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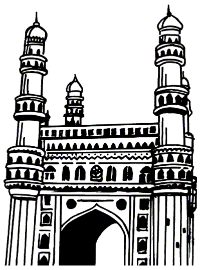

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(MATHEMATICS)

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UNIT I	**Errors in Numerical Calculations - Solutions of Equations in One Variable:** The Bisection Method - The Iteration Method - The Method of False Position- Newton's Method - Muller's Method - Solution of Systems of Nonlinear Equations.

1.1 Errors in Numerical Calcualtions

Q1. Define the types of errors.

Sol :

(i) Inherent Errors : Inherent errors are those that are present in the given data or the errors arise due to the limitations of the computing aids, mathematical tables, calculators or the digital computer.

(ii) Rounding Errors : Rounding errors arise from the process of rounding off the numbers during the calculations.

Rounding errors can be reduced by retaining atleast one or more significant figure at each step than that given in the data and rounding off at the last step.

(iii) Truncation Errors : These are the errors caused by using approximate formulae in calculations. Such as the one that arises when a function f(x) is calculated from an infinite seires for x after truncating it at a certain stage.

(iv) Absolute, Relative and Percentage Errors :

(a) Absolute error is the numerical difference between the true value of a quantity and its approximate value.

If X is the true value of a quantity X_1 is the its apprxomate value, then the absolute error is given by

$$E_A = X - X_1 = \delta X$$

(b) The relative errors E_R is defined by

$$E_R = \frac{E_A}{X} = \frac{X - X_1}{X} = \frac{\delta X}{X}$$

(c) The percentage error E_P is defined by

$$E_P = 100 \ E_R = \frac{X - X_1}{X} \times 100$$

Q2. Derive general error formulae.

Sol :

We derive a general formula for the error commited in using a functional relation

Let $u = f(x_1, x_2, ... x_n)$ be a function of several variables

$x_1, x_2, ... x_n$ ..... (1)

If $\Delta x_1, \Delta x_2, ... \Delta x_n$ be te the corresponding errors in $x_1, x_2, ... x_n$ respectively ten the error Δu in u is given by

$$u + \Delta u = f(x_1 + \Delta x_1, x_2 + \Delta x_2, ... x_n + \Delta x_n) \quad ..... (2)$$

Expanding the right side by Taylor's series we get

$$u + \Delta u = f(x_1, x_2, ... x_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i + \text{terms involving higher process of}$$

$$x_i \ (i = 1, 2, ... n) \quad ..... (3)$$

Assuming that the errors in x_i are so small that their squares and higher powers can be neglected.

Then equation (3) gives,

$$\Delta_u \approx \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i$$

$$= \frac{\partial f}{\partial x_i} \Delta x_i + \frac{\partial f}{\partial x_2} \Delta x_2 + ... + \frac{\partial f}{\partial x_n} \Delta x_n$$

The relative error in u is,

$$E_R = \frac{\Delta u}{u} = \frac{\partial u}{\partial x_1} \cdot \frac{\Delta x_1}{u} + \frac{\partial u}{\partial x_2} \cdot \frac{\Delta x_2}{u} + ... + \frac{\partial u}{\partial x_n} \cdot \frac{\Delta x_n}{u}$$

Q3. Fine the relative error if $\frac{2}{3}$ is approximate the 0.667.

Sol :

True value $X = \frac{2}{3}$

Approximate value $X_1 = 0.667$

Hence, absolute error $E_A = X - X_1 = \frac{2}{3} - 0.667$

Relative error, $E_R = \frac{E_A}{X} = \frac{\frac{2}{3} - 0.667}{\frac{2}{3}} = 0.0005$

Q4. Find the percentage error if 625.483 is approximated to three significant figures.

Sol :

True value X = 625.483

Approximate value X = 625

Hence, absolute error = $\left|\frac{X - X_1}{X}\right| \times 100$

$= \left|\frac{625.483 - 625}{625.483}\right| \times 100$

$= 0.007$

Q5. Round off the following numbers to four significant figures.

(i) 38.46235

(ii) 0.70029

(iii) 0.0022218

(iv) 2.36425

Sol :

Rounding off the number to four significant figures is,

(i) 38.406

(ii) 0.7003

(iii) 0.002222

(iv) 2.364

Q6. An approximate value of π is given by x_1 = 3.1428571 and its true value is x = 3.1415926. Find the absolute and relative errors.

Sol :

Given

$x_1 = 3.1428571$ and $x = 3.1415926$

$\therefore$ Absolute errors $E_A = |x_1 - x|$

$= |3.1428571 - 3.1415926|$

$= 0.0012645$

Relative error $E_R = \frac{E_A}{x} = \frac{0.0012645}{3.1415926}$

$= 0.0004025$

Q7. When the numbers x = 4.488 and y = 1.321 are rounded to two decimal places, find the value of $q = \frac{x}{y}$ and E_R (q)

Sol :

Given,

$x = 4.488$ and $y = 1.321$

$q = \frac{x}{y} = \frac{4.488}{1.321} = 3.397426 = X$ (true value)

If x and y are rounded to two decimals, then

$x = 4.49, \; y = 1.32$

$q = \frac{x}{y} = \frac{4.49}{1.32} = 3.40 = X_1$ (approximate value)

Absolute error $E_A(q) = |X - X_1| = |3.397426 - 3.4|$

$= 0.002574$

Relative error $E_R(q) = \frac{E_A}{X} = \frac{0.002574}{3.397426} = 0.000758$

---

8. If $y = \left[\frac{0.31x + 2.73}{x + 0.35}\right]$ where the co-efficient are rounded off, find the relative error in y when $x = 0.5 \pm 0.1$

Sol :

Given

$$y = \frac{0.31x + 2.73}{x + 0.35}$$

and $x = 0.5 \pm 0.1$

$\Rightarrow \; x = 0.6$ or $x = 0.4$

Ley Y be the true value of y and Y_1 be the approximate value of y.

If $x = 0.6$, $Y = \frac{(0.31)(0.6) + 2.73}{0.6 + 0.35} = \frac{2.916}{0.95} = 3.0694736$

If the co-efficient are rounded off, then

$$Y_1 = \frac{(0.3)(0.6) + 2.7}{0.6 + 0.4} = \frac{2.88}{1} = 2.88$$

Absolute error $E_A = |Y - Y_1| = |3.0694736 - 2.88| = 0.1894736 \cong 0.19$

Relative Error $E_R = \frac{E_A}{y} = \frac{0.19}{3.0694736} = 0.0618 \cong 0.06$

If $x = 0.4$, $Y = \frac{(0.31)(0.4) + 2.93}{0.4 + 0.35}$

$Y = 3.8053333$

If the co-efficients are round off their

$$Y_1 = \frac{(0.3)(0.4) + 2.7}{0.4 + 0.4} = 3.52$$

Absolute error $E_A = |Y - Y_1| = |3.8053333 - 3.52|$

$= 0.2853333 = 0.29$

Hence relative error $E_R = \frac{E_A}{Y} = \frac{0.29}{3.8053333} = 0.0762 \cong 0.08$

Q9. If $u = 3v^7 - 6v$ find the percentage error in u at v = 1 of the error in v is 0.05.

Sol :

Given $u = 3v^7 - 6v$

Error in v is $\delta v = -0.05$

$\therefore \quad \frac{\partial u}{\partial v} = 21v^6 - 6$

$\Rightarrow \quad \partial u = (21v^6 - 6)\, dv$ ... (1)

At0 $v = 1, \; u = 3(1)^7 - 6(1) = 3 - 6 = -3.$

$\partial u = (21(1)^6 - 6)\,(0.05) = 0.75$

$\therefore$ Percentage error in u $= E_P = \left|\frac{\partial u}{u}\right| \times 100$

$= \left|\frac{0.75}{-3}\right| \times 100 = 25$

Q10. Find the relative and percentage error in $u = 6v^5 - 3v^4$ at $v = 1.5 \pm 0.0025$.

Sol :

Given

$u = 6v^5 - 3v^4$

Error in v, $\partial v = 0.0025$

$\therefore \quad \frac{\partial u}{\partial v} = 30v^4 - 12v^3$

$\partial u = (30v^4 - 12v^3)\; \partial v$ ..... (1)

Given

$v = 1.5 \pm 0.0025$

$\Rightarrow \quad v = 1.5025, \; 1.4975$

(i) At $v = 1.5025$

Now $u = 6v^5 - 3v^4 = 6(1.5025)^5 - 3\,(1.5025)^4 = 30.6545$

and $\partial u = [30.(1.5025)^4 - 12(1.5025)^3]\;(0.0025)$

$= (152.8900 - 40.7028)\;(0.0025)$

$\partial u = 0.2805$

$\therefore$ Relative error $E_R = \left|\frac{\partial u}{u}\right| = \left|\frac{0.2805}{30.6545}\right| = 0.0092$

and percentage error in u, $E_P = \left|\frac{\partial u}{u}\right| \times 100 = 0.92$

(ii) At $v = 1.4975$

Now $u = 6v^5 - 3v^4 = 6(1.4975)^5 - 3(1.4975)^4 = 30.0976$

and $\partial u = \{30(1.4975)^4 - 12\ (1.4975)^3]\ (0.0025)$

$\partial u = 0.2764$

Relative error $E_R = \left|\frac{\partial u}{u}\right| = \frac{0.2764}{30.0976} = 0.0092$

and percentage error in u, $E_P = \left|\frac{\partial u}{u}\right| \times 100 = 0.92$

---

11. Find the relative error in u = $\frac{5xy^2}{z^3}$ with $\partial x = \partial y = \partial z = 0.001$ and x = y = z = 1.

Sol :

Given

$$u = \frac{5xy^2}{z^3} \qquad \text{..... (1)}$$

Differentiating partially w.r.t x, y, z

$$\frac{\partial u}{\partial x} = \frac{5y^2}{z^3},\ \frac{\partial u}{\partial y} = \frac{10xy}{z^3}\ ;\ \frac{\partial u}{\partial z} = \frac{-15xy^2}{z^4}$$

$$\therefore \quad \partial u = \frac{\partial u}{\partial x}\partial x + \frac{\partial u}{\partial y}\partial y + \frac{\partial u}{\partial z}\partial z$$

$$= \frac{5y^2}{z^3}\partial x + \frac{10xy}{z^3}\partial y + \frac{15xy^2}{z^4}\partial z$$

$$\partial u = \left|\frac{5y^2}{z^3}\partial x\right| + \left|\frac{10xy}{z^3}\partial y\right| + \left|\frac{15xy^2}{z^4}\partial z\right|$$

If we take $\partial x = \partial y = \partial z = 0.001$ and $x = y = z = 1$

Then,

$\partial u = 5(0.001) + 10(0.001) + 15(0.001)$

$\partial u = 0.03$ and $u = 5$

$\therefore$ Relative error $E_R = \frac{\partial u}{u} = \frac{0.03}{5} = 0.006.$

1.2 Solutions of Equations in One Variable

1.2.1 The Bisection Method

Q12. Write the procedure of Bisection Method.

Ans :

Step 1:

Choose two real numbers a and b

Such that f(a) . f(b) < 0

Step 2:

Set $x_r = \frac{a+b}{2}$

Step 3:

(a) If f(a). f(x_r) < 0, the root lies between(a, x_r)

Then, set b = x_r set b = x_r and go to step 2 above.

(b) If f(a). f(x_r) > 0, then root lies between(x_r, b)

Then, set a = x_r and go to step 2.

(c) If f(a) f(x_r) = 0 it means thet x_r is a root of the equations f(x) = 0 and the computation may be terminated.

Q13. Find a real root of the equation f(x) = $x^3 - x - 1 = 0$ by using bisection method.

Ans :

Given equation is $f(x) = x^3 - x - 1 = 0$

$f(1) = (1)^3 - 1 - 1 = -1 < 0$

$f(2) = (2)^3 - 2 - 1 = 5 > 0.$

$\therefore$ f(1) < 0 and f(2) > 0 then the root lies between 1 and 2

$\therefore$ Let a = 1, b = 2.

$$x_0 = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$\Rightarrow$ $f(x_0) = f(1.5) = (1.5)^3 - 1.5 - 1 = 0.8750 > 0.$

Since f(1.5) > 0 and f(1) < 0

Then the root lies between 1 and 1.5

$$x_1 = \frac{1+1.5}{2} = \frac{2.5}{2} = 1.25$$

$\Rightarrow$ $f(x_1) = f(1.25) = (1.25)^3 - 1.25 - 1 = -0.2969 < 0$

Since f(1.25) < 0 and f(1.5) > 0

Then the root lies between 1.5 and 1.25

$$x_2 = \frac{1.25 + 1.5}{2} = 1.375$$

$\Rightarrow$ $f(x_2) = f(1.375) = (1.375)^3 - 1.375 - 1 = 0.2246 > 0.$

Since $f(1.375) > 0$ and $f(1.25) < 0.$

Then the root lies between 1.25 and 1.375.

$$x_3 = \frac{1.25 + 1.375}{2} = 1.3125$$

$\Rightarrow$ $f(x_3) = f(1.3125) = -0.0515 < 0.$

Since $f(1.3125) < 0$ and $f(1.375) > 0.$

Then the root lies between 1.375 and 1.3125

$$x_4 = \frac{1.375 + 1.3125}{2} = 1.3438.$$

$\Rightarrow$ $f(x_4) = f(1.3438) = 0.0828 > 0.$

Since $f(1.3438) > 0$ and $f(1.3125) < 0$

Then the root lies between 1.3438 & 1.3125

$$x_5 = \frac{1.3438 + 1.3125}{2} = 1.3282.$$

$\Rightarrow$ $f(x_5) = f(1.3238) = 0.0149 > 0.$

Since $f(1.3282) > 0$ and $f(1.3125) < 0$

Then the root lies between 1.3125 and 1.3282

$$x_6 = \frac{1.3282 + 1.3125}{2} = 1.3204.$$

$\Rightarrow$ $f(x_6) = f(1.3204) = -0.0183 < 0$

Since $f(1.3204) < 0$ and $f(1.3282) > 0$

then the root lies between 1.3204 and 1.3282

$$x_7 = \frac{1.3204 + 1.3282}{2} = 1.3243.$$

$\Rightarrow$ $f(x_7) = f(1.3243) = -0.0018 < 0$

Since $f(1.3243) < 0$ and $f(1.3282) > 0$

Then the root lies between 1.3243 and 1.3282

$$x_8 = \frac{1.3243 + 1.3282}{2} = 1.3263$$

$\Rightarrow$ $f(x_8) = f(1.3263) = 0.0068 > 0$

Since f(1.3263) > 0 and f(1.3243) < 0 then the root lies

Between 1.3263 and 1.3243.

$$x_9 = \frac{1.3263+1.3243}{2} = 1.3253$$

$\Rightarrow$ $f(x_9) = f(1.3253) = 0.0025 > 0.$

Since f(1.3253) > 0 and f(1.3243) < 0 then the root lies

Between 1.3253 and 1.3243

$$x_{10} = \frac{1.3253+1.3243}{2} = 1.3248.$$

$\Rightarrow$ $f(x_{10}) = f(1.3248) = 0.0003 > 0.$

Since f(1.3248) > 0 and f(1.3243) < 0 then the root lies between 1.3248 and 1.3243

$$x_{11} = \frac{1.3248+1.3243}{2} = 1.3246$$

$\Rightarrow$ $f(x_{11}) = f(1.3246) = -0.0005 < 0.$

Since f(1.3246) < 0 and f(1.3248) > 0 then the root lies between 1.3246 and 1.3248

$$x_{12} = \frac{1.3246+1.3248}{2} = 1.3247$$

$\Rightarrow$ $f(x_{12}) = f(1.3247) = -0.0001 < 0.$

Since f(1.3247) < 0 and f(1.3248) > 0 then the root lies between 1.3247 and 1.3248.

$$x_{13} = \frac{1.3247+1.3248}{2} = 1.3248$$

We find

$$x_{13} - x_{12} = 1.3248 - 1.3247 = 0.0001$$

and $\left|\frac{x_{13}-x_{12}}{x_{13}}\right| \times 100 = \frac{0.0001}{1.3248} \times 100 = 0.01\%$

Hence a root, correct to three decinal placess is 1.324

Q14. Find a real root of the equation $x^3 - 2x - 5 = 0$ by using bisection method

Sol :

Given eqution $f(x) = x^3 - 2x - 5$

$f(1) = (1)^3 - 2(1) - 5 = -6 < 0$

$f(2) = (2)^3 - 2(2 - 5 = -1 < 0$

$f(3) = (3)^3 - 2(3) - 5 = 16 > 0.$

Since f(2) < 0 and f(3) > 0, then the root lies between 2 & 3

Let a = 2, b = 3.

$$x_1 = \frac{2+3}{2} = \frac{5}{2} = 2.5$$

$\Rightarrow$ $f(x_1) = f(2.5) = (2.5)^3 - 2(2.5) - 5 = 5.6250 > 0.$

Since f(2.5) > 0 and f(2) < 0 then the root lies between 2 and 2.5

$$x_2 = \frac{2+2.5}{2} = \frac{4.5}{2} = 2.25$$

$\Rightarrow$ $f(x_2) = f(2.25) = (2.25)^3 - 2(2.25) - 5 = 1.89063 > 0.$

Since f(2.25) > 0 and f(2) < 0 then the root lies between 2 and 2.25

$$x_3 = \frac{2+2.25}{2} = \frac{4.25}{2} = 2.125$$

$\Rightarrow$ $f(x_3) = f(2.125) = (2.125)^3 - 2(2.125) - 5 = 0.3457 > 0.$

Since f(2.125) > 0 and f(2) < 0 then the root lies between 2 and 2.125

$$x_4 = \frac{2+2.125}{2} = 2.0625$$

$\Rightarrow$ $f(x_4) = f(2.0625) = -0.35132$

Since f(2.0625) < 0 and f(2.125) > 0 then the root lies between 2.125 & 2.0625

$$x_5 = \frac{2.0625+2.125}{2} = 2.09375$$

$\Rightarrow$ $f(x_5) = f\,(2.09375) = (2.09375)^3 - 2(2.09375) - 5 = -\,000894 < 0$

Since f(2.09375) < 0 and f(2.125) > 0 then the root lies between 2.125 & 2.09375

$$x_6 = \frac{2.125+2.09375}{2} = 2.10938$$

$\Rightarrow$ $f(x_6) = f(2.10938) = 0.16689 > 0$

f(2.10938) > 0 and f(2.09375) < 0 then the root lies between 2.09375 and 2.10938.

$$x_7 = \frac{2.09375+2.10938}{2} = 2.10157$$

$\Rightarrow$ $f(x_7) = f(2.10157) = 0.07865 > 0.$

Since f(2.10157) > 0 and f(2.09375) < 0 then the root lies between 2.10157 and 2.09375.

$$x_8 = \frac{2.10157+2.09375}{2} = 2.09766$$

$\Rightarrow$ $f(x_8) = f(2.09766) = 0.03476 > 0.$

Since f(2.09766) > 0 and f(2.09375) < 0 then the root lies between 2.09766 and 2.09375

$$x_9 = \frac{2.09766 + 2.09375}{2} = 2.09571$$

$\Rightarrow$ $f(x_9) = f(2.09571) = 0.01294 > 0.$

Since f(2.09571) > 0 and f(2.09375) < 0 then the root lies between 2.09571 and 2.09375.

$$x_{10} = \frac{2.09571 + 2.09375}{2} = 2.09473$$

$\Rightarrow$ $f(x_{10}) = f(2.09473) = 0.00199 > 0.$

Since f(2.09473) > 0 and f(2.09375) < 0 then the root lies between 2.09473 and 2.09375.

$$x_{11} = \frac{2.09473 + 2.09375}{2} = 2.09424 \simeq 2.094$$

$f(x_{11}) = f(2.09424) = -0.00348 < 0.$

Since f(2.09424) < 0 and f(2.09473) > 0 then the root lies between 2.09424 and 2.09473.

$$x_{12} = \frac{2.09424 + 2.09473}{2} = 2.09449 \simeq 2.094$$

Hence, a root correct to three decimal places 2.094.

Q15. Find a real root of $f(x) = x^3 + x^2 + x + 7 = 0$ correct to three decimal places.

Sol :

The given equation is a cubic and the last term is positive.

Hence f(x) = 0 will have a negatine real root.

