frac{1}{x}\left[\frac{1}{1+\frac{y^2}{x^2}}\right]$$

Consider

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left\{\frac{1}{y}\left[\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\right] - \frac{y}{x^2}\left[\frac{1}{1+\frac{y^2}{x^2}}\right]\right\} + y\left\{\frac{-x}{y^2}\left[\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\right] + \frac{1}{x}\left[\frac{1}{1+y^2/x^2}\right]\right\}$$

$$= \frac{x}{y}\left[\frac{1}{\sqrt{y^2-x^2/y^2}}\right] - \frac{xy}{x^2}\left[\frac{1}{\frac{x^2+y^2}{x^2}}\right] - \frac{xy}{y^2}\left[\frac{1}{\sqrt{y^2-x^2/y^2}}\right] + \frac{y}{x}\left[\frac{1}{\frac{x^2+y^2}{x^2}}\right]$$

$$= \frac{xy}{y\sqrt{y^2-x^2}} - \frac{y}{x}\frac{x^2}{(x^2+y^2)} - \frac{x}{y}\frac{y}{\sqrt{y^2-x^2}} + \frac{yx}{(x^2+y^2)}$$

$$= \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{(x^2+y^2)}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

**19. If u = f(r) and $r = \sqrt{x^2+y^2}$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$**

*Sol :*

$$u = f(r)$$

$$r = \sqrt{x^2+y^2}$$

Partially differentiate with respect to x

$$\frac{\partial u}{\partial x} = f^1(r)\frac{1}{2\sqrt{x^2+y^2}}(2x)$$

$$= \frac{x - f'(r)}{\sqrt{x^2+y^2}}$$

$$= x f'(r)\frac{1}{\sqrt{x^2+y^2}}$$

Again, partially diff w.r to x

$$\frac{\partial^2 u}{\partial x^2} = x\cdot f'(r)\frac{-1}{2(x^2+y^2)^{3/2}}(2x) + \frac{1}{\sqrt{x^2+y^2}}\left[xf''(r)\frac{1}{2\sqrt{x^2+y^2}}(2x) + f'(r).1\right]$$

$$= \frac{-x^2 f'(r)}{\left(x^2+y^2\right)^{3/2}} + \frac{x^2 f''(r)}{x^2+y^2} + \frac{f'(r)}{\sqrt{x^2+y^2}}$$

$$= \frac{f'(r)}{\sqrt{x^2+y^2}}\left[\frac{-x^2}{x^2+y^2}+1\right] + \frac{x^2 f''(r)}{x^2+y^2}$$

$$= \frac{f'(r)}{\sqrt{x^2+y^2}}\left[\frac{y^2}{x^2+y^2}\right] + \frac{x^2 f''(r)}{x^2+y^2}$$

$$= \frac{y^2 f'(r)}{\left(x^2+y^2\right)^{3/2}} + \frac{x^2 f''(r)}{x^2+y^2}$$

Partially differentiate with respect to ‘y’

$$\frac{\partial u}{\partial y} = f^1(r)\,\frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}y)$$

$$= \frac{y\, f^1(r)}{\sqrt{x^2+y^2}}$$

Again partially differntiate with respect to ‘y’

$$\frac{\partial^2 u}{\partial y^2} = yf'(x)\frac{-1}{\cancel{2}\left(x^2+y^2\right)^{3/2}}(\cancel{2}y) + \frac{1}{\sqrt{x^2+y^2}}\left[yf''(r)\frac{1}{\cancel{2}\sqrt{x^2+y^2}}\cancel{2}y + f^1(r)1\right]$$

$$= \frac{f'(r)y^2}{\left(x^2+y^2\right)^{3/2}} + \frac{y^2 f''(r)}{x^2+y^2} + \frac{f'(r)}{\sqrt{x^2+y^2}}$$

$$= \frac{-f^1(r)}{\sqrt{x^2+y^2}}\left[\frac{-y^2}{x^2+y^2}+1\right] + \frac{y^2 f''(r)}{x^2+y^2}$$

$$= \frac{f^1(r)}{\sqrt{x^2+y^2}}\left[\frac{-\cancel{y^2}+x^2+\cancel{y^2}}{x^2+y^2}\right] + \frac{y^2 f''(x)}{x^2+y^2}$$

$$= \frac{f^1(r)x^2}{\left(x^2+y^2\right)^{3/2}} + \frac{y^2 f''(r)}{x^2+y^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 f^1(r)}{(x^2+y^2)^{3/2}} + \frac{x^2 f''(r)}{x^2+y^2} + \frac{f^1(r)x^2}{(x^2+y^2)^{3/2}} + \frac{y^2 f''(r)}{x^2+y^2}$$

$$= \frac{f^1(r)}{(x^2+y^2)^{3/2}}\left[x^2+y^2\right] + \frac{f''(r)}{x^2+y^2}(y^2+x^2)$$

$$= \frac{f^1(r)}{(x^2+y^2)^{1/2+1}}(x^2+y^2) + f''(r)$$

$$= \frac{f'(x)}{\left(\sqrt{x^2+y^2}\right)(x^2+y^2)}(x^2+y^2) + f''(r)$$

$$= \frac{f^1(r)}{\sqrt{x^2+y^2}} + f''(r)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f'(r)\frac{1}{r} + f''(r)$$

**20. If u = $e^x$ (x cosy – y siny). then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$**

*Sol :*

u = $e^x$ (x cosy – y siny)

Partial differentiate with respect to 'y'

$$\frac{\partial u}{\partial x} = e^x (x \cos y - y \sin y) + e^x (\cos y)$$

Again partially differentiate with respect to 'x'

$$\frac{\partial^2 u}{\partial x^2} = e^x (x \cos y - y \sin y) + e^x (\cos y) + \cos y \, e^x$$

$$= e^x (x \cos y - y \sin y) + 2 e^x \cos y$$

Consider

u = $e^x$ (x cos y – y siny)

Partially differentiate with respect to 'y'

$$\frac{\partial u}{\partial y} = e^x (-x \sin y - y \cos y - \sin y)$$

$$= e^x [-x \sin y - y \cos y - \sin y]$$

Again partially differentiate with respect to 'y'

$$\frac{\partial^2 u}{\partial y^2} = e^x [(-x \cos y) - y(-\sin y) - \cos y - \cos y]$$

$$= e^x [-x \cos y + y \sin y - 2 \cos y]$$

$$= e^x (-x \cos y + y \sin y) - 2e^x \cos y$$

Consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x (x \cos y - y \sin y) + 2e^x \cos y + e^x (-x \cos y + y \sin y) - 2e^x \cos y$$

$$= xe^x \cos y - e^x y \sin y + 2e^x \cos y - xe^x \cos y + ye^x \sin y - 2e^x \cos y$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

**21. If $u = x^2(y - z) + y^2 (z - x) + z^2(x - y)$**

**Then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.**

*Sol.:*

Given that $u = x^2(y - z) + y^2 (z - x) + z^2(x - y)$,

Partially differentiate with respect to x.

$$\frac{\partial u}{\partial x} = 2x(y - z) + y^2 (-1) + z^2(1)$$

$$= 2xy - 2xz - y^2 + z^2$$

$$\frac{\partial u}{\partial y} = x^2(1) + 2y(z - x) + z^2(-1)$$

$$= x^2 + 2yz - 2yx - z^2 \qquad \frac{\partial u}{\partial z} = x^2(-1) + y^2(1) + 2z(x - y)$$

$$= -x^2 + y^2 + 2zx - 2zy$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 2xy - 2xz - y^2 + z^2 + x^2 + 2yz - 2yx - z^2 - x^2 + y^2 + 2zx - 2zy$$

$$= 0.$$

**22. If $z = f(x + ay) + \phi(x - ay)$ then prove that $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$.**

*Sol.:*

Given that

$$z = f(x + ay) + \phi(x - ay)$$

Partially differentiative with respect to x

$$\frac{\partial z}{\partial x} = f'(x + ay)(1) + \phi'(x - ay)$$

Again differentiate with respect to x

$$\frac{\partial^2 z}{\partial x^2} = f''(x + ay)(1) + \phi''(x - ay)(1) \quad \text{....(1)}$$

Differentiate with respect to 'y'

$$z = f(x + ay) + \phi(x - ay)$$

$$\frac{\partial z}{\partial y} = f'(x + ay)\, a + \phi'(x - ay)(-a)$$

$$= af'(x + ay) + \phi f'(x - ay)$$

$$\frac{\partial^2 z}{\partial y^2} = a\, f''(x + ay)\, a - a\, \phi''(x - ay)(-a)$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x + ay) + a^2 \phi''(x - ay)$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 (f''(x + ay) + \phi''(x - ay) \quad \text{.....(2)}$$

Multiply $a^2$ on both sides .....(1)

$$a^2 \frac{\partial^2 z}{\partial x^2} = a^2[f''(x + ay) + f''(x - ay)] \quad \text{.....(3)}$$

$$\therefore \quad \frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$



**23. If $z = 3xy - y^3 + (y^2 - 2x)^{3/2}$ verify that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial z}$ and $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$.**

*Sol.:*

$$Z = 3xy - y^3 + (y^2 - 2x)^{3/2}$$

Partially differentiate w.r.t to 'x'

$$\frac{\partial z}{\partial x} = 3y + \frac{3}{2}(y^2 - 2x)^{3/2-1}(-2)$$

$$= 3y - 3(y^2 - 2x)^{1/2}$$

Again, partially differentiate with respect to y

$$\frac{\partial z}{\partial y} = 3x - 3y^2 + \frac{3}{2}(y^2 - 2x)^{3/2-1}(2y)$$

$= 3x - 3y^2 + 3y(y^2 - 2x)^{1/2}$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}[3x - 3y^2 + 3y(y^2 - 2x)^{1/2}]$$

$$= 3\frac{\partial}{\partial x}(x - y^2 + y(y^2 - 2x)^{1/2})$$

$$= 3\left[1 + y\frac{1}{2}\left(y^2 - 2x\right)^{\frac{1}{2}-1}(-2)\right]$$

$3 - 3y(y^2 - 2x)^{-1/2}$.

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(3y - 3(y^2 - 2x)^{1/2})$$

$$= 3\frac{\partial}{\partial y}\left(y - (y^2 - 2x)^{1/2}\right)$$

$$= 3\left[2 - \frac{1}{2}\left(y^2 - 2x\right)^{1/2-1}(2y)\right]$$

$$= 3 - 3y\left(y^2 - 2x\right)^{-1/2}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Consider

$$= \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(3y - 3\left(y^2 - 2x\right)^{1/2}\right)$$

$$= 3\frac{\partial}{\partial x}\left(y - \left(y^2 - 2x\right)^{1/2}\right)$$

$$= 3\left[0 - \frac{1}{\not{2}}\left(y^2 - 2x\right)^{1/2-1}\left(-\not{2}\right)\right]$$

$$= 3\left[\left(y^2 - 2x\right)^{-\frac{1}{2}}\right]$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\ (3x - 3y^2 + 3y(y^2 - 2x)^{1/2})$$

$$= 0 - 6y + 3y.\frac{1}{2}\ (y^2 - 2x)^{\frac{1}{2}-1}\ 3\,(y^2 - 2x)^{\frac{1}{2}}(1)$$

$$= -6y + 3y^2\ (y^2 - 2x)^{-\frac{1}{2}} + 3\left(y^2 - 2x\right)^{\frac{1}{2}}$$

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = 3(y^2 - 2x)^{-\frac{1}{2}}\left[-6y + 3y^2(y^2 - 2x)^{-\frac{1}{2}} + 3(y^2 - 2x)^{\frac{1}{2}}\right]$$

$$= -18\,y(y^2 - 2x)^{-\frac{1}{2}} + 9y^2\left[\left(y^2 - 2x\right)^{-1/2}\right]^2 + 9(y^2 - 2x)^{-\frac{1}{2}}(y^2 - 2x)^{\frac{1}{2}}$$

$$= -18\ y(y^2 - 2x)^{-\frac{1}{2}} + 9y^2\ (y^2 - 2x)^{-1} + 9$$

Consider

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = [3 - 3y\ (y^2 - 2x)^{-\frac{1}{2}}\ ]^2$$

$$= 9 + 9y^2\ (y^2 - 2x)^{-1} - 18y\ (y2 - 2x)–½$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2$$

**24. If Z = tan (y + ax) + (y – ax)$^{3/2}$ find the value of $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2}$.**

*Sol.:*

$$Z = \tan\,(y + ax) + (y - ax)^{3/2}$$

$$\frac{\partial z}{\partial x} = \sec^2\,(y + ax)\ (a) + \frac{3}{2}\ (y - ax)^{3/2-1}\ (-a)$$

$$= a\,\sec^2(y + ax) - \frac{3}{2}a\ (y - ax)^{\frac{1}{2}}$$

$$\frac{\partial^2 z}{\partial x^2} = a\ 2\sec(y+ax)\sec(y+ax)\tan(y+ax)(a) - \frac{3a}{2}\frac{1}{2}(y-ax)^{1/2-1}(-a)$$

$$= 2a^2\sec^2(y+ax)\tan(y+ax) + \frac{3a^2}{4}(y-ax)^{-1/2}$$

$$\frac{\partial z}{\partial y} = \sec^2(y+ax)(1) + \frac{3}{2}(y-ax)^{\frac{3}{2}-1}(1)\ \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$$

$$\frac{\partial^2 z}{\partial y^2} = 2\sec(y+ax)\sec(y+ax)\tan(y+ax) + \frac{3}{2}\cdot\frac{1}{2}(y-ax)^{1/2-1}(1)$$

$$= 2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}$$

$$= 2a^2\sec^2(y+ax)\tan(y+ax) + \frac{3a^2}{4}(y-ax)^{-1/2}\ \frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2}$$

$$= 2a^2\sec^2(y+ax)\tan(y+ax) + \frac{3a^2}{4}(y-ax)^{-1/2}$$

$$- 2a^2\sec^2(y+ax)\tan(y+ax) - \frac{3a^2}{4}(y-ax)^{-1/2}$$

$$= 0.$$

---

**25. If $u = f(x+2y) + g(x-2y)$ then show that $4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$.**

*Sol.:*

$$u = f(x+2y) + g(x-2y)$$

Partially differentiate with respect to x

$$\frac{\partial u}{\partial x} = f'(x+2y)(1) + g'(x-2y)(1)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x+2y) + g''(x-2y) \quad .....(1)$$

Partially differentiate with respect to y

$$\frac{\partial u}{\partial y} = f'(x+2y)(2) + g'(x-2y)(-2)$$

$$= 2f'(x+2y) - 2g'(x-2y)$$

$$\frac{\partial^2 u}{\partial y^2} = 2f''(x+2y)(2) - 2g''(x-2y)(-2)$$

$= 4\ f''(x + 2y) + 4\ g''(x - 2y)$

Multiply 4 on both side to equation (1).

$$4\frac{\partial^2 u}{\partial x^2} = 4[f''(x + 2y) + g''(x - 2y)]$$

$$\therefore 4\ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

$= 0$

## 1.7 HOMOGENEOUS FUNCTIONS.

**26. Define Homogenous function with example.**

*Sol :*

If the sum of indices of different variables contained in each term of an algebraic expression be n, it is called a homogenous function of degree n.

⇒ Let u = f(x, y) be a function of x and y. If this sum of the power of x and y in each term of f(x, y) be equal, then f(x, y) is called homogenous function.

⇒ **Eg.** $x^2 + y^3 + 3x^2 y$ is homogenous functio of order 3.

→ $x^4 + y^4 + 4x^2 y^2$ is homogenous function of 4th orders.

Consider the function

$f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + ..... + a_{n-1} xy^{n-1} + a_n y^n$ .......(1)

We see that the expression f(x,y) is polynomial in (x,y) such that the degree of each of the terms is the same f is called a homogenous function of degree n.

- An expression in (x,y) is homogenous of degree n, if it is expressible as

  $x^n f(y/x)$

- The polynomial function (1) which can be rewritten as

$$x^n\left[a_0 + a_1\frac{y}{x} + a_2\left(\frac{y}{x}\right)^2 + ...... + a_n\left(\frac{y}{x}\right)^n\right]$$

is a homogenous expression of order n

**27. Find the degree of given Homogenous function for $f(x,y) = x^n \sin(y/x)$.**

*Sol :*

Given that

$f(x,y) = x^n \sin(y/x)$

$f(x,y) = \left(\sqrt{y} + \sqrt{x}\right)\Big/\left(y + x\right)$

The degree of the expression $x^n \sin(y/x)$ is n

The degree of the expression $\Rightarrow \dfrac{\sqrt{y}+\sqrt{x}}{y+x} - \dfrac{\sqrt{x}\left[1+\dfrac{\sqrt{y}}{\sqrt{x}}\right]}{x\left[1+\dfrac{y}{x}\right]}$

$$= x^{1/2-1}\frac{\left[1+\dfrac{\sqrt{y}}{\sqrt{x}}\right]}{1+y/x}$$

$$= x^{-1/2}\frac{1+\sqrt{y}/\sqrt{x}}{1+y/z}$$

So, that it is of degree $\dfrac{-1}{2}$

**28. State and Prove Euler's theorem on Homogenous function.**

**Statement:**

If z = f(x, y) be a homogenous, function of x, y of degree n then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \ \forall x, y \in \text{domain of the function.}$$

**Proof:**

Since, z is homogenous function x and y of degree n,

Then we have $z = x^n f\left(y/x\right)$

$$\frac{\partial z}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= nx^{n-1} f\left(\frac{y}{x}\right) - y\,x^{n-2} f'\left(\frac{y}{x}\right)$$

also

$$\frac{\partial z}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x^{n-1} f'\left(\frac{y}{x}\right)$$

Now

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left[nx^{n-1}f\left(\frac{y}{n}\right) - y\,x^{n-2}f'\left(\frac{y}{n}\right)\right] + y\left[x^{n-1}f'\left(\frac{y}{n}\right)\right]$$

$$= nx^n f\left(\frac{y}{n}\right) - yx^{n-1} f'\left(\frac{y}{x}\right) + yx^{n-1} f'(y/n)$$

$$= nx^n f(y/n) = nz$$

$$\therefore x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = nz$$

**Note :**

Euler's theorem can be entended to a homogenous function of several variables thus, if u be the function of m independent variables $x_1, x_2, .... x_m$ of degree n.

Then Euler's theorem states that

$$x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} + ... + x_m\frac{\partial u}{\partial x_m} = nu.$$

**29. If z = f(x, y) is homogenous function of x, y of degree, then**

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)\,z$$

*Sol :*

Since z is homogenous function x and y of degree n.

Therefore by Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Differentiate partially with respect to 'x'

$$\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(nz)$$

$$x.\frac{\partial^2 z}{\partial x^2}.1 + \frac{\partial^2 z}{\partial x} + y\frac{\partial^2 z}{\partial y^2} + 0 = n\frac{\partial z}{\partial x}$$

$$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\,\partial y} = (n-1)\frac{\partial z}{\partial x} \quad .....(1)$$

Differentiate partially with respect to y .

$$\frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial n}\right) + \frac{\partial}{\partial y}\left(y\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(nz)$$

$$x\frac{\partial^2 z}{\partial x\partial y} + y\frac{\partial^2 z}{\partial y^2} + 1.\frac{\partial z}{\partial y} = n\frac{\partial z}{\partial y}$$

$$x\frac{\partial^2 z}{\partial x\,\partial y}+y\frac{\partial^2 z}{\partial y^2} = (n-1)\,\frac{\partial z}{\partial y} \qquad .....(2)$$

x × (1) + y x(2)

$$x^2\frac{\partial^2 z}{\partial x^2} + xy\,\frac{\partial^2 z}{\partial x\,\partial y} + yx\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2}$$

$$= x(n-1)\,\frac{\partial z}{\partial x} + y\,(x-1)\,\frac{\partial z}{\partial y}$$

$$= x^2\,\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} + 2xy\,\frac{\partial^2 z}{\partial x\partial y}$$

$$= (n-1)\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]$$

$$= (n-1)\,(n\,z)$$

$$= n(n-1)z$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

---

**30. If $\sin v = \frac{(x+2y+3z)}{\sqrt{x^8+y^8+z^8}}$ Show that $x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z} + 3\tan v = 0$**

*Sol :*

$$f = \sin v = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$$

$$= \frac{x\left[1+2\left(\frac{y}{x}\right)+3\left(\frac{2}{x}\right)\right]}{x^4\sqrt{1+\left(\frac{y}{x}\right)^8+\left(\frac{z}{x}\right)^8}}$$

$$= x^{-3}\left(\frac{1+2\left(\frac{y}{x}\right)+3\left(\frac{2}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^8+\left(\frac{z}{x}\right)^8}}\right)$$

f is a homogenous function of degree '–3'

By Enler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = -3\sin v.$$

$$\frac{\partial f}{\partial x} = \cos v \; ; \; \frac{\partial f}{\partial y} = \cos v \frac{\partial v}{\partial y} \; ; \; \frac{\partial f}{\partial z} = \cos v \frac{\partial v}{\partial z}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z}$$

$$= \cos v\left[x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}\right] = -3\sin v$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = -3\tan v$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} + 3\tan v = 0.$$

**31. If $u = \log_e\sqrt{x^2+y^2+z^2}$ Then $(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) = 1$**

*Sol :*

Partially differentiate with respect to 'x'

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}} \frac{1}{\not{2}\sqrt{x^2+y^2+z^2}}(\not{2}x)$$

$$= \frac{x}{x^2+y^2+z^2}$$

Again partially differentiating with respect to 'x'

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2)-x(2x)}{(x^2+y^2+z^2)} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

Partially differentiate with respect to 'y'

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{x^2+y^2+z^2}} \frac{1}{\not{2}\sqrt{x^2+y^2+z^2}}(\not{2}y)$$

$$= \frac{y}{x^2+y^2+z^2}$$

Again partially differentiate with respect to 'y'

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2+z^2)-y(2y)}{(x^2+y^2+z^2)} = \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2}$$

Partially differentiate with respect to 'z'

$$\frac{\partial u}{\partial z} = \frac{z}{x^2+y^2+z^2}$$

Again partially differentiate with respect to 'z'

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

Consider $(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$

$$= (x^2 + y^2 + z^2)\left[\frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}+\frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2}+\frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}\right]$$

$$= (x^2+y^2+z^2)\left[\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}\right] = 1$$

Hence proved

**32. If $u = \tan^{-1}\frac{xy}{\sqrt{1+x^2+y^2}}$ Show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$**

*Sol :*

$$u = \tan^{-1}\frac{xy}{\sqrt{1+x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left[\frac{xy}{\sqrt{1+x^2+y^2}}\right]^2}\left[\frac{\sqrt{1+x^2+y^2}\,(x)-xy\frac{1}{2\sqrt{1+x^2+y^2}}(2y)}{\left(\sqrt{1+x^2+y^2}\right)^2}\right]$$

$$= \frac{1+x^2+y^2}{1+x^2+y^2+x^2y^2}\left[\frac{x\sqrt{1+x^2+y^2}-\frac{xy^2}{\sqrt{1+x^2+y^2}}}{1+x^2+y^2}\right]$$

$$= \frac{x(1+x^2+y^2)-xy^2}{\sqrt{1+x^2+y^2}}\left(\frac{1}{(1+x^2)(1+y^2)}\right)$$

$$= \frac{x+x^3+\cancel{xy^2}-\cancel{xy^2}}{\sqrt{1+x^2+y^2}}\left(\frac{1}{(1+x^2)(1+y^2)}\right)$$

$$= \frac{x\cancel{(1+x^2)}}{\sqrt{1+x^2+y^2}}\left(\frac{1}{\cancel{(1+x^2)}(1+y^2)}\right)$$

$$= \frac{x}{(1+y^2)\sqrt{1+x^2+y^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\frac{x}{(1+y^2)\sqrt{1+x^2+y^2}}$$

$$= \frac{\partial}{\partial x}\ \frac{x}{(1+y^2)\sqrt{1+x^2+y^2}}$$

$$= \frac{(1+y^2)\sqrt{1+x^2+y^2}-x\cdot\frac{1+y^2}{\cancel{2}\sqrt{1+x^2+y^2}}(\cancel{2}x)}{\left(\sqrt{1+x^2+y^2}\right)^2(1+y^2)^2}$$

$$= \frac{(1+y^2)\sqrt{1+x^2+y^2}-\frac{x^2(1+y^2)}{\sqrt{1+x^2+y^2}}}{(1+y^2)^2(1+x^2+y^2)}$$

$$= \frac{(1+y^2)(1+x^2+y^2)-x^2(1+y^2)}{\sqrt{1+x^2+y^2}\,(1+y^2)^2(1+x^2+y^2)}$$

$$= \frac{(1+y^2)\left[1+\cancel{x^2}+y^2-\cancel{x^2}\right]}{\sqrt{1+x^2+y^2}\,(1+y^2)^2(1+x^2+y^2)}$$

$$= \frac{\cancel{(1+y^2)}\ \cancel{(1+y^2)}}{\sqrt{1+x^2+y^2}\,\cancel{(1+y^2)^2}\,(1+x^2+y^2)}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{\left(1+x^2+y^2\right)^{3/2}}$$

---

**33.** If z = xyf(y/x), Show that $\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ **= 2z, & if z a constant, then** $\frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x(y+xy')}{y(y-xy')}$

*Sol :*

Given that,

$$z = xyf\left(\frac{y}{x}\right)$$

$$z = x \cdot x\frac{y}{x} f\left(\frac{y}{x}\right)$$

$$z = x^2 t\left(\frac{y}{x}\right) \qquad \left[\text{where } t\left(\frac{y}{x}\right) = \frac{y}{x} f\left(\frac{y}{x}\right)\right]$$

z is a homogenous function of degree 2.

$$\frac{\partial z}{\partial x} = yf\left(\frac{y}{x}\right) + xy\, f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= yf\left(\frac{y}{x}\right) - \frac{y^2}{x} f'\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = xf\left(\frac{y}{x}\right) + \cancel{x} yf'\left(\frac{y}{x}\right)\left(\frac{1}{\cancel{x}}\right)$$

$$= xf\left(\frac{y}{x}\right) + yf'\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial^2}{\partial y} = xyf\left(\frac{y}{x}\right) - \cancel{y^2 f'\left(\frac{y}{x}\right)} + xyf\left(\frac{y}{x}\right) + \cancel{y^2 f'\left(\frac{y}{x}\right)}$$

$$= 2xyf\left(\frac{y}{x}\right)$$

$$= 2z$$

To find $= \frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)}$

Consider

$$z = xyf(y/x)$$

Differentiate z with respect to x, y

$$0 = xyf'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2}+\frac{1}{x}\frac{dy}{dx}\right] + f\left(\frac{y}{x}\right)x\frac{dy}{dx} + f\left(\frac{y}{x}\right)y.$$

$$0 = f'\left(\frac{y}{x}\right)\left[\frac{-y^2}{x}+y\frac{dy}{dx}\right] + f\left(\frac{y}{x}\right)\left[x\frac{dy}{dx}+y\right]$$

$$f'\left(\frac{y}{x}\right)\left[y\frac{dy}{dx}-\frac{y^2}{x}\right] = f\left(\frac{y}{x}\right)\left[x\frac{dy}{dx}+y\right]$$

$$\frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x[y+xy']}{[xyy'-y^2]}$$

$$= \frac{x[y+xy']}{y[-xy'+y^2]}$$

$$\therefore \quad \frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x[y+xy']}{y[y-xy']}$$

Hence proved

**34. If $v = r^m$ where $r^2 = x^2 + y^2 + z^2$; show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = m(m+1)\, r^{m-2}$.**

*Sol :*

Given,

$$v = r^m$$

$$r^2 = x^2 + y^2 + z^2$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$v = \left(x^2+y^2+z^2\right)^{m/2}$$

Differentiate partially with respect to 'x'

$$\frac{\partial v}{\partial x} = \frac{m}{\not{2}}\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1}(\not{2}x)$$

$$= mx\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1}$$

$$\frac{\partial^2 v}{\partial x^2} = m\left[\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1} + x\left(\frac{m-2}{\not{2}}\right)\left(x^2+y^2+z^2\right)^{\frac{m}{2}-2}\left(\not{2}x\right)\right]$$

$$= m\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1} + mx^2(m-2)\left(x^2+y^2+z^2\right)^{\frac{m}{2}-2}$$

Differentiating partially with respect to 'y'

$$\frac{\partial v}{\partial y} = \frac{m}{\not{2}}\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1}\left(\not{2}y\right)$$

$$= my\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1}$$

$$\frac{\partial^2 v}{\partial z^2} = m\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1} + my^2(m-2)\left(x^2+y^2+z^2\right)^{\frac{m}{2}-2}$$

Similarly

$$\frac{\partial^2 v}{\partial z^2} = m\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1} + m\,z^2(m-2)\left(x^2+y^2+z^2\right)^{\frac{m}{2}-2}$$

Consider

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 3m\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1} + m\,(m-2)\left(x^2+y^2+z^2\right)^{\frac{m}{2}-2}(x^2+y^2+z^2)$$

$$= 3m\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1} + m(m-2)\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1}$$

$$= m\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1}[3+m-2]$$

$$= m(m+1)\, r^{m-2}.$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = m(m+1)\, r^{m-2}.$$

**35. If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$; Show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$ (1 – 4 sin² u) sin 2u.**

*Sol :*

$$z = \tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\left(1+\left(\frac{y}{x}\right)^3\right)}{x\left[1-y\left(\frac{y}{x}\right)\right]}$$

$$= x^2\left[\frac{1+\left(\frac{y}{x}\right)^3}{1-\left(\frac{y}{x}\right)}\right]$$

z is a homogenous function of degree 2.

$$\frac{\partial z}{\partial x} = \sec^2 \frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$$

$$x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y} = \sec^2 u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right] = 2z.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cancel{\cos u}}\cdot\cancel{\cos u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

Differentiate partially with respect to 'x'

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u\frac{\partial u}{\partial x}$$

$$x\frac{x^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = (2\cos 2u - 1)\frac{\partial u}{\partial x} \qquad .....(1)$$

Differentiate partially with respect to 'y'

$$x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2\cos 2u\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\frac{\partial u}{\partial y} \qquad .....(2)$$

(1) × x + (2) × y

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} = [2\cos 2u - 1]\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$= [2[1 - 2\sin^2 u] - 1]\,[\sin 2u]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + y^2\frac{\partial^2 u}{\partial y^2} = [1 - u\sin^2 u]\,[\sin 2u]$$

**36. If $u = \left(1-2xy+y^2\right)^{-1/2}$ Prove that $\frac{\partial}{\partial x}\left(\left(1-x^2\right)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right) = 0$**

*Sol :*

Given,

$$u = \left(1-2xy+y^2\right)^{-1/2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left(1-2xy+y^2\right)^{-3/2}(-2y)$$

$$= y\left(1-2xy+y^2\right)^{-3/2}$$

$$= (1-x^2)\frac{\partial u}{\partial x} = (1-x^2)y\left(1-2xy+y^2\right)^{-3/2}$$

$$= \frac{\partial}{\partial x}\left(\left(1-x^2\right)\frac{2u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\left(1-x^2\right)y\left(1-2xy+y^2\right)^{-3/2}\right]$$

$$= y\left[\frac{-3}{2}\left(1-x^2\right)\left(1-2xy+y^2\right)^{-5/2}(-2y)+\left(1+2xy+y^2\right)^{-3/2}\right]$$

$$= 3y^2(1-x^2)\left(1-2xy+y^2\right)^{-5/2} - 2xy\left(1-2xy+y^2\right)^{-3/2} \quad .....(1)$$

$$\frac{\partial u}{\partial x} = \frac{-1}{2}\left(1-2xy+y^2\right)^{-3/2}(-2x+2y)$$

$$= \left(1-2xy+y^2\right)^{-3/2}(x-y)$$

$$y^2\frac{\partial u}{\partial y} = y^2(x-y)\left(1-2xy+y^2\right)^{-3/2}$$

$$\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left[y^2(x-y)\left(1-2xy+y^2\right)^{-3/2}\right]$$

$$= 2y(x-y)\left(1-2xy+y^2\right)^{-3/2} - y^2\left(1-2xy+y^2\right)^{-3/2} + y^2(x-y)\left(\frac{-3}{2}\right)\left(1-2xy+y^2\right)^{-3/2}(-2x+2y)$$

$$= 2y(x-y)\,2\left(1-2xy+y^2\right)^{-3/2} - y^2\left(1-2xy+y^2\right)^{-3/2} + 3y^2(x-y)^2\left(1-2xy+y^2\right)^{-5/2} \quad .....(2)$$

On adding (1) & (2)

$$= \frac{\partial}{\partial x}\left(\left(1-x^2\right)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right)$$

$$= 3y^2(1-x^2)\left(1-2xy+y^2\right)^{-5/2} - \cancel{2xy}\left(1-2xy+y^2\right)^{-3/2} + \cancel{2xy}\left(1-2xy+y^2\right)^{-3/2} - 2y^2\left(1-2xy+y^2\right)^{-3/2}$$

$$- y^2\left(1-2xy+y^2\right)^{-3/2} + 3y^2(x-y)^2\left(1-2xy+y^2\right)^{-5/2}$$

$$= 3y^2\left(1-2xy+y^2\right)^{-5/2}\left[1-\cancel{x^2}+\cancel{x^2}-2xy+y^2\right] - 3y^2\left(1-2xy+y^2\right)^{-3/2}$$

$$= 3y^2\left(1-2xy+y^2\right)^{-3/2}\left[\frac{y^2-2xy+1}{1-2xy+y^2}\right]$$

$$= 3y^2\left(1-2xy+y^2\right)^{-1/2}\left[\frac{y^2-\cancel{2xy}+\cancel{1}-\cancel{1}+\cancel{2xy}-\cancel{y^2}}{\cancel{1-2xy+y}}\right]$$

$$= 3y^2\left(1-2xy+y^2\right)^{-1/2}\ (0)$$

$$= 0$$

Hence proved $\therefore \frac{\partial}{\partial x}\left(\left(1-x^2\right)\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right) = 0$

**37. Verify that $\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial x\partial y}$ if u = log (y sin x + x sin y)**

*Sol :*

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{[u\cos x+\sin y]}{y\sin x + x\sin y}$$

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{(y\sin x + x\sin y)(\cos x+\cos y)-(y\cos x+\sin y)(\sin x + x\cos y)}{(y\sin x + x\sin y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{\sin x + x\cos y}{y\sin x + x\sin y}$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{(y\sin x + x\sin y)(\cos x+\cos y)-(\sin x + x\cos y)(y\cos x+\sin y)}{(y\sin x + x\sin y)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Hence verified

**38. Verify Euler's Theorem for $z = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + x^{1/5}}$**

*Sol :*

To verify, z is homogenous or not

Consider

$$z = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

$$= \frac{x^{1/4}\left[1 + \frac{y^{1/4}}{x^{1/4}}\right]}{x^{1/5}\left[1 + \frac{y^{1/5}}{x^{1/5}}\right]} = x^{1/4 - 1/5} \frac{\left[1 + \left(\frac{y}{x}\right)^{1/4}\right]}{1 + \left(\frac{y}{x}\right)^{1/5}}$$

$$= x^{1/20} \frac{\left[1 + \left(\frac{y}{x}\right)^{1/4}\right]}{\left(1 + \left(\frac{y}{x}\right)^{1/5}\right.}$$

z is a homogenous function of degree $\frac{1}{20}$ $\left[n = \frac{1}{20}\right]$

Consider

$$z = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$ Partially differentiate with respect to 'x'

$$\frac{\partial z}{\partial x} = x^{1/4} + y^{1/4}\left[\frac{-1}{\left(x^{1/5} + y^{1/5}\right)^2}\right]\frac{1}{5}\, x^{\frac{1}{5}-1} + \frac{1}{x^{1/5} + y^{1/5}}\left(\frac{1}{4} x^{1/4 - 1}\right)$$

$$= x^{1/4} + y^{1/4}\left[\frac{-1}{\left(x^{1/5} + y^{1/5}\right)^2}\right]\frac{1}{5}\, x^{-4/5} + \frac{1}{x^{1/5} + y^{1/5}}\left(\frac{1}{4} x^{-3/4}\right)$$

$$= \frac{-1}{5}x^{-4/5}\left[\frac{x^{1/4}+y^{1/4}}{\left(x^{1/5}+y^{1/5}\right)^2}\right] + \frac{1}{4}\,x^{-3/4}\left[\frac{1}{x^{1/5}+y^{1/5}}\right]$$

Application Partially differentiate with respect to 'y'

$$\frac{\partial z}{\partial y} = \left(x^{1/4}+y^{1/4}\right)\left[\frac{-1}{\left(x^{1/5}+y^{1/5}\right)^2}\right]\frac{1}{5}\,y^{-4/5} + \frac{1}{x^{1/5}+y^{1/5}}\left(\frac{1}{4}x^{-3/4}\right)$$

$$= \frac{-1}{5}y^{-4/5}\left[\frac{x^{1/4}+y^{1/4}}{\left(x^{1/5}+y^{1/5}\right)^2}\right] + \frac{1}{4}\,y^{-3/4}\left[\frac{1}{x^{1/5}+y^{1/5}}\right]$$

Consider

$$= x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{-1}{5}x^{1/5}\left[\frac{\left(x^{1/4}+y^{1/4}\right)}{\left(x^{1/5}+y^{1/5}\right)^2}\right] + \frac{1}{4}\,x^{1/4}\left[\frac{1}{x^{1/5}+y^{1/5}}\right] - \frac{1}{5}\,y^{1/5}\left[\frac{x^{1/4}+y^{1/4}}{\left(x^{1/5}+y^{1/5}\right)^2}\right]$$

$$+ \frac{1}{4}\,y^{1/4}\left[\frac{1}{x^{1/5}+y^{1/5}}\right]$$

$$= \frac{-1}{5}\left[\frac{x^{1/4}+y^{1/4}}{\left(x^{1/5}+y^{1/5}\right)^2}\right]\left(x^{1/5}+y^{1/5}\right) + \frac{1}{4}\left[\frac{1}{x^{1/5}+y^{1/5}}\right]\left(x^{1/4}+y^{1/4}\right)$$

$$= \frac{-1}{5}\,z + \frac{1}{4}\,z$$

$$= \frac{-4z+5z}{20} = \frac{z}{20} = \frac{1}{20}\,z.$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{20}z$$

**39. If $e^{xyz}$ such that $\dfrac{\partial^3 u}{\partial x\partial y\partial z} = (1 + 3xyz + x^2y^2z^2)\,e^{xyz}$.**

*Sol :*

Given that

$$u = e^{xyz}$$

$$\frac{\partial u}{\partial x} = e^{xyz}\,.\,yz$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[e^{xyz}(yz)\right]$$

$$= \frac{\partial}{\partial y}(yz).e^{xyz} + yz\frac{\partial}{\partial y}\left(e^{xyz}\right)$$

$$= yze^{xyz}(xz) + ze^{xyz}$$

$$\frac{\partial^2 u}{\partial y \partial x} = xyz^2 e^{xyz} + ze^{xyz}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial z}\left[\frac{\partial^2 u}{\partial y \partial x}\right]$$

$$= \frac{\partial}{\partial z}\left[xyz^2\ e^{xyz} + ze^{xyz}\right]$$

$$= \frac{\partial}{\partial z}\left[xyz^2\ e^{xyz}\right] + \frac{\partial}{\partial z}\left[ze^{xyz}\right]$$

$$= xyz^2 \frac{\partial}{\partial z}(e^{xyz}) + \frac{\partial}{\partial z}(xyz)^2 e^{xyz} + z\frac{\partial}{\partial z}(e^{xyz}) + \frac{\partial}{\partial z}(z)e^{xyz}$$

$$= xyz^2 e^{xyz}(xy) + 2zxy\ e^{xyz} + z\ e^{xyz}(xy) + e^{xyz}$$

$$= x^2 y^2 z^2 e^{xyz} + 2\ xyz\ e^{xyz} + xyz\ e^{xyz} + e^{xyz}$$

$$\therefore \quad \frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2 y^2 z^2)\ e^{xyz}$$



**40. Find the first order partial derivations of (i) $\tan^{-1}(x + y)$ (ii) $e^{an}$ sin by (iii) $\log(x^2 + y^2)$.**

*Sol :*

(i) Given function $z = f(x,y) = \tan^{-1}(x + y)$

Differentiating partially with respect to 'x'

$$\frac{\partial z}{\partial x} = \frac{1}{1+(x+y)^2} \quad (1)$$

$$\frac{\partial z}{\partial x} = \frac{1}{1+(x+y)^2}$$

Differentiating partially with respect to 'y'

$$\frac{\partial z}{\partial y} = \frac{1}{1+(x+y)^2}$$

(ii) $e^{ax}$ sin by

Given function $z = f(x,y) = e^{ax} \sin by$

Differentiating partially with respect to 'x'

$$\frac{\partial z}{\partial x} = ae^{ax} \sin by$$

Differentiating partially with respect to 'y'

$$\frac{\partial z}{\partial y} = e^{ax} \cos by\ (b)$$

$$\frac{\partial z}{\partial y} = be^{ax} \cos by$$

(iii) $\log(x^2+y^2)$

Given function is $z = f(x,y) = \log(x^2 + y^2)$

Differentiating partially with respect to 'x'

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2}(2x)$$

$$= \frac{2x}{x^2+y^2}$$

Differentiating partially with respect to 'y'

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2}(2y)$$

$$= \frac{2y}{x^2+y^2}$$

**41. Find the second order partially derivative of $e^{x-y}$.**

*Sol :*

$e^{x-y}$

Given function is $z = f(x,y) = e^{x-y}$

Differentiating partially with respect to 'x'

$$\frac{\partial z}{\partial x} = e^{x-y}$$

Again Differentiating partially with respect to 'x'

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^{x-y}\right)$$

$$= e^{x-y}$$

Differentiating partially with respect to 'y'

$$\frac{\partial z}{\partial y} = e^{x-y}(-1)$$

$$= -e^{x-y}$$

Again , Differentiate partially with respect to 'y'

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(-e^{x-y}\right)$$

$$= -e^{x-y}(-1)$$

$$\frac{\partial^2 z}{\partial y^2} = e^{x-y}$$

**42. If $z = (x + y) + (x + y)\,\phi\left(\frac{y}{x}\right)$ prove that $x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)$**

*Sol :*

Given function $z = (x + y) + (x + y)\,\phi\left(\frac{y}{x}\right)$

Partially differentiate with respect to 'x'

$$\frac{\partial z}{\partial x} = (1+0) + \left[(x+y)\phi^1\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + (1+0)\phi\left(\frac{y}{x}\right)\right]$$

$$= 1 + (x+y)\phi^1\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + \phi\left(\frac{y}{x}\right)$$

$$= 1 + \left(\frac{-xy}{x^2} - \frac{y^2}{x^2}\right)\phi^1\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial x} = 1 + \left(\frac{-y}{x} - \frac{y^2}{x^2}\right)\phi^1\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right)$$

Again, partially differentiate with respect to 'x'

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left[1 + \left(\frac{-y}{x} - \frac{y^2}{x^2}\right)\phi^1\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right)\right]$$

$$= \left[-y\left(\frac{-1}{x^2}\right) - y^2\left(\frac{-2}{x^3}\right)\right]\phi'\left(\frac{y}{x}\right) + \left(\frac{-y}{x} - \frac{y^2}{x^2}\right)\phi''\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + \phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= \left[\frac{y}{x^2} + \frac{2y^2}{x^3}\right]\phi'\left(\frac{y}{x}\right) + \left(\frac{y^2}{x^3} + \frac{y^3}{x^4}\right)\phi''\left(\frac{y}{x}\right) + \phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= \left[\frac{y}{x^2} + \frac{2y^2}{x^3} - \frac{y}{x^2}\right]\phi'\left(\frac{y}{x}\right) + \left[\frac{y^2}{x^3} + \frac{y^3}{x^4}\right]\phi''\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \left[\frac{2y^2}{x^3}\right]\phi'\left(\frac{y}{x}\right) + \left[\frac{y^2}{x^3} + \frac{y^3}{x^4}\right]\phi''\left(\frac{y}{x}\right) \quad .....(1)$$

Now partially differentiate with respect to 'y'

$$\frac{\partial z}{\partial y} = (0+1) + \left[(x+y)\phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + (0+1)\phi\left(\frac{y}{x}\right)\right]$$

$$= 1 + (x+y)\phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + \phi\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = 1 + \left(1 + \frac{y}{x}\right)\phi'\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right)$$

Again, partially differentiate with respect to 'y'

$$\frac{\partial^2 z}{\partial y^2} = 0 + \left(\frac{1}{x}\right)\phi'\left(\frac{y}{x}\right) + \left(1 + \frac{y}{x}\right)\phi''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + \phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\phi'\left(\frac{y}{x}\right) + \left(\frac{1}{x} + \frac{y}{x^2}\right)\phi''\left(\frac{y}{x}\right) + \frac{1}{x}\phi'\left(\frac{y}{x}\right)$$

$$= \left(\frac{1}{x} + \frac{1}{x}\right)\phi'\left(\frac{y}{x}\right) + \left(\frac{1}{x} + \frac{y}{x^2}\right)\phi''\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{2}{x}\right)\phi'\left(\frac{y}{x}\right) + \left(\frac{1}{x} + \frac{y}{x^2}\right)\phi''\left(\frac{y}{x}\right) \quad .....(2)$$

Now, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$

$$= \frac{\partial}{\partial y}\left(1 + \left(\frac{-y}{x} - \frac{y^2}{x^2}\right)\phi'\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right)\right)$$

$$= \left(\frac{-y}{x} - \frac{y^2}{x^2}\right)\phi''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + \left(\frac{-1}{x} - \frac{2y}{x^2}\right)\phi'\left(\frac{y}{x}\right) + \phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= \left(\frac{-y}{x^2}-\frac{y^2}{x^3}\right)\phi''\left(\frac{y}{x}\right)+\left(\frac{-1}{x}-\frac{2y}{x^2}+\frac{1}{x}\right)\phi'\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 z}{\partial y\partial x} = \left(\frac{-y}{x^2}-\frac{y^2}{x^3}\right)\phi''\left(\frac{y}{x}\right)-\frac{2y}{x^2}\phi'\left(\frac{y}{x}\right) \qquad .....(3)$$

Now $\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(1+\left(1+\frac{y}{x}\right)\phi'\left(\frac{y}{x}\right)+\phi\left(\frac{y}{x}\right)\right)$$

$$= \left(1+\frac{y}{x}\right)\phi''\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)+\left(\frac{-y}{x^2}\right)\phi'\left(\frac{y}{x}\right)+\phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= \left(\frac{-y}{x^2}-\frac{y^2}{x^3}\right)\phi''\left(\frac{y}{x}\right)+\left(\frac{-y}{x^2}-\frac{y}{x^2}\right)\phi'\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 z}{\partial x\partial y} = \left(\frac{-y}{x^2}-\frac{y^2}{x^3}\right)\phi''\left(\frac{y}{x}\right)+\left(\frac{-2y}{x^2}\right)\phi'\left(\frac{y}{x}\right) \qquad .....(4)$$

Now consider $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y\partial x}$

by equation (1) & (3) we have

$$= \frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y\partial x} = \frac{2y^2}{x^3}\phi'\left(\frac{y}{x}\right)+\left(\frac{y^2}{x^3}+\frac{y^3}{x^4}\right)\phi''\left(\frac{y}{x}\right)-\left[-\left(\frac{y}{x^2}+\frac{y^2}{x^3}\right)\phi''\left(\frac{y}{x}\right)-\left(\frac{2y}{x^2}\right)\phi'\left(\frac{y}{x}\right)\right]$$

$$= \frac{2y^2}{x^3}\phi'\left(\frac{y}{x}\right)+\left(\frac{y^2}{x^3}+\frac{y^3}{x^4}\right)\phi''\left(\frac{y}{x}\right)+\left(\frac{y}{x^2}+\frac{y^2}{x^3}\right)\phi''\left(\frac{y}{x}\right)+\left(\frac{2y}{x^2}\right)\phi'\left(\frac{y}{x}\right)$$

$$= \left(\frac{2y^2}{x^3}+\frac{2y}{x^2}\right)\phi'\left(\frac{y}{x}\right)+\left(\frac{y^2}{x^3}+\frac{y^3}{x^4}+\frac{y}{x^2}+\frac{y^2}{x^3}\right)\phi''\left(\frac{y}{x}\right)$$

$$= \left(\frac{2y^2}{x^3}+\frac{2y}{x^2}\right)\phi'\left(\frac{y}{x}\right)+\left(\frac{y}{x}+\frac{2y^2}{x^3}+\frac{y^3}{x^4}\right)\phi''\left(\frac{y}{x}\right)$$

$$\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y\partial x}\right) = x\left[\left(\frac{2y^2}{x^3}+\frac{2y}{x^2}\right)\phi'\left(\frac{y}{x}\right)+\left(\frac{y}{x^2}+\frac{2y^2}{x^3}+\frac{y^3}{x^4}\right)\phi''\left(\frac{y}{x}\right)\right]$$

$$= \left(\frac{2y^2}{x^2}+\frac{2y}{x}\right)\phi'\left(\frac{y}{x}\right)+\left(\frac{y}{x}+\frac{2y^2}{x^2}+\frac{y^3}{x^3}\right)\phi'\left(\frac{y}{x}\right) \qquad .....(5)$$

Now consider $\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}$

by equation (2) & (3) we have,

$$= \frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y} = \frac{2}{x}\phi'\left(\frac{y}{x}\right) + \left(\frac{1}{x} + \frac{y}{x^2}\right)\phi''\left(\frac{y}{x}\right) - \left[-\left(\frac{y}{x^2} + \frac{y^2}{x^3}\right)\phi''\left(\frac{y}{x}\right) - \frac{2y}{x^2}\phi'\left(\frac{y}{x}\right)\right]$$

$$= \frac{2}{x}\phi'\left(\frac{y}{x}\right) + \left(\frac{1}{x} + \frac{y}{x^2}\right)\phi''\left(\frac{y}{x}\right) + \left(\frac{y}{x^2} + \frac{y^2}{x^3}\right)\phi''\left(\frac{y}{x}\right) + \frac{2y}{x^2}\phi'\left(\frac{y}{x}\right)$$

$$= \left(\frac{2}{x} + \frac{2y}{x^2}\right)\phi'\left(\frac{y}{x}\right) + \left(\frac{1}{x} + \frac{2y}{x^2} + \frac{y^2}{x^3}\right)\phi''\left(\frac{y}{x}\right)$$

$$y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right) = y\left[\left(\frac{2}{x} + \frac{2y}{x^2}\right)\phi'\left(\frac{y}{x}\right) + \left(\frac{1}{x} + \frac{2y}{x^2} + \frac{y^2}{x^3}\right)\phi''\left(\frac{y}{x}\right)\right]$$

$$= \left(\frac{2y}{x} + \frac{2y^2}{x^2}\right)\phi'\left(\frac{y}{x}\right) + \left(\frac{y}{x} + \frac{2y^2}{x^2} + \frac{y^3}{x^3}\right)\phi''\left(\frac{y}{x}\right) \quad .....(6)$$

from (5) & (6)

$$\therefore \quad x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)$$

**43. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = x^3 + y^3 + z^3 - 3xyz$ prove that**

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0$$



*Sol :*

Given that $u = x + y + z$

Partial differentiate w.r.to x, y, and z

$$u_x = \frac{\partial u}{\partial x} = 1 + 0 + 0 = 1$$

$$u_y = \frac{\partial u}{\partial y} = 0 + 1 + 0 = 1$$

$$u_z = \frac{\partial u}{\partial z} = 0 + 0 + 1 = 1$$

$v = x^2 + y^2 + z^2$

Partial differentiate w.r.to x, y & z

$v_x = \frac{\partial v}{\partial x} = 2x + 0 + 0 = 2x$

$v_y = \frac{\partial v}{\partial y} = 0 + 2y + 0 = 2y$

$v_z = \frac{\partial v}{\partial z} = 0 + 0 + 2z = 2z$

$w = x^3 + y^3 + z^3 - 3xyz$

Partial differentiate with respect to x,y & z

$w_x = \frac{\partial w}{\partial x} = 3x^2 + 0 + 0 - 3yz$

$= 3x^2 - 3yz$

$w_y = \frac{\partial w}{\partial y} = 0 + 3y^2 + 0 - 3xz$

$w_y = 3y^2 - 3xz$

$w_z = \frac{\partial w}{\partial z} = 0 + 0 + 3z^2 - 3xy$

$= 3z^2 - 3xy$

Consider

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$= 1\left[2y\left(3z^2 - 3xy\right) - 2z\left(3y^2 - 3xz\right)\right] - 1\left[2x\left(3z^2 - 3xy\right) - 2z\left(3x^2 - 3yz\right)\right] + 1\left[2x\left(3y^2 - 3xz\right) - 2y\left(3x^2 - 3yz\right)\right]$$

$$= 6yz^2 - 6xy^2 - 6zy^2 + 6xz^2 - 6xz^2 + 6x^2y + 6zx^2 - 6yz^2 + 6xy^2 - 6x^2z - 6x^2y + 6y^2z$$

$= 0$

$$\therefore \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0$$

---

**44. If $z = x^m f(y/x) + x^n g(x/y)$ Prove that $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz = (m + n - 1)\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]$**

*Sol :*

Let $u = x^m f(y/x)$ and $v = x^n g(x/y)$

$\therefore z = u + v$

Now u = $x^m f(y/x)$ is a homogenous function in x and y degree of 'm'

$\therefore$ By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mu$$

Partially differentiate with respect to x & y

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = m\frac{\partial u}{\partial x} \qquad ...... (1)$$

$$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = m\frac{\partial u}{\partial y} \qquad ...... (2)$$

Multiplying (1) & (2) by x & y respectively and adding

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} + xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = m\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = m\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] - \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= (m-1)x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= (m-1)\, mu$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = m(m-1)u \qquad ..........(3)$$

Similarly we have v = $x^n$ y (x/y) is a homogenous function of x & y of degree n

$$\therefore x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)\, v \qquad .......... (4)$$

Adding (3) & (4) we have

$$x^2\frac{\partial^2}{\partial x^2}(u+v) + 2xy\frac{\partial^2}{\partial x \partial y}(u+v) + y^2\frac{\partial^2}{\partial y^2}(u+v) = m\,(m-1)u + n\,(n-1)v$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = m(m-1)\, u + n\,(n-1)\, v \qquad ............ (5)$$

Again from Euler's theorem we have for u & v which are homogeneous function in x & y of degree m & n respectively.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mu \qquad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$$

z = u + v

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = mu + nv$$

We have $m(m-1)u + n(n-1)v = m^2u - mu + n^2v - nv$

$= (m^2u + n^2v) - (mu + nv)$

Adding and subtracting mnu, mnv

$= (m^2u + n^2v) + mnu + mnv - mnu - mnv - (mu + nv)$

$= m(m + n)u + n(m + n)v - mn(u + v) - (mu + nv)$

$= (m + n)[mu + nv] - mn[u + v] - (mu + nv)$

$= [(m+n)-1)](mu + nv) - mnz$

$\therefore \quad m(m-1)u + n(n-1)v = [(m + n) - 1](mu + nv) - mnz$

$\therefore$ Sub the above value in (5)

$$x^2\frac{\partial^2 z}{\partial y^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (m + n - 1)(mu + nv) - mnz$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mnz = (m + n - 1)(mu + nv)$$

$$\therefore \quad x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mnz = (m + n - 1)\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]$$

**45. Verify Euler's Theorem for $Z = ax^2 + 2hxy + by^2$**

*Sol. :*

Given $Z = ax^2 + 2hxy + by^2$

By Euler's Theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n$$

To verify that z is homogenous or not, Consider

$$z = ax^2 + 2hxy + by^2$$

$$= x^2\left[a + 2h\frac{y}{x} + b\frac{y^2}{x^2}\right]$$

$$= x^2\left[b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a\right]$$

$$z = x^2 f\left(\frac{y}{x}\right)$$

z is a homogenous function of degree '2'.

Consider $z = ax^2 + 2hxy + by^2$

Partially differentiate with respect to 'x'

$$\therefore \quad \frac{\partial z}{\partial x} = 2ax + 2hy$$

Consider

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x[2ax + 2hy] + y[2hx + 2by]$$

$$= 2ax^2 + 2hxy + 2hxy + 2by^2$$

$$= 2ax^2 + 4hxy + 2by^2$$

$$= 2\left[ax^2 + 2hxy + by^2\right]$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

Hence, verified by Euler's Theorem.

**46. If $U = \cot^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{4}\sin 2u = 0$.**

*Sol. :*

If $\quad z = \cot u$

$$= \frac{x+y}{\sqrt{x}+\sqrt{y}} \qquad .....(1)$$

$$= \frac{x\left[1+\frac{y}{x}\right]}{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]} = x^{\frac{1}{2}}\left[\frac{1+\frac{y}{x}}{1+\sqrt{\frac{y}{x}}}\right]$$

$$z = x^{\frac{1}{2}} f\left(\frac{y}{x}\right)$$

$z$ is a homogenous function of $x$ and $y$ degree $\frac{1}{2}$

Therefore by Euler's Theorem.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}z$$

From (1)

$$\frac{\partial z}{\partial x} = -\operatorname{cosec}^2 u \frac{\partial u}{\partial x} \qquad ..(2) \qquad\qquad \frac{d}{du}\cot' x' = -\operatorname{cosec}^2 x$$

$$\frac{\partial z}{\partial y} = -\operatorname{cosec}^2 u \frac{\partial u}{\partial y} \qquad ...(3)$$

Substitute (2) (3) in (1)

$$\Rightarrow \quad -\operatorname{cosec}^2 u \frac{\partial u}{\partial x} - y \operatorname{cosec}^2 x \frac{\partial u}{\partial y} = \frac{1}{2}\cot u$$

$$\Rightarrow \quad -\operatorname{cosec}^2 u \left( x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) = \frac{1}{2}\cot u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{2}\frac{\cot u}{\operatorname{cosec}^2 u}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{4}\sin 2u = 0$$

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**47. If $U = \tan^{-1}\dfrac{x^3 + y^3}{x - y}$, $x \neq y$ Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$**

*Sol. :*

U is not a homogeneous function

$$z = \tan U = \frac{x^3 + y^3}{x - y}$$

$$= \frac{x^3\left[1 + \left(\frac{y}{x}\right)^3\right]}{x\left[1 - \left(\frac{y}{x}\right)\right]}$$

$$= x^2 \frac{\left[1 + \left(\frac{y}{x}\right)^3\right]}{1 - \left(\frac{y}{x}\right)}$$

So that z is a homogenous function of x, y of degree z.

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \qquad ...(1)$$

$$\text{But} \quad \frac{\partial z}{\partial x} = \sec^2 u \frac{\partial u}{\partial x} \qquad ...(2)$$

$$\frac{\partial z}{\partial y} = \sec^2 u \frac{\partial u}{\partial y} \qquad ...(3)$$

Sub (2) (3) in (1)

$$x\left(\sec^2 u\frac{\partial u}{\partial x}\right)+y\left(\sec^2 u\frac{\partial u}{\partial y}\right)=2z$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=2\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2\ \frac{\sin u}{\cos u}\ \frac{1}{\sec^2 u}$$

$$= 2\frac{\sin u}{\cos u}\cos^2 u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2\sin u\cos u$$

$$\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u$$

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**48. If Z = (x+y) $\phi$(y/x) where $\phi$ is any arbitrary function prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$.**

*Sol. :*

Z = (x+y) $\phi$(y/x)

Partially differentiate with respect to x

$$\frac{\partial z}{\partial x} = 1\phi\left(\frac{y}{x}\right)+(x+y)\phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$x\frac{\partial z}{\partial x} = x\left[\phi\left(\frac{y}{x}\right)-\frac{y}{x^2}(x+y)\phi'\left(\frac{y}{x}\right)\right]$$

$$x\frac{\partial z}{\partial x} = x\phi\left(\frac{y}{x}\right)-\frac{y}{x}(x+y)\ \phi'\left(\frac{y}{x}\right) \qquad ....(1)$$

Partially differentiate with respect to y

$$\frac{\partial z}{\partial y} = 1\phi\left(\frac{y}{x}\right)+(x+y)\phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$y\frac{\partial z}{\partial y} = y\left[\phi\left(\frac{y}{x}\right)+\frac{1}{x}(x+y)\phi'\left(\frac{y}{x}\right)\right]$$

$$y\frac{\partial z}{\partial y} = y\phi\left(\frac{y}{x}\right)+\frac{y}{x}(x+y)\phi'\left(\frac{y}{x}\right) \qquad ...(2)$$

Adding (1) (2)

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = x\phi\left(\frac{y}{x}\right)-\frac{y}{x}(x+y)\phi'\left(\frac{y}{x}\right)+y\phi\left(\frac{y}{x}\right)+\frac{y}{x}(x+y)\phi'\left(\frac{y}{x}\right)$$

$$= x\phi\left(\frac{y}{x}\right)+y\phi\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x}+y'\frac{\partial z}{\partial y} = (x+y)\phi\left(\frac{y}{x}\right)$$

$$\therefore \quad x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = z$$

**49. If $U = \log\left\{\frac{x^4+y^4}{x+y}\right\}$ show that Euler's Theorem that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$**

*Sol. :*

$$U = \log\left\{\frac{x^4+y^4}{x+y}\right\}$$

$$e^u = \frac{x^4+y^4}{x+y}=z$$

$$= \frac{x^4\left[1+\left(\frac{y}{x}\right)^4\right]}{x\left[1+\left(\frac{y}{x}\right)\right]}$$

$$= x^3\left[\frac{1+\left(\frac{y}{x}\right)^4}{1+\left(\frac{y}{x}\right)}\right]$$

$$e^u = z = x^3 f\left(\frac{y}{x}\right)$$

Z is a homogeneous function of degree 3.

Hence Euler's Theorem.

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=3z$$

$$x\frac{\partial}{\partial x}\left(e^{u}\right)+y\frac{\partial}{\partial y}(e^{u})=3e^{u}$$

$$e^{u}\cdot x\frac{\partial u}{\partial x}+e^{u}\cdot y\frac{\partial u}{\partial y}=3e^{u}$$

$$e^{x}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=e^{u}3$$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$$

**50. If $U = Ze^{ax+by}$, where $z$ is a homogenous function in $x$ and $y$ of degree n, prove that**

$$\mathbf{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(ax+by+n)\,u.}$$

*Sol. :*

Given that $z$ is a homogenous function of degree $n$ in $x$ and $y$. Then from Euler's theorem.

Then we get $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$ ...(1)

Now partially differentiate with respect to $u$.

$$\frac{\partial u}{\partial x} = \frac{\partial z}{\partial x}e^{ax+by}+z\frac{\partial}{\partial x}\ e^{ax+by}$$

$$= \frac{\partial z}{\partial x}e^{ax+by}+zae^{ax+by}$$

Partially differentiate with respect to $y$

$$\frac{\partial u}{\partial y} = \frac{\partial z}{\partial y}\ e^{ax+by}+z\frac{\partial}{\partial y}\ e^{ax+by}$$

$$= \frac{\partial z}{\partial y}e^{ax+by} + z\,b\,e^{ax+by}$$

Consider

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[\frac{\partial u}{\partial y}e^{ax+by}+zae^{ax+by}\right] + y\left[\frac{\partial z}{\partial y}e^{ax+by}+zbe^{ax+by}\right]$$

$$= x\frac{\partial z}{\partial x}e^{ax+by}+zxae^{ax+by}+y\frac{\partial z}{\partial y}e^{ax+by}+zybe^{ax+by}$$

$$= \left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) e^{ax+by} + z\left[x\,a\,e^{ax+by} + y\,b e^{ax+by}\right]$$

$$= n\,z\,e^{ax+by} + z\left[x\,a\,e^{ax+by} + y\,b\,e^{ax+by}\right] \text{ by (1)}$$

$$= (n + xa + yb)\,ze^{ax+by}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = [ax + by + n]u.$$

---

**51. Verify Euler's Theorem for $z = (x^2 + xy + y^2)^{-1}$**

*Sol. :*

Given that $Z = (x^2 + xy + y^2)^{-1}$

By Euler's theorem

We have $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$ ...(1)

To verify that z is homogenous or not.

Consider,

$$z = (x^2 + xy + y^2)^{-1}$$

$$= x^2\left[1 + \frac{y}{x} + \frac{y^2}{x^2}\right]^{-1}$$

$$= x^2\left[\frac{x^2 + xy + y^2}{x^2}\right]^{-1}$$

$$= \frac{1}{x^2 + xy + y^2}$$

$$= \frac{1}{x^2\left[1 + \left(\frac{y}{x}\right) + \frac{y^2}{x^2}\right]}$$

$$= x^{-2}\left[\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) + 1\right]^{-1}$$

$$z = x^{-2} f\left(\frac{y}{x}\right)$$

z is a homogenous function of degree '–2'

$$z = (x^2 + xy + y^2)^{-1}$$

Partially differentiate with respect to 'x'

$$\frac{\partial z}{\partial x} = -1\ (x^2 + xy + y^2)^{-2}\ (2x + y)$$

Partially differentiate with respect to 'y'

$$\frac{\partial z}{\partial y} = -1\,(x^2 + xy + y^2)^{-2}\ (2x + y)$$

Consider

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left[-(x^2 + xy + y^2)^{-2}(2x + y)\right] + y\left[-(x^2 + xy + y^2)^{-2}(x + 2y)\right]$$

$$= (x^2 + xy + y^2)^{-2}(2x^2 + xy) - (x^2 + xy + y^2)^{-2}(xy + 2y^2)$$

$$= -(x^2 + xy + y^2)^{-2}\left[2x^2 + xy + xy + 2y^2\right]$$

$$= -(x^2 + xy + y^2)^{-2}\ (2x^2 + 2xy + 2y^2)$$

$$= -2\ (x^2 + xy + y^2)^{-2}\ (x^2 + xy + y^2)$$

$$= -2\ (x^2 + xy + y^2)^{-2+1}$$

$$= -2\ (x^2 + xy + y^2)^{-1}$$

$$= -2z$$

$$\therefore\quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -2z$$

**52. If $u = \sin^{-1}\frac{x^2 + y^2}{x + y}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$**

*Sol. :*

$$\text{Given}\quad u = \text{Sin}^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$$

$$\sin u = \frac{x^2 + y^2}{x + y}$$

$$z = \sin u = \frac{x^2\left(1 + \left(\frac{y}{x}\right)^2\right)}{x\left[1 + \left(\frac{y}{x}\right)\right]}$$

$$= x\left[\frac{1+\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)}\right]$$

$$z = x^1 f\left(\frac{y}{x}\right)$$

$z$ = sin u is a homogeneous function $n = 1$

By Euler's Theorem we have $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

Consider $z$ = sin u

Partially differentiate with respect to 'x'

$$\frac{\partial z}{\partial x} = \cos u \frac{\partial u}{\partial x}.$$

Partially differentiate with respect to 'y'

$$\frac{\partial z}{\partial y} = \cos u \frac{\partial u}{\partial y}$$

Consider

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$\Rightarrow \quad x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$\cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u}$$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

**53. If $u = \sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$**

*Sol. :*

Given that $u = \sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

$$\sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

$$z = \sin u = \frac{\sqrt{x}-\left(1-\frac{\sqrt{y}}{\sqrt{x}}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}$$

$$= \frac{1-\sqrt{\frac{y}{x}}}{1+\sqrt{\frac{y}{x}}}$$

$$z = \sin u = x^0 f\left(\frac{y}{x}\right)$$

z = sin u is homogeneous function n = 0.

By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n \cdot z$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0 \cdot z \quad ...(1)$$

Consider z = sin u

Partially differentiate with respect to 'x'

$$\frac{\partial z}{\partial x} = \cos u \frac{\partial u}{\partial x} \quad ...(2)$$

Partially differentiate with respect to 'y'

$$\frac{\partial z}{\partial y} = \cos u \frac{\partial u}{\partial y} \quad ...(3)$$

Substitute (2) & (3) in (1)

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = 0. \sin u$$

$$\cos u \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 0$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

**54. If $u = \cos^{-1} \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}$ cost u = 0**

*Sol. :*

$$u = \cos^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$z = \cos u = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{x}\right)}$$

$$= x^{1/2}\left[\frac{y+\frac{y}{x}}{1+\frac{\sqrt{y}}{x}}\right]$$

$$z = x^{1/2} f\left(\frac{y}{x}\right)$$

$z = \cos u$ is homogenous function of degree $\dfrac{1}{2}$



By Euler's Theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z \quad ...(1)$$

Consider $z = \cos u$

Partially differentiate with respect to $x$ and $y$

$$\frac{\partial z}{\partial x} = -\sin u \frac{\partial u}{\partial x} \quad ...(2)$$

$$\frac{\partial z}{\partial y} = -\sin u \frac{\partial u}{\partial y} \quad ...(3)$$

Substitute (2) (3) in (1)

$$x\left[-\sin u\frac{\partial u}{\partial x}\right]+y\left[-\sin u\frac{\partial u}{\partial y}\right] = \frac{1}{2}\cos u$$

$$-\sin u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right] = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\cos u}{\sin u}$$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

---

**55. If $U = \tan^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$.**

*Sol. :*

Given $U = \tan^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$$\tan u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$z = \tan u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{x}\right)}$$

$$= x^{1/2}\frac{x\left(1+\frac{y}{x}\right)}{\left(1+\frac{\sqrt{y}}{x}\right)}$$

$$z = x^{1/2} f\left(\frac{y}{x}\right)$$

$z = \tan u$ is homogenous function of degree $\frac{1}{2}$

By Euler's Theorem

$$x\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=\frac{1}{2}z \qquad ...(1)$$

Consider $z = \tan u$

Partially differentiate with respect to 'x'

$$\frac{\partial z}{\partial x} = \sec^2 u\,\frac{\partial u}{\partial x} \qquad ...(2)$$

Partially differentiate with respect to 'y'

$$\frac{\partial z}{\partial y} = \sec^2 u\,\frac{\partial u}{\partial y} \qquad ..(3)$$

Substitute (2) & (3) in (1)

$$x\left[\sec^2 u\frac{\partial u}{\partial x}\right]+y\left[\sec^2 u\frac{\partial u}{\partial y}\right]=\frac{1}{2}\tan u$$

$$\sec^2 u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=\frac{1}{2}\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\tan u}{\sec^2 u}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\frac{\sin u}{\cos u}\cos^2 u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin u\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{2}\left[\frac{\sin 2u}{2}\right]$$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u.$$

---

**56. If $Z = \sec^{-1}\frac{x^3+y^3}{x+y}$, show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2\cot z$**

*Sol. :*

$$\text{Given} \quad z = \sec^{-1}\frac{x^3+y^3}{x+y}$$

$$\sec z = \frac{x^3+y^3}{x+y}$$

$$u = \sec z = \frac{x^3+y^3}{x+y}$$

$$= \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1+\frac{y}{x}\right]}$$

$$= x^2\left[\frac{1+\left(\frac{y}{x}\right)^3}{1+\left(\frac{y}{x}\right)}\right]$$

$$u = \sec z = x^2 f\left(\frac{y}{x}\right)$$

$u = \sec z$ is homogeneous function of degree 2

By Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \quad ...(1)$$

Consider $u = \sec z$

Partially differentiate with respect to 'x'

$$\frac{\partial u}{\partial x} = \sec z \tan z \frac{\partial z}{\partial x} \quad ...(2)$$

Partially differentiate with respect to 'y'

$$\frac{\partial u}{\partial y} = \sec z \tan z \frac{\partial z}{\partial y} \quad ...(3)$$

Substitute (2) & (3) in (1)

$$x\left[\sec z \tan z \frac{\partial z}{\partial x}\right] + y\left[\sec z \tan \frac{\partial z}{\partial y}\right] = 2 \sec z$$

$$\sec z \tan z\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right] = 2 \sec z$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2\sec z}{\sec z \tan z}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\frac{1}{\tan z}$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\cot z$$

**57. If $u = \log \dfrac{x^2+y^2}{x+y}$ prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$**

*Sol. :*

Given that $u = \log \dfrac{x^2+y^2}{x+y}$

$$e^u = \frac{x^2+y^2}{x+y}$$

$$z = e^u = \frac{x^2+y^2}{x+y}$$

$$= \frac{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}{x\left[1+\left(\frac{y}{x}\right)\right]} = x\left[\frac{1+\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)}\right]$$

$$z = e^u = x^1 f\left(\frac{y}{x}\right)$$

$e^u$ is a homogeneous function of degree n = 1

By Fuler's Theorem

We have $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1.z$

Consider $z = e^u$ ...(1)

Partially differentiate with respect to 'x'

$$\frac{\partial z}{\partial x} = e^u \frac{\partial u}{\partial x} \quad ...(2)$$

Partially differentiate with respect to 'y'

$$\frac{\partial z}{\partial y} = e^u \frac{\partial u}{\partial y} \quad ...(3)$$

Substitute (2) & (3) in (1)

$$x\left[e^u \frac{\partial u}{\partial x}\right] + y\left[e^u \frac{\partial u}{\partial y}\right] = 1.e^u$$

$$e^u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = e^u$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.$$

**58.** Verify Euler's Theorem for $z = \text{Sin}^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$

*Sol. :*

$z = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$

Consider

$u = \sin^{-1}\dfrac{x}{y}$

$\sin u = \dfrac{x}{y} \Rightarrow \dfrac{1}{\sin u} = \dfrac{y}{x}$

$\text{cosec}\, u = \dfrac{y}{x}$

$u = \text{cosec}^{-1}\dfrac{y}{x}$

Then, $z = \text{cosec}^{-1}\dfrac{y}{x} + \tan^{-1}\dfrac{y}{x}$

$z$ is a homogeneous function of degree $x = 0$

By Eulers Theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (0)z$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

Consider

$$z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

Partially differentiate with respect to 'x'

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{1}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)$$

$$= \frac{y}{\sqrt{y^2 - x^2}}\frac{1}{y} - \frac{yx^2}{x^2+y^2}\cdot\frac{1}{x^2}$$

$$= \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

Partially differentiate with respect to y

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}}\left(\frac{-x}{y^2}\right) + \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$$

$$= \frac{-yx}{\sqrt{y^2 - x^2}}\ \frac{1}{y^2} + \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x}$$

$$= \frac{-x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

By Euler's Theorem

$$x\left[\frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}\right] + y\left[\frac{-x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}\right] = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} - \frac{xy}{y\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2}$$

$$= 0$$

**59. If $u = \frac{x^2 y^2}{x^2 + y^2}$ then show that**

**i)** $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x}$

**ii)** $x\frac{\partial^2 u}{\partial y\,\partial x} + y \cdot \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y}$

**iii)** $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\,\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$

*Sol. :*

$$u = \frac{x^2 y^2}{x^2 + y^2}$$

$$u = \frac{x^2 y^2}{x^2\left[1 + \frac{y^2}{x^2}\right]}$$

$$u = \frac{y^2}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{x^2\left[\frac{y}{x}\right]^2}{\left[1+\left(\frac{y}{x}\right)^2\right]}$$

$$u = x^2 f\left(\frac{y}{x}\right)$$

u is homogeneous function of degree '2'

By Euler's Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \qquad ...(1)$$

Differentiate partially with respect to 'x'

$$\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(2u)$$

$$1.\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = 2\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = 2\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = 2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial u}{\partial x} \qquad ...(2)$$

Partially differentiate with respect to y

$$\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(2u)$$

$$x\cdot\frac{\partial^2 u}{\partial x\,\partial y} + y\cdot\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y}\frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \qquad ...(3)$$

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y}$$

Substitute (2), (3) in (1)

$$x\left[x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y}\right] + y\left[x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2}\right] = 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\,\partial y} + xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$$

Hence (i) (ii) and (iii) are proved.