$f(-1) = (-1)^3 + (-1)^2 + (-1) + 7 = 6 > 0$

$f(-2) = (-2)^3 + (-2)^2 + (-2) + 7 = 1 > 0.$

$f(-3) = (-3)^3 + (-3)^2 + (-3) + 7 = -14 < 0.$

Since f(–2) > 0 and f(–3) < 0 then the root lies between –2 and –3.

$$x_1 = \frac{-2-3}{2} = -2.5$$

$f(x_1) = f(-2.5) = (-2.5)^3 + (-2.5)^2 + (-2.5) + 7$

$= -4.8750 < 0$

Since f(–2.5) < 0 and f(–2) > 0 then the root lies between –2 and –2.5

$$x_2 = \frac{-2 + (-2.5)}{2} = -2.25$$

$f(x_2) = f(-2.25) = (-2.25)^3 + (-2.25)^2 + (-2.25) + 7$

$= -1.5781 < 0.$

Since f(–2.25) < 0 and f(–2) > 0 then the root lies between –2 and –2.25.

$$x_3 = \frac{-2-2.25}{2} = -2.125$$

$f(x_3) = f(-2.125) = -0.2051 < 0$

Since f(–2.125) < 0 and f(–2) > 0 then the root lies between –2 and –2.125.

$$x_4 = \frac{-2-2.125}{2} = -2.0625$$

$f(x_4) = f(-2.0625) = 0.4177 > 0.$

Since f(–2.0625) > 0. and f(–2.125) < 0 then the root lies between –2.0625 and –2.125

$$x_5 = \frac{-2.0625-2.125}{2} = -2.0938$$

$f(x_5) = f(-2.0938) = 0.1116 > 0.$

Since f(–2.0938) > 0 and f(–2.125) < 0 then the root lies between –2.0938 and -2.125

$$x_6 = \frac{-2.0938-2.125}{2} = -2.1094$$

$f(x_6) = f(-2.1094) = -11.9076 < 0$

Since f(–2.1094) < 0 and f(–2.0938) > 0 then the root lies between –2.1094 and –2.0938.

$$x_7 = \frac{-2.1094-2.0938}{2} = -2.1016$$

$f(x_7) = f(-2.1016) = 0.0329 > 0$

Since f(-2.1016) > 0 and f(-2.1094) < 0 then the root lies between –2.1016 and –2.1094.

$$x_8 = \frac{-2.1016-2.1094}{2} = -2.1055$$

$f(x_8) = f(-2.1055) = -0.0063 < 0$

Since f(–2.1055) < 0 and f(–2.1016) > 0 then the root lies between –2.1055 and –2.1016.

$$x_9 = \frac{-2.1055-2.1016}{2} = -2.1036$$

$f(x_9) = f(-2.1036) = 0.0135 > 0.$

Since f(–2.1036) > 0.and f(–2.1055) < 0. then the root lies between –2.1036 and –2.1055.

$$x_{10} = \frac{-2.1036-2.1055}{2} = -2.1046 \simeq -2.105$$

$f(x_{10}) = f(-2.1046) = 0.0027 > 0.$

Since f(–2.1046) > 0 and f(–2.1055) < 0 then the root lies between –2.1046 and –2.1055.

$$x_{11} = \frac{-2.1046 - 2.1055}{2} = -2.105 \mid \simeq -2.105$$

$\therefore$ x_{10} & x_{11} are same upto three decimal places

Hence the root of given equation $\boxed{x = -2.105}$

Q16. Find a root, correct to three decimal places and lying between 0 and 0.5 of the equation $4e^{-x} \sin x - 1 = 0$

Sol :

Let $f(x) = 4e^{-x} \sin x - 1$

We have f(0) = –1 and f(0.5) = 0.163145

Since f(0) = –1 < 0

f(0.5) = 0.163145 > 0

Then the root lies between 0 and 0.5

$$x_1 = \frac{0 + 0.5}{2} = 0.25$$

$f(x_1) = f(0.25) = 4e^{-0.25} \sin(0.25) - 1 = -0.2293 < 0$

Since f(0.25) < 0. and f(0.5) > 0

then the root lies between 0.25 and 0.5

$$x_2 = \frac{0.25 + 0.5}{2} = 0.375$$

$f(x_2) = f(0.375) = 4e^{0.375} \sin(0.375) - 1 = 1.1317 > 0$

Since f(0.375) > 0 and f(0.25) < 0 then the root lies between

0.375 and 0.25

$$x_3 = \frac{0.375 + 0.25}{2} = 0.3125$$

$f(x_3) = f(0.3125) = 0.6809 > 0.$

Since f(0.3125) > 0.and f(0.25) < 0 then the root lies between

0.3125 and 0.25

$$x_4 = \frac{0.3125 + 0.25}{2} = 0.2813$$

$f(x_4) = f(0.2813) = 0.4711 > 0$

Since f(0.2813) > 0 and f(0.25) < 0 then the root lies between

0.2813 and 0.25

$$x_5 = \frac{0.2813 + 0.25}{2} = 0.2657$$

$f(x_5) = f(0.2657) = 0.3700 > 0.$

Since f(0.2657) > 0 and f(0.25) < 0 then the root lies between

0.2657 and 0.25

$$x_6 = \frac{0.2657 + 0.25}{2} = 0.2579$$

$f(x_6) = f(0.2579) = 0.3204 > 0$

Since f(0.2579) > 0 and f(0.25) < 0 then the root lies between

0.2579 and 0.25

$$x_7 = \frac{0.2579 + 0.25}{2} = 0.2540$$

$f(x_7) = f(0.2540) = 0.2958 > 0.$

Since f(0.2540) > 0 and f(0.25) < 0 then the root lies between

0.2540 and 0.25

$$x_8 = \frac{0.2540 + 0.25}{2} = 0.2520$$

$f(x_8) = f(0.2520) = 0.2832 > 0$

Since f(0.2520) > 0 and f(0.25) < 0 then the root lies between

0.2520 and 0.25

$$x_9 = \frac{0.2520 + 0.25}{2} = 0.2510$$

$f(x_9) = f(0.2510) = 0.2769 > 0$

Since f(0.2510) > 0 and f(0.25) < 0 then the root lies between

0.2510 and 0.25

$$x_{10} = \frac{0.2510 + 0.25}{2} = 0.2505 \simeq 0.251$$

Hence, x_9 and x_{10} are repeating same upto three decinal place.

The required root is 0.251

Q17. Obtain a root of equation $x^3 - 4x - 9 = 0$ correct to three decimal places by using bisection method.

Sol :

Let the given function $f(x) = x^3 - 4x - 9$

$f(1) = 1^3 - 4(1) - 9 = -12 < 0$

$f(2) = 2^3 - 4(2) - 9 = -9 < 0$

$f(3) = 3^3 - 4(3) - 9 = 6 > 0$

Since f(2) < 0 and f(3) > 0 then the root lies between 2 and 3

$x_1 = \frac{2+3}{2} = \frac{5}{2} = 2.5$

$f(x_1) = f(2.5) = (2.5)^3 - 4(2.5) - 9 = -3.3750 < 0$

Since f(2.5) < 0 and f(3) > 0 then the root lies between

2.5 and 3.

$x_2 = \frac{2.5+3}{2} = 2.75$

$f(x_2) = f(2.75) = 0.7969 > 0$

Since f(2.75) > 0 and f(2.5) < 0 then the root lies between

2.75 and 2.5

$x_3 = \frac{2.75+2.5}{2} = 2.625$

$f(x_3) = f(2.625) = -1.4121 < 0$

Since f(2.625) < 0 and f(2.75) > 0 then the root lies between

2.625 and 2.75

$x_4 = \frac{2.625+2.75}{2} = 2.6875$

$f(x_4) = f(2.6875) = -0.3391 < 0$

Since f(2.6875) < 0 and f(2.75) > 0 then the root lies between

2.6875 and 2.75

$x_5 = \frac{2.6875+2.75}{2} = 2.7188$

$f(x_5) = f(2.7188) = 0.2207 > 0$

Since f(2.7188) > 0 and f(2.6875) < 0 then the root lies between

2.7188 and 2.6875

$x_6 = \frac{2.7188+2.6875}{2} = 2.7032$

$f(x_6) = f(2.7032) = -0.0608 < 0$

Since f(2.7032) < 0 and f(2.7188) > 0 then the root lies between

2.7032 and 2.7188

$x_7 = \frac{2.7032+2.7188}{2} = 2.7110$

$f(x_7) = f(2.7110) = 0.0806 > 0$

Since f(2.7110) > 0 and f(2.7032) < 0 then the root lies between

2.7110 and 2.7032

$$x_8 = \frac{2.7110 + 2.7032}{2} = 2.7071$$

$f(x_8) = f(2.7071) = 0.0130 > 0$

Since $f(2.7071) > 0$ and $f(2.7032) < 0$ then the root lies between

2.7071 and 2.7032

$$x_9 = \frac{2.7071 + 2.7032}{2} = 2.7052$$

$f(x_9) = f(2.7052) = -0.0250 < 0$

Since $f(2.7052) < 0$ and $f(2.7071) > 0$ then the root lies between

2.7052 and 2.7071

$$x_{10} = \frac{2.7052 + 2.7071}{2} = 2.7062$$

$f(x_{10}) = f(2.7062) = -0.0070 < 0$

Since $f(2.7062) < 0$ and $f(2.7071) > 0$ then the root lies between

2.7062 and 2.7071

$$x_{11} = \frac{2.7062 + 2.7071}{2} = 2.7067$$

$\therefore$ x_{10} & x_{11} are repeating same upto three decinal places

Hence root of the equation x = 2.706

Q18. Obtain a root of the equation $x^2 + x - \cos x = 0$ correct upto three decinal places by bisection method.

Sol :

Let the given function $f(x) = x^2 + x - \cos x$

$f(0) = 0^2 + 0 - \cos 0 = -1 < 0$

$f(1) = 1^2 + 1 - \cos 1 = 1.4597 > 0$

$f(0) < 0$ and $f(1) > 0$ then the root lies between 0 and 1

$$x_1 = \frac{0 + 1}{2} = 0.5$$

$f(x_1) = f(0.5) = -0.1276 < 0$

Since $f(0.5) < 0$ and $f(1) > 0$ then the root lies between

0.5 and 1.

$$x_2 = \frac{0.5 + 1}{2} = 0.75$$

$f(x_2) = f(0.75) = 0.5808 > 0$

Since f(0.75) > 0 and f(0.5) < 0 then the root lies between

0.5 and 0.75

$$x_3 = \frac{0.5+0.75}{2} = 0.625$$

$f(x_3) = f(0.625) = 0.2047 > 0$

Since f(0.625) > 0 and f(0.5) < 0 then the root lies between

0.625 and 0.5

$$x_4 = \frac{0.625+0.5}{2} = 0.5625$$

$f(x_4) = f(0.5625) = 0.0330 > 0$

Since f(0.5625) > 0 and f(0.5) < 0 then the root lies between

0.5625 and 0.5

$$x_5 = \frac{0.5625+0.5}{2} = 0.5313$$

$f(x_5) = f(0.5313) = -0.0486 < 0$

Since f(0.5313) < 0 and f(0.5625) > 0 then the root lies between

0.5313 and 0.5625

$$x_6 = \frac{0.5313+0.5625}{2} = 0.5469$$

$f(x_6) = f(0.5469) = -0.0081 < 0$

Since f(0.5469) < 0 and f(0.5625) > 0 then the root lies between

0.5469 and 0.5625

$$x_7 = \frac{0.5469+0.5625}{2} = 0.5547$$

$f(x_7) = f(0.5547) = 0.0123 > 0$

Since f(0.5547) > 0 and f(0.5469) < 0 then the root lies between

0.5547 and 0.5469

$$x_8 = \frac{0.5547+0.5469}{2} = 0.5508$$

$f(x_8) = f(0.5508) = +0.0021 > 0$

Since f(0.5508) > 0 and f(0.5469) < 0 then the root lies between

0.5508 and 0.5469

$$x_9 = \frac{0.5508+0.5469}{2} = 0.5489$$

$f(x_9) = f(0.5489) = -0.0030 < 0$

Since f(0.5489) < 0 and f(0.5508) > 0 then the root lies between

0.5489 and 0.5508

$$x_{10} = \frac{0.5489 + 0.5508}{2} = 0.5499$$

$f(x_{10}) = f(0.5499) = -0.0003 < 0$

Since f(0.5499) < 0 and f(0.5508) > 0 then the root lies between

0.5499 and 0.5508

$$x_{11} = \frac{0.5499 + 0.5508}{2} = 0.5504$$

$f(x_{11}) = f(0.5504) = -0.0222 < 0$

Since f(0.5504) < 0 and f(0.5508) > 0 then the root lies between

0.5504 and 0.5508

$$x_{12} = \frac{0.5504 + 0.5508}{2} = 0.5506$$

$\therefore$ x_{11} and x_{12} are repeating same upto three decimal places.

Hence, the root of the equation 0.550.

1.2.2 The Iteration Method

Q19. Write the steps Involved in Iteration Method.

Ans :

To decribe this method for finding a root of the equation f(x) = 0

Step 1 :

Find the Intial Interval for the given equation f(x) = 0

Let x_0 be the midpoinl of the Initial Interval.

Step 2 :

Convergent Condition :

- Convert the given equations in the form $x = \phi(x)$ and find first derivative of $\phi(x)$ with respect to 'x'. We get $\phi'(x)$.
- Substitute x_0 in $\phi^1(x)$

If $|\phi_1(x_0)| < 1$ then convergent condition is satisfies and this method is applicable.

Step 3 :

Let

$\phi(x) = x$

Substitute x_0 in above equation $\phi(x)$

i.e, $\phi(x_0) = x_1$

Step 4 :

Successive substitution give the approximation $\phi(x_1) = x_2$, $\phi(x_2) = x_3$ ..... $\phi(x_n) = x_{n+1}$

Q20. Find a real root, correct to three decimal places of the equation 2x – 3 = cos x in the Interval $\left[\frac{3}{2}, \frac{\pi}{2}\right]$ by whing Interation Method.

Sol :

Given equation 2x – 3 = cos x

Rewrile the given equation as,

$$x = \frac{1}{2}\left[\cos x + 3\right] \qquad ..... (1)$$

Equation (1) is in the form x = ϕ(x)

Where

$$\phi(x) = \frac{1}{2}\left[\cos x + 3\right]$$

Diff w.r to 'x'

$$\phi'(x) = \frac{1}{2}(-\sin x)$$

$$|\phi'(x)| = \frac{1}{2}|\sin x| < 1 \text{ in } \left[\frac{3}{2}, \frac{\pi}{2}\right]$$

$\therefore$ Convergent condition is satisfied

Choose $x_0 = \dfrac{\frac{3}{2}+\frac{\pi}{2}}{2} = 1.5354 \simeq 1.5$

Sub x_0 in $\phi(x)$, we get

$$x_1 = \phi(x_0) = \frac{1}{2}(\cos(1.5) + 3) = 1.5354$$

$$x_2 = \phi(x_1) = \frac{1}{2}(\cos(1.5354) + 3) = 1.5177$$

$$x_3 = \phi(x_2) = \frac{1}{2}(\cos(1.5177) + 3) = 1.5265$$

$$x_4 = \phi(x_3) = \frac{1}{2}(\cos(1.5265) + 3) = 1.5221$$

$$x_5 = \phi(x_4) = \frac{1}{2}(\cos(1.5221) + 3) = 1.5243$$

$$x_6 = \phi(x_5) = \frac{1}{2}(\cos(1.5243) + 3) = 1.5232$$

$$x_7 = \phi(x_6) = \frac{1}{2}(\cos(1.5232) + 3) = 1.5238$$

$\therefore$ x_6 and x_7 repeating same upto three decimal places

Hence, the root of equation x = 1.524.

Q21. Use the method of iteration to find a positive root of the equation $xe^x = 1$ given that a root lies between 0 and I.

Sol :

Given equation is $xe^x = 1$

We rewrite the equation as

$$x = \frac{1}{e^x} = e^{-x}$$

$$\therefore \quad x = e^{-x} \qquad \text{..... (1)}$$

Equation (1) is in the four $x = \phi(x)$

Where $\phi(x) = e^{-x}$

Given a root lies between 0 and I.

Choose $x_o = \frac{0+1}{2} = 0.5$

Since $\phi(x) = e^{-x}$

Diff w. r to 'x'

$$\phi'(x) = -e^{-x} \quad = -\frac{1}{e} \text{ for } x = 1$$

$\therefore$ $|\phi'(x)| < 1$

Convergent condition is satisfied

Now substitute $x_0 = 0.5$ in $\phi(x)$

We get,

$x_1 = \phi(x_0)$

$= \phi(0.5) = e^{-0.5} = 0.60653$

$x_2 = \phi(x_1)$

$= \phi(0.60653) = e^{0.60653} = 0.54524$

$x_3 = \phi(x_2)$

$= \phi(0.54524) = e^{0.54524} = 0.57970$

$x_4 = \phi(x_3)$

$= \phi(0.57970) = e^{0.57970} = 0.56007$

$x_5 = \phi(x_4)$

$= \phi(0.56007) = e^{0.56007} = 0.57117$

$x_6 = \phi(x_5)$

$= \phi(0.57117) = e^{0.57117} = 0.56486$

$x_7 = \phi(x_6)$

$= \phi(0.56486) = e^{0.56486} = 0.56844$

$x_8 = \phi(x_7)$

$= \phi(0.56844) = e^{0.56844} = 0.56641$

$x_9 = \phi(x_8)$

$= \phi(0.56641) = e^{0.56641} = 0.56756$

$x_{10} = \phi(x_9)$

$= \phi(0.56756) = e^{0.56756} = 0.56691$

$x_{11} = \phi(x_{10})$

$= \phi(0.56691) = e^{0.56691} = 0.56728$

$x_{12} = \phi(x_{11})$

$= \phi(0.56728) = e^{0.56728} = 0.56706$

$x_{13} = \phi(x_{12})$

$= \phi(0.56706) = e^{0.56706} = 0.56719$

$x_{14} = \phi(x_{13})$

$= \phi(0.56719) = e^{0.56719} = 0.56712$

x_{13} and x_{14} are repeating same upto 4 decimal places.

Hence, the root of the equation 0.5671.

Q22. Use the iteratine method to find a real root of the equation sinx = 10(x –1). Give your answer correct to three decimal places.

Sol :

Let $f(x) = \sin x - 10x + 10$.

We find from graph that a root lies between 1 and π.

Rewrite the given equation as,

$$x = 1 + \frac{\sin x}{10}$$

We have

$$\phi(x) = 1 + \frac{\sin x}{10}$$

$$|\phi'(x)| = \frac{1}{10}\cos x < 1 \text{ in } [1, \pi]$$

$\therefore$ Convergent condition is satisfied

Taking $x_0 = 1$, we obtaion succeessine iterates as,

$$x_1 = 1 + \frac{\sin 1}{10} = 1.0841$$

$$x_2 = 1 + \frac{\sin 1.0841}{10} = 1.0884$$

$$x_3 = 1 + \frac{\sin 1.0884}{10} = 1.0886$$

$$x_4 = 1 + \frac{\sin 1.0886}{10} = 1.0886$$

Hence, the required root is 1.089

Q23. Find a real root of $x = \frac{1}{(x+1)^2}$ by iteration method

Sol :

The given equation is of the form $x = \phi(x)$

where

$$\phi(x) = \frac{1}{(x+1)^2}$$

Given $f(x) = x(x+1)^2 - 1$

$f(0) = 0(0 + 1)^2 - 1 = -1 < 0$

$f(1) = 1(1 + 1)^2 - 1 = 3 > 0$

Hence, the root lies in between 0 and 1.