# *Choose the Correct Answers*

1. If z = xyf (x/y) then $x\frac{\partial z}{\partial x} + y\frac{\partial u}{\partial y} =$ [ c ]

   (a) 0 (b) $\frac{1}{z}$

   (c) 2z (d) z

2. If $\sin^{-1}\left(\frac{x-y}{\sqrt{x}+\sqrt{y}}\right)$ then the degree of homogenous function is [ d ]

   (a) 0 (b) $\frac{-1}{2}$

   (c) 2 (d) $\frac{1}{2}$

3. If z = f(y/x) then $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right)$ is [ d ]

   (a) 1 (b) 2

   (c) -2 (d) 0

4. If $f = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is [ d ]

   (a) f (b) 2f

   (c) sinf (d) tanx

5. If x = r cos $\phi$, y = r sin then $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2$ is [ a ]

   (a) 1 (b) r

   (c) –r (d) –1

6. If f(x,y) is homogenous function of x and y of degree n. then [ a ]

   (a) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$ (b) $y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = nf$

   (c) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n$ (d) None

7. If z = log($x^2$ + $y^2$) then $x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ is [a]

(a) 2 (b) 1

(c) 3 (d) 4

8. If $u = y^x$ then $\frac{\partial u}{\partial x}$ is [ b ]

(a) $xy^x$ (b) $y^x \log y$

(c) $y^x \log x - 1$ (d) $yx^{y-1}$

9. If u is a homogenous function of x and y of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is [ c ]

(a) n (b) n – 1

(c) nu (d) n (n – 1) u

10. If u be a homogenous function of degree n then $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} =$ [ b ]

(a) $n\frac{\partial u}{\partial x}$ (b) $(n-1)\frac{\partial u}{\partial x}$

(c) $(n+1)\frac{\partial u}{\partial x}$ (d) None

# *Fill in the Blanks*

1. $\frac{dy}{dx}$ = ________

2. $u = e^{x-y}$ then $\frac{\partial^2 u}{\partial x^2}$ = ________

3. First order partial derivative of $\tan^{-1}(x + y)$ = ________

4. If $z = \tan^{-1}(y/x)$ then $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$ = ________

5. $x^2 \frac{\partial z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$ = ________

6. If $u = \sin^{-1}x$ then $\frac{\partial u}{\partial x}$ = ________

7. The domain of the function $f(x,y) = \log(x+y)$ is ________

8. Second under derivation of $e^{x+y}$ = ________

9. If $z = \frac{x^3 + y^3}{x + y}$ then the degree of the function is ________

10. $z = \cos xy$ then $\frac{\partial z}{\partial x}$ = ________



## Answers

1. $\frac{-f_x}{f_y}$
2. $e^{x-y}$
3. $\frac{1}{1+(x+y)^2}$
4. 0
5. $n(n-1)z$
6. $\frac{-1}{\sqrt{1+x^2}}$
7. $\{(x,y) : 0 < x\ x + y\}$
8. $e^{x+y}$
9. 2
10. $-y \sin x$

| UNIT II | Theorem on Total Differentials - Composite Functions - Differentiation of Composite Functions - Implicit Functions - Equality of fxy(a, b) and fyz(a, b) - Taylor's theorem for a function of two Variables - Maxima and Minima of functions of two variables – Lagrange's Method of undetermined multipliers. |
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## 2.1 Theorem on Total Differentials - Composite Functions - Differentiation of Composite Functions

**1. State and prove composite functions.**

**Statement :**

**Let $z = (x, y)$ ... (1)**

**$x = \phi(t)$ ... (2)**

**$y = \phi(t)$ ... (3)**

**So that x, y are themselves functions of third variable 't' equation 1, 2 & 3 are said to define z as a composite function of 't'.**

**Differentiation of composite functions**

**Let $z = f(x, y)$**

**Possess continuous partial derivatives and let $x = \phi(t)$ $y = \varphi(t)$**

**Possess condition derivatives. Then**

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

*Sol :*

Let $t, t + \Delta t$ be any two values

Let $\Delta x, \Delta y, \Delta z$ be the changes in x, y, z consequent to the change $\Delta t$ in t,

We have

$$x + \Delta x = \phi(t + t);\ y + \Delta y = \varphi(t + \Delta t)$$

$$z + \Delta z = f(x + \Delta x, y + \Delta y)$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - z$$

$$\Delta z = f(x + \Delta x - y + \Delta Y) - f(x, y)$$

Adding & Subtracting $f(x, y + \Delta y)$

$$\Delta z = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x,y)]$$

$$\Delta z = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]$$

By applying legrange's mean values theorem to the two difference on the right and obtain.

$$\Delta z = \Delta x\, f_x (x + \theta_1 \Delta x, y + \Delta y) + \Delta y\, f_y (x, y + \theta_2 \Delta y)$$

Divide $\Delta t$ on both sides

$$\frac{\Delta z}{\Delta t} = \frac{\Delta x}{\Delta t} f_x (x + \theta_1 \Delta x, y + \Delta y) + \frac{\Delta y}{\Delta t} f_y(x, y + \theta_2 \Delta y) \quad \text{... (1)}$$

Let $\Delta t \to 0$ So that $\Delta x + \Delta y \to 0$

because of continuity of partial derivative,

We have

$$\lim_{(\Delta x, \Delta y) \to (0,0)} f_x(x + \theta_1 \Delta x, y + \Delta y) = f_x(x, y) = \frac{\partial z}{\partial x}$$

$$\lim_{\Delta y \to 0} f_y(x, y + \theta_2 \Delta y) = f_y (x, y) = \frac{\partial z}{\partial y}$$

by (1) become

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

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**2. Find dz / dt when**

**$z = xy^2 + x^2y$, $x = at^2$ & t $y = 2at$ verify by direct substitution**

*Sol :*

$x = at^2, \quad y = 2at$

Derivative with respect to x & y

$$\frac{dx}{dt} = 2at \,; \quad \frac{dy}{dt} = 2a$$

We know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$

Since $y = 2at$ & $x = at^2$

$$= [(2at)^2 + 2(2at)(at^2)](2at) + [2(at^2)(2at) + (at^2)^2]\, 2a$$

$$= (4a^2t^2 + 4a^2t^3)\, 2at + (4a^2t^3 + a^2t^4)\, 2a$$

$$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$$

$$= a^3[16t^3 + 10t^4]$$

again

$$z = xy^2 + x^2y$$

$$= (at^2)(2at)^2 + (at^2)(2at)$$

$$= (at^2)(4a^2t^2) + (a^2t^4)(2at)$$

$$= 4a^3t^4 + 2a^3t^5$$

$$\frac{dz}{dt} = 4.4a^3 t^3 + 5.2a^3 t^4$$

$$= 16a^3 t^3 + 10a^3 t^4$$

$$= a^3 [16t^3 + 10 z^4]$$

hence the verification

**3. Z is a function of x & y. Prove that if,**

$$\mathbf{x = e^u + e^{-v}, \ y = e^{-u} - e^v}$$

**Then** $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

*Sol :*

$$x = e^u + e^{-v} ; \quad y = e^{-u} - e^v$$

$$\frac{\partial x}{\partial u} = e^u \quad \frac{\partial y}{\partial u} = -e^{-u}$$

z as a composite function of u, v

We have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} e^u - \frac{\partial z}{\partial y} e^{-u} \qquad \ldots (1)$$

$$x = e^u + e^{-v}; \quad y = e^{-u} - e^v$$

$$\frac{\partial x}{\partial v} = -e^{-v} ; \quad \frac{\partial y}{\partial v} = -e^v$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$$

$$\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x} e^{-v} - \frac{\partial z}{\partial y} e^v \qquad \ldots (2)$$

Subtract (1) – (2)

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} e^u - \frac{\partial z}{\partial y} e^{-u} - \left[-\frac{\partial z}{\partial x} e^{-v} - \frac{\partial z}{\partial y} e^v\right]$$

$$= \frac{\partial z}{\partial x} e^u - \frac{\partial z}{\partial y} e^{-u} + \frac{\partial z}{\partial x} e^{-v} + \frac{\partial z}{\partial y} e^v$$

$$= \frac{\partial z}{\partial x}[e^u + e^{-v}] - \frac{\partial z}{\partial y}[-e^v + e^{-u}]$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$$\therefore \quad \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

**4. H is a homogenous function of x, y, z of degree n, prove that $x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} + z\frac{\partial H}{\partial z} = nz$**

*Sol :*

H is a homogenous function of x, y, z
then we have

$$H = x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$$

$$= x^n f(u, v)$$

where $u = \frac{y}{x}, \frac{z}{x}$

$$\frac{\partial H}{\partial x} = nx^{n-1} f(u, v) + x^n \left[\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}\right]$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2}; \frac{\partial v}{\partial x} = \frac{-z}{x^2}$$

Then

$$\frac{\partial H}{\partial x} = nx^{n-1} f(u, v) - x^{n-2}\left[y\frac{\partial f}{\partial u} + z\frac{\partial f}{\partial v}\right]$$

$$\frac{\partial H}{\partial y} = x^n\left[\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}\right]$$

$$\frac{\partial u}{\partial y} = \frac{l}{x}; \quad \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial H}{\partial y} = x^n\left[\frac{\partial f}{\partial u}\cdot\frac{1}{x} + \frac{\partial f}{\partial v}(0)\right] = x^{n-1}\frac{\partial f}{\partial u}$$

Similarly $\frac{\partial H}{\partial z} = x^{n-1}\frac{\partial f}{\partial v}$

Consider

$$\therefore \quad x\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + z\frac{\partial H}{\partial z} = x\left[nx^{n-1}f(u,v) - x^{n-2}y\frac{\partial f}{\partial u} + z\frac{\partial f}{\partial v}\right] + yx^{n-1}\frac{\partial f}{\partial u} + zx^{n-1}\frac{\partial f}{\partial v} = nz$$

**5. If H = f(y – z, z – x, x – y) Prove that $\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0$**

*Sol :*

Let $u = y - z,\ v = z - x,\ w = x - y$

$H = f(u, v, w)$

h is a composite function of x, y, z

we have

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$\frac{\partial u}{\partial x} = 0,\ \frac{\partial v}{\partial x} = -1\ ;\ \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}(0) + \frac{\partial H}{\partial v}(-1) + \frac{\partial H}{\partial w} \quad \text{... (1)}$$

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial y}$$

Partial different with respect to 'y'

$$\frac{\partial u}{\partial y} = 1,\ \frac{\partial v}{\partial y} = 0,\ \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}(1) + \frac{\partial H}{\partial v}(0) + \frac{\partial H}{\partial w}(-1)$$

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u} - \frac{\partial H}{\partial w} \quad \text{... (2)}$$

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial H}{\partial v}\cdot\frac{\partial u}{\partial v} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial z}$$

Partial different with respect to z

$$\frac{\partial u}{\partial z} = -1,\ \frac{\partial v}{\partial z} = 1,\ \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u}(-1) + \frac{\partial H}{\partial v}(1) + \frac{\partial H}{\partial w}(0)$$

$$\frac{\partial H}{\partial z} = \frac{-\partial H}{\partial u} + \frac{\partial H}{\partial v} \quad \text{... (3)}$$

(1) + (2) + (3)

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = -\frac{\partial H}{\partial v} + \frac{\partial H}{\partial w} + \frac{\partial H}{\partial u} - \frac{\partial H}{\partial w} - \frac{\partial H}{\partial u} + \frac{\partial H}{\partial v}$$

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0$$

## 2.2 Implicit Functions

**6. Define Implicit Function**

*Sol :*

Let f be a function of two variables since f(x, y) = 0 ... (1)

We can obtain y as function of x, the equation (1) defines y as an implicit function of x.

Assuming that the conditions under which the equation (1) defines y as a derivable function of x are satisfied.

We shall Now obtain the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terns of the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y^2}$ of 'f'

with respect to x & y

Then

$$\frac{dy}{dx} = \frac{-\partial f / \partial x}{\partial f / \partial y} = \frac{-fx}{fy} \text{ if } fy \neq 0$$

$$\frac{\partial^2 y}{dx^2} = \frac{f_{x^2}\left(f_y\right)^2 - 2f_{yx}f_x f_y + f_{y^2}\left(f_x\right)^2}{\left(f_y\right)^3}$$

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**7. Prove that if $y^3 - 3ax^2 + x^3 = 0$ then $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$**

*Sol :*

$$y^3 - 3ax^2 + x^3 = 0$$

$$y^3 = 3ax^2 - x^3$$

Differentiate with respect to 'x'

$$3y^2\frac{dy}{dx} = y^3 = 3ax^2 - x^3$$

Different with respect to 'x'

$$3y^2\frac{dy}{dx} = 6ax - 3x^2$$

$$\frac{dy}{dx} = \frac{6ax - 3x^2}{3y^2}$$

$$\frac{dy}{dx} = \frac{2ax - x^2}{y^2}$$

Again, different w.r. to x

$$\frac{d^2y}{dx^2} = \frac{y^2(2a-2x) - 2y(2ax-x^2)\frac{dy}{dx}}{(y^2)^2}$$

$$= \frac{(2a-2x)y^2 - 2y(2ax-x^2)\left[\frac{2ax-x^2}{y^2}\right]}{y^4}$$

$$= \frac{2(a-x)y^3 - 2(2ax-x^2)^2}{y^4 \cdot y}$$

$$= \frac{2(a-x)y^3 - 2(2ax-x^2)^2}{y^5}$$

$$= \frac{2(a-x)(3ax^2-x^3) - 2(2ax-x^2)^2}{y^5}$$

$$\frac{d^2y}{dx^2} = \frac{-2a^2x^2}{y^5}$$

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$$

**8. If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$ find $\frac{\partial u}{\partial r}$**

*Sol :*

Given

$u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$

Different with respect to 'r'

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

Consider $x = 2r - 3s - 4$ ; $y = -r + 8s - 5$

$$\frac{\partial x}{\partial r} = 2 ; \frac{\partial y}{\partial r} = -1$$

1. If $u = y^x$, then $\frac{\partial u}{\partial x}$ = ________

Consider $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial u}{\partial r} = 2x(2) + (-2y)(-1)$$

$$\frac{\partial u}{\partial r} = 4x + 2y$$

**9. If z = (cosy)/x and x = u² – v, y = eᵛ. Find $\frac{\partial z}{\partial v}$.**

*Sol :*

Given

$$z = \frac{\cos y}{x}, \quad x = u^2 - v, \quad y = e^v$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Consider

$$x = u^2 - v, \quad y = e^v$$

$$\frac{\partial x}{\partial v} = -1, \quad \frac{\partial y}{\partial v} = e^v$$

$$z = \frac{\cos y}{x}$$

$$\frac{\partial z}{\partial x} = \cos y\left(\frac{-1}{x^2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{-\sin y}{x}$$

$$\therefore \quad \frac{\partial z}{\partial v} = \cos y\left(\frac{-1}{x^2}\right)(-1) + \left(\frac{-\sin y}{x}\right)(e^v)$$

$$= \frac{\cos y}{x^2} - \frac{e^v \sin y}{x}$$

Since $y = e^v$

$$\frac{\partial z}{\partial v} = \frac{\cos y - xy \sin y}{x^2}$$

**10. Find $\frac{dy}{dx}$ for x sin(x – y) – (x + y) = 0**

*Sol :*

Given,

$$x \sin(x - y) - (x + y) = 0$$

with respect to $\frac{dy}{dx} = \frac{-fx}{fy}$

$$f(x, y) = x \sin(x - y)(x + y) = 0$$

$$x \sin(x - y) = x + y$$

$$\sin(x - y) = \frac{x + y}{x}$$

Partially different with respect to 'x' & 'y'

$$fy = \frac{\partial f}{\partial x}$$

$$= x\cos(x - y)(1) + 1.\sin(x - y) - 1$$

$$= x\cos(x - y) + \sin(x - y) - 1$$

$$fx = \frac{\partial f}{\partial y}$$

$$= x\cos(x - y)(-1) + 0.\sin(x - y) - (1)$$

$$= -x\cos(x - y) - 1$$

$$= -[x\cos(x - y) + 1]$$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$= \frac{-[x\cos(x-y) + \sin(s+y) - 1]}{-[x\cos(x-y) + 1]}$$

$$= \frac{x\cos(x-y) + \frac{x+y}{x} - 1}{x\cos(x-y) + 1}$$

$$= \frac{x^2\cos(x-y) + x + y - x}{x^2\cos(x-y) + x}$$

$$= \frac{x^2\cos(x-y) + y}{x^2\cos(x-y) + x}$$

**11. If u = $\frac{(x+y)}{1-xy}$; x = tan (2r – s²), y = cot (r² s), find $\frac{\partial u}{\partial s}$**

*Sol :*

Given,

$$u = \frac{x+y}{1-xy}$$

$$x = \tan(2r - s^2),\ y = \cot(r^2 s)$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

Consider

$$x = \tan(2r - s^2)$$

Partial different with respect to 's'

$$\frac{\partial x}{\partial s} = \sec^2(2r - s^2)(-2s)$$

$$= -2s\sec^2(2r - s^2)$$

Consider

$$y = \cot(r^2 s)$$

Partial differentiate with respect to 's'

$$\frac{\partial y}{\partial s} = -\operatorname{cosec}^2(r^2 s)(r^2)$$

$$= -r^2\operatorname{cosec}^2(r^2 s)$$

Consider

$$u = \frac{x+y}{1-xy} \Rightarrow (x+y).(1-xy)^{-1}$$

$$= (x+y)\left[\frac{-1}{(1-xy)^2}\right](-y) + (1-xy)(1)$$

$$\frac{\partial u}{\partial y} = (x+y)\left[\frac{-1}{(1-xy)^2}(-x)\right] + (1-xy)^{-1}(1)$$

Consider

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \left[x+y\left[\frac{-1}{(1-xy)^2}\right](-y)+\frac{1}{1-xy}\right]( -2s\ \sec^2 (2r - s^2) + x + y \left[\frac{-1}{(1-xy)^2}(-x)+\frac{1}{1-xy}\right] \left(-r^2 \operatorname{cosec}^2\right)\left(r^2 s\right)\Big]$$

$$= \left[\frac{1}{1-xy}+\frac{y(x+y)}{(1-xy)^2}\right][(-2\ \sec^2 (2r - s^2)] + \left[\frac{x(x+y)}{(1-xy)^2}+\frac{1}{1-xy}\right] [(-r^2\ \operatorname{cosec}^2 (r^2\ s)]$$

$$= \left[\frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}\right] [-2s\ \sec^2 92r - s^2) + \left[\frac{x^2+\cancel{xy}+1-\cancel{xy}}{(1-xy)^2}\right] [(-r^2\ \operatorname{cosec}^2 (r^2\ s)]$$

$$= \frac{\left(1+y^2\right)}{(1-xy)^2} - (2s\ (\sec^2 (2r - s^2)) + \frac{\left(1+x^2\right)}{(1-xy)^2} (-r^2\ \operatorname{cosec}^2 (r^2\ s))$$

$$\frac{\partial u}{\partial s} = \frac{1}{(1-xy)^2} [ (-2s)\ \sec^2 (2r - s^2)\ (1 + y^2) - r^2\ \operatorname{cosec}^2 (r^2\ s)\ (1 + x^2)].$$

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**12. If $x^5 + y^5 = 5a^3 xy$, Find $\frac{d^2y}{dx^2}$.**

*Sol :*

Given,

$x^5 + y^5 = 5a^3 xy$

We have

$$\frac{d^2y}{dx^2} = \frac{-\left[f_{x^2}\left(f_y\right)^2 - 2f_{yx}f_xf_y - f_{y^2}\left(f_x\right)^2\right]}{\left(f_y\right)^3}$$

Consider

$x^5 + y^5 = 5a^3 xy$

$f_x = 5x^4 - 5a^3 y$

$f_x^2 = 20x^3$

$f_y = 5y^4 - 5a^3 x$

$f_y^2 = 20y^3$

$$f_{yx} = \frac{\partial^2 f}{\partial y\ \partial x} = \frac{\partial}{\partial y}(5x^4 - 5a^3 y)$$

$= -5a^3$

$$\frac{\partial^2 y}{dx^2} = \frac{-\left[2x^3\left(5y^4-5a^3x\right)^2+20y^3\left(5x^4-5a^3y\right)^2-2\left(-5a^3\right)\left(5x^4-5a^3y\right)\left(5y^4-5a^3x\right)\right]}{\left(5y^4-5a^3x\right)^3}$$

$$= \frac{-\left[20x^3\left(25y^8+25a^6x^2-50xy^4a^3\right)\right]+10a^3\left[25x^4y^4-25a^3x^5-25a^3y^5+25a^6xy\right]+20y^3\left[25x^8+25a^6y^2-50a^3x^4y\right]}{125\left(y^4-a^3x\right)^3}$$

$$= \frac{-\left[125\left(4x^3y^8+4a^6x^5-8a^3x^4y^4+4x^8y^3\right)+4a^6y^5-8a^3x^4y^4-2a^6x^5+2a^3x^4y^4-2a^6x^5+2a^9xy\right]}{125\left(y^4-a^3x\right)^3}$$

$$= \frac{-\left[4\left(x^3y^8+x^8y^3\right)-14a^3x^4y^4+2a^6\left(2x^5+2y^5-x^5-y^5\right)+2a^9xy\right]}{\left(y^4-a^3x\right)^3}$$

$$= \frac{-\left[4x^3y^8+4x^8y^3-20a^3x^4y^4+6a^3x^4y^4+2a^6\left(x^5+y^5\right)+2a^9xy\right]}{\left(y^4-a^3x\right)^3}$$

$$= \frac{-\left[4x^3y^3\left[\left(y^5+x^5-5a^3xy\right)+6a^3x^4y^4+2a^6\left(x^5+y^5\right)+2a^9xy\right]}{\left(y^4-a^3x\right)^3}$$

$$= \frac{-\left[6a^3x^4y^4+2a^6\left(5a^3xy\right)+2a^9xy\right]}{\left(y^4-a^3x\right)^3}$$

$$= \frac{-\left[6a^3x^4y^4+10a^9xy+2a^9xy\right]}{\left(y^4-a^3x\right)^3}$$

$$= \frac{-\left[6a^3x^4y^4+12a^9xy\right]}{\left(y^4-a^3x\right)^3}$$

$$= \frac{-\left[6a^3xy\left(x^3y^3+2a^6\right)\right]}{\left(y^4-a^3x\right)^3}$$

$$= \frac{-\left[6a^3xy\left(2a^6+x^3y^3\right)\right]}{\left(y^4-a^3x\right)^3}$$

$$\frac{d^2y}{dx^2} = \frac{6a^3xy\left(2a^6+a^3y^3\right)}{\left(a^3x-y^4\right)^3}$$

**13. If $u = \tan^{-1}\left(\dfrac{xy}{\sqrt{1+x^2+y^2}}\right)$, Then Show that $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{1}{\left(1+x^2+y^2\right)^{3/2}}$.**

*Sol :*

$$u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$$

Partially differentiate with respect to 'y'

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\dfrac{xy}{\sqrt{1+x^2+y^2}}\right)^2}\left[y\frac{-1}{2\left(1+x^2+y^2\right)^{3/2}}2y + \frac{1}{\sqrt{1+x^2+y^2}}(1)\right]$$

$$= \frac{\left(1+x^2+y^2\right)x}{1+x^2+y^2+x^2y^2}\left[\frac{-y^2}{\left(1+x^2+y^2\right)^{3/2}} + \frac{1}{\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{x}{\left(1+y^2\right)+x^2\left(1+y^2\right)}\left[\frac{-y^2}{\left(1+x^2+y^2\right)^{1/2}} + \left(1+x^2+y^2\right)^{-1/2+1}\right]$$

$$= \frac{x}{\left(1+x^2\right)\left(1+y^2\right)}\left[\frac{-y^2}{\left(1+x^2+y^2\right)^{1/2}} + \left(1+x^2+y^2\right)^{1/2}\right]$$

$$= \frac{x}{\left(1+x^2\right)\left(1+y^2\right)}\left[\frac{-y^2+1+x^2+y^2}{\left(1+x^2+y^2\right)^{1/2}}\right]$$

$$= \frac{x}{\left(1+x^2\right)\left(1+y^2\right)}\left[\frac{1+x^2}{\left(1+x^2+y^2\right)^{1/2}}\right]$$

$$\frac{\partial u}{\partial y} = \frac{x}{\sqrt{1+x^2+y^2}\left(1+y^2\right)}$$

Consider

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{1+x^2+y^2}\left(1+y^2\right)}\right)$$

$$= \frac{1}{1+y^2}\left[\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{1+x^2+y^2}}\right]\right]$$

$$= \frac{1}{1+y^2}\left[x\cdot\frac{-1}{2\left(1+x^2+y^2\right)^{3/2}}(2x)+\frac{1}{\sqrt{1+x^2+y^2}}(1)\right]$$

$$= \frac{1}{1+y^2}\left[\frac{-x^2}{\left(1+x^2+y^2\right)^{3/2}}+\frac{1}{\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{1}{1+\cancel{y^2}}\left[\frac{-\cancel{x^2}+1+\cancel{x^2}+\cancel{y^2}}{\left(1+x^2+y^2\right)^{3/2}}\right]$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{\left(1+x^2+y^2\right)^{3/2}}$$

---

**14. If $z = xy\,f\left(\frac{y}{x}\right)$ and z is constant. Then show that $\frac{f'(y/x)}{f(y/x)} = \frac{x(y+xy')}{y(y-xy')}$.**

*Sol :*

$$z = xy\,f\left(\frac{y}{x}\right)$$

$$z = x\cdot x\frac{y}{x}f\left(\frac{y}{x}\right)$$

$$z = x^2\,\frac{y}{x}\,f\left(\frac{y}{x}\right)$$

$$= x^2\,z\left(\frac{y}{x}\right) \quad \text{where } t\left(\frac{y}{x}\right) = \frac{y}{x}f\left(\frac{y}{x}\right)$$

z is a homogenous function of degrees '2'

By Euler's theorem we have

$$x\frac{\partial z}{\partial r} + y\frac{\partial z}{\partial y} = 2z$$

Consider

$$z = xy\, f\left(\frac{y}{x}\right)$$

Differentiate with respect to 'x'

$$\Rightarrow \quad 0 = xy\, f'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2} + \frac{1}{x}\cdot\frac{dy}{dx}\right] + f\left(\frac{y}{x}\right)\cdot x\frac{dy}{dx} + y\cdot f\left(\frac{y}{x}\right)\cdot 1$$

$$= xy\, f'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2} + \frac{1}{x}\frac{dy}{dx}\right] + xf\left(\frac{y}{x}\right)\frac{dy}{dx} + y\cdot f\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow -f'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2} + y\frac{dy}{dx}\right] = f\left(\frac{y}{x}\right)\left[x\frac{dy}{dx} + y\right]$$

$$\Rightarrow \quad \frac{-f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x\frac{dy}{dx} + y}{\frac{y}{x}\left[-y + x\frac{dy}{dx}\right]}$$

$$\Rightarrow \quad \frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x\left(x\frac{dy}{dx} + y\right)}{y\left(y - x\frac{dy}{dx}\right)}$$

$$\therefore \quad \frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x\left(x\frac{dy}{dx} + y\right)}{y\left(y - x\frac{dy}{dx}\right)}$$

**15. Find $\frac{dy}{dx}$ for $(\cos x)^y - (\sin y)^x = 0$**

*Sol :*

Given,

$(\cos x)^y - (\sin y)^x = 0$

with respect to $\frac{dy}{dx} = \frac{-fx}{fy}$

$$f_x = \frac{\partial f}{\partial x}$$

do the partially differentiate with respect to 'x' & 'y'

$$\frac{\partial f}{\partial x} = f_x = y(\cos x)^{y-1}(-\sin x) - (\sin y)^x \log(\sin y)$$

$$f_x = -y(\cos x)^{y-1}\sin x - (\sin y)^x \log(\sin y)$$

$$\frac{\partial f}{\partial y} = f_y$$

$$= (\cos x)^y \log(\cos x) - x(\sin y)^{x-1}(\cos y)$$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$= \frac{-\left[-y(\cos x)^{y-1}\sin x - (\sin y)^x \log(\sin y)\right]}{(\cos x)^y \log(\cos x) - x(\sin y)^{x-1}(\cos y)}$$

$$= \frac{y(\cos x)^{y-1}\sin x + (\sin y)^x \log(\sin y)}{(\cos x)^y \log(\cos x) - x(\sin y)^{x-1}\cos y}$$

$$= \frac{(\cos)^y\left[y\frac{\sin x}{\cos x} + \log \sin y\right]}{(\cos x)^y\left[\log(\cos x) - x\frac{\cos y}{\sin y}\right]}$$

$$= \frac{\dfrac{y \sin x + \log \sin y \ \cos x}{\cos x}}{\dfrac{\log(\cos x)\sin y - x\cos y}{\sin y}}$$

$$\therefore \frac{dy}{dx} = \frac{\sin y[y\sin x + \log \sin y \cos x]}{\cos x[\sin y \ \log \cos x - x\cos y]}$$

**16. Find $\frac{dy}{dx}$ for $x^y = y^x$**

*Sol :*

$$x^y = y^x \quad \Rightarrow \quad x^y - y^x = 0$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

Partially different with respect to ‘x’ & ‘y’

$$\frac{\partial f}{\partial x} = f_x = yx^{y-1} - y^x \log y$$

$$\frac{\partial f}{\partial y} = f_y = x^y \log x - xy^{x-1}$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$= \frac{-\left[yx^{y-1} - y^x \log y\right]}{x^y \log x - xy^{x-1}}$$

$$= \frac{-\left[yx^{y-1} - y^x \log y\right]}{-\left[-x^y \log x + xy^{x-1}\right]}$$

$$= \frac{yx^y \cdot x^{-1} - y^x \log y}{xy^x \cdot y^{-1} - x^y \log x}$$

Since $x^y = y^x$

$$= \frac{y \cdot y^x \cdot x^{-1} - y^x \log y}{xy^x y^{-1} - y^x \log x}$$

$$= \frac{\cancel{y^x}\left[yx^{-1} - \log y\right]}{\cancel{y^x}\left[xy^{-1} - \log x\right]}$$

$$= \frac{\frac{y}{x} - \log y}{\frac{x}{y} - \log x}$$

$$= \frac{\frac{y - x\ \log y}{x}}{\frac{x - y\ \log x}{x}}$$

$$\frac{dy}{dx} = \frac{y(y - x\ \log y)}{x(x - y\ \log x)}$$

**17. If F(x, y, z) = 0, Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.**

*Sol :*

Given,

F = (x, y, z) = 0

a as function of x, y.

i.e., z = F(x, y)

Treating y as constant

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \Rightarrow \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

Treating 'x' as constant

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} \Rightarrow \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

**18. If f(x, y) = 0, $\phi$(x, y) = 0 show that $\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$**

*Sol :*

Given,

f(x, y) = 0, $\phi$(y, z) = 0

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$f(x, y) = 0 \Rightarrow \frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$= \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\phi(y, z) = 0 \qquad \frac{dz}{dy} = \frac{-\phi y}{\phi z}$$

$$= \frac{\frac{-\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$$

Consider

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$= \frac{\frac{-\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}} \cdot \frac{\frac{-\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\frac{dz}{dx} = \frac{\frac{\partial \phi}{\partial x} \cdot \frac{\partial f}{\partial x}}{\frac{\partial \phi}{\partial y} \cdot \frac{\partial f}{\partial y}}$$

$$\therefore \quad \frac{\partial \phi}{\partial z} \cdot \frac{\partial f}{\partial y} \cdot \frac{dz}{dx} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial f}{\partial x}$$

**19. If A, B, C are angles of triangle such that $\sin^2 A + \sin^2 B + \sin^2 C$ = constant prove that $\frac{dA}{dB} = \frac{\tan B - \tan C}{\tan C - \tan A}$.**

*Sol :*

Consider

$$\sin C = \sin (\pi - (A + B))$$

$$\sin C = \sin (A + B)$$

Let

$$f(A, B) = \sin^2 A + \sin^2 B + \sin^2 C - \text{constant}$$

$$= \sin^2 A + \sin^2 B + \sin^2 (A + B) - \text{constant}$$

$$f_{A(}(A, B) = 2 \sin A \cos A + 0 + 2 \sin(A + B) \cos(A + B)$$

$$f_{B(}(A, B) = 0 + 2 \sin B \cos B + 2 \sin (A + B) \cos(A + B)$$

$$\frac{dA}{dB} = \frac{-f_B}{f_A}$$

$$= \frac{[2\sin B \cos B + 2\sin(A+B)\cos(A+B)]}{[2\sin A \cos A + 2\sin(A+B)\cos(A+B)]}$$

$$= \frac{-2\sin B \cos B - 2\sin(A+B)\cos(A+B)}{-2\sin A \cos A - 2\sin(A+B)\cos(A+B)}$$

$$= \frac{-\sin 2B - \sin 2(A+B)}{-\sin 2A - \sin 2(A+B)}$$

$$= \frac{-\sin 2B - \sin 2C}{-\sin 2A - \sin 2C}$$

$$= \frac{-\cos A[\sin C \ \cos B - \cos C \ \sin B]}{-\cos B[\sin A \ \cos C - \cos A \ \sin C]}$$

$$= \frac{\cos A \ \sin B \ \cos C - \cos A \ \cos B \ \sin C}{\cos A \ \cos B \ \sin C - \sin A \ \cos B \ \cos C}$$

$$= \frac{\tan B - \tan C}{\tan C - \tan A}$$

**20. if $z = \frac{\sin u}{\cos v}$, $u = \frac{\cos y}{\sin x}$, $v = \frac{\cos x}{\sin y}$. Find $\frac{\partial z}{\partial x}$**

*Sol :*

Given that

$$z = \frac{\sin u}{\cos v},\ u = \frac{\cos y}{\sin z},\ v = \frac{\cos x}{\sin y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

Consider

$$u = \frac{\cos y}{\sin x}$$

Then

$$\frac{\partial u}{\partial x} = \cos y\ \frac{\partial}{\partial x}\left(\frac{1}{\sin x}\right)$$

$$\frac{\partial u}{\partial x} = \cos y\ (-\operatorname{cosec} x \cot x)$$

Consider

$$v = \frac{\cos x}{\sin y}$$

$$\frac{\partial v}{\partial x} = (-\sin x)\cdot\left(\frac{1}{\sin y}\right)$$

Consider

$$z = \frac{\sin u}{\cos v}$$

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}(\sin u)\cdot\frac{1}{\cos v} = \frac{\cos u}{\cos v}$$

$$\frac{\partial z}{\partial v} = \sin u\ \frac{\partial}{\partial v}\left(\frac{1}{\cos v}\right)$$

$$= \sin u\ (\sec v\cdot\tan v)$$

Now,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\cos u}{\cos v}\left(-\cos y\cdot\operatorname{cosec} x \cot x\right) - \sin u\ (\sec v\cdot\tan v)\ \frac{\sin x}{\sin y}$$

$$= \frac{-\cos u}{\cos v}\cdot \cos y \cdot \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} - \sin u\ \frac{1}{\cos u}\cdot\frac{\sin v}{\cos v}\cdot\frac{\sin x}{\sin y}$$

$$= \frac{-\cos u}{\cos v}\cdot \cos y \cdot \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} - \sin u\ \frac{1}{\cos u}\cdot\frac{\sin x}{\sin y}$$

$$= \frac{-\cos u}{\cos v}\cdot u \cdot \cot x - z\ \frac{\sin v}{\cos v}\cdot\frac{\sin x}{\sin y}$$

$$= \frac{-u\cot x\cos u\sin y - z\sin v\cdot \sin x}{\cos v\sin y}$$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{-\left(u\cot x\cdot\cos u\cdot\sin y + z\sin v\cdot\sin x\right)}{\cos v\cdot\sin y}$$

**21. find $\frac{d^2y}{dx^2}$ for $x^3 + y^3 = 3axy$**

*Sol :*

Given,

$$x^3 + y^3 = 3axy$$

$$\frac{d^2y}{dx^2} = \frac{-\left[f_x^2\left(f_y\right)^2 - 2f_{yx}f_xf_y + f_y^2\left(f_x\right)^2\right]}{\left(f_y\right)^3}$$

Consider

$$x^3 + y^3 = 3axy \Rightarrow x^3 + y^2 - 3axy$$

here

$$f_x = 3x^2 - 3ay$$

$$f_x^2 = 6x$$

$$f_y = 3y^2 - 3ax$$

$$f_{y^2} = 6y$$

$$f_{yx} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}(3x^2 - 3ay)$$

$$= 0 - 3a$$

$$= -3a$$

$$= \frac{-\left[(6x)(3y^2-3ax)^2 - 2(-3a)(3x^2-3ay)(3y^2-3ax) + (6y)(3x^2-3ay)^2\right]}{(3y^2-3ax)^3}$$

$$= \frac{-\left[6x\left[9y^4+9a^2x^2-18axy^2\right]+6a\left(9x^2y^2-9ax^3-9ay^3+9a^2xy\right)+6y\left(9x^4+9a^2y^2-18ax^2\right)\right]}{(3y^2-3ax)^3}$$

$$= \frac{-\left[54\left(xy^4+\cancel{a^2x^3}-2ax^2y^2+ax^2y^2-\cancel{a^2x^3}-\cancel{a^2y^3}+a^3xy+x^4y+\cancel{a^2y^3}-2ax^2y^2\right)\right]}{(3y^2-3ax)^3}$$

$$= \frac{-\left[54\left(xy^4+x^4y-3ax^2y^2+a^3xy\right)\right]}{(3y^2-3ax)^3}$$

$$= \frac{-\left[54\left(xy\left(y^3+x^3-3axy+a^3\right)\right)\right]}{(3y^2-3ax)^3}$$

$$= \frac{-54\left[xy\left(\cancel{y^3}+\cancel{x^3}-\cancel{x^3}-\cancel{y^3}+a^3\right)\right]}{27(y^2-ax)^3}$$

$$= \frac{-54\left[a^3xy\right]}{27(y^2-ax)^3}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{-2(a^3xy)}{(y^2-ax)^2}$$

**22. Find $\frac{d^2y}{dx^2}$ If $x^5 + y^5 = 5a^3x^2$**

*Sol :*

Given,

$x^5 + y^5 = 5a^3x^2 \Rightarrow x^5 + y^5 - 5a^3x^2 = 0$

$$\frac{d^2y}{dx^2} = \frac{-\left[f_x^2(f_y)^2 - 2f_{yx}f_xf_y + f_{y^2}(f_x)^2\right]}{(f_y)^3}$$

Consider

$f_x = 5x^4 - 10a^3x$

$f_x^2 = 20x^3 - 10a^3$

$f_y = 5y^4$

$f_y^2 = 20y^3$

$f_{xy} = 0$

$$\frac{d^2y}{dx^2} = \frac{-\left[\left(20x^3 - 10a^3\right)\left(5y^2\right) - 2(0)\left(5x^4 - 10a^3x\right)\left(5y^4\right) + (20y)^3\left(5x^4 - 10a^3x\right)^2\right]}{\left(5y^4\right)^3}$$

$$= \frac{-\left[\left(20x^3 - 10a^3\right)\left(25y^8\right) + 80y^3\left(25x^8 + 20a^6x^2 - 100a^3x^5\right)\right]}{\left(5y^4\right)^3}$$

$$= \frac{-\left[125\left(4x^3y^8 - 2a^3y^8 + 4x^8y^3 + 16a^6x^2y^3 - 16a^3x^5y^3\right)\right]}{125y^{12}}$$

$$= \frac{-\left[4x^3y^8 + 4x^8y^3 - 20a^3x^5y^3 + 4a^3x^5y^3 - 2a^3y^8 + 16a^6x^2y^3\right]}{y^{12}}$$

$$= \frac{-\left[4x^3y^3\left(y^5 + x^5 - 5a^3x^2\right) + 6a^3x^5y^3 - 2a^3x^5y^5 - 2a^3y^8 + 16a^2x^2y^3\right]}{y^{12}}$$

$$= \frac{-\left[6a^3x^5y^3 - 2a^3y^3\left(x^5 + y^5\right) + 16a^6x^2y^3\right]}{y^{12}}$$

$$= \frac{-\left[6a^3x^5 - 2a^3y^3\left(5a^3x^2\right) + 16a^6x^2y^3\right]}{y^{12}}$$

$$= \frac{-\left[6a^3x^5y^3 - 10a^6x^2y^3 + 16a^6x^2y^3\right]}{y^{12}}$$

$$= \frac{-\left[6a^3x^5y^3 + 6a^6x^2y^3\right]}{y^{12}}$$

$$= \frac{-\left[6a^3x^2y^3\left(x^3 + a^3\right)\right]}{y^{12}}$$

$$= \frac{y^3\left[6a^3x^2\left(x^3 + a^3\right)\right]}{y^{12}}$$

$$= \frac{-\left[6a^3x^2\left(x^3 + a^3\right)\right]}{y^9}$$

**23. If the curves f(x, y) = 0, $\phi$(x, y) = 0 touch each other, show that at the point of contact**

$$\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial x} = 0$$

*Sol :*

we have f(x, y) = 0

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \text{... (1)}$$

also we have $\phi$(x, y) = 0

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} \quad \text{... (2)}$$

At the point of contact, the slope of tangents to both curves must coincide, Then from (1) & (2)

$$\frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$$

$$\frac{\partial \phi}{\partial y}\cdot\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial x}$$

$$\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial x}\cdot\frac{\partial f}{\partial y} = 0.$$

**24. If $x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)} = a$ show that. $\frac{d^2y}{dx^2} = \frac{-a}{(1-x^2)^{3/2}}$**

*Sol :*

Let $f(x, y) = x\sqrt{1-y^2} + y\sqrt{1-x^2} - a = 0$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} \quad \text{...(1)}$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2} - a\right]$$

$$f_x = \sqrt{1-y^2} + y\,\frac{1}{\cancel{2}\sqrt{1-x^2}}\left(-\cancel{2}x\right) = \sqrt{1-y^2} - \frac{yx}{\sqrt{1-x^2}}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x - \sqrt{1-y^2} + y\sqrt{1-x^2} - a\right]$$

$$= x\cdot\frac{1}{\cancel{2}\sqrt{1-y^2}}\left(-\cancel{2}y\right) + \sqrt{1-x^2}$$

$$= \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$\therefore \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-\left[\sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}\right]}{\left[\frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}\right]}$$

$$= \frac{\dfrac{-\sqrt{1-y^2}\sqrt{1-x^2} + xy}{\sqrt{1-x^2}}}{\dfrac{-xy + \sqrt{1-x^2}\sqrt{1-y^2}}{\sqrt{1-y^2}}}$$

$$= \frac{xy - \cancel{\sqrt{1-y^2}}\sqrt{1-x^2}}{\sqrt{1-x^2}} \times \frac{\sqrt{1-y^2}}{-\left[\sqrt{1-x^2}\cancel{\sqrt{1-y^2}} + 2y\right]}$$

$$\frac{dy}{dx} = \frac{-\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Differentiate the above equation both the sides with respect to 'x'

$$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{1-x^2}\,\frac{1}{\cancel{2}\sqrt{1-y^2}}\left(-\cancel{2}y\right)\frac{dy}{dx} - \sqrt{1-y^2}\,\frac{1}{\cancel{2}\sqrt{1-x^2}}\left(-\cancel{2}x\right)}{\left(\sqrt{(1-x^2)}\right)^{\cancel{2}}}\right]$$

$$= \frac{-\left[\frac{-y\cancel{\sqrt{1-x^2}}}{\cancel{\sqrt{1-y^2}}}\left[\frac{-\cancel{\sqrt{1-y^2}}}{\cancel{\sqrt{1-x^2}}}\right] - \frac{x\sqrt{1-y^2}}{\sqrt{1-x^2}}\right]}{\left(1-x^2\right)}$$

$$= \frac{y - \frac{x\sqrt{1-y^2}}{\sqrt{1-x^2}}}{(1-x^2)} = \frac{y\sqrt{1-x^2} - x\sqrt{1-y^2}}{(1-x^2)\left(\sqrt{1-x^2}\right)}$$

$$= \frac{-\left[y\sqrt{1-x^2} + x\sqrt{1-y^2}\right]}{(1-x^2)(1-x^2)^{1/2}}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{-a}{(1-x^2)^{3/2}}$$

**25. If u and v are functions of x & y defined by $x = u + e^{-v}\sin u$, $y = v + e^{-v}\cos u$. Prove that $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.**

*Sol :*

$$x = u + e^{-v}\sin u \qquad y = v + e^{-v}\cos u$$

$$\frac{\partial x}{\partial v} = \sin u\,(-e^{-v}) \qquad \frac{\partial y}{\partial u} = e^{-v}(-\sin u)$$

$$\frac{1}{\frac{\partial x}{\partial v}} = \frac{-1}{\sin u\left(e^{-v}\right)}$$

$$\frac{\partial v}{\partial x} = \frac{-1}{\sin u e^{-v}} \qquad \frac{\partial u}{\partial y} = \frac{-1}{e^{-v}\sin u}$$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

## 2.3 EQUALITY OF $F_{XY}$(A, B) AND $F_{YX}$(A, B)

**26.** Define Equality of **$f_{xy}$(a, b), $f_{yx}$(a, b).**

*Sol :*

If has been seen that the two repeated second order partial derivatives are generally equal. They are not, however, always equal as is shown below.

We have $f_{xy}(a, b) = \lim\limits_{h\to 0}\frac{f_y(a+h,b) - f_y(a,b)}{h}$

also, $f_y(a + h, b) = \lim\limits_{k\to 0}\frac{f(a+h,b+k) - f(a+h,b)}{k}$

$$f_y(a, b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}$$

$$\therefore \quad f_{xy}(a, b) = \lim_{h \to 0} \lim_{k \to 0} \frac{\phi(h, k)}{hk}$$

$$\therefore \quad f_{xy}(a, b) = \lim_{h \to 0} \lim_{k \to 0} \frac{f(a+k, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{\phi(h,, k)}{hk}$$

It many similarly
show that

$$f_{yx}(a, b) = \lim_{k \to 0} \lim_{h \to 0} \frac{\phi(h, k)}{hk}.$$

---

**27. Prove that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ for the function f given by,**

$$\mathbf{f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}; (x, y) \neq (0, 0) \ \& \ f(0, 0) = 0}$$

*Sol :*

f is a function given by

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}; (x, y) \neq (0, 0)$$

As we know that

$$f_{xy}(a, b) = \lim_{h \to 0} \frac{f_y(a+h, b) - f_y(a, b)}{h}$$

where

$$f_y(a, b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}$$

$$\therefore \quad f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_y(0+h, 0) - f_y(0, 0)}{h} \quad \ldots (1)$$

where

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(a, b+k) - f(0, 0)}{k}$$

$$= \lim_{k \to 0} \frac{f(0, k) - 0}{k}$$

here

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

$$f_y(0, 0) = \lim_{k\to 0} \frac{0 \cdot k(0 - k^2)}{0 + k^2}$$

$$f_y(0, 0) = 0 \quad \text{... (2)}$$

and consider by (1)

$$f_y(h, 0) = \lim_{k\to 0} \frac{f(h, 0+k) - f(h, 0)}{k}$$

Since

$$f_y(a, b) = \lim_{k\to 0} \frac{f(a, b+k) - f(a, b)}{k}$$

$$= \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{hk(h^2 - k^2)}{k(h^2 + k^2)}$$

$$= \lim_{k\to 0} \frac{h\not{k}(h^2 - k^2)}{\not{k}(h^2 + k^2)}$$

$$f_y(h, 0) = h\left(\frac{h^2 - 0}{h^2 + 0}\right)$$

$$\Rightarrow \frac{h^3}{2} = h \quad \text{... (3)}$$

Let (2), (3) in (1)

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{h - 0}{h}$$

$$= 1$$

Consider

$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{f_x(0, 0+k) - f_x(0,0)}{k} \quad \text{... (4)}$$

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - 0}{h}$$

$$= \lim_{h\to 0} \frac{h \cdot 0(h^2 - 0)}{h^2 - 0}$$

$$= \frac{0}{h} = 0 \quad \text{... (5)}$$

$$f_x(0, 0 + k) = f_x(0, k)$$

$$= \lim_{h\to 0} \frac{f(0+h, k) - f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{hk(h^2 - k^2)}{h(h^2 + k^2)}$$

$$f_x(0, 0 + k) = \frac{k(0 - k^2)}{0 + k^2}$$

$$= \frac{-k^3}{k^2} = -k \quad \text{... (6)}$$

Sub (5), (6) in (4)

$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{-k - 0}{k} = -1$$

Thus,

$$f_{xy}(0, 0) = 1 \neq -1 = f_{yx}(0, 0).$$

**28. Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ where $f(x, y) = 0$ if $xy = 0$, $f(x, y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$ if $xy \neq 0$**

*Sol :*

Given that

$$f(x, y) = 0 \text{ if } xy = 0$$

$$f(x, y) = x^2 \tan^{-1}\frac{y}{x} - y^2 - \tan^{-1}\frac{x}{y}$$

if $xy \neq 0$

we have

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{f(0+h, 0) - f_y(0,0)}{h}$$

$$f_y(h,\ 0) = \lim_{k\to 0}\frac{f(h,0+k)-f(h,0)}{k}$$

$$= \lim_{k\to 0}\frac{f(h,k)-f(h,0)}{k}$$

$$= \lim_{k\to 0}\frac{1}{k}\left(h^2\tan\frac{k}{h}-k^2\tan^{-1}\frac{h}{k}\right)$$

$$= \lim_{k\to 0}\left[h\left(\frac{\tan^{-1}k/h}{k/h}\right)-k\tan^{-1}\frac{h}{k}\right]$$

$$= h\cdot 1-0 \text{ [ for } \tan^{-1}\frac{t}{z} \to 1 \text{ as } t \to 0]$$

$$= \lim_{k\to(0,0)}\tan^{-1}\frac{h}{k} = \frac{\pi}{2}$$

$$= \lim_{k\to(0,0)}\tan^{-1}\frac{h}{k} = \frac{-\pi}{2}$$

$$f_y(0,\ 0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = 0$$

$$f_{xy}(0,\ 0) = \lim_{h\to 0}\frac{h-0}{h} = 1$$

we many similarly show that

$$f_{yx}(0,\ 0) = -1$$

$\therefore \quad f_{xy}(0,\ 0) \neq f_{yx}(0,\ 0).$

**29. State and prove Equality of $f_{xy}$ and $f_{yx}$.**

**Statement :**

**If f(x, y) possess continuous second order partial derivative $f_{xy}$ & $f_{yx}$ then $f_{xy} = f_{yx}$.**

**Proof :**

f(x, y) possess continuous second order partial derivative $f_{xy}$ & $f_{yx}$ .

Consider the expression

$$\phi(h,\ k) = f(x+h,\ y+k) - f(x+h,\ y) - f(x,\ y+k) - f(x,\ y)$$

Let $\varphi(x) = f(x,\ y+k) - f(x,\ y)$

$$\varphi(x+h) = f(x+h,\ y+k) - f(x+h,\ y)$$

Then

$$\phi(h,\ k) = \varphi(x+h) - \varphi(x)$$

By legranges mean value theorem

$$\frac{\varphi(x+h)-\varphi(x)}{h} = \varphi'(x + \theta, h) \qquad 0 < \theta_1 < 1$$

$$\varphi(x + h) - \varphi(x) = h\varphi'(x + \theta_1, h)$$

$$\varphi'(x) = f_x(x, y + k) - f_x(x, y)$$

$$\phi(h - k) = h[f_x(x + \theta_1 h, y + k) - f_x(x + \theta_1 h, y)] \qquad \ldots (1)$$

Again applying the mean value theorems to the right side of (1)

$$\phi(h, k) = hk\, f_{yx}(x + \theta_1 h, y + \theta_2 k) \; 0 < \theta_2 < 1$$

Thus,

$$\frac{\phi(h,k)}{hk} = f_{yx}(x + \theta_1 h, y + \theta_2 k)$$

Again considering

$$F(y) = f(x + h, y) - f(x, y) \text{ in place of } \varphi(x)$$

$$\frac{\phi(h,k)}{hk} = f_{xy}\,[x + \theta_3 h; y + \theta_4 k]$$

$$f_{xy}(x + \theta_1 h, y + \theta_2 k) = f_{xy}(x + \theta_3 h, y + \theta_4 k)$$

Let $(h, k) \to (0, 0)$

Then because of the assumed continuity of the partial derivatives we obtain

$$f_{yx}(x, y) = f_{xy}(x, y)$$

$$f_{yx} = f_{xy}$$

## 2.4 Taylor's Theorem for a Function of two Variables

**30. Define Taylor's theorem for function of two variables.**

*Sol :*

If f is possesses continuous partial derivatives of the third order in a neighbourhood of a point (a, b) and if (a + h, b + k) be a point of this neighbourhood, then there exists a positive number $\theta$ which is less then 1. Such that

$$f(a + h, b + k) = f(a, b) + [h\, f_x(a, b) + k\, f_y(a, b)] + \frac{1}{2!}\left[h^2 \cdot f_{x^2}(a,b) + 2kh\, f_{xy}(a,b) + k^2 f_{y^2}(a,b)\right]$$

$$+ \frac{1}{3!}\left[h^2 f_{x^3}(u,v) + 3h^2 k f_{y^2x}(u,v) + 3hk^2 f_{y^2x}(u,v) + k^3 f_{y^2x}(u,v)\right]$$

---

**31. State and prove Taylor's theorem**

**Statement**

**If f possess continuous partial derivative of the third order in a neighbourhood of a point (a, b) and if (a + h, b + k) be a point of this neighbourhood then there exists a positive number $\theta$ which less than**

(a, h,, b + k) = f(a + b) + [hf$_x$ (a, b) + k f$_y$) + kf$_y$ (a, b)] + $\frac{1}{2!}\left[h^2 f_{x^2}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{y^2} = (a,b)\right]$

$$+ \frac{1}{3!}\left[h^3 f_{x^3}(u,v) + 3h^2 k f_{y^2x}(u,v) + 3hk^2 f_{x^2y}(u,v) + k^3 f_y 3(u,v)\right]$$

**Proof :**

When u = a + θh, v = b + θk

z = f(x, y) and x = a + ht, y = b + kt

So that z is a function of t which one denote

by g(t) i.e., z = g(t) = f(x, y)

$$\therefore \quad g'(t) = f_x(x, y)\frac{dx}{dt} + f_y(x, y)\frac{dy}{dt}$$

$$x = a + ht \Rightarrow \frac{dx}{dt} = h$$

$$y = b + kt \Rightarrow \frac{dy}{dt} = k$$

$$\Rightarrow \quad g'(t) = hf_x(x, y) + kf_y(x,y)$$

$$g''(t) = h\left[f_{x^2}(x,y)\frac{dx}{dt} + f_{yx}(x,y)\frac{dy}{dt}\right] + k\left[f_{xy}(x,y)\frac{dx}{dt} + f_{y^2}\frac{dy}{dt}\right]$$

$$= h\left[f_{x^2}(x,y)h + f_{yx}(x,y)k\right] + k\left[f_{xy}(x,y) + f_{y^2}(x,y)k\right]$$

$$= h^2 f_{x^2}(x, y) + hk f_{yx}(x,y) + hk f_{xy}(x,y) + k^2 f_{y^2}(x, y)$$

$$= h^2 f_{x^2}(x, y) + 2hkf_{xy}(x, y) + k^2 - f_{y^2}(x, y)$$

$$g'''(t) = h^2\left[f_{x^3}(x,y)\frac{dx}{dt} + \frac{dx}{dt} + f_{yx^2}(x,y)\frac{dy}{dt}\right] + 2hk\left[f_{x^2y}(x,y)\frac{dx}{dt} + f_{xy^2}(x,y)\frac{dy}{dt}\right]$$

$$+ k^2\left[f_{xy^2}(x,y)\frac{dx}{dt} + f_{y^3}(x,y)\frac{dy}{dt}\right]$$

$$= h^2\left[f_{x^3}(x,y)h + f_{yx^2}(x,y)k\right] + 2hk\left[f_{x^2y}(x,y)h + f_{xy^2}(x,y)k\right] + k^2\left[f_{xy^2}(x,y)h + f_{y^3}(x,y)k\right]$$

$$= h^3 f_{x^3}(x, y) + h^2k\, f_{yx^2}(x, y) + 2h^2k f_{x^2y}(x, y) + 2hk^2 f_{xy^2}(x, y) + k^2h f_{xy^2}(x, y) + k^3 f_{y^3}(x, y)$$

$$= h^3 f_{x^3}(x, y) + 3h^2k f_{x^2y}(x, y) + 3hk^2 f_{xy^2}(x, y) + k^3 f_{y^3}(x, y)$$

Now we have

$$g(t) = g(0) + t\,g'(0) + \frac{t^2}{2!}g''(0) + \frac{t^3}{3!}g'''(\theta t) \quad 0 < \theta < 1$$

$$g(1) = g(0) + g'(0) + \frac{1}{2!}g''(0) + \frac{1}{3!}(\theta)$$

Then we have

$$f(a\ h, b + k) = f(a + b) + [h\,f_x(a, b) + kf_y(a, b)] + \frac{1}{2!}\left[h^2 f_{x^2}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{y^2}(a,b)\right]$$

$$+ \frac{1}{3!}\left[h^3 f_{x^3}(u,v) + 3h^2 k f_{x^2y}(u,v) + 3hk^2 f_{y^2x}(u,v) + k^3 f_{y^3}(u,v)\right]$$

Where $u = a + \theta h$, $v = b + \theta k$, $0 < \theta < 1$.

**32. Write another form of Taylor's Theorem**

*Sol :*

Consider (x, y) i.e., f is a function of two variable x, y, then Taylor's (theorem) series of expansion of f(x, y) at the point (a, b) interms of (x – a) & (y – b) is given by

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]f(a, b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a, b)$$

$$+ \frac{1}{3!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^3 f(a, b)$$

$$+ \ldots\ldots + \frac{1}{(n-1)!}\left[x - a\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n-1} f(a, b) + R_n$$

where $R_n$ denotes the reminder term given by

$$R_n = \frac{1}{n!}f(a + (x - a)\,\theta, b + \theta(y - b))$$

Simplifying, we get

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]f(a, b) + \frac{1}{2!}\left[(x-a)^2\frac{\partial^2}{\partial x^2} + (y-b)^2\frac{\partial^2}{\partial y^2} + 2(x-a)(y-b)\frac{\partial^2}{\partial x\partial y}\right]f(a,b)$$

$$+ \frac{1}{3!}\left[(x-a)^3\frac{\partial^3}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3}{\partial x^2\partial y} + 3(x-a)(y-a)^2\frac{\partial^3}{\partial x\partial y^2} + (y-b)^3\frac{\partial^3}{\partial y^3}\right]f(a, b) + \ldots..$$

**33. Expand sin xy in powers of (x –1) of $\left(y-\frac{\pi}{2}\right)$ upto second degree term.**

*Sol :*

Here f(x, y) = sin xy

$$f\left(1,\frac{\pi}{2}\right) = \sin (1)\ \frac{\pi}{2} = 1$$

$$\frac{\partial f}{\partial x} = y \cos xy\ ; \quad \frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy$$

$$\frac{\partial f}{\partial y} = x \cos xy\ ; \quad \frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy$$

$$\frac{\partial f}{\partial x \partial y} = \cos xy - sy \sin xy$$

At the point $\left(1,\frac{\pi}{2}\right)$

$$\frac{\partial f}{\partial x} = \frac{\pi}{2} \cos (1)\ \frac{\pi}{2} = 0$$

$$\frac{\partial f}{\partial y} = 1 \cdot \cos(1)\frac{\pi}{2} = 0$$

$$\frac{\partial^2 f}{\partial x^2} = -\left(\frac{\pi}{2}\right)^2 \sin (1)\ \frac{\pi}{2} = \frac{-\pi^2}{4}$$

$$\frac{\partial^2 f}{\partial y^2} = -(1)^2 \sin (1)\frac{\pi}{2} = -1$$

$$\frac{\partial f}{\partial x \partial y} = \cos(1)\ \frac{\pi}{2} - 1\ \frac{\pi}{2}\ \sin(1)\ \frac{\pi}{2}$$

$$0 - \frac{\pi}{2} \sin \frac{\pi}{2}$$

hence by Taylor's theorem

$$f(x, y) = f\left(1,\frac{\pi}{2}\right)+\left[(x-1)\frac{\partial}{\partial x}+\left(y-\frac{\pi}{2}\right)\frac{\partial}{\partial y}\right]f\left(1,\frac{\pi}{2}\right)+\frac{1}{2!}\left[(x-1)^2\frac{\partial^2}{\partial x^2}+2(x-1)\left(y-\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)^2\frac{\partial^2}{\partial y^2}\right]f\left(1,\frac{\pi}{2}\right)$$

$$\sin x = 1 + \left[(x-1)\cdot 0+\left(y-\frac{\pi}{2}\right)\cdot 0\right]+\frac{1}{2}\left[(x-1)^2\left(\frac{x^2}{4}\right)+2(x-1)\left(y-\frac{\pi}{2}\right)\cdot\frac{\pi}{2}+\left(y-\frac{\pi}{2}\right)^2(-1)\right]$$

$$= 1-\frac{\pi^2}{8}(x-1)^2-\frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right)-\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2$$

**34. Obtain Taylor's formula for f(x, y) = cos (x + y) ; n = 3 at (0, 0)**

*Sol :*

n = 3, The Taylor's theorem is

$$f(x, y) = f(0, 0) + \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0, 0) + \frac{1}{2!}\left(x^2\frac{\partial^2}{\partial x^2}+2xy\frac{\partial^2}{\partial x\partial y}+y^2\frac{\partial^2}{\partial y^2}\right)f(0,0)+R_3$$

where

$$R_3 = \frac{1}{31}\left(x^3\frac{\partial^3}{\partial x^3}+3x^2y\frac{\partial^3}{\partial x^2\partial y}+3xy^2\frac{\partial^3}{\partial x\partial y^2}+y^3\frac{\partial^3}{\partial y^3}\right)f(\theta x, \theta y),\ 0 < \theta < 1$$

hence

$$f(x, y) = \cos (x + y)$$

$$f(0, 0) = \cos (0 + 0)$$

$$= 1$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = -\sin(x + y)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = -\cos (x + y)$$

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial x^2\partial y} = \frac{\partial^3 f}{\partial x\partial y^2} = \frac{\partial^2 f}{\partial y^3} = \sin(x + y)$$

$\therefore$ At (0, 0)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0,\ \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^3 f}{\partial x\,\partial y} = -1$$

and on substituting $\theta x$ for x.

and $\theta y$ for y

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^2 f}{\partial x^2\partial y} = \frac{\partial^3 f}{\partial x\,\partial y^2} = \frac{\partial^3 f}{\partial x^3} = \sin\theta(x + y)$$

$$\therefore \quad \cos (x + y) = 1 + \theta + \frac{1}{2!}(x^2 + 2xy + y^2)(-1) + \frac{1}{3!}(x^3 + 3x^2y + 3xy^2 + y^2)\sin \theta(x+y)$$

$$\cos (x + y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!}\sin \theta\,(x + y).$$

**35. Expand the function $f(x, y) = x^2 + xy - y^2$ by Taylor's theorem in power of (x–1) and (y+2)**

*Sol. :*

Given function is $f(x, y) = x^2 + xy - y^2$

$(a, b) = (1, -2)$

$$f(a,b) \Rightarrow f(1, -2) = (1)^2 + (1)(-2) - (-2)^2$$

$$= 1 - 2 + 4 = -5$$

By Taylor's Theorem, we have

$$f(x, y) = f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)] + \frac{1}{2!}\left[(x-a)^2 f_{x^2}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{y^2}(a,b)\right]$$

Now

$$f_x = \frac{\partial f}{\partial x} = 2x + y; \quad f_x(a, b) = f_x(1,-2)$$

$$= 2(1) + (-2) = 0$$

$$f_y = \frac{\partial f}{\partial y} = x - 2y; \quad f_y(a, b) = f_y(1, -2)$$

$$= 1 - 2(-2) = 5$$

$$f_{x^2} = \frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}(2x + y) = 2$$

$$f_{y^2} = \frac{\partial^2 f}{\partial y^2} = -2;$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x - 2y) = 1$$

sub above all values in equation (1)

$$x^2 + xy - y^2 = -5 + [(x-1)(0) + (y+2)\,5\,] + \frac{1}{2!}[(x-1)^2(2) + 2(x-1)(x+2) + (y+2)^2(-2)]$$

$$= -5 + [(y+2)5] + \frac{2}{2!}\left[(x-1)^2 + (x-1)(y+2) - (y+2)^2\right]$$

$$\therefore \quad x^2 + xy - y^2 = -5 + 5(y + 2) + (x-1)^2 + (x-1)(y+2) - (y+2)^2$$

**36. Expan`d $x^2y + 3y - 2$ in power of x–1 and y+2.**

*Sol. :*

Given that $f(x, y) = x^2y + 3y - 2$

In power of x–1 y+2 i.e. a = 1, b = –2

$\therefore \quad (a, b) = (1, -2)$

f(x, y) at (a, b) $\Rightarrow \quad x^2y + 3y - 2\Big|_{(1,-2)}$

$= (1)^2(-2) + 3(-2) - 2$

$= -2 - 6 - 2$

$= -10$

By Taylor's theorem

$$f(x, y) = f(a, b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{x^2}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b^2)f_{y^2}(a,b)\right] \quad \text{... (1)}$$

Here $f_x = \frac{\partial f}{\partial x} = 2xy$ ;

$f_x(a, b) = f_x(1, -2) = 2(1)(-2) = -4$

$f_y = \frac{\partial f}{\partial y} = x^3 + 3$

$f_y(a, b) = f_y(1, -2) = (1)^2 + 3$

$f_y(1, -2) = 4$

$f_{x^2} = \frac{\partial^2 f}{\partial x^2} = 2y$ ;

$f_{x^2}(a, b) = f_{x^2}(1, -2) = 2(-2) = -4$

$f_{y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(x^3 + 3) = 0$

$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x^2 + 3) = 2x$

$f_{xy}(a, b) = f_{xy}(1,-2) = 2(1)$

$\therefore \quad f_{xy} = 2$

Sub all above values in (1)

$$x^3y + 3y - 2 = -10 + [(x-1)(-4) + (y+2)(4)] + \frac{1}{2!}\left[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)(0)^2\right]$$

$$\therefore \quad x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2)$$

**37. Expand $f(x, y) = \log(x + e^y)$ by Taylor's series in powers of $(x-1)$ and $y$ such that it includes all terms up to second degree.**

*Sol. :*

Given that $f(x, y) = \log(x+e^y)$

$$(a, b) = (1, 0)$$

$$f(1, 0) = \log(1+e^0) = \log 2$$

$$f_x = \frac{\partial f}{\partial x} = \frac{1}{x+e^y}$$

$$\Rightarrow \quad f_x(1,0) = \frac{1}{1+e^0} = \frac{1}{1+1} = \frac{1}{2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{x+e^y}(e^y)$$

$$\Rightarrow \quad f_y(1,0) = \frac{1}{1+e^0}(e^0) = \frac{1}{2}$$

$$f_{y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{e^y}{x+e^y}\right)$$

$$= \frac{(x+e^y)e^y - e^y(e^y)}{(x+e^y)^2} = \frac{xe^y + e^y e^y - e^y e^y}{(x+e^y)^2}$$

$$f_{y^2} = \frac{xe^y}{(x+e^y)^2}$$

$$f_{y^2}(1, 0) = \frac{(1)e^0}{(1+e^0)} = \frac{1}{(2)^2} = \frac{1}{4}$$

$$\Rightarrow \quad x^2 = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x+e^y}\right) = \frac{-1}{(x+e^y)^2}$$

$$\Rightarrow \quad f_{x^2}(1,0) = \frac{-1}{(1+e^0)^2} = \frac{-1}{4}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{e^y}{x+e^y}\right) = \frac{(x+e^y)(0) - e^y(1)}{(x+e^y)^2}$$

$$f_{xy}(1,0) = \left(\frac{-e^0}{1+e^0}\right)^2 = \frac{-1}{(1+1)^2} = \frac{-1}{4}$$

$\therefore$ By Taylor's Theorem

$$f(x,y) = f(1,0) + \left[(x-1)f_x(1,0) + (y-0)f_y(1,0)\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2 f_{x^2}(1,0) + 2(x-1)(y-0)f_{xy}(1,0) + (y-0)^2 f_{y^2}(1,0)\right]$$
$$= \log 2 + \left[(x-1)\left(\frac{1}{2}\right) + y\left(\frac{1}{2}\right)\right] + \frac{1}{2!}\left[(x-1)^2\left(\frac{-1}{4}\right) + 2(x-1)y\left(\frac{-1}{4}\right) + y^2\left(\frac{1}{4}\right)\right]$$
$$= \log 2 + \frac{1}{2}(x-1) + \frac{1}{2}y - \left(\frac{1}{8}\right)(x-1)^2 - \left(\frac{1}{4}\right)(x-1)y + \frac{1}{8}y^2$$
$$\log(x+e^y) = \log 2 + \frac{1}{2}(x-1) + \frac{y}{2} - \frac{1}{8}(x-1)^2 - \frac{1}{4}(x-1)y + \frac{1}{8}y^2$$

**38. Expand $f(x, y) = x^2 + xy + y^2$ in power of $(x-2)$ t$(y-3)$**

*Sol. :*

Given that $f(x, y) = x^2 + xy + y^2$

$$(a, b) = (2, 3)$$
$$f(a, b) = f(2, 3)$$
$$= (2)^2 + 2(3) + 3^2$$
$$= 4 + 6 + 9 = 19$$
$$f_x = \frac{\partial f}{\partial x} = 2x + y$$
$$\Rightarrow f_x(2, 3) = 2(2) + 3 = 7$$
$$f_y \frac{\partial f}{\partial y} = x + 2y$$
$$\Rightarrow f_y(2, 3) = 2 + 2(3) = 8$$
$$f_{x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2x+y) = 2$$
$$f_{y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(x+2y)$$
$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x+2y) = 1$$

By Taylor's Theorem

$$f(x, y) = f(2, 3) + \left[(x-2)f_x(2,3) + (y-3)f_y(2,3)\right]$$
$$+\frac{1}{2!}\left[(x-2)^2 f_{x^2}(2,3) + 2(x-2)(y-3)f_{xy}(2,3) + (y-3)^2 f_{y^2}(2,3)\right]$$

Substitute all corresponding values into the above equation.

$$x^2 + xy + y^2 = 19 + [(x-2)7 + (y-3)8] + \frac{1}{2!}[(x-2)^2(2) + 2(x-2)(y-3)(2) + (y-3)^2(2)]$$

$$= 19 + 7(x-2) + 8(y-3) + (x-2)^2 + 2(x-2)(y-3) + (y-3)^2$$

$$\therefore x^2 + xy + y^2 = 19 + 7(x-2) + 8(y-3) + (x-2)^2 + 2(x-2)(y-3) + (y-3)^2$$

**39. Obtain Taylor's formula for the function $e^{x+y}$ at (0, 0) for n = 3**

*Sol. ;*

Given function is $(x, y) = e^{x+y}$ at $(a, b) = (0, 0)$ for $n = 3$

$$f(a, b) = e^{0+0} = e^0 = 1$$

$$f_x = \frac{\partial}{\partial x}(e^{x+y}) = e^{x+y}$$

$$\Rightarrow \quad f_x(0, 0) = e^{0+0} = 1$$

$$f_y = \frac{\partial}{\partial y}(e^{x+y}) = e^{x+y}$$

$$f_y(0,0) = e^{0+0} = 1$$

$$f_{x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(e^{x+y}) = e^{x+y}$$

$$f_{x^2}(0,0) = e^{0+0} = 1$$

$$f_{y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(e^{x+y}) = e^{x+y}$$

$$f_{y^2}(0,0) = e^{0+0} = 1$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(e^{x+y})$$

$$= e^{x+y}$$

$$f_{xy}(0,0) = e^{0+0} = 1$$

$$f_{x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial x}(e^{x+y}) = e^{x+y}$$

$$f_{x^3}(\theta x, \theta y) = e^{\theta(x+y)}$$

$$f_{x^2y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial y}(e^{x+y}) = e^{x+y}$$

$f_{x^2y}(\theta x, \theta y) = e^{\theta(x+y)}$

$$f_{xy^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial x}(e^{x+y}) = e^{x+y}$$

$f_{xy^2}(\theta x, \theta y) = e^{\theta(x+y)}$

$$f_{y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial y}(e^{x+y}) = e^{x+y}$$

$f_{y^3}(\theta x, \theta y) = e^{\theta(x+y)}$

$\therefore$ By Taylor's Theorem. at (a, b) = (0, 0)

$$f(x, y) = f(0, 0) + [2 f_x(0,0) + y f_2(0,0)]$$

$$+ \frac{1}{2!}\left[(x-0)^2 f_{x^2}(0,0) + 2(x-0)(y-0)f_{xy}(0,0) + (y-0)^2 f_{y^2}(0,0)\right]$$

$$+\frac{1}{3!}\left[(x-0)^3 f_{x^3}(\theta x, \theta y) + 3x^2 y f_{x^2y}(\theta x, \theta y) + 3xy^2 f_{xy}(\theta x, \theta y) + (y-0)^3 f_{y^3}(\theta x, \theta y)\right]$$

Substitute all corresponding values into the above equation.

$$= 1 + [x(1) + y(1)] + \frac{1}{2!}\left[x^2(1) + 2xy(1) + y^2(1)\right] + \frac{1}{3!}\left[x^3 e^{\theta(x+y)} + 3x^2 y e^{\theta(x+y)} + 3xy^2 e^{\theta(x+y)} + y^3 e^{\theta(x+y)}\right]$$

$$= 1 + x + y + \frac{1}{2!} + \left[x^2 + 2xy + y^2\right] + \frac{1}{3!}\left[x^3 + 3x^2 y + 3xy^2 + y^3\right] e^{\theta(x+y)}$$

$$= 1 + (x+y) + \frac{1}{2!}(x+y)^2 + \frac{1}{3!}(x+y)^3 e^{\theta(x+y)}$$

$$\therefore e^{x+y} = 1 + (x+y) + \frac{1}{2!}(x+y)^2 + \frac{1}{3!}(x+y)^3 e^{\theta(x+y)}$$

---

**40. Expand $f(x, y) = e^x \cos y$ by Taylor's series in power of $x$ and $y$ such that it includes all terms upto third degree.**

*Sol. :*

Given that

$f(x, y) = e^x \cos y$

$\Rightarrow$ (a, b) = (0, 0)

$f(0, 0) = e^0 \cos \theta = 1$

$$f_x = \frac{\partial f}{\partial x} = e^x \cos y$$

$$\Rightarrow f_x(0, 0) = e^0 \cos \theta = 1$$

$$f_y = \frac{\partial f}{\partial y} = -e^x \sin y$$

$$\Rightarrow f_y(0, 0) = -e^0 \sin 0 = 0$$

$$f_{x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^x \cos y\right)$$

$$= e^x \cos y$$

$$\Rightarrow f_{x^2}(0, 0) = e^0 \cos 0 = 1$$

$$f_{y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-e^x \sin y)$$

$$= -e^x \cos y$$

$$\Rightarrow f_{y^2}(0,0) = e^0 \cos 0 = -1$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-e^x \sin y)$$

$$= -e^x \sin y$$

$$\Rightarrow f_{xy}(0,0) = -e^0 \sin 0 = 0$$

$$f_{x^3} = \frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y$$

$$\Rightarrow f_{x^3}(0,0) = e^0 \cos 0 = 1$$

$$f_{y^3} = \frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial y}(-e^x \cos y)$$

$$= e^x \sin y$$

$$f_{y^3}(0,0) = 0\, e^0 \sin 0 = 0$$

$$f_{x^2y} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y$$

$$\Rightarrow f_{x^2y}(0,0) = -e^0 \sin 0 = 0$$

$$f_{xy^2} = \frac{\partial^3 f}{\partial x\,\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)$$

$$= \frac{\partial}{\partial x}(-e^x \cos y)$$

$$\Rightarrow f_{xy^2}(0,0) = -e^0 \cos(0) = -1$$

By Taylor's Theorem at (0, 0)

$$f(x, y) = f(\theta, \theta) + \left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{x^2}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{y^2}(0,0)\right]$$
$$+ \frac{1}{3!}\left[x^2 f_{x^3}(0,0) + 3x^2 y f_{x^2y}(0,0) + 3xy^2 f_{xy^2}(0,0) + y^3 f_{y^3}(0,0)\right]$$

$$e^x \cos y = 1 + [x(1) + y(0)] + \frac{1}{2!}\left[x^2(1) + 2xy(0) + y^2(-1)\right] + \frac{1}{3!}\left[x^3(1) + 3x^2 y(0) + 3xy^2(-1) + y^3(0)\right]$$

$$= 1 + (x + y) + \frac{1}{2!}[x^2 - y^2] + \frac{1}{3!}\left[x^3 - 3xy^2\right]$$

$$\therefore \quad e^x \cos y = 1 + x + \frac{1}{2!}[x^2 - y^2] + \frac{1}{3!}[x^3 - 3xy^2]$$

## 2.5 Maxima and Minima of Functions of two Variables

**41. Define Maxima and Minima of functions of two variables.**

*Sol :*

Let f(x, y) be a function of two independent variables x, y such that it is continuous and finite for all values of x and y in the neighbourhood of their values a & b.

The values of f(a, b) is called maximum or minimum value of f(x, y) according as f(a + h, b + k).

* Condition for the existence of maxima or minima.

We know by Taylor's expansion in two variables, that

$$f(x + h, y + k) = f(x, y) + h\left(\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots$$

(or)

$$f(x + h,, y + k) - f(x, y) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \text{(terms of second and higher order)}$$

**42. Write Lagrange's condition for maximum and minimum values of a function of two variables.**

*Sol :*

If r,s,t denote the values of $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$

When x = a, y = b then supposing that the necessary condition for the maximum & minimum are satisfied.

i.e $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ when x = a, y = b

We can write $f(a + h, b + k) - f(a,b) = \frac{1}{2!}[rh^2 + 2shk + tk^2] + R$

Where R consists of terms of higher order of h and k.

- Lagrange's condition for minimum is $rt - s^2 > 0$, and $r > 0$
- Lagrange's condition for maximum is $rt - s^2 > 0$ and $r < 0$

But it $rt - s^2 < 0$. then there is neither a maximum nor a minimum.

**43. Write Working Rule to find the maximum or minimum value of f (x,y)**

*Sol :*

**Step 1 :** Let the given function be f(x,y)

find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ and equation them to zero.

Solve the equation $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ for x and y.

Let the solution be (a,b), (c, d)

**Step 2 :** Calculate $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at (a,b) and (c,d). Calculate $rt - s^2$ in each care.

i.e., at (a,b) & (c,d)

**Step 3 :** If $rt - s^2 > 0$ and $r < 0$ at (a,b) then f has a maximum value at x = (a,b)

If or at (c,d) if $rt - s^2 > 0$ and $r < 0$ then f has maximum value at (c,d)

**Step 4 :** If $rt - s^2 > 0$, and $r > 0$ at (a,b) then f has a minimum value at (a,b) or if $rt - s^2 > 0$ and $r > 0$ at (c,d) & has a minimum at (c,d).

**Step 5 :** If $rt - s^2 < 0$. at (a,b) then f has neither , maximum , nor minimum . Then (a,b) is saddle point.

If $rt - s^2 < 0$ at (c,d), then (c,d) is saddle point.

**Step 6 :** If $rt - s^2 = 0$, we can't decide whether f, has maximum or minimum for the investigation is needed.

**44. Define Stationary points and Extreme points.**

***Sol :***

Points at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ are called stationary points for the function f(x,y)

If it is a maximum or a minimum is known as an extreme point and the value of the function at an extreme point is known as an extreme value.

**45. Discuss the maximum or minimum value of u, when $u = x^3 + y^3 - 3axy$.**

***Sol :***

$u = x^3 + y^3 - 3axy$

$\frac{\partial u}{\partial x} = 3x^2 - 3ay$ ; $\frac{\partial u}{\partial y} = 3y^2 - 3ax$.

for a max or min of u, we must have

$\frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0$

$3x^2 - 3ay = 0 \Rightarrow x^2 - ay = 0$ ... (1)

$3y^2 - 3ax = 0 \Rightarrow y^2 - ax = 0$ ... (2)

Solve (1) & (2)

from $x^2 = ay \Rightarrow y = \frac{x^2}{a}$

Sub y in (2)

$\Rightarrow \left(\frac{x^2}{a}\right) - ax = 0$

$\Rightarrow \frac{x^4}{a^2} - ax = 0$

$x^4 - a^3x = 0$

$x(x^3 - a^3) = 0$

$x = 0, x = a$

Similarly y = 0, y = a.

Similarly y = 0, y = a.

Thus (0,0) & (a,a) are the stationary points of u.

Now, $r = \frac{\partial^2 u}{\partial x^2} = 6x \quad t = \frac{\partial^2 u}{\partial y^2} = 6y$

$s = \frac{\partial^2 u}{\partial x \partial y} = -3a$

for x = 0, y = 0 ⇒ r = 0, t = 0, s = – 3a.

$rt - s^2 = (0)(0) - (-3a)^2$

$= 9a^2 < 0$

⇒ u is neither maximum nor minimum at x = 0 & y = 0

Also x = a, y = b

r = 6a, s = – 3a, t = 6a

Now, $rt - s^2 = (6a)(6a) - (-3a^2)$

$= 36a^2 - 9a^2$

$= 27a^2 > 0$

Also r = 6a which is positive if a > 0

(i) u is maximum at x = a, y = a, if r < 0

(ii) u is minimum at x = 0, y = 0 if r > 0.

**46. Show that minimum value of $u = xy + (a^3/x) + (a^3/y)$ is $3a^2$.**

***Sol :***

$u = xy + (a^3/x) + (a^3/y)$

we have $\frac{\partial u}{\partial x} = y - \frac{a^3}{x^2}$ ; $\frac{\partial u}{\partial y} = x - \frac{a^3}{y^2}$

$r = \frac{\partial^2 u}{\partial x^2} = \frac{2a^3}{x^3}$ ;

$s = \frac{\partial^2 u}{\partial x \partial y} = 1$;

$t = \frac{\partial^2 u}{\partial y^2} = \frac{2a^3}{y^3}$

Now, for maximum or minimum we must have

$\frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0.$

So from $\frac{\partial u}{\partial x} = 0 \Rightarrow y - \frac{a^3}{x^2} = 0$

$x^2 y = a^3$ ... (1)

from $\frac{\partial u}{\partial y} = 0 \Rightarrow x - \frac{a^3}{y^2} = 0$

$y^2 x = a^3$ ... (2)

From (1) & (2)

we get $x^2 y = y^2 x$

$x^2 y - y^2 x = 0 \Rightarrow xy (x - y) = 0$

$x = 0, y = 0, \& x = y$

from (1) & (2) we see that $x = 0$ & $y = 0$. do not hold as it gives $a = 0$

Hence we must have $x = y$ & from (1) we get

$x^2 y = a^3 \Rightarrow x^2 . x = a^3$

$x^3 = a^3 \Rightarrow x = a$

At $x = y = a$ we have

$$r = 2\frac{a^3}{x^3} = 2\frac{a^3}{a^3} = 2$$

$$s = 1; t = \frac{2a^3}{y^3} = \frac{2a^3}{a^3} = 2$$

$rt - s^2 = (2)(2) - 1^2 = 3 > 0$

Also, $r = 2 > 0$

Hence there is minimum at $x = y = a$.

hence the minimum value of,

$$u = a.a + \frac{a^3}{a} + \frac{a^3}{a}$$

$$= a^2 + \frac{a^3}{a} + \frac{a^3}{a}$$

$= a^2 + a^2 + a^2$

$= 3a^2$

**47. Discuss the maximum or minimum value of u given by $u = x^3 y^2 (1 - x - y)$.**

*Sol :*

$u = x^3 y^2 (1 - x - y)$

$$\frac{\partial u}{\partial x} = 3x^2 y^2 (1 - x - y) + x^3 y^2 (-1)$$

$= 3x^2 y^2 - 3x^3 y^2 - 3x^2 y^3 - x^3 y^2$

$$\frac{\partial u}{\partial x} = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$$

$$\frac{\partial u}{\partial y} = 2x^3 y (1 - x - y) + x^3 y^2 (-1)$$

$= 2x^3 y - 2x^4 y - 2x^3 y^2 - x^3 y^2$

$$\frac{\partial u}{\partial y} = 2x^3 y - 2x^4 y - 3x^3 y^2$$

$$r = \frac{\partial^2 u}{\partial x^2} = 6xy^2 - 12x^2 y^2 - 6xy^3$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2x^3 - 2x^4 - 6x^3 y$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 6x^2 y - 8x^3 y - 9x^2 y^2$$

Now, for maximum or minimum we must have

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0$$

from $\frac{\partial u}{\partial x} = 0 \Rightarrow 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = 0$

$x^2 y^2 (3 - 4x - 3y) = 0$

hence we get $x = 0, y = 0$

$4x + 3y = 3$ ... (2)

Also from $\frac{\partial u}{\partial y} = 0$, we get

$2xy^3 - 2x^4 y - 3x^3 y^2 = 0$

$x^3 y (2 - 2x - 3xy) = 0$

$x = 0, y = 0$ and $2x + 3y = 2$ ... (3)

By solving (1) & (2)

$$\begin{array}{cccccc} & x & & y & & 1 \\ 3 & \times & -3 & 4 & & 3 \\ 3 & & -2 & 2 & & 3 \end{array}$$

$$\frac{x}{-6+9} = \frac{y}{-6+8} = \frac{1}{12-6}$$

$$\frac{x}{3} = \frac{y}{2} = \frac{1}{6}$$

$$x = \frac{3}{6};\ y = \frac{2}{6}$$

$$x = \frac{1}{2};\ y = \frac{1}{3}$$

Hence the solution are

$$x = 0,\ y = 0,\ x = \frac{1}{2},\ y = \frac{1}{3},$$

when

$$r = 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3$$

$$= 3\frac{1}{9} - 3\left(\frac{1}{9}\right) - 3\left(\frac{1}{27}\right)$$

$$= \frac{1}{3} - \frac{1}{3} - \frac{1}{9} = -\frac{1}{9}$$

$$S = 6\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right) - 8\left(\frac{1}{3}\right)^3\left(\frac{1}{2}\right) - 9\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2$$

$$= 2\cdot\frac{1}{\not{4}_2} - 4\cdot\frac{1}{\not{8}_2} - \frac{1}{4} = \frac{1}{2} - \frac{1}{2} - \frac{1}{4} = -\frac{1}{4}$$

$$t = 2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{3}\right)^4 - 6\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} - \frac{2}{81} - \frac{1}{4} = \frac{-2}{81}$$

from there we have $rt - s^2 > 0$ and $r < 0$

So, there is maximum at $x = \frac{1}{2}$, $y = \frac{1}{3}$

**48. Find a point within a triangle such that the sum of the square of its distance from the three vertices is a minimum.**

*Sol :*

Let $(x_r, y_r)$, r = 1,2,3 be the vertices of the triangle and (x,y) be any point side the triangle.

Let $u = \sum_{r=1}^{3}\left[(x-x_r)^2 + (y-y_r)^2\right]$

For maximum or minimum of u, we have

$$\frac{\partial u}{\partial x} = \Sigma 2\,(x - x_r) = 0$$

$$(x - x_1) + (x - x_2) + (x - x_3) = 0$$

$$x - x_1 + x - x_2 + x - x_3 = 0$$

$$3x = x_1 + x_2 + x_3$$

$$x = \frac{x_1 + x_2 + x_3}{3}$$

Similarly $\frac{\partial u}{\partial y} = 0 \Rightarrow \Sigma 2(y - y_r) = 0$

$$(y - y_1) + (y - y_2) + (y - y_3) = 0$$

$$y - y_1 + y - y_2 + y - y_3 = 0$$

$$3y = y_1 + y_2 + y_3$$

$$y = \frac{y_1 + y_2 + y_3}{3}$$

$$r = \frac{\partial^2 u}{\partial x^2} = 6$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 0,\ t = \frac{\partial^2 u}{\partial y^2} = 6$$

so that $rt - s^2 \Rightarrow (6)(6) - 0 = 36 > 0$

Hence u is minimum when

$$x = \frac{x_1 + x_2 + x_3}{3},\ y = \frac{y_1 + y_2 + y_3}{3}$$

Thus, the required point is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

**49. Find the maximum value of (ax + by + cz) $e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$ .**

*Sol :*

u = (ax + by + cz) $e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$

log u = log (ax + by + cz) – $(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)$

Differentiating partially w.r.to 'x'

We get

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{a}{ax+by+cz}(-2\alpha^2 x) = 0$$

Similarly $\frac{1}{u}\frac{\partial u}{\partial y} = \frac{b}{ax+by+cz} - 2\beta^2 y = 0$

$$\frac{1}{u}\frac{\partial u}{\partial z} = \frac{c}{ax+by+cz} - 2\gamma^2 \beta = 0$$

$$x(ax + by + cz) = \frac{a}{2\alpha^2} \quad \text{... (1)}$$

$$y(ax + by + cz) = \frac{b}{2\beta^2} \quad \text{... (2)}$$

$$z(ax + by + cz) = \frac{c}{2\gamma^2} \quad \text{... (3)}$$

Multiplying (1), (2), (3) by a,b,c and adding. we get

$$(ax + by + cz)^2 = \frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)$$

$$(ax + by + cz) = \sqrt{\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)} = A$$

$$x = \frac{a}{2\alpha^2 A},\ y = \frac{b}{2\beta^2 A},\ z = \frac{c}{2\gamma^2 A}$$

Again $\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{a^2}{(ax+by+cz)^2} - 2\alpha^2$

$$\frac{\partial^2 u}{\partial x^2} = -u\left[\frac{a^2}{(ax+by+cz)^2} + 2\alpha^2\right]$$

Since $\frac{\partial u}{\partial \alpha} = 0$

Hence for these values of x, y, z will be maximum, maximum value of u is given by

$$= \sqrt{\left[\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right]e^{\frac{-1}{4a^2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)}}$$

$$= \sqrt{\left[\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right]e^{1/2}}$$

$$\therefore \quad U = \sqrt{\frac{1}{2e}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)}$$

**50. Discuss the maxima or minima of u = $x^4 + y^4 - 2x^2 + 4xy - 2y^2$.**

*Sol :*

Given $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y$$

$$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y$$

for max and min of 'u' we must have $\frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0$

$4x^3 - 4x + 4y = 0$ ... (1)

$4y^3 + 4x - 4y = 0$ ... (2)

Solve (1) & (2)

$x^3 + y^3 = 0$

$(x + y)(x^2 - xy + y^2) = 0$

$x + y = 0 \Rightarrow x = -y$

Substituting y = – x in (1) we get

$4x^3 - 8x = 0 \Rightarrow 4x(x^2 - 2x) = 0$

$x = 0, \pm\sqrt{2}$

The values of y are 0, $-\sqrt{2}$, $\sqrt{2}$

Therefore the stationary points are (0,0), $(\sqrt{2},\sqrt{2})$, $(-\sqrt{2},-\sqrt{2})$

Now, $r = \dfrac{\partial^2 u}{\partial x^2} = 12x^2 - 4$

$s = \dfrac{\partial^2 u}{\partial x \partial y} = 4$

$t = \dfrac{\partial^2 u}{\partial y^2} = 12y^2 - 4$

At (0,0) = $rt - s^2$

$= (12\ (0) - 4)\ (12\ (0) - 4) - 16$

$= (-4)\ (-4) - 16$

$= 16 - 16 = 0$

i.e, no conclusion can be drawn about max or min.

At $(\sqrt{2},-\sqrt{2})$

$rt - s^2 = (12(2)-4)\ (12(2) - 4) - (4)^2$

$= 400 - 16$

$= 384 > 0$

$rt - s^2 > 0,\ r < 0.$

$\therefore$ f has minimum value at $(\sqrt{2},\sqrt{2})$, $(-\sqrt{2},-\sqrt{2})$

Now, $r = \dfrac{\partial^2 u}{\partial x^2} = 12x^2 - 4$

$s = \dfrac{\partial^2 u}{\partial x \partial y} = 4$

$t = \dfrac{\partial^2 u}{\partial y^2} = 12y^2 - 4$

At (0,0)

$rt - s^2 = (12(0) - 4)\ (12\ (0) - 4) - 16$

$= 0$

i.e., no. conclusion can be shown about max or min.

At $(\sqrt{2},-\sqrt{2})$

$rt - s^2\ (2) - 4)\ (12(2) - 4) - (4)^2$

$400 - 16$

$384 > 0$

$rt - s^2 > 0,\ r > 0$

$\therefore$ f has minimum value at $(\sqrt{2},-\sqrt{2})$.

**51. Discuss the maximum and minimum of $x^4 + 2x^2y - x^2 + 3y^2$.**

*Sol :*

Let $u = x^4 + 2x^2y - x^2 + 3y^2$

$\dfrac{\partial u}{\partial x} = 4x^3 + 4xy - 2x$

$\dfrac{\partial u}{\partial y} = 2x^2 + 6y$

for max and min value of 'u'

we must have $\dfrac{\partial u}{\partial x} = 0$, and $\dfrac{\partial u}{\partial y} = 0$

$4x^3 + 4xy - 2x = 0$

$\Rightarrow\ x\ (4x + 4y - 2) = 0$ ... (2)

$2x^2 + 6y = 0$ ... (2)

Solving (1) & (2)

$x = \dfrac{\sqrt{3}}{2}, \dfrac{-\sqrt{3}}{2},\ y = \dfrac{1}{4}, \dfrac{-1}{4}$

The stationary points are $\left(\dfrac{\sqrt{3}}{2}, \dfrac{-1}{4}\right)$ & $\left(\dfrac{-\sqrt{3}}{2}, \dfrac{-1}{4}\right)$.

Now, $r = \dfrac{\partial^2 u}{\partial x^2} = 12x^2 - 4y - 2$

$s = \dfrac{\partial^2 u}{\partial x \partial y} = 4x,\ t = \dfrac{\partial^2 u}{\partial y^2} = 6$

At $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$

$$rt - s^2 = 12\left(\frac{\sqrt{3}}{2}\right)^2 + 4\left(\frac{-1}{4}\right) - 2 - \left[4\left(\frac{\sqrt{3}}{2}\right)\right]^2 = 33 > 0$$

$rt - s^2 > 0, r > 0$

$\therefore$ f has a minimum values at $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$

At $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$

$$rt - s^2 = \left[12\left(\frac{\sqrt{3}}{2}\right)^2 + 4\left(\frac{-1}{4}\right) - 2 - \left[4\left(\frac{\sqrt{3}}{2}\right)\right]\right]^2$$

$$r = \frac{\partial^2 u}{\partial x^2} = 12x^2 - 4y - 2$$

$$= 12\left(\frac{\sqrt{3}}{2}\right)^2 + 4\left(-\frac{1}{4}\right) - 2 = 8 > 0$$

$rt - s^2 > 0, r > 0$

$\therefore$ f has a minimum value at $\left(\frac{-\sqrt{3}}{2}, \frac{-1}{4}\right)$.

**52. Find the minimum & maximum values of the function z = sinx siny sin (x + y).**

*Sol :*

We have z = sinx siny sin(x + y)

$$\frac{\partial z}{\partial x} = \sin y\,[\sin x \cos(x + y) + \cos x \sin(x + y)]$$

$$\frac{\partial z}{\partial y} = \sin x\,[\sin y \cos(x + y) + \cos y \sin(x + y)]$$

for a maxima & minima of 'z'
we must have

$$\frac{\partial z}{\partial x} = 0,\ \frac{\partial z}{\partial y} = 0$$

siny [ sinx cos (x + y) + cosx sin(x + y)] = 0 ... (1)

sin x [siny cos(x + y) cosy sin(x + y)] = 0 ... (2)

from equation (1) & (2)

$\tan(x + y) = -\tan x$

$\tan(x + y) = -\tan y - \tan x = \tan y \Rightarrow x = y$

from (1) & (2)

we have $\tan 2x = -\tan x = \tan(\pi - x)$

$2x = \pi - x$

$3x = \pi$

$$x = \frac{\pi}{3} = y$$

$\therefore$ Siny = 0 $\Rightarrow$ Siny = Sin0

$\Rightarrow y = 0$

Sinx = 0 $\Rightarrow x = 0$

$\therefore$ The stationary points are (0,0) & $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

Now, $r = \frac{\partial^2 z}{\partial x^2} = \sin y\,[\cos x \cos(x + y) - \sin x \sin(x + y)] + [(-\sin x)\sin(x+y) + \cos x \cos(x+y)]$

$= \sin y\,[\cos x \cos(x + y) - \sin x \sin(x + y)] + [\cos x \cos(x + y) - \sin x \sin(x + y)]$

$= 2 \sin y \cos(2x + y)$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \sin 2(x + y)$$

$$t = \frac{\partial^2 z}{\partial y^2} = 2 \sin x \cos(2y + x)$$

At (0,0) $rt - s^2 = 0 \Rightarrow r = 0$ is

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$rt - s^2 = \left(2\sin\frac{\pi}{3}\cos\pi\right)\left(2\sin\frac{\pi}{3}\cos\pi\right) - \left(\sin\frac{4\pi}{3}\right)^2$$

$$= \left(-\sqrt{3}\right)\left(-\sqrt{3}\right) - \frac{3}{4} = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

$rt - s^2 > 0$, $r = -\sqrt{3} < 0$

$\therefore$ The function z has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

**53. If x, y and z are angles of triangle, then find the maximum value of sin x siny sin z.**

*Sol :*

We have sinx siny sin z

$x + y + z = \pi$

$= \sin x \sin y \sin (\pi - (x = y))$

$= \sin x \sin y \sin (x + y)$

$\therefore \quad f(x, y) = \sin x \sin y \sin(x + y).$

---

**54. Find the three positive numbers whose sum is 30 and whose product is maximum.**

*Sol :*

Let three positive number x, y & z

$x + y + z = 30, f(x, y,z) = xyz$

$z = 30 - x - y$

$f(x,y) = xy (30 - x - y)$

$= 30 xy - x^2y - xy^2$

$$\frac{\partial f}{\partial x} = 30 y - 2xy - y^2 , \frac{\partial f}{\partial y} = 30x - x^2 - 2xy$$

for a max & min value of 'f' must have

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$$

$y(30 - 2x - y) = 0$ ... (1)

$x (30 - x - 2y) = 0$ ... (2)

Solving (1) , (2)

$x = 10, y = 10$

$x = 10, y = 10, z = 0$

$$r = \frac{\partial^2 f}{\partial x^2} = -2y, s = \frac{\partial^2 z}{\partial x \partial y} = 30 - x^2 - 2y$$

$t = -2z$

at $x = y = z = 10$

$r = -20 < 0$

$rt - s^2 > 0$

$\therefore$ f is maximum.

$\therefore$ The product is maximum, when all positive three numbers are equal.

**55. Examine for maximum and minima values of the function $Z = x^2 - 3xy + y^2 + 2x$.**

*Sol. :*

Given,

$$z = x^2 - 3xy + y^2 + 2x$$

$$\frac{\partial z}{\partial x} = 2x - 3y + 2$$

$$\frac{\partial z}{\partial y} = -3x + 2y$$

For maximum or minima value of $z$ we have $\frac{\partial z}{\partial y} = 0, \ \frac{\partial z}{\partial y} = 0$

$$2x - 3y + 2 = 0 \Rightarrow 2x - 3y = -2 \quad ...(1)$$

$$-3x - 2y = 0 \Rightarrow -3x + 2y = 0 \quad ... (2)$$

Solving (1), (2)

$$6x - 9 = -6$$

$$-6x + 4y = 0$$

$$-5y = -6$$

$$y = \frac{6}{5}$$

sub $y = \frac{6}{5}$ in (2)

$$-3x + 2\left(\frac{6}{5}\right) = 0$$

$$-3x = \frac{-12}{5}$$

$$-x = \frac{-12}{5} = \frac{4}{5} \qquad \therefore \ x = \frac{4}{5}$$

**56. Find a point within a triangle such that the sum of the squares of its distances from the angular points may be maximum.**

*Sol. :*

Let A $(x_1, y_1)$ B$(x_2, y_2)$, C$(x_3, y_3)$ be the vertices of the triangle.

Let $(x, y)$ be an arbitrary point.

The sum of squares of the distance from (x, y) and the three vertices of the triangle is given by

$$f(x, y) = (AP)^2 + (BP)^2 + (CP)^2$$

$$f(x, y) = (x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 + (x - x_3)^2 + (y - y_3)^2$$

$$\frac{\partial f}{\partial x} = 2(x-x_1) + 2(x - x_2) + 2(x - x_3)$$

$$\frac{\partial f}{\partial y} = 2(y - y_1) + 2(y - y_2) + 2(y - y_3)$$

For a maximum and minima values of ‘f’ we must have

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$2x - 2x_1 + 2x - 2x_2 + 2x - 2x_3 = 0$$

$$6x - (2x_1 + 2x_2 + 2x_3) = 0$$

$$3x - (x_1 + x_2 + x_3) = 0$$

$$x = \frac{x_1 + x_2 + x_3}{3}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y - 2y_1 + 2y - 2y_2 + 2y - 2y_3 = 0$$

$$6y - (2y_1 + 2y_2 + 2y_3) = 0$$

$$3y - (y_1 + y_2 + y_3) = 0$$

$$y = \frac{y_1 + y_2 + y_3}{3}$$

Now

$$r = \frac{\partial^2 f}{\partial x^2} = 2 + 2 + 2 = 6$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = 0$$

$$rt - s^2 = (6)(6) - 0 = 36 > 0, \quad r > 0$$

f is minium when $x = \frac{x_1 + x_2 + x_3}{y_3}$

$$y = \frac{y_1 + y_2 + y_3}{3}$$

**57. Show that the maxima and minima of the function** $\dfrac{ax^2+by^2+c+2hxy+2gx+2fy}{a'x^2+b'y^2+c'+2h'xy+2g'x+2f'y}$ **are given by are given by the roots of the equation.**

$$\begin{vmatrix} a-a'u & h-h'u & g-g'u \\ h-h'u & b-b'u & f-f'u \\ g-g'u & f-f'u & c-c'u \end{vmatrix} = 0$$

*Sol :*

We have

$$\frac{ax^2+by^2+c+2hxy+2gx+2fy}{ax^2+b^1y^2+c^1+2h'xy+2g'x+2f'y}$$

$$u(a'x^2+b'y^2+c'+2h'xy+2g'x+2f'y) = ax^2+by^2+c = 2hxy + 2gx + 2fy \quad \text{... (1)}$$

Differentiating partially w.r.t to 'x' and 'y'

Then we have

$$\frac{\partial u}{\partial x}(a'x^2+b'y^2+c'+2h'xy+2g'x+2f'y) + u(2a'x+2h'y+2g') = 2ax + 2hy + 2g \quad \text{... (2)}$$

$$\frac{\partial u}{\partial y}(a'x^2+b'y^2+c'+2h'xy+2g'x+2f'y) + u(2b'y+2h'x+2f') = 2by + 2hx + 2f \quad \text{... (3)}$$

For maxima and minima of 'u' we must have

$$\frac{\partial u}{\partial x}=0 \quad \Rightarrow \quad u(a'x+h'y+g') = ax+hy+g \quad \text{... (4)}$$

$$\frac{\partial u}{\partial y}=0 \quad \Rightarrow \quad u(h'x+b'y+f') = hx + by + f \quad \text{... (5)}$$

Equation (4) x r + equation (5) x y,

$$u(a'x^2+h'xy+g'x+h'xy+b'y^2+f'y) = ax^2+by^2+2hxy+gx+fy$$

$$u(a'x^2+2h'xy+g'x+b'y^2+f'y) = ax^2+by^2+2hxy+gx+fy \quad \text{... (6)}$$

Sub equation (6) from equation (1),

Then, we get

$$u(g'x+f'y+c') = gx + fy + c \quad \text{... (7)}$$

Now, from (4), (5) & (7)

$$(a-a'u)x+(h-h'u)y+(g-g'u)=0 \quad \text{... (8)}$$

$$(h-h'u)x+(b-b'u)yf(f-f'u)=0 \quad \text{... (9)}$$

$$(g-g'u)x+(f-f'u)y+(c-c'u)=0 \quad \text{... (10)}$$

Eliminating x and y (8), (9), (10)

$$\begin{vmatrix} a-a'u & h-h'u & g-g'u \\ h-h'u & b-b'u & f-f'u \\ g-g'u & f-f'u & c-c'u \end{vmatrix} = 0$$

This is cubic equation in 'u'.

## 2.6 Lagrange's Method of Undetermined Multipliers

**58. Write working rule of Lagrange's Method of Undetermined Multiplier.**

*Sol. :*

Suppose f(x, y, z) is a function of three variables x, y, z which are connected by the relation.

$\phi(x, y, z) = 0$

'z' value from (2) can be solved and substituted in (1). The maximum or minimum of f can be formed by

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0,$$

We use lagrange's method

Writing $u = f(x, y, z)$ ... (1)

Given that $\phi(x, y, z) = 0$ ... (2)

Differentiate partially with respect to x and y

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}$$

For 'u' to have maximum or minimum

$$\frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \qquad ... (3)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = 0 \qquad ... (4)$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) = 0 \qquad ... (5)$$

From (2)

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad ... (6)$$

Multiplying (6) by 'λ' and add (5)

Then we have

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right)dz$$

Equation (7) will be satisfied if

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \qquad \text{... (8)}$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0 \qquad \text{... (9)}$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \qquad \text{... (10)}$$

The value of x, y, z which gives extremum of 'u' satisfy (2), (3), (4) and also satisfy (5) & (6), (7) (8), (8) & (10).

Using the equation (8), (9), (10) & (2) x, y, z, λ can be solved.

Here, 'λ' is called a Legrange's multipliers.

---

**59. Determine the maximum and minima of $x^2 + y^2 + z^2$ when $ax^2 + by^2 + cz^2 = 1$.**

*Sol. :*

Let $U = x^2 + y^2 + z^2$ ... (1)

$\phi = ax^2 + by^2 + cz^2 - 1 = 0$ ... (2)

The conditions are

$$\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0, \qquad \text{... (3)}$$

$$\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial x}=0 \qquad \text{... (4)}$$

$$\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \qquad \text{... (5)}$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial \phi}{\partial x} = 2\,ax$$

$$\frac{\partial u}{\partial y} = 2y, \quad \frac{\partial \phi}{\partial y} = 2\,by$$

$$\frac{\partial u}{\partial z} = 2z, \quad \frac{\partial \phi}{\partial z} = 2cz$$

By (3) $2x + \lambda\,(2ax) = 0$ ... (6)

By (4) $2y + \lambda\,(2by) = 0$ ... (7)

By (5) $2z + \lambda\,(2cz) = 0$ ... (8)

Consider

$$x\left(\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x}\right)+y\left(\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}\right)+z\left(\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right)=0$$

$$= x\left(2x+\lambda(2ax)\right)+y\left(2y+\lambda(2by)\right)+z\left(2z+\lambda(2cz)\right)=0$$

$$2x^2+\lambda(2ax^2)+2y^2+\lambda(2by^2)+2z^2+\lambda(2cz^2)=0$$

$$2(x^2+y^2+z^2)+2\lambda\ (ax^2+by^2+cz^2)=0$$

$$2u+2\lambda(1)=0$$

$$2u=-2\lambda$$

$$u=-\lambda \Rightarrow \lambda=-u$$

From (6) $2x - u\,(2ax) = 0$

$x - uax = 0$

$x\,(1-ua) = 0$

$1 - ua = 0 \Rightarrow u=\frac{1}{a}$

Similarly, $u=\frac{1}{b},\ u=\frac{1}{c},$

$\therefore\ u=\frac{1}{a},\frac{1}{b},\frac{1}{c}$

**60. Find the maximum and minima of $x^2 + y^2 + z^2$ subject to $ax + by + cz = p$.**

*Sol. :*

Suppose $f(x, y, z) = x^2 + y^2 + z^2$ ... (1)

$\phi(x, y, z) = ax + by + cz = p$ ... (2)

Suppose '$\lambda$' is the Lagrange's multiplies

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0$$

$\Rightarrow 2x + \lambda a = 0$ ... (3)

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0$$

$\Rightarrow 2y + \lambda b = 0$ ... (4)

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$$

$\Rightarrow 2z + \lambda c = 0$ ... (5)

From (3) $x = \frac{-\lambda a}{2} \Rightarrow \frac{x}{a} = \frac{-\lambda}{2}$

From (4) $y = \frac{-\lambda b}{2} \Rightarrow \frac{y}{b} = \frac{-\lambda}{2}$

From (5) $z = \frac{-\lambda c}{2} \Rightarrow \frac{z}{2} = \frac{-\lambda}{2}$

$\therefore \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{-\lambda}{2}$

From (1)

$$\frac{ax+by+cz}{a^2+b^2+c^2} = \frac{p}{a^2+b^2+z^2}$$

$$x = \frac{pa}{a^2+b^2+c^2},\ y = \frac{Pb}{a^2+b^2+c^2},\ z = \frac{Pc}{a^2+b^2+c^2}$$

Substitute in (1), we get

$$\frac{P^2a^2+P^2b^2+P^2c^2}{(a^2+b^2+c^2)^2} = \frac{P^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}$$

$\therefore$ Extremium of $x^2 + y^2 + z^2$ is $\frac{P^2}{a^2+b^2+c^2}$

**61. Determine the minima value of $x^2 + y^2 + z^2$ subject to the condition $x + 2y - 4z = 5$**

*Sol. :*

Suppose $f(x, y, z) = u = x^2 + y^2 + z^2$ ... (1)

$\phi(x, y, z) = x + 2y - 4z - 5 = 0$ ... (2)

Suppose $\lambda$ is the lagrenage's multipliers

$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 2r + \lambda(1) = 0$ ... (3)

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda 2 = 0$ ... (4)

$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z - 4\lambda = 0$ ... (5)

Multiply (3) by x, (4) by y, and (5) by z and adding we get

$$x(2x+\lambda)+y(2y+2\lambda)+z(2z-4\lambda)=0$$

$2x^2 + 2\lambda + 2y^2 + 2\lambda y + 2z^2 - 4z\lambda = 0$

$2(x^2 + y^2 + z^2) + \lambda(x + 2y - 4z) = 0$

$2u + 5\lambda = 0$

$\lambda = \frac{-2u}{5}$

From (3) $2x + \lambda = 0 \Rightarrow x = -\frac{\lambda}{2} \quad \frac{-\left(\frac{-2u}{5}\right)}{2} = \frac{u}{2}$

From (4) $2y + 2\lambda = 0 \Rightarrow y = -\lambda = -\left(\frac{-2u}{5}\right) = \frac{2u}{5}$

From (5) $2z - 4\lambda = 0 \Rightarrow 2z = 4\lambda$

$z = 2\lambda \Rightarrow 2\left(\frac{-2u}{5}\right) = \frac{-44}{5}$

From

$u = x^2 + y^2 + z^2 \Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{-4}{5}\right)^2$

$= \frac{1}{4} + \frac{4}{25} + \frac{16}{25}$

$u = \frac{25}{21}$

$\therefore$ Minimum value of $x^2 + y^2 + z^2$ is $\frac{25}{21}$.

**62. Find the minima value of $x^2 + y^2 + z^2$ when $yz + zx + xy = 3a^2$.**

*Sol. :*

$f(x, y, z)\ u = x^2 + y^2 + z^2$ ... (1)

$yz + zx + xy = 3a^2$ ... (2)

$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + \lambda(z + y) = 0$ ... (3)

$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow y + \lambda(x + z) = 0$ ... (4)

$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow z + \lambda(y + x) = 0$ ... (5)

Multiplying (3) by x (4) by y and (5) by z, and adding we get

$x(x + \lambda(z + y)) + y(y + \lambda(x + z)) + z(z + \lambda(y + x)) = 0$

$x^2 + \lambda xz + \lambda xy + y^2 + \lambda xy + \lambda yz + z^2 + \lambda zy + \lambda zx = 0$

$$(x^2+y^2+z^2)+\lambda(2xy+2zx+2xy)=0$$

$$u+\lambda 2(3a^2)=0$$

$$u+6\lambda a^2=0$$

$$6\lambda a^2=-u$$

$$\lambda = \frac{-u}{6a^2}$$

From (3), (4) & (5)

$$x-\frac{u}{6a^2}(y+z) = 0 \;\Rightarrow\; 6a^2x-uy-uz=0 \qquad ...(6)$$

$$y-\frac{u}{6a^2}(x+z)=0 \;\Rightarrow\; 6a^2y-ux-uz=0 \qquad ...(7)$$

$$z-\frac{u}{6a^2}(x+y) = 0 \;\Rightarrow\; 6a^2z-ux-uy=0 \qquad ...(8)$$

By Solving (6), (7) & (8)

$x = z = y = a$

From (1), maximum value of $u = a^2 + a^2 + a^2 = 3a^2$.

**63. In a plane triangle find the maximum value of u = cos A cos B cos C.**

*Sol. :*

Let f = cos A cos B cos C

In $\Delta ABC = A + B + C = \pi$

$C = \pi - (A+B)$

$\text{Cos } C = \cos(\pi - (A+B))$

$= -\cos(A + B)$

$f = -\cos A \cos B \cos(A+B)$

$$\frac{\partial f}{\partial A} = \sin A \cos B \cos(A+B) + \cos A \cos B \sin(A+B)$$

$= \cos B (\sin A \cos(A+B) + \cos A \sin(A+B)$

$= \cos B \sin(2A+B)$

$$\frac{\partial f}{\partial B} = \cos A \sin B \cos(A+B) + \cos A \cos B \sin(A+B)$$

$= \cos A (\sin B \cos(A+B)) + \cos B \sin(A+B)$

$= \cos A \sin(A + 2B)$

The condition for f to have maximum is

$$\frac{\partial f}{\partial A} = 0 \Rightarrow \cos B \sin(2A + B) = 0$$

$$\frac{\partial f}{\partial B} = 0 \Rightarrow \cos A \sin(A + 2B) = 0$$

$\cos B \sin(2A+B) = 0 \Rightarrow \cos B = 0$

$$B = \frac{\pi}{2}$$

$\sin(2A + B) = 0 \quad \Rightarrow \quad 2A + B = \pi$

$\cos A \sin(A + 2B) = 0 \Rightarrow \quad A+2B = \pi \ (\because \cos A \neq 0)$

But $\frac{\pi}{2}$ not possible since $A+2B > \pi$

$2A + b = \pi$

$A + 2B = \pi$

$$A = \frac{\pi}{3} = B$$

$$r = \frac{\partial^2 f}{\partial A^2} = \cos B \cos(2A + B)$$

$$= 2\cos\frac{\pi}{2}\cos\pi = -1$$

$$S = \frac{\partial^2 f}{\partial A \partial B} = \frac{-1}{2}$$

$$t = \frac{\partial^2 f}{\partial B^2} = 1$$

$$rt - s^2 = 1 + \frac{1}{4} > 0, \ r < 0$$

$\therefore$ f is maximum at $A = \frac{\pi}{3} = B$

$$C = \pi - (A + B) = \frac{\pi}{3}$$

$$A = B = C = \frac{\pi}{3}$$

$$U = \cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{\pi}{3} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$U = \frac{1}{8}$$

$\therefore$ The maximum value of $U = \frac{1}{8}$

**64. If U = $x^2 + y^2 + z^2$ where $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$ find the maximum value of u.**

*Sol. ;*

Let $u = x^2 + y^2 + z^2$ ... (1)

$\phi = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - 1 = 0$ ... (2)

For stationary values, the conditions are

$$\Rightarrow \frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0;$$

$2x + \lambda\,(2ax + 2gz + 2hy)$ ... (3)

$$\Rightarrow \frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$2y + \lambda\,(2by + 2fz + 2hx) = 0$ ... (4)

$$\Rightarrow \frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

$2z + \lambda\,(2fy + 2gz + 2cz) = 0$ ... (5)

Multiply (3) by x, (4) by y (5) by z and adding,

$x\,[2x + \lambda(2ax + 2gz + 2hy)] + y\,[2y + \lambda\,(2by + 2fz + 2hx)] + z\,[2z + \lambda\,(2fy + 2gz + 2cz)] = 0$

$$2(x^2 + y^2 + z^2) + 2\lambda\;(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0$$

$$2u + 2\lambda(1) = 0$$

$u = -\lambda$

$\boxed{\lambda = -u}$

From (3), (4) & (5) we get

$(1 - ua)\,x - ugz - uhy = 0$

$$-u\left[\left(a - \frac{1}{u}\right)x + hy + gz\right] = 0 \qquad \text{... (6)}$$

Similarly $hx + \left(b - \frac{1}{u}\right)y + fz = 0$ ... (7)

$$gx + fy + \left(c - \frac{1}{u}\right)z = 0 \qquad \text{... (8)}$$

Eliminate x, y, and z from (6), (7) & (8)

$$\begin{vmatrix} a - \frac{1}{u} & h & g \\ h & b - \frac{1}{u} & f \\ g & f & c - \frac{1}{u} \end{vmatrix} = 0$$

**65. If two variables x and y are connected by the relation $ax^2 + by^2 = ab$. Show that the maximum and minimum values of the function $x^2 + y^2 + xy$ will be the values of u given by the equation $4(u-a)(u-b) = ab$.**

*Sol. :*

$$U = x^2 + y^2 + xy \quad \text{... (1)}$$

$$\phi = ax^2 + by^2 - ab = 0 \quad \text{... (2)}$$

For stationary values, the conditions are

$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0, \quad 2x + y + \lambda\,(2ax) = 0 \quad \text{... (3)}$$

$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0, \quad 2y + x + \lambda\,(2by) = 0 \quad \text{... (4)}$$

Now, (3) x $x$ + (4) × y, gives

$$x(2x + y + \lambda(2ax)) + y(2y + x + \lambda(2by)) = 0$$

$$2x^2 + xy + \lambda\, 2ax^2 + 2y^2 + xy + \lambda\, 2by^2 = 0$$

$$2x^2 + 2y^2 + 2xy + 2\lambda(ax^2 + by^2) = 0$$

$$2(x^2 + y^2 + xy) + 2\lambda(ax^2 + by^2) = 0$$

$$2u + 2\lambda(ab) = 0$$

$$u + \lambda(ab) = 0$$

$$\lambda = -\frac{u}{ab}$$

From (3), $2x + y - \frac{2ux}{b} = 0$

$$2\left(1 - \frac{u}{b}\right)x + y = 0 \quad \text{... (5)}$$

From (4) $x + 2\left(1 - \frac{u}{b}\right)y = 0$ ... (6)

Equation (5) × $2\left(1 - \frac{u}{a}\right)$ equation ... (7)

$$4\left(1 - \frac{u}{b}\right)\left(1 - \frac{u}{a}\right)x - x = 0$$

$$\left[4\left(1 - \frac{u}{b}\right)\left(1 - \frac{u}{a}\right) - 1\right] x = 0$$

$$4(b - u)(a - u) - ab = 0 \quad x \neq 0$$

$$4(b - u)(a - u) = ab$$

**66. Find the point such that sum of squares of its distances form four faces of given tetrahedron shall be minimum.**

*Sol. :*

$$u(x, y) = (x-x_1)^2+(y-y_1)^2+(x-x_2)^2+(y-y_2)^2+(x-x_3)^2+(y-y_3)^2+(x-x_4)^2+(y-y_4)^2$$

$$\frac{\partial u}{\partial x} = 2(x-x_1)+2(x-x_2)+2(x-x_3)+2(x-x_4)$$

$$\frac{\partial u}{\partial y} = 2(y-y_1)+2(y-y_2)+2(y-y_3)+2(y-y_4)$$

Condition for 'f' be minimum is $\frac{\partial u}{\partial x}=0$ and $\frac{\partial u}{\partial y}=0$

$$\frac{\partial u}{\partial x}=0 \Rightarrow 4x-(x_1+x_2+x_3+x_4)=0$$

$$x = \frac{x_1+x_2+x_3+x_4}{4}$$

$$\frac{\partial u}{\partial y}=0 \Rightarrow 4y-(y_1+y_2+y_3+y_4)=0$$

$$y = \frac{y_1+y_2+y_3+y_4}{4}$$

$$r = \frac{\partial^2 u}{\partial x^2}=8,\quad S = \frac{\partial^2 u}{\partial x\,\partial y}=0,\quad t = \frac{\partial^2 u}{\partial y^2}=8$$

$rt - s^2 = (8)(8) - 0 = 64 > 0,\ r > 0$

u is minimum.

**67. Find the maxima and minima of $x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$.**

*Sol :*

Let $u = x^2 + y^2 + z^2$ ... (1)

and alsθ given condition $ax^2 + by^2 + cz^2 = 1$ ... (2)

and $lx + my + nz = 0$ ... (3)

from (1)

$\Rightarrow du = 2xdx + 2ydy + 2zdz$

$\Rightarrow xdx + ydy + zdz = 0$ ... (4)

by (2)

$\Rightarrow 2axdx + 2by\,dy + 2czdz = 0$

$axdx + bydy + czdz = 0$ ... (5)

by (3)

$\Rightarrow$ $l dx + mdy + ndz = 0$ ... (6)

Multiplying (4), (5), (6) by 1, $\lambda_1$, $\lambda_2$ and adding we get

$(xdx + ydy + zdz) + \lambda_1 (axdx + bydy + czdz) + \lambda_2 (l dx + mdy + ndz) = 0$

$xdx + ydy + zdz + \lambda_1 axdx + \lambda_1 bydy + \lambda_1 czdz + \lambda_2 l dx + \lambda_2 mdy + \lambda_2 ndz = 0$

$(x + \lambda_1 ax + \lambda_2 l) dx + (y + \lambda_1 by + \lambda_2 m) dy + (z + \lambda_1 cz + \lambda_2 n)dz = 0$

Equating to zero the coefficients of dx, dy & dz

we get

$x + a\lambda_1 x + l\lambda_2 = 0$ ... (7)

$y + b\lambda_1 y + m\lambda_2 = 0$ ... (8)

$z + c\lambda_1 z + n\lambda_2 = 0$ ... (9)

Multiplying (7), (8) & (9) by x, y, z and adding, we get

$x(x + a\lambda_1 x + l\lambda_2) + y(y + b\lambda_1 y + m\lambda_2) + z(z + c\lambda_1 z + n\lambda_2) = 0$

$x^2 + a\lambda_1 x^2 + lx\lambda_2 + y^2 + b\lambda_1 y^2 + my\lambda_2 + z^2 + c\lambda_1 z + zn\lambda_2 = 0$

$(x^2 + y^2 + z^2) + \lambda_1 (ax^2 + by^2 + cz^2) \lambda_2 (lx + my + nz) = 0$

$u + \lambda_1(1) + \lambda_2(0) = 0$

$u + \lambda_1 = 0$

$\lambda_1 = -u$

Sub $\lambda_1 = -u$ in (7)

$x + a(-u)x + l\lambda_2 = 0$

$x - aux + l\lambda_2 = 0$

$x(1 - au) + l\lambda_2 = 0$

$$x = \frac{-l\lambda_2}{1 - au}$$

$$x = \frac{l\lambda_2}{au - 1}$$

From (8) $y + b(-u)y + m\lambda_2 = 0$

$y - buy + m\lambda_2 = 0$

$y(1 - bu) = -m\lambda_2$

$$y = \frac{-m\lambda_2}{1-bu}$$

$$y = \frac{-m\lambda_2}{-(bu-1)}$$

$$y = \frac{m\lambda_2}{bu-1}$$

From (9) $z + c(-u)z + n\lambda_2 = 0$

$$z(1-cu) + n\lambda_2 = 0$$

$$z = \frac{-n\lambda_2}{1-cu}$$

$$z = \frac{-n\lambda_2}{-(cu-1)}$$

$$z = \frac{n\lambda_2}{cu-1}$$

Sub x, y, z in equation (3) we get

$$l\left(\frac{l\lambda_2}{au-1}\right) + m\left(\frac{m\lambda_2}{bu-1}\right) + n\left(\frac{n\lambda_2}{cu-1}\right) = 0$$

$$\frac{l^2\lambda_2}{au-1} + \frac{m^2\lambda_2}{bu-1} + \frac{n^2\lambda_2}{cu-1} = 0$$

$$\lambda_2\left[\frac{l^2}{au-1} + \frac{m^2}{bu-1} + \frac{n^2}{cu-1}\right] = 0$$

$$\frac{l^2}{au-1} + \frac{m^2}{bu-1} + \frac{n^2}{cu-1} = 0.$$

**68. Show that the maximum and minimum of radii vectors of the section of the surface $(x^2+y^2+z^2)^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$ by the $\lambda x + \mu y + \nu z = 0$ are given by the equation $\frac{a^2\lambda^2}{1-a^2r^2} + \frac{b^2\mu^2}{1-b^2r^2} + + \frac{c^2\nu^2}{1-c^2r^2} = 0$ we have to find the maximum value of radius vector's where $r^2 = x^2 + y^2 + z^2$.**

*Sol. :*

Given radius vector r is

$$r^2 = x^2 + y^2 + z^2 \quad \text{... (1)}$$

and vector of section of the surface

$$(r^2)^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

$$r^4 = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \quad \text{... (2)}$$

and also, $\lambda x + \mu y + \nu z = 0$ ... (3)

Differentiating there equations and putting the condition dr = 0

From (1) $\Rightarrow$ 0 = 2x dx + 2y dy + 2z dz

$$x dx + y\,dy + z\,dz = 0 \qquad \text{... (4)}$$

From (2) $\Rightarrow$ $0 = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz$

$$\frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz = 0 \qquad \text{... (5)}$$

From (3) $\Rightarrow$ $\lambda\, dx + \mu\, dy + \nu\, dz = 0$ ... (6)

Multiplying (4), (5) & (6) by 1, $\lambda_1$, $\lambda_2$

Respectively and adding

Then we get

$$(x\,dx + y\,dy + z\,dz) + \lambda_1\left(\frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz\right) + \lambda_2(\lambda\, dx + \mu\, dy + \nu\, dz) = 0$$

$$x\,dx + y\,dy + z\,dz + \lambda_1\frac{x}{a^2} + \lambda_1\frac{y}{b^2}dy + \lambda_1\frac{z}{c^2}dz + \lambda_2\lambda\, dx + \lambda_2\mu\, dy + \lambda_2\nu\, dz = 0$$

$$\left(x + \lambda_1\frac{x}{a^2} + \lambda_2\lambda\right)dx + \left(y + \lambda_1\frac{y}{b^2} + \lambda_2\mu\right)dy + \left(z + \lambda_1\frac{z}{c^2} + \lambda_2\nu\right)dz = 0$$

Equating to zero the co-efficients of dx dy, dz, we get

$$x + \lambda_1\frac{x}{a^2} + \lambda_2\lambda = 0 \qquad \text{... (7)}$$

$$y + \lambda_1\frac{y}{b^2} + \lambda_2\mu = 0 \qquad \text{... (8)}$$

$$z + \lambda_1\frac{z}{c^2} + \lambda_2\nu = 0 \qquad \text{... (9)}$$

Multiplying (7), (8) & (9) by x, y, z and adding

$$x^2 + \lambda_1\frac{x^2}{a^2} + \lambda_2\lambda x + y^2 + \lambda_1\frac{y^2}{b^2} + \lambda_2\mu y + z^2 + \lambda_1\frac{z^2}{c^2} + \lambda_2\nu z = 0$$

$$(x^2 + y^2 + z^2) + \lambda_1\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \lambda_2\,(\lambda x + \mu y + \nu z) = 0$$

By (1), (2) & (3)

$$\Rightarrow \quad r^2 + \lambda_1(r^4) + \lambda_2(0) = 0$$

$$r^2 + r^4\lambda_1 = 0$$

$$r^4\lambda_1 = -r^2$$

$$\lambda_1 = \frac{-1}{r^2}$$

Sub $\lambda_1 = \frac{-1}{r^2}$ in (7), (8) & (9)

by (7) $\Rightarrow$ $x + \left(\frac{x}{a^2}\right)\left(\frac{-1}{r^2}\right) + \lambda\lambda_2 = 0$

$$x - \frac{x}{a^2r^2} + \lambda\lambda_2 = 0$$

$$x\left(1 - \frac{1}{a^2r^2}\right) + \lambda\lambda_2 = 0$$

$$x\left(1 - \frac{1}{a^2r^2}\right) = -\lambda\lambda_2$$

$$x = \frac{-\lambda\lambda_2}{\left(1 - \frac{1}{a^2r^2}\right)}$$

$$= \frac{-a^2r^2\lambda\lambda_2}{a^2r^2 - 1}$$

$$x = \frac{-a^2r^2\lambda\lambda_2}{1 - a^2r^2}$$

by (8)

$$y + \frac{y}{b^2}\left(\frac{-1}{r^2}\right) + \mu\lambda_2 = 0$$

$$y - \frac{y}{b^2r^2} + \mu\lambda_2 = 0$$

$$y\left(1 - \frac{y}{b^2r^2}\right) + \mu\lambda_2 = 0$$

$$y = \frac{-\mu\lambda_2}{1 - \frac{1}{b^2r^2}}$$

$$= \frac{-\mu\lambda_2}{\frac{b^2r^2 - 1}{b^2r^2}} = \frac{-b^2r^2\mu\lambda_2}{b^2r^2 - 1}$$

$$y = \frac{-b^2r^2\mu\lambda_2}{-(1 - b^2r^2)}$$

$$y = \frac{b^2r^2\mu\lambda_2}{1 - b^2r^2}$$

by (9)

$$z + \frac{z}{c^2}\left(\frac{-1}{r^2}\right) + \nu\lambda_2 = 0$$

$$z - \frac{z}{c^2r^2} + \nu\lambda_2 = 0$$

$$z\left(1 - \frac{1}{c^2r^2}\right) + \nu\lambda_2 = 0$$

$$z = \frac{-\nu\lambda_2}{1 - \frac{1}{c^2r^2}}$$

$$= \frac{-c^2r^2\nu\lambda_2}{c^2r^2 - 1}$$

$$= \frac{-c^2r^2\nu\lambda_2}{-(1 - c^2r^2)}$$

$$z = \frac{c^2r^2\nu\lambda_2}{1 - c^2r^2}$$

Sub, x, y & z in (3)

i.e. $\lambda x + \mu y + \nu z = 0$

$$\Rightarrow \lambda\left(\frac{a^2r^2\lambda\lambda_2}{1 - a^2r^2}\right) + \mu\left(\frac{b^2r^2\mu\lambda_2}{1 - b^2r^2}\right) + \nu\left(\frac{c^2r^2\nu\lambda_2}{1 - c^2r^2}\right)$$

$$\Rightarrow \frac{a^2r^2\lambda^2\lambda_2}{1 - a^2r^2} + \frac{b^2r^2\mu^2\lambda_2}{1 - b^2r^2} + \frac{c^2r^2\nu^2\lambda_2}{1 - c^2r^2} = 0$$

$$\frac{a^2\lambda^2}{1 - a^2r^2} + \frac{b^2\mu^2}{1 - b^2r^2} + \frac{c^2\nu^2}{1 - c^2r^2} = 0$$

**69. Find the maximum and minimum values of $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$ when lx + my + nz = 0 and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.**

*Sol. :*

Given $U = \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$ ... (1)

and also lx + my + nz = 0 ... (2)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad ... (3)$$

Differentiating these equation and putting the condition du = 0

By (1) $\Rightarrow$ U = $\frac{2x}{a^4}dx+\frac{2y}{b^4}dy+\frac{2z}{c^4}dz$

$$\frac{x}{a^4}dx+\frac{y}{b^4}dy+\frac{z}{c^4}dz=0 \quad ... (4)$$

By (2) $\Rightarrow$ ldx + m dy + n dz = 0 ... (5)

By (3) $\Rightarrow$ $\frac{2x}{a^2}dx+\frac{2y}{b^2}dy+\frac{2z}{c^2}dz=0$

$$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0 \quad ... (6)$$

Multiplying (4), (5) & (6) by 1, $\lambda_1$, $\lambda_2$, and adding.

$$\frac{x}{a^4}dx+\frac{y}{b^4}dy+\frac{z}{c^4}dz+\lambda_1 \text{ (ldx + m dy + n dz)}+\lambda_2\left(\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz\right)=0$$

$$\left(\frac{x}{a^4}+l\lambda_1+\lambda_2\frac{x}{a^2}\right)dx+\left(\frac{y}{b^4}+m\lambda_1+\lambda_2\frac{y}{b^2}\right)dy+\left(\frac{z}{c^4}+n\lambda_1+\lambda_2\frac{z}{c^2}\right)dz=0$$

Equating to zero the coefficients of dx, dy, dz we get

$$\frac{x}{a^4}+l\lambda_1+\lambda_2\frac{x}{a^2}=0 \quad ...(7)$$

$$\frac{y}{b^4}+m\lambda_1+\lambda_2\frac{y}{b^2}=0 \quad ...(8)$$

$$\frac{z}{c^4}+n\lambda_1+\lambda_2\frac{z}{c^2}=0 \quad ...(9)$$

Multiplying (7), (8), & (9) by x, y, z & adding

$$x\left(\frac{x}{a^4}+l\lambda_1+\lambda_2\frac{x}{a^2}\right) + y\left(\frac{y}{b^4}+m\lambda_1+\lambda_2\frac{y}{b^2}\right) + z\left(\frac{z}{c^4}+n\lambda_1+\lambda_2\frac{z}{c^2}\right)=0$$

$$\frac{x^2}{a^4}+xl\lambda_1+\frac{x^2}{a^2}\lambda_2+\frac{y^2}{b^4}+ym\lambda_1+\frac{y^2}{b^2}\lambda_2+\frac{z^2}{c^4}+zn\lambda_1+\lambda_2\frac{z^2}{c^2}=0$$

$$\left[\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right]+\lambda_1[\,lx + my + nz\,] +\lambda_2\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right]=0$$

By (1), (2) & (3)

$\Rightarrow$ $U + \lambda_1 (0) + \lambda_2 (1) = 0$

$U + \lambda_2 = 0$

$\lambda_1 = -U$

Sub $\lambda_1 = -U$ in (7), (8) & (9)

(7) $\Rightarrow$ $\dfrac{x}{a^4}+l\lambda_1-U\dfrac{x}{a^2}=0$

$$\frac{x}{a^2}\left[\frac{1}{a^2}-U\right]+l\lambda_1=0$$

$$\frac{x}{a^2}\left[\frac{1}{a^2}-U\right]=-l\lambda_1$$

$$\frac{x}{a^2} = \frac{-a^2 l\lambda_1}{1-Ua^2}$$

$$x = \frac{-a^4 l\lambda_1}{1-Ua^2}$$

(8) $\Rightarrow$ $\dfrac{y}{b^4}+m\lambda_1-U\dfrac{y}{b^2}=0$

$$\frac{y}{b^2}\left[\frac{1}{b^2}-U\right]+m\lambda_1=0$$

$$\frac{y}{b^2} = \frac{-m\lambda_1}{\frac{1}{b^2}-U}$$

$$\frac{y}{b^2} = \frac{-b^2 m\lambda_1}{1-b^2U}$$

$$y = \frac{-b^4 m\lambda_1}{1-b^2U}$$

(9) $\Rightarrow \quad \frac{z}{c^4} + n\lambda_1 - U\frac{z}{c^2} = 0$

$$\frac{z}{c^2}\left[\frac{1}{c^2} - U\right] + n\lambda_1 = 0$$

$$\frac{z}{c^2} = \frac{-c^2 n\lambda_1}{1 - c^2 U}$$

$$z = \frac{-c^4 n\lambda_1}{1 - c^2 U}$$

Sub x, y & z in lx + my + nz = 0 we get

$$l\left[\frac{-a^4\, l\lambda_1}{1 - Ua^2}\right] + m\left[\frac{-b^4\, m\lambda_1}{1 - b^2 U}\right] + n\left[\frac{-c^4\, n\lambda_1}{1 - c^2 U}\right] = 0$$

$$-\lambda_1\left[\frac{a^4\, l^2\, \lambda_1}{1 - a^2 U} + \frac{b^4\, m^2\, \lambda_1}{1 - b^2 U} + \frac{c^2\, n^2\, \lambda_1}{1 - c^2 U}\right] = 0$$

$$\frac{a^4\, l^2}{1 - a^2 U} + \frac{b^4\, m^2}{1 - b^2 U} + \frac{c^4\, n^2}{1 - c^2 U} = 0$$

**70. Show that the point within a triangle for which the sum of squares of its perpendicular distance from the sides is least is the centre of the cosine - circle.**

*Sol. :*

Let x, y, z be the perpendicular distance of sides BC, CA, AB respectively the sum of square of the distance is given by

$$U = x^2 + y^2 + z^2 \qquad \text{... (1)}$$

$$\Delta ABC = \Delta OBC + \Delta OCA + \Delta OAB$$

$$\Delta ABC = \frac{1}{2}(ax + by + cz)$$

$\Rightarrow$ ax + by + cz = k ... (2)

Differentiating equation (1) and (2)

By (1) $\Rightarrow$ du = 2x dx + 2y dy + 2z . dz

For maximum or minimum du = 0

2x dx + 2y dy + 2z dz = 0

x dx + y dy + zdz = 0 ... (3)

By (2) $\Rightarrow$ adx + b dy + c dz = 0 ... (4)

Multiplying (3) & (4) by 1, $\lambda_1$ and adding

$(x\,dx + y\,dy + z\,dz) + \lambda_1(a dx + b\,dy + c\,dz) = 0$

$(x + a\lambda_1)dx + (y + \lambda_1 b)\,dy + (z + \lambda_1 c)\,dz = 0$

Equating the coefficients dx, dy, dz to zero.

$$x + a\lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = \frac{-x}{a}$$

$$y + b\lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = \frac{-y}{b}$$

$$z + c\lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = \frac{-z}{c}$$

$$\Rightarrow \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

Consider x, y as independent variables and z is a function of x and y.

Differentiating equation (2) partially w.r.t to x

$$a + 0 + c \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-a}{c} \qquad \text{... (5)}$$

Partially differentiating w. r. t to 'y'

$$0 + b + c \cdot \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-6}{c} \qquad \text{... (6)}$$

Partially differentiate (1) w.r.t to 'x'

$$\frac{\partial u}{\partial x} = 2x + 0 + 2z\frac{\partial z}{\partial x}$$

$$= 2x + 2z\frac{\partial z}{\partial x} \qquad \text{Since } \frac{\partial z}{\partial x} = \frac{-a}{c}$$

$$= 2x + 2z\left(\frac{-a}{c}\right)$$

$$\frac{\partial u}{\partial x} = 2x - \frac{2za}{c} \qquad \text{... (7)}$$

Differentiating equation (1) partially with respect to 'y'

$$\frac{\partial u}{\partial y} = 0 + 2y + 2z\frac{\partial z}{\partial y} \qquad \text{Since } \frac{\partial z}{\partial y} = \frac{-b}{c}$$

$$= 2y + 2z\left(\frac{-b}{c}\right)$$

$$\frac{\partial u}{\partial y} = 2y - \frac{2zb}{c}$$

Now

$$r = \frac{\partial^2 u}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x - \frac{2az}{c}\right)$$

$$= \frac{\partial}{\partial x}(2x) - \frac{2a}{c}\frac{\partial}{\partial x}(z) = 2 - \frac{2a}{c}\frac{\partial z}{\partial x}$$

$$= 2 - \frac{2a}{c}\left(\frac{-a}{c}\right) \qquad [\because \text{ by } (5)]$$

$$r = 2 + \frac{2a^2}{c^2} > 0$$

$$S = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(2y - \frac{2zb}{c}\right)$$

$$= 0 - \frac{2b}{c}\frac{\partial z}{\partial x} = \frac{-2b}{c}\left(\frac{-a}{c}\right) \qquad [\because \text{ by } (5)]$$

$$S = \frac{2ab}{c^2}$$

$$t = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(2y - \frac{2zb}{c}\right) = 2 - \frac{2b}{c}\frac{\partial z}{\partial y}$$

$$= 2 - \frac{2b}{c}\left(\frac{-b}{c}\right)$$

$$t = 2 + \frac{2b^2}{c^2}$$

Consider

$$rt - s^2 = \left(2+\frac{2a^2}{c^2}\right)\left(2+\frac{2b^2}{c^2}\right) - \left(\frac{2ab}{c^2}\right)^2$$

$$= 4+\frac{4b^2}{c^2}+\frac{4a^2}{c^2}+\frac{4a^2b^2}{c^4}-\frac{4a^2b^2}{c^4}$$

$$= 4+\frac{4a^2}{c^2}+\frac{4b^2}{c^2} > 0$$

$\therefore$ $rt - s^2 > 0$ and $r > 0$

$\therefore$ The function u is minimum

$\therefore$ Maximum value of u at $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ where (x, y, z) is the centre of cosine circle.