Since, $\phi(x) = \frac{1}{(x+1)^2}$

Diff w. r to 'x'

$$|\phi'(x)| = \left|\frac{2}{(x+1)^3}\right| < 1 \ \forall \ x \in (0,1)$$

$\therefore$ Convergent condition is satisfied

Now $x_0 = \frac{0+1}{2} = \frac{1}{2} = 0.5$

Substitute x_0 in $\phi(x)$,

We set successively,

$$x_1 = \phi(x_0) = \phi(0.5) = \frac{1}{(0.5+1)^2} = 0.4444$$

$$x_2 = \phi(x_1) = \phi(0.4444) = \frac{1}{(0.4444+1)^2} = 0.4793$$

$$x_3 = \phi(x_2) = \phi(0.4793) = \frac{1}{(0.4793+1)^2} = 0.4570$$

$$x_4 = \phi(x_3) = \phi(0.4570) = \frac{1}{(0.4570+1)^2} = 0.4711$$

$$x_5 = \phi(x_4) = \phi(0.4711) = \frac{1}{(0.4711+1)^2} = 0.4621$$

$$x_6 = \phi(x_5) = \phi(0.4621) = \frac{1}{(0.4621+1)^2} = 0.4678$$

$$x_7 = \phi(x_6) = \phi(0.4678) = \frac{1}{(0.4678+1)^2} = 0.4642$$

$$x_8 = \phi(x_7) = \phi(0.4642) = \frac{1}{(0.4642+1)^2} = 0.4664$$

$$x_9 = \phi(x_8) = \phi(0.4664) = \frac{1}{(0.4664+1)^2} = 0.4650$$

$$x_{10} = \phi(x_9) = \phi(0.4650) = \frac{1}{(0.4650+1)^2} = 0.4659$$

$$x_{11} = \phi(x_{10}) = \phi(0.4659) = \frac{1}{(0.4659+1)^2} = 0.4654$$

$$x_{12} = \phi(x_{11}) = \phi(0.4654) = \frac{1}{(0.4654+1)^2} = 0.4657$$

$$x_{13} = \phi(x_{12}) = \phi(0.4657) = \frac{1}{(0.4657+1)^2} = 0.4655$$

$$x_{14} = \phi(x_{13}) = \phi(0.4655) = \frac{1}{(0.4655+1)^2} = 0.4656$$

$$x_{15} = \phi(x_{14}) = \phi(0.4656) = \frac{1}{(0.4656+1)^2} = 0.4656$$

Since x_{14} and x_{15} are equal.

Hence, the root of the equation 0.4656.

Q24. Use the method of iteration correct to 4 decimal places, find a root of the equation $e^x = 3x$.

Sol :

Given equation $f(x) = e^x - 3x$

We rewrite the equation $x = \frac{e^x}{3}$

It is in the form $x = \phi(x)$.

Where

$$\phi(x) = \frac{e^x}{3}$$

$f(0) = e^0 - 3(0) = 1 > 0$

$f(1) = e^1 - 3(1) = -0.2817 < 0.$

Hence, the root lies between 0 and 1.

Since $\phi(x) = \frac{e^x}{3}$

$$|\phi'(x)| = \left|\frac{e^x}{3}\right| < 1 \text{ in } (0,1)$$

Convergent condition is satisfied

Now $x_0 = \frac{0+1}{2} = 0.5$

Taking $x_0 = 0.5$, we obtoin successine iteates as,

$x_1 = \phi(x_0) = \phi(0.5) = 0.5496.$

$x_2 = \phi(x_1) = \phi(0.5496) = 0.5775.$

$x_3 = \phi(x_2) = \phi(0.5775) = 0.5939.$

$x_4 = \phi(x_3) = \phi(0.5939) = 0.6037.$

$x_5 = \phi(x_4) = \phi(0.6037) = 0.6096.$

$x_6 = \phi(x_5) = \phi(0.6096) = 0.6132.$

$x_7 = \phi(x_6) = \phi(0.6132) = 0.6154.$

$x_8 = \phi(x_7) = \phi(0.6154) = 0.6168.$

$x_9 = \phi(x_8) = \phi(0.6168) = 0.6177.$

$x_{10} = \phi(x_9) = \phi(0.6177) = 0.6182.$

$x_{11} = \phi(x_{10}) = \phi(0.6182) = 0.6185.$

$x_{12} = \phi(x_{11}) = \phi(0.6185) = 0.6187.$

$x_{13} = \phi(x_{12}) = \phi(0.6187) = 0.6188.$

$x_{14} = \phi(x_{13}) = \phi(0.6188) = 0.6189.$

$x_{15} = \phi(x_{14}) = \phi(0.6189) = 0.6190.$

$x_{16} = \phi(x_{15}) = \phi(0.6190) = 0.6190.$

x_{15} and x_{16} are repeating same.

$\therefore$ The required root is $x = 0.6190$.

Q25. Use the method of iteration. Find correct to four decimal places for the equation $x - \sin x = \frac{1}{2}$

Sol :

Let the given equation $f(x) = x - \sin x - \frac{1}{2}$

We rewrite the equation $x = \sin x + \frac{1}{2}$

It is in the four $x = \phi(x)$

Where

$$\phi(x) = \sin x + \frac{1}{2}$$

$$f(0) = 0 - \sin(0) - \frac{1}{2} = -0.5 < 0$$

$$f(1) = 1 - \sin(1) - \frac{1}{2} = -0.3415 < 0$$

$$f(2) = 2 - \sin(2) - \frac{1}{2} = +0.5907 > 0$$

Hence the root lies between 1 and 2

Since, $\phi(x) = \sin x + \frac{1}{2}$

$$|\phi'(x)| = |\cos x| < 1$$

Convergent condition is satisfied.

Now, taking $x_0 = 1.5$, we obtain successine iterates as,

$$x_1 = \phi(x_0) = \phi(1.5) = \sin(1.5) + \frac{1}{2} = 1.4975$$

$$x_2 = \phi(x_1) = \phi(1.4975) = \sin(1.4975) + \frac{1}{2} = 1.4973$$

$$x_3 = \phi(x_2) = \phi(1.4973) = \sin(1.4973) + \frac{1}{2} = 1.4973$$

x_2 and x_3 are repeating same,

Hence, the root of the equation is 1.4973.

1.2.3 The Method of False Position

Q26. Write the method of false position (or) Regular False Method.

Ans :

We choose two points a and b such that f(a) and f(b) are of opposite signs. Hence a root must lie in between these points.

Now, the equationof the chord joining the two points $[a, f(a)]$ and $[b, f(b)]$ is given by,

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} \qquad ..... (1)$$

The method consists in replacing the part of the curve between the points $[a, f(a)]$ and $[b, f(b)]$ by means of the chord joining these points, and taking the point of intersection of the chord with the x - axis as an approximation to the root.

The point of intersection in the present case is obtained by putting y = 0 in (1),

Thus, we obtain

$$x_1 = a - \frac{f(a)}{f(b) - f(a)}(b - a)$$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} \quad \text{-------} (2)$$

Which is the first approximation to the root of f(x) = 0

If now $f(x_1)$ and f(a) are of opposite signs, then the root lies between a and x_1, and we replace b by x_1 in (2) and we obtoin the next approximation.

Otherwise we replace a by x_1 and generate the next approximation

The procedure is repeated till the root is obtained to the desired accuracy.

Q27. Find a real root of the equation $f(x) = x^3 - 2x - 5 = 0$.

Sol :

The given function $f(x) = x^3 - 2x - 5$

$$f(1) = (1)^3 - 2(1) - 5 = -6 < 0$$

$$f(2) = (2)^3 - 2(2) - 5 = -1 < 0$$

$$f(3) = (3)^3 - 2(3) - 5 = +16 > 0$$

Since f(2) < 0 and f(3) > 0 then the root lies between 2 and 3 By Method of false position,

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{32 + 3}{17} = \frac{35}{17} = 2.05882$$

Now $f(x_1) = f(2.05882) = (2.05882) - 2(2.05882) - 5 = -0.39079 < 0$

= f(2.05882) < 0 and f(3) > 0 then the root lies between

2.05882 and 3.

$$x_2 = \frac{2.05882(16) - 3(-0.39079)}{16 - (-0.39079)}$$

$$= \frac{32.94112 + 1.17237}{16.39079} = 2.08126$$

Now $f(x_2) = f(2.08126) = -0.14724 < 0.$

Since f(2.08126) < and f(3) > 0 then the root lies between 2.08126 and 3.

$$x_3 = \frac{2.08126(16) - 3(-0.14724)}{16 - (-0.14724)}$$

$$= \frac{33.30016 + 0.44172}{16.14724}$$

$x_3 = 2.08964$

Now $f(x_3) = f(2.08964) = -0.05470 < 0$

Since f(2.08964) < 0 and f(3) > 0 then the root lies between 2.08964 and 3.

$$x_4 = \frac{2.08964(16) - 3(-0.05470)}{16 - (-0.05470)}$$

$= 2.09274$

$f(x_4) = f(2.09274) = -0.02018 < 0$

Since f(2.09274) < 0 and f(3) > 0 then the root lies between 2.09274 and 3.

$$x_5 = \frac{2.09274(16) - 3(-0.02018)}{16 - (-0.02018)}$$

$x_5 = 2.09388$

$f(x_5) = f(2.09388) = -0.00745 < 0$

Since f(2.09388) < 0 and f(3) > 0 then the root lies between 2.09388 and 3.

$$x_6 = \frac{2.09388(16) - 3(-0.00745)}{16 - (-0.00745)}$$

$x_6 = 2.09430$

$f(x_6) = f(2.09430) = -0.00278 < 0$

Since f(2.09430) < 0 and f(3) > 0 then the root lies between 2.09430 and 3.

$$x_7 = \frac{2.09430(16) - 3(-0.00278))}{16 - (-0.00278)}$$

$x_7 = 2.09446$

$f(x_7) = f(2.09446) = -0.00106 < 0$

Since f(2.09446) < 0 and f(3) > 0 then the root lies between 2.09446 and 3.

$$x_8 = \frac{2.0094(16) - 3(-0.00106)}{16 - (-0.00106)}$$

$x_8 = 2.09452$

$f(x_8) = f(2.09452) = -0.00035 < 0$

Since f(2.09452) < 0 and f(3) > 0 then the root lies between 2.09452 and 3.

$$x_9 = \frac{2.09542(16) - (0.00035)3}{16 - (-0.00035)}$$

$x_9 = 2.09454$

x_8 and x_9 are repeating same upto 4 decinal places.

Hence, the root of the equation 2.0945.

Q28. Given that the equation $x^{2\cdot 2} = 69$ has a root between 5 and 8. Use the method of regula - falsi to determine it.

Sol :

Let $f(x) = x^{2\cdot 2} - 69$

$f(5) = -34.50678$ and $f(8) = 28.00586$

Since f(5) < 0 and f(8) > 0 then the root lies between 5 and 8

$$x_1 = \frac{5(28.00586) - 8(-34.50678)}{28.00586 - (-34.50678)}$$

$x_1 = 6.65599$

$f(x_1) = f(6.65599) = -4.27562 < 0$

Since f(6.65599) < 0 and f(8) > 0 then the root lies between 6.65599 and 8.

$$x_2 = \frac{6.65599(28.00586) - 8(-4.27562)}{28.00586 - (-4.27562)}$$

$$x_2 = \frac{220.61168}{32.28148} = 6.83400$$

$f(x_2) = f(6.83400) = -0.40619 < 0$

Since f(6.83400) < 0 and f(8) > 0 then the root lies between 6.83400 and 8.

$$x_3 = \frac{6.83400(28.00586) - 8(-0.40619)}{28.00586 - (-0.40619)}$$

$$= \frac{194.64157}{28.41205} = 6.85067$$

$f(x_3) = f(6.85067) = -0.03755 < 0$

Since f(6.85067) < 0 and f(8) > 0 then the root lies between

6.85067 and 8.

$$x_4 = \frac{6.85067(28.00586) - 8(-0.03755)}{28.00586 - (-0.03753)}$$

$$= \frac{192.15930}{28.04341} = 6.85221$$

$f(x_4) = f(6.85221) = -0.00344 < 0$

Since f(6.85221) < 0 and f(8) > 0 then the root lies between 6.85221 and 8.

$$x_5 = \frac{6.85221(28.00586) - 8(-0.00344)}{28.00586 - (-0.00344)}$$

$$x_5 = \frac{191.92955}{28.00930} = 6.85235$$

x_4 and x_5 are repeating some upto three decimal places.

Hence, the root ot the equation 6.852.

Q29. The equation $2x = \log_{10}x + 7$ has a root between 3 and 4. Find this root correct to three decinal placess by Regula falso method.

Sol :

Let $f(x) = 2x - \log_{10}^{x} - 7$

$a = 3$, and $b = 4$.

$f(3) = 2(3) - \log 3_{10} - 7 = -1.4771 < 0.$

$f(4) = 2(4) - \log_{10} 4 - 7 = 0.3979 > 0$

Hence

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{3(0.3979) - 4(-1.4771)}{0.3979 - (-1.4771)} = \frac{1.1937 + 5.9084}{1.875}$$

$$= \frac{7.1021}{1.8750} = 3.7888$$

$f(x_1) = f(3.7888) = -0.0009 < 0$

Since f(3.7888) < 0 and f(4) > 0 then the root lies between 3.7888 and4.

$$x_2 = \frac{3.7888(0.3979) - 4(-0.0028)}{0.3979 - (-0.0028)} = 3.7893$$

$f(x_2) = f(3.7893) = 0.00004 > 0$

Since f(3.7893) > 0 and f(3.7878) < 0 then the root lies between 3.7878 and 3.7893

$$x_3 = \frac{3.7878(0.00004) - 3.7893(-0.0009)}{0.0004 - (-0.0009)} = 3.7893$$

x_2 and x_3 are repeating same

Hence, the required root is 3.789

Q30. Find a root of the equation $4e^{-x}\sin x - 1 = 0$ by regula - falsi method given that the root lies between 0 and 0.5

Sol :

Let $f(x) = 4e^{-x}\sin x - 1$

$a = 0$ and $b = 0.5$

We have $f(0) = 4e^{-0}\sin(0) - 1 = -1 < 0$

$f(0.5) = 4e^{-0.5}\sin(-0.5) - 1 = 0.1631 > 0$

$$\therefore \quad x_1 = \frac{0(0.1631) - 0.5(-1)}{0.1631 - (-1)}$$

$x_1 = 0.4299$

$f(x_1) = f(0.4299) = 0.0846 > 0$

Since f(0) < 0 and f(0.4299) > 0 then the root lies between 0 and 0.4299

$$x_2 = \frac{0(0.0846) - 0.4299(-1)}{0.0846 - (-1)}$$

$$x_2 = \frac{0.4299}{1.0846} = 0.3964$$

Now $f(x_2) = f(0.3964) = 0.0390 > 0$

Since f(0) < 0 and f(0.3964) > 0 then the root lies between 0 and 0.3964

$$x_3 = \frac{0(0.0390) - 0.3964(-1)}{0.0390 - (-1)}$$

$x_3 = 0.3815$

$f(x_3) = f(0.3815) = 0.0169 > 0$

Since f(0.3815) > 0 and f(0) < 0 then the root lies between 0 and 0.3815

$$x_4 = \frac{0(0.0169) - 0.3815(-1)}{0.0169 - (-1)} = 0.3752$$

$f(x_4) = f(0.3752) = 0.0073 > 0$

Since f(0.3752) > 0 and f(0) < 0 then the root lies between 0 and 0.3752

$f(x_4) = f(0.3752) = 0.0073 > 0$

Since f(0.3752) > 0 and f(0) < 0 then the root lies between 0 and 0.3752

$$x_5 = \frac{0(0.0073) - 0.3752(-1)}{0.0093 - (-1)}$$

$x_5 = 0.3725$

$f(x_5) = f(0.3725) = 0.0030 > 0$

Since f(0.3725) > 0 and f(0) < 0 then the root lies between 0 and 0.3725

$$x_6 = \frac{0(0.0030) - 0.3725(-1)}{0.0030 - (-1)}$$

$x_6 = 0.3714$

$f(x_6) = f(0.3714) = 0.0013 > 0$

Since f(0.3714) > 0 and f(0) < 0 then the lies between 0 and 0.3714

$$x_7 = \frac{0(0.0013) - 0.3714(-1)}{0.0013 - (-1)} = 0.3709 \simeq 0.371$$

$\therefore$ $f(x_7) = f(0.3709) = 0.0005 > 0$

Since f(0.3709) > 0 and f(0) < 0 then the root lies between 0 and 0.3709

$$x_8 = \frac{0(0.0005) - 0.3709(-1)}{0.0005 - (-1)} = 0.3709 \simeq 0.371$$

$\therefore$ x_7 and x_8 are repeating same upto three decimal places

$\therefore$ The root of the equation x = 0.371

Q31. Use the method of false position to find a real root correct to three decimal places for the equation $x^3 + x^2 + x + 7 = 0$.

Sol :

Let the function $f(x) = x^3 + x^2 + x + 7$

$f(-1) = (-1)^3 + (-1)^2 + (-1) + 7 = 6 > 0$

$f(-2) = (-2)^3 + (-2)^2 + (-2) + 7 = 1 > 0$

$f(-3) = (-3)^3 + (-3)^2 + (-3) + 7 = -14 < 0$

Since f(–2) > 0 and f(-3) < 0 then the root lies between –2 and –3

a = –2, b = –3

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{-2(-14) - (-3)(1)}{-14 - 1} = \frac{28 + 3}{-15} = \frac{31}{-15} = -2.0667$$

$f(x_1) = f(-2.0667) = (-2.0667)^3 + (-2.0667)^2 + (-2.0667) + 7$

$= 0.3772 > 0$

Since f(–2.0667) > 0 and f(–3) < 0 then the root lies between –2.0667 and –3.

$$x_2 = \frac{-2.0667(-14)-(-3)(0.3772)}{-14-0.3772}$$

$$= \frac{30.0654}{-14.3772} = -2.0912$$

$f(x_2)$ = f(-2.0912) = 0.1369 > 0

Since f(-2.0912) > 0 and f(-3) < 0 then the root lies between –2.0912 and –3.

$$x_3 = \frac{-2.0912(-14)-(-3)(0.1369)}{-14-0.1369} = \frac{29.6875}{-14.1369} = -2.1$$

$f(x_3)$ = f(–2.1) = 0.049 > 0

Since f(–2.1) > 0 and f(-3) < 0 then the root lies between –2.1 and –3

$$x_4 = \frac{-2.1(-14)-(-3)(0.049)}{-14-0.049} = -2.1031$$

$f(x_4)$ = f(-2.1031) = 0.0179 > 0

Since f(–2.1031) > 0 and f(-3) < 0 then the root lies between –2,1031 and –3

$$x_5 = \frac{-2.1031(-14)-(-3)(0.0179)}{-14-0.0179} = -2.1042$$

$f(x_5)$ = f(-2.1042) = 0.0068 > 0

f(–2.1042) > 0 and f(–3) < 0 then the root lies between –2.1042 and –3.

$$x_6 = \frac{-2.1042(-14)-(-3)(0.0068)}{-14-0.0068} = -2.\ 1046$$

$f(x_6)$ = f(-2.1046) = 0.0027 > 0

Since f(-2.1046) > 0 and f(-3) < 0 then the root lies between –2.1046 and –3

$$x_7 = \frac{-2.1046(-14)-(-3)(0.0027)}{-14-0.0027} = -2.1048$$

∴ x_6 and x_7 are repeating same upto three decimal places.

Hence, the real root of the equation –2.105.

Q32. Use the method of false postion to find a real root correct to three decimal places for the equation $x^3 - x - 4 = 0$.

Sol :

Given function is,

$f(x) = x^3 - x - 4$

$f(0) = (0)^3 - (0) - 4 = -4 < 0$

$f(1) = (1)^3 - (1) - 4 = -4 < 0$

$f(2) = (2)^3 - (2) - 4 = 2 > 0$

Since f(1) < 0 and f(2) > 0 then the root lies between 1 & 2

Let a = 1 & b = 2

$$x_1 = \frac{1(2)-2(-4)}{2-(-4)} = 1.666$$

$f(x_1) = f(1.666) = (1.666)^3 - (1.666) - 4 = -1.042 < 0$

Since f(1.666) < 0 and f(2) > 0.

Now, the root lies between 1.666 and 2.

$$x_2 = \frac{1.666(2)-2(-1.042)}{2-(-1.042)} = 1.780$$

$f(x_2) = f(1.780) = (1.780)^3 - 1.780 - 4 = -0.1402 < 0.$

Since f(1.780) < 0 and f(2) > 0.

Now, the root lies between 1.780 and 2.

$$x_3 = \frac{1.780(2)-2(-0.1402)}{2-(-0.1042)} = 1.794$$

$f(x_3) = f(1.794) = -0.0201 < 0$

Since f(1.794) < 0 and f(2) > 0

Now, the root lies between 1.794 and 2.

$$x_4 = \frac{1.794(2)-2(-0.0201)}{2-(-0.0201)} = 1.796$$

$f(x_4) = f(1.796) = -0.0027 < 0$

Since f(1.796) < 0 and f(2) > 0

Now, the root lies between 1.796 and 2

$$x_5 = \frac{1.796(2)-2(-0.0027)}{2-(-0.0027)} = 1.796$$

$\therefore$ x_4 and x_5 are repeating some upto three decimal places.