---

**71. Discuss the maxima and minima of the function U = sinx Sin y Sin z. where x, y, z are the angle of triangle.**

*Sol. :*

Given U = Sin x Sin y Sin z ... (1)

x, y, z are the angle of triangle $x + y + z = \pi$ ... (2)

Differentiating equation (1) on both sides

$$du = \sin y \sin z \frac{d}{dx}(\sin x) + \sin x \sin z \frac{d}{dy}(\sin y) + \sin x \sin y \frac{d}{dz}(\sin z)$$

$$du = \cos x \sin y \sin z + \sin x \cos y \sin z + \sin x \sin y \cos z$$

For a maximum or minimum of u, du = 0

$\cos x \sin y \sin z\, dx + \sin x \cos y \sin z\, dy + \sin x \sin y \cos z\, dz = 0$ ... (3)

From (2)

$dx + dy + dz = 0$ ... (4)

Multiplying 1, $\lambda$ to equation (3) & (4) and adding

$\cos x \sin y \sin z\, dx + \sin x \cos y \sin z\, dy + \sin x \sin y \cos z\, dz + \lambda (dx + dy + dz) = 0$

$(\cos x \sin y \sin z + \lambda)dx + (\sin x \cos y \sin z + \lambda)\, dy + (\sin x \sin y \cos z + \lambda)\, dz = 0$

Equating the co-efficients of dx, dy, dz to zero.

$\cos x \sin y \sin z + \lambda = 0$

$\Rightarrow$ $-\lambda = \cos x \sin y \sin z$

$\sin x \cos y \sin z + \lambda = 0$

$\Rightarrow \quad -\lambda = \sin x \cos y \sin z$

$\sin x \sin y \cos z + \lambda = 0$

$\Rightarrow \quad -\lambda = \sin x \cos y \cos z$

$\therefore \quad \cos x \sin y \sin z = \sin x \cos y \sin z = \sin x \sin y \cos z$

Dividing by $\sin x \sin y \sin z$ on both sides

$$\frac{\cos x \sin y \sin z}{\sin x \sin y \sin z} = \frac{\sin x \cos y \sin z}{\sin x \sin y \sin z} = \frac{\sin x \sin y \cos z}{\sin x \sin y \sin z}$$

$\cot x = \cot y = \cot z$

$\Rightarrow \quad x = y = z = \frac{\pi}{3}$

Consider $x, y$ as independent variables and $z$ as a function of $x$ and $y$.

Differentiating equation (2) partially w.r.t to $x, y$

$$1+0+\frac{\partial z}{dx} = 0$$

$$\frac{\partial z}{dx} = -1 \qquad ...(5)$$

$$0 + 1 + \frac{\partial z}{dy} = 0$$

$$\frac{\partial z}{dy} = -1 \qquad ...(6)$$

Now,
Differentiating equation (1) partially w.r. to

$$x\frac{\partial u}{dx} = \sin y \sin z \frac{\partial}{\partial x}(\sin x) + \sin x \sin z \frac{\partial}{\partial x}(\sin y) + \sin x \sin y \frac{\partial}{\partial x}(\sin z)$$

$$\frac{\partial u}{\partial x} = \sin y \sin z \cos x + 0 + \sin x \sin y \cos z \frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial x} = \sin y \sin z \cos x + \sin x \sin y \cos z \frac{\partial z}{\partial x}$$

From (5)

$$\frac{\partial u}{\partial x} = \sin y \sin z \cos x + \sin x \sin y \cos z (-1)$$

$$\frac{\partial u}{\partial x} = \sin y \sin z \cos x - \sin x \sin y \cos z \qquad ... (7)$$

Now we will find r, s, and t

$$r = \frac{\partial^2 u}{\partial x^2} = \sin y\left[\sin z\frac{\partial}{\partial x}\cos x + \cos x\frac{\partial}{\partial x}(\sin z)\right] - \sin y\left[\sin x\frac{\partial}{\partial x}(\cos z) + \cos z\frac{\partial}{\partial x}(\sin x)\right]$$

$$= \sin y\left[\sin z(-\sin x) + \cos x\cos z\frac{\partial z}{\partial x}\right] - \sin y\left[\sin x(-\sin z)\frac{\partial z}{\partial x} + \cos z\cos x\right]$$

by (5)

$$= \sin y\left[-\sin x\sin z + \cos x\cos z(-1)\right] - \sin y\left[-\sin x\sin z(-1) + \cos z\cos x\right]$$

$$= -\sin x\sin y\sin z - \sin y\cos x\cos z - \sin y\sin x\sin z - \sin y\cos z\cos x$$

$$= -2\sin x\sin y\sin z - 2\cos x\cos z\sin y$$

r at $\frac{\pi}{3}$

$$r = -2\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) - 2\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)$$

$$= -2\left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{-3\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = \frac{-4\sqrt{3}}{4} = -\sqrt{3}$$

$\therefore$ r at $\frac{\pi}{3} = \sqrt{3} < 0$

Similarly $t = \frac{\partial^2 u}{\partial y^2}$

$$t = -2\sin x\sin y\sin z - 2\cos x\cos z\sin y$$

t at $\frac{\pi}{3} = -\sqrt{3}$

$$S = \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left[\sin y\sin z\cos x - \sin x\sin y\cos z\right]$$

$$= \cos x\left[\sin z\frac{\partial}{\partial y}(\sin y) + \sin y\frac{\partial}{\partial y}(\sin z)\right] - \sin x\left[\sin y\frac{\partial}{\partial y}(\cos z) + \cos z\frac{\partial}{\partial y}(\sin y)\right]$$

$$= \cos x\left[\sin z(\cos y) + \sin y\cos z\frac{\partial z}{\partial y}\right] - \sin x\left[\sin y(-\sin z)\frac{\partial z}{\partial y} + \cos z\cos y\right]$$

by (6)

$$= \cos x\,[\sin z \cos y + \sin y \cos z\,(-1)] - \sin x\,[\sin y\,(-\sin z)(-1) + \cos y \cos z]$$

$$S = \cos x \cos y \sin z - \cos x \cos z \sin y - \sin x \sin y \sin z - \sin x \cos y \cos z$$

S at $\frac{\pi}{3}$

$$S = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{3\sqrt{3}}{8} - \frac{\sqrt{3}}{8}$$

$$= \frac{-4\sqrt{3}}{8} = \frac{-\sqrt{3}}{2}$$

$$S \text{ at } \frac{\pi}{3} = \frac{-\sqrt{3}}{2}$$

Now $rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{-\sqrt{3}}{2}\right)^2$

$$= 3 - \frac{3}{4}$$

$$= \frac{9}{4} > 0$$

$rt - s^2 > 0$ and $r < 0$ then the function U is maximum.

Maximum value $= \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)$

$$= \frac{\sqrt{3}}{2}\,\frac{\sqrt{3}}{2}\,\frac{\sqrt{3}}{2}$$

Maximum value of u is $= \frac{3\sqrt{3}}{8}$

# Choose the Correct Answers

1. The function has neither maximum nor minimum value if [ b ]
   (a) $rt - s^2 > 0$ (b) $rt - s^2 < 0$
   (c) $r > 0$ (d) $r < 0$

2. If f(x,y) has continuous second order partial derivatives $f_{xy}$ and $f_{yx}$ then [ a ]
   (a) $f_{xy} = f_{yx}$ (b) $f_{xy} \neq f_{yx}$
   (c) $f_{xy} < f_{yx}$ (d) $f_{xy} > f_{yx}$

3. The function is maximum value if [ a ]
   (a) $rt - s^2 > 0,\ r < 0$ (b) $rt - s^2 > 0, r > 0$
   (c) $rt - s^2 < 0, r < 0$ (d) $rt - s^2 < 0, r > 0$

4. The conclution for maximum or minimum values are [ b ]
   (a) $h\dfrac{\partial f}{\partial x} = 0,\ k\dfrac{\partial f}{\partial y} = 0$ (b) $\dfrac{\partial f}{\partial x} = 0,\ \dfrac{\partial f}{\partial y} = 0$
   (c) $\dfrac{\partial^2 f}{\partial x^2} = 0,\ \dfrac{\partial^2 f}{\partial y^2} = 0$ (d) None

5. Maximum value of (log x) / x [ a ]
   (a) $x = \pi/3$ (b) $x = \pi$
   (c) $x = \pi/2$ (d) $x = 0$

6. The maximum value of sin x + cos x is [ d ]
   (a) 2 (b) 1
   (c) $\sqrt{2}$ (d) $1+\sqrt{2}$

7. Sinx (1 + cos x) is maximum at [ a ]
   (a) $x = \pi/3$ (b) $x = \pi$
   (c) $x = \pi/2$ (d) $x = 0$

8. If $y^5 - 3ax^2 + x^5 = 0$ then $\dfrac{d^2y}{dx^2} =$ [ c ]
   (a) $\dfrac{-2(x-a)}{y^3}$ (b) $\dfrac{2ax - x^2}{y^2}$
   (c) $\dfrac{-2a^2x^2}{y^3}$ (d) None

9. Euler's theorem on homogenous function if 'F' is homogenous x, y, z of degree n, then [ d ]

(a) $x\frac{\partial F}{\partial x} - y\frac{\partial F}{\partial y} = nz$

(b) $x\frac{\partial F}{\partial y} + x\frac{\partial F}{\partial x} = nz$

(c) $x\frac{\partial F}{\partial x} - y\frac{\partial F}{\partial y} - z\frac{\partial F}{\partial z} = nF$

(d) $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = nF$

10. If f(x,y) = c then $\frac{dy}{dx}$ [ d ]

(a) $\frac{\partial f / \partial x}{\partial f / \partial y}$

(b) $\frac{\partial f}{\partial x}$

(c) $\frac{\partial f}{\partial y}$

(d) $\frac{-fx}{fy}$

# Fill in the Blanks

1. If $u = y^x$, then $\frac{\partial u}{\partial x}$ = ________

2. The value of f(a,b) is called ________ value of f(x,y)

3. If $\Delta x$ is increment in x then $\Delta y$ is ________ in y

4. Legrange's condition for maximum are ________ and ________

5. Legrange's condition for maximum are ________ and ________

6. If $u = (\tan x)^y + y \cot x$ then $\frac{dy}{dx}$ = ________

7. If $u = x^2 - y^2$ then $\frac{\partial u}{\partial x}$ = ________

8. If $z = (\cos y) / x$ and $x = u^2 - v$, $y = e^v$ then $\frac{\partial z}{\partial v}$ = ________

9. $\frac{dz}{dt}$ by composite function ________

10. If $x = e^u + e^{-v}$ then $\frac{\partial u}{\partial x}$ = ________

## Answers

1. $y^x \log x$
2. Maxima or minima
3. Consequent increment
4. $rt - s^2 > 0$, $r < 0$
5. $rt - s^2 > 0$ and $r < 0$
6. a
7. 2x
8. $(\cos y - xy \sin y) / x^2$
9. $\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$
10. $e^u$

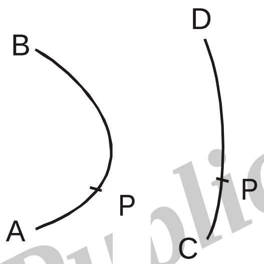
| | |
|---|---|
| UNIT III | **Curvature and Evolutes:** Introduction - Definition of Curvature - Radius of Curvature - Length of Arc as a Function, Derivative of arc - Radius of Curvature - Cartesian Equations - Newtonian Method - Centre of Curvature - Chord of Curvature.<br>**Evolutes:** Evolutes and Involutes - Properties of the evolute.<br>**Envelopes:** One Parameter Family of Curves - Consider the family of straight lines - Definition - Determination of Envelope. |

## 3.1 Introduction - Definition of Curvature Radius of Curvature

### Introduction Definition of Curvature

Let AB and CD be two curves we observe that the curve AB bends more sharply than curve CD we say that curvature of AB in greater than curvature of CD

i.e., nature of curve in called curvature

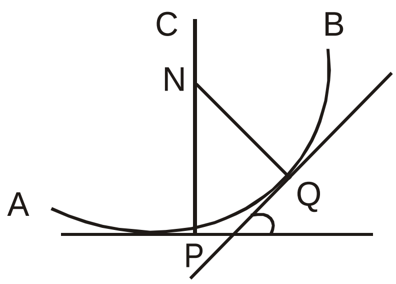





- **Radius of Curvature**

Let AB be a curve and P and Q are any two points on the curve. Let N is point in point if intersection of normals of P and Q.

If N tends to a definite position C, as Q tends to P, then the point C in called the Centre of Curvature of the curve at the point. The length CP is called the Radius of curvature of the curve it P in denoted by $\rho$




**1. Find the radius of curvature at any point of the following**

**(i) $s = c \tan \psi$ (catenary).**

*Sol :*

$s = c \tan \psi$

Differentiating, the above equation with respect to $\psi$ we get $\frac{ds}{d\psi} = c \sec^2\psi$.

**(ii) s = 4a sin ψ (cycloid)**

*Sol :*

$$s = 4a \sin \psi$$

Differentiating the above equation s with respect to ψ we get,

$$\frac{ds}{d\psi} = 4a \cos \psi$$

**(iii) s = 4a sin $\frac{1}{3}$ ψ (cardioide)**

*Sol :*

Differentiating the above equation s with respect to ψ we get,

$$\frac{ds}{d\psi} = 4a \cos \frac{1}{3}\psi \cdot \frac{1}{3} = \frac{4}{3} a \cos \frac{1}{3}\psi$$

**(iv) s = c log sec ψ (tractrix)**

*Sol :*

Differentiating the above equation s with respect to ψ we get,

$$\frac{ds}{d\psi} = c \cdot \frac{1}{\sec\psi} \cdot \sec\psi \tan\psi = c \tan\psi$$

**(v) s = a log (tan ψ + sec ψ) + a tan ψ sec ψ. (parabola)**

*Sol :*

Differentiating the above equation s with respect to ψ we get,

$$\frac{ds}{d\psi} = a \cdot \frac{1}{\tan\psi + \sec\psi} \cdot (\sec^2\psi + \sec\psi \tan\psi) + a(\sec\psi \sec^2\psi + \tan\psi \sec\psi \tan\psi)$$

$$= \frac{a \sec\psi(\sec\psi + \tan\psi)}{\tan\psi + \sec\psi} + a \sec\psi(\sec^2\psi + \tan^2\psi)$$

$$= a \sec\psi + a \sec\psi (\sec^2\psi + \sec^2\psi - 1)$$

$$= a \sec\psi + a \sec\psi (2\sec^2\psi - 1)$$

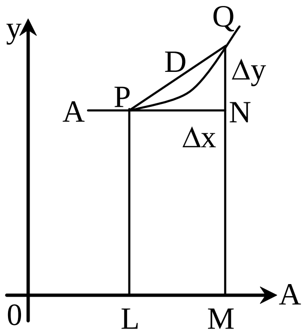
$$= a \sec\psi + 2a \sec^3\psi - a \sec\psi$$

$$\frac{ds}{d\psi} = 2a \sec^3\psi$$

## 3.2 Length of arc as a Function, Derivative of arc

Let y = f(x) be the equation of a given curve on which we take a fixed point A. To any given value of x corresponds a value of y i.e., f(x) and to this pair of numbers x and f(x) corresponds a point p(x, y) on the curve, and this point p has some actual length 's' from A. Thus, we have a function s of x for the curve y = f(x) we need to prove that

$$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$




We take a point Q(x + Δx, y + Δy) on the curve near P.

Let arc AQ = s + Δs so that arc

PQ = Δs

From the rt. angled ΔPQN, we have

$$PQ^2 = PN^2 = NQ^2$$

$$= (\Delta x)^2 + (\Delta y)^2$$

$$\Rightarrow \quad \left(\frac{PQ}{\Delta x}\right)^2 = 1+\left(\frac{\Delta y}{\Delta x}\right)^2$$

$$\Rightarrow \quad \left[\frac{\text{chord } PQ}{\text{arc } PQ}\right]^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1+\left(\frac{\Delta y}{\Delta x}\right)^2$$

Assuming, on an intuitive basis that

$$\lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} = 1$$

we obtain in the limit

$$\left(\frac{ds}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2$$

We make a convention that for the curve y = f(x) 's' is measured positively in the direction of x increasing, so that, s, increase with x.

Hence ds/dx is positive

Thus we have

$$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$

**2. Find $\frac{ds}{dx}$ for the curves y = cos h (x/c)**

*Sol :*

y = cos h (x/c)

We know that

$$\frac{ds}{dx} = \sqrt{\left(1+\frac{dy}{dx}\right)^2}$$

y = cos h(x/c)

Differentiating the above equation with respect to x we get,

$$\frac{dy}{dx} = \sin h\,(x/c)\,.\,1/c$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \quad \frac{ds}{dx} = \sqrt{\left(1+\frac{1}{c^2}\sinh^2(x/c)\right)}$$

---

**3. Find $\frac{ds}{dx}$ for the curves y = a log (a²/a² – x²)**

*Sol :*

$y = a \log (a^2/a^2 - x^2)$

Differentiating the above equation with respect to x we get,

$$\frac{dy}{dx} = a\,.\,\frac{1}{\frac{a^2}{a^2-x^2}}\,.\,-a^2(a^2-x^2)^{-2}\,.\,-2x$$

$$= a.\,\frac{a^2-x^2}{a^2}\cdot\frac{2a^2x}{(a^2-x^2)^2}$$

$$\frac{dy}{dx} = \frac{2ax}{a^2-x^2}$$

$$\frac{ds}{dt} = \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}$$

$$\frac{ds}{dt} = \sqrt{\left(1+\left(\frac{2ax}{a^2-x^2}\right)^2\right)}$$

$$= \sqrt{\left(1+\frac{4a^2x^2}{(a^2-x^2)^2}\right)}$$

$$= \sqrt{\left(\frac{(a^2-x^2)+4a^2x^2}{(a^2-x^2)^2}\right)}$$

$$= \sqrt{\frac{(a^2)^2+(x^2)^2-2a^2x^2+4a^2x^2}{(a^2-x^2)^2}}$$

$$= \sqrt{\frac{(a^2)^2+(x^2)^2+2a^2x^2}{(a^2-x^2)^2}}$$

$$\therefore \frac{ds}{dx} = \sqrt{\frac{(a^2+x^2)^2}{(a^2-x^2)^2}} = \frac{a^2+x^2}{a^2-x^2}$$

---

**4. Find $\frac{ds}{dt}$ for $x = a\cos^3 t,\ y = b\sin^3 t$**

*Sol :*

Given that

$x = a\cos^3 t,\ y = b\sin^3 t$

We know that for the parametric equation

$x = f(t),\ y = F(t)$

$$\frac{ds}{dt} = \sqrt{\left(\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right)}$$

$x = a\cos^3 t$

Differentiating the above equation with respect to t we get,

$$\frac{dx}{dt} = a.\ 3\cos^2 t\,(-\sin t)$$

$$\Rightarrow \frac{dx}{dt} = -3a\sin t\cos^2 t$$

$y = b\sin^3 t$

Differentiating the above equation with the respect to t we get,

$$\frac{dy}{dt} = b3\sin^2 t.\cos t = 3b\sin^2 t\cos t$$

$$\frac{ds}{dt} = \sqrt{[(-3a\sin t\cos^2 t)^2 + (3b\sin^2 t\cos t)^2]}$$

$$= \sqrt{[(9a^2\sin^2 t\cos^4 t + 9b^2\sin^4 + \cos^2 t]}$$

$$= \sqrt{9\sin^2 + \cos^2 t + (a^2\cos^2 t + b^2\sin^2 t)}$$

$$\therefore \quad \frac{ds}{dt} = 3 \text{ sint cost } \sqrt{(a^2\cos^2 t + b^2\sin^2 t)}$$

**5. Find $\frac{ds}{dx}$ for x = a(t – sin t), y = a(1 – cos t)**

*Sol :*

x = a(t – sin t), y = a(1 – cos t)

x = a(t – sin t)

Differentiating the above equation with respect to t we get,

$$\frac{dx}{dt} = a(1 - \cos t)$$

y = a (1 – cost)

Differentiating the above equation with respect to t we get,

$$\frac{dy}{dt} = a(0 + \sin t) = a \text{ sint}$$

$$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{[(a(1-\text{cost}))^2 + (a\sin t)^2]}$$

$$= \sqrt{a^2(1-\text{cost})^2 + a^2\sin^2 t}$$

$$= \sqrt{a^2[1+\cos^2 t - 2\cos t + \sin^2 t]}$$

$$= \sqrt{a^2[1+1-2\cos t]}$$

$$= \sqrt{2a^2(1-\cos t)}$$

$$\therefore \quad \frac{ds}{dt} = \sqrt{2}\, a \sqrt{(1-\cos t)}$$

**6. Find $\frac{ds}{dx}$ for $x = ae^t \sin t$, $y = ae^t \cos t$**

*Sol :*

$x = ae^t \sin t,\ y = ae^t \cos t$

$x = ae^t \sin t$

Differentiating the above equation with respect to t we get,

$$\frac{dx}{dt} = a(e^t \cos t + e^t \sin t)$$

$$\Rightarrow \quad \frac{dx}{dt} = ae^t(\cos t + \sin t)$$

$y = ae^t \cos t$

Differentiating the above equations with respect to t we get,

$$\frac{dy}{dt} = a(e^t. - \sin t + e^t \cos t)$$

$$\frac{dy}{dt} = ae^t(-\sin t + \cos t)$$

$$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{[(ae^t(\cos t + \sin t))^2 + (ae^t(-\sin t + \cos t)^2]}$$

$$= \sqrt{a^2e^{2t}[\cos^2 t + \sin^2 t + 2\cos t \sin t + \sin^2 t + \cos^2 t - 2\sin t \cos t}$$

$$= \sqrt{a^2e^{2t}(1+1)}$$

$$= \sqrt{2a^2e^{2t}}$$

$$\therefore \quad \frac{ds}{dt} = \sqrt{2}\, ae^t$$

---

**7. Find $\frac{ds}{d\theta}$ for $r = a(1 + \cos\theta)$**

*Sol :*

$r = a(1 + \cos\theta)$

We know that for polar equation $r = f(\theta)$

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}$$

$r = a(1 + \cos\theta)$

Squaring on both sides we get,

$r^2 = a^2(1 + \cos\theta)^2 = a^2(1 + \cos^2\theta + 2\cos\theta)$

$r = a(1 + \cos\theta)$

Differentiating the above equation with respect to θ we get

$$\frac{dr}{d\theta} = a(0 - \sin\theta) = -a\sin\theta$$

$$\frac{ds}{d\theta} = \sqrt{[(a^2(1+\cos^2\theta+2\cos\theta)+(-a\sin\theta)^2]}$$

$$= \sqrt{a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta}$$

$$= \sqrt{a^2(1+\cos^2\theta+\sin^2\theta+2\cos\theta)}$$

$$= \sqrt{a^2(1+1+2\cos\theta)}$$

$$= \sqrt{a^2(2+2\cos\theta)} = \sqrt{2a^2(1+\cos\theta)}$$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{2}\, a\sqrt{1+\cos\theta}$$

**8.** Find $\frac{ds}{d\theta}$ **for $r^2 = a^2 \cos 2\theta$**

*Sol :*



$r^2 = a^2\cos 2\theta$

Differentiating the above equation with respect to θ we get,

$$2r\frac{dr}{d\theta} = a^2. - \sin 2\theta \,.\, 2$$

$$\Rightarrow \quad r\frac{dr}{d\theta} = -a^2\sin 2\theta$$

$$\Rightarrow \quad \frac{dr}{d\theta} = \frac{-a^2}{r}\sin 2\theta$$

$$\Rightarrow \quad \frac{dr}{d\theta} = \frac{-a^2\sin 2\theta}{a\sqrt{\cos 2\theta}} = -a\sin 2\theta(\cos 2\theta)^{-1/2}$$

$$\frac{ds}{d\theta} = \sqrt{\left(r^2 + \frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{a^2 \cos 2\theta + a^2 \sin^2 2\theta (\cos 2\theta)^{-1}}$$

$$= \sqrt{a^2\left(\cos 2\theta + \frac{\sin 2\theta}{\cos 2\theta}.\sin 2\theta\right)}$$

$$= a\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}$$

$$= a\sqrt{\frac{1}{\cos 2\theta}}$$

$$\therefore \frac{ds}{d\theta} = a\sqrt{\sec 2\theta}$$

---

**9. In the Curve $r^m = a^m \cos m\theta$. Prove that $\frac{ds}{d\theta} = a\sec^{\frac{m-1}{m}} m\theta$ and $a^{2m}\frac{d^2r}{ds^2} + mr^{2m-1} = 0$**

*Sol :*

Given curve is

$$r^m = a^m \cos m\theta \qquad .....(1)$$

Taking log on both sides

$$\log r^m = \log (a^m \cos m\theta)$$

$$m \log r = \log a^m + \log \cos m\theta$$

$$m \log r = m \log a + \log \cos m\theta$$

Differentiating on boths sides with respect to 'θ'

$$m\frac{d}{d\theta}(\log r) = m\frac{d}{d\theta}(\log a) + \frac{d}{d\theta}(\log \cos m\theta)$$

$$m\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos m\theta}\frac{d}{d\theta}(\cos m\theta)$$

$$\frac{m}{r}\frac{dr}{d\theta} = \frac{-m\sin m\theta}{\cos m\theta}$$

$$\frac{dr}{d\theta} = -m\tan m\theta.\frac{r}{m}$$

$$\frac{dr}{d\theta} = -r\tan m\theta \qquad .....(2)$$

We know that $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

$$= \sqrt{r^2 + (-r\tan m\theta)^2}$$

$$= \sqrt{r^2 + r^2\tan^2 m\theta}$$

$$= r\sqrt{1 + \tan^2 m\theta}$$

$$\frac{ds}{d\theta} = r\sec m\theta \qquad .....(3)$$

Sub equation (1) in equation (3)

But from (1) $r^m = a^m \cos m\theta$

$$r = \left(a^m \cos m\theta\right)^{1/m}$$

$$r = a\left(\cos m\theta\right)^{1/m}$$

$$\therefore \text{ by (3)} \Rightarrow \frac{ds}{d\theta} = a\left(\cos m\theta\right)^{1/m}\sec m\theta$$

$$= \frac{a\left(\cos m\theta\right)^{1/m}}{\cos m\theta}$$

$$= a\left(\cos m\theta\right)^{\frac{1}{m}-1}$$

$$= a\left(\cos m\theta\right)^{\frac{1-m}{m}}$$

$$= a\left(\cos m\theta\right)^{-\frac{(m-1)}{m}}$$

$$= a\,\frac{1}{\left(\cos m\theta\right)^{m-1/m}}$$

$$\frac{ds}{d\theta} = a\sec^{(m-1)/m} m\theta$$

from (3) $\dfrac{ds}{d\theta} = r\sec m\theta = \dfrac{r}{\cos m\theta}$

$$= \frac{r}{r^m/a^m}$$

$$\frac{ds}{d\theta} = a^m r^{1-m} \qquad .....(4)$$

Consider

$$\frac{dr}{ds} = \frac{dr}{d\theta} \cdot \frac{d\theta}{ds}$$

$$\frac{dr}{ds} = -r \tan m\theta \frac{1}{a^m r^{1-m}} \qquad [\because \text{ from } (2)]$$

$$\frac{dr}{ds} = -a^{-m} r\, r^{-1+m} \tan m\theta \qquad .....(5)$$

Differentiating equation (5) with respect to 'x'

$$\frac{d^2r}{ds^2} = \frac{d}{ds}\left(\frac{dr}{ds}\right) = \frac{d}{dr}\left(\frac{dr}{ds}\right)\frac{dr}{ds}$$

$$= \frac{d}{dr}\left(-a^{-m}r^m \tan m\theta\right)\left(-a^{-m}r^m \tan\theta\right)$$

$$= -a^{-m}r^m \tan m\theta\left(-a^{-m}\right)\frac{d}{dr}\left(r^m \tan m\theta\right)$$

$$= -a^{-2m}r^m \tan m\theta\left[mr^{m-1}\tan m\theta + mr^m \sec^2 m\theta \frac{d\theta}{dr}\right]$$

$$= ma^{-2m}r^m \tan m\theta\left[r^{m-1}\tan m\theta - r^m \sec^2 m\theta \frac{1}{r}\cot m\theta\right]$$

$$= ma^{-2m}r^{2m-1}\left[\tan^2 m\theta - \tan m\theta \sec^2 m\theta \cot m\theta\right]$$

$$= ma^{-2m}r^{2m-1}\left[\tan^2 m\theta - \cancel{\tan} m\theta \sec^2 m\theta \sec^2 m\theta \frac{1}{\cancel{\tan} m\theta}\right]$$

$$= ma^{-2m}r^{2m-1}\left[\tan^2 m\theta - \sec^2 m\theta\right]$$

$$= ma^{-2m}r^{2m-1}(-1)$$

$$\frac{d^2r}{ds^2} = -ma^{-2m}r^{2m-1}$$

$$\frac{d^2r}{ds^2} + ma^{-2m}r^{2m-1} = 0$$

$$\therefore \quad a^{2m}\frac{d^2r}{ds^2} + mr^{2m-1} = 0$$

**10. Prove that for any curve $\sin^2\phi \dfrac{d\phi}{d\theta} + r\dfrac{d^2r}{ds^2} = 0$.**

*Sol :*

We know that $\dfrac{dr}{ds} = \cos\phi$ ......(1)

differentiating equation (1) with respect. to 's'

$$\frac{d}{ds}\left(\frac{dr}{ds}\right) = \frac{d}{ds}(\cos\phi)$$

$$\frac{d^2r}{ds^2} = -\sin\phi\,\frac{d\phi}{ds}$$

$$\frac{d^2r}{ds^2} = -\sin\phi\frac{d\theta}{d\theta}.\frac{d\theta}{ds}$$

$$\frac{d^2r}{ds^2} = -\sin\phi\frac{d\phi}{d\theta}\frac{\sin\phi}{r} \qquad \left[\because r\frac{d\theta}{ds} = \sin\phi\right]$$

$$\frac{d^2r}{ds^2} = \frac{-\sin^2\phi}{r}\frac{d\phi}{d\theta}$$

$$-\sin^2\phi\frac{d\phi}{d\theta} = r\frac{d^2r}{ds^2}$$

$$-\sin^2\phi\frac{d\theta}{d\theta} - r\frac{d^2r}{ds^2} = 0$$

$$\therefore \quad r\frac{d^2r}{ds^2} + \sin^2\phi\frac{d\phi}{d\theta} = 0$$



## 3.3 RADIUS OF CURVATURE - CARTESIAN EQUATIONS

- **Radius of curvature Cartesian Equations**

Radius of curvature $\rho = \dfrac{ds}{d\psi} = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$

- **Radius of curvature – Parametric equations**

$x = f(t),\ y = F(t)$

Radius of curvature $\rho = \pm \dfrac{[f'^2(t) + F'^2(t)]^{3/2}}{f'(t)F''(t) - F'(t)f''(t)]}$

- **Radius of curvature - Polar equations**

Let $r = f(\theta)$ be the given curve in polar co-ordinates.

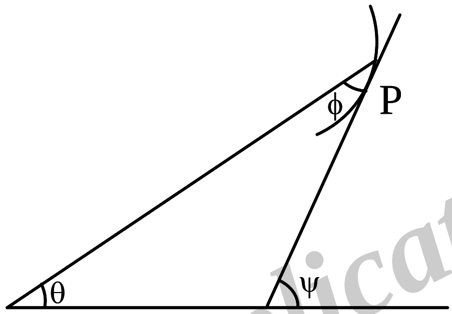
$x = r\cos\theta,\ y = r\sin\theta$ where $r = f(\theta)$

Radius of curvature $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

where $r_1 = f'(\theta)$, $r_2 = f''(\theta)$

- **Radius of curvature - Pedal equations $\rho = r\dfrac{dr}{d\rho}$**

From the figure, $\psi = \theta + \phi$




Differentiating both sides with respect to s, we obtain

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\theta}{ds} + \frac{d\phi}{ds}\cdot\frac{dr}{ds}$$

$$\therefore \quad \frac{1}{\rho} = \frac{1}{r}\sin\phi + \cos\phi\,.\,\frac{d\phi}{dr} \qquad \left[\because \frac{d\theta}{ds} = \frac{1}{r}\sin\phi \text{ and } \frac{dr}{ds} = \cos\phi\right]$$

$$\therefore \quad \frac{1}{\rho} = \frac{1}{r}\left(\sin\phi + r\cos\phi.\frac{d\phi}{dr}\right)$$

$$= \frac{1}{r}\,\frac{d}{dr}\,(r\sin\phi)$$

$$= \frac{1}{r}\frac{d}{dr}$$

$$\therefore \quad \rho = r\,\frac{dr}{dp}$$

**11. Radius of curvature when the equation of the curve is given in p and $\psi$ or to prove that**

$$\rho = p + \frac{d^2p}{d\psi^2}$$

*Sol :*

We have

$$\frac{dp}{d\psi} = \frac{dp}{dr}.\frac{dr}{ds}.\frac{ds}{d\psi}$$

$$= \frac{dp}{dr} . \cos\phi . \rho$$

$$= \frac{dp}{dr}\cos\phi . r\frac{dr}{dp}$$

$$= r\cos\phi$$

Now again

$$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2\sin^2\phi + r^2\cos^2\phi = r^2$$

Differentiating with respect to p, we obtain

$$2p + 2.\frac{dp}{d\psi}.\frac{d^2p}{d\psi^2}.\frac{d\psi}{dp} = 2r\frac{dr}{dp}$$

or $$p + \frac{d^2p}{d\psi^2} = \rho$$

This is known as tangential polar formula.



**12. For the cylcorid x = a(t + sint), y = a(1 – cost) prove that $\rho = 4a\cos\left(\frac{1}{2}t\right)$**

*Sol :*

x = a(t + sin t)

Differentiating the above equations with respect to t we get,

$$\frac{dx}{dt} = a(1 + \cos t)$$

y = a(1 – cost)

Differentiating the above equation with respect to t we get,

$$\frac{dy}{dt} = a(0 + \sin t) = a\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin t/2\cos t/2}{2\cos^2 t/2} = \tan t/2$$

$$\frac{d^2y}{dx^2} = \sec^2 t/2.\frac{1}{2}\ \frac{dt}{dx}$$

$$= \frac{1}{2}\sec^2 t/2 . \frac{1}{a(1+\cos t)} = \frac{1}{2}\ \sec^2 t/2\ \frac{1}{2a.\cos^2 t/2}$$

$$= \frac{1}{4a}.\frac{1}{\cos^2 t/2.\cos^2 t/2} = \frac{1}{4a}.\frac{1}{\cos^4 t/2}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\tan^2\frac{1}{2}t\right)^{3/2}}{\frac{1}{4a}.\frac{1}{\cos^4 t/2}}$$

$$= \frac{(\sec^2 t/2)^{3/2}}{\frac{1}{4a}\frac{1}{\cos^4 t/2}} = \frac{\frac{1}{\cos^3 t/2}}{\frac{1}{4a\cos^4 t/2}}$$

$\therefore\quad \rho = 4a\cos t/2$

**13. Find the Radius of curvature art any point on the curve $y = c\cosh\frac{x}{c}$ (Catenary).**

*Sol :*

The given curve is $y = c\cosh\frac{x}{c}$.

Differentiating it with respect to x, we have

$$\frac{dy}{dx} = c.\sin h\ \frac{x}{c}.\frac{1}{c} = \sinh\frac{x}{c}$$

and $$\frac{d^2y}{dx^2} = \cosh\frac{x}{c}.\frac{1}{c}$$

$$\therefore\quad \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{c\left[1+\sinh^2\frac{x}{c}\right]^{3/2}}{\cosh\frac{x}{c}}$$

$$= c\cosh^2 x/c = c.\ \frac{y^2}{c^2} = \frac{y^3}{c}$$

Hence $\rho\ \alpha$ (ordinate)$^2$

**14. Find the Radius of curvature art any point on the curve x = a (cost + t sint ) y = a (sint) – t (cos t).**

*Sol :*

$$x = a(\cos t + t\sin t), \quad y = (a\sin t - t\cos t)$$

Let $x = a(\cos t + t\sin t)$

$$\frac{dx}{dt} = a\left[(-\sin t) + t(\cos t) + \sin t\cdot 1\right]$$

$$= a[t\cos t]$$

$$y = a(\sin t - t\cos t)$$

$$\frac{dy}{dt} = a\left[\cos t - t(-\sin t) + \cos t\cdot 1\right]$$

$$= a[t\sin t]$$

$$\frac{dy}{dx} = \frac{at\cos t}{at\sin t} = \tan t$$

$$\frac{d^2y}{dx^2} = \sec^2 t\cdot\frac{dt}{dx}$$

$\therefore$ Radius of Curvature in $\rho = \dfrac{\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\dfrac{dy^2}{dx^2}}$

$$= \frac{\left(1+\tan^2 t\right)^{\frac{3}{2}}}{\sec^2 t\cdot\dfrac{dt}{dx}}$$

$$= \frac{\sec^3 t}{\sec^2 t \cdot \frac{dt}{dx}} \quad = \sec t \times \frac{dx}{dt}$$

$$= \sec t \cdot a[t \cos t]$$

$\therefore \; \rho = at$

---

**15. For the curve $r^m = a^m \cos m\theta$, Prove that $P = \dfrac{a^m}{(m+1)\, r^{m-1}}$.**

*Sol :*

We have

$$r^m = a^m \cos m\theta$$

Apply log on both sides

$$\log r^m = \log(a^m \cos m\theta)$$

$$m \log r = \log a^m + \log \cos m\theta$$

$$m \log r = m \log a + \log \cos m\theta$$

Differentiate with respect to $\theta$

$$m \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{1}{\cos m\theta}(-m \sin m\theta)$$

$$\frac{m}{r} \cdot \frac{dr}{d\theta} = -m \frac{\sin m\theta}{\cos m\theta}$$

$$\frac{dr}{d\theta} = -\cancel{m} - \tan m\theta \times \frac{r}{\cancel{m}}$$

$$\Rightarrow \quad r_1 = \frac{dr}{d\theta} = -r \tan m\theta$$

again differentiate with respect to '$\theta$'

$$r_2 = \frac{d^2 r}{d\theta^2} = -r\, m \sec^2 m\theta - \tan m\theta \frac{dr}{d\theta}$$

$$= -rm \sec^2 m\theta - \tan m\theta(-r \tan m\theta)$$

$$= -rm \sec^2 m\theta + r \tan^2 m\theta$$

Here,

$$P = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - r\, r^2}$$

$$= \frac{\left[r^2 + (-r\tan m\theta)^2\right]^{3/2}}{r^2 + 2(-r\tan m\theta)^2 - \left[r\left(-rm\sec^2\theta + r\tan^2 m\theta\right)\right]}$$

$$= \frac{\left(\cancel{r^2}\right)^{3/\cancel{2}}\left[4\tan^2 m\theta\right]^{3/2}}{\left[r^2 + r^2\tan^2 m\theta + r^2 m\sec^2\right]}$$

$$= \frac{r^3\left[\sec^2 m\theta\right]^{3/2}}{r^2\left[1+\tan^2 m\theta\right] + mr^2\sec^2 m\theta}$$

$$= \frac{r^3\sec^3 m\theta}{r^2\sec^2 m\theta + mr^2\sec^2 m\theta}$$

$$= \frac{\cancel{r^3}\sec^3 m\theta}{\cancel{r^2}\sec^2 m\theta(m+1)}$$

$$= \frac{r\sec m\theta}{(m+1)}$$

$$= \frac{r}{(m+1)\cos m\theta} \Rightarrow \frac{1}{m+1}\cdot\frac{r}{\cos m\theta}$$

$$= \left(\frac{1}{m+1}\right)\frac{a^m \cdot r}{r^m}$$

$$\therefore \quad r = \frac{a^m}{(m+1)r^{m-1}}$$

**16 Show that the curvature of the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the family $x^3 + y^3 = 3axy$ is $\frac{-8\sqrt{2}}{3a}$**

*Sol :*

Given that

$$x^3 + y^3 = 3axy$$

Differentiating

$$3x^2 + 3y^2\frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx} \quad .....(1)$$

$$3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$$

$$\frac{dy}{dx}\left(3y^2 - 3ax\right) = 3ay - 3x^2$$

$$\frac{dy}{dx} = \frac{\not{3}\left(ay - x^2\right)}{\not{3}\left(y^2 - ax\right)}$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)}$$

$$= \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}}$$

$$= \frac{6a^2 - 9a^2}{9a^2 - 6a^2} \Rightarrow \frac{-\left[9a^2 - 6a^2\right]}{9a^2 - 6a^2}$$

$$\therefore \quad = \left.\frac{dy}{dx}\right|_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -1$$

again differentiate (1)

$$\Rightarrow \quad (y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$y^2\frac{dy}{dx} - ax\frac{dy}{dx} - ay - x^2 = 0$$

$$2y\frac{dy}{dx} + y^2\frac{d^2y}{dx^2} - a\frac{dy}{dx} - ax\frac{d^2y}{dx^2} - a\frac{dy}{dx} - 2x = 0$$

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y2\frac{d^2y}{dx^2} = a\frac{dy}{dx} + a\frac{dy}{dx} + ax\frac{d^2y}{dx^2}$$

Substituting $\frac{3a}{2}, \frac{3a}{2}, -1$ for $x, y, \frac{dy}{dx}$ respectively,

$$\left.\frac{d^2y}{dx^2}\right|_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{-32}{3a}$$

Hence the curvature at $\left(\frac{3a}{2},\frac{3a}{2}\right)$

$$= \frac{d^2y/dx^2}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{\frac{-32}{3a}}{\left[1+(-1)^2\right]^{3/2}}$$

$$= \frac{\frac{-3a}{3a}}{(2)^{3/2}} = \frac{-8\sqrt{2}}{3a}$$

---

**17. If a curve is defined by the equation x = f(t) ; y = $\phi$(t) Prove that $\rho = \frac{\left(x'^2+y'^2\right)^{3/2}}{x'y''-y'x''}$**

*Sol :*

Given that, $x = f(t)$ $y = \phi(t)$

differentiating with respect to 't'

$$\frac{dx}{dt} = f'(t) \qquad \frac{dy}{dx} = \phi'(t)$$

i.e., $x' = f'(t)$ $y' = f'(t)$

Now

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'} \text{ where y' and x' are function of t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx}$$

$$= \frac{x'y''-y'x''}{(x')^2}\cdot\frac{1}{x'}$$

$$= \frac{x'y''-y'x''}{x'^3}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\left(\frac{y'}{x'}\right)^2\right)^{3/2}}{\frac{x'y''-y'x''}{x'^3}}$$

$$= \frac{\frac{\left(x'^2-y'^2\right)^{3/2}}{\left(x'^2\right)^{3/2}}}{\frac{x'y''-y'x''}{x'^3}}$$

$$= \frac{\left(x'^2-y'^2\right)^{3/2}}{(x')^3} \times \frac{x'^3}{x'y''-y'x''}$$

$$\therefore \quad \rho = \frac{\left(x'^2-y'^2\right)^{3/2}}{x'y''-y'x''}$$



---

**18. In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that the radius of curvature at an end of the major axis is equal to semi latus return of the ellipse.**

*Sol :*

The given curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Differentiating with respect to 'x' we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$

$$\frac{2^4}{b^2}\cdot\frac{dy}{dx}=\frac{-2x}{a^2}$$

$$\frac{dy}{dx}=\frac{-xb^2}{a^2y}$$

Differentiating with respect to 'x'

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\left[\frac{y - x\frac{dy}{dx}}{y^2}\right]$$

$$= \frac{-b^2}{a^2}\left[\frac{y - x\left(\frac{-b^2x}{a^2y}\right)}{y^2}\right]$$

$$= \frac{-b^2}{a^2}\left[\frac{\left(a^2y^2 + b^2x^2\right)}{a^2y^3}\right]$$

$$= \frac{-b^2}{a^4y^3}a^2b^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$$

$$= \frac{-b^4a^2}{a^4y^3} \ (1)$$

$$\frac{d^2y}{dx^2} = \frac{-b^4}{a^2y^3}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + \left(\frac{-xb^2}{a^2y}\right)^2\right]^{3/2}}{\frac{-b^4}{a^2y^3}}$$

$$= \frac{\frac{\left(a^4y^2 + x^2b^4\right)^{3/2}}{\left(a^4y^2\right)^{3/2}}}{\frac{-b^4}{a^2y^3}}$$

$$= -\frac{\left(a^4y^2 + b^4x^2\right)^{3/2}}{a^6y^3} \cdot \frac{a^2y^3}{b^2}$$

$$= -\frac{\left(a^4y^2 + b^4x^2\right)^{3/2}}{a^4b^4}$$

$\rho$ at one end of major axis (a, 0)

$$= \frac{\left(a^4(0) + b^4a^2\right)^{3/2}}{a^4b^4}$$

$$= \frac{\left(b^4a^2\right)^{3/2}}{a^4b^4} \Rightarrow \frac{b^6a^3}{a^4b^4} = \frac{b^2}{a} \text{ Semi latus rectum}$$

**19.** Show that for any curve $\mathbf{\frac{r}{\rho} = \sin\phi\left(1 + \frac{d\phi}{d\theta}\right)}$.

*Sol :*

Consider

$$\sin\phi\left(1 + \frac{d\theta}{d\theta}\right)$$

$$= r\frac{d\theta}{ds}\left(1 + \frac{d\theta}{d\theta}\right)$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\theta}{ds}\cdot\frac{d\phi}{d\theta}\right)$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right)$$

$$= r\frac{d}{ds}(\theta + \phi)$$

$$= r\frac{d\varphi}{ds}$$

$$= \frac{r}{\rho}$$

**20. Find the radius of curvature for the curve r = a(1 – cos$\theta$).**

*Sol :*

$r = a(1 - \cos\theta) \Rightarrow r = a - a\cos\theta$

differentiate with respect to '$\theta$'

$$\frac{dr}{d\theta} = -(-a\sin\theta) = a\sin\theta$$

again differentiate with respect to to θ

$$\frac{d^2r}{d\theta^2} = a\cos\theta$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)}$$

$$= \frac{\left[r^2 + \left(a^2\sin^2\theta\right)\right]^{3/2}}{r^2 + 2(a\sin\theta)^2 - r(a\cos\theta)}$$

$$= \frac{\left\{\left[a(1-\cos\theta)\right]^2 + \left[a^2\sin^2\theta\right]\right\}^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - ra\cos\theta}$$

$$= \frac{\left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a^2\sin^2\theta + 2a^2\sin^2\theta - \left[a(1-\cos\theta)a\cos\theta\right]}$$

$$= \frac{a^3\sin^2\theta + a^3\sin^2\theta}{3a^2\sin^2\theta - a^2\cos\theta + a^2\cos^2\theta}$$

$$= \frac{2a^3\sin^2\theta}{3a^2\sin^2\theta}$$

$$= \frac{2}{3}a\sin\theta$$

$$\therefore \rho = \frac{2}{3}\sqrt{2ar}$$

---

**21. Show that for any curve r = f(θ) the curvature is given by $\left(u\frac{d^2u}{d\theta^2}\right)\sin^3\phi$, where $u = \frac{1}{r}$.**

*Sol :*

The curve $r = f(\theta)$ the curvature is given by $\left(u + \frac{d^2u}{d\theta^2}\right)\sin^3\phi$.

$$\rho = \frac{\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]^{3/2}}{u^3\left[u + \frac{d^2u}{d\theta^2}\right]}$$

$$\frac{1}{\rho} = \frac{u^3\left(u + \frac{d^2u}{d\theta^2}\right)}{\left[u^2 + \left(\frac{du}{d\theta}\right)\right]^{3/2}}$$

$$= \frac{u^3}{\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]^{3/2}}\left[u + \frac{d^2u}{d\theta^2}\right]$$

$$\therefore \quad u = \frac{1}{r}$$

$$= \frac{\frac{1}{r^3}}{\left[\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}\left[u + \frac{d^2u}{d\theta^2}\right]$$

$$= \frac{\frac{1}{r_3}}{\frac{1}{p^3}}\left[u + \frac{d_u^2}{d\theta^2}\right] = \frac{p^3}{r^3}\left[u + \frac{d^2u}{d\theta^2}\right]$$

$$= \frac{r^3 \sin^3\phi}{r^3}\left[u + \frac{d^2u}{d\theta^2}\right]$$

$$= \left[u + \frac{d^2u}{d\theta^2}\right]\sin^3\phi$$

**22. Prove that for any curve $\frac{1}{\rho} = \frac{d}{x}\left(\frac{dy}{ds}\right)$**

*Sol :*

Consider

$$\frac{d}{dx}\left(\frac{dy}{ds}\right) = \frac{d}{ds}(\sin\varphi)$$

$$= \frac{d}{d\varphi}(\sin\varphi)\frac{d\varphi}{dx}$$

$$= \cos\varphi\,\frac{d\varphi}{dx}\cdot\frac{ds}{dx}$$

$$\frac{d\varphi}{dx} = \frac{1}{\rho}$$

**23. Find the radius of curvature at any point on the curves x = (a cost t) / t, y = (a sin t) / t**

*Sol :*

$$x = \frac{a\cos t}{t}, \quad y = \frac{a\sin t}{t}$$

$$\frac{dx}{dt} = \frac{1}{t}(-a\sin t) + a\cos t\left[\frac{-1}{t^2}\right]$$

$$= \frac{-at\sin t - a\cos t}{t^2}$$

$$\frac{dy}{dt} = \frac{1}{t}(a\cos t) + a\sin t\left[\frac{-1}{t^2}\right]$$

$$= \frac{-at\cos t - a\sin t}{t^2}$$

$$\frac{dy}{dx} = \frac{-(at\cos t - a\sin t)}{at\sin t + a\cos t}$$

Consider

$$\left(1+y_1^2\right) = 1 + \left[\frac{-(at\cos t - a\sin t)}{at\sin t + a\cos t}\right]^2$$

$$= 1 + \frac{(at\cos t - a\sin t)^2}{(at\sin t + a\cos t)^2}$$

$$= \frac{a^2t^2\sin^2 t + a^2\cos^2 t + 2a^2t\sin t\cos t + a^2t^2\cos^2 t + a^2\sin^2 t - 2a^2t\sin t\cos t}{(at\sin t + a\cos t)^2}$$

$$= \frac{a^2t^2\left(\sin^2 t + \cos^2 t\right) + a^2\left(\sin^2 t + \cos^2 t\right)}{(at\sin t + a\cos t)^2}$$

$$= \frac{a^2t^2 + a^2}{(at\sin t + a\cos t)^2}$$

$$= \frac{a^2\left(1+t^2\right)}{(at\sin t + a\cos t)^2}$$

Consider

$$y_2 = \frac{-1}{at\sin t + a\cos t}(-a\,t\sin t + a\cos t - a\cos t)\frac{dt}{dx} - (a\,t\cos t\,t$$

$$- a\sin t)\frac{-1}{(at\sin t + a\cos t)^2}(at\cos t + a\sin t - a\sin t)\frac{dt}{dx}$$

$$= \frac{-at\sin t}{a+\sin t + a\cos t}\cdot\frac{t^2}{at\sin t + a\cos t} + \frac{(at\cos t - a\sin t)at\cos t}{(at\sin t + a\cos t)^2},\frac{t^2}{(at\sin t + a\cos t)^2}$$

$$= \frac{-at^3\sin t(at\sin t + a\cos t) + t^2\left(a^2t^2\cos^2 t - a^2t\sin t\cos t\right)}{(at\sin t + a\cos t)^2}$$

$$= \frac{-a^2t^4\sin^2 t - a^2t^3\sin t\cos t + a^2t^4\cos^2 t - a^2t^3\sin t\cos t}{(at\sin t + a\cos t)^2}$$

$$= \frac{-a^2t^4}{(at\sin t + a\cos t)^3}$$

Consider

$$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$$

$$= \left[\frac{a^2\left(t^2+1\right)}{(at\sin t + a\cos t)^2}\right]^{3/2}\cdot\frac{(at\sin t + a\cos t)^3}{-a^2t^4}$$

$$= \frac{a^3\left(t^2+1\right)^{3/2}}{-a^2t^4} = \frac{-a\left(t^2+1\right)^{3/2}}{t^4}$$

$$\therefore\ \rho = \frac{-a\left(t^2+1\right)^{3/2}}{t^4}$$

**24. Show that for the curve x = a cos θ(1 + sin θ), y = a sin θ(1 + cosθ). The radius of curvature is a, at the point for which the value of the parameter θ is $\frac{-\pi}{4}$.**

*Sol :*

$x = a\cos\theta(1+\sin\theta), \quad y = a\sin\theta(1+\cos\theta)$

$x' = \frac{dx}{d\theta} = a\cos\theta(\cos\theta) + a(1+\sin\theta)(-\sin\theta)$

$= a\cos^2\theta - a\sin\theta - a\sin^2\theta$

$= a(\cos^2\theta - \sin^2\theta) - a\sin\theta$

$= a\cos 2\theta - \theta\sin\theta$

$x'' = \frac{d^2x}{d\theta^2} = -2a\sin 2\theta - a\cos\theta$

$y' = a\sin\theta(-\sin\theta) + a(1+\cos\theta)\cos\theta$

$= -a\sin^2\theta + a\cos\theta + a\cos^2\theta$

$= a(\cos^2\theta - \sin^2) + a\cos\theta$

$= a\cos 2\theta + a\cos\theta$

$y'' = -2a\sin 2\theta - a\sin\theta$

Now,

$$\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{x'y'' - y'x''}$$

$$= \frac{\left[(a\cos 2\theta - a\sin\theta)^2 + (a\cos 2\theta + a\cos\theta)^2\right]^{3/2}}{(a\cos 2\theta - a\sin\theta)(-2a\sin 2\theta - a\sin\theta) - (a\cos 2\theta + a\cos\theta)(-2a\sin 2\theta - a\cos\theta)}$$

$$= \frac{\left[\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{\sqrt{2}}\right)^2\right]^{3/2}}{\frac{a}{\sqrt{2}}\left(-2a + \frac{a}{\sqrt{2}}\right) - \left(\frac{a}{\sqrt{2}}\right)\left(-2a - \frac{a}{\sqrt{2}}\right)}$$

$$= \frac{\left[\frac{a^2}{2} + \frac{a^2}{2}\right]^{3/2}}{\frac{-2a^2}{\sqrt{2}} + \frac{a^2}{2} + \frac{2a^2}{\sqrt{2}} + \frac{a^2}{2}}$$

$$= \frac{a^3}{a^2} = a.$$

$\therefore \quad \rho = a$

**25.** Show that for the curve y = $\dfrac{\mathbf{ax}}{\mathbf{a+x}}\left(\dfrac{\mathbf{2\rho}}{\mathbf{a}}\right)^{\frac{3}{2}} = \left(\dfrac{\mathbf{y}}{\mathbf{x}}\right)^{2} + \left(\dfrac{\mathbf{x}}{\mathbf{y}}\right)^{2}$

*Sol :*

$$y = \frac{ax}{a+x}$$

$$y' = \frac{1}{a+x}(a) + ax\left(\frac{-1}{(a+x)^2}\right)$$

$$= \frac{a-ax}{(a+x)^2} \Rightarrow \frac{a(a+x)-ax}{(a+x)^2}$$

$$= \frac{a^2+a\cancel{x}-a\cancel{x}}{(a+x)^2}$$

$$= \frac{a^2}{(a+x)^2}$$

$$y'' = a^2\left[\frac{-2}{(a+x)^3}\right]$$

Consider

$$\rho = \frac{\left[1+\left(\dfrac{a^2}{(a+x)^2}\right)^2\right]^{3/2}}{\dfrac{-2aa^2}{(a+x)^3}}$$

$$= \left[\frac{(a+x)^4+a^4}{(a+x)^4}\right]^{3/2} \cdot \frac{(a+x)^3}{-2a^2}$$

$$= \frac{\left((a+x)^4+a^4\right)^{3/2}}{(a+x)^6} \cdot \frac{(a+r)^3}{-2a^2}$$

$$= \frac{\left((a+x)^4 + a^4\right)^{3/2}}{-2a^2(a+x)^3}$$

$= -2a^2(a+x)^3 \rho$

$$= \left((a+x)^4 + a^4\right)^{3/2}$$

$$= \left(-2a^2(a+x)^3 \rho\right)^{3/2} = (a+r)^4 + a^4$$

$$= (2\rho)^{2/3}\left(a+x^2\right)a^{4/3} = (a+x)^4 + a^4$$

$$\frac{(2\rho)^{2/3}}{a^{2/3}} a^2(a+x)^2 = (a+x)^4 + a^4$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \frac{(a+x)^2}{a^2} + \frac{a^2}{(a+x)^2}$$

$$\left(\frac{2\rho}{a}\right)^{3/2} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

---

**26. Prove that for the cardioid r = a(1 + cos θ), $\frac{\rho^2}{r}$ is constant.**

*Sol :*

Given,

$r = a(1 + \cos\theta)$

$r_1 = \frac{dr}{d\theta} = -a\sin\theta$

$r_2 = \frac{d^2r}{d\theta^2} = -a\sin\theta$

We know that

$$r = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{\left[\left[a(1+\cos\theta)\right]^2 + (-a\sin\theta)^2\right]^{3/2}}{\left(a(1+\cos\theta)\right)^2 + 2(-a\sin\theta)^2 - \left[(1+\cos\theta)(-a\cos\theta)\right]}$$

$$= \frac{\left[a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta\right]^{3/2}}{a^2 + a^2\cos^2\theta + 2a\cos\theta + 2a^2\sin^2\theta + a^2\cos\theta + a^2\cos^2\theta}$$

$$= \frac{\left(a^2 + a^2(1) + 2a^2\cos\theta\right)^{3/2}}{a^2 + 2a^2\left(\cos^2\theta + \sin^2\theta\right) + a^2\cos^2\theta + 2a\cos\theta}$$

$$= \frac{\left[2a^2(1+\cos\theta)\right]^{3/2}}{3a^2 + 3a^2\cos\theta}$$

$$= \frac{2^{3/2}a^3(1+\cos\theta)^{3/2}}{3a^2(1+\cos\theta)}$$

$$= \frac{2^{3/2}}{2}a(1+\cos\theta)^{1/2}$$

$$= \frac{2^{3/2}}{3}a(1+\cos\theta)^{1/2}$$

$$= \frac{2^{3/2}}{3}a\sqrt{\frac{r}{a}}$$

$$\frac{\rho}{\sqrt{r}} = \frac{2^{3/2}}{3}\sqrt{a}$$

$$\frac{\rho^2}{r} = \frac{8a}{a} \text{ which is constant}$$



**27. Find the radius of curvature at the point of the curve r = a cos nθ. Also show that at the point where r = a its values is $\frac{a}{1+n^2}$.**

*Sol :*

Given, r = a cos n θ

$$r_1 = \frac{dr}{d\theta} = -\,an\sin\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = -\,an^2\cos\theta$$

We know that

$$\rho = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 = 2r_1^2 - rr_2}$$

$$= \frac{\left[(a\cos n\theta)^2 + (-an\sin n\theta)^2\right]^{3/2}}{(a\cos n\theta)^2 + 2(-an\sin n\theta)^2 - (a\cos n\theta)\left(-an^2\cos n\theta\right)}$$

$$= \frac{\left[a^2\cos n^2\theta + a^2n^2\sin n^2\theta\right]^{3/2}}{a^2\cos n^2\theta + 2a^2n^2\sin n^2\theta + a^2n^2\cos n^2\theta}$$

$$= \frac{a^2\left[r^2 + a^2n^2\sin n^2\theta\right]^{3/2}}{r^2 + n^2r^2 + 2a^2n^2\sin n^2\theta}$$

$$= \frac{\left[r^2 + a^2n^2\left(1 - \cos n^2\theta\right)\right]^{3/2}}{r^2 + n^2r^2 + 2a^2n^2\left(1 - \cos n^2\theta\right)}$$

$$= \frac{\left[r^2 + a^2n^2 - a^2n^2\cos n^2\theta\right]^{3/2}}{r^2 + n^2r^2 + 2a^2n^2 - 2a^2n^2\cos n^2\theta}$$

$$= \frac{\left[r^2 + a^2n^2 - n^2r^2\right]^{3/2}}{r^2 + n^2r^2 + 2a^2n^2 - 2n^2r^2}$$

$$= \frac{\left(r^2 + a^2n^2 - n^2r^2\right)^{3/2}}{r^2 - n^2r^2 + 2a^2n^2}$$

$$\rho_{r=a} = \frac{\left(a^2 + a^2n^2 - n^2a^2\right)^{3/2}}{a^2 - n^2a^2 + 2a^2n^2}$$

$$= \frac{\left(a^2\right)^{3/2}}{a^2 + a^2n^2} = \frac{a^2}{a^2\left(1 + n^2\right)} = \frac{a}{1 + n^2}$$

**28. Find the radius of curvature of the curve r = a(1 + cosθ) at the point where the tangent is parallel to the initial line.**

*Sol :*

Given $r = a(1 + \cos\theta)$

We know that

$$\rho = \frac{2^{3/2}}{3} a(1+\cos\theta)^{1/2}$$

$$\rho = \frac{2^{3/2}}{3} a\left(1+\cos\frac{\pi}{3}\right)^{1/2}$$

$$= \frac{2^{3/2}}{3} a\left(1+\frac{1}{2}\right)^{1/2}$$

$$= \frac{2^{3/2}}{3} a\left(1+\frac{1}{2}\right)^{1/2}$$

$$= \frac{2^{3/2}}{3} a\frac{3^{1/2}}{2^{1/2}} \qquad \therefore\ \rho = \frac{2\sqrt{3}}{3}a$$

**29. Find the radius of curvature at the point of the curve r = a sin nθ at the origin.**

*Sol :*

Given $r = a \sin n\theta$

$$r_1 = \frac{dr}{d\theta} = -an \cos n\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = an \sin n\theta$$

$$\rho = \frac{\left(r^2+r_1^2\right)^{3/2}}{r^2+2r_1^2-rr_2} = \frac{\left[(a\sin n\theta)^2+(-an\cos n\theta)^2\right]^{3/2}}{(a\sin n\theta)^2+2(-an\cos n\theta)^2-(a\sin n\theta)(an\sin n\theta)}$$

$$= \frac{\left[a^2\sin^2 n\theta+a^2n^2\cos^2 n\theta\right]^{3/2}}{a^2\sin^2\theta+2a^2n^2\cos n\theta-a^2n\sin^2\theta}$$

$\rho$ at origin $\theta = 0$

$$= \frac{\left[a^2\sin^2 n(0)+a^2n^2\cos^2 n(0)\right]^{3/2}}{a^2\sin^2 n(0)+2a^2n^2\cos(0)-a^2n\sin^2 n(0)}$$

$$= \frac{\left[a^2(0)+a^2n^2(1)\right]^{3/2}}{a^2(0)+2a^2n^2(1)-a^2n(0)}$$

$$= \frac{(a^2n^2)^{3/2}}{2a^2n^2} = \frac{a^2n^3}{2a^2n^2} = \frac{an}{2}$$

$\therefore$ Radius of curvature curve $r = a \sin n\theta$. at origin is $\rho = \frac{an}{2}$

---

**30. Find the points in the parabola $y^2 = 8x$ which the radius of curvature is $7\frac{13}{6}$.**

*Sol :*

Given that $y^2 = 8x$

$$2y\frac{dy}{dx} = 8$$

$$\frac{dy}{dx} = \frac{8}{2y}$$

$$y_1 = \frac{dy}{dx} = \frac{4}{\sqrt{8x}} = \frac{4}{2\sqrt{2x}} = \frac{2}{\sqrt{2x}} = \frac{\sqrt{2}}{x}$$

$$\frac{dy}{dx} = y_1 = \frac{\sqrt{2}}{x}$$

$$\frac{d^2y}{dx^2} = \sqrt{2}\frac{-1}{2}x^{-3/2}$$

$$= \frac{-1}{\sqrt{2x}}x^{-3/2}$$

$$\therefore \quad \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\left[\sqrt{2}\left(\frac{1}{\sqrt{x}}\right)\right]^2\right]^{3/2}}{\frac{-1}{\sqrt{2}}x^{-3/2}} = \frac{\left(1+\frac{2}{x}\right)^{3/2}}{\frac{-1}{\sqrt{2}x^{3/2}}} = \left(\frac{x+2}{x}\right)^{3/2}\sqrt{2}x^{3/2}$$

$$= -(x+2)^{3/2}\sqrt{2}$$

$$\rho = 7\frac{13}{16} = \frac{125}{16}$$

$$(x+2)^{3/2}\sqrt{2} = \frac{125}{16} \Rightarrow \left[(x+2)^{3/2}\right]^{2/3}\left(\sqrt{2}\right)^{2/3} = \left(\frac{125}{16}\right)^{2/3}$$

$$(x+2)2^{1/3} = \frac{5^2}{4^{4/3}}$$

$$(x+2)2^{1/3} = 5^2 \cdot 2^{-8/3}$$

$$(x + 2) = 5^2 \cdot 2^{-8/3} \cdot 2^{-1/3}$$

$$(x + 2) = 25.2^{-3}$$

$$x + 2 = \frac{25}{8}$$

$$x = \frac{25}{8} - 2 = \frac{25-16}{8} = \frac{9}{8}$$

Sub $x = 9/8$ in $y^2 = 8x$

$$y^2 = 8\left(\frac{9}{8}\right) = 9$$

$$y = \pm 3$$

$\therefore$ Points are $\left(\frac{9}{8}, 3\right), \left(\frac{9}{8}, -3\right)$



**31. Prove that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The radius of curvature is $\rho = \frac{a^2b^2}{p^3}$, p being the length of perpendicular from centre on the tangent at point (x, y).**

*Sol :*

Given, curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2} \Rightarrow \frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y}$$

$$\frac{dy}{dx} = \frac{-xb^2}{a^2y}$$

$$\frac{d^2y}{dr^2} = \frac{-b^2}{a^2}\left[\frac{1}{y}(1)+x\left(\frac{-1}{y^2}\right)\frac{dy}{dx}\right]$$

$$= \frac{-b^2}{a^2}\left[\frac{1}{y}-\frac{x}{y^2}\left[\frac{-b^2x}{a^2y}\right]\right]$$

$$= \frac{-b^2}{a^2}\left[\frac{a^2y^2+b^2x^2}{a^2y^3}\right] = \frac{-b^2}{a^4y^3}\left[a^2b^2\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}\right]\right]$$

$$= \frac{-b^2}{a^4y^3}\, a^2\, b^2\,(1)$$

$$\frac{d^2y^2}{dx^2} = \frac{-b^4}{a^2y^3}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{-xb^2}{a^2y}\right)^2\right]^{3/2}}{\frac{-b^4}{a^2y^3}} = \frac{-\left[a^4y^2+x^2b^4\right]^{3/2}}{a^6y^3}\times\frac{a^2y^3}{b^4}$$

$$= \frac{-\left[a^4b^2+b^4x\right]^{3/2}}{a^4b^4}$$

P is perpendicular distance from centre on the tangent at the point (x, y)

$$\rho = \frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}}} = \frac{1}{\sqrt{\frac{b^4x^2+a^4y^2}{a^4b^4}}} = \frac{a^6b^6}{\sqrt{b^4x^2+a^4y^2}}$$

$$p^3 = \left[\frac{a^2b^2}{\sqrt{b^4x^2+a^4y^2}}\right]^3 = \frac{a^6b^6}{\left(b^4x^2+a^4y^2\right)^{3/2}}$$

We have

$$\rho = \frac{-\left[a^4y^2+b^4x^2\right]^{3/2}}{a^4b^4}$$

by $p^3 = \frac{a^6b^6}{\left(b^4x^2+a^4y^2\right)^{3/2}} \Rightarrow \left(b^4x^2+a^4y^2\right)^{3/2} = \frac{a^6b^6}{p^3}$

Then,

$$\rho = \frac{-\frac{a^6b^6}{p^3}}{a^4b^4} = \frac{-a^6b^6}{p^3a^4b^4} = \frac{a^2b^2}{p^3}$$

$$\therefore \quad \rho = \frac{a^2b^2}{p^3}$$

---

**32. Find the radius of curvature of $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the points where the line y = x cuts it,**

*Sol :*

Given curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Differentiate with respect to 'x'

$$2\sqrt{x} + 2\sqrt{y}\frac{dy}{dx} = 0$$

Here $\frac{dy}{dx} = y_1$

$$2\sqrt{x} + 2\sqrt{y} \quad y_1 = 0$$

$$\sqrt{x} = -\sqrt{y}\,y_1$$

$$y_1 = \frac{-\sqrt{y}}{\sqrt{x}}$$

Differentiate with respect to 'x'

$$\frac{d^2y}{dx^2} = y_2$$

$$y_2 = \frac{-1}{\sqrt{x}}\left[\frac{1}{2\sqrt{y}}\cdot y_1\right] - \sqrt{y}\left[\frac{-1}{2x^{3/2}}\right]$$

$$= -\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{y}}\cdot\frac{-\sqrt{y}}{\sqrt{x}} + \frac{\sqrt{y}}{2x^{3/2}}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{1}{2x} + \frac{\sqrt{y}}{2x^{3/2}}$$

We know that radius of curvature $\rho = \frac{\left[1+\left(\frac{dy}{dx^2}\right)\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$\rho = \frac{\left[1+\left(\frac{-\sqrt{y}}{\sqrt{x}}\right)^2\right]^{3/2}}{\frac{1}{2x}+\frac{\sqrt{y}}{2x^{3/2}}} = \frac{\left(1+\frac{y}{x}\right)^{3/2}}{\frac{1}{2x}+\frac{\sqrt{y}}{2x^{3/2}}}$$

The point where the line y = x cuts the given curve is $\left(\frac{a}{4},\frac{a}{4}\right)$

$$\rho|_{(a/4,a/4)} = \frac{\left(1+\frac{\frac{a}{4}}{\frac{a}{4}}\right)^{3/2}}{\frac{1}{2\left(\frac{a}{4}\right)}+\frac{\sqrt{\frac{a}{4}}}{2\left(\frac{a}{4}\right)^{3/2}}} = \frac{\left[\frac{2\frac{a}{4}}{\frac{a}{4}}\right]}{\frac{2}{a}+\frac{1}{2}\cdot\frac{4}{a}}$$

$$= \frac{2^{3/2}}{\frac{4}{a}} = \frac{a}{\sqrt{2}}$$

$$\therefore \rho = \frac{a}{\sqrt{2}}$$



**33. Prove that the radius of curvature at any point of the catenary y = c cosh $\frac{x}{c}$ varies as the square of the ordinate.**

*Sol :*

Given curve is $y = c\cosh\frac{x}{c}$ .....(1)

differentiate equation (1) with respect to 'x'

$$\frac{dy}{dx} = c\,\frac{d}{dx}\left(\cosh\frac{x}{c}\right)$$

$$= c\sinh\frac{x}{c}\left(\frac{1}{c}\right)$$

$$\frac{dy}{dx} = \sinh\frac{x}{c} \qquad .....(2)$$

Differentiate equation (2) with respect to 'x'

$$\frac{d^2y}{dx^2} = \cosh\frac{x}{c} \cdot \frac{1}{c}$$

$$\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\frac{x}{c} \quad .....(3)$$

The expression for Radius of curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \quad .....(4)$$

sub (2) & (3) in (4)

$$\rho = \frac{\left[1+\left(\sinh\frac{x}{c}\right)^2\right]^{3/2}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)}$$

$$= \frac{c\left[\cosh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\cosh\left(\frac{x}{c}\right)}$$

$$= c\cosh^3\left(\frac{x}{c}\right)\Big/\cosh\left(x/c\right)$$

$$= c\cos h^2\left(x/c\right)$$

$$\rho = c\left(\frac{y^2}{c^2}\right)$$

$$\rho = \frac{y^2}{c}$$

**34. Find the value of $\rho$ for $r = a e^{\theta \cot \propto}$ and show that the radius of curvature subtends a right angle at the pole.**

*Sol :*

Given that

$$r = ae^{\theta \cot \propto} \quad .....(1)$$

differentiate equation (1) with respect to ‘$\theta$’

$$\frac{dr}{d\theta} = a\frac{d}{d\theta}\left(e^{\theta\cot\alpha}\right)$$

$$= a\, e^{\theta\cot\alpha}\left(\cot\alpha\right) \quad .....(2)$$

$$\frac{dr}{d\theta} = r\cot\alpha \quad \text{by (1)} \quad .....(3)$$

$$\frac{d\theta}{dr} = \frac{1}{r}\frac{1}{\cot\alpha} \quad .....(4)$$

$$r\frac{d\theta}{dr} = \tan\alpha$$

$$r\frac{d\theta}{dr} = \tan\phi$$

$$r.\frac{1}{r\cot\alpha} = \tan\phi \quad [\because \text{ by (3)}]$$

$$\frac{1}{\cot\alpha} = \tan\phi$$

$$\tan\alpha = \tan\phi$$

$\therefore \quad \theta = \alpha$

The chord of curvature through the pole $= 2\rho\sin\phi$ .....(5)

Where

$$\rho = r\frac{dr}{d\rho}$$

Consider

$$p = r\,.\,\sin\phi$$

$$= r\sin\alpha$$

$$\frac{d\rho}{dr} = \sin\alpha\,(1)$$

$$\therefore \quad \rho = \frac{r}{\sin\alpha}$$

Substituting the corresponding values in equation (4)

$$= 2\left(\frac{r}{\sin\alpha}\right)\sin\phi$$

$$= 2\frac{r}{\cancel{\sin}\alpha}\cancel{\sin}\alpha$$

$$= 2r$$

$\therefore$ The chord of curvature through the pole $= 2r$

**35. Prove that if ρ be the radius of curvature at any point ρ on the parabola $y^2 = 4ax$ and S be its focal, then $\rho^2$ various as $(Sp)^3$ .**

*Sol :*

Given parbola is $y^2 = 4ax$ and we know that, the radius of curvature

$$\rho = \frac{\left[1+\left[\frac{dy}{dx}\right]^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \quad .....(1)$$

Since $y^2 = 4ax$

$$y = \sqrt{4ax} = 2\sqrt{ax}$$

$$y = 2\sqrt{ax} \quad .....(2)$$

Differentiating equation (2) with respect to x

$$\frac{dy}{dx} = 2\sqrt{a}\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}} \quad .....(3)$$

Differentiating equation (3) with respect to 'x'

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\sqrt{a}/\sqrt{x}\right]$$

$$= \frac{d}{dx}\left[\sqrt{a}x^{-1/2}\right]$$

$$= \sqrt{a}\left(\frac{d}{dx}\left(x^{-1/2}\right)\right)$$

$$= \sqrt{a}\left(\frac{-1}{2}x^{-1/2-1}\right)$$

$$\frac{d^2y}{dx^2} = \sqrt{a}\left(\frac{-1}{2}x^{-3/2}\right) \quad .....(4)$$

Substituting the coordinating values equation (1)

$$\text{by (1)} \Rightarrow \rho = \frac{\left[1+\left(\sqrt{\frac{a}{x}}\right)^2\right]^{3/2}}{-\sqrt{a}\frac{1}{2}x^{-3/2}} = \frac{-2\left[1+\frac{a}{x}\right]^{3/2}x^{3/2}}{\sqrt{a}}$$

$$= \frac{-2x^{3/2}(x+a)^{3/2}}{x^{3/2}\sqrt{a}}$$

$$\rho = \frac{-2}{\sqrt{a}}(x+a)^{3/2} \qquad .....(5)$$

any point P(x,y) on the parabola, the total distance of the point P is,

$$SP = \sqrt{(x-a)^2 + y^2}$$

We have, Distance between the points

A ($x_1$, $y_2$) and B ($x_1$, $y_2$) is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{x^2 + y^2 + a^2 - 2ax}$$

$$= \sqrt{(x+a)^2}$$

$$AB = x + a \qquad .....(6)$$

Focal distance (Sp) = (x + a)

from (5) & (6)

$$\rho = \frac{-2}{\sqrt{a}}[SP]^{3/2}$$

$$\rho^2 = \frac{4}{a}[SP]^{3/2}$$

$\therefore$ Square of the radius of curvature ($\rho^2$) at any point of the curve varies as the cube of the focal distance.

**36. Show that $\frac{3\sqrt{3}}{2}$ is the Least value of $|\rho|$ for y = log x.**

*Sol :*

Given,

$$y = \log x$$

differentiating with respect to 'x'

$$\frac{dy}{dx} = \frac{1}{x} \qquad .....(1)$$

Again differentiate with respect to 'x'

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= -x^{-2}$$

$$= \frac{-1}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2} \qquad .....(2)$$

Radius of curvature (ρ) is given

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \qquad .....(3)$$

Sub $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ values from (1) & (2) in (3)

$$\rho = \frac{\left(1+\left(\frac{1}{x}\right)^2\right)^{3/2}}{\frac{-1}{x^2}}$$

$$= \frac{\left[1+\frac{1}{x^2}\right]^{3/2}}{\frac{-1}{x^2}}$$

$$= \left(\frac{x^2+1}{x^2}\right)^{3/2}\left(\frac{-x^2}{1}\right)$$

$$= \frac{\left(x^2+1\right)^{3/2}}{\left(x^2\right)^{3/2}}\left(\frac{-x^2}{1}\right)$$

$$= \frac{-\left(x^2+1\right)^{3/2}}{x^3}\left(\frac{x^2}{1}\right)$$

$$\rho = \frac{-\left(x^2+1\right)^{3/2}}{x}$$

By Neglecting – ve sign $\rho = \frac{\left(x^2+1\right)^{3/2}}{x}$ .....(4)

Differentiating equation (4) with respect to x

$$\frac{d\rho}{dx} = \frac{d}{dx}\left[\frac{(x^2+1)^{3/2}}{x}\right]$$

$$= \frac{x\left(\frac{3}{2}\right)(x^2+1)^{\frac{3}{2}-1}(2x)-(x^2+1)^{3/2}(1)}{x^2}$$

$$= \frac{3x^2(x^2+1)^{1/2}-(x^2+1)^{3/2}}{x^2}$$

$$= \frac{(x^2+1)^{1/2}\left[3x^2-(x^2+1)\right]}{x^2}$$

$$\frac{d\rho}{dx} = \frac{(x^2+1)^{1/2}(2x^2-1)}{x^2} \qquad .....(5)$$

For $\frac{d\rho}{dx}$ to be maxima or minima

$$\frac{d\rho}{dx} = 0$$

$$\Rightarrow \quad \frac{(x^2+1)^{1/2}(2x^2-1)}{x^2} = 0$$

$$2x^2 - 1 = 0$$

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{2}$$

Again differentiate equation (5) with respect to 'x'

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{(x^2+1)^{1/2}\left[2x^2-1\right]}{x^2}\right]$$

$$= \frac{x^2\frac{d}{dx}\left[(x^2+1)^{1/2}(2x^2-1)\right]-\left[(x^2+1)^{1/2}(2x^2-1)\right]2x}{(x^2)^2}$$

$$= \frac{x^2\left[(x^2+1)^{1/2}(4x)+(2x^2-1)\frac{1}{2}(x^2+1)^{\frac{-1}{2}}(2x)\right]-\left[(x^2+1)^{1/2}(2x^2-1)(2x)\right]}{x^4}$$

$$= \frac{(x^2)(x^2+1)^{1/2}(4x)+x(2x^2-1)(x^2+1)^{-1/2}-(x^2+1)^{1/2}(2x^2-1)(2x)}{x^4}$$

$$= \frac{x(x^2+1)^{1/2}\left[4x^2+(2x^2-1)(x^2+1)^{-1}-(2x^2-1)(2x)\right]}{x^4}$$

$$= \frac{x(x^2+1)^{1/2}\left[4x^2+\frac{(2x^2-1)}{x^2+1}-(2x^2-1)2x\right]}{x^4}$$

$$= \frac{x(x^2+1)^{1/2}\left[4x^2(x^2+1)+2x^2-1-2(2x^2-1)(x^2+1)\right]}{x^4(x^2+1)}$$

$$= \frac{4x^4+4x^2+2x^4-x^2-4x^4-4x^2+2x^2+2}{x^3\sqrt{x^2+1}}$$

$$\frac{d^2\rho}{dx^2} = \frac{2x^4+x^2+2}{x^3\sqrt{x^2+1}}$$

At $x = \frac{1}{\sqrt{2}}$

$$\left.\frac{d^2\rho}{dx^2}\right|_{x=\frac{1}{\sqrt{2}}} = \frac{2\left(\frac{1}{\sqrt{2}}\right)^4+\left(\frac{1}{\sqrt{2}}\right)^2+2}{\left(\frac{1}{\sqrt{2}}\right)^3\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2+1}}$$

$$= \frac{2\left(\frac{1}{4}\right)+\frac{1}{2}+2}{\frac{1}{2\sqrt{2}}}$$

$$= \frac{3}{\sqrt{3}/4}$$

$$= 4\sqrt{3} > 0$$

$\rho$ has minimum value at $x = \frac{1}{\sqrt{2}}$

Minimum value of $\rho$ is

$$\rho\Big|_{x=\frac{1}{\sqrt{2}}} = \frac{\left[1+\left(\frac{1}{\sqrt{2}}\right)^2\right]^{3/2}}{\frac{1}{\sqrt{2}}} = \frac{\left(1+\frac{1}{2}\right)^{3/2}}{\frac{1}{\sqrt{2}}}$$

$$= \left(\frac{3}{2}\right)^{3/2}(2)^{1/2}$$

$$= (3)^{3/2}(2)^{-3/2}(2)^{1/2}$$

$$\therefore\ \rho = \frac{3\sqrt{3}}{2}$$

**37. Find the radius of curvature at the origin for the curve $x^3 + y^3 - 2x^2 + 6y = 0$.**

*Sol :*

The Given curve is $x^3 + y^3 - 2x^2 + 6y = 0$

The curve passes through (0,0) and x-axis is a tangent to the curve at origin then radius of curvature at the origin is given by,

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{y} = 2\rho \qquad .....(1)$$

Dividing equation (1) by 'y'

$$\frac{x^3}{y} + \frac{y^3}{y} - \frac{2x^2}{y} + \frac{6y}{y} = 0$$

$$\frac{x^3}{y} + y^2 - \frac{2x^2}{y} + 6 = 0$$

Apply $\lim\limits_{\substack{x\to 0\\ y\to 0}}$ on both sides

$$\lim_{\substack{x\to 0\\ y\to 0}}\left[x\frac{x^2}{y} + y^2 - \frac{2x^2}{y} + 6\right] = 0$$

$$\lim_{\substack{x\to 0\\ y\to 0}} x\left(\frac{x^2}{y}\right) + \lim_{\substack{x\to 0\\ y\to 0}}\left(y^2\right) - 2\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x^2}{y}\right) + \lim_{\substack{x\to 0\\ y\to 0}}(6) = 0$$

$$0(2\rho) + 0 - 2(2\rho) + 6 = 0$$

$$-4\rho + 6 = 0 \Rightarrow -4\rho = -6$$

$$\rho = 3/2$$

## 3.4 Newtonian Method

**38. Define Neewtanian Method**

*Sol :*

If a curve passes through the origin and the axis of x is tangent at the origin.

Then $\lim \frac{x^2}{2y}$, as $x \to 0$. gives the radius of curvature at the origin.

Here $y_1(0) = \left(\frac{dy}{dx}\right)_{(0,0)} = 0$

Now, $\frac{x^2}{2y}$, assume the indeterminate form $\frac{0}{0}$ as $x \to 0$

$$\lim_{x\to 0} \frac{x^2}{2y} = \lim \frac{2x}{2y_1} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1}{y_2} = \frac{1}{y_2(0)}$$

The origin where x - axis is a tangent $\rho = \lim_{x\to 0}\left(\frac{x^2}{2y}\right)$

Similarly the origin where y - axis is tangent $\rho = \lim_{x\to 0}\left(\frac{y^2}{2x}\right)$

**39. Find the radius of curvature at the origin of the curve $x^3 - 2x^2y + 3xy^2 - 4y^2 + 5x^2 - 6xy + 7y^2 - 8y = 0$**

*Sol :*

The given curve

$$x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0$$

It is see that x-axis is the tangent to the origin. Divide by 'y' we get

$$\frac{x^3}{y} - \frac{2x^2y}{y} + \frac{3xy^2}{y} - \frac{4y^3}{y} + \frac{5x^2}{y} - \frac{6xy}{y} + \frac{7y^2}{y} - \frac{8y}{y} = 0$$

$$x \cdot \frac{y^2}{y} - 2x^2 + 3xy - 4y^2 + 5\frac{x^2}{y} - 6x + 7y - 8 = 0$$

Let $x \to 0$ so that $\underset{x\to 0}{\text{im}} \frac{x^2}{y} = 2\rho$

$0.(2\rho) - 2(0) + 5(2\rho) - 8 = 0$

$10\rho - 8 = 0$

$10\rho = 8$

$\rho = \frac{8}{10} = \frac{4}{5}$

$\therefore \rho = \frac{4}{5}$

**40. Find $\rho$ at the pole for the curve $r = a \sin n\theta$**

*Sol :*

The curve $r = a \sin n\theta$

$r_1$ $\theta$ are '0'

Intial line is tangent to the curve at origin

$$\rho = \lim_{\theta \to 0} \frac{r}{2\theta} = \lim_{\theta \to 0} \frac{a \sin n\theta}{2\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin n\theta}{n\theta} \cdot \frac{n\theta}{2} = \frac{n\theta}{2}$$

$$\therefore \rho = \frac{n\theta}{2}$$

**41. Find $\rho$ at origin of the curve $2x^4 + 3y^4 + 4x^2y + xy - y^2 + 2x = 0$**

*Sol :*

Given that $2x^4 + 3y^4 + 4x^2y + xy - y^2 + 2x = 0$

Since, the curve passes through the origin equating to zero, the lowest degree term

$2x = 0$

$\Rightarrow x = 0$

$\therefore$ y axis is tangent at the origin

Dividing by 2x.

$$\frac{2x^4}{2x} + \frac{3y^4}{2x} + \frac{4x^2y}{2x} + \frac{xy}{2x} - \frac{y^2}{2x} = 0$$

$$x^3 + \frac{3}{2}\frac{y^4}{x} + 2xy + \frac{2y}{2} - \frac{y^2}{2x} + 1 = 0$$

Applying the limits as $x \to 0, y \to 0$

we get

$$-\lim_{x \to 0} \frac{y^2}{2x} + 1 = 0 \quad \Rightarrow -\rho + 1 = 0$$

$\therefore \rho = 1$

**42. Find the r at origin of the curve $y = x^4 - 4x^3 - 18x^2$**

*Sol :*

$y = x^4 - 4x^3 - 18x^2 = 0$

The curve passes through the origin equating to zero, the lowest degree term y = 0

x-axis is tangent at the origin

Dividing by 2y

$$\frac{y}{2y} - \frac{x^4}{2y} + \frac{4x^3}{2y} + \frac{18x^2}{2y} = 0$$

$$\frac{1}{2} - \frac{x^4}{2y} + 2\frac{x^3}{2y} + 18\frac{x^2}{2y} = 0$$

Applying the limits as $x \to 0, y \to 0$

$$\frac{1}{2} - x^2 \cdot \frac{x^2}{2y} + 2x \cdot \frac{x^2}{2y} + 18\frac{x^2}{2y} = 0$$

$$\frac{1}{2} - 0(\rho) + 0(\rho) + 18\rho = 0$$

$$18\rho = -\frac{1}{2} \quad \Rightarrow \rho = -\frac{1}{36}$$

$$\therefore \quad \rho = \frac{1}{36}$$

**43. Find $\rho$ at the origin of the curve $3x^3 + y^3 + 5y^2 + 3y x^2 + 2x = 0$**

*Sol :*

$$3x^3 + y^3 + 5y^2 + 3y x^2 + 2x = 0$$

The curve passes through the origin, equating zero, the lowest degree term 2x = 0, x = 0

$\therefore$ y-axis is tangent at the origin

Dividing by 2x

$$\frac{3x^3}{2x} + \frac{y^3}{2x} + \frac{5y^2}{2x} + \frac{3yx^2}{2x} + \frac{2x}{2x} = 0$$

$$\frac{3}{2}x^2 + \frac{y^3}{2x} + \frac{5y^2}{2x} + \frac{3}{2}yx + 1 = 0$$

Applying the limits as $x \to 0, y \to 0$

$$0 + 0 + \frac{5}{2}(\rho) + 0 + 1 = 0$$

$$5\rho = -1$$

$$\rho = \frac{-1}{5} \Rightarrow \rho = \frac{1}{5}$$

**44. Find the $\rho$ at origin of the curve $x^4 - y^4 + x^3 - y^3 + x^2 - y^2 - y = 0$**

*Sol :*

The curve passes through the origin equating to zero, the lowest degree terms y = 0

$\therefore$ x-axis is tangent at the origin.

Dividing by 2y

$$\frac{x^4}{2y} - \frac{y^4}{2y} + \frac{x^3}{2y} - \frac{y^3}{2y} + \frac{x^2}{2y} - \frac{y^2}{2y} + \frac{y}{2y} = 0$$

$$0 - 0 + 0 - 0 + \rho - 0 + \frac{1}{2} = 0$$

$$\rho + \frac{1}{2} = 0 \quad \Rightarrow \rho = \frac{-1}{2} \quad \Rightarrow \rho = \frac{1}{2}.$$

**45. Find the radius of curvature of the cardioid $r = a(1 - \cos\theta)$ at the pole (origin)**

*Sol :*

Given equation $r = a(1 - \cos\theta)$

Differentiating with respect to '$\theta$'

$$\frac{dr}{d\theta} = a\sin\theta, \quad \frac{d^2r}{d\theta^2} = a\cos\theta$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{\left[[a(1+\cos\theta)]^2 + [a\sin\theta]^2\right]^{3/2}}{[a(1-\cos\theta)]^2 + 2[a\sin\theta]^2 - r[a\cos\theta]}$$

$$= \frac{\left[a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta\right]^{3/2}}{a^2\left(1+\cos^2\theta - 2\cos\theta\right) + 2a^2\sin^2\theta - \left(a(1-\cos\theta)\right)(a\cos\theta)}$$

$$= \frac{a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta}{a^2 + a^2\cos^2\theta - 2a^2\cos\theta + 2a^2\sin^2\theta - a^2\cos\theta + a^2\cos^2\theta}$$

At origin

$\rho = 0$

**46. Apply Newton's formula to find the radius of curvature at the origin for the cycloid $x = a(\theta + \sin\theta)$ , $y = a(1 - \cos\theta)$**

*Sol :*

Given Cycloids are

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

$$\frac{dx}{d\theta} = a(1 + \cos\theta);\ \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$= \tan\theta/2$$

$$\frac{dy}{dx} = 0, \quad \text{when } \theta = 0$$

Hence initial line is tangent to origin $\rho = \lim_{\theta\to 0}\frac{x^2}{2y} = \lim_{\theta\to 0}\frac{a^2(\theta+\sin\theta)^2}{2a(1-\cos\theta)}$

$$= \lim_{\theta\to 0}\frac{a}{2}\left[\frac{2(\theta+\sin\theta)(1+\cos\theta)}{\sin\theta}\right]$$

$$= \lim_{\theta\to 0}\frac{a}{2}\left[\frac{2(1+\cos\theta)^2 - \sin\theta(\theta+\sin\theta)}{\cos\theta}\right]$$

$$\therefore \rho = 4a$$

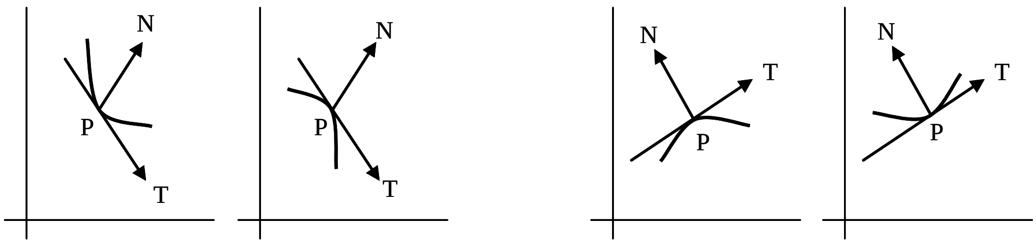


## 3.5 Centre of Curvature

The centre of curvature at any point P on a curve is the point which lies on the positive direction of the normal at P and is at a distance $\rho$ from it.

The centre of curvature at any point of a curve lies on the side towards which side the curve is concave.

How can we decide the positive direction of the normal? The direction of the tangent in the x increasing direction on the curve is considered to be positive. The positive direction of the normal is obtained by the rotating this positive direction of the tangent through $\pi/2$ in the antic lock wise direction.



At P, $\frac{d^2y}{dx^2}$ will be + ve and At P, $\frac{d^2y}{dx^2}$ will be – ve and

$\rho$ will be – ve. Centre of curvature

curvature is on PN will be on NP produced

**47. Find the coordinates of the centre of curvature of the parabola $y^2 = 4ax$.**

*Sol :*

Given parabola is $y^2 = 4ax$ .....(1)

differentiating with respect to 'x'

$$2y\, y_1 = 4a$$

$$y_1 = \frac{4a}{2y} = \frac{2a}{y} \quad .....(2)$$

Again differentiate with respect to x.

$$y_2 = \frac{-2a}{y^2}\, y_1 = \frac{-2a}{y^2}\left(\frac{2a}{y}\right)$$

$$y_2 = \frac{-4a^2}{y^3}$$

$$1+y_1^2 = 1+\left[\frac{2a}{y}\right]^2$$

$$1+y_1^2 = \frac{y^2+4a^2}{y^2}$$

The coordinate of the centre of curvature is given as,

$$X = x - \frac{y_1\left(1+y_1^2\right)}{y^2};$$

$$X = x - \frac{\frac{2a}{y}\left[\frac{y^2+4a^2}{y^2}\right]}{\frac{-4a^2}{y^3}}$$

$$X = x + \frac{y^3}{4a^2}\frac{2a}{y^3}\left(y^2+4a^2\right)$$

$$X = x + \frac{1}{2a}(y^2 + 4a^2)$$

$$X = 3x + 2a \quad [\because y^2 = 4ax]$$

$$Y = y + \frac{1+y_1^2}{y_2}$$

$$= y + \frac{\frac{y^2 + 4a^2}{y^2}}{\frac{-4a^2}{y^3}}$$

$$= y - \frac{y^3}{4a^2} \frac{(y^2 + 4a^2)}{y^2}$$

$$= y - \frac{y}{4a^2} (y^2 + 4a^2)$$

$$= \frac{y}{4a^2} [4a^2 - y^2 - 4a^2]$$

$$= \frac{-[y^2][y]}{4a^2}$$

$$= \frac{-4ax\sqrt{4ax}}{4a^2}$$

$$= \frac{-2x^{3/2}\sqrt{a}}{a}$$

$$= \frac{-2x^{3/2}}{\sqrt{a}}$$

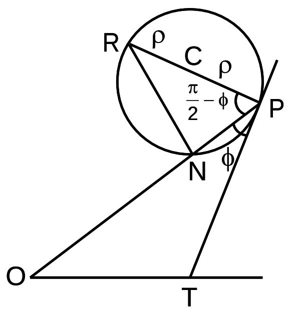
$\therefore$ The centre of curvature is (x,y) $= \left(3x + 2a, \frac{-2x^{3/2}}{\sqrt{a}}\right)$



## 3.6 Chord of Curvature

If there is any point P at the given curve and a circle having the radius of curvature r is drawn passing through P, then any chord PN is called the chord of curvature. If C be the centre of the circle then

$$PN = RP \cos RPN = 2\rho \cos RPN$$
$$= 2\rho \cos \alpha$$




**Figure**

Chord of curvature through pole (origin). In the figure PN is the required chord of curvature

$$PN = RP \cos RPN = 2\rho \cos\left(\frac{\pi}{2} - \phi\right)$$

Hence chord of curvature through pole = $2\rho \sin \phi$

Chord of curvature through the pole for the curve p = f(r)

We know r = r sin $\phi$ ; $\therefore$ $\sin\phi = \frac{p}{r}$

Hence chord of curvature $= 2\rho \,.\, \frac{p}{r}$

$$= 2 \,.\, r\, \frac{dr}{dp} \cdot \frac{p}{r} = 2p\frac{dr}{dp}$$

$$= 2f(r)\, \frac{dr}{dp}, \text{ i.e., } \frac{2f(r)}{f'(r)}$$

**48. Define Circle of Curvature.**

*Sol :*

The circle with radius equal to radius of curvature $\rho$ and its centre the centre of curvature (X, Y) is called the circle of curvature.

Then the equation of the circle of curvature is

$$(x - X)^2 + (y - Y)^2 = \rho^2$$

**49. Find the centre of curvature at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$ of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$. Find also the equation of the circle of curvature at that point.**

*Sol :*

The given curve is $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Differentiating with respect to 'x' we get,

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \qquad \therefore \left(\frac{dy}{dx}\right) = \left(\frac{\sqrt{y}}{\sqrt{x}}\right)$$

$$\frac{d^2y}{dx^2} = -\left(\frac{\frac{\sqrt{x}}{2\sqrt{y}}\frac{dy}{dx} - \frac{\sqrt{y}}{2\sqrt{x}}}{x}\right)$$

$\therefore$ At $\left(\frac{a}{4}, \frac{a}{4}\right)$, $y_1 = -1$; $y_2 = \frac{4}{a}$

If (X, Y) is the centre of curvature at (x, y)

$$X = x - \frac{y_1(1+y_1^2)}{y_2};\ Y = y + \frac{1+y_1^2}{y_2}$$

$\therefore$ At $\left(\frac{a}{4},\frac{a}{4}\right)$, $X = \frac{a}{4} + \frac{2}{(4/a)} = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$ and

$$Y = \frac{a}{4} + \frac{2}{(4/a)} = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

$\therefore$ $\left(\frac{3a}{4},\frac{3a}{4}\right)$ is the centre of the curvature.

At $\left(\frac{a}{4},\frac{a}{4}\right)$, $\rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)} = \dfrac{(2)^{3/2}}{(4/a)} = \dfrac{2\sqrt{2}.a}{4} = \dfrac{a}{\sqrt{2}}$

Hence equation of the circle of curvature at $\left(\frac{a}{4},\frac{a}{4}\right)$ is $\left(x-\frac{3a}{4}\right)^2 + \left(y-\frac{3a}{4}\right)^2 = \frac{a^2}{2}$

**50. Find the centre of curvature of the four cusped hypocycloid.**
**$x = a\cos^3\theta$, $y = a\sin^3\theta$, i.e., $x^{2/3}+y^{2/3} = a^{2/3}$.**

*Sol :*

We have $x = a\cos^3\theta$, then $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$

$y = a\sin^3\theta$, then $\frac{dy}{d\theta} = -3a\sin^2\theta\cos\theta$

Then $y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\tan\theta$

and $y_2 = \frac{d^2y}{dx^2} = -\sec^2\theta\frac{d\theta}{dx}$

$$= \frac{1}{3a}\sec4\theta\ \text{cosec}\ \theta$$

Now, if (X, Y) be the coordinates of the centre of curvature, then

$$X = x - \frac{y_1(1+y_1^2)}{y_2} = a\cos^2\theta + \frac{\tan\theta(1+\tan^2\theta)}{\frac{1}{3a}\sec^4\theta\cdot\text{cosec}\,\theta}$$

$= a\cos^3\theta + 3a\sin^2\theta\cos\theta$

and $Y = y + \frac{1+y_1^2}{y_2} = a\sin^3\theta + \frac{(1+\tan^2\theta)\,3\alpha}{\sec^4\theta\cdot\operatorname{cosec}\theta}$.

---

**51. In the curve $y = a\log\sec\left(\frac{x}{a}\right)$ prove that the chord of curvature parallel to the axis of y is of constant length**

*Sol :*

Given curve is, $y = a\log\sec\left(\frac{x}{a}\right)$ .....(1)

Differentiating equation (1) with respect to 'x'

$$\frac{dy}{dx} = a\frac{1}{\sec\left(\frac{x}{a}\right)}\frac{d}{dx}\left(\sec\left(\frac{x}{a}\right)\right)$$

$$= \frac{a}{\sec\left(\frac{x}{a}\right)}\sec\left(\frac{x}{a}\right)\tan\left(\frac{x}{a}\right).\frac{1}{a}$$

$$\frac{dy}{dx} = \tan\left(\frac{x}{a}\right) \quad .....(2)$$

Differentiating equation (2) with respect to 'x'

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\left(\frac{x}{a}\right)\right)$$

$$= \sec^2\left(\frac{x}{a}\right).\frac{1}{a}$$

$$\frac{d^2y}{dx^2} = \frac{1}{a}.\text{Sec}^2\left(\frac{x}{a}\right)$$

The chord of curvature parallel to y - axis

$$= 2\rho\cos\phi$$

$$= \frac{2\rho}{\sec\phi}$$

$$= \frac{2\rho}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$= \frac{2.\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \cdot \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$= \frac{2.\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

$$= \frac{2\left[1+\tan^2\left(\frac{x}{a}\right)\right]}{\frac{1}{a}\sec^2\left(\frac{x}{a}\right)}$$

$$= \frac{2a\sec^2\left(\frac{x}{a}\right)}{\sec^2\left(\frac{x}{a}\right)}$$

$= 2a$

$\therefore$ The chord of curvature parallel to y - axis is constant length.



## 3.7 Evolutes and Involutes - Properties of the evolute

**52. Define Evolutes and Involutes.**

*Sol :*

If the centre of curvature for each point on a curve be taken, we get nene curve called a evolute of the original one.

Also the original curve, when considered with respect to its evolute is called an involute.

➢ To determine the equation of an evolute

We know that the coordinates of the centre of curvature are given (x, y) where as

$$X = x - \frac{y_1\left(1+y_1^2\right)}{y_2}$$

$$Y = y + \frac{\left(1+y_1^2\right)}{y_2}$$

Now are eliminate x & y and then relation between X & Y is required equation of the evolute.

**53. Show that the evolute of the ellipse x = a cosθ, y = b sinθ is**

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

*Sol :*

Given x = a cosθ, y = b sinθ

$$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = y_1 \Rightarrow \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = \frac{-b}{a}\cot\theta$$

$$\frac{d^2y}{dx^2} = y_2 = \frac{b}{a}\text{cosec}^2\theta\frac{d\theta}{dx}$$

$$= \frac{b}{a}\text{cosec}^2\theta\cdot\frac{-1}{a\sin\theta} = \frac{-b}{a^2}\text{cosec}^3\theta$$

Let (x, y) be the coordinates of centre of curvature

Where,

$$X = a\cos\theta + \frac{\frac{b}{a}\cot\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{\frac{-b}{a^2}\text{cosec}^3\theta}$$

$$= a\cos\theta + \frac{b}{a}\cot\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)\cdot\frac{-a^2}{b}\sin^3\theta$$

$$= a\cos\theta - a\cot\theta\sin^3\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)$$

$$= a\cos\theta - a\cos\theta\sin^2\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta(1-\sin^2\theta) - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta(\cos^2\theta) - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$X = \cos^3\theta\left(\frac{a^2-b^2}{a}\right) \qquad .....(1)$$

Consider

$$Y = b\sin\theta + \frac{\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= b\sin\theta + \left(1+\frac{b^2}{a^2}\cot^2\theta\right)-\frac{a^2\sin^3\theta}{b}$$

$$= b\sin\theta - \frac{a^2\sin^3\theta}{b} - b\cos^2\theta\sin\theta$$

$$= b\sin\theta(1-\cos^2\theta) - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin^3\theta - \frac{a^2}{b}\sin^3\theta$$

$$= \frac{b^2-a^2}{b}\sin^3\theta$$

Now we have to elimate '$\theta$' from x & y to obtain an evolute from (1)

$$ax = \cos^3\theta(a^2-b^2)$$

$$(ax)^{2/3} = (a^2-b^2)^{2/3}\cos^2\theta$$

From (2) $\quad by = (a^2-b^2)\sin^3\theta$

$$(by)^{2/3} = (a^2-b^2)^{2/3}\sin^2\theta$$

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{3/2}$$

Which is required evolute

**54. Show that the parabolas $y = -x^2 + x + 1$, $x = -y^2 + y + 1$ have the same circle of curvature at the point (1, 1)**

*Sol :*

Given curve are

$$x = -y^2 + y + 1,$$
$$y = -x^2 + x + 1$$

Consider

$x = -y^2 + y + 1$

$1 = -2yy_1 + y_1$

$1 = (-2y + 1)\, y_1$

$$y_1 = \frac{1}{-2y+1}$$

$$y_2 = \frac{2y_1}{(-2y+1)^2} = \frac{2}{(-2y+1)^2}\,\frac{1}{-2y+1} = \frac{2}{(-2y+1)^3}$$

We know that circle of curvature is

$(x - X)^2 + (y - Y)^2 - p_2$

$$X = x - \frac{y_1(1+y_1^{\,2})}{y_2}$$

$$x - \frac{\frac{1}{-2y+1}\left[1+\frac{1}{(-2y+1)}\right]}{\frac{2}{(-2y+1)^3}}$$

$$X \text{ at } (1, 1) = 1 - \frac{-1(1+1)}{\frac{1}{-1}} \qquad = 1 - \frac{-(2)}{-2} = 1 - 1 = 0$$

$$Y = y + \frac{1+y_1^{\,2}}{y_2}$$

$$= y + \frac{1+\frac{1}{(-2y+1)^2}}{\frac{2}{(-2y+1)^3}}$$

$$Y \text{ at } (1, 1) = 1 + \frac{1+1}{\frac{2}{-1}} = 1 - 1 = 0$$

Consider $\rho = \dfrac{(1+y_1^{\,2})^{y_2}}{y_2}$

$$= \frac{\left[1+\frac{1}{(-2y+1)^2}\right]^{3/2}}{\frac{2}{(-2y+1)^3}}$$

$\rho$ at (1, 1) = $\frac{(1+1)^{3/2}}{-2} = -2^{3/2} \cdot -2^{-1}$

$= -2^{1/2} = -\sqrt{2}$

$\rho = -\sqrt{2} \Rightarrow \rho^2 = 2$

Consider

$y = -x^2 + x + 1$

$y_1 = -2x + 1$

$y_2 = -2$

We know that the circle of curvature is

$(x - X)^2 + (y - Y)^2 = \rho^2$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{(-2x+1)[1+(-2x+1)^2]}{2}$$

$$= x + \frac{(-2x+1)[1+(-2x+1)^2]}{2}$$

$$X \text{ at } (1, 1) = 1 + \frac{(-2(1)+1)\ [1+(-2(1)+1)^2]}{2}$$

$$= 1 + \frac{(-1)\ [1+1]}{2}$$

$$= 1 + \frac{2}{2}$$

$$= 1 - 1 = 0$$

Now $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{[1+(-2x+1)^2]^{3/2}}{2}$$

$$\rho \text{ at } (1, 1) = \frac{[1+(-2(1)+1)^2]^{3/2}}{2}$$

$$= \frac{(1+1)^{3/2}}{2}$$

$$= 2^{3/2} \cdot 2^{-1} \Rightarrow 2^{1/2}$$

$\rho^2 = 2$

$\therefore$ Circle of curvature is $x^2 + y^2 = 2$ for the curve $x = -y^2 + y + 1$ and the curve have same circle of curvature.

**55. Show that $\left(x-\frac{3}{4}a\right)^2 + \left(y-\frac{3}{4}a\right)^2 = \left(\frac{a}{2}\right)^2$ is the circle of curvature of the curve $\sqrt{x} + \sqrt{y} = y$ at the point $\left(a/4, a/4\right)$**

*Sol :*

Given curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Differentiating the above curve

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} y_1 = 0$$

$$\frac{1}{2\sqrt{x}} = \frac{-y_1}{2\sqrt{y}}$$

$$\frac{1}{\sqrt{x}} = \frac{-2y_1}{2\sqrt{y}} = \frac{-y_1}{\sqrt{y}}$$

$$\frac{1}{\sqrt{x}} = \frac{-y_1}{\sqrt{y}} \Rightarrow y_1 = \frac{-\sqrt{y}}{\sqrt{x}}$$

Again differentiate the above

$$y_2 = \frac{-1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{y}} y_1 - \sqrt{y}\left(\frac{-1}{2x^{3/2}}\right)$$

$$y_2 = \frac{-1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{y}} \cdot \frac{-\sqrt{y}}{\sqrt{x}} + \frac{\sqrt{y}}{2x^{3/2}}$$

$$y_2 = \frac{1}{2x} + \frac{y}{2x^{3/2}}$$

We know that equation of circle of curvature is

$$(x - X)^2 + (y - Y)^2 = \rho^2$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$Y = y + \frac{1+y_1^2}{y_2}$$

Consider

$$X = x = \frac{y_1(1+y_1^2)}{y_2}$$

$$\Rightarrow \qquad x - \frac{-\frac{\sqrt{y}}{\sqrt{x}}\left(1-\frac{y}{x}\right)}{\frac{1}{2x}+\frac{y}{2x^{3/2}}}$$

Now

$$X \text{ at } \left(\frac{a}{4}, \frac{a}{4}\right) = \frac{a}{4} + \frac{\frac{\sqrt{a/4}}{\sqrt{a/4}}\left[1+\frac{a/4}{a/4}\right]}{\frac{1}{2\left(2/4\right)}+\frac{\sqrt{a}/4}{2\left(a/4\right)^{3/2}}}$$

$$= \frac{a}{4} + \frac{2}{\frac{2}{a}+\frac{1}{\alpha}\left(\frac{a}{4}\right)^{1/2-3/2}}$$

$$= \frac{a}{4} + \frac{2}{\frac{2}{a}+\frac{1}{2}\left(\frac{a}{4}\right)^{-2/2}}$$

$$= \frac{a}{4} + \frac{2}{\frac{2}{a}+\frac{1}{2}\left(\frac{a}{4}\right)^{-1}}$$

$$= \frac{a}{4} + \frac{2}{\frac{2}{a}+\frac{1}{2}\left(\frac{4}{a}\right)}$$

$$= \frac{a}{4} + \frac{2}{\frac{4}{a}} = \frac{a}{4} + \frac{2a}{4} = \frac{3a}{4}$$

$$Y = y + \frac{1+y_1^{\,2}}{y_2} = y + \frac{1+\frac{y}{x}}{\frac{1}{2x}+\frac{y^{1/2}}{2x^{3/2}}}$$

$$Y \text{ at } \left(a/4,\ a/4\right) = \frac{a}{4} + \frac{1+\frac{a/4}{a/4}}{\frac{1}{2\left(\frac{a}{4}\right)}+\frac{\left(a/4\right)^{1/2}}{2\left(\frac{a}{4}\right)^{3/2}}}$$

$$= \frac{a}{4} + \frac{2}{\frac{2}{a} + \frac{1}{2\left(a/4\right)}}$$

$$= \frac{a}{4} + \frac{2a}{4}$$

$$= \frac{3a}{4}$$

$$\rho = \frac{\left(1 + y_1^2\right)^{3/2}}{y_2} = \frac{1 + \left(\frac{y}{x}\right)^{3/2}}{\frac{1}{2x} + \frac{\sqrt{y}}{2x^{3/2}}}$$

$$\rho \text{ at } \left(a/4, \ a/4\right) = \frac{1 + \left(\frac{a/4}{a/4}\right)^{3/2}}{\frac{1}{2\left(a/4\right)} + \frac{\sqrt{a/4}}{2.\left(a/4\right)^{3/2}}}$$

$$= \frac{2}{\frac{2}{a} + \frac{2}{a}} = 2 . \frac{a}{4}$$

$$= \frac{a}{2}$$

$\therefore$ Circle of curvature is

$(x - X)^2 + (y - Y)^2 = \rho^2$

$$\left(x - \frac{3}{4}a\right)^2 + \left(y - \frac{3}{4}a\right)^2 = \left(\frac{a}{2}\right)^2$$

**56 If (X, Y) be the co-ordinate of the centre of curvature if the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at (x, y) then prove that X + Y = 3(x + y)**

*Sol :*

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} y_1 = 0$$

$$\frac{1}{2\sqrt{y}}\ y_1 = \frac{-1}{2\sqrt{x}}$$

$$y_1 = \frac{-2\sqrt{y}}{2\sqrt{x}} = \frac{-\sqrt{y}}{\sqrt{x}}$$

$$y_2 = \frac{-1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{y}} \cdot y_1 - \sqrt{y} \cdot \frac{-1}{2x^{3/2}}$$

$$= \frac{-1}{\sqrt{x}}\ \frac{1}{2\sqrt{y}}\left(\frac{-\sqrt{y}}{\sqrt{x}}\right) + \frac{\sqrt{y}}{2x^{3/2}}$$

$$y_2 = \frac{1}{2x} + \frac{\sqrt{y}}{2x^{3/2}}$$

We know that

$$X = x - \frac{y_1(1+y_1^{\,2})}{y_2},\ Y = y + \frac{1+y_1^{\,2}}{2}$$

Consider

$$X = x - \frac{-\sqrt{y}/\sqrt{x}\left(1+\left(-\sqrt{y}/\sqrt{x}\right)^2\right)}{\frac{1}{2x}+\frac{\sqrt{y}}{2x^{3/2}}}$$

$$= x + \frac{\frac{\sqrt{y}}{\sqrt{x}}\left(1+\frac{y}{x}\right)}{\frac{1}{2x}+\frac{\sqrt{y}}{2x^{3/2}}}$$

$$= x + \frac{\frac{\sqrt{y}(x+y)}{\sqrt{x}.x}}{\frac{1}{2x}+\frac{\sqrt{y}}{2x\sqrt{x}}}$$

$$= x + \frac{\frac{\sqrt{y}\ (x+y)}{x^{3/2}}}{\frac{\sqrt{x}+\sqrt{y}}{2x^{3/2}}}$$

$$X = x + \frac{2\sqrt{y}\ (x+y)}{\sqrt{x}+\sqrt{y}}$$

Consider

$$Y = y + \frac{1+\left(\frac{-\sqrt{y}}{\sqrt{x}}\right)^2}{\frac{1}{2x}+\frac{\sqrt{y}}{2x^{3/2}}}$$

$$= y + \frac{1+y/x}{\frac{1}{2x}+\frac{\sqrt{y}}{2x^{3/2}}}$$

$$= y + \frac{\frac{x+y}{x}}{\frac{\sqrt{x}+\sqrt{y}}{2x^{3/2}}}$$

$$= y + \frac{(x+y)\ (2\sqrt{x})}{\sqrt{x}+\sqrt{y}}$$

Consider

$$X + Y = x + \frac{2\sqrt{y}\ (x+y)}{\sqrt{x}+\sqrt{y}} + y + \frac{2\sqrt{x}\ (x+y)}{\sqrt{x}+\sqrt{y}}$$

$$= x + y + \frac{2\sqrt{y}\ (x+y) + 2\sqrt{x}\ (x+y)}{\sqrt{x}+\sqrt{y}}$$

$$= x + y + 2\ (x + y)\ \frac{\left[\sqrt{y}+\sqrt{x}\right]}{\sqrt{x}+\sqrt{y}}$$

$$= x + y + 2x + 2y$$

$$= 3x + 3y$$

$$X - Y = 3(x + y)$$

**57. In the curve $y = C \cosh\left(\frac{x}{C}\right)$, show that co-ordinates of the centre of curvature are**

$$\mathbf{X = x - y\left(\frac{y^2}{C^2}-1\right)^{1/2},\ Y = 2y}$$

*S0l :*

Given curve = C = cos h $\left(\frac{x}{C}\right)$

$$y_1 = C \sinh \frac{x}{C} . \left(\frac{1}{C}\right)$$

$$y_1 = \sinh \frac{x}{C}$$

$$y_2 = \cosh \frac{x}{C} . \frac{1}{C}$$

We know that the coordinates of the centre of curvature are

$$X = x - \frac{y_1(1+y_1^2)}{y_2}, \quad Y = y + \frac{(1+y_1^2)}{y_2}$$

$$X = x - \frac{\sinh\frac{x}{C}\left(1+\sinh^2\frac{x}{C}\right)}{\frac{1}{C}\cosh\frac{x}{C}}$$

$$X = x - \frac{\sinh\frac{x}{C}\left(\cosh^2\frac{x}{C}\right)}{\frac{1}{C}\cosh\frac{x}{C}}$$

$$X = x - C\cosh\frac{x}{C} \sinh\frac{x}{C}$$

$$X = x - y\left(\sqrt{\cosh^2\frac{x}{C} - 1}\right)$$

$$X = x - y\sqrt{\frac{y^2}{C^2} = -1}$$

$$= y + \frac{1+\sinh^2\frac{x}{C}}{\frac{1}{C}\cosh\frac{x}{C}}$$

$$= y + \frac{\cosh^2\frac{x}{C}.C}{\cosh\frac{x}{C}}$$

.

$$= y + C\cosh\frac{x}{C}$$

$$Y = y + y$$

$$Y = 2y$$

$$\therefore \quad X = x - y\sqrt{y/_{C^2}} - 1 \;\; \& \;\; Y = 2y.$$

**58.** Find the equation of circle of curvature at the point (1, –1) of the curve y = $x^3$ **– 6x² + 3x + 1.**

*Sol :*

Given

$$y = x^3 - 6x^2 + 3x + 1$$

$$y_1 = 3x^2 - 12x + 3$$

$$y_2 = 6x - 12$$

We know that equation of circle is

$$(x - X)^2 + (y - Y)^2 = \rho^2$$

Where

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{(3x^2-12x+3)(1+(3x^2-12x+3)^2)}{6x-12}$$

$$X \text{ at } (1, 1) = 1 - \frac{\left(3(1)-12(1)+3\right)\left(1+(3(1)^2-12(1)+3)\right)^2}{6(1)-12}$$

$$= 1 - \frac{(6-12)(1+(3-12+3)^2)}{-6}$$

$$= 1 - \frac{(6)\,(1-(-6)^2)}{-6}$$

$$= 1 - \frac{(-6)\,(37)}{-6} = 1 - 37 = 36$$

$$Y = y + \frac{1+y_1^2}{y_2} = y + \frac{\left(1+(3x^2-12x+3)^2\right)}{6x-12}$$

$$Y \text{ at } (1, -1) = -1 + \frac{\left[1+(3(1)^2-12(1)+3)^2\right]}{6(1)-12}$$

$$= -1 + \frac{\left[1+(3-12+3)^2\right]}{-6}$$

$$= -1 + \frac{1+36}{-6}$$

$$= -1 + \frac{37}{-6} \Rightarrow \frac{-43}{6}$$

$$\rho = \left[\frac{1+y_1{}^2}{y_2}\right]^{3/2}$$

$$= \frac{\left[1+(3x^2-12x+3)^2\right]^{3/2}}{6x-12} = \frac{\left[1+(3-12+3)^2\right]^{3/2}}{-6}$$

$$= \frac{(37)^{3/2}}{-6}$$

$$\rho^2 = \frac{(37)^3}{36} \quad \frac{(37)^3(37)^{1/2}}{-6} \Rightarrow \frac{(37)^3}{36}$$

$\therefore$ Circle of curvature is

$$(x + 36)^2 + \left(y+\frac{43}{6}\right)^2 = \frac{37^3}{36}$$

**59. If $C_x$ and $C_y$ be the chords of curvature parallel to the axes at any point of the curve $y = ae^{x/a}$. Prove that $\frac{1}{C_{x^2}} + \frac{1}{C_{y^2}} = \frac{1}{2aC_x}$.**

*Sol :*

Given

$$y = ae^{x/a}$$

$$y_1 = ae^{x/a}(\tfrac{1}{a})$$

$$y_1 = e^{x/a}$$

$$y_2 = e^{x/a} \cdot \frac{1}{a}$$

$C_x$ is the chord of curvature parallel to x-axis.

$$C_x = 2\rho \sin\phi$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

.

Consider

$$\rho = \frac{\left[1+\left(e^{x/a}\right)^2\right]^{3/2}}{\dfrac{e^{x/a}}{a}}$$

$$= \left[1+\left(e^{x/a}\right)^2\right]^{3/2} \cdot \frac{a}{x^{x/a}}$$

We know that

$$\tan\phi = \frac{dy}{dx} = e^{x/a}$$

Now

$$\cos\phi = \frac{1}{\sqrt{1+\tan^{-1}\phi}} = \frac{1}{\sqrt{1+(e^{x/a})^2}}$$

and $$\sin\phi = \sqrt{1-\cos^2\phi} = \sqrt{1-\left[\frac{1}{\sqrt{1+(e^{x/a})^2}}\right]}$$

$$= \sqrt{\frac{1+(e^{x/a})^2-1}{1+(e^{x/a})^2}} = \sqrt{\frac{(e^{x/a})^2}{1+(e^{x/a})^2}}$$

$$= \frac{e^{x/a}}{\sqrt{1+(e^{x/a})^2}}$$



Consider

$$C_x = 2\rho \sin\phi$$

$$= 2\left[1+(e^{x/a})^2\right]^{3/2} \cdot \frac{a}{e^{x/a}}\left[\frac{e^{x/a}}{\sqrt{1+(e^{x/a})^2}}\right]$$

$$= 2a\left[1+(e^{x/a})^2\right]^{3/2}\left(1+(e^{x/a})^2\right)^{1/2}$$

$$C_x = 2a\left[1+(e^{x/a})^2\right]$$

Let $C_y$ be the chord of curvature parallel to y-axis

$$C_y = 2\rho \cos\rho\phi$$

$$= 2\,\frac{a}{e^{x/a}}\left[1+(e^{x/a})^2\right]^{3/2}\frac{1}{\sqrt{1+(e^{x/a})^2}}$$

$$= 2\,\frac{a}{e^{x/a}}\left[1+(e^{x/a})^2\right]$$

Consider

$$\frac{1}{C_{x^2}}+\frac{1}{C_{y^2}} = \frac{1}{4a^2\left[1+(e^{x/a})^2\right]^2}+\frac{(e^{x/a})^2}{4a^2\left[1+(e^{x/a})^2\right]^2}$$

$$= \frac{1+(e^{x/a})^2}{4a^2\left[1+(e^{x/2})^2\right]^2} = \frac{1}{4a^2\left[1+(e^{x/a})^2\right]}$$

$$= \frac{1}{2a\,.\,2a\left[1+(e^{x/a})^2\right]}$$

$$= \frac{1}{2a\ C_x}$$

$$\therefore\quad \frac{1}{C_{x^2}}+\frac{1}{C_{y^2}} = \frac{1}{2aC_x}$$



**60. Show that the chord of the curvature through the pole of the curve $r^n = a^n \cos n\theta$ is $\frac{2r}{n+1}$.**

*Sol :*

Given

$$r^n = a^n \cos n\theta$$

$$nr^{n-1}\frac{dr}{d\theta} = a^n(-n\sin\theta)$$

$$\frac{dr}{d\theta} = \frac{-na^n\sin\theta}{nr^{n-1}}$$

$$\frac{dr}{d\theta} = \frac{-a^n\sin n\theta}{r^{n-1}}$$

Consider

$$\tan\phi = r\,\frac{d\theta}{dr}$$

$$= \frac{-r\, r^{n-1}}{a^n \sin n\theta}$$

$$= \frac{-r^n}{a^n \sin n\theta} = \frac{-a^n \cos n\theta}{\sin n\theta}$$

$$\tan\phi = -\cot n\theta = \tan\left(\frac{\pi}{2} + n\theta\right)$$

$$\Rightarrow \quad \phi = \frac{\pi}{2} + n\theta$$

$$\rho = r \sin\phi$$

$$= r \sin\left(\frac{\pi}{2} + n\theta\right)$$

$$= r \cos n\theta = r\, \frac{r^n}{a^n}$$

$$\rho = \frac{r^{n-1}}{a^n}$$

$$\frac{dp}{dr} = \frac{1}{a^n}\,(n + 1)r^n$$

We know that chord of curvature is given by

$$2\rho\, \frac{dr}{dp} = 2\, \frac{r^{n+1}}{a^n} \cdot \frac{a^n}{(n+1)r^n}$$

$$= \frac{2r}{n+1}$$

$\therefore$ Chord of curvature through the pole of curve $r^n = a^n \cos n\theta$ is $\frac{2r}{n+1}$

**61. Show that the chord of curvature through the pole of curve $r^2 \cos 2\theta = a^2$.**

*Sol :*

Given curve $r^2 \cos 2\theta = a^2$

$$r^2 = a^2\, \frac{1}{\cos 2\theta}$$

$$r^2 = a^2 \sec 2\theta$$

$$2r\, \frac{dr}{d\theta} = a^2\, 2\sec 2\theta \,.\, \tan 2\theta$$

$$\frac{dr}{d\theta} = \frac{a^2}{2r}\, 2\sec 2\theta \,.\, \tan 2\theta$$

$$= \frac{a^2}{r} \sec 2\theta \,.\, \frac{\sin 2\theta}{\cos 2\theta} = \frac{a^2}{r} \frac{1}{\cos 2\theta} \frac{\sin 2\theta}{\cos 2\theta} = \frac{a^2}{r} \frac{\sin 2\theta}{\cos^2 2\theta}$$

$$r^2 \cos 2\theta = a^2 \Rightarrow \cos 2\theta = \frac{a^2}{r^2}$$

$$\frac{1}{\cos 2\theta} = \frac{r^2}{a^2}$$

$$= \frac{a^2}{r} \,.\, \frac{\sin 2\theta}{\cos 2\theta} \,.\, \frac{r^2}{a^2}$$

$$\frac{dr}{d\theta} = r \frac{\sin 2\theta}{\cos 2\theta} = r \tan 2\theta$$

We know that

$$\tan\phi = r \frac{dr}{d\theta}$$

$$= r \frac{1}{r \tan 2\theta}$$

$$= \cot 2\theta$$

$$\tan\phi = \tan\left(\frac{\pi}{2} - 2\theta\right)$$

$$\phi = \frac{\pi}{2} - 2\theta$$

Now

$$\rho = r \sin\phi$$

$$= r \sin\left(\frac{\pi}{2} - 2\theta\right)$$

$$= r \cos 2\theta$$

$$= r \frac{a^2}{r^2} = \frac{a^2}{r}$$

$$\frac{dp}{dr} = a^2\left(\frac{-1}{r^2}\right) = \frac{-a^2}{r^2}$$

We know that chord of curvature is

$$2p \frac{dr}{dp} = -2 \frac{a^2}{r} \frac{r^2}{a^2} = -2r$$

$\therefore$ Chord of curvature through the pole of curve
$r^2 \cos 2\theta = a^2$ is $-2r$

**62. Show that for hyperbola $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ a $^4$x = (a$^2$ + b$^2$) x$^3$, b $^4$y = (a$^2$ + b$^2$) y$^3$ and the equation of evolute is $(ax)^{2/3} - (ay)^{2/3} = (a^2 + b^2)^{2/3}$.**

*Sol :*

Given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$

A point on hyperbola is $(a\sec\theta,\ b\tan\theta)$

$x = a\sec\theta, \qquad y = b\tan\theta$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{b\sec^2\theta}{a\sec\theta\,.\tan\theta}$$

$$= \frac{b}{a}\frac{\sec\theta}{\tan\theta}$$

$$= \frac{b}{a}\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

$$= \frac{b}{a}\frac{1}{\sin\theta}$$

$$= \frac{b}{a}\operatorname{cosec}\theta$$

$$y_1 = \frac{dy}{dx} = \frac{b}{a}\operatorname{cosec}\theta$$

$$y_2 = -\frac{b}{a}\operatorname{cosec}\theta\cot\theta\frac{d\theta}{dx}$$

$$= \frac{-b}{a}\operatorname{cosec}\theta\,.\cot\theta\,\frac{1}{a\sec\theta\,.\,\tan\theta}$$

$$= \frac{-b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}$$

(X, Y) be the coordinate of centre of curvature

Where

$$X = x - \frac{y_1(1+y_1^2)}{y_2}, \quad Y = y + \frac{(1+y_1^2)}{y_2}$$

Consider

$$X = a\sec\theta - \frac{\frac{b}{a}\ \text{cosec}\theta\left(1+\frac{b^2}{a^2}\text{cosec}^2\theta\right)}{\frac{-b\cos^3\theta}{a^2\sin^3\theta}}$$

$$= a\sec\theta + \frac{b}{a}\ \text{cosec}\theta\left(1+\frac{b^2}{a^2}\cos^2\theta\right)\frac{a^2\sin^3\theta}{b\cos^3\theta}$$

$$= a\sec\theta + a\frac{\sin^2\theta}{\cos^3\theta} + \frac{b^2}{a}\sec^3\theta$$

$$= a\sec\theta\,(1 + \tan^2\theta) + \frac{b^2}{a}\sec^3\theta$$

$$= a\sec\theta\,(\sec^2\theta) + \frac{b^2}{a}\sec^3\theta$$

$$= a\sec^3\theta + \frac{b^2}{a}\sec^3\theta$$

$$= \frac{(a^2+b^2)\sec^3\theta}{a}$$

$$X = \frac{(a^2+b^2)x^3/a^3}{a}$$

$$= \frac{(a^2+b^2)x^3}{a^4}$$

$$a^4 X = (a^2 + b^2)\,x^3$$

Consider

$$Y = b\tan\theta + \frac{\left(1+\frac{b^2}{a^2}\text{cosec}^2\theta\right)}{\frac{-b\cos^3\theta}{a^2\sin^3\theta}}$$

$$= b\tan\theta - \frac{a^2\sin^3\theta}{b\cos^3\theta}\left(1+\frac{b^2}{a^2}\ \text{cosec}^2\theta\right)$$

$$= b\tan\theta - \frac{a^2}{b}\tan^3\theta - b\tan\theta\sec^2\theta$$

$$= b\tan\theta\,(1 - \sec^2\theta) - \frac{a^2}{b}\tan^3\theta$$

$$= -b\tan^3\theta - \frac{a^2}{b}\tan^3\theta$$

$$= -\tan^3\theta\,\frac{(b^2 - a^2)}{b}$$

$b^4 y = -y3\,(a^2 + b^2)$

Now, we have to eliminate $\theta$ from X + Y to obtain an evolute.

$$aX = (a^2 + b^2)\sec^3\theta$$

$$(aX)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}(\sec^3\theta)^{\frac{2}{3}}$$

$$(aX)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}\sec^2\theta$$

$$bY = (a^2 + b^2)\tan^3\theta$$

$$bY^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}(\tan^3\theta)^{\frac{2}{3}}$$

$$(bY)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}\tan^2\theta$$

$$(aX)^{\frac{2}{3}} - (bY)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}\sec^2\theta - (a^2 + b^2)^{\frac{2}{3}}\tan^2\theta$$

$$= (a^2 + b^2)^{\frac{2}{3}}\left[\sec^2\theta - \tan^2\theta\right]$$

$(aX)^{\frac{2}{3}} - (bY)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$ is the required evolute.

$b^4 y = -y3\,(a^2 + b^2)$

Now, we have to eliminate $\theta$ from X + Y to obtain an evolute.

$$aX = (a^2 + b^2)\sec^3\theta$$

$$(aX)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}(\sec^3\theta)^{\frac{2}{3}}$$

$$(aX)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}\sec^2\theta$$

$$bY = (a^2 + b^2)\tan^3\theta$$

$$bY^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}(\tan^3\theta)^{\frac{2}{3}}$$

$$(bY)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}\tan^2\theta$$

$$(aX)^{\frac{2}{3}} - (bY)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}\sec^2\theta - (a^2 + b^2)^{\frac{2}{3}}\tan^2\theta$$

$$= (a^2+b^2)^{\frac{2}{3}}\left[\sec^2\theta-\tan^2\theta\right]$$

$(aX)^{\frac{2}{3}}-(bY)^{\frac{2}{3}} - (a^2+b^2)^{\frac{2}{3}}$ is the required evolute.

---

**63. Find the evolute of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Deduce the evolute of a rectangular hyperbola.**

*Sol :*

Given hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ... Eqn. (1)

Difference with respect to 'x', we get

$$\frac{2x}{a^2}-\frac{2yy_1}{b^2}=0 \Rightarrow y_1=\frac{b^2}{a^2}\frac{x}{y} \quad \text{... Eqn. (2)}$$

Difference $y_1$ with respect to 'x'; we get

$$y_2 = \frac{b_2}{a^2}\left(\frac{y-xy_1}{y^2}\right)=\frac{b^2}{a^2}\left(\frac{y-x\frac{b^2}{a^2}\frac{x}{y}}{y^2}\right)$$

$$= \frac{b^2}{a^4y^3}(a^2y^2-x^2b^2) = \frac{b^2}{a^4y^3}(-b^2a^2) \quad \text{[From eqn. (1)]}$$

$$= \frac{-b^4}{a^2y^3}$$

We know that centre of curvature is given by

$$X = x-\frac{y_1(1+y_1^2)}{y_2},\ Y = y + \frac{1+y_1^2}{y_2}$$

$$= x-\frac{b^2x}{a^2y}\left(1+\frac{b^4x^2}{a^4y^2}\right)\left(\frac{a^2y^3}{-b^4}\right) = x^3(a^2+b^2)\left(\frac{1}{a^4}\right)$$

and $Y = y + \left(1+\frac{b^4x^2}{a^4y^2}\right)\left(\frac{-a^2y^3}{b^4}\right)\Big\} = \frac{-y^3(a^2+b^2)}{b^4}$

i.e., $x^2 = \left(\frac{a^4X}{a^2+b^2}\right)^{2/3}$, $y^2 = \left(\frac{b^4Y}{a^2+b^2}\right)^{2/3}$

From (1), $\left(\frac{a^4X}{a^2+b^2}\right)^{2/3} \cdot \frac{1}{a^2} - \left(\frac{b^4Y}{a^2+b^2}\right)^{2/3} \frac{1}{b^2} = 1$

$\Rightarrow \quad (aX)^{2/3} - (bY)^{2/3} = (a^2 + b^2)^{2/3}$

Hence the locus of (x, y) i.e., the evolute of the hyperbola is

$(aX)^{2/3} - (bY)^{2/3} = (a^2 + b^2)^{2/3}$

For rectangular hyperbola, a = b.

$\therefore$ Evolute of the rectangular hyperbola is $x^{2/3} - y^{2/3} = (2a)^{2/3}$

**64. Show that the evolute of the cycloid x = a (θ – sin θ), y = a (1 – cosθ) is another cycloid.**

*Sol :*

We have $y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \cot\frac{\theta}{2}$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cot\frac{\theta}{2}\right) = \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\frac{d\theta}{dx} = -\frac{1}{4a}\ \text{cosec}^4\frac{\theta}{2}$$

If (X, Y) is the centre of curvature at any point of the curve, then

$$X = x - \frac{y_1(1+y_1^2)}{y_2} = a(\theta - \sin\theta) + \frac{4a\cot(\theta/2)+(1+\cot^2(\theta/2))}{\text{cosec}^4(\theta/2)}$$

$$= a(\theta - \sin\theta) + \frac{4a\cot(\theta/2)}{\text{cosec}^4(\theta/2)} = a(\theta - \sin\theta) + \frac{4a\cot(\theta/2)}{1+\cot^2(\theta/2)}$$

$$= a(\theta - \sin\theta) + 2a \cdot \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} = a(\theta - \sin\theta) + 2a\sin\theta$$

$$= a(\theta + \sin\theta)$$

and $\quad Y = y + \frac{1+y_1^2}{y_2}$

$$= a(1-\cos\theta) - 4a \cdot \frac{1+\cot^2(\theta/2)}{\text{cosec}^4(\theta/2)} = a(1-\cos\theta) - 4a\sin^2(\theta/2)$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta) = -a(1-\cos\theta)$$

$\therefore \quad X = a(\theta + \sin\theta)$ and $Y = -a(1-\cos\theta)$

The evolute is given by $x = a(\theta + \sin\theta)$, $y = -a(1-\cos\theta)$ which is another equal cycloid.

**65. Find the evolute of the astroid x = a cos³ θ, y = a sin³ θ**

*Sol :*

We have

$$\frac{dy}{dx} = -\tan\theta\ ; \quad \frac{d^2y}{dx^2} = \frac{1}{3a}\sec^4\theta\ \text{cosec}\ \theta$$

$$X = a\cos^3\theta + \frac{\tan\theta(1+\tan^2\theta)}{\sec^4\theta\,\mathrm{cosec}\,\theta}.3a$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos\theta \quad \text{... (i)}$$

$$Y = a\sin^3\theta + \frac{1+\tan^2\theta}{\sec^4\theta\,\mathrm{cosec}\,\theta}.3a$$

$$= a\sin^3\theta + 3a\cos^2\theta\sin\theta \quad \text{... (ii)}$$

To eliminate $\theta$, we separately add and subtract (i), (ii) so that we have

$$(X + Y) = a(\cos\theta + \sin\theta)^3 \Leftrightarrow (X + Y)^{1/3} = a^{1/3}(\cos\theta + \sin\theta)$$

$$(X - Y) = a(\cos\theta - \sin\theta)^3 \Leftrightarrow (X - Y)^{1/3} = a^{1/3}(\cos\theta - \sin\theta)$$

On squaring and adding, we obtain

$$(X + Y)^{2/3} + (X - Y)^{2/3} = 2a^{2/3}$$

as the required evolute.

**66. Obtain the evolute of the parabola $y^2 = 4ax$**

*Sol :*

Here the point (x, y) may be taken as $x = at^2$, $y = 2at$, which satsify the equation of the parabola.

Hence $\frac{dx}{dt} = 2at$ and $\frac{dy}{dt} = 2a$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{t} \text{ and } \frac{d^2y}{dx^2} = \frac{1}{t^2}\frac{dt}{dx}$$

$$= -\frac{1}{2at^3}$$

Then if (X, Y) be the coordinates of the centre of curvature, we have

$$X = x - \frac{y_1(1+y_1^2)}{y_2} = at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}}$$

$$= at^2 + 2at^2\left(1 - \frac{1}{t^2}\right) = 3at^2 + 2a$$

and $$Y = y + \frac{(1+y_1^2)}{y_2} = 2at + \frac{\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}}$$

$$= 2at + (t^2 + 1)(-2at)$$

$$= -2at^3$$

Now $X = 3at^2 + 2a$,

$Y = -2at^3$

Eliminating t between these relations, we obtain

$$\left(\frac{X-2a}{3a}\right)^{1/2} = \left(-\frac{Y}{2a}\right)^{1/3}$$

or $\quad 4(X-2a)^3 = 27\ aY^2$

Hence the locus is

$$27\ ay^2 = 4(x-2a)^3$$

**67. Find the radius of curvature at the point $(r,\theta)$ on the cordiode $r = a(1+\cos\theta)$**

*Sol :*

Given curve is $r = a(1+\cos\theta)$

differentiating w.r.t. to '$\theta$'

$$\frac{dr}{d\theta} = a + (0 + (-\sin\theta)) \Rightarrow -a\sin\theta$$

$$\frac{d^2y}{d\theta} = -a\cos\theta$$

$\therefore$ Radius of curvature $\rho = \dfrac{\left(r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right)^{\frac{3}{2}}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\left(\dfrac{d^2r}{d\theta^2}\right)}$

$$\rho = \frac{(a(1+\cos\theta))^2 + ((-a\sin\theta)^2)^{\frac{3}{2}}}{(a(1+\cos\theta))^2 + 2(-a\sin\theta)^2 - a(1+\cos\theta)(-a\cos\theta)}$$

$$= \frac{a^2(1+\cos\theta)^2 + (a^2\sin^2\theta)^{\frac{3}{2}}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + a^2\cos\theta + a^2\cos^2\theta}$$

$$= \frac{(a^2(1+\cos^2\theta + 2\cos\theta) + a^2\sin^2\theta)^{\frac{3}{2}}}{a^2(1+\cos^2\theta + 2\cos\theta) + 2a^2\sin^2\theta + a^2\cos\theta + a^2\cos^2\theta}$$

$$\rho = \frac{(a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta)^{\frac{3}{2}}}{a^2 + a^2\cos\theta + 2a^2\cos\theta + 2a^2\sin^2\theta + a^2\cos\theta + a^2\cos\theta}$$

$$= \; = \frac{\left(a^2 + a^2\left(\sin^2\theta + \cos^2\theta\right) + 2a^2\cos\theta\right)^{\frac{3}{2}}}{a^2 + 2a^2\cos\theta + 2a^2\left(\sin^2\theta + \cos^2\theta\right) + a^2\cos\theta}$$

$$= \frac{\left(a^2 + a^2 + 2a^2\cos\theta\right)^{\frac{3}{2}}}{a^2 2a^2\cos\theta + 2a^2 + a^2\cos\theta}$$

$$= \frac{\left(2a^2 + 2a^2\cos\theta\right)^{\frac{3}{2}}}{3a^2 + 3a^2\cos^2\theta}$$

$$= \frac{\left(2a^2\left(1+\cos\theta\right)\right)^{\frac{3}{2}}}{3a^2\left(1+\cos\theta\right)} = \frac{\left(2a^2\right)^{\frac{3}{2}}\left(1+\cos\theta\right)^{\frac{3}{2}}}{3a^2\left(1+\cos\theta\right)}$$

$$= \frac{2^{\frac{3}{2}} \cdot \cancel{a^3}\left(1+\cos\theta\right)^{\frac{1}{2}}}{3\cancel{a^2}}$$

$$= \frac{2^{\frac{3}{2}} \cdot a \cdot \left(\frac{r}{a}\right)^{\frac{1}{2}}}{3}$$

$$= \frac{= 2^{\frac{3}{2}} \cdot a \frac{r^{\frac{1}{2}}}{a^{\frac{1}{2}}}}{3}$$

$$= \frac{2\sqrt{2}\sqrt{a}\sqrt{r}}{3}$$

$$\therefore \; \rho = \frac{2}{3}\sqrt{2ar}$$

## 3.8 Envelopes: One Parameter Family of Curves - Consider the family of straight lines - Definition - Determination of Envelope

➢ **One parameter family of curves**

An equation in two variables X & Y if the form F(x, y, $\alpha$) = 0 where $\alpha$ is any constant, is know as a curve.

If $\alpha$ takes all real values, then the equation F(x, y, $\alpha$) = 0 is known as a family of curves with one parameter $\alpha$.

An equation in two variables x & y of the form F(x, y, $\alpha$) = 0 is known as family of curve with two parameter $\alpha$ & $\beta$ if $\alpha$ & $\beta$ takes all real values.

**For example**

- The equation $x^2 + y^2 - 2ax = 0$ determines a family of cirles with their centre on x-axis which pass through the origin. Hence 'a' is the parameter.
- The equation $y = mx + c$ represents a family of straight line with one to parabola $y^2 - 4ax$ with one parameter 'm'.
- Consider the family of straight lines.

$$y = mx + \frac{a}{m} \qquad .....(1)$$

where 'm' is the parameter and 'a' is some given constant

The two members of this family corresponding to the vaues $m_1$ & $m_1 + \Delta m$ of the parameter m are

$$y = m_1 x + \frac{a}{m_1} \qquad .....(2)$$

$$y = (m_1 + \Delta m)x + \frac{a}{m_1 + \Delta m} \qquad ...(3)$$

We shall keep $m_1$, fixed and regard $\Delta m$ as a variable which tends towards 'O' so tht the line (3) tends to coincide wit the line (2)

The two lines (2) (3) are interested at the point x, y where

x, y where

$$x = \frac{a}{m_1(m_1 + \Delta m)} \cdot \; y = \frac{a(2m_1 + \Delta m)}{m_1(m_1 + \Delta m)}$$

As $\Delta m \to 0$ this point of intersection goes on changing its position on the line (2) and in the limit tends to the point $\left(\frac{a}{m_1}, \frac{2a}{m_1}\right)$ which lies on (2)

The point is the limiting position of the point of intersection of the line (2) with another line of the family when the latter tends to coincide with the former.

There will be a point similarly obtained on every line of the family. The locus of such points is called the envelope of the given family of lines.

To find this focus for the family of line (2) we notice that the coordinate (x, y) of such a point lying on the line 'm' are given by $x = \frac{a}{m^2}, \; y = \frac{2a}{m}$

Eliminating m, we obtain $y^2 = 4ax$

As the envelope of the given family of lines.

- **Defintion**

The envelope of a one parameter of family of curves is the lows of the limiting position of the prints of interesection of any two curves of the family when one of them tends to coincide with the other which is kept fixed.