Hence, the root of the equation 1.796.

Q33. Use the method of false position find a real root of the equation $x = 3e^{-x}$ correct to three decimal places.

Sol :

Let $f(x) = x - 3e^{-x}$

$f(1) = 1 - 3e^{-1} = -0.1036 < 0$

$f(2) = 2 - 3e^{-2} = 1.5940 > 0.$

Since f(1) < 0 and f(2) > 0

Now the root lies between 1 and 2

Let $a = 1, b = 2$

$$x_1 = \frac{1(1.5940) - 2(-0.1036)}{1.5940 - (-0.1036)}$$

$x_1 = 1.0610$

$f(x_1) = f(1.0610) = 0.0227 > 0$

Since $f(1.0610) > 0$ and $f(1) < 0$

Now the root lies between 1 and 1.0610

$$x_2 = \frac{1(0.0227) - 1.0610(-0.1036)}{0.0227 - (-0.1036)} = 1.0500$$

$f(x_2) = f(1.05) = 0.0002 > 0$

Since $f(1.05) > 0$ and $f(1) < 0$

Now the root lies between 1 and 1.05

$$x_3 = \frac{1(0.0002) - 1.05(-0.1036)}{0.0002 - (-0.1036)} = \frac{0.1090}{0.1038} = 1.0501$$

x_2 and x_3 are repiating same upto three decimal places,

Hence, the root of the equation $x = 1.05$

1.2.4 Newton-Raphson Method (Newton's Method)

Q34. Derive Newton - Raphson formula.

Ans :

Let the given equation be $f(x) = 0$

Let x_0 be an approximate root of $f(x) = 0$ and let

$x_1 = x_0 + h$ be the correct root so that $f(x_1) = f(x_0 + h) = 0$

Expanding $f(x_0 + h)$ by Taylor's Series,

We obtain

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + ... = 0$$

Neglecting the second and higher order derivatives,

We have

$f(x_0) + hf'(x_0) = 0$

$hf'(x_0) = -f(x_0)$

$$h = -\frac{f(x_0)}{f^1(x_0)}$$

Substituting this in x_1 we get

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x_1 is a better approximation than x_0.

Successive approximations are given by $x_2, x_3, \ldots\ldots x_{n+1}$.

Where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Q35. Find a real root of the equation $x^3 - 2x - 5 = 0$ by Newton – Raphson's method.

Sol:

Let $f(x) = x^3 - 2x - 5$

Then $f'(x) = 3x^2 - 2$

By Newton – Raphson's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We have

$f(1) = (1)^3 - 2(1) - 5 = -6 < 0$

$f(2) = (2)^3 - 2(2) - 5 = -1 < 0$

$f(3) = (3)^3 - 2(3) - 5 = +16 > 0$

$\therefore$ $f(x) = 0$ has a root lies between 2 and 3

$$x_0 = \frac{2+3}{2} = \frac{5}{2} = 2.5$$

Now,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{(x_0)^3 - 2(x_0) - 5}{3(x_0) - 2}$$

$$x_1 = 2.5 - \frac{(2.5)^3 - 2(2.5) - 5}{3(2.5)^2 - 2} = 2.1642$$

Now,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = x_1 - \frac{x_1^2 - 2x_1 - 5}{3x_1^2 - 2}$$

$$x_2 = 2.1642 - \frac{(2.1642)^3 - 2(2.1642) - 5}{3(2.1642)^2 - 2} = 2.0971$$

Next approximation,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= x_2 - \frac{x_2^3 - 2x_2 - 5}{3{x_2}^2 - 2} = 2.0971 - \frac{(2.0971)^3 - 2(2.0971) - 5}{3(2.0971)^2 - 2}$$

$= 2.0946$

Next approximation,

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= x_3 - \frac{{x_3}^3 - 2x_3 - 5}{3{x_3}^2 - 2}$$

$$= 2.0946 - \frac{2.0946^3 - 2(2.0946) - 5}{3(2.0946)^2 - 2}$$

$x_4 = 2.0946$

x_3 and x_4 are repeating same.

Hence, the real root of the equation is 2.0946

Q36. Find a root of the equation xsinx + cosx = 0 by Newton's method with $x_0 = \pi$.

Sol :

Given equation xsinx + cosx = 0.

Let $f(x) = x\sin x + \cos x$

$f'(x) = x\cos x$.

Since $x_0 = \pi$

By Newon's method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{f(\pi)}{f'(\pi)}$$

$$x_1 = \pi - \left(\frac{\pi\sin(\pi) + \cos\pi}{\pi\cos\pi}\right)$$

$x_1 = 2.8233$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.8233 - \frac{f(2.8233)}{f^1(2.8233)}$$

$$x_2 = 2.8233 - \frac{(-0.0662)}{(-2.6815)} = 2.7984$$

$$x_3 = x_2 = - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7984 - \frac{f(2.7984)}{f'(2.7984)}$$

$x_3 = 2.7984$

x_2 and x_3 are repeating same.

Thus, the required root is 2.7984

Q37. Find a real root of the equation $x = e^{-x}$ using the Newton - Raphson method.

Sol :

Given equation is $x - e^{-x} = 0$

Let $f(x) = x - e^{-x}$

$f'(x) = 1 - (-e^{-x}) = 1 + e^{-x}$

$f(0) = 0 - e^{-(0)} = -1 < 0$

$f(1) = 1 - e^{-1} = 0.63212 > 0$

The root lies between 0 and 1.

$$x_0 = \frac{0+1}{2} = \frac{1}{2} = 0.5$$

Next approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.5 - \left[\frac{(0.5) - e^{-(0.5)}}{1 + e^{-(0.5)}}\right] = 0.5 - \left[\frac{-0.1065}{1.6065}\right]$$

$x_1 = 0.5663$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.5663 - \frac{f(0.5663)}{f'(0.5663)}$$

$$= 0.5663 - \left(\frac{0.5663 - e^{-0.5663}}{1 + e^{-0.5663}}\right)$$

$$= 0.5663 + 0.0008$$

$$x_2 = 0.5671$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.5671 - \frac{f(0.5671)}{f^1(0.5671)}$$

$$= 0.5671 - \left(\frac{0.5671 - e^{-0.5671}}{1 + e^{-0.5671}}\right)$$

$$= 0.5671 + 0.0001$$

$$x_3 = 0.5672$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.5672 - \frac{f(0.5672)}{f^1(0.5672)}$$

$$= 0.5672 - \left(\frac{0.5672 - e^{-0.5672}}{1 + e^{-0.5672}}\right)$$

$$= 0.5672 - 0.00005$$

$$x_4 = 0.5671$$

Hence, the required root is 0.5671.

Q38. Using Newton - Raphson method find a real root correct to 3 decimal places of the equation $\sin x = \frac{x}{2}$ given that the root lies between $\frac{\pi}{2}$ and π.

Sol :

Let $f(x) = \sin x - \frac{x}{2}$.

Then $f'(x) = \cos x - \frac{1}{2}$

Choosing $x_0 = \frac{\pi}{2}$, we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$$x_1 = \frac{\pi}{2} - \frac{f\left(\frac{\pi}{2}\right)}{f^1\left(\frac{\pi}{2}\right)} = \frac{\pi}{2} - \left(\frac{\sin\frac{\pi}{2} - \frac{\pi}{4}}{\cos\frac{\pi}{2} - \frac{1}{2}}\right)$$

$$= 2.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2 - \frac{f(2)}{f^1(2)} = 2 - \left(\frac{\sin 2 - \frac{2}{2}}{\cos 2 - \frac{1}{2}}\right) = 1.9010$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.9010 - \frac{f(1.9010)}{f'(1.9010)}$$

$$= 1.9010 - \left(\frac{\sin 1.9010 - 0.9505}{\cos 1.9010 - 0.5}\right)$$

$$x_3 = 1.8955$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 1.8955 - \frac{f(1.8955)}{f'(1.8955)}$$

$$x_4 = 1.8955 - \left(\frac{\sin 1.8955 - \frac{1.8955}{2}}{\cos 1.8955 - \frac{1}{2}}\right) = 1.8954$$

x_3 and x_4 are repeating same upto three decimal places

Hence, the requried root is $x = 1.895$

Q39. Given the equation $4e^{-x} \sin x - 1 = 0$. Find the root between 0 and 0.5 correct to three decimal places.

Sol :

Let the equation $f(x) = 4e^{-x} \sin x - 1$

$$f'(x) = 4e^{-x}(\cos x - \sin x)$$

Let $x_0 = \frac{0+0.5}{2} = 0.25$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.25 - \frac{f(0.25)}{f'(0.25)}$$

$$= 0.25 - \left(\frac{4e^{-0.25}\sin 0.25 - 1}{4e^{-0.25}(\cos 0.25 - \sin 0.25)}\right)$$

$$= 0.25 - \left(\frac{-0.2293}{2.2476}\right) = 0.3520$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.3520 - \frac{f(0.3520)}{f'(0.3520)}$$

$$= 0.3520 - \left(\frac{4e^{-0.3520}\sin 0.3520 - 1}{4e^{-0.3520}(\cos 0.3520 - \sin 0.3520)}\right)$$

$$x_2 = 0.3520 - \left(\frac{-0.0301}{1.6707}\right)$$

$$x_2 = 0.3720$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.3700 - \frac{f(0.3700)}{f'(0.3700)}$$

$$= 0.3700 - \left(\frac{4e^{-0.3700}\sin 0.3700 - 1}{4e^{-0.3700}(\cos 0.3700 - \sin 0.3700)}\right)$$

$$x_3 = 0.3700 - \left(\frac{-0.0009}{1.5768}\right) = 0.3706.$$

x_2 and x_3 are repeating same upto three decimal places.

Hence, the required root is 0.370.

Q40. Use the Newton - Raphson method to obtain a root correct to three decimal places of the equation $e^x = 4x$.

Sol :

Let the given equation

$f(x) = e^x - 4x.$

$f'(x) = e^x - 4$

$f(0) = e^0 - 4(0) = 1 > 0$

$f(1) = e^1 - 4(1) = -1.2817 < 0$

Hence, the root lies between 0 and 1

$$x_0 = \frac{0+1}{2} = \frac{1}{2} = 0.5$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{f(0.5)}{f^1(0.5)}$$

$$= 0.5 - \left(\frac{e^{0.5} - 4(0.5)}{e^{0.5} - 4}\right) = 0.5 - \left(\frac{-0.3513}{-2.3513}\right) = 0.3506$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.3506 - \frac{f(0.3506)}{f^1(0.3506)} = 0.3506 - \left(\frac{0.0175}{-2.5801}\right) = 0.3574$$

$$x_3 = x_2 - \frac{f(x_2)}{f^1(x_2)}$$

$$= 0.3574 - \frac{f(0.3574)}{f^1(0.3574)} = 0.3574 - \frac{(0.0000)}{(-2.5704)} = 0.3574$$

x_2 and x_3 are repeating same.

Hence, the required root is 0.3574.

Q41. Find a root of the equation $x^3 - 5x + 3 = 0$ by Newton - Raphson method.

Sol :

Let the given equation

$f(x) = x^3 - 5x + 3$

$f'(x) = 3x^2 - 5$

$f(0) = 0^3 - 5(0) + 3 = 3 > 0$

$f(1) = (1)^3 - 5(1) + 3 = -1 < 0$

Then, the root lies between 0 and 1.

$$x_0 = \frac{0+1}{2} = \frac{1}{2} = 0.5$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \left(\frac{(0.5)^3 - 5(0.5) + 3}{3(0.5)^2 - 5}\right) = 0.6471$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.6471 - \frac{f(0.6471)}{f'(0.6471)} = 0.6471 - \left(\frac{(0.6471)^3 - 5(0.6471) + 3}{3(0.6471)^2 - 5}\right) = 0.6566$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6566 - \frac{f(0.6566)}{f'(0.6566)} = 0.6566 - \left(\frac{(0.6566)^3 - 5(0.6566) + 3}{3(0.6566)^2 - 5}\right) = 0.6566$$

x_2 and x_3 are repeating same.

Hence the required root is 0.657.

1.2.4.1 Generalized Newton's Method

Q42. Describe the Generalized Newton's Method

Sol :

If ξ is a root of f(x) = 0 with multiplicity p, then the iteration formula will be,

$$x_{n+1} = x_n - p\,\frac{f(x_n)}{f^1(x_n)}$$

Which means that $\frac{1}{p}$ $f'(x_n)$ is the slope of the straight line passing through (x_n, y_n) and intersecting the x - axis at the point $(x_{n+1}, 0)$

Since ξ is a root of f(x) = 0 with multiplicity p, it implies that is ξ also a root of $f^1(x) = 0$ with multiplicity (p – 1), and it is a root of f"(x) = 0 with multiplicity (p – 2), and so on.

Therefore,

$$x_0 - \frac{f(x_0)}{f'(x_0)},\ x_0 - (p-1)\frac{f'(x_0)}{f''(x_0)},\ x_0 - (p-2)\frac{f''(x_0)}{f'''(x_0)}$$

Must have the same value if there is a roof with multiplicity p, the initial approsimation x_0 is choosen sufficiently close to the root.

Q43. Find a double root of the equation $f(x) = x^3 - x^2 - x + 1 = 0$ choosing $x_o = 0.8$.

Sol :

The given equation

$f(x) = x^3 - x^2 - x + 1 = 0$

$f'(x) = 3x^2 - 2x - 1$

$f''(x) = 6x - 2$

Choosing $x_0 = 0.8$ we obtain

$$x_0 - 2\,\frac{f(x_0)}{f'(x_0)} = 0.8 - 2\,\frac{0.072}{(-0.68)} = 1.012$$

and $x_0 - \frac{f'(x_0)}{f''(x_0)} = 0.8 - \frac{(-0.68)}{2.8} = 1.043$

The closeness of these values indicates that there is a double root neal to unity (1).

For the next approximation, we choose $x_1 = 1.01$ and obtain

$$x_1 - 2\frac{f(x_1)}{f'(x_1)} = 1.01 - 0.0099 = 1.0001$$

and $x_1 - \frac{f'(x_1)}{f''(x_1)} = 1.01 - 0.0099 = 1.0001$

We conclude, therefore, that there is a double root at x = 1.0001 which is suffiueutly close the actual root unity.

It is clear that the generalized Newton's Method converges more rapidly than the Newton - Raphson procedure.

1.2.5 Muller's Method

Q44. Derive Muller's Method.

Ans :

Muller's Method: The roots of the Quadratic are then assumed to be the approximations to the roots of the equation

$$f(x) = 0.$$

Let x_{i+2}, x_{i-1}, x_i be three distinct approcimations to a root of f(x) = 0.

Let y_{i-2}, y_{i-1} and y_i be the corresponding values of y = f(x).

Assuming that

$$p(x) = A(x-x_i)^2 + B(x - x_i) + y_i \quad ..... (1)$$

Is the parabola passing through the points (x_{i-2}, y_{i-2}), (x_{i-1}, y_{i-1}) and (x_i, y_i)

we have,

$$y_{i-1} = A(x_{i-1} - x_i)^2 + B(x_{i-1} - x_i) + y_i \quad ..... (2)$$

and $y_{i-2} = A(x_{i-2} - x_i)^2 + B(x_{i-2} - x_i) + y_i$ ..... (3)

From equations (2) & (3), we obtain.

$$y_{i-1} - y_i = A(x_{i-1} - x_i)^2 + B(x_{i-1} - x_i) \quad ..... (4)$$

and $y_{i-2} - y_i = A(x_{i-2} - x_i)^2 + B(x_{i-2} - x_i)$ ..... (5)

Solutions of equations (4) and (5) gives,

$$A = \frac{(x_{i-2}-x_i)(y_{i-1}-y_i)-(x_{i-1}-x_i)(y_{i-2}-y_i)}{(x_{i-1}-x_{i-2})(x_{i-1}-x_i)(x_{i-2}-x_i)} \quad ..... (6)$$

$$B = \frac{(x_{i-2}-x_i)^2(y_{i-1}-y_i)-(x_{i-1}-x_i)^2(y_{i-2}-y_i)}{(x_{i-2}-x_{i-1})(x_{i-1}-x_i)(x_{i-2}-x_i)} \quad ..... (7)$$

With the values of A and B given in (6) & (7) ,

The Quadratic equation $P(x) = A(x - x_i)^2 + B(x - x_i) + y_i = 0$.

Now gives the next approximation x_{i+1} :

$$x_{i+1} - x_i = -\frac{-B \pm \sqrt{B^2 - 4Ay_i}}{2A} \quad ..... (8)$$

A direct solution from (8) leads to inaccurate results and therefore it is usually written the form

$$x_{i+1} - x_i = \frac{-2y_i}{B \pm \sqrt{B^2 - 4Ay_i}}$$

In equation (9), the sign in the denominator should be choosen so that the denominator will be largest in magnitude. with this choice, equation (9) then gives the next approximate to the root.

Q45. Find the root of the equation $y(x) = x^3 - 2x - 5 = 0$ which lies between 2 and 3 by Muller's method

Sol :

Let $x_{i-2} = 1, x_{i-1} = 2$ and $x_i = 3$.

Then $y_{i-2} = (1)^3 - 2(1) - 5 = -6$

$y_{i-1} = (2)^3 - 2(2) - 5 = -1$

and $y_i = (3)^3 - 2(3) - 5 = 16$

Now we find A and B

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$A = \frac{(1-3)(-1-16) - (2-3)(-6-16)}{(2-1)(2-3)(1-3)} = \frac{(-2)(-17) - (-1)(-22)}{(1)(-1)(-2)}$$

$y_{i-2} = -6$, $y_{i-1} = -1$, and $y_i = -0.0861$

Now we find A and B :

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$A = \frac{(1-2.0868)(-1-(-0.0861)) - (2-2.0868)(-6-(-0.0861))}{(2-1)(2-2.0868)(1-2.0868)}$$

$$A = \frac{(-8.0868)(-0.9139) - (-0.0868)(-5.9139)}{(1)(-0.0868)(-1.0868)}$$

$$A = \frac{0.9932 - 0.5133}{0.0943} = \frac{0.4799}{0.0943} = 5.0891$$

$$B = \frac{(x_{i-2}-x_i)^2(y_{i-1}-y_i)-(x_{i-1}-x_i)^2(y_{i-2}-y_i)}{(x_{i-2}-x_{i-1})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

$$= \frac{(1-2.0868)^2(-1-(-0.0861))-(2-(2.0868))^2(-6-(-0.0861))}{(2-1)(2-2.0868)(1-2.0868)}$$

$$= \frac{(-1.0868)^2(-0.9139))-(-0.0868)^2(-5.9139)}{1(-0.0868)(-1.0868)}$$

$$= \frac{-1.0794+0.0446}{0.0943} = -10.9735.$$

$$= \frac{34-(22)}{2} = \frac{12}{2} = 6.$$

$$B = \frac{(x_{i-2}-x_i)^2(y_{i-1}-y_i)-(x_{i-1}-x_i)^2(y_{i-2}-y_i)}{(x_{i-2}-x_{i-1})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

$$= \frac{(1-3)^2(-1-16)-(2-3)^2(-1-16)}{(1-2)(2-3)(1-3)}$$

$$= \frac{(-2)^2(-17)-(-1)^2(-22)}{(-1)(-1)(-2)} = \frac{-68+22}{-2} = \frac{-46}{-2} = 23$$

The Quadratic equation is ,

$A(x–xi)^2+B(x–xi)+yi = 0$

$C(x–3)2+23(x–3)+16 = 0$

The next approximation to the desired root is,

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2-4AY}}$$

$$x = 3 - \frac{2(16)}{23+\sqrt{23^2-4(6)(16)}}$$, Since B is positive

$$= 3 - \frac{32}{23+\sqrt{145}}$$

$= 2.0868$

The proceduce can now be repeated with the three approximations as 1,2 and 2.0868 respechrely

Now, let $x_{i-2} = 1$, $x_{i-1} = 2$, $x_i = 2.0868$.