- **Determination of Envelope**

Let $f(x, y, \alpha) = 0$ be any given family of curves. .....(1)

Consider the two curves

$f(x, y, \alpha) = 0$ and $(x, y, \alpha + \Delta\alpha) = 0$ .....(2)

Corresponding to the values $\alpha$ and $\alpha + \Delta\alpha$ of the parameter.

The points common to the two curves satisfy the equation

$f(x, y, \alpha + \Delta\alpha) - f(x, y, \alpha) = 0$

$$\Leftrightarrow \frac{f(x,y,\alpha+\Delta\alpha)-f(x,y,\alpha)}{\Delta\alpha} = 0 \quad .....(3)$$

Let $\Delta\alpha \to 0$

$\therefore$ The limiting positions of the points of intersection of the curves (1) satisfying the equation which is the limit of (3).

$$f_\alpha(x, y, \alpha) = 0 \quad .....(4)$$

Thus, the coordinate of the points of the envelope satisfy the equations.

$$f(x, y, \alpha) = 0, \text{ and } f_\alpha(x, y, \alpha) = 0$$

Let the elimination of $\alpha$ between (1) & (4) lead to an equation.

$\phi(x, y) = 0$

This is, the required envelope.

**68. Find the envelope of the straight lines $x \cos\alpha + y \sin\alpha = l \sin\alpha \cos\alpha$, '$\alpha$' being the parameter.**

*Sol :*

The given equation of the family of straight lines can be written as

$x \cos\alpha + y \sin\alpha = l \sin\alpha \cos\alpha$

$$x\frac{\cos\alpha}{\sin\alpha\cos\alpha} + y\frac{\sin\alpha}{\sin\alpha\cos\alpha} = l$$

$$\frac{x}{\sin\alpha} + \frac{y}{\cos\alpha} = l \quad ...(1)$$

Equation (1) Differentiating partially with respect to '$\alpha$' we have

$$-\frac{x}{\sin^2\alpha}\cos\alpha + \frac{y}{\cos^2\alpha}\cdot\sin\alpha = 0$$

$$\tan^3\alpha = \frac{x}{y} \quad \text{or} \quad \tan\alpha = \frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}$$

So that

$$\sin\alpha = \frac{x^{1/3}}{\sqrt{x^{2/3}+y^{2/3}}} \quad \text{and} \quad \cos\alpha = \frac{y^{1/3}}{\sqrt{x^{2/3}+y^{2/3}}}$$

Substituting the values of sin $\alpha$, and cos $\alpha$ in (1), we get the required envelope as

$$\frac{x\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}} + \frac{y\sqrt{x^{2/3}+y^{2/3}}}{y^{1/3}} = l$$

(or) $\left(x^{2/3}+y^{2/3}\right)^{3/2} = l$

i.e. $x^{2/3}+y^{2/3}=l^{2/3}$

**69. Find the envelope of the parabola $y^2 = m^2(x - m)$ 'm' being the parameter.**

*Sol :*

The given parabola is $y^2 = m^2(x - m)$ .....(1)

'm' being the parameter

Differentiating equation (1) partially with respect to 'm' we have

$= 2mx - 3m^2$

$m(2x - 3m) = 0$

$m = 0, \ 3, = 2x$

$$m = \frac{2x}{3}$$

If we substute $m = 0$ equation (1)

we obtain $y = 0$ which is not a part of the envelope since it touches none of the given parabolas.

Substituting $m = \frac{2x}{3}$ equation (1)

we obtain

$y^2 = (x - m)$

$= \left(\frac{2x}{3}\right)^2 \left(x - \frac{2x}{3}\right)$

$= \frac{4x^2}{9}\left(\frac{3x-2x}{3}\right)$

$y^2 = \frac{4x^2}{9}\left(\frac{x}{3}\right)$

$27y^2 = 4x^3$

Which is the required envelope of the given parabolas

Differentiating equation (1) partially with respect to 'm' we have

$0 = 2mx - 3m^2$

$m(2x - 3m) = 0$

$m = 0, \ 3m = 2x$

$$m = \frac{2x}{3}$$

If we substitute $m = 0$ equation (1)

We obtain $y = 0$ which is not a part of the envelope since it touches none of the given parabolas.

Substituting $m = \frac{2x}{3}$ in equation (1)

We obtain

$$y^2 = m^2 (x - m)$$

$$= \left(\frac{2x}{3}\right)^2 \left(x - \frac{2x}{3}\right)$$

$$= \frac{4x^2}{9}\left(\frac{3x - 2x}{3}\right)$$

$$y^2 = \frac{4x^2}{9}\left(\frac{x}{3}\right)$$

$$27y^2 = 4x^3$$

which is the required envelope of the given parabolas.

**70. Find the envelope of $x^2 \sin\alpha + y^2 \cos\alpha = a^2$ a is a parameter.**

*Sol :*

The given curve is

$$x^2 \sin\alpha + y^2 \cos\alpha = a^2$$

Differentiating partially with respectg to $\alpha$ we have

$$x^2 \cos\alpha + y^2(-\sin\alpha) = 0$$

$$x^2 \cos\alpha - y^2 \sin\alpha = 0$$

$$x^2 \cos\alpha = y^2 \sin\alpha$$

$$\frac{x^2}{y^2} = \frac{\sin\alpha}{\cos\alpha}$$

$$\frac{x^2}{y^2} = \tan\alpha \quad \text{(or)}$$

$$\frac{x^2}{y^2} = \frac{\sin x}{\cos x}$$

$$\frac{x^2}{\sin\alpha} = \frac{y^2}{\cos\alpha} = \sqrt{\frac{(x^2)^2 + (y^2)^2}{\cos^2\alpha + \sin^2\alpha}}$$

$$= \sqrt{\frac{x^4 + y^4}{1}}$$

Putting the value of sin $\alpha$ and cos $\alpha$ in $x^2 \sin\alpha + y^2 \cos\alpha = a^2$

$$\frac{x^4}{(x^4+y^4)^{1/2}} + \frac{y^4}{(x^4+y^4)^{1/2}} = a^2$$

which is required envelope.

**71. Find the envelope of the curve $y = mx + \frac{a}{m}$ where m is parameter.**

*Sol :*

$$y = mx + \frac{a}{m} \qquad .....(1)$$

Partially differentiating with respect to 'm'

$$0 = x + a\left(\frac{-1}{m^2}\right)$$

$$\frac{-a}{m^2} + x = 0$$

$$\frac{a}{m^2} = x \quad \Rightarrow \quad \frac{m^2}{a} = \frac{1}{x}$$

$$m^2 = \frac{a}{x}$$

$$m = \pm\sqrt{\frac{a}{x}}$$

Substituting $m = \pm\sqrt{\frac{a}{x}}$ in equation (1)

$$y = \left(\pm\sqrt{\frac{a}{x}}\right)x \pm \frac{a}{\sqrt{\frac{a}{x}}}$$

$$= \left(\pm\sqrt{\frac{a}{x}}\right)x \pm a\frac{x}{a}$$

Squaring on both sides

$$y^2 = \left[\pm\sqrt{\frac{a}{x}}(x) \pm a\sqrt{\frac{x}{a}}\right]^2$$

$$= \left(\sqrt{\frac{a}{x}}\right)^2 x^2 + a^2\left(\sqrt{\frac{a}{x}}\right)^2 + 2x\sqrt{\frac{a}{x}}\, a\sqrt{\frac{x}{a}}$$

$= \frac{a}{x}x^2 + a^2\frac{x}{a} + 2xa$

$= ax + xa + 2ax$

$= 2ax + 2ax$

$y^2 = 4ax$

$\therefore \quad y^2 = 4ax$ is the equation of envelope.

**72. Find the envelope of the curve $y = mx + am^3$ where is a parameter.**

*Sol :*

$y = mx + am^3$ .....(1)

Differentiating partially with respect to 'm'

$0 = x + 3am^2$

$x = -3am^2$

$-3am^2 = x$

$m^2 = \frac{-x}{3a}$

$m = \pm\sqrt{\frac{-x}{3a}}$

Substituting $m = \pm\sqrt{\frac{-x}{3a}}$ in (1)

$y = \left[\pm\sqrt{\frac{-x}{3a}}\right]x + a\left[\pm\sqrt{\frac{-x}{3a}}\right]^3$

Squaring on both sides

$y^2 = -\frac{x}{3a}x^2 + a^2\frac{-x^2}{27a^3} + \left(2\frac{-x}{3a}\right)\left(a\sqrt{\frac{-x}{3a}}\right)^3$

$y^2 = \frac{-x^3}{3a} - \frac{x^3}{27a} + 2ax\left(\frac{x^2}{9a^2}\right)$

$y^2 = \frac{-9x^3 - x^3 + 6x^3}{27a}$

$y^2 = \frac{-10x^3 + 6x^3}{27a}$

$y^2 = \frac{-4x^3}{27a}$

Therefore $27ay^2 + 4x^3 = 0$ is the equation of the envelope.

**73. Find the envelope of the curve $y = mx + am^p$**

*Sol. :*

$y = mx + am^p$

Partially differentiating with respect to 'm'

$0 = x + Pam^{P-1}$

$0 = x + p\frac{am^p}{m}$

$aPm^P = -mx$

$y = mx - \frac{mx}{P}$

$Py = Pmx - mx$

$Py = mx(P-1)$

$(Py)^P = m^P x^P (P-1)^P$

$(Py)^P = \frac{-mx}{aP^x}(P-1)^P$

$P^P y^P = \frac{-1}{ap}\frac{y}{P-1}x^P (P-1)^P$

$P^P y^P = \frac{-y}{a}x^P (P-1)^P$

$\therefore \quad aP^P y^{P-1} = x^P (P-1)^{P-1}$ is the required envelope



**74. Find the envelope of the curve $\frac{a^2}{x}\cos\alpha - \frac{b^2}{y}\sin\alpha = k$**

*Sol. :*

The given curve $\frac{a^2}{x}\cos\alpha - \frac{b^2}{y}$ sin a = k partially differentiating w.r.t to 'α'..(1)

$\frac{a^2}{x}(-\sin\alpha) - \frac{b^2}{y}\cos\alpha = 0$ .....(2)

Squarring and adding equation (1) and equation (2)

$$\left(\frac{a^2}{x}\cos\alpha - \frac{b^2}{y}\sin\alpha\right)^2 + \left(\frac{a^2}{x}(-\sin\alpha) - \frac{b^2}{y}\cos\alpha\right)^2 = k^2$$

$$\frac{a^4}{x^2}\cos^2\alpha + \frac{b^4}{y^2}\sin^2\alpha - \frac{2a^2b^2}{xy}\cos\alpha\ \sin\alpha + \frac{a^4}{x^2}\sin^2\alpha + \cos^2\alpha + \frac{2a^2b^2}{xy}\cos\alpha\sin\alpha = k^2$$

$$\frac{a^4}{x^2}(\sin^2\alpha + \cos\alpha) + \frac{b^4}{y^2}(\sin^2\alpha + \cos^2\alpha) = k^2$$

$$\frac{a^4}{x^2}+\frac{b^4}{y^2} = k^2$$ is the required equation of envelope.

**75. Find the envelope of the curve $y = t^2(x - 1)$ where 'm' is parameter.**

*Sol :*

Given curve is $y = t^2 (x - t)$

Partially differentiating with respect to 't'

$$0 = t^2 (-1) + (x - t)\, 2t$$

$$-t^2 + 2xt - 2t^2$$

$$0 = -3t^2 + 2tx$$

$$2tx = 3t^2$$

$$2x = 3t$$

$$t = \frac{2x}{3}$$

Substituting

$$t = \frac{2}{3}x \quad \text{in} \quad y = t^2 (x-t)$$

$$y = \left(\frac{2}{3}x\right)^2\left[x-\frac{2}{3}x\right]$$

$$= \frac{4}{9}x^2\left[\frac{3x-2x}{3}\right]$$

$$= \frac{4}{9}x^2\left(\frac{x}{3}\right)$$

$$y = \frac{4x^3}{27}$$

$27y = 4x^3$ which is required equation of envelope.

**76. Find the envelope of the curve $tx^3 + t^2 y = a$ where 'm' is paramter t**

*Sol :*

The given curve is $tx^3 + t^2y = a$

Partially differentiate with respect to 't'

$$x^3 + 2ty = 0$$

$$2ty = -x^3$$

$$t = \frac{-x^3}{2y}$$

Substituting

$$t = \frac{-x^3}{2y} \text{ in } tx^3 + ty^2 = a$$

$$\frac{-x^3}{2y}(x^3) + 2\left(\frac{-x^3}{2y}\right)^2 y = a$$

$$\frac{-x^6}{2y} + \frac{-x^6}{4y^2}y = a$$

$$\frac{-x^6}{2y} + \frac{x^6}{4y} = a$$

$$\frac{-2x^6 + x^6}{4y} = a$$

$$\frac{-x^6}{4y} = a \Rightarrow -x^6 = 4ay$$

$$x^6 = -4ay$$

$$x^6 + 4ay = 0$$

which is required equation of envelope.

# *Choose the Correct Answers*

1. Radius of curvature at (x, y) of the curve $y = c \cosh (x/(c)$ is [ c ]

   (a) cy (b) $y^2/c$

   (c) y/c (d) $y^3/c$

2. Locus of centre of curvature is known as [ a ]

   (a) envelope (b) chord of curvature

   (c) circle of curvature (d) evolutes

3. Normal of curve are [ a ]

   (a) Tangents to its evolute (b) normals to its envolute

   (c) chords to its evolute (d) none of the above

4. The ratios of curvature of any point (x, y) on the curve y = f(x) is given by [ b ]

   (a) $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^3}{\dfrac{d^2y}{dx^2}}$ (b) $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$

   (c) $\dfrac{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]^{2/3}}{\dfrac{d^2y}{dx^2}}$ (d) $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{2/3}}{\dfrac{d^2y}{dx^2}}$

5. The pedal formula for the radius of curvature is [ c ]

   (a) $\rho = r + \dfrac{dr}{dp}$ (b) $\rho = r\dfrac{dp}{dr}$

   (c) $\rho = r\dfrac{dr}{dp}$ (d) $\rho = \dfrac{1}{r}\dfrac{dr}{dp}$

6. The radius of curvature of the origin if y-axis is the tangent at the origin, is given by [ d ]

   (a) $\lim\limits_{x\to 0}\dfrac{x^2}{2y}$ (b) $\lim\limits_{x\to 0}\dfrac{y^2}{x}$

   (c) $\lim\limits_{x\to 0}\dfrac{x^2}{y}$ (d) $\lim\limits_{x\to 0}\dfrac{y^2}{2x}$

7. The radius of curvature at any point on the hyperbola $pr = a^2$ is [ a ]

(a) $r^3/a^2$ (b) $r^2/a^2$

(c) $r/a^2$ (d) r

8. Radius of curvature of the curve $y = e^x$ at the point (a, 1) is [ b ]

(a) $2\sqrt{2}$ (b) $3\sqrt{2}$

(c) 0 (d) None

9. For any curve $\frac{d}{dx}\left(\frac{dx}{ds}\right)$: [ a ]

(a) $\frac{1}{\rho}$ (b) $\frac{-1}{\rho}$

(c) $\rho$ (d) $\rho^2$

10. The intrinsic formula for the radius of curvature is [ b ]

(a) $\rho = \frac{dy}{dx}$ (b) $\rho = \frac{ds}{d\psi}$

(c) $\rho = \frac{1}{s}\frac{ds}{d\psi}$ (d) $\rho = s\frac{d\psi}{ds}$

# Fill in the Blanks

1. The envelope of the family of straight line $y = mx +$ unital is ________ .

2. The equation of the envelope of the family of curve $F(x, y, \alpha)$ where '$\alpha$' is a parameter, is obtained by eliminating $\alpha$ between the equation $F(x, y, \alpha) = 0$ and ________

3. The chord of curvature ________ .

4. The reciprocal of the curvature at that point is defined as the ________ .

5. The whole length of the envelope of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ is ________ .

6. Envelope of $x^2 \sin\alpha + y^2 \cos\alpha = a^2$ is ________ .

7. Envelope of the family of curve $y^2 = t^2(x - t)$ is ________ .

8. Envelope of the system of circles $(x - \alpha)^2 + y^2 = 4\alpha$ is ________ .

9. Envelope of the family of curve $tx^2 + t^2y = a$ is ________ .

10. The evolute of a curve is the ________ of its normals.



**ANSWERS**

1. $x^2 = 4ay$
2. $\frac{\partial}{\partial\alpha}F(x, y, \alpha) = 0$
3. $\tan\frac{t}{2}$
4. Radius of curvature
5. 12a
6. $x^3 + y^3 = a^3$
7. $4x^3 = 27y^2$
8. $y^2 - 4x - 4 = 0$
9. $x^2 + 4ay = 0$
10. Envelope

| UNIT IV | **Lengths of Plane Curves:** Introduction - Expression for the lengths of curves y = f(x) - Expressions for the length of arcs x = f(y); x = f(t), y = ?(t); r = f(?) **Volumes and Surfaces of Revolution:** Introduction - Expression for the volume obtained by revolving about either axis - Expression for the volume obtained by revolving about any line - Area of the surface of the frustum of a cone - Expression for the surface of revolution - Pappus Theorems - Surface of revolution. |
|---|---|

## 4.1 Rectification

**1. What is Rectification**

*Sol :*

The length of area of plane curve whose equations are given in the cartesian, is parametric cartesian or polar form. This process is known as rectification.

## 4.2 Expression for the Lengths of curves y = f(x)

➢ Cartisian equations y = f(x)

The length of arc of the curve y = f(x) included between two points whose abscissae are a and b is

$$\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx = \int_a^b \sqrt{\left[1+f'^2(x)\right]}dx$$

➢ Cartisian equation x = f(x)

The length of the arc of the curve x = f(y) included between two points whose ordinates are c,d is

$$\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{\left[1+f'^2(y)\right]}dy$$

**2. Find the length of arc of the parabola $x^2$ = 4ay measured from the vertex to one extremily of the latus rectum.**

*Sol :*

Given curve is $x^2 = 4ay$

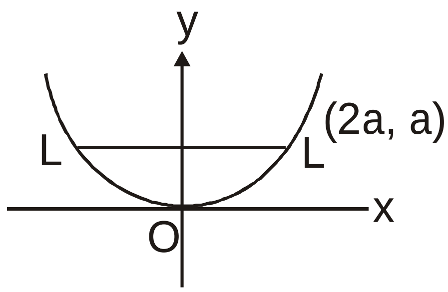
$$y = \frac{x^2}{4a} \qquad ........(1)$$

Differentiating with respect to 'x'

$$\frac{dy}{dx} = \frac{2x}{4a}$$

$$\frac{dy}{dx} = \frac{x}{2a}$$

The length of arc is $\int_a^b \left[\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right]dx$




$$\Rightarrow \int_0^{2a}\left[\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right]dx = \int_0^{2a}\left[\sqrt{1+\left(\frac{x}{2a}\right)^2}\right]dx$$

$$= \int_0^{2a}\sqrt{\frac{4a^2+x^2}{4a^2}}\; dx$$

$$= \frac{1}{2a}\int_0^{2a}\left(\sqrt{4a^2+x^2}\right)dx$$

$$= \frac{1}{2a}\left[\frac{x\sqrt{x^2+4a^2}}{2}+2a^2\sinh^{-1}\frac{x}{2a}\right]_0^{2a}$$

$$= \frac{1}{2a}\left[2\sqrt{2a^2+2a^2\sinh^{-1}1}\right]$$

$$= a\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]$$

**3. Find the length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum.**

*Sol :*

Given equation is $y^2 = 4ax$ .........(1)

Differentiating equation (1) w.r.t to 'x'

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y}$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

by equation (1) $y^2 = 4ax \Rightarrow y = \sqrt{4ax}$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{\sqrt{4ax}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{\sqrt{4}\sqrt{a}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{2\sqrt{a}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a}{\sqrt{a}\sqrt{x}} \times \frac{\sqrt{a}}{\sqrt{a}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a\sqrt{a}}{a\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{a}{x}}$$

The length of the curve

$$S = \int_0^{2a}\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$$

$$= 2\int_0^a\sqrt{1+\left(\sqrt{\frac{a}{x}}\right)^2}\,dx$$

$$= 2\int_0^a\sqrt{1+\frac{a}{x}}\,dx$$

$$= 2\int_0^a\sqrt{\frac{x+a}{x}}\,dx$$

Let $\sqrt{x} = t \Rightarrow x = t^2$

$$\frac{1}{2\sqrt{x}}dx = dt \Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$$

$$dx = 2\sqrt{x}\,dt$$

But $\sqrt{x} = t$

$$dx = 2t\,dt$$

If $x = a \Rightarrow t = \sqrt{a}$

If $x = 0 \Rightarrow t = 0$

$$= 2\int_0^{\sqrt{a}}\frac{\sqrt{t^2+a}}{\sqrt{x}}\,dx$$

$$= 2\int_0^{\sqrt{a}}\sqrt{t^2+a}\cdot 2dt$$

$$= 4\int_0^{\sqrt{a}}\sqrt{t^2+a}\,dt$$

$$= 4\int_0^{\sqrt{a}}\sqrt{t^2+\left(\sqrt{a}\right)^2}\,dt$$

$$= 4\left[\frac{t}{2}\sqrt{t^2+a}+\frac{a}{2}\log\left|\frac{t+\sqrt{a+t^2}}{\sqrt{a}}\right|\right]_0^{\sqrt{a}}$$

$$= 4\left\{\left[\frac{\sqrt{a}}{2}\sqrt{\left(\sqrt{a}\right)^2 + a} + \frac{a}{2}\log\left|\frac{\sqrt{a}+\sqrt{a+\left(\sqrt{a}\right)^2}}{\sqrt{a}}\right|\right] - \left[\frac{0}{2}\sqrt{0+a} + \frac{a}{2}\log\left|\frac{0+\sqrt{a+0}}{\sqrt{a}}\right|\right]\right\}$$

$$= 4\left\{\left[\frac{\sqrt{a}}{2}\sqrt{a+a} + \frac{a}{2}\log\left|\frac{\sqrt{a}+\sqrt{a+a}}{\sqrt{a}}\right|\right] - \left[\frac{a}{2}\log\left|\frac{\sqrt{a}}{\sqrt{a}}\right|\right]\right\}$$

$$= 4\left\{\left[\frac{\sqrt{a}}{2}\sqrt{2a} + \frac{a}{2}\log\left|\frac{\sqrt{a}+\sqrt{2a}}{\sqrt{a}}\right|\right] - \left[\frac{a}{2}\log(1)\right]\right\}$$

$$= 4\left\{\left[\frac{\sqrt{a}\sqrt{a}.\sqrt{2}}{2} + \frac{a}{2}\log\left|\frac{\sqrt{a}\left(1+\sqrt{2}\right)}{\sqrt{a}}\right|\right] - \left[\frac{a}{2}(0)\right]\right\}$$

$$= 4\left\{\left[\frac{a\sqrt{2}}{2} + \frac{a}{2}\log\left(1+\sqrt{2}\right)\right] - 0\right\}$$

$$= \frac{4a}{2}\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]$$

$$= 2a\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]$$

$\therefore$ The length of the arc is $2a\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]$

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**4. Find the length of the arc of the caterx y = c cosh$\left(x/c\right)$ measured from the vertex (0,c) to any point (x,y)**

*Sol :*

Given curve is $y = c\cosh\left(\frac{1}{c}\right)$ ....(1)

differentiating equation (1) w.r.to 'x'

$$\frac{dy}{dx} = c\left[\frac{d}{dx}\left(\cosh\left(\frac{x}{c}\right)\right)\right]$$

$$= c\left[\sinh\left(\frac{x}{c}\right).\frac{1}{c}(1)\right]$$

$$= c\left[\sinh\left(\frac{x}{c}\right)\right]\frac{1}{c}$$

$$\frac{dy}{dx} = \sinh\left(\frac{x}{c}\right)$$

and we have the catenary $y = \cosh\left(\frac{x}{c}\right)$ measured from the vertex (0,c) to any point (x,y)

$\therefore$ the length of the arc is $S = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$S = \int_0^x \sqrt{1+\left[\sinh\left(\frac{x}{c}\right)\right]^2}\, dx$$

$$= \int_0^x \sqrt{1+\sinh^2\frac{x}{c}}\, dx \qquad [\because 1 + \sinh^2 x = \cosh^2 x]$$

$$= \int_0^x \sqrt{\cosh^2\frac{x}{c}}\, dx$$

$$= \int_0^x \cosh\frac{x}{c}\, dx$$

$$= \left[\frac{\sinh\frac{x}{c}}{\frac{1}{c}}\right]_0^x$$

$$= c\left[\sinh\frac{x}{c}\right]_0^x$$

$$= c\left[\sinh\frac{x}{c} - \sinh\frac{0}{c}\right]$$

$$= c\left[\sinh\frac{x}{c} - 0\right]$$

$$S = c\sinh\left(\frac{x}{c}\right)$$

$\therefore$ the length of the arc is $S = c\sinh\left(\frac{x}{c}\right)$

**5. Find the length of the arc of the curve y = log sec x from x = 0 to x = $\pi/3$.**

*Sol :*

Given curve is y = log sec x ........ (1)

differentiating equation (1) w.r.to "x"

$$\frac{dy}{dx} = \frac{1}{\sec x}\frac{d}{dx}(\sec x)$$

$$\frac{dy}{dx} = \frac{1}{\sec x}\sec x \tan x$$

$$\frac{dy}{dx} = \tan x$$

∴ The length of the curve is

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = \int_0^{\pi/3} \sqrt{1+(\tan x)^2}\,dx \quad [\because 1+\tan^2 x = \sec^2 x]$$

$$= \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx$$

$$= \int_0^{\pi/3} \sec^2 x\,dx$$

$$= \left[\log(\sec x + \tan x)\right]_0^{\pi/3}$$

$$= \left\{\log\left[\sec\left(\pi/3\right)+\tan\left(\pi/3\right)\right]\right\} - \left\{\log\left[\sec(0)+\tan(0)\right]\right\}$$

$$= \left[\log\left(2+\sqrt{3}\right) - \log(1+0)\right]$$

$$= \log\left(2+\sqrt{3}\right) - \log 1$$

$$= \log\left(2+\sqrt{3}\right) - 0$$

$$S = \log\left(2+\sqrt{3}\right)$$

∴ The length of the arc is $S = \log\left(2+\sqrt{3}\right)$

**6. Show that the length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ measured from (0,a) to the point (x,y) is is given by $S = \frac{3}{2} 3\sqrt{ax^2}$**

*Sol :*

Given curve is $x^{2/3} + y^{2/3} + a^{2/3}$ ......... (1)

differentiating equation (1) w.r.to 'x'

$$\frac{2}{3}x^{\frac{2}{3}-1} + \frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx} = 0$$

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

$$\frac{2}{3}\left[x^{-1/3} + y^{-1/3}\frac{dy}{dx}\right] = 0$$

$$x^{-1/3} = -y^{-1/3}\frac{dy}{dx}$$

$$\frac{-x^{-1/3}}{y^{-1/3}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$\frac{dy}{dx} = \frac{-\frac{1}{(x)^{1/3}}}{\frac{1}{(y)^{1/3}}} = -\frac{1}{(x)^{1/3}} \times \frac{(y)^{1/3}}{1}$$

$$= -\left(\frac{y}{x}\right)^{1/3}$$

$$\therefore \quad \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

$\therefore$ The length of the arc $S = \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$S = \int_0^a \sqrt{1+\left[\left(\frac{-y}{x}\right)^{1/3}\right]^2}\, dx$$

$$= \int_0^a \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\, dx$$

$$= \int_0^a \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}}\, dx$$

$$= \int_0^a \sqrt{\frac{a^{2/3}}{x^{2/3}}}\, dx \quad \text{[by equation (1)}$$

$$x^{2/3} + y^{2/3} = a^{2/3}]$$

$$= \int_0^a \sqrt{\left(\frac{a^{1/3}}{x^{1/2}}\right)^2}\, dx$$

$$= \int_0^a \left(\frac{a}{x}\right)^{1/3} dx$$

$$= a^{1/3}\int_0^a \frac{1}{x^{1/3}}\, dx$$

$$= a^{1/3}\int_0^a x^{-1/3}$$

$$= a^{1/3}\left[\frac{x^{-1/3+1}}{\frac{-1}{3}+1}\right]_0^a$$

$$= a^{1/3}\left[\frac{x^{2/3}}{\frac{2}{3}}\right]_0^a$$

$$= \frac{3}{2}a^{1/3}\left[x^{2/3}\right]_0^a$$

$$= \frac{3}{2}a^{1/3}\left[x^{2/3} - 0\right]$$

$$= \frac{3}{2}\sqrt[3]{ax^2}$$

## 4.3 Expression for the Length of arcs x = f(y); x = f(t), y = f(t); r = f(q)

* The expression for length of arcs from cartesion equations x = f(y)

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

To determine the length of arcs (6)

* Cartesian equation x = f(y)
* The length of the arc of the curve x = f(y) included between two points whose ordinates are c,d is

$$\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{\left[1+f'^2(y)\right]}\, dy$$

* Parametric cartesian equation x = f(t), y = $\phi$(t),

The length of the arc of the curve x = f(t), y = $\phi$(t) included between two points whose parametric values are $\alpha$, $\beta$ is

$$\int_\alpha^\beta \left[\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\right] d\theta = \int_\alpha^\beta \left[f'^2(t)+\phi'^2(t)\right] dt$$

* Polar equations r = f($\theta$)

The length of the arc of the curve r = f($\theta$) included between two points whose vectorial angles are $\alpha$, $\beta$ is

$$\int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{\left[f'^2(\theta)+f'^2(\theta)\right]}\, d\theta$$

**7. Prove that whole length of the curve $x^2(a^a - x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$**

*Sol :*

Given curve is $x^2(a^2 - x^2) = 8a^2y^2$

$x^2a^2 - x^4 = 8a^2y^2$ .....(1)

Differentiating equation (1) w. r. to 'x'.

$$a^2(2x) - 4x^3 = 8a^2(2y)\frac{dy}{dx}$$

$$\frac{2xa^2 - 4x^3}{8a^2(2y)} = \frac{dy}{dx}$$

$$\frac{2x\left(a^2-2x^2\right)}{16a^2y} = \frac{dy}{dx} \Rightarrow \frac{x\left(a^2-2x^2\right)}{8x^2y} = \frac{dy}{dx}$$

$\therefore$ The length of curve

$$s = 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\int_0^a \sqrt{1+\left[\frac{x\left(a^2-2x^2\right)}{8a^2y}\right]^2}\,dx$$

$$= 4\int_0^a \sqrt{1+\left(\frac{x^2\left(a^2-2x^2\right)^2}{64a^4y^2}\right)}\,dx$$

from (1) $\Rightarrow x^2a^2 - x^4 = 8a^2y^2$

$$y^2 = \frac{x^2a^2-x^4}{8a^2}$$

$$= 4\int_0^a \sqrt{1+\left[\frac{x^2\left(a^2-2x^2\right)^2}{64a^4\left(\frac{x^2a^2-x^4}{8a^2}\right)}\right]}\,dx$$

$$= 4\int_0^a \sqrt{1+\frac{x^2\left(a^2-2x^2\right)^2 8a^2}{64a^4x^2\left(a^2-x^2\right)}}\,dx$$

$$= 4\int_0^a \sqrt{1+\frac{\left(a^2-2x^2\right)^2}{8a^2\left(a^2-x^2\right)}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{8a^2\left(a^2-x^2\right)+\left(a^2-2x^2\right)^2}{8a^2\left(a^2-x^2\right)}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{8a^4-8a^2x^2+a^4+4x^4-4x^2a^2}{8a^2\left(a^2-x^2\right)}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{9a^4-12a^2x^2+4x^4}{8a^2\left(a^2-x^2\right)}}\,dx$$

$$= 4\int_0^a \frac{\sqrt{\left(3a^2-2x^2\right)^2}}{\sqrt{8a^2\left(a^2-x^2\right)}}\,dx$$

$$= 4\int_0^a \frac{3a^2-2x^2}{2\sqrt{2}a\sqrt{a^2-x^2}}\,dx$$

$$= \frac{4}{2\sqrt{2a}}\int_0^a \frac{2a^2+a^2-2x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= \frac{2}{\sqrt{2a}}\int_0^a \frac{2\left(a^2-x^2\right)+a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= \frac{\sqrt{2}}{a}\int_0^a \left[\frac{2\left(a^2-x^2\right)}{\sqrt{a^2-x^2}}+\frac{a^2}{\sqrt{a^2-x^2}}\right]dx$$

$$= \frac{\sqrt{2}}{a}\left[\int_0^a \left(2\sqrt{a^2-x^2}\right)dx+\int_0^a \frac{a^2}{\sqrt{a^2-x^2}}\,dx\right]$$

$$= \frac{\sqrt{2}}{a}\left[2\int_0^a \sqrt{a^2-x^2}\,dx+a^2\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,dx\right]$$

$$= \frac{\sqrt{2}}{a}\left[2\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]+a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{\sqrt{2}}{a}\left\{\left[2\left(\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}\frac{a}{a}\right)+a^2\sin^{-1}\left(\frac{a}{a}\right)\right]-\left[2\left(\frac{0}{2}\sqrt{a^2-0}+\frac{a}{2}\sin^{-1}(0)\right)+a^2\sin^{-1}(0)\right]\right\}$$

$$= \frac{\sqrt{2}}{a}\left\{\left[2\left(\frac{a}{2}\left(\sqrt{0}\right)+\frac{a^2}{2}\sin^{-1}(1)\right)+a^2\sin^{-1}(1)\right]\left[2\left(0+\frac{a^2}{2}\sin^{-1}(0)+a^2\sin^{-1}(0)\right)\right]\right\}$$

$$= \frac{\sqrt{2}}{a}\left[2\left(0+\frac{a^2}{2}\left(\frac{\pi}{2}\right)\right)+a^2\left(\frac{\pi}{2}\right)\right]-0-0$$

$$= \frac{\sqrt{2}}{a}\left[2\frac{a^2}{2}\left(\frac{\pi}{2}\right)+a^2\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{\sqrt{2}}{a}\left[\frac{2a^2\pi}{4}+\frac{a^2\pi}{2}\right]$$

$$= \frac{\sqrt{2}}{a}\left[\frac{2a^2\pi + 2a^2\pi}{4}\right]$$

$$= \frac{\sqrt{2}}{a}\frac{4a^2\pi}{4}$$

$$= \frac{\sqrt{2}a^2\pi}{a}$$

$$= \sqrt{2}a\pi$$

The length of the given arc is $\pi a\sqrt{2}$

**8. Find the perimeter of the loop of the curve $9ay^2 = (x - 2a)(x - 5a)^2$.**

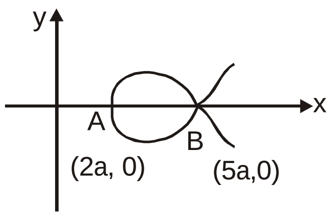
*Sol :*

Given curve is $9ay^2 = (x - 2a)(x - 5a)^2$ ......... (1)

The loop lies between the limits x = 2a and x = 5a.

The curve is symmertical about x axis

∴ The perimeter of the loop is double of the length of its part lying about x-axis.




differentiating (1) with respect to 'x'

$$9a\left(2y\frac{dy}{dx}\right) = (x - 2a)\frac{d}{dx}(x - 5a)^2 + (x - 5a)^2\frac{d}{dx}(x - 2a)$$

$$18ay\frac{dy}{dx} = (x - 2a)[2(x - 5a)](1) + (x - 5a)^2(1)$$

$$18ay\frac{dy}{dx} = 2(x - 2a)(x - 5a) + (x - 5a)^2$$

$$18ay\frac{dy}{dx} = (x - 5a)[2(x - 2a) + (x - 5a)]$$

$$18ay\frac{dy}{dx} = (x - 5a)[2x - 4a + x - 5a]$$

$$18ay\frac{dy}{dx} = (x - 5a)[3x - 9a]$$

$$\frac{dy}{dx} = \frac{(x - 5a)(3x - 9a)}{18ay}$$

$$\frac{dy}{dx} = \frac{3(x-5a)(x-3a)}{18ay}$$

$$\frac{dy}{dx} = \frac{(x-5a)(x-3a)}{6ay}$$

$$\frac{dy}{dx} = \frac{3(x-5a)(x-3a)}{18ay}$$

$\therefore$ The length of the given are $S = 2\int_{2a}^{5a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$S = 2\int_{2a}^{5a}\sqrt{1+\left[\frac{(x-5a)(x-3a)}{6ay}\right]^2}\,dx$$

$$= 2\int_{2a}^{5a}\sqrt{1+\frac{(x-5a)^2(x-3a)^2}{36a^2y^2}}$$

by equation (1) $\Rightarrow$ $9ay^2 = (x-2a)(x-5a)^2$

$$y^2 = \frac{(x-2a)(x-5a)^2}{9a}$$

$$= 2\int_{2a}^{5a}\sqrt{1+\frac{(x-5a)^2(x-3a)^2}{36a^2\left[\frac{(x-2a)(x-5a)^2}{9a}\right]}}\,dx$$

$$= 2\int_{2a}^{5a}\sqrt{1+\left[\frac{9a(x-5a)^2(x-3a)^2}{36a^2(x-2a)(x-5a)^2}\right]}\,dx$$

$$= 2\int_{2a}^{5a}\sqrt{1+\left[\frac{(x-3a)^2}{4a(x-2a)}\right]}\,dx$$

$$=2\int_{2a}^{5a}\sqrt{\frac{4a(x-2a)+(x-3a)^2}{4a(x-2a)}}\,dx$$

$$=2\int_{2a}^{5a}\sqrt{\frac{4ax-8a^2+x^2+9a^2-6xa}{4a(x-2a)}}\,dx$$

$$= 2\int_{2a}^{5a}\sqrt{\frac{x^2+a^2-2xa}{4a(x-2a)}}\,dx$$

$$= 2\int_{2a}^{5a}\frac{\sqrt{(x-a)^2}}{2\sqrt{a}\sqrt{x-2a}}\,dx$$

$$= \frac{2}{2\sqrt{a}}\int_{2a}^{5a}\frac{x-9}{\sqrt{x-2a}}\,dx$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\frac{x-2a+a}{\sqrt{x-2a}}\,dx$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\left(\frac{x-29}{\sqrt{x-2a}}+\frac{a}{\sqrt{x-2a}}\right)dx$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\left(\sqrt{x-2a}+\frac{a}{\sqrt{x-2a}}\right)dx$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\sqrt{x-2a}\,dx + a\int_{2a}^{5a}\frac{1}{\sqrt{x-2a}}\,dx$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}(x-2a)^{1/2}\,dx + a\int_{2a}^{5a}(x-2a)^{-1/2}\,dx$$

$$= \frac{1}{\sqrt{a}}\left\{\left[\frac{(x-2a)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{2a}^{5a} + a\left[\frac{(x-2a)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}\right]_{2a}^{5a}\right\}$$

$$= \frac{1}{\sqrt{a}}\left\{\left[\frac{(x-2a)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{2a}^{5a} + a\left[\frac{(x-2a)^{\frac{1}{2}}}{\frac{1}{2}}\right]_{2a}^{5a}\right\}$$

$$= \frac{1}{\sqrt{a}}\left\{\frac{2}{3}\left[(x-2a)^{\frac{3}{2}}\right]_{2a}^{5a} + 2a.\left[(x-2a)^{1/2}\right]_{2a}^{5a}\right\}$$

$$= \frac{1}{\sqrt{a}}\left\{\frac{2}{3}\left[(5a-2a)^{3/2}-(2a-2a)^{3/2}\right]+2a\left[(5a-2a)^{1/2}-(2a-2a)^{1/2}\right]\right\}$$

$$= \frac{1}{\sqrt{a}}\left\{\frac{2}{3}\left[(3a)^{3/2}-0\right]+2a\left[(3a)^{1/2}-0\right]\right\}$$

$$= \frac{1}{\sqrt{a}}\left\{\frac{2}{3}3^{3/2}a^{3/2} + 2a\left(3^{1/2}a^{1/2}\right)\right\}$$

$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}3\sqrt{3}\ a\sqrt{a} + 2a\sqrt{3}\sqrt{a}\right]$$

$$= \frac{a\sqrt{3}\sqrt{a}}{\sqrt{a}}\left[\frac{2}{3}(3) + 2\right]$$

$$= \sqrt{3}\sqrt{a}(4)$$

$$S = 4a\sqrt{3}$$

$\therefore$ The length of curve is $4a\sqrt{3}$

**9. A curve is given by the equations x = a(cos$\theta$ + $\theta$ sin $\theta$), y = a(sin$\theta$ – $\theta$ cos $\theta$). Find the length of the arc from $\theta$ = 0 to $\theta$ = $\alpha$.**

*Sol :*

Given curves $x = a(\cos\theta + \theta \sin\theta)$ .....(1)

$y = a(\sin\theta - \theta\cos\theta)$ .....(2)

differentiating equation (1) and (2) with respect to '$\theta$' by

by (1) $\Rightarrow$ $\frac{dx}{d\theta} = a(-\sin\theta - \theta\cos\theta + (1)\sin\theta)$

$$= -a\sin\theta + a\theta\cos\theta + a\sin\theta$$

$$\frac{dx}{d\theta} = a\,\theta\cos\theta \quad .....(3)$$

by (2) $\Rightarrow$ $\frac{dy}{d\theta} = a\left(\cos\theta - \left(\theta(-\sin\theta) + (1)\cos\theta\right)\right)$

$$= a\cos\theta + a\theta\sin\theta - a\cos\theta$$

$$\frac{dy}{d\theta} = a\,\theta\sin\theta \quad ........(4)$$

$\therefore$ the length of the arc from $\theta = 0$ to $\theta = \alpha$

$$S = \int_0^{\alpha}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{\alpha}\sqrt{(a\theta\cos\theta)^2 + (a\theta\sin\theta)^2}\,d\theta$$

$$= \int_0^{\alpha}\sqrt{a^2\theta^2\cos^2\theta + a^2\theta^2\sin^2\theta}\ d\theta$$

$$= \int_0^{\alpha} \sqrt{a^2\theta^2\left(\cos^2\theta + \sin^2\theta\right)}\, d\theta$$

$$= \int_0^{\alpha} \sqrt{a^2\theta^2(1)}\, d\theta$$

$$= \int_0^{\alpha} a\theta d\theta$$

$$= a\int_0^{\alpha} \theta d\theta$$

$$= a\left[\frac{\theta^2}{2}\right]_0^{\alpha}$$

$$= a\left[\frac{\alpha^2}{2} - 0\right]$$

$$S = \frac{a\alpha^2}{2}$$

$\therefore$ The length of given curve is $\frac{1}{2}a\alpha^2$

**10. Show that the arc of the Upper half of the curve r = a (1 – cos θ) is bisected by $\theta = \frac{2\pi}{3}$.**

*Sol :*

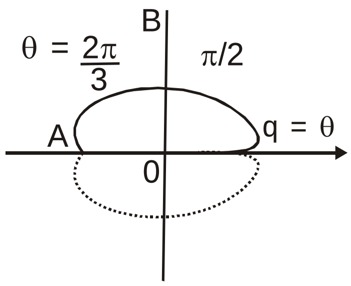
Given curve is r = a (1 – cos θ) .............. (1)

differentiating (1) with respect to θ r = a – a cos θ)

$$\frac{dr}{d\theta} a - a(-\sin\theta)$$

$$\frac{dr}{d\theta} = a\sin\theta$$

$\therefore$ The length of the arc



$$S = \int_0^{2\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= \int_0^{2\pi/3} \sqrt{\left[a(1-\cos\theta)\right]^2 + (a\sin\theta)^2} d\theta$$

$$= \int_0^{2\pi/3} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\ d\theta$$

$$= \int_0^{2\pi/3} \sqrt{a^2\left(1+\cos^2\theta - 2\cos\theta\right) + a^2\sin^2\theta}\ d\theta$$

$$= \int_0^{2\pi/3} \sqrt{a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta}\ \ d\theta$$

$$= \int_0^{2\pi/3} \sqrt{a^2 + a^2\left(\sin^2\theta + \cos^2\theta\right) - 2a^2\cos\theta}\ \ d\theta$$

$$= \int_0^{2\pi/3} \sqrt{a^2 + a^2(1) - 2a^2\cos\theta}\ d\theta$$

$$= \int_0^{2\pi/3} \sqrt{2a^2 - 2a^2\cos\theta}\ \ d\theta$$

$$= \int_0^{2\pi/3} \sqrt{2a^2(1-\cos\theta)}\ d\theta$$

$$= \int_0^{2\pi/3} \sqrt{2}\sqrt{a^2(1-\cos\theta)}\ d\theta$$

$$= \sqrt{2a} \int_0^{2\pi/3} \sqrt{1-\cos\theta}\ d\theta$$

$$= \sqrt{2}a \int_0^{2\pi/3} \sqrt{2\sin^2\frac{\theta}{2}}\ d\theta$$

$$= \sqrt{2}a.\sqrt{2} \int_0^{2\pi/3} \sqrt{\sin^2\frac{\theta}{2}}\ d\theta$$

$$= \sqrt{2}a.\sqrt{2} \int_0^{2\pi/3} \sin\frac{\theta}{2} d\theta$$

$$= 2a\left[\frac{-\cos\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{2\pi/3}$$

$$= 2a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi/3}$$

$$= -4a\left[\cos\frac{\theta}{2}\right]_0^{2\pi/3}$$

$$= -4a\left[\cos\frac{\left(2\frac{\pi}{3}\right)}{2} - \cos\left(\frac{\theta}{2}\right)\right]$$

$$= -4a\left[\cos\left(\frac{2\pi}{6}\right) - \cos 0\right]$$

$$= -4a\left[\cos\frac{\pi}{3} - 1\right]$$

$$= -4a\left[\frac{1}{2} - 1\right]$$

$$= -4a\left[\frac{-1}{2}\right]$$

$$= \frac{4a}{2}$$

$$S = 2a$$

$\therefore$ The length of the upper half plane is a 2a.

Hence $\theta = \frac{2\pi}{3}$ is bisects the uper half of the curve.

**11. Find the length of an arc of the curve $r = e^{\theta \cot \alpha}$ taking S = 0 when $\theta = 0$.**

*Sol :*

Given curve is $r = e^{\theta \cot \alpha}$ .....(1)

differentiating (1) w.r.to '$\theta$'

$$\frac{dr}{d\theta} = ae^{\theta \cot \alpha} \frac{d}{d\theta}(\theta \cot \alpha)$$

$$= ae^{\theta \cot \alpha}(1) \cot\alpha$$

$$\frac{dr}{d\theta} = a\cot\alpha e^{\theta\cot\alpha}$$

$$\frac{dr}{d\theta} = ae^{\theta\cot\alpha}\cot\alpha$$

By equation (1) we know that $r = ae^{\theta\cot\alpha}$

$\therefore \quad \frac{dr}{d\theta} = e\cot\alpha$

$\therefore$ The length of the arc is $S = \int_{\alpha}^{\beta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,ds$

Which $\alpha = 0$ & $\beta = \theta$

$$S = \int_0^{\theta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{\theta}\sqrt{r^2+(r\cot\alpha)^2}\,d\theta$$

$$= \int_0^{\theta}\sqrt{\left(r^2+r2\cot^2\alpha\right)}\,d\theta$$

$$= \int_0^{\theta}\sqrt{r^2}\sqrt{1+\cot^2\alpha}\,d\theta$$

$$= r\int_0^{\theta}\sqrt{1+\cot^2\alpha}\,d\theta \qquad \left[\because 1+\cot^2\alpha = \operatorname{cosec}^2\alpha\right]$$

$$= r\int_0^{\theta}\sqrt{\operatorname{cosec}^2\alpha}\,d\theta$$

$$= r\int_0^{\theta}\operatorname{cosec}\alpha\,d\theta$$

$$= \operatorname{cosec}\alpha\int_0^{\theta}r\,d\theta$$

$$= \operatorname{cosec}\alpha\int_0^{\theta}ae^{\theta\cot\alpha}d\theta$$

$$= a\operatorname{cosec}\alpha\int_0^{\theta}e^{\theta\cot\alpha}d\theta$$

$$= a \text{ cosec } \alpha \left[\frac{e^{a\cot\alpha}}{\cot\alpha}\right]_0^{\theta}$$

$$= \frac{a \text{ cosec } \alpha}{\cot\alpha}\left[e^{\theta\cot\alpha} - e^{0}\right]$$

$$= \frac{a\dfrac{1}{\sin\alpha}}{\dfrac{\cos\alpha}{\sin\alpha}}\left[e^{\theta\cot\alpha} - 1\right]$$

$$= a\frac{1}{\cos\alpha}\left[e^{\theta-\cot\alpha} - 1\right]$$

$$S = a\sec\alpha\left[e^{\theta\cot\alpha} - 1\right]$$

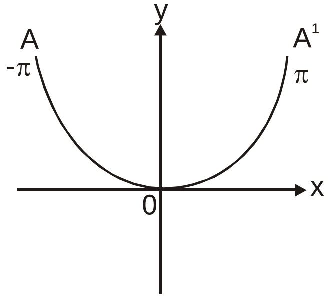
$\therefore$ The length of the arc is $a \sec \alpha \left[e^{\theta\cot\alpha} - 1\right]$

**12. Rectify the curve x = a(θ + sinθ), y = a (1–cosθ).**

*Sol :*

Given curve is $x = a(\theta + \sin\theta)$ .....(1)

$y = a(1 - \cos\theta)$ .....(2)




A point moves from one end A[1] to another end A of the one arc, the parameter $\theta$ increases from $-\pi$ to $\pi$.

As the arc is symmetrical about OY are AOA[1] = 2 arc OA.

i.e., $$2\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Now differentiate equation (1) & (2) with respect to 'θ'

by (1) $\Rightarrow$ $\dfrac{dx}{d\theta} = a(1 + \cos\theta)$

by (2) $\Rightarrow$ $\dfrac{dy}{d\theta} = a(0 - (\sin\theta)$

$\dfrac{dy}{d\theta} = a\sin\theta$

$\therefore$ The length of the arc

$$S = 2\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{\left[a(1+\cos\theta)\right]^2+\left[a\sin\theta\right]^2}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{a^2\left(1+\cos^2\theta+2\cos\theta\right)+a^2\sin^2\theta}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{a^2+a^2\cos^2\theta+2a^2\cos\theta+a^2\sin^2\theta}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{a^2+a^2\left(\sin^2\theta+\sin^2\theta\right)+2a^2\cos\theta}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{a^2+a^2(1)+2a^2\cos\theta}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{2a^2(1+\cos\theta)}\,d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi}\sqrt{1+\cos\theta}\,d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi}\sqrt{2\cos^2\frac{\theta}{2}}\,d\theta$$

$$= 4a\int_0^{\pi}\left[\cos\frac{\theta}{2}\right]d\theta$$

$$= 4a\left[\frac{\sin\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{\pi}$$

$$= 8a\left[\sin\frac{\theta}{2}\right]_0^{\pi}$$

$$= 8a\left[\sin\frac{\pi}{2}-\sin 0\right]$$

$$= 8a\,(1-0)$$

$$S = 8a$$

$\therefore$ The length of the curve is 8a.

**13. Find the perimeter of the cardiode r = a (1 – cos θ).**

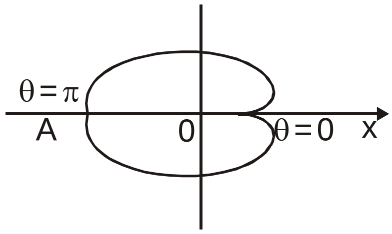
*Sol :*

The Given curve is $r = a(1 - \cos\theta)$ .....(1)

differentiating (1) w.r.to $\theta$

$$\frac{dr}{d\theta} = a(0 - \sin\theta)$$

$$\frac{dr}{d\theta} = a\sin\theta$$




The curve is symmetrical about the initial line, and its perimeter is double the length of the arc of the curve.

i.e., the length of the arc $S = 2\int_0^{\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$$S = 2\int_0^{\pi}\sqrt{r^2 + (a\sin\theta)^2}\,d\theta$$

by equation (1) we have $r = a(1 - \cos\theta)$

$$= 2\int_0^{\pi}\sqrt{(a(1-\cos\theta))^2 + (a\sin\theta)^2}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{a^2\left(1+\cos^2\theta - 2\cos\theta\right) + a^2\sin^2\theta}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin\theta}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{a^2 + a^2\left(\cos^2\theta + \sin^2\theta\right) - 2a^2\cos\theta}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{2a^2 - 2a^2\cos\theta}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{2a^2(1-\cos\theta)}\ d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi}\sqrt{2\sin^2\frac{\theta}{2}}\ d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi}\sqrt{2}\sqrt{\sin^2\frac{\theta}{2}}\ d\theta$$

$$= 2\sqrt{2}a.\sqrt{2}\int_0^{\pi}\sin\frac{\theta}{2}d\theta$$

$$= 4a\left[\frac{-\cos\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{\pi}$$

$$= 8a\left[-\cos\frac{\pi}{2}+\cos 0\right]$$

$$= 8a\ [0 + 1]$$

$$S = 8a$$

$\therefore$ The perimeter of cardiode is 8a

**14. Show that the length of the loop of the curve $3ay^2 = x(x - a)^2$ is $4a/\sqrt{3}$.**

*Sol :*

Given curve is $3ay^2 = x(x - a)^2$ ........... (1)

Differentiating (1) w.r.t to 'x'

$$3a\ (2y)\frac{dy}{dx} = x\ \frac{d}{dx}(x-a)^2 + (x-a)$$

$$= x\ (2(x - a)\ (1) + (x - a)^2$$

$$= 2x(x - a) + (x - a)^2$$

$$= 2x(x - a) + (x - a)^{2|}$$

$$= (x - a)\ (2x + x - a)$$

$$6ay\ \frac{dy}{dx} = (x - a)\ (3x - a)$$

$$\frac{dy}{dx} = \frac{(x-a)(3x-a)}{6ay}$$

$\therefore$ The length of the curve $S = 2\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$

$$S = 2\int_0^a \sqrt{1+\left[\frac{(x-a)(3x-a)}{6ay}\right]^2}\,dx$$

$$= 2\int_0^a \sqrt{1+\frac{(x-a)^2(3x-a)^2}{36a^2y^2}}\,dx$$

But equation (1) we have,

$$3ay^2 = x(x-a)^2$$

$$y^2 = \frac{x(x-a)^2}{3a}$$

$$S = 2\int_0^a \sqrt{1+\frac{(x-a)^2(3x-a)^2}{36a^2\left(\frac{x(x-a)^2}{3a}\right)}}\,dx$$

$$= 2\int_0^a \sqrt{1+\frac{3a(x-a)^2(3x-a)^2}{36a^2\left(x(x-a)^2\right)}}\,dx$$

$$= 2\int_0^a \sqrt{1+\frac{(x-a)^2(3x-a)^2}{12a\left(x(x-a)^2\right)}}\,dx$$

$$= 2\int_0^a \sqrt{1+\frac{(3x-a)^2}{12ax}}\,dx$$

$$= 2\int_0^a \sqrt{\frac{12ax+9x^2+a^2-6xa}{12ax}}\,dx$$

$$= 2\int_0^a \sqrt{\frac{6ax+9x^2+a^2}{12ax}}\,dx$$

$$= 2\int_0^a \sqrt{\frac{(3x+a)^2}{12ax}}\,dx$$

$$= 2\int_0^a \frac{\sqrt{(3x+a)^2}}{2\sqrt{3}\sqrt{a}\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{3}\sqrt{a}}\int_0^a \frac{3x+a}{\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{3}\sqrt{a}}\int_0^a \frac{3x}{\sqrt{x}}\,dx + \int_0^a \frac{a}{\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{3}\sqrt{a}}\int_0^a 3\sqrt{x}\,dx + \int_0^a a\,x^{-1/2}dx$$

$$= \frac{1}{\sqrt{3}\sqrt{a}}\left\{\left[3\frac{x^{1/2+1}}{\frac{1}{2}+1}\right]_0^a + a\left[\frac{x^{-1/2+1}}{-1/2+1}\right]_0^a\right\}$$

$$= \frac{1}{\sqrt{3}\sqrt{a}}\left[\cancel{3}.\frac{2}{\cancel{3}}\left(x^{3/2}\right)_0^a + 2a\left(x^{1/2}\right)_0^a\right]$$

$$= \frac{4}{\sqrt{3}\sqrt{a}}\left[a^{3/2}+a^{3/2}\right]$$

$$= \frac{4}{\sqrt{3}\sqrt{a}}\left[a^{3/2}\right]$$

$$S = \frac{4a}{\sqrt{3}}$$

$\therefore$ The length of the loop of curve is $\frac{4a}{\sqrt{3}}$

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**15.** Prove that the loop of the curve $x = t^2$ **, $y = t - \frac{1}{3}t^3$ is of length $4\sqrt{3}$.**

*Sol :*

Given curve is $x = t^2$ .....(1)

$y = t - \frac{1}{3}t^3$ .....(2)

differentiating equation (1), (2) w.r.to 'x'

(1) $\Rightarrow \frac{dx}{dt} = 2t$

(2) $\Rightarrow \frac{dy}{dt} = 1 - \frac{1}{3}(3t^2)$

$\frac{dy}{dt} = 1 - t^2$

by (2) $\Rightarrow$ $y = t - \frac{1}{3}t^3$

$$y = t\left(1 - \frac{t^2}{3}\right)$$

$$y^2 = t^2\left(1 - \frac{t^2}{3}\right)^2$$

$$y^2 = x\left(1 - \frac{x}{3}\right)^2$$

Patting y = 0 in equation on (2)

$\therefore$ $0 = t - \frac{t^3}{3}$

$$t = \frac{t^3}{3}$$

$3t - t^3 = 0$

$t(3 - t^2) = 0$

$t = 0, 3 - t^2 = 0$

$t = 0, t = \pm\sqrt{3}$

$\therefore$ The length of the arc is twice the length of loop which 't' varies from '0' to '$\sqrt{3}$'

$\therefore$ The length of arc is $S = 2\int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= 2\int_0^{\sqrt{3}} \sqrt{(2t)^2 + (1 - t^2)^2}\, dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{4t^2 + 1 + t^4 - 2t^2}\, dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{2t^2 + t^4 + 1}\, dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{(t^2 + 1)^2}\, dt$$

$$= 2\int_0^{\sqrt{3}} t^2 + 1\, dt$$

$$= 2\left[\frac{t^3}{3} + t\right]_0^{\sqrt{3}}$$

$$= 2\left\{\left[\frac{\left(\sqrt{3}\right)^3}{3} + \sqrt{3}\right] - [0]\right\}$$

$$= 2\left[\frac{3\sqrt{3}}{3} + \sqrt{3}\right]$$

$$= 2\left[\sqrt{3} + \sqrt{3}\right]$$

$$= 2\left(2\sqrt{3}\right)$$

$$= 4\sqrt{3}$$

$\therefore$ The length of the loop of curves is $4\sqrt{3}$

**16. Prove that the length of the arc of hyperbolic spiral $r\theta = a$, taken from the point $r = \alpha$ a to is $a\left\{\sqrt{5} - \sqrt{2} + \log\frac{2+\sqrt{8}}{1+\sqrt{5}}\right\}$**

*Sol :*

Given curve is $r\theta = a$

$$\theta = \frac{a}{r} \qquad .....(1)$$

differentiating (1) with respect to 'r'

$$\frac{d\theta}{dr} = \frac{-a}{r^2}$$

The length of the arc is $S = \int_0^{2a} \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}\, dr$

$$S = \int_a^{2a} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

$$= \int_a^{2a} \sqrt{1 + r^2\left(\frac{-a}{r^2}\right)^2}\, dr$$

$$= \int_a^{2a} \sqrt{1 + \frac{a^2r^2}{r^4}}\, dr$$

$$= \int_a^{2a} \sqrt{1+\frac{a^2}{r^2}}\ dr$$

$$= \int_a^{2a} \frac{\sqrt{r^2+a^2}}{\sqrt{r^2}} dr$$

$$= \int_a^{2a} \frac{\sqrt{r^2+a^2}}{r}\ dr$$

Let, $r^2 + a^2 = t^2 \Rightarrow r^2 = r^2 - a^2 \Rightarrow r = \sqrt{t^2 - a^2}$

$2r\ dr = 2t\ dt$

$dr = \frac{2t}{2r}\ dt$

$dr = \frac{t}{r}\ dt$

It $r = 2a \Rightarrow t^2 = (2a)^2 + a^2$

$= 4a^2 + a^2$

$t^2 = 5a^2$

$t = \sqrt{5a^2}$

$t = \sqrt{5}a$

It $r = a \Rightarrow \quad t^2 = a + a^2$

$t^2 = 2a^2$

$t = a\sqrt{2}$

$$= \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{\sqrt{t^2}}{\sqrt{t^2-a^2}}.\frac{t}{r} dt$$

$$= \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{t.t}{\sqrt{t^2-a^2}\sqrt{t^2-a^2}}\ dt$$

$$= \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{t^2}{t^2-a^2}\ dt$$

$$= \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{t^2-a^2+a^2}{t^2-a^2} dt$$

$$= \int_{a\sqrt{2}}^{a\sqrt{5}} \left[\frac{t^2-a^2}{t^2-a^2}+\frac{a^2}{t^2-a^2}\right] dt$$

$$= \int_{a\sqrt{2}}^{a\sqrt{5}} \left(1 + \frac{a^2}{t^2 - a^2}\right) dt$$

$$= \int_{a\sqrt{2}}^{a\sqrt{5}} dt + a^2 \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{1}{t^2 - a^2} dt$$

$$= [t]_{a\sqrt{2}}^{a\sqrt{5}} + a^2 \left[\frac{1}{2a} \log\left[\frac{t-a}{t+a}\right]\right]_{a\sqrt{2}}^{a\sqrt{5}}$$

$$= \left(a\sqrt{5} - a\sqrt{2}\right) + \frac{a^2}{2a}\left[\log\left(\frac{a\sqrt{5}-a}{a\sqrt{5}+a}\right) - \log\left(\frac{a\sqrt{2}-a}{a\sqrt{2}+a}\right)\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right) - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log\left(\frac{\sqrt{5}-1}{\sqrt{5}+1} \times \frac{\sqrt{5}+1}{\sqrt{5}+1}\right) - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}\right)\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log\left(\frac{\left(\sqrt{5}\right)^2 - (1)^2}{\left(\sqrt{5}+1\right)^2}\right) - \log\left(\frac{\left(\sqrt{2}\right)^2 - (1)^2}{\left(\sqrt{2}+1\right)^2}\right)\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log\left(\frac{5-1}{\left(\sqrt{5}+1\right)^2}\right) - \log\left(\frac{2-1}{\left(\sqrt{2}+1\right)^2}\right)\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log\frac{4}{\left(\sqrt{5}+1\right)^2} - \log\frac{1}{\left(\sqrt{2}+1\right)^2}\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log\left(\frac{\frac{4}{\left(\sqrt{5}+1\right)^2}}{\frac{1}{\left(\sqrt{2}+1\right)^2}}\right)\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log\frac{4\left(\sqrt{2}+1\right)^2}{\left(\sqrt{5}+1\right)^2}\right]$$

$$= a\left(\sqrt{5}-\sqrt{2}\right)+\frac{a}{2}\left[\log\frac{2\left(\sqrt{2}+1\right)}{\left(\sqrt{5}+1\right)}\right]^2$$

$$= a\left(\sqrt{5}-\sqrt{2}\right)+\frac{2a}{2}\log\left(\frac{2\left(\sqrt{2}+1\right)}{\sqrt{5}+1}\right)$$

$$= a\left(\sqrt{5}-\sqrt{2}\right)+a\log\left(\frac{2\sqrt{2}+2}{\sqrt{5}+1}\right)$$

$$= a\left(\sqrt{5}-\sqrt{2}\right)+a\log\left(\frac{2+\sqrt{8}}{\sqrt{5}+1}\right)$$

$$S = a\left\{\sqrt{5}-\sqrt{2}+\log\frac{2+\sqrt{8}}{1+\sqrt{5}}\right\}$$

$\therefore$ The length of the arc is $a\left\{\sqrt{5}-\sqrt{2}+\log\frac{2+\sqrt{8}}{1+\sqrt{5}}\right\}$

---

**17. Find the length of the arc of the curve $y = \log\frac{e^x - 1}{e^x + 1}$ from x = 1 to x = 2.**

*Sol :*

Given curve is $y = \log\frac{e^x-1}{e^x+1}$ .....(1)

from x = 1 to x = 2

differentiating (1) w.r. to 'x'

$$\frac{dy}{dx} = \frac{1}{\frac{e^x-1}{e^x+1}}\frac{d}{dx}\left[\frac{e^x-1}{e^x+1}\right]$$

$$= \frac{e^x+1}{e^x-1}\left[\frac{\left(e^x+1\right)\left(e^x\right)-\left(e^x-1\right)e^x}{\left(e^x+1\right)^2}\right]$$

$$= \frac{\left(e^x+1\right)\left(e^x+1\right)e^x-\left(e^x+1\right)\left(e^x-1\right)e^x}{\left(e^x+1\right)^2\left(e^x-1\right)}$$

$$= \frac{e^x\left(e^x+1\right)\left[\left(e^x+1\right)-\left(e^x-1\right)\right]}{\left(e^x+1\right)^2\left(e^x-1\right)}$$

$$= \frac{e^x(e^x+1)\left[\cancel{e^x}+1-\cancel{e^x}+1\right]}{(e^x+1)^2(e^x-1)}$$

$$= \frac{2e^x(e^x+1)}{(e^x+1)^2(e^x-1)} = \frac{2e^x}{(e^x-1)(e^x+1)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2e^x}{(e^x+1)(e-1)} = \frac{2e^x}{(e^x-1)^2}$$

The length of the curve is $S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$S = \int_1^2 \sqrt{1+\left[\frac{2e^x}{(e^x-1)^2}\right]^2}\,dx$$

$$= \int_1^2 \sqrt{1+\left(\frac{2e^x}{e^{2x}-1}\right)^2}\,dx$$

$$= \int_1^2 \sqrt{1+\frac{4e^{2x}}{(e^{2x}-1)^2}}\,dx$$

$$= \int_1^2 \sqrt{\frac{(e^{2x}-1)^2+4e^{2x}}{(e^{2x}-1)^2}}\,dx$$

$$= \int_1^2 \sqrt{\frac{e^{4x}+1-2e^{2x}+4e^{2x}}{(e^{2x}-1)^2}}\,dx$$

$$= \int_1^2 \sqrt{\frac{(e^{4x}+2e^{2x}+1)}{(e^{2x}-1)^2}}\,dx$$

$$= \int_1^2 \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}\,dx$$

$$= \int_1^2 \frac{\sqrt{\left(e^{2x}+1\right)^2}}{\sqrt{\left(e^{2x}-\right)^2}}\, dx$$

$$= \int_1^2 \frac{e^{2x}+1}{e^{2x}-1}\, dx$$

$$= \int_1^2 \frac{e^{x}.e^{x}+1}{e^{x}.e^{x}-1}\, dx$$

$$= \int_1^2 \frac{\frac{e^{x}}{e^{-x}}+1}{\frac{e^{x}}{e^{-x}}-1}\, dx$$

$$= \int_1^2 \frac{\frac{e^{x}+e^{-x}}{\cancel{e^{-x}}}}{\frac{e^{x}-e^{-x}}{\cancel{e^{-x}}}}\, dx$$

$$= \int_1^2 \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\, dx \qquad \left[\because \int \frac{f^1(x)}{f(x)}dx = \log f(x)\right]$$

$$= \left[\log\left(e^{x}-e^{-x}\right)\right]_1^2$$

$= \log (e^2 - e^{-2}) - \log(e^1 - e^{-1})$

$$= \log\left(\frac{e^2-e^{-2}}{e^1-e^{-1}}\right)$$

$$= \log\left(\frac{\left(e+e^{-1}\right)\left(e-e^{-1}\right)}{e-e^{-1}}\right)$$

$S = \log (e + e^{-1})$

$\therefore$ Length of the curve is $\log (e + e^{-1})$

**18. Find the length of the curve $x = e^{\theta} \sin \theta$, $y = e^{\theta} \cos \theta$**

*Sol :*

Given curves $x = e^{\theta} \sin \theta$ .....(1)

$y = e^{\theta} \cos \theta$ .....(2)

differentiating (1) & (2) with respect to '$\theta$'

$$\frac{dx}{d\theta} = e^{\theta}\left(\cos\theta\right) + e^{\theta}\sin\theta$$

$$\frac{dx}{d\theta} = e^{\theta}(\sin\theta + \cos\theta)$$

$$\frac{dy}{d\theta} = e^{\theta}(-\sin\theta) + e^{\theta}\cos\theta$$

$$= -\ e^{\theta}\sin\theta + e^{\theta}\cos\theta$$

$$\frac{dy}{d\theta} = e^{\theta}(\cos\theta - \sin\theta)$$

The length of the arc is given by

$$s = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{\left(e^{\theta}(\sin\theta + \cos\theta)\right)^2 + \left(e^{\theta}(\cos\theta - \sin\theta)\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{e^{2\theta}(\cos\theta + \sin\theta)^2 + e^{2\theta}(\cos\theta - \sin\theta)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{e^{2\theta}\left(\cos^2\theta + \sin^2\theta + 2\cancel{\sin\theta}\cos\theta + \cos^2\theta + \sin^2\theta\right) - 2\cancel{\sin\theta}\cos\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{e^{2\theta}(1+1)}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{2e^{2\theta}}$$

$$= \sqrt{2}\int_0^{\pi/2} \sqrt{e^{2\theta}}\, d\theta$$

$$= \sqrt{2}\int_0^{\pi/2} \sqrt{\left(e^{\theta}\right)^2}\, d\theta$$

$$= \sqrt{2}\int_0^{\pi/2} e^{\theta}\, d\theta$$

$$= \sqrt{2}\left[e^{\theta}\right]_0^{\pi/2}$$

$$= \sqrt{2}\left[e^{\pi/2} - e^{\theta}\right]$$

$$s = \sqrt{2}\left[e^{\pi/2} - 1\right]$$

$\therefore$ The length of the curve is $\sqrt{2}\left[e^{\pi/2} - 1\right]$

## 4.4 VOLUMES AND SURFACE OF REVOLUTION

**Introduction**

Expression for the volume obtained by revolving about either axis

**Volume of a Solid of Revolution**

- The volume obtained by revolving about x-axis the arc of the curve y = f(x) intercepted between the points whose abseissae are a, b is

$\int_a^b \pi y^2 dx$ i.e., $\int_a^b \pi [f(x)]^2$ dx. If being assumed that the are does not cut x-axis.

- x = f(y) about y-axis between the point whose ordinates are a, b is

$\int_a^b \pi x^2\, dy$

**19. Find the volume of the solid obtained by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the axis of x.**

*Sol :*

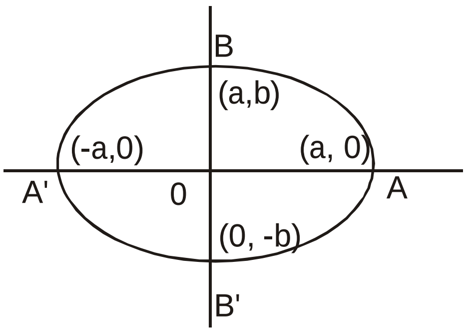
The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \left[\frac{a^2 - x^2}{a^2}\right] \quad .....(1)$$

It is easy to see that the solid obtained by revolving the arc ABA[1] about x-axis is same as the solid obtained by revolving the whole ellipse.



Also, the volume of the solid is double the volume of the solid revolving the arc AB.

$\therefore$ The volume of solid is $V = 2\left[\int_0^a \pi y^2\right]dx$

$$= 2\pi\int_0^a y^2 dx$$

$$= 2\pi\int_0^a b^2\left(\frac{a^2-x^2}{a^2}\right)dx \quad [\because \text{ by (1)}]$$

$$= \frac{2\pi b^2}{a^2}\int_0^a \left(a^2-x^2\right)dx$$

$$= \frac{2\pi b^2}{a^2}\left[\int_0^a a^2 dx - \int_0^a x^2 dx\right]$$

$$= \frac{2\pi b^2}{a^2}\left[a^2\int_0^a dx - \int_0^a x^2 dx\right]$$

$$= \frac{2\pi b^2}{a^2}\left[a^2[x]_0^a - \left[\frac{x^3}{3}\right]_0^a\right]$$

$$= \frac{2\pi b^2}{a^2}\left[a^2(a-0) - \left(\frac{a^3}{3}-0\right)\right]$$

$$= \frac{2\pi b^2}{a^2}\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{2\pi b^2}{a^2}\left[\frac{3a^3-a^3}{3}\right]$$

$$= \frac{2\pi b^2}{a^2}\left(\frac{2a^3}{3}\right)$$

$$V = \frac{4\pi b^2 a}{3}$$

$\therefore$ Volume of solid is $V = \dfrac{4\pi b^2 a}{3}$

**20. Find the volume of the solid obtained by resolving the cordiode r = a (1 + cos θ) about the intial line.**

*Sol :*

Given curve is $r = a(1 + \cos\theta)$

Let $x = r\cos\theta,\ y = r\sin\theta.$

$x = a(1 + \cos\theta)\cos\theta$ .....(1)

$y = a(1 + \cos\theta)\sin\theta$ .....(2)

differentiating (1), (2) with respect to 'θ'

$$\frac{dx}{d\theta} = a\left[(1+\cos\theta)\frac{d}{d\theta}(\cos\theta) + \frac{d}{d\theta}(1+\cos\theta)\cos\theta\right]$$

$$= a[(1 + \cos\theta)(-\sin\theta) + (0 - \sin\theta)\cos\theta]$$

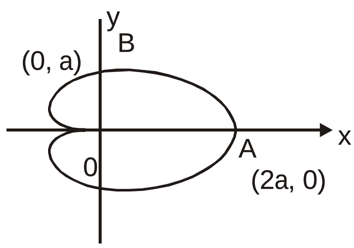
$$= a[-\sin\theta - \sin\theta\cos\theta - \sin\theta\cos\theta]$$

$$\frac{dx}{d\theta} = a[\sin\theta - 2\sin\theta\cos\theta] \quad .....(2)$$

∴ The required volume is $v = \pi\int_0^{2a} y^2\,dx$ for 0A

$\theta = \pi$ when $x = 0$

$\theta = 0$ when $x = 2a$




∴ $v = \int_0^{2a} y^2 dx$

$$= \pi\int_\pi^0 y^2 dx$$

by (2) ⇒ $dx = a[-\sin\theta - 2\sin\theta\cos\theta]\,d\theta$

$$= \pi\int_A^0 [a(1 + \cos\theta)\sin\theta]^2\,[a(-\sin\theta - 2\sin\theta\cos\theta]\,d\theta$$

$$= \pi\int_A^0 a^2(1 + \cos\theta)^2\sin^2\theta(-a\sin\theta - 2a\sin\theta\cos\theta]\,d\theta$$

$$= -\pi\int_\pi^0 a^2(1+\cos\theta)^2\sin^2\theta(a\sin\theta + 2a\sin\theta\cos\theta)\,d\theta$$

$$= -\pi\int_{\pi}^{0} a^2(1+\cos\theta)^2 \sin^2\theta \; a\sin\theta(1+2\cos\theta)d\theta$$

$$= -\pi\int_{\pi}^{0} a^3\sin^3\theta(1+\cos\theta)^2(1+2\cos\theta)d\theta$$

$$= \pi a^3\int_{0}^{\pi} \sin^3\theta(1+\cos\theta)^2(1+2\cos\theta)d\theta$$

$$= \pi a^3\int_{0}^{\pi}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^3\left(2\cos^2\frac{\theta}{2}\right)^2\left(1+2\left(2\cos^2\frac{\theta}{2}-1\right)\right)d\theta$$

$$= \pi a^3\int_{0}^{\pi} 8\sin^3\frac{\theta}{2}\cos^3\frac{\theta}{2}4\cos^4\frac{\theta}{2}\left(1+4\cos^2\frac{\theta}{2}-2\right)d\theta$$

$$= 32\pi a^3\int_{0}^{\pi}\sin^3\frac{\theta}{2}\cos^2\frac{\theta}{2}\left(4\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$= 32\pi a^3\int_{0}^{\pi}\left(4\sin^3\frac{\theta}{2}\cos^9\frac{\theta}{2}-\sin^3\frac{\theta}{2}\cos^7\frac{\theta}{2}\right)d\theta$$

Let $\frac{\theta}{2}=\phi$

$d\theta = 2d\phi$

If $\theta=\pi \Rightarrow \phi=\frac{\pi}{2}$

If $\theta=0 \Rightarrow \phi=0$

$$V = 32\pi a^3\int_{0}^{\pi/2}\left(4\sin^3\phi\cos^9\phi-\sin^3\phi\cos^7\phi\right)2d\phi$$

$$= 64\pi a^3\int_{0}^{\pi/2}\left(4\sin^3\phi\cos^9\phi-\sin^3\phi\cos^7\right)d\theta$$

$$= 64\pi a^3\int_{0}^{\pi/2}4\sin^3\phi\cos^9\phi d\phi - 64\pi a^3\int_{0}^{\pi/2}\sin^3\phi\cos^7\phi d\phi$$

$$= 256\pi a^3\int_{0}^{\pi/2}\sin^3\phi\cos^9\phi\, d\phi - 64\pi a^3\int_{0}^{\pi/2}\sin^3\cos^7\phi d\phi$$

$$\left[\because \int_{0}^{\pi/2}\sin^m x\cos^n x dx = \left(\frac{n-1}{m+n}\right)\left(\frac{n-3}{m+n-2}\right)...\left(\frac{2}{m+3}\right)\left(\frac{1}{m+1}\right)\right]$$

$$= 256\pi a^3\left(\frac{9-1}{3+9}\right)\left(\frac{9-3}{9+3-2}\right)\left(\frac{9-5}{9+3-4}\right)\left(\frac{9-7}{9+3-6}\right)\left(\frac{1}{3+1}\right)$$

$$= 64\pi a^3\left(\frac{7-3}{3+4}\right)\left(\frac{7-3}{3+7-2}\right)\left(\frac{1}{3+1}\right)$$

$$= 256\pi a^3\left[\left(\frac{8}{12}\right)\left(\frac{6}{10}\right)\left(\frac{4}{8}\right)\left(\frac{2}{6}\right)\left(\frac{1}{4}\right)\right]-64\pi a^3\left[\left(\frac{6}{10}\right)\left(\frac{4}{8}\right)\left(\frac{2}{6}\right)\left(\frac{1}{4}\right)\right]$$

$$= 256\pi a^3\left(\frac{1}{60}\right)-64\pi a^3\left(\frac{1}{40}\right)$$

$$= \frac{64\pi a^3}{15}-\frac{8\pi a^3}{5}$$

$$= \frac{64\pi a^3-24\pi a^3}{15}$$

$$= \frac{40\pi a^3}{15}$$

$$V = \frac{8\pi a^3}{3}$$

$\therefore$ The volume of the solid is $\frac{8\pi a^3}{5}$

**21. Find the volume of the spindle shaped solid generated by revolving the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis.**

*Sol :*

Given hypocycloid is $x^{2/3}+y^{2/3}=a^{2/3}$

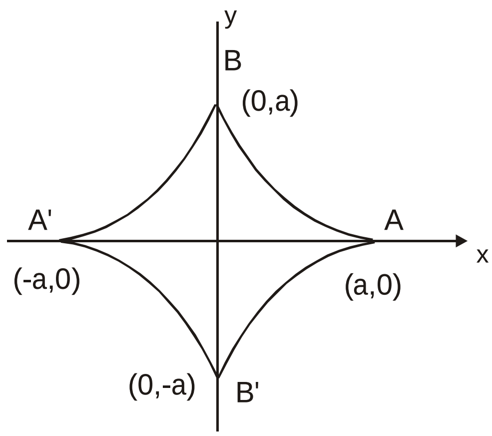
$$y^{2/3}=a^{2/3}-x^{2/3}$$

Cube on both sides $\left(y^{2/3}\right)^3=\left(a^{2/3}\right)-\left(x^{2/3}\right)^3$

$$y^2 = \left(a^{2/3}\right)^3-\left(x^{2/3}\right)^3-3\left(a^{2/3}\right)\left(x^{2/3}\right)\left(a^{2/3}-x^{2/3}\right)$$

$$y^2 = a^2-x^2-3a^{4/3}x^{2/3}+3a^{2/3}x^{4/3}$$

The volume of hypocycloid is double of the volume generated by revolving the area lying in the 1st Quadrant.




$\therefore$ The volume of the solid is $V = 2\int_0^9 \pi y^2 dx$

$$= 2\pi\int_0^a y^2 dx$$

$$= 2\pi\left[\int_0^a \left(a^2x^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3}\right)\right]dx$$

$$= 2\pi\left[\int_0^a a^2dx - \int_0^a x^2dx - 3a^{4/3}\int_0^a x^{2/3}dx + 3a^{2/3}\int_0^a x^{4/3}dx\right]$$

$$= 2\pi\left[a^2\int_0^a dx - \int_0^a x^2dx - 3a^{4/3}\int_0^a x^{2/3} + 3a^{2/3}\int_0^9 x^{4/3}dx\right]$$

$$= 2\pi\left[a^2(x)_0^a - \left[\frac{x^3}{3}\right]_0^a - 3a^{4/3}\left[\frac{x^{2/3+1}}{\frac{2}{3}+1}\right]_0^a + 3a^{2/3}\left[\frac{x^{4/3+1}}{\frac{4}{3}+1}\right]_0^a\right]$$

$$= 2\pi\left[a^2(a-0) - \left(\frac{a^3}{3} - 0\right) - 3a^{4/3}\left[\frac{x^{5/3}}{\frac{5}{3}}\right]_0^a + 3a^{2/3}\left[\frac{x^{7/3}}{\frac{7}{3}}\right]_0^a\right]$$

$$= 2\pi\left[a^3 - \frac{a^3}{3} - 3a^{4/3}.\frac{3}{5}\left[a^{\frac{5}{3}} - 0\right] + 3a^{2/3}.\frac{3}{7}\left[a^{7/3} - 0\right]\right]$$

$$= 2\pi\left[\frac{3a^3 - a^3}{3} - \frac{9a^{4/3}}{5}a^{5/3} + \frac{9a^{2/3}}{7}a^{7/3}\right]$$

$$= 2\pi\left[\frac{2a^3}{3} - \frac{9a^{a/3}}{5} + \frac{9}{7}a^{a/3}\right]$$

$$= 2\pi\left[\frac{2a^3}{3} - \frac{9a^3}{5} + \frac{9}{7}a^3\right]$$

$$= 2\pi\left[\frac{70a^3 - 189a^3 + 135a^3}{105}\right]$$

$$= 2\pi\left[\frac{16a^3}{105}\right]$$

$$= \frac{32\pi a^3}{105}$$

$\therefore$ Volume generated by revolving hypocycloid is $\frac{32\pi a^3}{105}$

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**22. Prove that the volume of the solid generated by the revolving of the curve $y = \frac{a^3}{a^2 + x^2}$ about its asymptote is $\frac{\pi^2 a^3}{2}$**

*Sol :*

Given curve $y = \frac{a^3}{a^2 + x^2}$

The curve is symmetrical about y - axis.

$\therefore$ The volume generated by the revolution of whole curve is double the volume generated by the revolution of arc.

$\therefore$ The volume of solid is $V = 2\int_0^\infty \pi y^2 dx$

$$V = 2\pi\int_0^\infty \left(\frac{a^3}{a^2 + x^2}\right)^2 dx$$

$$= 2\pi\int_0^\infty \frac{a^6}{\left(a^2 + x^2\right)^2} dx$$

Let $x = a\tan\theta$

$dx = a\sec^2\theta \, d\theta$

If $x = \infty$, $a\tan\theta = \infty$

$\tan\theta = \infty$

$\theta = \frac{\pi}{2}$

If $x = 0, \quad a\tan\theta = 0$

$\tan\theta = 0$

$\theta = 0$

$$V = 2\pi\int_0^{\pi/2}\frac{a^6}{\left(a^2+(a\tan\theta)^2\right)^2}a\sec^2\theta d\theta$$

$$= 2\pi a^6\int_0^{\pi/2}\frac{a\sec^2\theta}{\left(a^2+a^2\tan^2\theta\right)^2}d\theta$$

$$= 2\pi a^6\int_0^{\pi/2}\frac{a\sec^2\theta}{\left(a^2(1+\tan^2\theta)\right)^2}d\theta \quad \left[\because 1+\tan^2\theta=\sec^2\theta\right]$$

$$= 2\pi a^7\int_0^{\pi/2}\frac{\sec^2\theta}{a^4\left(\sec^2\theta\right)^2}d\theta$$

$$= \frac{2\pi a^7}{a^4}\int_0^{\pi/2}\frac{1}{\sec^2}d\theta$$

$$= 2\pi a^3\int_0^{\pi/2}\left(\cos^2\theta\right)d\theta$$

$$= 2\pi a^3\int_0^{\pi/2}\frac{(1+\cos 2\theta)}{2}d\theta$$

$$= \frac{2\pi a^3}{2}\int_0^{\pi/2}(1+\cos 2\theta)d\theta$$

$$= \pi a^3\left[\int_0^{\pi/2}1d\theta+\int_0^{\pi/2}\cos 2\theta d\theta\right]$$

$$= \pi a^3\left\{[x]_0^{\pi/2}+\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}\right\}$$

$$= \pi a^3\left\{\left[\frac{\pi}{2}-0\right]+\frac{1}{2}\left(\sin 2\left(\frac{\pi}{2}\right)0\sin(0)\right)\right\}$$

$$= \pi a^3\left\{\left[\frac{\pi}{2}+\frac{1}{2}(0-0)\right]\right\}$$

$$= \pi a^3\left[\frac{\pi}{2}+(0)\right]$$

$$V = \frac{\pi^2 a^3}{2}$$

$\therefore$ The volume of solid is $\frac{\pi^2 a^3}{2}$

**23. Show that the voulme of the solod obtained by revolving the area included between the curves $y^2 = x^3$ and $x^2 = y^3$ and $x^2 = y^2$ about x-axis is $\frac{5\pi}{28}$.**

*Sol :*

The given curves are $y^2 = x^3$ .....(1)

$x^2 = y^3$ .....(2)

by (1) $\Rightarrow$ $y^2 = x^3$

$y = (x^3)^{1/2}$ .....(3)

sub in (2) $\Rightarrow$ $x^2 = \left[(x^3)^{1/2}\right]^3$

$x^2 = (x^3)^{3/2}$

$x^2 = x^{9/2}$

$x^2 - x^{9/2} = 0$

$x^2\left[1-x^{5/2}\right] = 0$

$x^2 = 0;\ 1-x^{5/2} = 0$

$x = 0;\ x^{5/2} = 1$

$x = 0;\ x = 1$

$\therefore$ The volume of solid generated by $y^3 = x^2$

$$V_1 = \int_0^1 \pi y^2 dx$$

$$= \pi\int_0^1 y^2 dx$$

But $y^3 = x^2$

Then $y = (x^2)^{1/3}$

$$y = x^{2/3}$$

$$V_1 = \pi\int_0^1 \left(x^{2/3}\right)^2 dx$$

$$= \pi\int_0^1 x^{4/3} dx$$

$$= \pi\left[\frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1}\right]_0^1$$

$$= \pi\left[\frac{x^{\frac{7}{3}}}{\frac{7}{3}}\right]_0^1$$

$$= \frac{3\pi}{7}\left[1^{7/3} - 0\right]$$

$$V_1 = \frac{3\pi}{7}$$

Now, Volume of solid which is generated by $y^2 = x^3$

$$y = \left(x^3\right)^{1/2}$$

$$y = x^{3/2}$$

$$V_2 = \int_0^1 \pi y^2 dx$$

$$= \pi\int_0^1 \left(x^{3/2}\right)^2 dx$$

$$= \pi\int_0^1 \left(x^{6/2}\right) dx$$

$$= \pi\left[\frac{x^{6/2}+1}{\frac{6}{2}+1}\right]_0^1$$

$$= \pi\left[\frac{x^{8/2}}{\frac{8}{2}}\right]_0^1$$

$$= \frac{2\pi}{8}\left[x^{8/2}\right]_0^1$$

$$= \frac{\pi}{4}\left[1^{8/2} - 0\right]$$

$$V_2 = \frac{\pi}{4}$$

V = (The volume of solid generated by area $y^3 = x^2$) – (Volume of solid generated by area $y^2 = x^3$)

i.e. $v = v_1 - v_2$

$$V = \frac{3\pi}{7} - \frac{\pi}{4}$$

$$= \frac{12\pi - 7\pi}{28}$$

$$V = \frac{5\pi}{28}$$

**24. Prove that the volume of the reel formed by the revolution of the cycloid x = a (θ + sinθ), y = a (1 – cosθ) about the x-axis is $\pi^2 a^3$**

*Sol :*

Given curve is $x = a(\theta + \sin\theta)$ .....(1)

$y = a(1 - \cos\theta)$ .....(2)

Differentiating (1) & (2) with respect to 'θ'

$$\frac{dr}{d\theta} = a(1 + \cos\theta)$$

$$\frac{dy}{d\theta} = a(0 - (-\sin\theta))$$

$$\frac{dy}{d\theta} = a\sin\theta$$

The volume of the reel is twice the volume of are

$$V = 2\int_0^{\pi} \pi y^2 \frac{dx}{d\theta} d\theta$$

$$= 2\pi\int_0^{\pi} y^2 \frac{dx}{d\theta} d\theta$$

$$= 2\pi\int_0^{\pi} \left[a(1-\cos\theta)\right]^2 \left[a(1+\cos\theta)\right] d\theta$$

$$= 2\pi\int_0^{\pi} a^2(1-\cos\theta)^2 a(1+\cos\theta)d\theta$$

$$= 2\pi a^3\int_0^{\pi}(1-\cos\theta)(1-\cos\theta)(1+\cos\theta)d\theta$$

$$= 2\pi a^3\int_0^{\pi}(1-\cos\theta)(1^2-\cos^2\theta)d\theta$$

$$= 2\pi a^3\int_0^{\pi}(1-\cos\theta)(1-\cos^2\theta)d\theta$$

$$= 2\pi a^3\int_0^{\pi}(1-\cos^2\theta-\cos\theta+\cos^3\theta)d\theta$$

$$= 2\pi a^3\int_0^{\pi}\left[1-\left(\frac{1+\cos 2\theta}{2}\right)-\cos\theta+\left(\frac{1}{4}\cos 3\theta+\frac{3}{4}\cos\theta\right)\right]d\theta$$

$$= 2\pi a^3\int_0^{\pi}\left(1-\frac{1}{2}-\cos\frac{2\theta}{2}-\cos\theta+\frac{1}{4}\cos 3\theta+\frac{3}{4}\cos\theta\right)d\theta$$

$$= 2\pi a^3\int_0^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta+\frac{1}{4}\cos 3\theta-\frac{1}{4}\cos\theta\right)d\theta$$

$$= 2\pi a^3\left[\frac{1}{2}\int_0^{\pi}d\theta-\frac{1}{2}\int_0^{\pi}\cos 2\theta d\theta+\frac{1}{4}\int_0^{\pi}\cos 3\theta d\theta-\frac{1}{4}\int_0^{\pi}\cos\theta d\theta\right]$$

$$= 2\pi a^3\left[\frac{1}{2}(\pi-0)-\frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)_0^{\pi}+\frac{1}{4}\left(\frac{\sin 3\theta}{3}\right)_0^{\pi}-\frac{1}{4}(\sin\theta)_0^{\pi}\right]$$

$$= 2\pi a^3\left[\frac{\pi}{2}-\frac{1}{2}\left(\frac{\sin 2\pi}{2}-\frac{\sin 0}{2}\right)+\frac{1}{4}\left(\frac{\sin 3\pi}{3}-\frac{\sin 0}{3}\right)-\frac{1}{4}(\sin\pi-0)\right]$$

$$= 2\pi a^3\left[\frac{\pi}{2}-\frac{1}{2}(0-0)+\frac{1}{4}(0-0)-\frac{1}{4}(0-0)\right]$$

$$= 2\pi a^3\left[\frac{\pi}{2}-\frac{1}{2}(0)+\frac{1}{4}(0)-\frac{1}{4}(0)\right]$$

$$= 2\pi a^3\left[\frac{\pi}{2}-0+0-0\right]$$

$V = \pi^2 a^3$

Hence proved the volume of reel formed by the revolution of cycloid is $\pi^2 a^3$

**25. Show that the volume of the solid generated by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the y-axis is $\pi a^3\left(\frac{3}{2}\pi^2 - \frac{8}{3}\right)$**

*Sol :*

The Given curves is $x = a(\theta + \sin\theta)$ .....(1)

$y = a(1 - \cos\theta)$ .....(2)

Differentiating (1) & (2) with respect to 'θ'

Then $\frac{dx}{d\theta} = a(1+\cos\theta)$

$\frac{dy}{d\theta} = a(-1-\sin\theta)$

$\frac{dy}{d\theta} = a\sin\theta$

∴ The volume of solid is generated by revolution of the cycloid is y - axis

i.e. $V = \int_0^{\pi} \pi x^2 \frac{dy}{d\theta} d\theta$

$= \int_0^{\pi} \pi\left(a(\theta+\sin\theta)\right)^2 (a\sin\theta) d\theta$

$= \int_0^{\pi} \pi a^2 (\theta+\sin\theta)^2 a\sin\theta d\theta$

$= \int_0^{\pi} \pi a^2 \left(\theta^2+\sin^2\theta+2\theta\sin\theta\right) a\sin\theta\, d\theta$

$= \int_0^{\pi} \pi a^3 \left(\theta^2+\sin^2\theta+2\theta\sin\theta\right)\sin\theta\, d\theta$

$= \pi a^3 \int_0^{\pi} \theta^2\sin\theta + \sin^3\theta + 2\theta\sin^2 d\theta$

$= \pi a^3 \left[\int_0^{\pi} a^2\sin\theta d\theta + \int_0^{\pi}\sin^3\theta d\theta + 2\int_0^{\pi}\theta\sin^2\theta\, d\theta\right]$ .....(3)

A B C

$A = \int_0^{\pi} \theta^2 \sin\theta d\theta$

$= \theta^2\int_0^{\pi}\sin\theta d\theta - \int_0^{\pi}\frac{d}{d\theta}(\theta)^2 \int\sin\theta d\theta . d\theta$

$$= \theta^2(-\cos\theta)_0^\pi - \int_0^\pi 2\theta(-\cos\theta)d\theta$$

$$= -\left[\pi^2(\cos\pi - \cos 0)\right] + 2\int_0^\pi \theta\cos\theta d\theta$$

$$= -\left[\pi^2(-1) - 0\right] + 2\left[\theta\int_0^\pi \cos\theta d\theta - \int_0^\pi \frac{d}{d\theta}(\theta)\int\cos\theta d\theta\cdot d\theta\right]$$

$$= \pi^2 + 2\theta\left[(\sin\theta)_0^\pi - \int_0^\pi 1(\sin\theta)d\theta\right]$$

$$= \pi^2 + 2\left[\pi\sin\pi - \sin 0 - (-\cos)_0^\pi\right]$$

$$= \pi^2 + 2[0 - 0 + \cos\pi - \cos 0]$$

$$= \pi^2 + 2(-1 -1)$$

$$= \pi^2 + 2(-2)$$

$$A = \pi^2 - 4$$

$$B = \int_0^\pi \sin^3\theta d\theta$$

$$= \int_0^\pi\left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right)d\theta$$

$$= \frac{3}{4}\int_0^\pi \sin d\theta - \frac{1}{4}\int_0^\pi \sin 3\theta d\theta$$

$$= \frac{3}{4}(-\cos\theta)_0^\pi - \frac{1}{4}\left(\frac{-\cos 3\theta}{3}\right)_0^\pi$$

$$= \frac{-3}{4}(\cos\pi - \cos 0) + \frac{1}{12}(\cos 3\pi - \cos 0)$$

$$= \frac{-3}{4}(-1 -1) + \frac{1}{12}(-1 -1)$$

$$= \frac{-3}{4}(-2) + \frac{1}{12}(-2)$$

$$= \frac{6}{4} - \frac{2}{12}$$

$$= \frac{3}{2} - \frac{1}{6}$$

$$= \frac{9-1}{6}$$

$$= \frac{8}{3} = \frac{4}{3}$$

$$\therefore \quad B = \frac{4}{3} \qquad .....(5)$$

$$C = \int_0^{\pi} \theta \sin^2 \theta d\theta$$

$$= \int_0^{\pi} \theta\left(\frac{1-\cos\theta}{2}\right)d\theta$$

$$= \int_0^{\pi} \frac{\theta}{2}d\theta - \int_0^{\pi} \frac{\theta\cos 2\theta}{2}d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\theta d\theta - \frac{1}{2}\left[\int_0^{\pi}\theta\cos 2\theta d\theta\right]$$

$$= \frac{1}{2}\left[\frac{\theta^2}{2}\right]_0^{\pi} - \frac{1}{2}\left[\theta\int_0^{\pi}\cos 2\theta d\theta - \int_0^{\pi}\frac{d}{d\theta}\int\cos 2\theta d\theta d\theta\right]$$

$$= \frac{1}{4}\left[\pi^2 - 0\right] - \frac{1}{2}\left[\theta\left(\frac{\sin 2\theta}{2}\right)_0^{\pi}\int_0^{\pi} 1\cdot\frac{\sin 2\theta}{2}d\theta\right]$$

$$= \frac{\pi^2}{4} - \frac{1}{2}\left[\frac{1}{2}(\theta\sin 2\theta)_0^{\pi} - \frac{1}{2}\left(\frac{-\cos 2\theta}{2}\right)\right]_0^{\pi}$$

$$= \frac{\pi^2}{4} - \frac{1}{2}\left[\frac{1}{2}(\pi\sin 2\pi - 0) + \frac{1}{4}(\cos 2\pi - \cos(0))\right]$$

$$= \frac{\pi^2}{4} - \frac{1}{2}\left(\frac{1}{2}(0-0) + \frac{1}{4}(1-1)\right)$$

$$= \frac{\pi^2}{4} - \frac{1}{2}(0 + 0)$$

$$C = \frac{\pi^2}{4} \qquad .....(6)$$

Substituting (4), (5), (6) in (8)

$$V = \pi a^3\left[\pi^2 - 4 + \frac{4}{3} + 2\left(\frac{\pi^2}{4}\right)\right]$$

$$V = \pi a^3\left[\frac{3\pi^2}{2} - \frac{8}{3}\right]$$

$\therefore$ The volume of the solid genrated by revolution of the cycloid about

y – axis is $\pi a^3\left[\frac{3\pi^2}{2} - \frac{8}{3}\right]$

**26. Find the volume of the solid obtained by revolving one arc of the cycloid x = a(θ + sinθ), y = a(1+ cosθ) about x -axis.**

*Sol :*

The given curve $x = a(\theta + \sin\theta)$ .....(1)

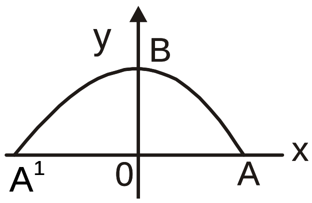
$y = a(1 + \cos\theta)$ .....(2)

Differentiating (1) & (2) with respect to 'θ'

by (1) $\Rightarrow \frac{dx}{d\theta} = a(1 + \cos\theta)$

by (2) $\Rightarrow \frac{dy}{d\theta} = a(0 + (-\sin\theta)$

$$\frac{dy}{d\theta} = -a\sin\theta$$




The volume of solid is given as

$$V = 2\pi\int_0^{\pi} y^2 \frac{dx}{d\theta} d\theta$$

$$= 2\pi\int_0^{\pi}\left[a(1+\cos\theta)\right]^2.\left[a(1+\cos\theta)\right]d\theta$$

$$= 2\pi\int_0^{\pi} a^3(1+\cos\theta)^2(1+\cos\theta)d\theta$$

$$= 2\pi a\int_0^{\pi}(1+\cos\theta)^3 d\theta$$

$$= 2\pi a^3\int_0^{\pi}\left(1+\cos^3\theta + 3\cos\theta + 3\cos^2\theta\right)d\theta$$

$$= 2\pi a^3 \int_0^{\pi} \left[1 + \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta + 3\cos\theta + 3\left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta$$

$$\left[\because \cos 3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta\right]$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= 2\pi a^3 \int_0^{\pi} \left(1 + \frac{1}{4}\cos 3\theta + \frac{15}{4}\cos\theta + \frac{3}{2} + \frac{3}{2}\cos 2\theta\right) d\theta$$

$$= 2\pi a^3 \int_0^{\pi} \left[\frac{5}{2} + \frac{15}{4}\cos\theta + \frac{3}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta\right] d\theta$$

$$= 2\pi a^3 \left[\frac{5}{2}\theta + \frac{15}{4}(\sin\theta) + \frac{3}{2}\left(\frac{\sin 2\theta}{2}\right) + \frac{1}{4}\left(\frac{\sin 3\theta}{3}\right)\right]_{\theta}^{\pi}$$

$$= 2\pi a^3 \left[\frac{5}{2}(\pi - 0) + \frac{15}{4}(\sin\pi - \sin 0) + \frac{3}{2}\left(\frac{\sin 2\pi}{2} - \frac{\sin 2(0)}{2}\right) + \frac{1}{4}\left(\frac{\sin 3\pi}{3} - \frac{\sin 3\theta}{3}\right)\right]$$

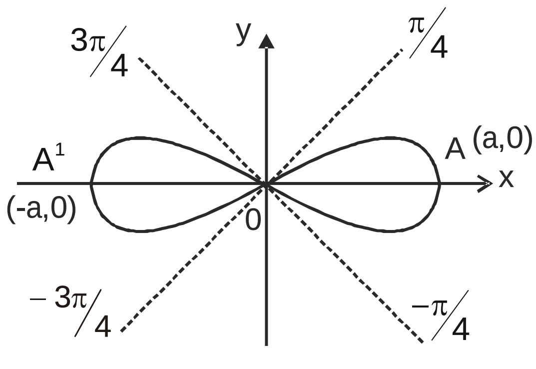
$$= 2\pi a^3 \left[\frac{5}{2}\pi + \frac{15}{4}(0-0) + \frac{3}{2}(0-0) + \frac{3}{2} + \frac{1}{4}(0-0)\right]$$

$$= 2\pi a^3 \left[\frac{5}{2}\pi\right]$$

$= 5\pi^2 a^3$

$\therefore$ The volume of solid obtained by revolving one are of cycloid is $5\pi^2 a^3$

**27. Find the volume of the solid obtained by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line.**




*Sol :*

The given curve is $r^2 = a^2 \cos 2\theta$ .....(1)

differentiating with respect to '$\theta$'

$$2r \frac{dr}{d\theta} = a^2(-\sin 2\theta).2$$

$$2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta$$

$$r\frac{dr}{d\theta} = -a^2 \sin 2\theta$$

Squaring on Both sides

$$r^2\left(\frac{dr}{d\theta}\right)^2 = a^4 \sin^2 2\theta$$

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4 \sin^2 2\theta}{a^2 \cos 2\theta}$$

$$= \frac{a^4 \sin^2 2\theta}{a^2 \cos 2\theta}$$

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{a^2 \sin^2 2\theta}{\cos 2\theta}$$

$\therefore$ The polar equation is $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

$$= \sqrt{a^2 \cos 2\theta + \left(\frac{a^2 \sin^2 2\theta}{\cos 2\theta}\right)}$$

$$= \sqrt{\frac{a^2 \cos^2 2\theta + a^2 \sin^2 2\theta}{\cos 2\theta}}$$

$$= \frac{\sqrt{a^2\left(\cos^2 2\theta + \sin^2 2\theta\right)}}{\sqrt{\cos 2\theta}}$$

$$= \frac{\sqrt{a^2(1)}}{\sqrt{\cos 2\theta}}$$

$$\frac{ds}{d\theta} = \frac{a}{\sqrt{\cos 2\theta}}$$

Now, the surface are of lemniscate is

$$S = 2\int_0^{\pi/4} 2\pi y \frac{ds}{d\theta}.d\theta$$

$$= 4\pi \int_0^{\pi/4} y.\frac{ds}{d\theta}.d\theta$$

In polar $r = (\theta)$

$x = r\cos\theta$

$y = r\sin\theta$

$$= 4\pi\int_0^{\pi/4} r\sin\theta.\frac{a}{\sqrt{\cos 2\theta}}d\theta$$

$r^2 = a^2\cos 2\theta$

$\Rightarrow \quad r = \sqrt{a^2\cos 2\theta}$

$r = a\sqrt{\cos 2\theta}$

$$= 4\pi\int_0^{\pi/4}\frac{a\sqrt{\cos 2\theta}.\sin\theta.a}{\sqrt{\cos 2\theta}}\,d\theta$$

$$= 4\pi a^2\int_0^{\pi/4}\sin\theta\,d\theta$$

$$= 4\pi a^2(-\cos\theta)_0^{\pi/4}$$

$$= 4\pi a^2\left(-\cos\frac{\pi}{4}-(-\cos 0)\right)$$

$$= 4\pi a^2\left(-\frac{1}{\sqrt{2}}+1\right)$$

$$= 4\pi a^2\left(1-\frac{1}{\sqrt{2}}\right)$$

$\therefore$ The surface area of lemniscate $4\pi a^2\left(1-\frac{1}{\sqrt{2}}\right)$

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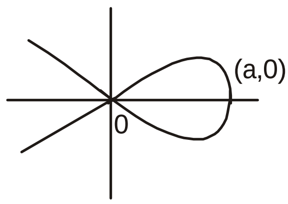
**28. Find the volume formed by the revolution of the loop of the curve $y^2 = \frac{x^2(a-x)}{a+x}$ about the x-axis.**

*Sol :*

Given curve is $y^2 = \frac{x^2(a-x)}{a+x}$ about x -axis & meets x-axis at (0,0) and (a,0)



$\therefore$ The volume of solid is

$$V = \int_0^a \pi y^2 dx$$

$$= \pi\int_0^a y^2 dx$$

$$= \pi\int_0^a \frac{x^2(a-x)}{a+x}dx$$

$$= \pi\int_0^a \frac{x^2a-x^3}{a+x}dx$$

$$= \pi\int_0^a \left(-x^2+2ax-2a^2+\frac{2a^3}{x+a}\right)dx$$

Re writing $\dfrac{ax^2-x^3}{a+x}$

$$= -x^2 + 2ax - 2a^2 + \frac{2a^3}{x+a}$$

$$= \frac{-x^3-x^2a+2ax^2+\cancel{2a^2x}-\cancel{2a^2x}-\cancel{2a^3}+\cancel{2a^3}}{x+a}$$

$$= \frac{-x^3+ax^2}{x+a}$$

$$= \pi\left[\int_0^a -x^2dx + 2a\int_0^a xdx - 2a^2\int_0^a dx + 2a^3\int_0^a \frac{1}{x+a}dx\right]$$

$$= \pi\left[-\left(\frac{x^3}{3}\right)_0^a + 2a\left(\frac{x^2}{2}\right)_0^a - 2a^2(x)_0^a + 2a^3\left(\log(x+a)\right)_0^a\right]$$

$$= \pi\left[\frac{-a^3}{3} + 2a.\frac{a^2}{2} - 2a^2.a + 2a^3\left[\log(0+a)-\log(0+a)\right]\right]$$

$$= \pi\left[\frac{-a^3}{3} + \frac{\cancel{2}a^3}{\cancel{2}} - 2a^3 + 2a^3(\log 2a - \log a)\right]$$

$$= \pi\left[\frac{-a^3}{3} + a^3 - 2a^3 + 2a^3\log\left(\frac{2a}{a}\right)\right]$$

$$= \pi\left[\frac{-4a^3}{3} + 2a^3 \log 2\right]$$

$$= \frac{2a^3\pi}{3}[-2 + 3\log 2]$$

$$= \frac{2a^3\pi}{3}\left[-\log e^2 + \log_2{}^3\right]$$

$$= \frac{2\pi a^3}{3}\ [\log 8 - \log e^2]$$

$$= \frac{2\pi a^3}{3}\ \left[\log \frac{8}{e^2}\right]$$

$\therefore$ Volume formed by revolving the loop is $\frac{2\pi a^3}{3} \log \left(\frac{8}{e^2}\right)$

**29. Show that the volume of the solid obtained by revolving about x-axis the area enclosed by the parabola $y^2 = 4ax$ & its evolute $27ay^2 = 4(x - 2a)^3$ is $80\pi a^3$**

*Sol :*

Given curve $y^2 = 4ax$ .....(1)

Its evolute is $27ay^2 = 4(x - 2a)^3$

$$y^2 = \frac{4(x-2a)^3}{2+a} \qquad .....(2)$$

equating (1) & (2)

$$4ax = \frac{4(x-2a)^3}{27a}$$

$$27a(\not{4}ax) = \not{4}(x-2a)^3$$

$27a^2x = x^3 - 8a^3 - 6ax^2 + 12a^2 x$

$x^3 - 8a^3 - 6ax^2 + 12a^2 x - 27a^2 x = 0$

$x^3 - 8a^3 - 6ax^2 - 15a^2 x = 0$

$(x + a)^2 (x - 8a) = 0$

$(x + a)^2 = 0$ ; $x - 8a = 0$

$x + a = 0$; $x = 8a$

$x = -a$

Since reglecting $x = -a$ on the curve

$\therefore x = 8a$

The volume of the solid obtained by revolving about x - axis the area enclosed by parabola and its evolutes.

$\therefore$ Volume generated by area of parabola $(V_1)$

$$V_1 = \int_0^{8a} \pi y^2 dx \qquad y^2 = 4ax$$

$$= \int_0^{8a} \pi(4ax)dx$$

$$= 4\pi a \int_0^{8a} x\, dx$$

$$= 4\pi a \left[\frac{x^2}{2}\right]_0^{8a}$$

$$= 4\pi a \left[\frac{(8a)^2}{2} - 0\right]$$

$$= 4\pi a \left[\frac{64a^2}{2}\right]$$

$$V_1 = 128\,\pi a^3$$

Now, volume generated by its evolutes ($V_2$)

$$V_2 = \int_{2a}^{8a} \pi y^2 dx$$

by (2) $\Rightarrow$ $y^2 = \dfrac{4(x-2a)^3}{27a}$

$$= \int_{2a}^{8a} \pi \left[\frac{4(x-2a)^3}{27a}\right] dx$$

$$= \frac{4\pi}{27a} \int_{2a}^{8a} (x-2a)^3 dx$$

$$= \frac{4\pi}{27a} \left[\frac{(x-2a)^4}{4}\right]_{2a}^{8a}$$

$$= \frac{\not{4}\pi}{\not{4}.27a}\left[(8a-2a)^4 - (2a-2a)^4\right]$$

$$= \frac{\pi}{27a}\left[(6a)^4 - 0\right]$$

$$= \frac{\pi}{27a}\left[1296a^4\right]$$

$V_2 = 48\,\pi\,a^3$

$\therefore$ The volume of solid is $V = V_1 - V_2$

$V = 128\,\pi\,a^3 - 48\,\pi\,a^3$

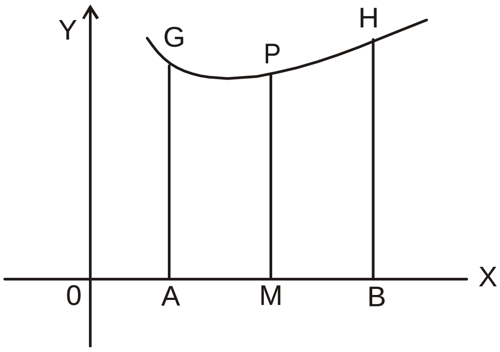
$V = 80\,\pi\,a^3$

## 4.5 Expression for the Volume Obtained by Revolving about any Line

**30. Write the Expression for the volume obtained by $\mu$.**

*Sol :*

Let y = f(x) be curve, P be point on the curve. G, H, are the extreme ends of the arc. PM be the length of the perpendicular to x-axis of revolution. X denote the distance of the foot of the perpendicular M from a fixed point O on the x-axis. and A, B are the perpendicular from the entreme ends G, H of the arc.



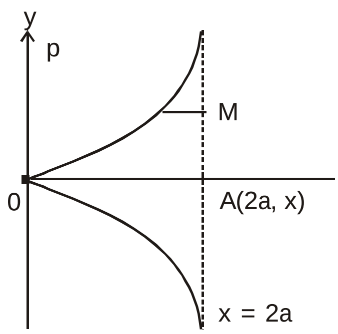

Then the volume obtained by revolving the are GH abent line AB is $\int_{OA}^{OB} \pi\,(MP)^2\,d(OM)$

**31. Find the volune of the solid generated by the revolution of the ciossoid $y^2\ (2a-x) = x^3$ about its asymptote.**

*Sol :*

The given ciossoids $y^2\ (2a-x) = x^3$ .......... (1)

The line x = 2a is the asymptote of the curve. The perpendicular distance MP of any point P(x,y) on the curve 2a –x.




$2a - x = 0 \Rightarrow x - 2a = 0$

$\therefore$ The asymptate parallel to y-axis is x – 2a = 0 the length of the perpendicular from P(x,y) to line x – 2a = 0

$MP = x - 2a$

$(MP)^2 = (x - 2a)^2$

'A' is point where the asymptote meets the x-axis is fined point on the axis of revolution.

The volume of the solid obtained by revolving the whole curve about the asuymptate is double of the volume obtained by revolving the part of its lying in the 1st Quadrant.

i.e. $AM = y$

but $y^2 (2a - x) = x^3$

$$y^2 = \frac{x^3}{2a - x}$$

$$AM = y = \sqrt{\frac{x^3}{2a-x}} \quad .....(2)$$

$\therefore$ The required volume is $V = 2\pi \int_0^{2a} (MP)^2 \, d(AM)$

$$V = 2\pi \int_0^{2a} (x-a)^2 \frac{d}{dx}\left(\sqrt{\frac{x^3}{2a-x}}\right)$$

Now, we will find $\frac{d}{dx}(AM) = \frac{d}{dx}\left(\sqrt{\frac{x^3}{2a-x}}\right)$

$$\frac{d}{dx}\left(\sqrt{\frac{x^3}{2a-x}}\right) = \frac{1}{2\sqrt{\frac{x^3}{2a-x}}} \frac{d}{dx}\left(\frac{x^3}{2a-x}\right)$$

$$= \frac{1}{2\sqrt{\frac{x^3}{2a-x}}}\left[\frac{(2a-x)(3x^2) - x^3(-1)}{(2a-x)^2}\right]$$

$$= \frac{1}{2\sqrt{\frac{x^3}{2a-x}}}\left[\frac{6ax^2 - 3x^3 + x^3}{(2a-x^2)^2}\right]$$

$$= \frac{1}{\frac{2\sqrt{x^3}}{\sqrt{2a-x}}}\left[\frac{6ax^2 - 2x^3}{(2a-x)^2}\right]$$

$$= \frac{\sqrt{2a-x}}{2\sqrt{x^3}}\left(\frac{6ax^2 - 2x^3}{(2a-x)^2}\right)dx$$

$$= \frac{\sqrt{2a-x}\ \cancel{2}x^{\cancel{2}}(3a-x)}{\cancel{2}x\sqrt{x}(2a-x)^2}$$

$$= \frac{\sqrt{2a-x}\sqrt{x}(3a-x)}{(2a-x)^2}$$

$$\frac{d}{dx}(AM) = \frac{\sqrt{2a-x}\sqrt{x}(3a-x)}{(2a-x)^2}$$

$$d(AM) = \frac{\sqrt{2a-x}\sqrt{x}(3a-x)}{(2a-x)^2}dx$$

$$\therefore \quad V = 2.\pi\int_0^{2a} \cancel{(x-2a)^2}\frac{\sqrt{2a-x}\sqrt{x}(3a-x)}{\cancel{(2a-x)^2}}dx$$

$$= 2\pi\int_0^{2a}\sqrt{2a-x}\sqrt{x}(3a-x)dx$$

Let $x = 2a\sin^2\theta$

$dx = 2a(2\sin\theta\cos\theta)\,d\theta$

$dx = 4a\sin\theta\cos\theta\,d\theta$

If $x = 2a,\ 2a\sin^2\theta = 2a$

$\sin^2\theta = 1$

$\theta = \pi/2$

If $x = 0 \Rightarrow 2a\sin^2\theta = 0$

$\sin^2\theta = 0$

$\theta = 0$

$$\therefore \quad V = 2\pi\int_0^{\pi/2}\sqrt{2a-2a\sin^2\theta}\sqrt{2a\sin^2\theta}\left(3a-2a\sin^2\theta\right)\ 4a\sin\theta\cos\theta\,d\theta$$

$$= 2\pi\int_0^{\pi/2}\sqrt{2a\left(1-\sin^2\theta\right)}\sqrt{2a}\sin\theta\ a\left(3-2a\sin^2\theta\right)\ 4a\sin\theta\cos\theta\,d\theta$$

$$= 2\pi\int_0^{\pi/2}4a^2\sqrt{2a}\sqrt{2a}\cos\theta\sin\theta(3-2\sin^2\theta)\sin\theta\cos\theta d\theta$$

$$= 4a^2\cdot 2a.2\pi\int_0^{\pi/2}\left(3-2.\sin^2\theta\right)\sin^2\theta\cos^2\theta d\theta$$

$$= 16\pi a^3\left[\int_0^{\pi/2}3\sin^2\theta\cos^2\theta-2\sin^4\theta\cos^2\theta d\theta\right]$$

$$= 16\pi a^3\left[3\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta - 2\int_0^{\pi/2}\sin^4\theta\cos^2\theta d\theta\right]$$

Since $\int_0^{\pi/2} \text{Sin}^m \cos^n\theta d\theta = \left(\frac{n-1}{m+n}\right)\left(\frac{n-3}{m+n-2}\right)......\left(\frac{1}{m+2}\right)\left(\frac{m-1}{m}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$ if m is even

$$= 16\pi a^3\left[3\left[\left(\frac{2-1}{2+2}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) - 2\left(\left(\frac{4-1}{4+2}\right)\left(\frac{4-3}{4+2-2}\right)\frac{1}{2}.\frac{\pi}{2}\right]\right]$$

$$= 16\pi a^3\left[3\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) - 2\left(\frac{3}{6}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)\right]$$

$$= 16\pi a^3\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\frac{\pi}{3}\left[3-2\left(\frac{3}{6}\right)\right]$$

$$= \frac{16\pi a^3}{16}[3-1]$$

$= 2a^3\pi$

$\therefore \quad V = 2a^3\pi$

Volume of the solid is $2a^3\pi$

**32. The smaller segment of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cut off by the chord $\frac{x}{a}+\frac{y}{b}=1$ revolves completely about this chord. Show that the volume generated is $\frac{\pi}{6}(10-3\pi)a^2b^2(a^2+b^2)^{-1/2}$**

*Sol :*

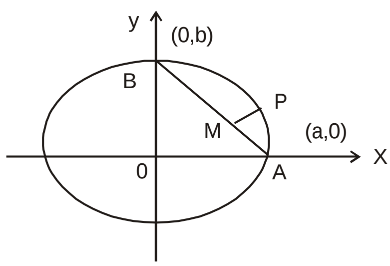
Given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Given chord is $\frac{x}{a}+\frac{y}{b}=1$

$\Rightarrow \quad \frac{bx+ay}{ab}=1$

$\Rightarrow \quad bx + ay = ab$

$\Rightarrow \quad bx + ay - ab = 0$



The chord $\frac{x}{a}+\frac{y}{b}=1$ joins the point A, B.

Take any point P (a cosθ, b sinθ) on the ellipse. M be the perpendicular from P to the chord AB.

$A = (a,0)$ & $P = (a\cos\theta, b\sin\theta)$

$AP^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$= (a\cos\theta - a)^2 + (b\sin\theta - 0)^2$

$AP^2 = (a\cos\theta - a)^2 + (b\sin\theta)^2$

The length of the perpendicular from (a cosθ, b sinθ) to bx + ay – ab = 0

$$\Rightarrow \quad \frac{b(a\cos\theta)+a(b\sin\theta)-ab=0}{\sqrt{a^2+b^2}}$$

$$MP = \frac{ab\cos\theta + ab\sin\theta - ab}{\sqrt{a^2+b^2}}$$

$$MP = \frac{ab(\sin\theta+\cos\theta-1)}{\sqrt{a^2+b^2}}$$

$$(MP)^2 = \left[\frac{ab(\sin\theta+\cos\theta-1)}{\sqrt{a^2+b^2}}\right]^2$$

$$(MP)^2 = \frac{a^2b^2(\sin\theta-\cos\theta-1)^2}{a^2+b^2}$$

From figure

$AM^2 + MP^2 = AP^2$

$AM^2 = AP^2 - MP^2$

$$= (a\cos\theta-a)^2+(b\sin\theta)^2-\frac{a^2b^2(\sin\theta+\cos\theta-1)^2}{a^2+b^2}$$

$$= a^2(\cos\theta-1)^2+b^2\sin^2\theta-a^2b^2\frac{(\cos\theta+\sin\theta-1)^2}{a^2+b^2}$$

$$= \frac{(a^2+b^2)\left[a^2(\cos\theta-1)^2\right]+(a^2+b^2)\left[b^2\sin^2\theta\right]-a^2b^2\left[(\cos\theta-1)^2+\sin^2\theta+2\sin\theta(\cos\theta-1)\right]}{a^2+b^2}$$

$$= \frac{a^4(1-\cos\theta)^2+\cancel{a^2b^2(1-\cos\theta)^2}+\cancel{a^2b^2\sin^2\theta}+b^4\sin^2\theta-\cancel{a^2b^2(1-\cos\theta)^2}-\cancel{a^2b^2\sin^2\theta}-2a^2b^2\sin\theta(\cos\theta-1)}{a^2+b^2}$$

$$= \frac{a^4(1-\cos\theta)^2+b^4\sin^2\theta-2a^2b^2\sin\theta(\cos\theta-1)}{a^2+b^2}$$

$$= \frac{a^4(1-\cos\theta)^2+b^4\sin^2\theta+2a^2b^2\sin\theta(1-\cos\theta)}{a^2+b^2}$$

$$AM^2 = \frac{\left[a^2(1-\cos\theta)+b^2\sin\theta\right]^2}{a^2+b^2}$$

$$AM = \sqrt{\frac{\left(a^2(1-\cos\theta)+b^2\sin\theta\right)^2}{a^2+b^2}}$$

$$AM = \frac{a^2(1-\cos\theta)+b^2\sin\theta}{\sqrt{a^2+b^2}} \qquad \text{......(2)}$$

differentiating (1) with respect to '$\theta$'

$$\frac{d}{d\theta}(AM) = \frac{d}{d\theta}\left[\frac{a^2(1-\cos\theta)+b^2\sin\theta}{\sqrt{a^2+b^2}}\right]$$

$$= \frac{1}{\sqrt{a^2+b^2}}\frac{d}{d\theta}\left[a^2(1-\cos\theta)+b^2\sin\theta\right]$$

$$= \frac{1}{\sqrt{a^2+b^2}}\left[a^2\left(0-(\sin\theta)\right)+b^2\cos\theta\right]$$

$$= \frac{1}{\sqrt{a^2+b^2}}\left[a^2\sin\theta+b^2\cos\theta\right]$$

$$\frac{d}{d\theta}(AM) = \frac{1}{\sqrt{a^2+b^2}}\left[a^2\sin\theta+b^2\cos\theta\right]$$

$$d(AM) = \frac{a^2\sin\theta+b^2\cos\theta}{\sqrt{a^2+b^2}}\,d\theta$$

Now, the volume generated $V = \int_0^{AB}\pi(MP)^2\,d(AM)$

$$V = \pi\int_0^{\pi/2}\frac{a^2b^2(\sin\theta+\cos\theta-1)^2}{a^2+b^2}.\frac{a^2\sin\theta+b^2\cos\theta}{\sqrt{a^2+b^2}}\,d\theta$$

$$= \frac{\pi a^2b^2}{(a^2+b^2)^{3/2}}\int_0^{\pi/2}\left[\left(\cos^2\theta+\sin^2\theta+1+2\sin\theta\cos\theta-2\cos\theta-2\sin\theta\right)\left(a^2\sin\theta+b^2\cos\theta\right)\right]d\theta$$

$$= \frac{\pi a^2b^2}{(a^2+b^2)^{3/2}}\int_0^{\pi/2}\left[\left(1+1+2\sin\theta\cos\theta-2\sin\theta-2\cos\theta\right)\left(a^2\sin\theta+b^2\cos\theta\right)\right]d\theta$$

$$= \frac{\pi a^2b^2}{(a^2+b^2)^{3/2}}\int_0^{\pi/2}2\left(1+\cos\theta\sin\theta-\sin\theta-\cos\theta\right)\left(a^2\sin\theta+b^2\cos\theta\right)d\theta$$

$$= \frac{2\pi a^2 b^2}{\left(a^2+b^2\right)^{3/2}} \int_0^{\pi/2} (1+\cos\theta\sin\theta-\sin\theta-\cos\theta)\left(a^2\sin\theta+b^2\cos\theta\right)d\theta$$

$$= \frac{2\pi a^2 b^2}{\left(a^2+b^2\right)^{3/2}} \int_0^{\pi/2} a^2\sin\theta+b^2\cos\theta+a^2\sin^2\theta\cos\theta+b^2\cos^2\theta\sin\theta$$

$$- a^2\sin^2\theta - b^2\sin\theta\cos\theta - a^2\cos\theta\cos\theta - b^2\cos^2 d\theta$$

$$= \frac{2\pi a^2 b^2}{\left(a^2+b^2\right)^{3/2}} \int_0^{\pi/2} \left(a^2\sin\theta+b^2\cos\theta+a^2\sin^2\theta\cos\theta+b^2\cos^2\theta\sin\theta-\left(a^2+b^2\right)\sin\theta\cos\theta-a^2\sin^2\theta-a^2\cos^2\theta\right)d\theta$$

$$= \frac{2\pi a^2 b^2}{\left(a^2+b^2\right)^{3/2}} \Bigg[ \underset{(a)}{a^2\int_0^{\pi/2}\sin\theta d\theta} + \underset{(b)}{b^2\int_0^{\pi/2}\cos\theta d\theta} + \underset{(c)}{a^2\int_0^{\pi/2}\sin^2\theta\cos\theta} + \underset{(d)}{b^2\int_0^{\pi/2}\cos^2\theta\sin\theta d\theta} - \underset{(e)}{\left(a^2+b^2\right)\int_0^{\pi/2}\sin\theta\cos\theta d\theta}$$

$$\underset{(f)}{-b^2\int_0^{\pi/2}\cos^2\theta d\theta} - \underset{(g)}{a^2\int_0^{\pi/2}\sin^2\theta d\theta}\Bigg] \text{......(3)}$$

(a) $\Rightarrow \int_0^{\pi/2}\sin\theta d\theta \Rightarrow [-\cos\theta]_0^{\pi/2}$

$= -\cos\pi/2 - (-\cos\theta)$

$= 0 + 1$

$= 1$

(b) $\Rightarrow \int_0^{\pi/2}\cos\theta d\theta = [\sin\theta]_0^{\pi/2}$

$= \sin\pi/2 - \sin 0$

$= 1$

(c) $\Rightarrow \int_0^{\pi/2}\sin^2\theta\cos\theta d\theta$

Let $t = \sin\theta$

$dt = \cos\theta\, d\theta$

$= \int_0^{\pi/2} t^2.dt$

$= \left[\frac{t^3}{3}\right]_0^{\pi/2} \Rightarrow \left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2}$

$$= \frac{1}{3}\left[\sin^3\left(\frac{\pi}{2}\right) - \sin^3(0)\right]$$

$$= \frac{1}{3}\left[(1) - 0\right] = \frac{1}{3}$$

(d) $\Rightarrow \int_0^{\pi/2} \cos^2\theta \sin\theta d\theta$

Let $t = \cos\theta$

$dt = -\sin d\theta$

$$\int_0^{\pi/2} t^2.(-dt) = -\left[\frac{t^3}{3}\right]_0^{\pi/2}$$

$$= \frac{-1}{3}\left[\cos^3\theta\right]_0^{\pi/2}$$

$$= \frac{-1}{3}\left[\cos^3\left(\pi/2\right) - \cos^3(0)\right]$$

$$= \frac{-1}{3}(0-1)$$

$$= \frac{1}{3}$$

(e) $\Rightarrow \int_0^{\pi/2} \sin^2\theta \cos\theta d\theta$

Let $t = \sin\theta$

$dt = \cos\theta\, d\theta$

$$\int_0^{\pi/2} t.dt \Rightarrow \left[\frac{t^2}{2}\right]_0^{\pi/2} = \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\sin^2\left(\frac{\pi}{2}\right) - \sin^2(0)\right]$$

$$= \frac{1}{2}\left[(1) - 0\right]$$

$$= \frac{1}{2}$$

(f) $\Rightarrow$ $\int_0^{\pi/2} \cos^2\theta d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta$

$$= \frac{1}{2}\left[\int_0^{\pi/2} 1 d\theta + \int_0^{\pi/2} \cos 2\theta d\theta\right]$$

$$= \frac{1}{2}\left[[\theta]_0^{\pi/2} + \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}\left(\sin 2\left(\frac{\pi}{2}\right) - \sin(0)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}(0-0)\right]$$

$$= \frac{\pi}{4}$$

(g) $\Rightarrow$ $\int_0^{\pi/2} \sin^2\theta d\theta = \int_0^{\pi/2} \frac{1-\cos 2\theta}{2} d\theta$

$$= \frac{1}{2}\left[\int_0^{\pi/2} 1 d\theta - \int_0^{\pi/2} \cos 2\theta d\theta\right]$$

$$= \frac{1}{2}\left[[\theta]_0^{\pi/2} - \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\left(\sin 2\left(\frac{\pi}{2}\right) - \sin(0)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}(0-0)\right]$$

$$= \frac{\pi}{4}$$

Sub (a), (b), (c), (d), (e), (f) & (g) in (3)

$$V = \frac{2\pi a^2 b^2}{(a^2+b^2)^{3/2}}\left[a^2(1) + b^2(1) + a^2\left(\frac{1}{3}\right) + b^2\left(\frac{1}{3}\right) - (a^2+b^2)\left(\frac{1}{2}\right) - b^2\left(\frac{\pi}{4}\right) - a^2\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{2\pi a^2 b^2}{(a^2+b^2)^{3/2}}\left[a^2 + b^2 + \frac{a^2}{3} + \frac{b^2}{3} - \frac{a^2}{2} - \frac{b^2}{2} - \frac{b^2\pi}{4} - \frac{a^2\pi}{4}\right]$$

$$= \frac{2\pi a^2 b^2}{\left(a^2+b^2\right)^{3/2}}\left[\left(\left(a^2+b^2\right)\left[1+\frac{1}{3}-\frac{1}{2}-\frac{\pi}{4}\right]\right)\right]$$

$$= \frac{2\pi a^2 b^2\left(a^2+b^2\right)}{\left(a^2+b^2\right)^{3/2}}\left(\frac{10-3\pi}{12}\right)$$

$$= \frac{\pi a^2 b^2}{\left(a^2+b^2\right)^{1/2}}\frac{(10-3\pi)}{6}$$

$$V = \frac{\pi}{6}(10-3\pi)a^2b^2\left(a^2+b^2\right)^{-1/2}$$

$\therefore$ Volume generated by $\frac{\pi}{6}(10-3\pi)a^2b^2\left(a^2+b^2\right)^{-1/2}$

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**33. The ellipse $b^2 x^2 + a^2y^2 = a^2 b^2$ is divided into two parts by the line $x = \frac{a}{2}$ and the smaller part is rotated through four right angles about line. Prove that the volume generated is $\pi a^2 b\left(\frac{3}{4}\sqrt{3}-\frac{1}{3}\pi\right)$**

*Sol :*

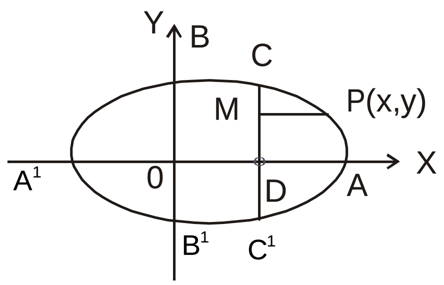
The given ellipse is, $b^2 x^2 + a^2 y^2 = a^2 b^2$ .....(1)

dividing by $a^2 b^2$ $\frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Which is divided into two parts of line $x = \frac{1}{2}a$.

$x = \frac{1}{2}a \Rightarrow 2x - a = 0!$

Let P(x,y) be any point on are of the ellipse and M be the part of the perpendicular drawn from P to the line such that $PM \perp CC^1$




The length of the perpendicular from P (x,y) to line 2x – a is,

$$PM = \frac{2x-a}{\sqrt{2^2}}$$

$$PM = \frac{2x-a}{2} \Rightarrow (PM)^2 = \frac{(2x-a)^2}{2^2}$$

Let $DM = y$ by (1) $\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$

$$a^2y^2 = a^2b^2 - b^2x^2$$

$$y^2 = \frac{b^2(a^2-x^2)}{a^2}$$

$$y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y = \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}$$

$DM = y$

$$= \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}$$

$$= \sqrt{b^2}\left(\frac{\sqrt{a^2-x^2}}{a^2}\right)$$

$$= b\frac{\sqrt{a^2-x^2}}{a} = \frac{b}{a}\sqrt{a^2-x^2}$$

Differentiating (DM) with respect to 'x'

$$\frac{d}{dx}(DM) = \frac{d}{dx}\left[\frac{b}{a}\sqrt{a^2-x^2}\right]$$

$$= \frac{b}{a}\frac{d}{dx}\sqrt{a^2-x^2}$$

$$= \frac{b}{a}\frac{1}{2\sqrt{a^2-x^2}}(0-2x)$$

$$= \frac{b}{a}\frac{-2x}{2\sqrt{a^2-x^2}}$$

$$\frac{d}{dx}(DM) = \frac{-bx}{a\sqrt{a^2 - x^2}}$$

$$d\,(DM) = \frac{-x}{a\sqrt{a^2 - x^2}}dx$$

$\therefore$ The volume generated by revolving the are CAC[1] is twice the volume generated by revolving the are CA

The volume of solid is given

$$V = 2\int_{a/2}^{a} \pi(MP)^2\, d(DM)$$

$$= 2\pi\int_{a/2}^{a} \frac{(2x-a)^2}{4}\left(\frac{-xb}{a\sqrt{a^2 - x^2}}\right)dx$$

$$= \frac{-2\pi b}{4a}\int_{a/2}^{a} (2x-a)^2 \frac{x}{\sqrt{a^2 - x^2}}dx$$

Let $x = a \sin\theta$

$dx = a \cos\theta\, d\theta$

If $x = a \Rightarrow a \sin\theta = a$

$\sin\theta = \frac{a}{a}$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}\sin^{-1}(1)$

$\theta = \frac{\pi}{2}$

If $x = \frac{a}{2} \Rightarrow a \sin\theta = \frac{a}{2}$

$\sin\theta = \frac{a}{2a}$

$\sin\theta = \frac{1}{2}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right)$

$\theta = \frac{\pi}{6}$

$$V = \frac{-2\pi b}{4a}\int_{\pi/6}^{\pi/2}\left((2a\sin\theta - a)^2 . \frac{a\sin\theta}{\sqrt{a^2-(a\sin\theta)^2}} a\cos\theta d\theta\right)$$

$$= \frac{-\pi b}{2a}\int_{\pi/6}^{\pi/2} a^2(2\sin\theta-1)^2 . \frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}} .\cos\theta \, d\theta$$

$$= \frac{-a^4\pi b}{2a}\int_{\pi/6}^{\pi/2}(2\sin\theta-1)^2 \frac{\sin\theta\cos\theta}{a\sqrt{1-\sin^2\theta}} d\theta$$

$$= \frac{-\pi b a^3}{a2}\int_{\pi/6}^{\pi/2}\left(4\sin 2\theta + 1 - 4\sin^2\theta\right)\frac{\sin\theta\cos\theta}{\sqrt{1-\sin^2\theta}} d\theta$$

$$= \frac{-\pi b a^2}{2}\int_{\pi/6}^{\pi/2}\left(4\sin 2\theta + 1 - 4\sin^2\theta\right)\frac{\sin\theta.\cos\theta}{\sqrt{\cos^2\theta}} d\theta$$

$$= \frac{-\pi b a^2}{2}\int_{\pi/6}^{\pi/2}\left(4\sin 2\theta + 1 - 4\sin^2\theta\right)\frac{\sin\theta.\cancel{\cos\theta}}{\cancel{\cos\theta}} d\theta$$

$$= \frac{-\pi b a^2}{2}\int_{\pi/6}^{\pi/2}\left(4\sin^3\theta + \sin\theta - 4\sin^3\theta\right) d\theta$$

$$= \frac{-\pi b a^2}{2}\left[\int_{\pi/6}^{\pi/2} 4\sin^3\theta d\theta + \int_{\pi/6}^{\pi/2}\sin\theta d\theta - \int_{\pi/6}^{\pi/2} 4\sin^3\theta d\theta\right]$$

$$= \frac{-\pi b a^2}{2}\left[\int_{\pi/6}^{\pi/2}(3\sin\theta - \sin 3\theta) d\theta + \int_{\pi/6}^{\pi/2}\sin\theta d\theta - \int_{\pi/6}^{\pi/2} 4\left(\frac{1-\cos 2\theta}{2}\right) d\theta\right]$$

$$= \frac{-\pi b a^2}{2}\left[\int_{\pi/6}^{\pi/2}\left(3\sin\theta - \sin 3\theta + \sin\theta - \frac{4}{2} + \frac{4\cos 2\theta}{2}\right) d\theta\right]$$

$$= \frac{-\pi b a^2}{2}\left[\int_{\pi/6}^{\pi/2}(4\sin\theta - \sin 3\theta - 2 + 2\cos 2\theta) d\theta\right]$$

$$= \frac{-\pi b a^2}{2}\left[4(-\cos\theta) - \left(\frac{-\cos 3\theta}{3}\right) - 2\theta + 2\left(\frac{\sin 2\theta}{2}\right)\right]_{\pi/6}^{\pi/2}$$

$$= \frac{-\pi ba^2}{2}\left[-4\cos\theta + \frac{\cos 3\theta}{3} - 2\theta + \sin 2\theta\right]_{\pi/6}^{\pi/2}$$

$$= \frac{-\pi ba^2}{2}\left[-4\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{6}\right) + \frac{1}{3}\left(\cos 3\left(\frac{\pi}{2}\right) - \cos 3\left(\frac{\pi}{6}\right)\right) - 2\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + \left(\sin 2\frac{\pi}{2} - \sin 2\frac{\pi}{6}\right)\right]$$

$$= \frac{-\pi ba^2}{2}\left[-4\left(0 - \frac{\sqrt{3}}{2}\right) + \frac{1}{2}(0-0) - 2\left(\frac{2\pi}{6}\right) + \left(0 - \frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{-\pi ba^2}{2}\left[\frac{4\sqrt{3}}{3} - \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right]$$

$$= \frac{-\pi a^2 b}{2}\left[\frac{3\sqrt{3}}{2} - \frac{2\pi}{3}\right]$$

$$= -\pi b\, a^2\left[\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\right]$$

$$V = \pi b\, a^2\left[\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\right] \quad \text{by Neglecting negative sign}$$

$\therefore$ Volume generated by solid is $\pi a^2 b\left[\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\right]$

**34. A quadrant of circle of radius a revolves about its chord. Show that the volume of the spindle thus generated is $\frac{\pi}{6\sqrt{2}}(10 - 3\pi)a^3$.**

*Sol :*

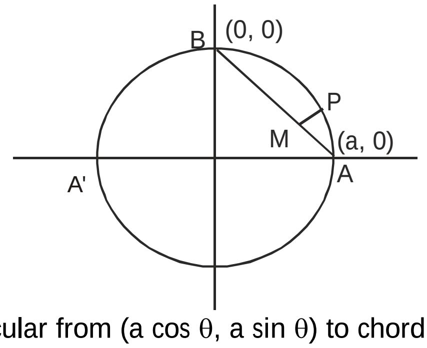
Given circle is $x^2 + y^2 = a^2$

$x + y = a$ is a chord of circle

$x + y - a = 0$

Let $x = a\cos\theta$, $y = a\sin\theta$

Then p(a cos $\theta$, a sin $\theta$) be point on the circle and M be the foot of the perpendicular drawn from P to the chord AB.