The quadratic equation is

$A(x - x_i)^2 + B(x - x_i) + y_i = 0$

$5.0891(x - 2.0868)^2 - 10.9735 (x - 2.0868) - 0.0861 = 0$

The next approximation to the desired roof is,

$$x_{i+1} = x_i - \frac{2yi}{B \pm \sqrt{B^2 - 4AYi}}$$

$$= 2.0868 - \frac{2(-0.0861)}{(-10.9735) \pm \sqrt{(-10.9735)^2 - 4(5.0891)(-0.0861)}}$$

$$= 2.0868 - \frac{0.1722}{-10.9735 - 11.0531} = 2.0868 + \frac{0.1722}{-22.0266}$$

$$= 2.0868 - 0.0078$$

$$x_{i+1} = 2.0790$$

$$x_{i+1} = 2.1$$

Q46. Use Mulles's method to find a root of the equation $x^3 - x^2 - x - 1 = 0$.

Sol :

Let the three initial approximations be

$x_{i-2} = 0$, $x_{i-1} = 1$ and $x_i = 2$

Then $y_{i-2} = -1$, $y_{i-1} = -2$ and $y_i = 1$

Now we find A and B

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-2} - x_{i-1})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$= \frac{(0-2)(-2-1) - (1-2)(-1-1)}{(1-0)(1-2)(0-2)} = \frac{-2(-3) - (-1)(-2)}{(1)(-1)(-2)}$$

$$\frac{6-2}{2} = \frac{4}{2} = 2$$

$$B = \frac{(x_{i-2} - x_i)^2(y_{i-1} - y_i) - (x_{i-1} - x_i)^2(y_{i-2} - y_i)}{(x_{i-2} - x_{i-1})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$= \frac{(0-2)^2(-2-1) - (1-2)^2(-1-1)}{(0-1)(1-2)(0-2)}$$

$$= \frac{4(-3) - (-1)^2(-2)}{(-1)(-1)(-2)} = \frac{-12+2}{-2} = \frac{-10}{-2} = 5$$

The Quadratic equation is

$$A(x - x_i)^2 + B(x - x_i) + y_i = 0$$

$$2(x-2)^2 + 5(x-2) + 1 = 0$$

The next approximation is,

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2 - 4AY_i}}$$

$$x_{i+1} = 2 - \frac{2(1)}{5 + \sqrt{5^2 - 4(2)(1)}}$$

$$= 2 - \frac{2}{5 + \sqrt{25 - 8}} = 2 - \frac{2}{5 + \sqrt{17}} = 2 - 0.2192. = 1.7808$$

The procedrer can now be repeated with the three approximatuems as 1, 2, 1.7808 respecticely

Now let $x_{i-2} = 1$, $x_{i-1} = 2$, and $x_i = 1.7808$.

$y_{i-2} = -2$, $y_{i-1} = 1$, $y_i = 0.30407$

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-2} - x_{i-1})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$= \frac{(1 - 1.7808)(1 - (-0.30407)) - (2 - 1.7808)(-2 + 0.30407)}{(1 - 2)(2 - 1.7808)(1 - 1.7808)}$$

$$= \frac{(-0.7808)(1.3041)) - (0.2192)(-1.6959)}{(1 - 2)(2 - 1.7808)(1 - 1.7808)}$$

$$= \frac{-1.0182 + 0.3717}{0.1712} = -3.7763$$

$$B = \frac{(x_{i-2} - x_i)^2 (y_{i-1} - y_i) - (x_{i-1} - x_i)^2 (y_{i-2} - y_i)}{(x_{i-2} - x_{i-1})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$= \frac{(1 - 1.7808)^2 (1 - (-0.30407)) - (2 - 1.7808)(-2 + 0.30407)}{(1 - 2)(2 - 1.7808)(1 - 1.7808)}$$

$$= \frac{(0.6096)(1.3041) - (0.2192)(-1.6959)}{0.1712} = \frac{0.7950 + 0.3717}{0.1712} = 6.8148$$

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2 - 4AY_i}}$$

$$x_{i+1} = 1.7808 - \frac{2(-0.30407)}{6.8148 + \sqrt{6.8148^2 - 4(-3.7763)(-0.30407)}}$$

$$= 1.7808 + \frac{0.6081}{6.8148 + 6.4690}$$

$x_{i+1} = 1.8266.$

The procedure can now be repeated with the three approximations as 2,1.7808 and 1.8266.

Now let $x_{i-2} = 2$, $x_{i-1} = 1.7808$ and $x_i = 1.8266$.

$y_{i-2} = 1$, $y_{i-1} = -0.30407$ and $y_i = -0.0687$.

$$A = \frac{(x_{i-2}-x_i)(y_{i-1}-y_i)-(x_{i-1}-x_i)(y_{i-2}-y_i)}{(x_{i-2}-x_{i-1})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

$$= \frac{(2-1.7808)(-0.30407+0.0687)-(1.7808-1.8266)(1+0.0687)}{(2-1.7808)(1.7808-1.8266)(2-1.8266)}$$

$$= \frac{-0.0516-(-0.0489)}{(0.2192)(-0.0458)(0.1734)} = \frac{-0.0027}{-0.0017} = 1.5882$$

$$B = \frac{(x_{i-2}-x_i)^2(y_{i-1}-y_i)-(x_{i-1}-x_i)^2(y_{i-2}-y_i)}{(x_{i-2}-x_{i-1})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

$$= \frac{(2-1.7808)(-0.30407+0.0687)-(1.7808-1.8266)^2(1+0.0687)}{(2-1.7808)(1.7808-1.8266)(2-1.8266)}$$

$$= \frac{-0.0113-0.0022}{-0.0017}$$

$$B = 7.9412$$

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2-4AY_i}}$$

$$= 1.8266 - \frac{2(-0.0687)}{7.9412+\sqrt{(7.9412)^2-4(1.5882)(-0.0687)}}$$

$$x_{i+1} = 1.8266 + \frac{0.1374}{7.9412+7.9685} = 1.8352$$

The required root is 1.83.

1.2.6 Solution of Systems of Non - Linear Equations

Q47. Write the methods of solutions of system of non-linear equations.

Ans :

We consider two methods for the solution of simultaneous non linear equations :

1. The Method of Iteration
2. Newton - Raphson Method

We consider a system of two equations :

$$\left.\begin{aligned} f(x,y) &= 0 \\ \text{and } g(x,y) &= 0 \end{aligned}\right\} \quad \text{..... (1)}$$

Method of Interation :

As in the case of a single equation,

We assume that eq^n (1) may be written in the form

$x = F(x, y), y = G(x, y)$ ..... (2)

Where the functions F and G sastisfy the following conditions in a closed neighbourhood R of the root (α, β)

i) F and G are their first partial derivatives are contanuous in R

ii) $\left|\frac{\partial F}{\partial x}\right| + \left|\frac{\partial F}{\partial y}\right| < 1$ and $\left|\frac{\partial G}{\partial x}\right| + \left|\frac{\partial G}{\partial y}\right| < 1$ ..... (3)

For all (x,y) in R.

If (x_0, y_0) is an Inital approximation to the root (α, β)

Then equation (2) give the sequence

$$\left.\begin{array}{l} x_1 = F(x_0, y_0), \quad y_1 = G(x_0, y_0) \\ x_2 = F(x_1, y_1), \quad y_2 = G(x_1, y_1) \\ \vdots \\ x_{n+1} = F(x_n, y_n), \quad y_{n+1} = G(x_n, y_n) \end{array}\right\} \quad ..... (4)$$

For faster convergence, recently computed values x_i may be used in the evalution of y_i in (4), conditions in equation (3) are sufficent for convergence and in the limit, we obtain

$\alpha = F(\alpha, \beta)$ and $\beta = G(\alpha, \beta)$ .....(5)

Hence , α and β are the roots of the equation. (2)

and therefore , also of the equation (1)

The method can obviously be generalized to any number equations.

Newton - Raphson Method

We consider a system of two equations :

$$\left.\begin{array}{r} f(x,y)=0 \\ \text{and}\ g(x,y)=0 \end{array}\right\} \quad ... (1)$$

Let (x_0, y_0) be an intial approximation to the root of equation (1)

If (x_0+h, y_0+k) is the root of the system, then we must have,

$f(x_0 + h, y_0 + k) = 0,$

$g(x_0 + h, y_0 + k) = 0$ ... (2)

Assuming that f and g are sufficiently differentiable, we expand both the functions in equation (2) by Taylor’s series to obtain

$$\left.\begin{aligned} f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} + ... = 0 \\ g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} + ... = 0 \end{aligned}\right\} \quad ... (3)$$

Where

$$\frac{\partial f}{\partial x_0} = \left(\frac{\partial f}{\partial x}\right)_{x=x_0}, \; f_0 = f(x_0, y_0) \text{ etc.}$$

Neglecting the second and higher - order derivative terms, we obtain the following system of linear equations:

$$\left.\begin{aligned} h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} = -f_0 \\ \text{and} \quad h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} = -g_0 \end{aligned}\right\} \quad ... (4)$$

Equation (4) possesses a unique solution If

$$D = \begin{vmatrix} \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial y_0} \\ \frac{\partial g}{\partial x_0} & \frac{\partial g}{\partial y_0} \end{vmatrix} \neq 0$$

By cramer's rule, the solution of equation (4) is given by,

$$h = \frac{1}{D}\begin{vmatrix} -f_0 & \frac{\partial f}{\partial y_0} \\ -g_0 & \frac{\partial g}{\partial y_0} \end{vmatrix} \text{ and } k = \frac{1}{D}\begin{vmatrix} \frac{\partial f}{\partial x_0} & -f_0 \\ \frac{\partial g}{\partial x_0} & -g_0 \end{vmatrix} \quad .... (5)$$

The new approximations are,

$x_1 = x_0 + h$ and $y_1 = y_0 + k$

The process is to be repeated till we obtain the roots to the desired accuracy

Q48. Find a root of the equations $y^2-5y+4=0$ and $3yx^2-10x+7=0$ using the iteration method.

Sol :

Given system of equations,

$$\left.\begin{aligned} y^2 - 5y + 4 = 0 \\ 3yx^2 - 10x + 7 = 0 \end{aligned}\right\} \quad .... (1)$$

clearly,a real root is $x=1$ and $y=1$

To apply the iteration method,

we rewrite the equation (1) as,

$x = \frac{1}{10}(3yx^2 + 7)$....(2)

and $y = \frac{1}{5}(y^2 + 4)$ ....(3)

Here,

$F(X,Y) = \frac{1}{10}(3yx^2 + 7)$

$\frac{\partial F}{\partial X} = \frac{6xy}{10}$; $\frac{\partial F}{\partial Y} = \frac{3x^2}{10}$

$G(x,y) = \frac{1}{5}(y^2 + 4)$

$\frac{\partial G}{\partial x} = 0$; $\frac{\partial G}{\partial y} = \frac{2y}{5}$

Let (0.5 ,0.5) be an approximate root . Then ,

$\left|\frac{\partial G}{\partial x}\right| + \left|\frac{\partial G}{\partial y}\right| = \left|\frac{6xy}{10}\right|_{(0.5,0.5)} + \left|\frac{3x^2}{10}\right|_{(0.5,0.5)}$

$= 0.15 + 0.075 < 1$

and $\left|\frac{\partial G}{\partial x}\right| + \left|\frac{\partial G}{\partial y}\right| = \left|\frac{2y}{5}\right|_{(0.5)} = 0.2 < 1$

Hence the conditions for convergence are satisfied and the approximations are given by ,

$x_{n+1} = \frac{1}{10}\left[3y_n x^2_n + 7\right]$ and $y_{n+1} = \frac{1}{5}\left[y^2_n + 4\right]$

we obtain successfully with $x_0 = 0.5$ and $y_0 = 0.5$

$x_1 = \frac{1}{10}\left[\frac{3}{8} + 7\right] = 0.7375$ $\qquad y_1 = \frac{1}{5}\left[\frac{1}{4} + 4\right] = 0.85$

$x_2 = \frac{1}{10}\left[3(0.85)(0.7375)^2 + 7\right]$ $\qquad y_2 = \frac{1}{5}\left[(0.85)^2 + 4\right] = 0.9445$

$x_3 = \frac{1}{10}\left[3(0.8387)^2(0.9445) + 7\right]$ $\qquad y_3 = \frac{1}{5}\left[(0.9445)^2 + 4\right] = 0.9784$

$x_4 = \frac{1}{10}\left[3(0.8993)^2(0.9784)^2 + 7\right]$ $\qquad y_4 = \frac{1}{5}\left[(0.9784)^2 + 4\right] = 0.9914$

$= 0.9374$

$x_5 = \frac{1}{10}\left[3(0.9374)(0.9914)+7\right]$ $y_5 = \frac{1}{5}\left[(0.9914)^2+4\right]$

$= 0.9613$ $= 0.9966$

$x_6 = 0.9763$ $y_6 = 0.9986$

$x_7 = 0.9855$ $y_7 = 0.9994$

Converges to the root (1, 1) is obvious.

49. Solve the system of equations

$y^2 - 5y + 4 = 0$ and $3yx^2 - 10x + 7 = 0$ by Newton - Raphson Method.

Sol :

We have

$f(x) = 3yx^2 - 10x + 7 = 0$

$g(x) = y^2 - 5y + 4 = 0$

Then, $\frac{\partial f}{\partial x} = 6yx - 10$; $\frac{\partial f}{\partial y} = 3x$

$\frac{\partial g}{\partial x} = 0$; $\frac{\partial g}{\partial y} = 2y - 5.$

Taking $x_0 = y_0 = 0.5$

We obtain

$\frac{\partial f}{\partial x_0} = -8.5,\ \frac{\partial f}{\partial y_0} = 0.75,\ f_0 = 2.375$

$\frac{\partial g}{\partial x_0} = 0,\ \frac{\partial g}{\partial y_0} = -4,\ g_0 = 1.75$

Hence,

$$D = \begin{vmatrix} -8.5 & 0.75 \\ 0 & -4 \end{vmatrix} = 34$$

Therefore,

$$h = \frac{1}{34}\begin{vmatrix} -2.375 & 0.75 \\ -1.75 & -4 \end{vmatrix} = 0.3180$$

and $k = \frac{1}{34}\begin{vmatrix} -8.5 & -2.375 \\ 0 & -1.75 \end{vmatrix} = 0.4375$

If follows that,

$x_1 = 0.5 + 0.3180 = 0.8180$

$y_1 = 0.5 + 0.4375 = 0.\ 9375$

For the second approximation, we have

$$f_1 = 0.7019, \quad ; \quad g_1 = 0.1914$$

$$\frac{\partial f}{\partial x_1} = -5.3988 \quad ; \quad \frac{\partial f}{\partial y_1} = 2.0074$$

$$\frac{\partial g}{\partial x_1} = 0 \quad ; \quad \frac{\partial g}{\partial y_1} = -3.125$$

Therefore,

$$D = \begin{vmatrix} -5.3988 & 2.0074 \\ 0 & -3.125 \end{vmatrix} = 16.8712$$

Hence

$$h = \frac{1}{16.8712}\begin{vmatrix} -0.7019 & 2.0074 \\ -0.1914 & -3.125 \end{vmatrix} = 0.1528$$

$$k = \frac{1}{16.8712}\begin{vmatrix} -5.3918 & -0.7019 \\ 0 & -0.1914 \end{vmatrix} = 0.0612$$

Then

$$x_2 = 0.8180 + 0.1528 = 0.9708$$

$$y_2 = 0.94375 + 0.0612 = 0.9987$$

Q50. Solve the system $x^2 + y^2 = 1$ and $y = x^2$ by Newton - Raphson method.

Sol :

Let

$$f = x^2 + y^2 - 1 \text{ and } g = y - x^2$$

From the graphs of the curves, we find that there are two points of intersection, one each in the first and second Quadrants. we shall approximate to the solution to the first Quadrant.

We have

$$\frac{\partial f}{\partial x} = 2x, \; \frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial g}{\partial x} = -2x, \; \frac{\partial g}{\partial y} = 1$$

We start with $x_0 = y_0 = 0.7071$ obtained from the approximation $y = x$.

$$\frac{\partial f}{\partial x_0} = 1.4142, \quad \frac{\partial f}{\partial y_0} = 1.4142$$

$$\frac{\partial g}{\partial x_0} = -1.4142, \quad \frac{\partial g}{\partial y_0} = 1.$$

Therefore,

$$D = \begin{vmatrix} 1.4142 & 1.4142 \\ -1.4142 & 1 \end{vmatrix} = 3.4142,\ f_0 = 0$$

Hence,

$$h = \frac{1}{3.4142}\begin{vmatrix} 0 & 1.4142 \\ -0.2071 & 1 \end{vmatrix} = 0.0858$$

and

$$k = \frac{1}{3.4142}\begin{vmatrix} 1.4142 & 0 \\ -1.4142 & -0.2071 \end{vmatrix} = -0.0858$$

$\therefore$ $x_1 = 0.7071 + 0.0858 = 0.7858$

and $y_1 = 0.7071 - 0.0858 = 0.6213$

For the second approximation,

$f_1 = 0.0035, \quad g_1 = 0.0038$

$$\frac{\partial f}{\partial x_1} = 1.5716, \quad \frac{\partial f}{\partial y_1} = 1.2426$$

$$\frac{\partial g}{\partial x_1} = -1.5716, \quad \frac{\partial g}{\partial y_1} = 1.$$

Now,

$$D = \begin{vmatrix} 1.5716 & 1.2426 \\ -1.5716 & 1 \end{vmatrix} = 3.5245,$$

Hence,

$$h = \frac{1}{3.5245}\begin{vmatrix} -0.0035 & 1.2426 \\ -0.0038 & 1 \end{vmatrix} = 0.0003$$

$$k = \frac{1}{3.5245}\begin{vmatrix} 1.5716 & -0.0035 \\ -1.5716 & -0.0038 \end{vmatrix} = -0.0033$$

$\therefore$ $x_2 = 0.7858 + 0.0003 = 0.7861$

$y_2 = 0.6213 - 0.0033 = 0.6180$

Q51. Solve the system

$\sin x - y = -0.9793$

$\cos y - x = -0.6703$ with $x_0 = 0.5$ and $y_0 = 1.5$ as

The initial approximation by Newton - Raphson method

Sol :

We have

$f(x,y) = \sin x - y + 0.9793$

$g(x,y) = \cos y - x + 0.6703$

1st interation :

$f_0 = -0.0413 \quad ; \quad g_0 = 0.2410$

$$\frac{\partial f}{\partial x} = \cos x \quad ; \quad \frac{\partial f}{\partial y} = -1$$

$$\frac{\partial g}{\partial x} = -1 \quad ; \quad \frac{\partial g}{\partial y} = -\sin y.$$

We start with $x_0 = 0.5$ and $y_0 = 1.5$

$$\frac{\partial f}{\partial x_0} = 0.8776 \quad ; \quad \frac{\partial f}{\partial y_0} = -1$$

$$\frac{\partial g}{\partial x_0} = -1 \quad ; \quad \frac{\partial g}{\partial y_0} = -0.9975$$

$$\therefore \quad D = \begin{vmatrix} 0.8776 & -1 \\ -1 & -0.9975 \end{vmatrix} = -1.8754$$

Hence,

$$h = \frac{1}{-1.8754}\begin{vmatrix} 0.0413 & -1 \\ -0.2410 & -0.9975 \end{vmatrix} = 0.1505$$

$$k = \frac{1}{-1.8754}\begin{vmatrix} 0.8776 & 0.0413 \\ -1 & -0.2410 \end{vmatrix} = 0.0908$$

$x = 0.5 + 0.1505 = 0.6505$

$y = 1.5 + 0.0908 = 1.5908$

For the second iteration,

$x_0 = 0.6505 \quad ; \quad y_0 = 1.5908$

$f_0 = -0.0059 \quad ; \quad g_0 = -0.0002$

$$\frac{\partial f}{\partial x_0} = 0.7958 \quad ; \quad \frac{\partial f}{\partial y_0} = -1$$

$$\frac{\partial g}{\partial x_0} = -1 \ ; \quad \frac{\partial g}{\partial y_0} = -0.9998$$

$$D = \begin{vmatrix} 0.7958 & -1 \\ -1 & -0.9998 \end{vmatrix} = -1.7956$$

$$h = \frac{1}{-1.7956}\begin{vmatrix} 0.0059 & -1 \\ 0.0002 & -0.9998 \end{vmatrix} = 0.0032$$

$$k = \frac{1}{-1.7956}\begin{vmatrix} 0.7958 & 0.0059 \\ -1 & 0.0002 \end{vmatrix} = -0.0034$$

Now

$$x = 0.6505 + 0.0032 = 0.6537$$

$$y = 1.5908 - 0.0034 = 1.5874$$

For the third iteration,

$$x_0 = 0.6537 \quad ; \quad y_0 = 1.5874$$

$$f_0 = 0 \quad ; \quad g_0 = 0$$

$$\frac{\partial f}{\partial x_0} = 0.7938 \quad ; \quad \frac{\partial f}{\partial y_0} = -1$$

$$\frac{\partial g}{\partial x_0} = -1 \quad ; \quad \frac{\partial g}{\partial y_0} = -0.9999$$

$$D = \begin{vmatrix} 0.7938 & -1 \\ -1 & -0.9999 \end{vmatrix} = -1.7937$$

$$h = \frac{1}{-1.7937} \begin{vmatrix} 0 & -1 \\ 0 & -0.9999 \end{vmatrix} = 0$$

$$k = \frac{1}{-1.7937} \begin{vmatrix} 0.7938 & 0 \\ -1 & 0 \end{vmatrix} = 0$$

Now

$$x = 0.6537 + 0 = 0.6537$$

$$y = 1.5874 + 0 = 1.5874$$

In second and third iteration the values of x and y are same.