The length of the perpendicular from (a cos $\theta$, a sin $\theta$) to chord $x + y - a = 0$

$$MP = \frac{a\cos\theta + a\sin\theta - a}{\sqrt{1^2+1^2}}$$

$$MP = \frac{a(\cos\theta + \sin\theta - 1)}{\sqrt{2}}$$

$$\Rightarrow \quad MP^2 = \frac{a^2(\cos\theta + \sin\theta - 1)^2}{(\sqrt{2})^2}$$

$$MP^2 = \frac{a^2(\cos\theta + \sin\theta - 1)^2}{2}$$

Here,

$A = (a, 0) \quad P = (a\cos\theta, a\sin\theta)$

$$AP = \sqrt{(a\cos\theta - a)^2 + (a\sin\theta - 0)^2}$$

$(AP)^2 = (a\cos\theta - a)^2 + (a\sin\theta)^2$

$(AP)^2 = a^2(\cos\theta - 1)^2 + a^2\sin^2\theta$

$\therefore \quad AM^2 = PM^2 - AP^2$

$AM^2 = PM^2 - AP^2$

$$= a^2(\cos\theta - 1)^2 + a^2\sin^2\theta - \frac{a^2(\cos\theta + \sin\theta - 1)^2}{2}$$

$$= \frac{2a^2(\cos\theta - 1)^2 + 2a^2\sin^2\theta - a^2\left[(\cos\theta - 1) + \sin\theta\right]^2}{2}$$

$$= \frac{2a^2(\cos\theta - 1)^2 + 2a^2\sin^2\theta - a^2\left[(\cos\theta - 1)^2 + \sin^2\theta + 2(\cos\theta - 1)(\sin\theta)\right]}{2}$$

$$= \frac{2a^2(\cos\theta - 1)^2 + 2a^2\sin^2\theta - a^2(\cos\theta - 1)^2 - a^2\sin^2\theta - 2a^2(\cos\theta - 1)\sin\theta}{2}$$

$$= \frac{a^2(\cos\theta - 1)^2 + a^2\sin^2\theta - \left(-2a^2\sin\theta(1 - \cos\theta)\right)}{2}$$

$$= \frac{a(1 - \cos\theta)^2 + a^2\sin^2\theta + 2a^2\sin\theta(1 - \cos\theta)}{2}$$

$$AM^2 = \frac{\left[a(1 - \cos\theta) + a\sin\theta\right]^2}{2}$$

$$AM = \sqrt{\frac{(a(1 - \cos\theta) + a\sin\theta)^2}{2}}$$

$$AM = \frac{a(1-\cos\theta)+a\sin\theta}{\sqrt{2}}$$

Differentiating AM with respected to 'θ'

$$\frac{d}{d\theta}(AM) = \frac{d}{d\theta}\left(\frac{a(1-\cos\theta)+a\sin\theta}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\frac{d}{d\theta}(a(1-\cos\theta) + a\sin\theta)$$

$$= \frac{1}{\sqrt{2}}(a(0-(-\sin\theta)) + a\cos\theta)$$

$$\frac{d}{d\theta}(AM) = \frac{a\sin\theta + a\cos\theta}{\sqrt{2}}$$

$$d(AM) = \frac{a\sin\theta + a\cos\theta}{\sqrt{2}}d\theta$$

∴ The volume of the spindle is given as

$$V = \pi\int_0^{AB}(MP)^2 d(AM)$$

$$= \pi\int_0^{\pi/2}\frac{a^2(\cos\theta+\sin\theta-1)^2}{2}\cdot\frac{a\sin\theta+a\cos\theta}{\sqrt{2}}d\theta$$

$$= \frac{\pi a^2}{2\sqrt{2}}\int_0^{\pi/2}(\cos\theta+\sin\theta-1)^2\cdot a(\sin\theta+\cos\theta)d\theta$$

$$= \frac{\pi a^3}{2\sqrt{2}}\int_{2\sqrt{2}}^{\pi/2}(\cos^2\theta+\sin^2\theta+1+2\sin\theta\cos\theta-2\sin\theta-2\cos\theta)(\sin\theta+\cos\theta)d\theta$$

$$= \frac{\pi a^3}{2\sqrt{2}}\int_{2\sqrt{2}}^{\pi a^3}(1+1+2\sin\theta\cos\theta-2\sin\theta-2\cos\theta)(\sin\theta+\cos\theta)d\theta$$

$$= \frac{\pi a^3}{2\sqrt{2}}\int_0^{\pi/2}(2+2\sin\theta\cos\theta-2\sin\theta-2\cos\theta)(\sin\theta+\cos\theta)d\theta$$

$$= \frac{2\pi a^3}{2\sqrt{2}}\int_{2\sqrt{2}}^{\pi a^3}(1+\sin\theta\cos\theta-\sin\theta\cos\theta)(\sin\theta+\cos\theta)d\theta$$

$$= \frac{\pi a^3}{\sqrt{2}}\int_0^{\pi/2}(\text{scin}\theta+\cos\theta+\sin^2\theta\cos\theta+\sin\theta\cos^2\theta-\sin^2\theta-\sin\theta\cos\theta-\sin\theta\cos\theta-\cos^2\theta)d\theta$$

$$= \frac{\pi a^3}{\sqrt{2}}\left[\int_0^{\pi/2} \sin\theta + \cos\theta + \sin^2\theta\cos\theta + \sin\theta\cos^2\theta - 2\sin\theta\cos\theta - \left(\sin^2\theta + \cos^2\theta\right)\right]d\theta$$

$$= \frac{\pi a^3}{\sqrt{2}}\left[\underbrace{\int_0^{\pi/2}\sin\theta d\theta}_{(a)} + \underbrace{\int_0^{\pi/2}\cos\theta d\theta}_{(b)} + \underbrace{\int_0^{\pi/2}\sin^2\theta\cos\theta d\theta}_{(c)} + \underbrace{\int_0^{\pi/2}\sin\theta\cos^2\theta d\theta}_{(d)} - \underbrace{2\int_0^{\pi/2}\sin\theta\cos\theta d\theta}_{(e)} - \underbrace{\int_0^{\pi/2}\left(\sin^2\theta + \cos^2\theta\right)}_{(f)}\right]d\theta$$

(a) $\int_0^{\pi/2} \sin\theta d\theta \Rightarrow \left[(-\cos\theta)\right]_0^{\pi/2}$

$= -\cos\frac{\pi}{2} - (-\cos 0)$

$= 0 + 1$

$= 1$

(b) $\int_0^{\pi/2} \cos\theta d\theta \Rightarrow [\sin\theta]_0^{\pi/2}$

$\Rightarrow \sin\frac{\pi}{2} - \sin 0$

$= 1 - 0$

$= 1$

(c) $\int_0^{\pi/2} \sin^2\theta\cos\theta d\theta$

Let $\sin\theta = t$

$\cos\theta \, d\theta = dt$

$\int_0^{\pi/2} t^2 \cdot dt \Rightarrow \left[\frac{t^3}{3}\right]_0^{\pi/2}$

$= \frac{1}{3}\left[\sin^3\left(\frac{\pi}{2}\right) - \sin^3(0)\right]$

$= \frac{1}{3}[1 - 0] = \frac{1}{3}$

(d) $\int_0^{\pi/2} \sin\theta\cos^2\theta d\theta$

Let $\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$$-\int_0^{\pi/2} t^2 \cdot dt = \left[\frac{t^3}{3}\right]_0^{\pi/2}$$

$$= \frac{-1}{3}\left[\cos^3\left(\frac{\pi}{2}\right) - \cos^3(0)\right]$$

$$= \frac{-1}{3}(0 - 1) = \frac{1}{3}$$

(e) $\int_0^{\pi/2} \sin\theta\cos\theta d\theta$

Let $\sin\theta = t$

$\cos\theta \, d\theta = dt$

$$\int_0^{\pi/2} t \cdot dt \Rightarrow \left[\frac{t^2}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\sin^2\left(\frac{\pi}{2}\right) - \sin^2(0)\right]$$

$$= \frac{1}{2}[1 - 0]$$

$$= \frac{1}{2}$$

(f) $\int_0^{\pi/2} \left(\sin^2 + \cos^2\theta\right) d\theta$

$$= \int_0^{\pi/2} (1) d\theta \Rightarrow [\theta]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Sub a, b, c, d, e, & f values in (1)

$$= \frac{\pi a^3}{\sqrt{2}}\left[1 + 1 + \frac{1}{3} - \frac{1}{3} - 2\left(\frac{1}{2}\right) - \frac{\pi}{2}\right]$$

$$= \frac{\pi a^3}{\sqrt{2}}\left[2 + \frac{1}{3} + \frac{1}{3} - 1 - \frac{\pi}{2}\right]$$

$$= \frac{\pi a^3}{\sqrt{2}}\left[1 + \frac{2}{3} - \frac{\pi}{2}\right]$$

$$= \frac{\pi a^3}{\sqrt{2}}\left[\frac{5}{3} - \frac{\pi}{2}\right]$$

$$= \frac{\pi a^3}{\sqrt{2}}\left(\frac{10 - 3\pi}{6}\right)$$

$\therefore$ The volume of spindle is $\frac{\pi a^3}{2\sqrt{2}}(10 - 3\pi)$

**35. Find the volume of the solid generated by the revolution of the curve (a – x) $y^2$ = $a^2x$. about the curve.**

*Sol :*

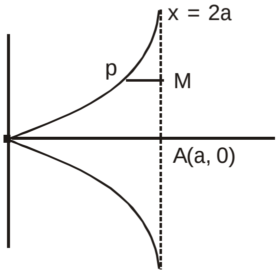
Given curve is $(a - x)\, y^2 = a^2x$. .....(1)

The asymptotes for the given curve can be obtained by equating the coefficients of higher power of x & y to zero

$a - x = 0$

$\Rightarrow$ $x - a = 0$

$\therefore$ The asymptote parallel to y-axis is $x = a$




Let P(x, y) be any point on the curve M be the foot of the perpendicular to A which drawn from P.

The length of perpendicular from P(x, y) to line $x - a = 0$ is

$MP = x - a$

$(MP)^2 = (x - a)^2$

$AM = y$

by (1) $(a - x)\, y^2 = a^2x$

$$y^2 = \frac{a^2x}{a - x}$$

$$y = \sqrt{\frac{a^2x}{a - x}}$$

$$y = a\sqrt{\frac{x}{a - x}}$$

$$\therefore \quad AM = a\sqrt{\frac{x}{a - x}}$$

differentiating AM with respect to

$$\frac{d}{dx}(AM) = a\frac{d}{dx}\sqrt{\frac{x}{a-x}}$$

$$= a\left[\frac{1}{2\sqrt{\frac{x}{a-x}}}\frac{d}{dx}\left(\frac{x}{a-x}\right)\right]$$

$$= \frac{a}{\frac{2\sqrt{x}}{\sqrt{a-x}}}\left[\frac{(a-x)(1)-x(-1)}{(a-x)^2}\right]$$

$$= \frac{a\sqrt{a-x}}{2\sqrt{x}}\cdot\frac{a-x+x}{(a-x)^2}$$

$$d(AM) = \frac{a\sqrt{a-x}}{2\sqrt{x}}\cdot\frac{a}{(a-x)^2}\ \frac{a}{(a-x)^2}\,dx$$

The volume of the solid is given as

$$V = 2\int_0^a \pi(MP)^2\, d(AM)$$

$$= 2\pi\int_0^a (x-a)^2\cdot\frac{a\sqrt{a-x}}{2\sqrt{x}}\cdot\frac{a}{(a-x)^2}dx$$

$$= \frac{2\pi a^2}{2}\int_0^a \cancel{(x-a)^2}\cdot\frac{\sqrt{a-x}}{\sqrt{x}}\cdot\frac{1}{\cancel{(x-a)^2}}dx$$

$$= \pi a^2\int_0^a \frac{\sqrt{a-x}}{\sqrt{x}}dx$$

Let $x = a\sin^2\theta$

$dx = a(2\sin\theta\cos\theta)\,d\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

If $x = a \Rightarrow a\sin^2\theta = a$

$\sin^2\theta = 1$

$\sin\theta = 1$

$\theta = \sin^{-1}(1)$

$$\theta = \frac{\pi}{2}$$

If $x = 0 \Rightarrow a \sin^2 \theta = 0$

$$\sin^2 \theta = 0$$

$$\theta = \sin^{-1}(0)$$

$$= 0$$

$$V = \pi a^2 \int_0^{\pi/2} \frac{\sqrt{a - a\sin^2 \theta}}{\sqrt{a \sin^2 \theta}} \, 2a \sin\theta \cos\theta \, d\theta$$

$$= \pi a^2 \int_0^{\pi/2} \frac{\sqrt{a\left(1 - \sin^2 \theta\right)}}{\sqrt{a \sin^2 \theta}} \, 2a \sin\theta \cos\theta \, d\theta$$

$$= 2\pi a^3 \int_0^{\pi/2} \frac{\cancel{\sqrt{a}}\sqrt{\cos^2 \theta}}{\cancel{\sqrt{a}} \sin\theta} \sin\theta \cos\theta \, d\theta$$

$$= 2\pi a^3 \int_0^{\pi/2} \frac{\cos\theta}{\cancel{\sin\theta}} \cancel{\sin\theta} \cos\theta \, d\theta$$

$$= 2\pi a^3 \int_0^{\pi/2} \cos^2 \theta \, d\theta$$

$$= 2\pi a^3 \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \frac{2\pi a^3}{2} \int_0^{\pi/2} \left(1 + \cos 2\theta\right) d\theta$$

$$= \pi a^3 \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \pi a^3 \left[\frac{\pi}{2} - 0 + \frac{1}{2}\left(\sin 2\left(\frac{\pi}{2}\right) - \sin(0)\right)\right]$$

$$= \pi a^3 \left[\frac{\pi}{2} + \frac{1}{2}(0 - 0)\right]$$

$$V = \frac{\pi^2 a^3}{2}$$

$\therefore$ Volume of solid is $\frac{\pi^2 a^3}{2}$

**36. Prove that the volume of the solid generated by the revolution of an ellipse round its minor axis, is a mean proportional between that generated by the revolution of the ellipse and of its auxiliary circle round the major axis.**

*Sol :*

We have equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .....(1)

Auxillary circle is $x^2 + y^2 = a^2$ .....(2)

The volume of solid generated by the revolution of an ellipse about its minor axis

from (1) $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} = \frac{1-y^2}{b^2}$

$$\frac{x^2}{a^2} = \frac{b^2 - y^2}{b^2}$$

$$x^2 = a^2\left[\frac{b^2 - y^2}{b^2}\right]$$

$$x^2 = \frac{a^2}{b^2}\left[b^2 - y^2\right]$$

The volume of solid is $V_1 = \int_a^b 2\pi x^2 dy$

$$V_1 = 2\pi\int_a^b x^2 dy$$

$$= 2\pi\int_0^b \frac{a^2}{b^2}\left[b^2 - y^2\right]dy$$

$$= \frac{2\pi a^2}{b^2}\int_0^b \left(b^2 - y^2\right)dy$$

$$= \frac{2\pi a^2}{b^2}\int_0^b \left(b^2 - y^2\right)dy$$

$$= \frac{2\pi a^2}{b^2}\left[\int_0^b b^2 dy - \int_0^b y^2 dy\right]$$

$$= \frac{2\pi a^2}{b^2}\left[b^2\left[y\right]_0^b - \frac{1}{3}\left[b^3 - 0\right]\right]$$

$$= \frac{2\pi a^2}{b^2}\left[b^2\left[b - 0\right] - \frac{1}{3}\left[b^3 - 0\right]\right]$$

$$= \frac{2\pi a^2}{b^2}\left[b^3 - \frac{b^3}{3}\right]$$

$$= \frac{2\pi a^2}{b^2}\left[\frac{3b^3 - b^3}{3}\right]$$

$$= \frac{2\pi a^2}{b^2}\left[\frac{2b^3}{3}\right]$$

$$V_1 = \frac{4\pi a^2 b}{3}$$

Volume Generated by revolution of an ellipse about Major axis.

The volume of solid is $V = \int_0^a 2\pi y^2 dx$

$$V = 2\pi\int_0^a y^2 dx$$

by (1) $\Rightarrow$ $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$$y^2 = b^2\left[\frac{a^2 - x^2}{a^2}\right]$$

$$y^2 = \frac{b^2}{a^2}[a^2 - x^2]$$

$$\Rightarrow \quad V = 2\pi\int_0^a \frac{b^2}{a^2}\left[a^2 - x^2\right]dx$$

$$= \frac{2\pi b^2}{a^2}\int_0^a \left(a^2 - x^2\right)dx$$

$$= \frac{2\pi b^2}{a^2}\left[\int_0^a a^2 dx - \int_0^a x^2 dx\right]$$

$$= \frac{2\pi b^2}{a^2}\left[a^2[x]_0^a - \left[\frac{x^3}{3}\right]_0^a\right]$$

$$= \frac{2\pi b^2}{a^2}\left[a^2(a-0) - \frac{1}{3}\left(a^3 - 0\right)\right]$$

$$= \frac{2\pi b^2}{a^2}\left(a^3 - \frac{1}{3}a^3\right)$$

$$= \frac{2\pi b^2}{a^2}\left(\frac{3a^3 - a^3}{3}\right)$$

$$= \frac{2\pi b^2}{a^2}\left(\frac{2a^3}{3}\right)$$

$$V = \frac{4\pi b^2 a}{3}$$

Now, volume generated by revolution of Auxillary circle about it major axis.

by (2) $\Rightarrow x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2$

$\therefore$ The Revolution of arc about x-axis is same as the revolution of the entire solid.

$\therefore$ The volume of solid is twice of the volume of arc.

$\therefore$ The volume of solid is $V_2 = \int_0^a 2\pi y^2 dx$

$$V_2 = 2\pi\int_0^a y^2 dx$$

$$= 2\pi\int_0^a \left(a^2 - x^2\right)dx$$

$$= 2\pi\left[\int_0^a a^2 dx - \int_0^a x^2 dx\right]$$

$$= 2\pi\left[a^3 - \frac{1}{3}\left(a^3\right)\right]$$

$$= 2\pi\left(\frac{2a^3}{3}\right)$$

$$V_2 = \frac{4\pi a^3}{3}$$

$\therefore$ Mean proportion between the volumes generated by revolution of ellipse and auxiliary circle bound it major axis.

$$\sqrt{VV_2} = \sqrt{\frac{4\pi ab^2}{3} \times \frac{4\pi a^3}{3}}$$

$$= \sqrt{\left(\frac{4\pi a^2 b}{3}\right)^2}$$

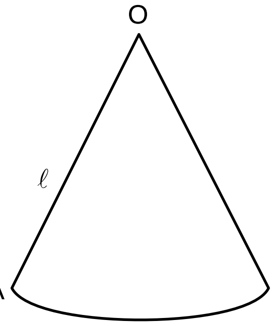
$$= \frac{4\pi a^2 b}{3}$$

$$\sqrt{VV_2} = V_1$$

## 4.6 Area of the Surface of the Frustum of a Cone Expression for the Surface of Revolution

We can easily see that the area of the surface or right circular cone, the radius of whose circular base is r, and slant height is $\ell$ is $\pi r\ell$.

If we tear a right circular cone along one of its generators, we get a circular sector whose radius OA is equal to the circumference of the circular base of the cone.

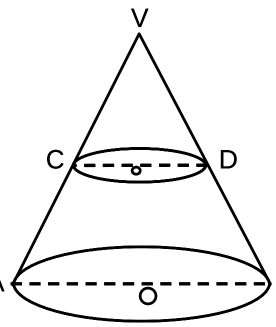



The area of this sector and, therefore also the surface of the cone is equal to $\pi r\ell$.

If $\alpha$ be the semi vertical angle of the cone, we have $\frac{r}{\ell} = \sin\alpha$.

$\therefore$ The surface of the cone is also equal to $\pi r\ell \Rightarrow \pi r\,(r \sin\alpha)$

$$= \pi p^2 \sin\alpha \qquad \left[\frac{r}{\ell} = \sin\alpha \quad r = \ell\sin\alpha\right]$$




Consider the frustum CDBA of cone VAB, let the radil O'A and OC of its circular base be $r_1$ and $r_2$ respectively and let its slant height CA be $l_1$.

The area of the surface of this frustum

$$= \pi(AV^2 - CV^2)\sin\alpha$$

$$(\text{i.e., } r^2 = r_1^2 - r_2^2$$

$$= AV^2 - CV^2)$$

$$= \pi(AV - CV)\,(AV \sin\alpha + CV \sin\alpha)$$

$$= \pi AC(O'A + OC)$$

$$= \pi\ell\,(r_1 + r_2)$$

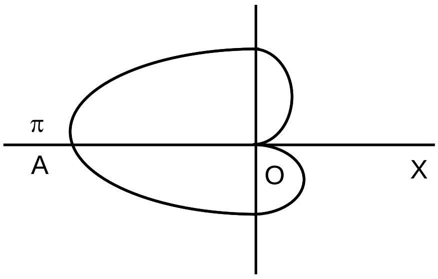
$= \pi \times$ slant height X sum of the radil of two bases.

**37. Find the surface of the solid formed by revolving the cardioid r = a(1 + cos θ) about the initial line.**

*Sol.:*

Given that

$r = a(1 + \cos\theta)$




The surface of the solid formed by revolving the cardioide is

$$S = \int_0^{\pi} 2\pi y \frac{ds}{d\theta}\, d\theta$$

We know that

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$\therefore \quad r = a(1 + \cos\theta)$

$$\frac{dr}{d\theta} = a(0 + \sin\theta)$$

$$= -a\sin\theta$$

$$\Rightarrow \quad \frac{ds}{d\theta} = \sqrt{[a(1+\cos\theta)]^2 + [-a\sin\theta]^2}$$

$$\frac{ds}{d\theta} = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \sqrt{a^2(1+\cos^2\theta + 2\cos\theta) + a^2\sin^2\theta}$$

$$= \sqrt{a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta}$$

$$= \sqrt{2a^2 + 2a^2\cos\theta}$$

$$= \sqrt{2a^2(1+\cos\theta)}$$

$$= \sqrt{2a^2\left(2\cos^2\frac{\theta}{2}\right)}$$

$$= \sqrt{4a^2\cos^2\frac{\theta}{2}}$$

$$\frac{ds}{d\theta} = 2\,a\cos\frac{\theta}{2}$$

$$\therefore \quad S = \int_0^{\pi} 2\pi y \frac{ds}{d\theta}\,d\theta \quad [\because\ y = r\sin\theta]$$

$$= \int_0^{\pi} 2\pi(r\sin\theta)\,2a\cos\theta/2\,d\theta$$

$$= \int_0^{\pi} 2\pi[a(1+\cos\theta)\sin\theta]\,2\,a\cos\frac{\theta}{2}\,d\theta$$

$$= 4\pi a^2 \int_0^{\pi} (1+\cos\theta)\sin\theta.\cos\frac{\theta}{2}d\theta$$

$$= 4\pi a^2 \int_0^{\pi} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.2\cos^2\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$= 4 \times 4\,\pi a^2 \int_0^{\pi} \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}d\theta$$

$$= 16\,\pi\,a^2\,(-2) \int_0^{\pi} \cos^4\frac{\theta}{2}\left(\frac{-1}{2}\right)\left(\sin\frac{\theta}{2}\right)d\theta$$

$$= -32\,\pi\,a^2 \int_0^{\pi} \left(\cos\frac{\theta}{2}\right)^4\left(-\sin\frac{\theta}{2}.\frac{1}{2}\right)d\theta$$

$$= -32\,\pi\,a^2 \left[\frac{\cos^4 + \theta/2}{4+1}\right]_0^{\pi}$$

$$\left[\because \int f^n(\theta)f'(\theta)d\theta = \frac{f^{n+1}(\theta)}{n+1}\right]$$

$$= -\frac{32\pi a^2}{5}\left[\cos^5\phi\right]_0^{\pi} \quad \text{where } \frac{\theta}{2} = \phi$$

$$= -\frac{32\pi a^2}{5}\left[\cos^5\left(\frac{\pi}{2}\right) - \cos^5(0)\right]$$

$$= -\frac{32\pi a^2}{5}[-1]$$

$$S = \frac{32\,\pi\,a^2}{5}$$

$\therefore$ The surface of solid formed by revolving the cardioid is $\frac{32\,\pi\,a^2}{5}$.

**38. Evaluate the surface area of the solid generated by revolving the cycloid x = a(θ – sin θ), y = a(1 – cos θ) about the line y = 0.**

*Sol.:*

Given cycloid's are

$x = a(\theta - \sin\theta)$ ......(1)

$y = a(1 - \cos\theta)$ .....(2)

We know that $\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$

Differentiate (1) with respect to 'θ'

1. $\Rightarrow \frac{dx}{d\theta} = a(1 - \cos\theta)$

Differentiate (2) with respect to 'θ'

2. $\Rightarrow \frac{dy}{d\theta} = a(0 - (-\sin\theta)$

$= a\sin\theta$

$\therefore \frac{ds}{d\theta} = \sqrt{\left[a(1-\cos\theta)^2 + [a\sin\theta]^2\right]}$

$= \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$

$= \sqrt{a^2(1+\cos^2\theta + 2\cos\theta) + a^2\sin^2\theta}$

$= \sqrt{a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta}$

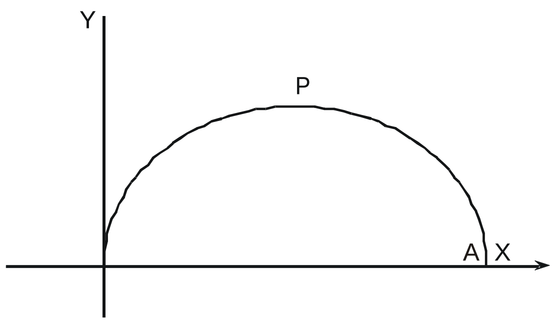
$= \sqrt{2a^2 - 2a^2\cos\theta}$

$= \sqrt{2a^2(1-\cos\theta)}$

$= \sqrt{2a^2 2\sin^2\frac{\theta}{2}}$

$= \sqrt{4a^2\sin^2\frac{\theta}{2}}$

$\frac{ds}{d\theta} = 2a\sin\frac{\theta}{2}$



$\therefore$ The required surface

$$S = \int_0^{2\pi} 2\pi y \frac{ds}{d\theta}.d\theta$$

$$= 2\pi \int_0^{2\pi} a(1-\cos\theta)\, 2a\sin\frac{\theta}{2} d\theta$$

$$= 4\pi a^2 \int_0^{2\pi} (1-\cos\theta)\sin\frac{\theta}{2} d\theta$$

$$= 4\pi a^2 \int_0^{2\pi} 2\sin^2\frac{\theta}{2}.\sin\frac{\theta}{2} d\theta$$

$$= 4\pi a^2 \int_0^{2\pi} 2\sin^2\frac{\theta}{2}.\sin\frac{\theta}{2} d\theta$$

$$= 8\pi a^2.2\int_0^{\pi} \sin^3\frac{\theta}{2} d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \sin^3\frac{\theta}{2} d\theta$$

Let $\frac{\theta}{2} = \phi$

$d\theta = 2d\phi$

If $\theta = \pi \Rightarrow \phi = \frac{\pi}{2}$

If $\theta = 0 \Rightarrow \phi = 0$

$$= 16\pi a^2 \int_0^{\pi/2} \sin^3\phi\, 2d\phi$$

$$= 32\pi a^2 \int_0^{\pi/2} \sin^3\phi$$

$$= 32\pi a^2\left(\frac{2}{3}\right)$$

$$S = \frac{64}{3}\pi a^2$$

$$\because \int_0^{\pi/2} \sin^n x\, dx = \left(\frac{n-1}{n}\right)$$

$$\Rightarrow \int_0^{\pi/2} \sin^3 x dx = \left(\frac{3-1}{3}\right) = \frac{2}{3}$$

Which is required the surface area of the cycloid.

**39. Find the surface of the solid generated by revolving the arc of the parabola $y^2$ = 4a x bounded by its latus return about x-axis.**

*Sol.:*

Given arc of the parabola is

$$y^2 = 4ax$$

To find the surface area of solid is

$$S = \int_0^a 2\pi y \frac{ds}{dx} dx$$

Now, to find $\frac{ds}{dx}$

Differentiate (1) with respect to 'x'.

$$2y\frac{dy}{dx} = 4a(1)$$

$$\frac{dy}{dx} = \frac{4a}{2y}$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1+\left(\frac{2a}{y}\right)^2}$$

$$= \sqrt{1+\left(\frac{2a}{4ax}\right)^2}$$

$$= \sqrt{1+\frac{4a^2}{4ax}}$$

$$= \sqrt{1+\frac{a}{x}}$$

$$\frac{ds}{dx} = \sqrt{\frac{x+a}{x}}$$

Now surface area of solid is,

$$S = \int_0^a 2\pi y \frac{ds}{dx} dx$$

$$= 2\pi\int_0^a \sqrt{4ax}\sqrt{\frac{x+a}{x}}\, dx$$

$$= 2\pi\int_0^a 2\sqrt{ax}\sqrt{\frac{x+a}{x}}\, dx$$

$$= 4\pi\sqrt{a}\int_0^a \sqrt{x}\, \frac{\sqrt{x+a}}{\sqrt{x}}\, dx$$

$$= 4\pi\sqrt{a}\int_0^a (x+a)^{1/2} dx$$

$$= 4\pi\sqrt{a}\left[\frac{(x+a)^{1/2+1}}{\frac{1}{2}+1}\right]_0^a$$

$$= \left(\frac{2}{3}\right)\times 4\pi\sqrt{a}\ [(x+a)^{3/2}]_0^a$$

$$= \frac{8}{3}\pi\sqrt{a}\left[(a+a)^{3/2} - (0+a)^{3/2}\right]$$

$$= \frac{8}{3}\pi\ \sqrt{a}\left[(2a)^{3/2} - a^{3/2}\right]$$

$$= \frac{8}{3}\pi\ a^{1/2}(2^{3/2}\ a^{3/2} - a^{3/2})$$

$$= \frac{8}{3}\pi\ a^{1/2}.a^{3/2}\left(\ 2^{3/2} - 1\right)$$

$$= \frac{8}{3}\pi\ a^{4/2}(2\sqrt{2} - 1)$$

$$S = \frac{8}{3}\pi\ a^2(2\sqrt{2} - 1)$$

$\therefore$ Surface area of the solid $= \frac{8}{3}\pi a^2(2\sqrt{2} - 1)$

**40. Find the surface of the solid generated by the revolution of the astroid**

**$x = a\cos^3 t$, $y = a\sin^3 t$ about the axis of x**

*Sol.:*

Given astroid $x = a\cos^3 t$ ... (1)

$y = a\sin^3 t$ ... (2)

Differentiating (1) and (2) with respect to 't'.

$$\frac{dx}{dt} = a\,(3\cos^2 t)\ \frac{d}{dt}\ (\cos t)$$

$$= 3a\cos^2 t\ (-\sin t)$$

$$\frac{dx}{dt} = -3a\cos^2 t\sin t$$

$$\frac{dy}{dt} = a(3\sin^2 t)\ \frac{d}{dt}\ (\sin t)$$

$$= 3a\sin^2 t\ (\cos t)$$

$$\frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\therefore \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2}$$

$$= \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}$$

$$= \sqrt{9a^2\cos^4 t\sin^2 t\,(\cos^2 t + \sin^2 t)}$$

$$= 3a\cos t\sin t.$$

$\therefore$ The surface area of the solid.

$$S = 2\int_0^{\pi/2} 2\pi y\frac{ds}{dt}\, dt$$

$$= 4\pi\int_0^{\pi/2} (a\sin^3 t)\,(3a\sin t\cos t)dt$$

$$= 12\pi a^2\int_0^{\pi/2} \sin^4 t\cos t\, dt$$

$$= 12\pi a^2\left[\frac{\sin t^{4+1}}{4+1}\right]_0^{\pi/2}$$

$$\left[\because \int_0^{\pi/2} f^n(\theta)\, f'(\theta)\, d\theta = \frac{f^{n+1}(\theta)}{n+1}\right]$$

$$= \frac{12\,\pi a^2}{5}\left[\sin^5\left(\frac{\pi}{2}\right) - \sin^5(0)\right]$$

$$= \frac{12\pi a^2}{5}[(1) - 0]$$

$$S = \frac{12\pi a^2}{5}$$

Which is required surface area of asteroid.

**41.** Find the surface oft he solid obtained by revolving the cardiodie r = a(1 – cos θ**) about the initial line.**

*Sol.:*

Given cardiodie is $r = a(1 - \cos\theta)$

Differentiating (1) with respect to 'θ' ... (1)

$$\frac{dr}{d\theta} = a\frac{d}{d\theta}(1 - \cos\theta)$$

$$= a(-(-\sin\theta))$$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{[a(1-\cos\theta)]^2 + (a\sin\theta)^2}$$

$$= \sqrt{a^2(1-\cos\theta)^2 + (a\sin\theta)^2}$$

$$= \sqrt{a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta}$$

$$= \sqrt{2a^2 - 2a^2\cos\theta}$$

$$= \sqrt{2a^2(1-\cos\theta)}$$

$$= \sqrt{2a^2\, 2\sin^2\frac{\theta}{2}}$$

$$\frac{ds}{d\theta} = 2a\sin\frac{\theta}{2}$$

The surface of solid is $S = \int_0^{\pi} 2\pi y\frac{ds}{d\theta}.d\theta$

$$= 2\pi\int_0^{\pi} y\frac{ds}{d\theta}\,d\theta$$

$$= 2\pi\int_0^{\pi} r\sin\theta.2a.\sin\frac{\theta}{2}d\theta \quad [\because y = r\sin\theta]$$

$$= 2\pi\int_0^{\pi}[a(1-\cos\theta)\sin\theta.\ 2a\sin\frac{\theta}{2}d\theta$$

$$= 4\pi a\int_0^{\pi} a(1-\cos\theta)\sin\theta\ .\ \sin\frac{\theta}{2}\ d\theta$$

$$= 4\pi a\int_0^{\pi} 2\sin^2\frac{\theta}{2}.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.\sin\frac{\theta}{2}d\theta$$

$$[\because (1-\cos\theta) = a\sin^2\frac{\theta}{2}\quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}]$$

$$= 4\pi a\int_0^{\pi} 2\sin^2\frac{\theta}{2}.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.\sin\frac{\theta}{2}d\theta$$

$$= 4\pi a^2\int_0^{\pi} 2\sin^2\frac{\theta}{2}\ 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$= 16\pi a^2\int_0^{\pi}\sin^4\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$= 16\pi a^2\ .\ 2\int_0^{\pi}\left(\sin\frac{\theta}{2}\right)^4\left(\frac{1}{2}\cos\frac{\theta}{2}\right)d\theta$$

$$= 32\pi a^2\left[\frac{\sin^{4+1}\frac{\theta}{2}}{4+1}\right]_0^{\pi}$$

$$= \frac{32\ \pi\ a^2}{5}\left[\sin^5\frac{\theta}{2}\right]_0^{\pi}$$

$$= \frac{32\ \pi\ a^2}{5}\left[\sin^5\left(\frac{\pi}{2}\right)-\sin^5(0)\right]$$

$$= \frac{32\pi a^2}{5}[1-0]$$

$$S = \frac{32\pi a^2}{5}$$

$\therefore$ Surface area of solid is $S = \dfrac{32\pi a^2}{5}$

**42. Prove that the surface generated by the revolution of the tractrix.**

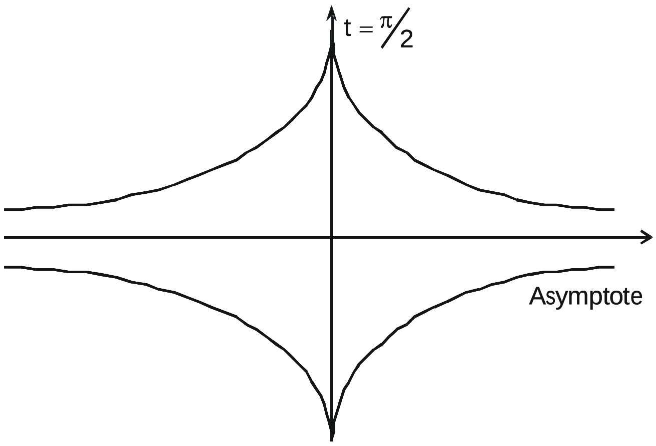
$$x = a\cos t + \frac{1}{2}\, a \log \tan^2 \frac{t}{2},\ y = a \sin t$$

**about its asymptote is equal to the surface of a sphere of radius a.**

*Sol.:*

Here $x = a\cos t + \frac{a}{2} \log \tan^2 \frac{t}{2}$ .....(1)

$y = a \sin t$ .....(2)




Differentiate with respect to 't' to equation (1)

$$\frac{dx}{dt} = a(-\sin t) + \frac{a}{2} \cdot \frac{1}{\tan^2 \frac{t}{2}} \frac{d}{dt}\left(\tan^2 . \frac{t}{2}\right)$$

$$= -a\sin t + \frac{a}{2} \frac{1}{\tan^2 \frac{t}{2}} \cdot 2\tan\frac{t}{2} \cdot \mathrm{Sec}^2 \frac{t}{2} \cdot \frac{1}{2}$$

$$= -a\sin t + \frac{2a}{4} \frac{\sec^2 \frac{t}{2}}{\tan \frac{t}{2}}$$

$$= -a\sin t + \frac{1}{2}\, a \frac{1}{\cos^2 \frac{t}{2}} \times \frac{\cos t/2}{\sin t/2}$$

$$= -a\sin t + \frac{a}{2} \frac{1}{\sin t/2 \cos t/2}$$

$$= -a\sin t + \frac{a}{\sin 2\left(\frac{t}{2}\right)}$$

$$= -a\,.\sin t + \frac{a}{\sin t}$$

$$= -\frac{a\sin^2 t + a}{\sin t}$$

$$= \frac{a(1-\sin^2 t)}{\sin t}$$

$$\frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$$

Differentiate (2) with respect to 't'

$$\frac{dy}{dt} = a\cos t$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{\left(\frac{a\cos^2 t}{\sin t}\right)^2 + (a\cos t)^2}$$

$$= \sqrt{\frac{a^2\cos^4 t}{\sin^2 t} + a^2\cos^2 t}$$

$$= \frac{\sqrt{a^2\cos^4 t + a^2\cos^2 t\sin^2 t}}{\sin t}$$

$$= \frac{\sqrt{a^2\cos^2 t\,(\cos^2 t + \sin^2 t)}}{\sin t}$$

$$= \frac{\sqrt{a^2\cos^4 t + (\cos^2 t + \sin^2 t)}}{\sin t}$$

$$= \frac{\sqrt{a^2\cos^2 t}}{\sin t}$$

$$\frac{ds}{dt} = \frac{a\cos t}{\sin t}$$

$\therefore$ The surface area of tractrix.

$$S = 2\int_0^{\pi/2} 2\pi y\frac{ds}{dt}\,dt$$

$$= 4\pi\int_0^{\pi/2} a\sin t\, a\,\frac{\cos t}{\sin t}\,dt$$

$$= 4\pi a^2\int_0^{\pi/2}\cos t\,dt$$

$$= 4\pi a^2\int_0^{\pi/2}\cos t\,dt$$

$$= 4\pi a^2(\sin t)_0^{\pi/2}$$

$$= 4\pi a^2\left(\sin\frac{\pi}{2} - \sin(0)\right)$$

$$= 4\pi a^2(1-0)$$

$$= 4\pi a^2$$

$\therefore$ The surface area of sphere of a radius.

**43. Prove that the surface of the solid obtained by revolving the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about the axis of x is $2\pi ab\left[\sqrt{1-e^2} + \frac{1}{e}\sin^{-1}e\right]$.**

*Sol.:*

Given ellipse is $b^2x^2 + a^2y^2 = a^2b^2$ ... (1)

Differentiating (1) with respect to 'x'

$$b^2(2x) + a^2 2y.\frac{dy}{dx} = 0$$

$$2b^2x + 2a^2y\frac{dy}{dx} = 0$$

$$b^2x + a^2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

$\therefore \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$= \sqrt{1+\left(\frac{-b^2x}{a^2y}\right)^2}$

$\frac{ds}{dx} = \frac{\sqrt{a^4y^2+b^4x^2}}{a^2y}$

$\therefore$ The surface area of solid is

$S = 2\int_0^a 2\pi y \frac{ds}{dx} dx$

$= 4\pi\int_0^a y.\frac{1}{a^2y}\sqrt{a^4y^2+b^4x^2}\ dx.$

$= \frac{4\pi}{a^2}\int_0^a \sqrt{a^4y^2+b^4x^2}\ dx$

by (1) $\Rightarrow$ $b^2x^2 + a^2y^2 = a^2b^2$

$a^2y^2 = a^2b^2 - b^2x^2$

$y^2 = \frac{b^2(a^2-x^2)}{a^2} = b^2 - \frac{b^2x^2}{a^2}$

$= \frac{4\pi}{a^2}\int_0^a \sqrt{a^4\left(\frac{b^2(a^2-x^2)}{a^2}\right)+b^4x^2}\ dx$

$= \frac{4\pi}{a^2}\int_0^a \sqrt{a^2(a^2-b^2x^2)+b^4x^2}\ dx$

$= \frac{4\pi}{a^2}\int_0^a \sqrt{a^2b^2(a^2-x^2+\frac{b^2x^2}{a^2})}\ dx$

$= \frac{4\pi}{a^2}\int_a^b ab\sqrt{a^2-x^2\left(1-\frac{b^2}{a^2}\right)}\ dx$

$= \frac{4\pi}{a^2}ab\int_0^a \sqrt{a^2-x^2\left(\frac{a^2-b^2}{a^2}\right)}\ dx$

$= \frac{4\pi b}{a}\int_0^a \sqrt{a^2-e^2x^2}\ dx$

$\left[\because \frac{a^2-b^2}{a^2} = e^2\right]$

$$= \frac{4\pi b}{a}\int_0^a \sqrt{e^2\left(\frac{a^2}{e^2}-x^2\right)}\,dx$$

$$= \frac{4\pi be}{a}\int_0^a \sqrt{\frac{a^2}{e^2}-x^2}\,dx$$

$$= \frac{4\pi be}{a}\left[\frac{x}{2}\sqrt{\frac{a^2}{e^2}-x^2}+\frac{\frac{a^2}{e^2}}{2}\sin^{-1}\left(\frac{x}{\frac{a}{e}}\right)\right]_0^a$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{n}{a}\right)\right]$$

$$= \frac{4\pi be}{a}\left[\frac{a^2}{2}\sqrt{\frac{a^2}{e^2}-a^2}+\frac{a^2}{2e^2}\sin^{-1}\left(\frac{ea}{a}\right) - \frac{0}{2}\sqrt{\frac{a^2}{e^2}-0}-\frac{a^2}{2e^2}\sin^{-1}\frac{e(0)}{a}\right]$$

$$= \frac{4\pi be}{a}\left[\frac{a}{2}\sqrt{\frac{a^2-e^2a^2}{e^2}}+\frac{a^2}{2e^2}\sin^{-1}(e)\right]$$

$$= \frac{4\pi be}{a}\left[\frac{a}{2e}\sqrt{a^2-e^2a^2}+\frac{a^2}{2e^2}\sin^{-1}(e)\right]$$

$$= \frac{4\pi be}{a}\left[\frac{a^2}{2e}a\sqrt{1-e^2}+\frac{a^2}{2e^2}\sin^{-1}(e)\right]$$

$$= \frac{4\pi be}{a}\left[\frac{a^2}{2e}\sqrt{1-e^2}+\frac{a^2}{2e^2}\sin^{-1}(e)\right]$$

$$= \frac{4\pi be}{a}\ \frac{a^2}{2e}\left[\sqrt{1-e^2}+\frac{1}{e}\sin^{-1}e\right]$$

$$= \frac{4\pi ba}{2}\left[\sqrt{1-e^2}+\frac{1}{e}\sin^{-1}e\right]$$

$$= 2\pi ba\left[\sqrt{1-e^2}+\frac{1}{e}\sin^{-1}e\right]$$

$\therefore$ The surface are a of ellipse is

$$2\pi ba\left[\sqrt{1-e^2}+\frac{1}{e}\sin^{-1}e\right]$$

## 4.7 Pappus Theorem - Surfaces of Revolution

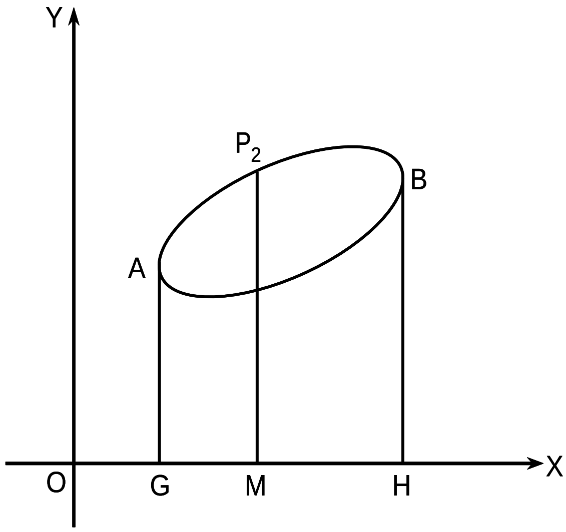
**44. State and prone pappus theorem for volume of revolution.**

**Statement**

**If a closed plane curve revolves about a straight line in its plane, (the straight line not intersecting the curve), then the volume of the solid of revolution thus formed is obtained on multiplying the area of the region enclosed by the curve with the length of the path described by the centroid of the region.**

*Proof :*

Suppose that the curve is such that every line parallel to y-axis and lying between the co-ordinate x = a, x = b of two points A, B meets the curve in two and only two points $P_1$, $P_2$.




Let MP = $y_1$, MQ = $y_2$

The volume of solid of revolution is given as

$$V = \pi \int_a^b y_2^2\, dx - \pi \int_a^b y_1^2\, dx$$

$$= \pi \int_a^b \left(y_2^2 - y_1^2\right) dx \quad .....(1)$$

The ordinate of the centroid of the region is given as,

$$\bar{y} = \frac{\int_a^b \frac{1}{2}(y_1 + y_2)(y_2 - y_1)\, dx}{A}$$

$$\bar{y} = \frac{1}{2A} \int_a^b (y_2^2 - y_1^2)\, dx$$

$$2A\bar{y} = \int_a^b y_2^2 - y_1^2\, dx \quad ....(2)$$

Substituting equation (2) in equation (1)

$$V = 2\pi\,(2A\,\bar{y})$$

$$V = 2\pi\, A\,\bar{y}$$

Where $2\pi\bar{y}$ length of the path described by the centroid.

**45. State and Prove Pappus Theorem for Surface area.**

**Statement**

**If a closed plane curve revolves about a straight line in its plane (the straight line not intersecting the curve) then the surface of the solid of revolution thus formed is obtained on multiplying the length of the curve what that of the path described by the centroid of the curve.**

*Proof :*

Suppose a curve lying between the coordinates.

x = a and x = b

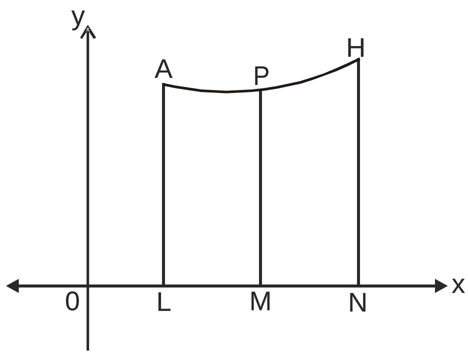
revolvecy about x-axis such that every line is parallel to y-axis.

Let MP = y

The surface of solid of revolution

$$S = \int_0^1 2\pi y\; ds$$

$$= 2\pi \int_0^1 y\, ds$$




The ordinate of the centroid of the curve is given as

$$\bar{y} = \int_0^1 \frac{yds}{l}$$

$$\bar{y}\ell = \int_0^{\ell} y\, d$$

.....(2)

from equation (1) and (2)

$$S = 2\pi\bar{y}\,\ell$$

where $\ell$ = length o the are of the curve

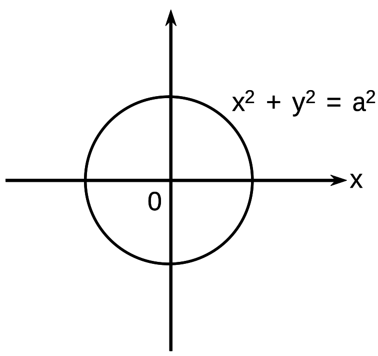
$2\pi\bar{y}$ = Length of the path described by the centroid.

**46. Find the surface of a sphere of radius 'a'.**

*Sol :*

Given that, the radius of sphere is a equation of circle is $x^2 + y^2 = a^2$ .... (1)

Differentiating (1) with respect to 'x'.




$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1+\left(\frac{-x}{y}\right)^2}$$

$$= \sqrt{1+\frac{x^2}{y^2}}$$

$$= \sqrt{\frac{y^2+x^2}{y^2}}$$

$$\frac{ds}{dx} = \frac{1}{y}\sqrt{y^2+x^2}$$

by (1) $\Rightarrow \frac{1}{y}\sqrt{a^2}$

$$\frac{ds}{dx} = \frac{a}{y}$$

$\therefore$ The surface area of sphere is

$$S = 2\int_0^a 2\pi y\frac{ds}{dx}dx$$

$$= 4\pi\int_0^a y.\frac{a}{y}dx$$

$$= 4\pi\, a\int_0^a 1\,dx$$

$$= 4\pi\, a(a)$$

$$= 4\pi a^2$$

$\therefore$ Surface area of sphere $4\pi a^2$.

**47. An arc of a circle of radius a revolves about its chord, it the length of the arc is $2a\alpha(\alpha < \pi/2)$. Show that the area of surface generated is $4\pi a^2(\sin\alpha - \alpha\cos\alpha)$.**

*Sol :*

The equation of circle is, $x^2 + y^2 = a^2$

Let $x = a\cos\theta$, $y = a\sin\theta$

Differentiate 'x' and 'y' with respect to 'θ'

$$\frac{dx}{d\theta} = a(-\sin\theta)$$

$$= -a\sin\theta$$

$$\frac{dy}{d\theta} = a(\cos\theta)$$

$$= a\cos\theta$$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2}$$

$$= \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}$$

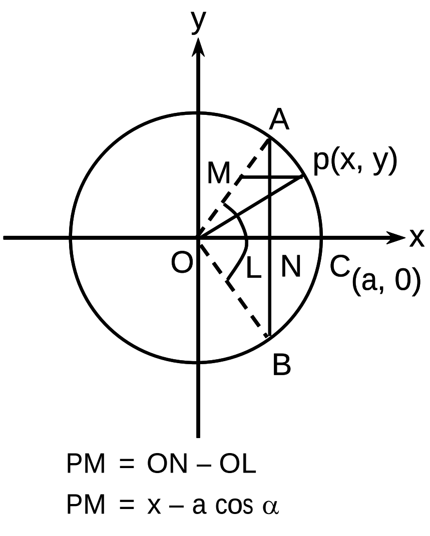
$$= \sqrt{a^2(\sin^2\theta + \cos^2\theta)}$$

$$= \sqrt{a^2}$$

$$\frac{ds}{d\theta} = a$$

The surface area of solid is given as

$$S = 2\int_0^{\infty} 2\pi \text{ (PM) } \frac{ds}{d\theta}\, d\theta.$$




$$PM = ON - OL$$

$$PM = x - a\cos\alpha$$

$$= 4\pi \int_0^{\alpha} (x - a\cos\alpha)\, a\, .\, d\theta$$

$$= 4\pi a^2 \int_0^{\alpha} (a\cos\theta - a\cos\alpha)\, d\theta$$

$$= 4\pi a\, a \int_0^{\alpha} (\cos\theta - \cos\alpha)\, d\theta$$

$$= 4\pi a^2 \left[\int_0^{\alpha} \cos\theta\, d\theta - \int_0^{\alpha} \cos\alpha\, d\theta\right]$$

$$= 4\pi a^2 \left[\left[\sin\theta\right]_0^{\alpha} - \cos\alpha \int_0^{\alpha} d\theta\right]$$

$$= 4\pi a^2 [\sin\alpha - \sin(0) - \cos\alpha(\alpha - 0)]$$

$$= 4\pi a^2 [\sin\alpha - \alpha\cos\alpha]$$

$\therefore$ Surface are of solid $4\pi a^2$ [$\sin\theta - \cos\alpha$].

**48. Find the surface of the solid generated by revolution of the curve $x^2 + 4y^2 = 16$ about the x-axis.**

*Ans :*

The given curve $x^2 + 4y^2 = 16$ ... (1)

$\Rightarrow$ divide by 16

$$\frac{x^2}{16} + \frac{4y^2}{16} = 1$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

Now, differentiating with respect to 'x'

$$\frac{2x}{16} + \frac{2y}{4}\frac{dy}{dx} = 0$$

$$\frac{x}{8} + \frac{y}{2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{8} \times \frac{2}{y}$$

$$= \frac{-x}{4y}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1 + \left(\frac{-x}{16y}\right)^2}$$

$$= \sqrt{1 + \frac{x^2}{16y^2}}$$

$$= \sqrt{\frac{16y^2 + x^2}{16y^2}} = \sqrt{\frac{16y^2 + x^2}{4y}}$$

by (1) $x^2 + 4y^2 = 16$

$4y^2 = 16 - x^2$

$$y^2 = \frac{16 - x^2}{4}$$

$$= \frac{16}{4} - \frac{x^2}{4}$$

$$y^2 = 4 - \frac{x^2}{4} = \sqrt{\frac{16\left(4 - \frac{x^2}{4}\right) + x^2}{4y}}$$

$$= \sqrt{\frac{64 - \frac{16x^2}{4} + x^2}{4y}}$$

$$= \sqrt{\frac{64 - 4x^2 + x^2}{4y}}$$

$$= \sqrt{\frac{64 - 3x^2}{4y}}$$

Since $y^2 = 4 - \frac{x^2}{4}$

$$y = \sqrt{4 - \frac{x^2}{4}} = 2 - \frac{x}{2} = \frac{4 - X}{2}$$

$$= \sqrt{\frac{64 - 3x^2}{4\left(\frac{4 - x}{2}\right)}}$$

$$= \frac{1}{2}\sqrt{\frac{64 - 3x^2}{16 - 4x^2}}$$

$$= \frac{1}{2}\sqrt{\frac{64 - 3x^2}{4y^2}}$$

$$\frac{ds}{dx} = \frac{1}{2}\frac{\sqrt{64 - 3x^2}}{2y}$$

The required surface generated by the cone about x-axis.

$$S = 2\pi \int_{-4}^{4} y \frac{ds}{dz} dx$$

$$= 2\pi \int_{-4}^{4} y, \frac{1}{2}\frac{\sqrt{64 - 3x^2}}{2y} dx$$

$$= \frac{2\pi}{4}\int_{-4}^{4} \sqrt{64 - 3x^2}\, dx$$

$$= \frac{\pi}{2}\int_{-4}^{4} \sqrt{\frac{64}{3} - x^2}\, dx$$

$$= \frac{\pi\sqrt{3}}{2}\left[\frac{1}{2}x\sqrt{\frac{64}{3} - x^2} + \frac{1}{2}.\frac{64}{3}\sin^{-1}\frac{x\sqrt{3}}{8}\right]_{-4}^{4}$$

$$= \frac{\pi\sqrt{3}}{2}\left[2\sqrt{\frac{64}{3} - 16} + \frac{32}{3}\sin^{-1}\frac{\sqrt{3}}{2} + 2\right.$$

$$= \left.\sqrt{\frac{64}{3} - 16} + \frac{32}{3}\sin^{-1}\frac{\sqrt{3}}{2}\right]$$

$$= \frac{\pi\sqrt{3}}{2}\left[\frac{16}{\sqrt{3}} + \frac{64}{3}.\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[16\pi + \frac{64\pi^2\sqrt{2}}{9}\right]$$

$$S = 8\pi + \frac{32\sqrt{3}}{9}\pi^2$$

$\therefore$ The surface of solid is $8\pi + \frac{32\sqrt{3}}{9}\pi^2$

# *Choose the Correct Answers*

1. The volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ = 1 about the initial line is ________. [ d ]

(a) $\pi ab^2$ (b) $\frac{2}{3}\pi ab^2$

(c) $\frac{1}{3}\pi\, ab^2$ (d) $\frac{4}{3}\pi ab^2$

2. The volume of the solid generated by revolving of the loop of the curve $y^2 = x^2 (a - x)$ about the x-axis is [ b ]

(a) $\pi a^4$ (b) $\frac{1}{12}\pi a^4$

(c) $\frac{3}{4}\pi a^4$ (d) $\frac{4}{3}\pi a^4$

3. Area of the surface of a cone whose semi vertical angle is $\alpha$ and base a circle of radius r is [ d ]

(a) $\pi r^2 \sin \alpha$ (b) $\pi r^2 \cos \alpha$

(c) $\pi r^2$ (d) $\pi r^2 \operatorname{cosec} \alpha$

4. The surface area of the solid of revolution of he circle $x^2 + y^2 = a^2$ about the diameters is [ a ]

(a) $4\pi a^2$ (b) $3\pi a^2$

(c) $4\pi a$ (d) $2\pi a^2$

5. The surface area of the solid generated by revolving the asteroid. $x = a \cos^3 t$, $y = a \sin^3 t$ about the x-axis is [ a ]

(a) $\frac{12\pi a^2}{5}$ (b) $\frac{12}{5}\pi a$

(c) $\frac{12\pi\, a^3}{5}$ (d) None

6. The volume of a come of height ‘h’ and base radius ‘r’ is [ c ]

(a) $\frac{\pi r^2 h}{2}$ (b) $\frac{\pi rh}{3}$

(c) $\frac{\pi r^2 h}{3}$ (d) $\frac{\pi rh}{2}$

7. Surface area of the segment of a sphere of radius a and height ‘h’ is [ a ]

(a) $2\pi ah$ (b) $\pi ah$

(c) $\frac{\pi ah}{2}$ (d) None

8. If $x = a(\theta + \sin\theta)$ them $\frac{dx}{d\theta}$ [ b ]

(a) $a(\theta + \cos\theta)$ (b) $a(1 + \cos\theta)$

(c) $a\cos\theta$ (d) None

9. The arc length of the curve $y = f(x)$ lying between two points for which $x = a$ and $x = b$ is [ c ]

(a) $\int_a^b y\,dx$ (b) $\pi\int_a^b y^2 dx$

(c) $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ (d) None

10. Perimeter of the curve $r = 2\cos\theta$ is [ d ]

(a) $\pi$ (b) $\frac{\pi}{2}$

(c) $\frac{3}{2}\pi$ (d) $2\pi$

# Fill in the Blanks

1. The volume of the solid generated by revolution of the loop of the curve $y^2 = x^2 (a - x)$ about x-axis is ________.
2. Area of the surface of revolution formed by revolving the curve $r = 2a \cos\theta$ about the intital line is ________.
3. Surface area of the sphere ________.
4. The volume of the solid generated by the revolving of the curve $(a - x) y^2 = a^2x$ ________.
5. The volume of the solid obtained by revolving the cardiode $r = a(1 + \cos\theta)$ about initial line is ________.
6. Intrinsic equation of the catenary $y = c \cosh\left(\frac{x}{a}\right)$ is ________.
7. Area of the surface of the frustum of a cone ________.
8. Surface generated by the revolution of an arc of the catenary $y = c \cosh\frac{x}{c}$ about the axis of x ________.
9. The surface of the solid formed by revolving the cardiodie $r = a(1 + \cos\theta)$ about the initial line ________.
10. $\frac{ds}{dt}$ = ________.

## Answers

1. $\frac{1}{12}\pi a^4$
2. $4\pi a^2$
3. $4\pi a^2$
4. $\pi^2 a^3/2$
5. $8\pi a^3/5$
6. $s = a \tan\varphi$
7. $\pi \ell (r_1 + r_2)$
8. $\pi C\left[x + \frac{C}{2}\sinh\left(\frac{x}{C}\right)\right]$
9. $\frac{32}{5}\pi a^2$
10. $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

## FACULTY OF SCIENCE
## B.Sc. I - Semester (CBCS) Examination
## DIFFERENTIAL AND INTEGRAL CALCULUS
## MODEL PAPER - I

Time : 3 Hours ] [Max. Marks : 80

**PART – A (8 × 4 = 32 Marks)**
**[Short Answer Type]**

**Note : Answer any Eight of the following questions**

**ANSWERS**

1. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ Show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$ **(Unit - I, Q.No. 6)**

2. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. **(Unit - I, Q.No. 18)**

3. Expand $f(x, y) = e^x \cos y$ by Taylor's series in power of x and y such that it includes all terms upto third degree. **(Unit - II, Q.No. 40)**

4. Find $\frac{dy}{dx}$ for $x^y = y^x$ **(Unit - II, Q.No. 16)**

5. Expand $f(x, y) = \log(x + e^y)$ by Taylor's series in powers of (x–1) and y such that it includes all terms up to second degree. **(Unit - II, Q.No. 37)**

6. Find $\frac{ds}{dx}$ for $x = ae^t \sin t$, $y = ae^t \cos t$ **(Unit - III, Q.No. 6)**

7. Find the points in the parabola $y^2 = 8x$ which the radius of curvature is $7\frac{13}{6}$. **(Unit - III, Q.No. 30)**

8. If $C_x$ and $C_y$ be the chords of curvature parallel to the axes at any point of the curve $y = ae^{x/a}$. Prove that $\frac{1}{C_{x^2}} + \frac{1}{C_{y^2}} = \frac{1}{2aC_x}$. **(Unit - III, Q.No. 59)**

9. Find the length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \pi/3$. **(Unit - IV, Q.No. 5)**

10. Rectify the curve $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$. **(Unit - IV, Q.No. 12)**

11. If $u = 3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$ & $l^2 + m^2 + n^2 = 1$

Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ **(Unit - I, Q.No. 8)**

12. Find the length of an arc of the curve $r = e^{\theta \cot \alpha}$ taking $S = 0$ when $\theta = 0$. **(Unit - IV, Q.No. 11)**

**PART – B (4 × 12 = 48 Marks)**

**[Essay Answer Type]**

**Note : Answer any Four of the following questions**

13. (a) State and Prove Euler's theorem on Homogenous function. **(Unit - I, Q.No. 28)**

**(OR)**

(b) If $u = \tan^{-1}\frac{xy}{\sqrt{1+x^2+y^2}}$ Show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$ **(Unit - I, Q.No. 32)**

14. (a) State and prove Equality of $f_{xy}$ and $f_{yx}$. **(Unit - II, Q.No. 29)**

**(OR)**

(b) Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ where $f(x, y) = 0$ if $xy = 0$,

$f(x, y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$ if $xy \neq 0$ **(Unit - II, Q.No. 28)**

15. (a) Find the radius of curvature at any point on the curves

$x = (a \cos t) / t$, $y = (a \sin t) / t$ **(Unit - III, Q.No. 23)**

**(OR)**

(b) Show that $\left(x - \frac{3}{4}a\right)^2 + \left(y - \frac{3}{4}a\right)^2 = \left(\frac{a}{2}\right)^2$ is the circle of

curvature of the curve $\sqrt{x} + \sqrt{y} = y$ at the point $\left(a/4, a/4\right)$ **(Unit - III, Q.No. 55)**

16. (a) Find the length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum. **(Unit - IV, Q.No. 3)**

**(OR)**

(b) Find the length of the arc of the curve $y = \log \frac{e^x - 1}{e^x + 1}$

from $x = 1$ to $x = 2$. **(Unit - IV, Q.No. 17)**

# FACULTY OF SCIENCE
## B.Sc. I - Semester (CBCS) Examination
## DIFFERENTIAL AND INTEGRAL CALCULUS
## MODEL PAPER - II

Time : 3 Hours ] [Max. Marks : 80

**PART – A (8 × 4 = 32 Marks)**

**[Short Answer Type]**

**Note : Answer any Eight of the following questions**

**ANSWERS**

1. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ when $u = \sin^{-1}\frac{x}{y}$ **(Unit - I, Q.No. 16)**
2. Verify Euler's Theorem for $Z = ax^2 + 2hxy + by^2$ **(Unit - I, Q.No. 45)**
3. find $\frac{d^2y}{dx^2}$ for $x^3 + y^3 = 3axy$ **(Unit - II, Q.No. 21)**
4. Discuss the maximum and minimum of $x^4 + 2x^2y - x^2 + 3y^2$. **(Unit - II, Q.No. 51)**
5. Find a point within a triangle such that the sum of the square of its distance from the three vertices is a minimum. **(Unit - II, Q.No. 48)**
6. Show that the curvature of the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the family $x^3 + y^3 = 3axy$ is $\frac{-8\sqrt{2}}{3a}$ **(Unit - III, Q.No. 16)**
7. Find the radius of curvature at the origin of the curve $x^3 - 2x^2y + 3xy^2 - 4y^2 + 5x^2 - 6xy + 7y^2 - 8y = 0$ **(Unit - III, Q.No. 39)**
8. Show that the chord of curvature through the pole of curve $r^2 \cos 2q = a^2$. **(Unit - III, Q.No. 61)**
9. Find the surface of the solid formed by revolving the cardioid $r = a(1 + \cos q)$ about the initial line. **(Unit - IV, Q.No. 37)**
10. Show that the voulme of the solod obtained by revolving the area included between the curves $y^2 = x^3$ and $x^2 = y^3$ and $x^2 = y^2$ about x-axis is $\frac{5\pi}{28}$. **(Unit - IV, Q.No. 23)**

11. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. **(Unit - I, Q.No. 18)**

12. A curve is given by the equations $x = a(\cos\theta + \theta \sin\theta)$, $y = a(\sin\theta - \theta \cos\theta)$. Find the length of the arc from $\theta = 0$ to $\theta = \alpha$. **(Unit - IV, Q.No. 9)**

## PART – B (4 × 12 = 48 Marks)

### [Essay Answer Type]

**Note : Answer any Four of the following questions**

13. (a) If $z = f(x, y)$ is homogenous function of x, y of degree,

then $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)\,z$ **(Unit - I, Q.No. 29)**

**(OR)**

(b) If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$; Show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y}$

$+ y^2\frac{\partial^2 u}{\partial y^2}$ $(1 - 4\sin^2 u)\sin 2u$. **(Unit - I, Q.No. 35)**



14. (a) Discuss the maxima and minima of the function U = sinx Sin y Sin z. where x, y, z are the angle of triangle. **(Unit - II, Q.No. 71)**

**(OR)**

(b) In a plane triangle find the maximum value of $u = \cos A \cos B \cos C$. **(Unit - II, Q.No. 63)**

15. (a) Find the coordinates of the centre of curvature of the parabola $y^2 = 4ax$. **(Unit - III, Q.No. 47)**

**(OR)**

(b) Show that the evolute of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ is **(Unit - III, Q.No. 53)**

16. (a) Find the perimeter of the loop of the curve $9ay^2 = (x - 2a)(x - 5a)^2$. **(Unit - IV, Q.No. 8)**

**(OR)**

(b) Find the volume of the solid obtained by revolving one arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about x -axis. **(Unit - IV, Q.No. 26)**

# FACULTY OF SCIENCE
## B.Sc. I - Semester (CBCS) Examination
## DIFFERENTIAL AND INTEGRAL CALCULUS
## MODEL PAPER - III

Time : 3 Hours ] [Max. Marks : 80

**PART – A (8 × 4 = 32 Marks)**
**[Short Answer Type]**

**Note : Answer any Eight of the following questions**

**ANSWERS**

1. If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 - \tan^{-1}\frac{x}{y}$ ; $xy \neq 0$. prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$$ **(Unit - I, Q.No. 4)**

2. If $V = At^{-\frac{1}{2}} e^{-x^2/4a^2t}$ prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$ **(Unit - I, Q.No. 10)**
3. Obtain Taylor's formula for the function $e^{x+y}$ at (0, 0) for n = 3 **(Unit - II, Q.No. 39)**
4. Find the maxima and minima of $x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. **(Unit - II, Q.No. 67)**
5. Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is another cycloid. **(Unit - III, Q.No. 64)**
6. Find the Radius of curvature art any point on the curve $y = c\cosh\frac{x}{c}$ (Catenary). **(Unit - III, Q.No. 13)**
7. Find the radius of curvature at the origin for the curve $x^3 + y^3 - 2x^2 + 6y = 0$. **(Unit - III, Q.No. 37)**
8. Find the length of the curve $x = e^{\theta}\sin\theta$, $y = e^{\theta}\cos\theta$ **(Unit - IV, Q.No. 18)**
9. Show that the volume of the solid obtained by revolving about x-axis the area enclosed by the parabola $y^2 = 4ax$ & its evolute $27ay^2 = 4(x - 2a)^3$ is $80\pi a^3$ **(Unit - IV, Q.No. 29)**
10. Find the volume of the spindle shaped solid generated by revolving the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis. **(Unit - IV, Q.No. 21)**
11. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$ show that $u_x + u_y + u_z = 0$ **(Unit - I, Q.No. 15)**
12. If $z = xy\, f\left(\frac{y}{x}\right)$ and z is constant. Then show that $\frac{f'(y/x)}{f(y/x)} = \frac{x(y + xy')}{y(y - xy')}$. **(Unit - II, Q.No. 14)**

## PART – B (4 × 12 = 48 Marks)

### [Essay Answer Type]

**Note : Answer any Four of the following questions**

13. (a) If $U = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, $x \neq y$ Show that

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$ **(Unit - I, Q.No. 47)**

**(OR)**

(b) If $U = Ze^{ax+by}$, where z is a homogenous function in x and y of degree n, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = (ax+by+n)u.$ **(Unit - I, Q.No. 50)**

14. (a) State and prove Taylor's theorem **(Unit - II, Q.No. 31)**

**(OR)**

(b) Obtain Taylor's formula for f(x, y) = cos (x + y) ; n = 3 at (0, 0) **(Unit - II, Q.No. 34)**

15. (a) In the Curve $r^m = a^m \cos m\theta$. Prove that $\dfrac{ds}{d\theta} = a\sec^{\frac{m-1}{m}} m\theta$

and $a^{2m}\dfrac{d^2}{ds^2} + mr^{2m-1} = 0$ **(Unit - III, Q.No. 9)**

**(OR)**

(b) Find the radius of curvature of $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the points where the line y = x cuts it, **(Unit - III, Q.No. 32)**

16. (a) Prove that the volume of the reel formed by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the x-axis is $\pi^2 a^3$ **(Unit - IV, Q.No. 24)**

**(OR)**

(b) The smaller segment of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ cut off by the chord $\dfrac{x}{a} + \dfrac{y}{b} = 1$ revolves completely about this chord. Show that the volume generated is $\dfrac{\pi}{6}(10-3\pi)a^2b^2\left(a^2+b^2\right)^{-1/2}$ **(Unit - IV, Q.No. 32)**

FACULTY OF SCIENCE

**B.Sc. I - Semester (CBCS) Examination**

**OCTOBER / NOVEMBER -2020**

DIFFERENTIAL AND INTEGRAL CALCULUS

Time : 2 Hours ] [Max. Marks : 80

**PART – A (5 × 4 = 20 Marks)**

**Note : Answer any Five questions**

1. If $f(x,y) = \frac{2y}{y+\cos x}$ then evaluate $\frac{\partial^2 f}{\partial x \partial y}$.

2. If $f(x,y) = \log(x^2 + xy + y^2)$ then evaluate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

3. If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

4. If $z = \frac{\cos y}{x}$ and $x = u^2 - v$, $y = e^y$ the evaluate $\frac{\partial z}{\partial t}$

5. If $z = x^2 + y^3$, $x = at^3$, $y = 2at^2$ then evaluate $\frac{\partial z}{\partial t}$

6. Expand $f(x,y) = 6x^3 y^2$ as Taylor's series interms of $(x - 1)$ and $(y - 1)$.

7. Find the radius of curvature of the curve $y = c \cosh \frac{x}{c}$ at any point $P_{(x,y)}$

8. Find the envelope of the family of straight lines $y = mx + \frac{1}{m}$

9. Using Newton's method, find the radius of curvature for the curve $y = x^4 - 4x^3 - 18x^2$ at O (0,0).

10. Find the length of the curve $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$.

11. Find the length of the arc of the curve $x = t^2$, cot, $y = t^2 \sin t$.

12. Find the volume of the solid. Obtained by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ about X – axis.

**PART – B (3 × 20 = 60 Marks)**

**Note : Answer three the questions**

13. If u = f(r), r = $\sqrt{x^2 + y^2}$ then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$

14. If z (x + y) = $x^2 + y^2$ then show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

15. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition x + 2y – 4z = 5

16. Find the maximum and minimum values of f(x,y) = $x^4 + 2x^2 y - x^2 + 3y^2$

17. Find the evolute of the ellipse x = a cos θ, y = b sin θ.

18. Find the envelope of the family of parabolas $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$ where ab = $c^2$

19. Prove that the length of the arc of the curve x = a sin 2θ (1 + cos 2θ) y = a cos 2θ (1 – cos 2θ) means used from O(0,0) to P(x,y) is $\frac{4}{3}$ a sin 3θ

20. Find the volume of the solid. Obtained by revolving the cardiod r = a( 1+ cosθ) about the initial line.

FACULTY OF SCIENCE

**B.Sc. I - Semester (CBCS) Examination**

**NOVEMBER / DECEMBER - 2019**

DIFFERENTIAL AND INTEGRAL CALCULUS

Time : 3 Hours ] [Max. Marks : 80

**PART – A (8 × 4 = 32 Marks)**

**[Short Answer Type]**

**Note : Answer any Eight of the following questions**

1. If f(x,y) = y cos xy then evaluate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

2. If f(x,y) = $\frac{xy}{x^2+y^2}$ then evaluate $\frac{\partial^2 f}{\partial x \partial y}$

3. If u = $\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ = tan u.

4. If H = f(y –z, z –x, x – y) then show that $\frac{\partial H}{\partial x}+\frac{\partial H}{\partial y}+\frac{\partial H}{\partial z} = 0$

5. If $z = x^2 + y^2$, $x = at^2$, $y^2 = 2at$ then evaluate $\frac{dz}{dt}$

6. Expand f(x,y) = $x^2 + 2xy - ye$ as a Taylor's series in powers of (x –1) and (y–2).

7. Find the radius of curvature for the curve $y=\frac{30}{x}$ at P(3,10).

8. Find the envelope of the family of curves $y = mx + am^3$.

9. Using Newton's method, find the radius of curvature for the curve $x^3 + y^3 - 2x^2 + 6y = 0$ at the origin O(0,0).

10. Find the length of the curve $y = x^{3/2}$ from x = 0 to x = 4.

11. Find the length of the curve $x = e^{\theta}\sin\theta,\ y = e^{\theta}\cos\theta$ from $\theta=0$ to $\theta=\frac{\pi}{2}$

12. Find the volume of the region generated by revolving the curve y = cosx, y = 0 from x = 0 to $x = \frac{\pi}{2}$ about x – axis.

**PART – B (4 × 12 = 48 Marks)**
**[Essay Answer Type]**

**Note : Answer ALL the questions**

13. (a) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, $(x \neq y)$ then show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u$

OR

(b) If $u(x,y) = \frac{y^3 - x^3}{y^2 + x^2}$ then using Euler's theorem show that, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

14. (a) If f(x,y) possesses continuous second order partial derivatives $f_{xy}$ and $f_{yx}$ then show that $f_{xy}$ and $f_{yx}$.

OR

(b) Show that the minimum value of $u(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$ is $3a^2$.

15. (a) Find the evolute of the hyperbola $2xy = a^2$

OR

(b) Find the envelope of the curve $\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = !$ where $a^n + b^n = c^n$

16. (a) Show that the length of the curve $x^2 = a^2(1 - e^{y/a})$ measured from O(0,0) to P(x,y) is $a\log\left(\frac{a+x}{a-x}\right) - x$

OR

(b) Find the volume of the solid obtained by revolving one arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about X – axis.