Hence, the required values of x, y are

$$x = 0.6537$$

$$y = 1.5874$$

Choose the Correct Answers

1. The number 1.6583 round off to four significant figure is ________ [c]
   (a) 1.65 (b) 1.6
   (c) 1.658 (d) None

2. The difference $\sqrt{6.37} - \sqrt{6.35}$ to three significant figures is ________ [b]
   (a) 0.0198 (b) 0.00198
   (c) 0.198 (d) 198

3. The Absolute error E_A is ________ [a]
   (a) $X - X_1$ (b) $X_1 - X$
   (c) Both (d) None

4. The Method of false position is also called as ________ [c]
   (a) Bisection Method (b) Newton's Method
   (c) Regular Falsi Method (d) None

5. In Iteration Method we express the equation f(x) = 0 in the form $x = \phi(x)$ must be such that [b]
   (a) $|\phi^1(x)| = 1$ (b) $|\phi^1(x)| < 1$
   (c) $|\phi^1(x)| \neq 1$ (d) None

6. Aitken's Δ^2 method can be used to ________ [c]
   (a) Accelerate the convergence of sequence that is convergent
   (b) Not convergent
   (c) Accelerate the convergence of a sequence that is linearly convergent
   (d) None

7. The fixed points for the function $f(x) = x^2$ is ________ [a]
   (a) 0, 1 (b) 2, 0
   (c) 1, 1 (d) None of these

8. The first approximation in Muller's Method is determined as ________ [d]
   (a) $x_3 = x_2 + \frac{2c}{b + \text{sgn}(b)\sqrt{b^2 - 4ac}}$ (b) $x_3 = x_2 + \frac{2c}{b - \text{sg}(b)\sqrt{b^2 - 4ac}}$
   (c) $x_3 = x_2 - \frac{2c}{b - \text{sgn}(b)\sqrt{b^2 - 4ac}}$ (d) $x_3 = x_2 - \frac{2c}{b + \text{sgn}(b)\sqrt{b^2 - 4ac}}$

9. The solution of simultaneous non-linear equations solved by ________ [c]
   (a) Iteration method (b) Newton - Raphson Method
   (c) Both (d) None

10. $f(x) = c_1e^x + c_2e^{-x} = 0$ is a ________ equation [a]
    (a) Transcendental (b) Algebraic
    (c) Both (d) None

Fill in the Blanks

1. Round - off the number 52.275 to two decimal places is ________.
2. The value & ln 3 correct to find decimal places is ________._
3. The percentage error (E_p) by ________.
4. A number α is called a root of an equation f(x) = 0 if ________.
5. The method of generating better and better approximation from an initial guess is called ________.
6. The newton Raphson formula is ________.
7. Approximate value of $\sqrt{3}$ = ________.
8. If $f(x) = e^x - x - 1$ and $x_0 = 1$ then by Newton's Raphson method $x_1 \approx$ ________.
9. The number x is a find point for a function 'g' if ________.
10. The sequence is said to be linearly convergent if ________.

Answers

1. 52.28
2. 1.09861
3. 100 E_R
4. $f(\alpha) = 0$
5. Iteration method
6. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
7. 1.732051
8. 0.58198
9. $\phi(x) = x$
10. $\alpha = 1$

UNIT II

Interpolation and Polynomial Approximation: Interpolation - Finite Differences - Differences of Polynomials - Newton's formula for Interpolation-Gauss's central differences formulae - Stirling's and Bessel's formula - Lagrange's Interpolation Polynomial - Divided Differences - Newton's General Interpolation formula - Inverse Interpolation.

2.1 Interpolation Finite Differences - Differences of Polynomials

Q1. Define interpolation.

Ans :

We consider the statement $y = f(x)$, $x_0 \leq x \leq x_n$, that means we can fine the value of y, corresponding to every value of x in the range $x_0 \leq x \leq x_n$. If the function f(x) is single valued and continuous that is know explicity then the value corresponding of f(x) for certain values of x like x_0, x_1, ... x_n can be calculated and tabulated. The setof tabulor values (x_0, y_0), (x_1, y_1), ... (x_n, y_n) satisfying the relation $y = f(x)$ where the explicit defination of f(x) is not known, it is required to find a simpler function, say $\phi(x)$, such that f(x) and $\phi(x)$ agree at the set of tabulated points. Such process is called interpolation. If $\phi(x)$ is a polynomial, then the process is called polynomial interpotation and $\phi(x)$ is called interpolating polynomial.

Finite Differences

Consider a function

$$y = f(x)$$

Let y_0, y_1, y_2 ...... y_n be the values of y corresponding to the values of x_0, x_1, x_2 .... x_n of x respectinely.

Then the differences $y_1 - y_0$, $y_2 - y_1$ .... are called the first forward differences of y.

We denote them by Δy_0, Δy_1 ...... i.e., $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$, $\Delta y_2 = y_3 - y_2$ ......

In general,

$\Delta y_{n1} = y_n - y_{n-1}$, n = 0, 1, 2 .....

Where Δ is called the forward difference operator. and Δy_0, Δy_1, ... are called first forward defferences.

The differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0$, $\Delta^2 y$, ...

Similarly we can define third forward differences, fourte forward differences, etc.

Thus

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$
$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0)$$
$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.$$

Forward Difference Table

x	y	First Differences	Second Differences	Third Differences	Fourth Differences	Fifth Differences
x_0	y_0					
		$\Delta y_0 = y_1 - y_0$				
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$			
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$		
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$	
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$		$\Delta^5 y_0 = \Delta^4 y_1 - \Delta^4 y_0$
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		$\Delta^4 y_1 = \Delta^3 y_2 - \Delta^3 y_0$	
		$\Delta y_3 = y_4 - y_3$		$\Delta^3 y_2 = \Delta^3 y_3 - \Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3 = \Delta y_4 - \Delta y_3$			
		$\Delta y_4 = y_5 - y_4$				
x_5	y_5					

Backward Differences

The differences $y_1 - y_0$, $y_2 - y_1$, ... $y_n - y_{n-1}$ are called first backward differences. If they are denoted by $\nabla y_1, \nabla y_2$, ... ∇y_n respectively,

So that

$$\nabla y_1 = y_1 - y_0$$
$$\nabla y_2 = y_2 - y_1$$
.
.
.
$$\nabla y_n = y_n - y_{n-1}$$

Where ∇ is called the backward difference opercitor.

In a similar way, one can define backward difference of higher orders.

Thus, we obtain,

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$

$= y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$

$= y_3 - 3y_2 + 3y_1 - y_0$ etc

Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
x_1	y_1	∇y_1				
x_2	y_2	∇y_2	$\nabla^2 y_2$			
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$		
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$	
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$

Central Differences

y_0, y_1 ...... y_n as the values of a function $y = f(x)$ corresponding to the values x_0, x_1, ... x_n of x

We define the first Central Differences $\delta y_{1/2}$, $\delta y_{3/2}$, $\delta y_{5/2}$ ...... as follows.

$\delta y_{1/2} = y_1 - y_0,$

$\delta y_{3/2} = y_2 - y_1,$

$\delta y_{5/2} = y_3 - y_2$

:

.

$\delta y_{n-1/2} = y_n - y_{n-1}$ ..... (1)

The symbol δ is called the central difference operator. This operator is a linear operator.

Comparing expressions (1) above with expansions earlier used on forward and backward differences, we get

$\delta y_{1/2} = \Delta y_0 = \nabla y_1$

$\delta y_{3/2} = \Delta y_1 = \nabla y_2$

$\delta y_{5/2} = \Delta y_2 = \nabla y_3$ ......

In general, $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}$, n = 0, 1, 2, ...

Central Differences Table

x	y	δ	δ^2	δ^3	δ^4	δ^5	δ^6
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

2.1.1 Symbolic Relations and Separation of Symbols

2Q. Define averageing, shift, inverse operators.

Ans :

Averaging operator

The averaging operator μ is defined by the equation

$$\mu y_r = \frac{1}{2}\left(y_{r+1/2} + y_{r-1/2}\right).$$

Shift Operator

The shifted operatar E is defined by the equation $Ey_r = y_{r+1}$

This shows that the effect of E is to shift the functional value y_r, to the next higher value y_{r+1}.

A second operation with E gives,

$$E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$$

In general,

$$E^n y_r = y_{r+n}$$

Inverse Operator

E^{-1} is defined as $E^{-1} y_r = y_{r-1}$

In general $E^{-n} y_r = y_{r-n}$

Q3. Prove that $\Delta = E - 1$

Sol :

We have

$$\Delta y_0 = y_1 - y_0$$
$$= Ey_0 - y_0$$
$$\Delta y_0 = (E - 1)\, y_0$$
$$\Rightarrow \Delta = E - 1 \text{ or } E = I + \Delta$$

Q4. Prove that $\nabla \equiv 1 - E^{-1}$

Sol :

$$\nabla (y_r) = y_r - y_{r-1}$$
$$= 1 \,.\, y_r - E^{-1}(y_r)$$
$$\nabla (y_r) = (1 - E^{-1})y_r$$
$$\nabla = 1 - E^{-1}$$

Q5. Prove that $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

Sol :

$$\delta y_r = y_{\left(r+\frac{1}{2}\right)} - y_{\left(r-\frac{1}{2}\right)}$$
$$= E^{\frac{1}{2}} y_r - E^{-\frac{1}{2}} y_r$$
$$\delta y_r = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) y_r$$
$$\therefore \quad \delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

Q6. Prove that $\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$.

Sol :

$$\mu y_r = \frac{1}{2}\left(y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right)$$
$$= \frac{1}{2}\left(E^{\frac{1}{2}} y_r + E^{-\frac{1}{2}} y_r\right)$$
$$= \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) y_r$$
$$\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right).$$

Q7. Prove that $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

Sol

$$\mu y_r = \frac{1}{2}\left(y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(E^{\frac{1}{2}}y_r + E^{-\frac{1}{2}}y_r\right)$$

$$\mu y_r = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)y_r$$

$$\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right).$$

$$\mu^2 = \frac{1}{4}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)^2$$

$$= \frac{1}{4}(E + E^{-1} + 2)$$

$$= \frac{1}{4}\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 + 4\right]$$

$$\boxed{\mu^2 = \frac{1}{4}\left(\delta^2 + 4\right)}$$

Q8. Prove that $\delta = E^{\left(-\frac{1}{2}\right)}\Delta = \nabla E^{\frac{1}{2}}$

Sol :

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

$$= E^{-\frac{1}{2}}(E-1)$$

$$\delta = E^{-\frac{1}{2}}\Delta$$

Also $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} = E^{\frac{1}{2}}\left(1 - E^{-1}\right) = \nabla E^{\frac{1}{2}}$

Hence $\delta = E^{-\frac{1}{2}}\Delta = \nabla E^{\frac{1}{2}}$

Q9. Using the method of separation of symbols, show that

$$\Delta^n u_{x-n} = u_x - nu_{x-1} + \frac{n(n-1)}{2} u_{x-2} + \ldots\ldots + (-1)^n u_{x-n}$$

Sol :

Consider

$$u_x - nu_{x-1} + \frac{n(n-1)}{2} u_{x-2} + \ldots\ldots + (-1)^n u_{x-n}$$

$$= u_x - n E^{-1} u_{x-1} + \frac{n(n-1)}{2} E^{-2} u_x + \ldots\ldots + (-1)^n E^{-n} u_x$$

$$= [1 - n E^{-1} + \frac{n(n-1)}{2} E^{-2} + \ldots\ldots + (-1)^n E^{-n}] u_x$$

$$= (1 - E^{-1})^n u_x$$

$$= \left(1 - \frac{1}{E}\right)^n u_x = \left(\frac{E-1}{E}\right)^n u_x$$

$$= \frac{\Delta^n}{E^n} u_x$$

$$= \Delta^n E^{-n} u_x$$

$$= \Delta^n u_{x-n}$$

$$\therefore \quad \Delta^n u_{x-n} = u_x - nu_{x-1} + \frac{n(u-1)}{2} u_{x-2} + \ldots\ldots\ldots + (-1)^n u_{x-n}$$

Q10. Show that $e^x(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \ldots\ldots) = u_0 + u_1 x + u^2 \frac{x^2}{2!} + \ldots\ldots$

Sol :

Now

$$e^x \left[u_0 + x\Delta u_0 + \frac{x^2}{2!} \Delta^2 u_0 + \ldots\ldots \right] = e^x \left[1 + x\Delta + \frac{x^2}{2!} \Delta^2 + \ldots\ldots \right] u_0$$

$$= e^x e^{x\Delta} u_0$$

$$= e^{x + x\Delta} u_0$$

$$= e^{x(1+\Delta)} u_0$$

$$= e^{xE} u_0$$

$$= \left(1 + xE + \frac{x^2 E^2}{2!} + \ldots\ldots\right) u_0$$

$$= u_0 + xu_1 + \frac{x^2}{2!} u_2 + \ldots\ldots\ldots$$

which is required result.

Q11. Find the missing term in the following data

x	0	1	2	3	4
y	1	3	9	-	81

Sol :

Consider $\Delta^4 y_0 = 0$ (We are given only four values)

Since $\Delta^4 y_0 = 0$

$\Rightarrow \quad y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Substitute given values we get

$81 - 4y_3 + 54 - 12 + 1 = 0$

$\Rightarrow \quad y_3 = 31$

2.1.2 Differences of a Polynomial

Q12. Theorem :

If y(x) is a polynomial of degree n and the values of x are equally spaced then Δ^n f(x) is a constant.

Proof :

Let $y(x) = a_0 x_n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n.$

Where

$a_0, a_1, a_2, \ldots\ldots\ a_n$ are constants and $a_0 \neq 0$.

We know the formula for the first forward difference

$$\Delta y(x) = y(x+h) - y(x)$$

$$= [a_0(x+h)^n + a_1(x+h)^{n-1} + \ldots\ldots + a_{n-1}(x+h) + a_n] - [a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1}x + a_n]$$

$$= a_0\left[\left\{x^n + n.x^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}.h^2 + \ldots\ldots\right\} - x^n\right]$$

$$+ a_1\left[\left\{x^{n-1} + (n-1)x^{n-2}h + \frac{(n-1)(n\ 2)}{2!}x^{n-3}h^2 \ldots\ldots\right\} - x^{n-1}\right] + \ldots\ldots + a_{n-1}h.$$

$$= a_0 nhx^{n-1} + a'_1 x^{n-2} + a'_2 x^{n-3} + \ldots\ldots + a'_{n-2}x + a'_{n-1}$$

Where

$a'_1, a'_2, \ldots\ldots\ a'_{n-1}$ are constants

Here this is polynomial is of degree (n – 1)

Thus, the first difference of a polynomial of n^{th} degree is a polynomial of degree (n – 1).

Now, $\Delta^2 y(x) = \Delta\ [\Delta y(x)]$

$$= \Delta[a_0 nh\ .\ x^{n-1} + a^1{}_1 x^{n-2} + a^1{}_2 x^{n-3} - \ldots\ldots + a^1{}_{n-1}]$$

$= a_0 nh\,[(x + h)^{n-1} - x^{n-1}] + a^1_1\,[(x + h)^{n-2} - x^{n-2}] + ... + a'_{n-3}[(x + h) - x]$

$= a_0 n\,(n - 1)h^2 x^{n-2} + a^{11}_1\, x^{n-3} + ... + a^{11}_{n-2}\, x + a^{11}_{n-1}$

Where

$a_1'', a_2'', ... - a_{n-1}''$ are constants.

This is a polynomial of degree (n – 2)

Thus, the second difference of a polynomial of degree n is a polynomial of degree (n – 2).

Continuing like this, we get

$\Delta^n y(x) = a_0 n\,(n - 1)\,(n - 2)\,......\,2.1.h^n = a_0\, h^n\,(n!)$

Which is a constant

Hence the result.

2.2 Newton's Formula for Interpolation

Q13. Derive Newton's forward difference interpolation formula.

Ans :

Let $y = f(x)$ be a polynomial of degree n.

i.e, $y = f(x) = a_0 + a_1(x - x_0) + a_2\,(x - x_0)\,(x - x_1) + a_3\,(x - x_0)\,(x - x_1)(x - x_2) + ...$

$+ a_n\,(x - x_0)\,(x - x_1)\,......\,(x - x_{n-1})$ ..... (1)

This polynomial passes through all the points (x_i, y_i) for $i = 0, 1, 2$ ......n

Therefore, we can obtain the y_i's by substituting the corresponding x_i's as :

at $x = x_0$, $y_0 = a_0$.

at $x = x_1$, $y_1 = a_0 + a_1(x_1 - x_0)$

at $x = x_2$, $y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$ and so on. ..... (2)

Let h be the length of the internal such that x_i's represents.

$x_i = x_0 + ih, \quad i = 0, 1, 2 ...... n.$

$\Rightarrow\ x_1 - x_0 = h, \quad x_2 - x_0 = 2h, \quad x_3 - x_0 = 3h\ ....\ x_n - x_0 = nh$ ..... (3)

From (2) and (3), we get

$y_0 = a_0$

$y_1 = a_0 + a_1 h$

$y_2 = a_0 + a_1\, 2h + a_2\,(2h)h$

$y_3 = a_0 + a_1\, 3h + a_2\,(3h)\,(2h)h + a_3\,(3h)\,(2h)h$

. . . .

. . . .

$y_n = a_0 + a_1\,(nh) + a_2\,(nh)(n-1)hn + ... + a_n\,(nh)\,[(n - 1)h]\,[(n - 2)h]$ ..... (4)

Solving these equations for $a_0, a_1, a_2\ .....\ a_n$ we get

$a_0 = y_0.$

$$a_1 = \frac{y_1 - a_0}{h} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

$$a_2 = \frac{y_2 - a_0 - a_1 2h}{2h^2} = \frac{y_2 - y_0 - \left(\frac{y_1 - y_0}{h}\right)2h}{2h^2}$$

$$= \frac{y_2 - y_0 - 2y_1 - 2y_0}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2}$$

$$= \frac{\Delta^2 y_0}{2h^2}$$

$$\therefore \quad a_2 = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly, we can obtain

$$a_3 = \frac{\Delta^3 y_0}{3!h^3}, \; a_4 = \frac{\Delta^4 y_0}{4!h^4}, \; .... \; a_n = \frac{\Delta^n y_0}{n!h^n}$$

$$\therefore \quad y = f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + $$

$$..... + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1) ..... (x - x_{n-1}) \qquad ...... (5)$$

Setting $x = x_0 + ph \Rightarrow x - x_0 = ph$ where $p = 0, 1, 2 .... n.$

Then $x - x_1 = x - (x_0 + h) = x - x_0 - h = ph - h = (p - 1)h.$

$x - x_2 = x - (x_1 + h) = (x - x_1) - h = (p - 1)h - h = (p - 2)h.$

. . . .

. . . .

$x - x_i = (p - i)h$

. . . .

. . . .

$x - x_{n-1} = [p - (n - 1)]h.$

$\therefore$ Equation (5) becomes,

$$y = f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ...$$

$$+ \frac{p(p-1)(p-2).......[p-(n-1)]}{n!}\Delta^n y_0 \qquad ..... (6)$$

Which is Newton's forward difference interpolation formula.

This is useful for interpolation near the beginnig of a set of tabular values.

Q14. Derive Newton's Backward Interpolation

Ans :

If we consider

$y_n(x) = a_0 + a_1 (x - x_n) + a_2 (x - x_n)(x - x_{n-1}) + a_3 (x - x_n)(x - x_{n-1})(x - x_{n-2}) + ...$

$+ a_n (x - x_n)(x - x_{n-1})(x - x_{n-2}) .... (x - x_1)$

And then impose the condition that y and $y_n(x)$ should agree at the tabulated points

$x_n, x_{n-1}, ... x_2, x_1, x_0$.

We obtain,

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + ... \frac{p(p+1)......(P+n-1)}{n!}\nabla^n y_n.$$

Where

$$p = \frac{x - x_n}{h}$$

This is Newton's backward differnece interpolation formula.

If uses tabular values to the left of y_n.

Thus, this formula useful for interpolation near the and of the tabular values.

Q15. Find the cubic polynomial which takes the following values y(1) = 24, y(3) = 120, y(5) = 336 and y(7) = 720.

Hence obtain the value of y(8).

Sol :

The difference table is :

x	y	Δ	Δ^2	Δ^3
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		324		
7	720			

Hence h = 2, x_0 = 1.

$$p = \frac{x - x_0}{h} = \frac{x-1}{2}$$

Now, the newton's forward interpolation formula is,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(P-1)}{2!}\Delta^2 y_0 + \frac{p(P-1)(p-2)}{3!}\Delta^3 y_0 + ...$$

$$= 24 + \frac{x-1}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2}(120) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{6}(48)$$

$$y(x) = x^3 + 6x^2 + 11x + 6.$$

Hence, $y(8) = 8^3 + 6(8)^2 + 11(8) + 6 = 990$

Q16. Values of x (in degrees) and sinx are given in the following table.

x (In degrees)	**15**	**20**	**25**	**30**	**35**	**40**
Sinx	**0.2588190**	**0.3420201**	**0.4226183**	**0.5**	**0.5735764**	**0.6427876**

Determine the value of sin 38°.

Sol :

The difference table is

x	**sinx**	**Δ**	**Δ²**	**Δ³**	**Δ⁴**	**Δ⁵**
15	0.25883190					
		0.0832011				
20	0.3420201		–0.0026029			
		0.0805982		–0.0006136		
25	0.4226183		–0.0032165		0.0000248	
		0.0773817		–0.0005888		0.0000041
30	0.5		–0.0038053		0.0000289	
		0.0735764		–0.0005599		
35	0.5735764		–0.0043652			
		0.0692112				
40	0.6427876					

To find sin 38°.

We use Newton's backward difference formula, with $x_n = 40$ and $x = 38$.

$$p = \frac{x-x_n}{h} = \frac{38-40}{5} = -\frac{2}{5} = -0.4$$

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n + ...$$

$$y(38) = 0.6427876 - 0.4\ (0.0692112) + \frac{-0.4(-0.4+1)}{2}(-0.0043652)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)}{6} \quad (-0.\ 0005599)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{24} \quad (0.0000289)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)(0.4+4)}{120} \quad (0.0000041)$$

= 0.6427876 – 0.02768448 + 0.00052382 + 0.00003583 – 0.00000120

y(38) = 0.6156614

Q17. The table below gives the values of tanx for

0.10 ≤ x ≤ 0.30

x	y = tanx
0.10	**0.1003**
0.15	**0.1511**
0.20	**0.2027**
0.25	**0.2553**
0.30	**0.3093**

Find (a) tan 0.12 (b) tan 0.26

Sol :

The table of difference is

x	y	Δ	Δ^2	Δ^3	Δ^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

(a) To find tan (0.12)

Here x = 0.12, x_0 = 0.10 and h = 0.05

$$p = \frac{x - x_0}{h} = \frac{0.12 - 0.10}{0.05} = 0.4$$

By Newton's forward interpolation formula,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + ...$$

$$\tan(0.12) = 0.1003 + 0.4\,(0.0508) + \frac{0.4(0.4-1)}{2}(0.0008) +$$

$$\frac{0.4(0.4-1)(0.4-2)}{6}(0.0002) +$$

$$\frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(0.0002)$$

$\tan(0.12) = 0.1205$

(b) To find tan (0.26)

Here $x = 0.26$, $x_n = 0.30$ and $h = 0.05$

$$p = \frac{x - x_n}{h} = \frac{0.26 - 0.30}{0.05} = -0.8$$

By Newton's Backward interpolation formula,

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + ...$$

$$\tan(0.26) = 0.3093 - 0.8\,(0.0540) + \frac{-0.8(-0.8+1)}{2}(0.0014) +$$

$$\frac{(-0.8)(-0.8+1)(-0.8+2)}{6}(0.0004) + \frac{(-0.8)(-0.8+1)(-0.8+2)(-0.8+3)}{24}(0.0002)$$

$\tan(0.26) = 0.2662$

Q18. Find f(0.23) and f(0.29) from the following table :

x	**0.20**	**0.22**	**0.24**	**0.26**	**0.28**	**0.30**
f(x)	**1.6596**	**1.6698**	**1.6804**	**1.6912**	**1.7024**	**1.7139**

Sol :

The difference table is,

x	y	Δy	Δ²y	Δ³y	Δ⁴y	Δ⁵y
0.20	1.6596					
		0.0102				
0.22	1.6698		0.0004			
		0.0106		–0.0002		
0.24	1.6804		0.0002		0.0004	
		0.0108		0.0002		–0.0007
0.26	1.6912		0.0004		–0.0003	
		0.0112		–0.0001		
0.28	1.7024		0.0003			
		0.0115				
0.30	1.7139					

(a) To find f(0.23)

Here $x = 0.23,\ x_0 = 0.20$ and $h = 0.02$

$$p = \frac{x - x_0}{h} = \frac{0.23 - 0.20}{0.20} = \frac{0.03}{0.02} = 1.5$$

By Newton's forward interpolation is,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0 + ...$$

$$y(0.23) = 1.6596 + 1.5\ (0.0102) + \frac{1.5(1.5-1)}{2}\ (0.0004) + \frac{1.5(1.5-1)(1.5-2)}{6}(-0.0002)$$

$$+ \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{24}\ (0.0004) +$$

$$\frac{1.5(1.5-1)(1.5-2)(1.5-3)(1.5-4)}{120}(-0.0007)$$

$$= 1.6596 + 0.0153 + 0.00015 + 0.000125 + 0.000009375 + 0.000008203$$

$$y(0.23) = 1.6752.$$

(b) To find f(0.29)

Here $x = 0.29,\ x_n = 0.30$ and $h = 0.02$

$$p = \frac{x - x_n}{h} = \frac{0.29 - 0.30}{0.02} = -0.5$$

By Newton's Backward interpolation is,

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$+ \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n + ...$$

$$= 0.7139 + (-0.5)\ (0.0115) + \frac{(-0.5)(-0.5+1)}{2}(0.0003)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(-0.0001)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{24}(-0.\ 0003)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)}{120}\ (-0.0007)$$

$= 0.7139 - 0.00575 - 0.0000375 + 0.00000625 + 0.000011719 + 0.000019141$

$= 0.70815$

$y(0.29) = 0.7082.$

Q19. Find the Newton's forward difference interpolating polynomial for the data :

x	0	1	2	3
f(x)	1	3	7	13

Sol :

The difference table is :

x	f(x)	Δ	Δ^2	Δ^3
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

By Newton's forward interpolation formula,

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ...$$

Here

$x_0 = 0, h = 1$

$$p = \frac{x - x_0}{h} = \frac{x-0}{1} = x$$

Thus,

$$y = f(x) = 1 + x(2) + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)(x-2)}{6}(0)$$

$$= 1 + 2x + x^2 - x$$

$$y = f(x) = x^2 + x + 1$$

Q20. From the following table of values of f(x) computer f(0.63)

x	0.30	0.40	0.50	0.60	0.70
f(x)	0.6179	0.6554	0.6915	0.7257	0.7580

Sol :

The difference table is,

x	y=f(x)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.30	0.6179				
		0.0375			
0.40	0.6554		–0.0014		
		0.0361		–0.0005	
0.50	0.6915		–0.0019		0.0005
		0.0342		0	
0.60	0.7257		–0.0019		
		0.0323			
0.70	0.7580				

Here

$x_n = 0.70, x = 0.63$ and $h = 0.10$

$$p = \frac{x - x_n}{h} = \frac{0.63 - 0.70}{0.10} = -0.7$$

From, Newton's backward interpolation formula

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + ...$$

$$= 0.7580 + (-0.7)(0.0323) + \frac{(-0.7)(-0.7+1)}{2}(-0.0014) +$$

$$\frac{(-0.7)(-0.7+1)(-0.7+2)}{6}(0) + \frac{(-0.7)(-0.7+1)(-0.7+2)(-0.7+3)}{24}(0.0005)$$

$y = 0.7356$

Q21. Find the polynomial which approximates the following values :

x	3	4	5	6	7	8	9
y	13	21	31	43	57	73	91

If the number 31 is fifth term of the series, find the first and the tenth terms of the series.

Sol :

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
3	13						
		8					
4	21		2				
		10		0			
5	31		2		0		
		12		0		0	
6	43		2		0		0
		14		0		0	
7	57		2		0		
		16		0			
8	73		2				
		18					
9	91						

From the difference table, it will be seen that second differences are constant and hence tabulated function represents a polynomial of second degree. we conclude that both interpolation and extra polation would yield entract results.

To obtain tenth term,

We use the formula with

$x_0 = 3, x = 10, h = 1$

$$p = \frac{x - x_0}{h} = \frac{10-3}{1} = 7.$$

$$y(10) = 13 + 7(8) + \frac{7(7-1)}{2!}(2) = 13 + 56 + 42$$

$$= 111$$

To obtain first term,

We use the formula with

$x_n = 9, x = 1$, and $h = 1$

$$p = \frac{x - x_n}{h} = \frac{1-9}{1} = -8.$$

$$y(1) = 91 + (-8)(18) + \frac{(-8)(-8+1)}{2}(2)$$

$$= 91 - 144 + 56$$

$$y(1) = 3.$$

2.3 Gauss's Central Differences Formulae - Stirling's and Bessel's Formula

Q22. Derine Gauss forward formula.

Ans :

We consider the following difference table in which central ordinate is takes for convenience as y_0 corresponding to $x = x_0$

Where

G_1, G_2 ...... have to be determined.

The y_p on the left side can be expressed in terms of y_0, Δy_0 and higher order differences of y_0 as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_3	y_{-3}						
		Δy_{-3}					
x_2	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_3$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

From the table, we note the following:

$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1,}\ \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1,}\ \Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$

$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$ ..... (1) and so on

and $\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2}$,

$\Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2}$,

$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$,

$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$, and so on ...... (2)

By using the expressions (1) and (2)

We now obtain two versions of the following

Newton's forward interpolation formula:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots\ldots \quad (3)$$

Here y_p is the value of y at $x = x_0 + ph$

Substituting for Δy_0, $\Delta^2 y_0$ ...... (1) in (3) we get,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots\ldots]$$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \ldots\ldots$$

Substituting for $\Delta^4 y_{-1}$ from (2), this becomes

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)(p-1)p(p-2)}{4!}\Delta^4 y_{-2} \ldots\ldots(4)$$

This is knows as the Gauss' Forwared interpolation formula.

Q23. Derive Gauss' Backward interpolation formula.

Sol :

Newton's forward interpolation formula :

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 \ .....(1)$$

From the difference table (above table) we have

$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$; $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$; $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$;

$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$; .... (2) and so on.

and $\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2}$; $\Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2}$; $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$;

$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$; $\Delta^5 y_{-1} = \Delta^5 y_{-2} + \Delta^6 y_{-2}$ .... (3) and so on.

Substiute the equation (2) values in (1)

We obtain

$$y_p = y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + ...$$

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + ...$$

Substituting

$\Delta^3 y_{-1}$ and $\Delta^4 y_{-1}$ from (3) this become,

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-2})$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + ...$$

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + ...$$

This is know as Gauss' backward interpolation formula.

Q24. Derive stirling's formula.

Sol :

The Gauss' forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + ...$$

The Gauss' Backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + ...$$

Taking the mean of Gauss' forward and backward formula, we get

$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{1}{2}\left[\frac{p(p-1)}{2!} + \frac{(p+1)p}{2!}\right]\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] +$$

$$+\frac{1}{2}\left[\frac{(p+1)p(p-1)(p-2)}{4!} + \frac{(p-2)(p-1)p(p+1)}{4!}\right]\Delta^4 y_{-2} + ...$$

$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + ...$$

This formula is known as stirling's formula.

Q25. From the following table, find the value of $e^{1.17}$ using Gauss forward formula.

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Sol :

Here

$x = 1.17,\ x_0 = 1.15$ and $h = 0.05$

$$p = \frac{x - x_0}{h} = \frac{1.17 - 1.15}{0.05} = \frac{0.02}{0.05} = \frac{1}{4}$$

The difference table is :

x	e^x	Δ	Δ^2	Δ^3	Δ^4
$x_{-3}=1.00$	$y_{-3}=2.7183$				
		$\Delta y_{-3}=0.1394$			
$x_{-2}=1.05$	$y_{-2}=2.8577$		$\Delta^2 y_{-3}=0.0071$		
		$\Delta y_{-2}=0.1465$		$\Delta^3 y_{-3}=0.0004$	
$x_{-1}=1.10$	$y_{-1}=3.0042$		$\Delta^2 y_{-2}=0.0075$		$\Delta^4 y_{-3}=0$
		$\Delta y_1=0.1540$		$\Delta^3 y_{-2}=0.0004$	
$x_0=1.15$	$y_0=3.1582$		$\Delta^2 y_{-1}=0.0079$		$\Delta^4 y_{-2}=0$
		$\Delta y_0=0.1619$		$\Delta^3 y_{-1}=0.0004$	
$x_1=1.20$	$y_1=3.3201$		$\Delta^2 y_0=0.0083$		$\Delta^4 y_{-1}=0.0001$
		$\Delta y_1=0.1702$		$\Delta^3 y_0=0.0005$	
$x_2=1.25$	$y_2=3.4903$		$\Delta^2 y_1=0.0086$		
		$\Delta y_2=0.1790$			
$x_3=1.30$	$y_3=3.6693$				

The Gaues' forward formula is,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + ...$$

$$e^{1.17} = 3.1582 + \frac{1}{4}(0.1619) + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2}(0.0079) + \frac{\left(\frac{1}{4}+1\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)^2}{6}(0.0004)$$

$$e^{1.17} = 3.1582 + 0.0405 - 0.0002 - 0.0003.$$

$$e^{1.17} = 3.1983$$

Q26. Using Gauss' Backward formula, find the value of f(32) given that f(25) = 0.2707; f(30) = 0.3027; f(35) = 0.3386 and f(40) = 0.3794.

Sol :

Let us take $x_0 = 35$ and construct the difference table

x	y=f(x)	Δy	Δ²y	Δ³y
$x_{-2}=25$	$y_{-2}=0.2707$			
		$\Delta y_{-2}=0.032$		
$x_{-1}=30$	$y_{-1}=0.3027$		$\Delta^2 y_{-2}=0.0039$	
		$\Delta y_{-1}=0.0359$		$\Delta^3 y_{-2}=0.0010$
$x_0=35$	$y_0=0.3386$		$\Delta^2 y_{-1}=0.0049$	
		$\Delta y_0=0.0408$		
$x_1=40$	$y_1=0.3794$			

Let

$x = 32,\ x_0 = 35$ and $h = 5$

$$p = \frac{x-x_0}{h} = \frac{32-35}{5} = -0.6$$

The Gauss' backward formula is

$$y = f(x) = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!} + \Delta^4 y_{-2} + ...$$

$$y = f(32) = 0.3386 + (-0.6)(0.0359) + \frac{(-0.6+1)(-0.6)}{2}(0.0049)$$
$$+ \frac{(-0.6+1)(-0.6)(-0.6-1)}{2}(0.0010)$$

$f(32) = 0.3165$

Q27. State Gauss' backward formula and use it to find the value of $\sqrt{12525}$. given that $\sqrt{12500}$ =111.8034, $\sqrt{12510}$ = 111.8481 $\sqrt{12520}$ = 111.8928, $\sqrt{12530}$ = 111.9375 and $\sqrt{12540}$ =111.9822.

Sol :

The Gauss' backward formula is,

$$y = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)p(p-1)(p-2)}{4!} + \Delta^4 y_{-2} + ...$$

The difference table is,

x	$y = \sqrt{x}$	Δy	$\Delta^2 y$
$x_{-2} = 12500$	$y_{-2} = 111.8034$		
		$\Delta y_{-2} = 0.0447$	
$x_{-1} = 12510$	$y_{-1} = 111.8481$		$\Delta^2 y_{-2} = 0$
		$\Delta y_{-1} = 0.0447$	
$x_0 = 12520$	$y_0 = 111.8928$		$\Delta^2 y_{-1} = 0$
		$\Delta y_0 = 0.0447$	
$x_1 = 12530$	$y_1 = 111.9375$		$\Delta^2 y_0 = 0$
		$\Delta y_1 = 0.0447$	
$x_2 = 12540$	$y_2 = 111.9822$		

$x = 12525, x_0 = 12520$ and $h = 10$

$$p = \frac{x - x_0}{h} = \frac{12525 - 12520}{10} = \frac{5}{10} = 0.5$$

By Gauss Backward formula is

$y_p = f(x) = 111.8928 + 0.5(0.0447)$

$= 111.8928 + 0.0224$

$y_p = f(x) = 111.9152$

i.e, $\sqrt{12525} = 111.9152.$

Q28. From the following table values of x and y = e^x interpolate values of y when x = 1.91.

x	1.7	1.8	1.9	2.0	2.1	2.2
e^x	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

Sol :

The difference table is

x	$y = e^x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$x_{-2} = 1.7$	$y_{-2} = 5.4739$					
		$\Delta y_{-2} = 0.5757$				
$x_{-1} = 1.8$	$y_{-1} = 6.0496$		$\Delta^2 y_{-2} = 0.0606$			
		$\Delta y_{-1} = 0.6363$		$\Delta^3 y_{-2} = 0.0063$		
$x_0 = 1.9$	$y_0 = 6.6859$		$\Delta^2 y_{-1} = 0.0669$		$\Delta^4 y_{-2} = 0.0007$	
		$\Delta y_0 = 0.7032$		$\Delta^3 y_{-1} = 0.0070$		$\Delta^5 y_{-2} = 0.0001$
$x_1 = 2.0$	$y_1 = 7.3891$		$\Delta^2 y_0 = 0.0739$		$\Delta^4 y_{-1} = 0.0008$	
		$\Delta y_1 = 0.7771$		$\Delta^3 y_0 = 0.0078$		
$x_2 = 2.1$	$y_2 = 8.1662$		$\Delta^2 y_1 = 0.0817$			
		$\Delta y_2 = 0.8588$				
$x_3 = 2.2$	$y_3 = 9.0250$					

$x = 1.91,\ x_0 = 1.90$ and $h = 0.1$

$$p = \frac{x - x_0}{h} = \frac{1.90 - 1.90}{0.1} = \frac{0.01}{0.1} = 0.1$$

By striling 's formula,

$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{p^2(p^2-1)}{4!}(\Delta^4 y_{-2}) + \ldots$$

$$= 6.6859 + 0.1\left[\frac{0.7032 + 0.6363}{2}\right] + \frac{(0.1)^2}{2}(0.0669)$$

$$+ \frac{(0.1)(0.1^2-1)}{6}\left[\frac{0.0070 + 0.0063}{2}\right] + \frac{(0.1)^2(0.1^2-1)}{24}(0.0007)$$

$$= 6.6859 + 0.0670 + 0.0003 - 0.0001 - 0.$$

$e^{1.91} = 6.7531$

Q29. Using stirling's formula find cos (0.17) given that cos0 = 1, cos (0.05) = 0.9988, cos(0.10) = 0.9950, cos (0.15) = 0.9888, cos(0.20) = 0.9801, cos(0.25) = 0.9689 and cos(0.30) = 0.9553.

Sol :-

Here

$x = 0.17,\ x_0 = 0.15$ and $h = 0.05$

$$p = \frac{x - x_0}{h} = \frac{0.17 - 0.15}{0.05} = 0.4$$

$p = 0.4$

The difference table is,

x	cosx = y	Δy	Δ²y	Δ³y	Δ⁴y	Δ⁵y
0	$y_{-3} = 1$					
		$\Delta y_{-3} = -0.0012$				
0.05	$y_{-2} = 0.9988$		$\Delta^2 y_{-3} = -0.0026$			
		$\Delta y_{-2} = -0.0038$		$\Delta^3 y_{-3} = +0.0002$		
0.10	$y_{-1} = 0.9950$		$\Delta^2 y_{-2} = -0.0024$		$\Delta^4 y_{-3} = -0.0003$	
		$\Delta y_{-1} = -0.0062$		$\Delta^3 y_{-2} = -0.0001$		$\Delta^5 y_{-3} = 0.0004$
$x_0 = 0.15$	$y_0 = 0.9888$		$\Delta^2 y_{-1} = -0.0025$		$\Delta^4 y_{-2} = +0.0001$	
		$\Delta y_0 = -0.0087$		$\Delta^3 y_{-1} = 0$		$\Delta^5 y_{-2} = 0$
0.20	$y_1 = 0.9801$		$\Delta^2 y_0 = -0.0025$		$\Delta^4 y_{-1} = +0.0001$	
		$\Delta y_1 = -0.0112$		$\Delta^3 y_0 = +0.0001$		
0.25	$y_2 = 0.9689$		$\Delta^2 y_1 = -0.0024$			
		$\Delta y_2 = -0.0136$				
0.30	$y_3 = 0.9553$					

By stirling's formula

$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots$$

$$0.9888 + (0.4)\left[\frac{-0.0087-0.0062}{2}\right] + \frac{(0.4)^2}{2}(-0.0025)$$

$$+ \frac{(0.4)(0.4^2-1)}{6}\left[\frac{0-0.0001}{2}\right] + \ldots\ldots$$

$$= 0.9888 - 0.0030 - 0.0002$$

cos(0.17) = 0.9856.

Q30. From the following table find y when x = 38,

x	30	35	40	45	50
y	15.9	14.9	14.1	13.3	12.5

Sol :

The diffenence table is,

x	y	Δy	Δ²y	Δ³y	Δ⁴y
30	15.9				
		–1.0			
35	14.9		0.2		
		–0.8		–0.2	
40	14.1		0		0.2
		–0.8		0	
45	13.3		0		
		–0.8			
50	12.5				

Here

$$x = 38,\ x_0 = 40 \text{ and } h = 5$$

$$p = \frac{x - x_0}{h} = \frac{38-40}{5} = -0.4$$

According to Gauss Backward formula,

$$y = y_{(38)} = f(38) = 14.1 + (-0.4)\ (-0.8) + \frac{(-0.4)(-0.4+1)}{2} + (-0) + \frac{(-0.4-1)(-0.4)(-0.4+1)}{3!} \times (0)$$

$$+ \frac{(-0.4-1)(-0.4)(-0.4+1)(-0.4+2)}{4!} \times (0.2)$$

$$= 14.4245.$$

Q31. Use Gauss' forward interpolation formula to find f(30) given that f(21) = 18.4708, f(25) = 17.8144, f(29) = 17.1070, f(33) = 16.3432, f(37) = 15.5154.

Sol :

Let $x_0 = 29$

The difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 21$	$y_{-2} = 18.4708$				
		$\Delta y_{-2} = -0.6564$			
$x_{-1} = 25$	$y_{-1} = 17.8144$		$\Delta^2 y_{-2} = -0.0510$		
		$\Delta y_{-1} = -0.7074$		$\Delta^3 y_{-2} = -0.0064$	
$x_0 = 29$	$y_0 = 17.1070$		$\Delta^2 y_{-1} = -0.0574$		$\Delta^4 y_{-2} = +0.0018$
		$\Delta y_0 = -0.7648$		$\Delta^3 y_{-1} = -0.0046$	
$x_1 = 33$	$y_1 = 16.3422$		$\Delta^2 y_0 = -0.0620$		
		$\Delta y_1 = -0.8268$			
$x_2 = 37$	$y_2 = 15.5154$				

Here

$x = 30,\ x_0 = 29$ and $h = 4$.

$$p = \frac{x - x_0}{4} = \frac{30 - 29}{4} = 0.25$$

By Gauss Forward Fornula,

$$f(30) = 17.1070 + (0.25)(-0.7648) + \frac{(0.25)(0.25-1)}{2}(-0.0574)$$

$$+ \frac{(0.25+1)(0.25)(0.25-1)}{6}(-0.0046) + \frac{(0.25+1)(0.25)(0.25-1)(0.25-2)}{24}(0.0018)$$

$f(30) = 16.921.$

Q32. Using Gauss Backward difference formula, find y(8) from the following table.

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Sol :

The difference table is

x	y	Δy	Δ²y	Δ³y	Δ⁴y	Δ⁵y
$x_{-2}=0$	$y_{-2}=7$					
		$\Delta y_{-2}=4$				
$x_{-1}=5$	$y_{-1}=11$		$\Delta^2 y_{-2}=-1$			
		$\Delta y_{-1}=3$		$\Delta^3 y_{-2}=2$		
$x_0=10$	$y_0=14$		$\Delta^2 y_{-1}=1$		$\Delta^4 y_{-2}=-1$	
		$\Delta y_0=4$		$\Delta^3 y_{-1}=1$		$\Delta^5 y_{-2}=0$
$x_1=15$	$y_1=18$		$\Delta^2 y_0=2$		$\Delta^4 y_{-1}=-1$	
		$\Delta y_1=6$		$\Delta^3 y_0=0$		
$x_2=20$	$y_2=24$		$\Delta^2 y_1=2$			
		$\Delta y_2=8$				
$x_3=25$	$y_3=32$					

By Gauss Backward formula,

$$y = f(x) = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + ...$$

Here

$x = 8$, $x_0 = 10$ and $h = 5$

$$p = \frac{x-x_0}{h} = \frac{8-10}{5} = \frac{-2}{5} = -0.4.$$

$$y = f(8) = 14 + (-0.4)(3) + \frac{(-0.4)(-0.4+1)}{2}(1) + \frac{(-0.4+1)(-0.4)(-0.4-1)}{6}(2)$$

$$+ \frac{(-0.4+1)(-0.4)(-0.4-1)(-0.4-2)}{24}(-1)$$

$f(8) = 14 - 1.2 + 0.1112 + 0.0836 - 0.12$

$f(8) = 12.7024$

Q33. Derive Bessel's Formula.

Sol :

The Bessel's formula is very useful formula for practical interpolation, and it uses the difference in the following table, where the brackets mean that the average of the values has to be taken.

x	y	Δy	Δ²y	Δ³y	Δ⁴y	Δ⁵y	Δ⁶y	....
⋮								
x_{-1}	y_{-1}							
x_0	$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$	Δy_0	$\begin{pmatrix} \Delta^2 y_{-4} \\ \Delta^2 y_0 \end{pmatrix}$	$\Delta^3 y_{-4}$	$\begin{pmatrix} \Delta^4 y_{-2} \\ \Delta^4 y_{-1} \end{pmatrix}$	$\Delta^5 y_{-2}$	$\begin{pmatrix} \Delta^6 y_{-3} \\ \Delta^6 y_{-2} \end{pmatrix}$	
x_1								
⋮								

The Bersel's formula can be assumed in the form

$$y_p = \frac{y_0 + y_1}{2} + B_1 \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1} + B_4 \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + ...$$

$$= y_0 + \left[B_1 + \frac{1}{2}\right] \Delta y_0 + B_2 \left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + B_3 \Delta^3 y_{-1} + B_4 \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + ...$$

Using Gauss forward formula, we obtain

$$B_1 + \frac{1}{2} = p,\ B_2 = \frac{p(p-1)}{2!},\ B_3 = \frac{p(p-1)(p-\frac{1}{2})}{3!}\ B_4 = \frac{(p+1)p(p-1)(p-2)}{4!}\ .....$$

Hence, Bessel's interpolation formula

May be written as,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)(p-\frac{1}{2})}{3!} \Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!} \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + ...$$

Q34. The following table gives the values of e^x for certain equidistant values of x. Find the value of e^x when x = 0.644.

x	**0.61**	**0.62**	**0.63**	**0.64**	**0.65**	**0.66**	**0.67**
e^x	**1.840431**	**1.858928**	**1.877610**	**1.896481**	**1.915541**	**1.934792**	**1.954237**

Sol :

The difference table is:

x	**y = e^x**	**Δ**	**Δ^2**	**Δ^3**	**Δ^4**
0.61	1.840431				
		0.018497			
0.62	1.858928		0.000185		
		0.018682		0.000004	
0.63	1.877610		0.000189		–0.000004
		0.018871		0	
0.64	1.896481		0.000189		0.000002
		0.019060		0.000002	
0.65	1.915541		0.000191		0.000001
		0.01975		0.00003	
0.66	1.934792		0.000194		
		0.019445			
0.67	1.954237				

Here

$x = 0.644, \; x_0 = 0.64$ and $h = 0.01$

$$p = \frac{0.644 - 0.64}{0.01} = 0.4$$

By Bessel's formula,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + \frac{p(p-1)(p-\frac{1}{2})}{3!}\,\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\,\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots\ldots$$

$$y(0.644) = 1.896481 + 0.4(0.019060) + \frac{0.4(0.4-1)}{2}\,\frac{(0.000189 + 0.000111)}{2}$$

$$= 1.896481 + 0.0076240 - 0.0000228$$

$$y(0.644) = 1.904082.$$

Q35. The values of x and e^{-x} are given in the following table. Find the value of e^{-x} when x = 1.7475.

x	1.72	1.73	1.74	1.75	1.76	1.77	1.78
e^{-x}	0.1790661479	0.1772844100	0.1755204006	0.1737739435	0.1720448638	0.1703329888	0.1686381473

Sol :

Here

$x = 1.7475, \; x_0 = 1.74$ and $h = 0.01$

$$p = \frac{1.7475 - 1.74}{0.01} = \frac{3}{4}.$$

x	y = e^x	Δ	Δ^2	Δ^3	Δ^4
1.72	0.1790661479				
		–17817379			
1.73	0.1772844100		177285		
		–17640094		–1762	
1.74	0.1755204006		175523		13
		–17464571		–1749	
1.75	0.1737739435		173774		22
		–17290797		–1727	
1.76	0.1720448638		172047		15
		–17118750		–1712	
1.77	0.1703329888		170335		
		–16948415			
1.78	0.168638143				

By Bessel's formula,

$$y(1.7475) = 0.17552044006 - \frac{3}{4}(0.0017464571)$$

$$+ \frac{\left(\frac{3}{4}\right)\left(\frac{3}{4}-1\right)}{2} \frac{(0.0000175523+0.0000173774)}{2}$$

$= 0.1755204006 - 0.00130984284 - 0.00000163734 + 0.000000007$

$= 0.1742089218.$

Q36. Using Bessel's formula, find y(5) given that y(0) = 14.27, y(4) = 15.81, y(8) = 17.72, y(12) =19.96

Sol :

Here

$x = 5, \; x_0 = 4$ and $h = 4$

$$p = \frac{x-x_0}{h} = \frac{5-4}{4} = \frac{1}{4}$$

The difference table is,

x	y	Δy	Δ²y	Δ³y
0	14.27			
		1.54		
4	15.81		0.37	
		1.91		–0.04
8	17.72		0.33	
		2.24		
12	19.96			

Besse's Formla is,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + \frac{p(p-1)(p-\frac{1}{2})}{3!}\Delta^3 y_{-1} + ...$$

$$y(5) = 15.81 + \frac{1}{4}(1.91) + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2}\left[\frac{0.37+0.33}{2}\right]$$

$= 15.81 + 0.4775 - 0.1313$

$y(5) = 16.1563$

Q37. Evaluate sin (0.20) By using Bessel's formula, given that sin(0.15) = 0.1494, sin(0.17) = 0.1692, sin (0.19) = 0.1889, sin (0.21) = 0.2085, sin (0.23) = 0.2280.

Sol :

Here

$x = 0.20,\ x_0 = 0.19$ and $h = 0.02.$

$$p = \frac{x - x_0}{h} = \frac{0.20 - 0.19}{0.02} = 0.5$$

The difference table is:

x	y = sin x	Δy	$\Delta^2 y$	$\Delta^3 y$
0.15	0.1494			
		0.0198		
0.17	0.1692		–0.0001	
		0.0197		0
0.19	0.1889		–0.0001	
		0.0196		0
0.21	0.2085		–0.0001	
		0.0195		
0.23	0.2280			

Bessel's formla is,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + ...$$

$$\sin(0.20) = 0.1889 + 0.5(0.0196) + \frac{0.5(0.5-1)}{2}\left[\frac{-0.0001 - 0.0001}{2}\right]$$

$$= 0.1889 + 0.0098$$

$$\sin(0.20) = 0.1987.$$

2.4 Lagrange's Interpolation Polynomial

Q38. Derive Lagrange's formula.

Ans :

Let $x_0, x_1, x_2, ... x_n$ be the (n + 1) values of x which are not necessarihy equally spaced.

Let $y_0, y_1, y_2 .... y_n$ be the corresponding values of y = f(x).

Let the polynomial of degree n for the function y = f(x)

Passing throngh the (n + 1) points $(x_0, y_0), (x_1, y_1), (x_2, y_2) .... (x_n, y_n)$ be in the following form.

$$y = f(x) = a_0(x - x_1)(x - x_2) ....(x - x_n) + a_1(x - x_0)(x - x_2) ...... (x - x_n)$$
$$+ a_2(x - x_0)(x - x_1)(x - x_3) ...... (x - x_n) + ...$$
$$+ a_n(x - x_0)(x - x_1) .... (x - x_{n-1}) \quad ..... (1)$$

Where

$a_0, a_1 ...... a_n$ are constants.

Since the polynomial passes through $(x_0, y_0), (x_1, y_1) ...... (x_n, y_n)$ the constants can be determined by substituting

One of the values of $x_0, x_1, x_2 ..... x_n$ for x in the a bove equation

Putting $x = x_0$ in (1) we get,

$y_0 = f(x_0) = a_0(x_0 - x_1)(x_0 - x_2) .... (x_0 - x_n)$

$$\Rightarrow \quad a_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)}$$

Putting $x = x_1$ in (1) we get ,

$f(x_1) = a_1(x_1 - x_0)(x_1 - x_2) ...... (x_1 - x_n)$

$$\Rightarrow \quad a_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)}$$

Continuing in this manner and putting $x = x_n$ in (1) we get

$$a_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1).....(x_n-x_{n-1})}$$

Substituting

$a_0, a_1 ...... a_n$ equation (1) we get

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3).....(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3).....(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)} f(x_1) + ...$$

$$+ \frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).....(x_n-x_{n-1})} f(x_n)$$

(or)

$$y = \frac{(x-x_1)(x-x_2).....(x-x_n)}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)} y_1 + ...$$

$$+ \frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).....(x_n-x_{n-1})} y_n$$

This is known as Lagranges's interpolation formula

Q39. Find y(2) from the following data using Lagrange's formula.

x	**0**	**1**	**3**	**4**	**5**
y	**0**	**1**	**81**	**256**	**625**

Sol :

Here

$x_0 = 0,\ x_1 = 1,\ x_2 = 3,\ x_3 = 4,\ x_4 = 5$ and $y_0 = 0,\ y_1 = 1,\ y_2 = 81, y_3 = 256,\ y_4 = 625.$

The lagrage's formula is,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3).....(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3).....(x_0-x_n)}\ y_0 + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)}\ y_1 + ...$$

$$+ \frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).....(x_n-x_{n-1})}\ y_n.$$

Now

$$y = \frac{(x-1)(x-3)(x-4)(x-5)}{(0-1)(0-3)(0-4)(0-5)}(0) + \frac{(x-0)(x-3)(x-4)(x-5)}{(1-1)(1-3)(1-4)(1-5)}(1) + \frac{(x-0)(x-1)(x-4)(x-5)}{(3-0)(3-1)(3-4)(3-5)}(81)$$

$$+ \frac{(x-0)(x-1)(x-3)(x-5)}{(4-0)(4-1)(4-3)(4-5)}(256) + \frac{(x-0)(x-1)(x-3)(x-4)}{(5-0)(5-1)(5-3)(5-4)}(625)$$

Here

$x = 2$

$$y(2) = \frac{(2-1)(2-3)(2-4)(2-5)}{(-1)(-3)(-4)(-5)}(0) + \frac{(2-0)(2-3)(2-4)(2-5)}{(1)(-2)(-3)(-4)}(1)$$

$$+ \frac{(2-0)(2-1)(2-4)(2-5)}{(3)(2)(-1)(-2)}(81) + \frac{(2-0)(2-1)(2-3)(2-5)}{(4)(3)(2)(-1)(1)}(256)$$

$$+ \frac{(2-0)(2-1)(2-3)(2-4)}{(5)(4)(2)(1)}(625)$$

$$= 0 + \frac{(-12)}{(-24)} + \frac{12}{12}(81) + \frac{6}{-12}(256) + \frac{4}{40}(625)$$

$$= 0 + 0.5 + 81 - 128 + 62.5$$

$y(2) = 16$

Q40. Using Lagrange's interpolation formula, find the value of y(10) from the following table:

x	5	6	9	11
y	12	13	14	16

Find y(10), Given that y(5) = 12, y(6) = 13, y(9) = 14, y(11) = 16 using Lagrange's formula.

Sol :

Lagrange's interpolation formula is given by

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\ y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\ y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\ y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\ y_3$$

Here

$x_0 = 5, \quad x_1 = 6, \quad x_2 = 9, \quad x_3 = 11$

and $y_0 = 12, \quad y_1 = 13, \quad y_2 = 14, \quad y_3 = 16$

Here

$x = 10.$

$$y = f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) = \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3}$$

$y(10) = f(10) = 14.6666$

Q41. Represent the function f(x) approximately by a polynomial of degree 2 from the following data

x	1	2	–4
f(x)	3	–5	4

Sol :

Lagrange's interpolation formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

Here

$x_0 = 1, \quad x_1 = 2, \quad x_2 = -4$

and $y_0 = 3, \quad y_1 = -5, \quad y_2 = 4$

$$f(x) = \frac{(x-2)(x+4)}{(1-2)(1+4)}(3) + \frac{(x-1)(x+4)}{(2-1)(2+4)}(-5) + \frac{(x-1)(x-2)}{(-4-1)(-4-2)}(4)$$

$$\frac{-3}{5}(x^2 + 2x - 8) - \frac{5}{6}(x^2 + 3x - 4) + \frac{4}{30}(x^2 - 3x + 2)$$

$$f(x) = \frac{-1}{10}(13x^2 + 41x - 84).$$

Q42. Using Lagrange's interpolation formula, find the form of the function y(x) from the following table :

x	0	1	3	4
y	–12	0	12	24

Sol :

From the table we observe

$x = 1, \; y = 0$

Thus

$x - 1$ is a factor,

Let $y(x) = (x - 1)\, R(x) \Rightarrow R(x) = \dfrac{y}{x-1}$

Tabulating the values of x and R(x) and R(x) we get

x	0	3	4
R(x)	12	6	8

Using the Lagrange's interpolation formula,

$$R(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-3)(x-4)}{(-3)(-4)} (12) + \frac{(x-0)(x-4)}{(3-0)(3-4)} (6) + \frac{(x-0)(x-3)}{(4-0)(4-3)} (8)$$

$$= (x - 3)(x - 4) - 2x(x - 4) + 2x(x - 3)$$

$$= x^2 - 5x + 12.$$

Hence, the required poly approximate to y(x) is given by $y(x) = (x - 1)(x^2 - 5x + 12)$

Q43. Find the parabola passing through points (0, 1) (1, 3) and (3, 55) using lagrange's interpolation formula.

Sol :

Given points are,

$x_0 = 0,\quad x_1 = 1,\quad x_2 = 3$ and $y_0 = 1,\quad y_1 = 3$ and $y_2 = 55$

Lagrange's interpolation formula is,

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-1)(x-3)}{(0-1)(0-3)} (1) + \frac{(x-0)(x-3)}{(1-0)(1-3)} (3) + \frac{(x-0)(x-1)}{(3-0)(3-1)} (55)$$

$$= \frac{x^2-4x+3}{3} + \frac{x^2-3x}{-2} (3) + \frac{x^2-x}{6} (55)$$

$$= \frac{2x^2-8x+6-9x^2+27x+55x^2-55x}{6}$$

$$= \frac{1}{6}(48x^2 - 36x + 6)$$

$$f(x) = 8x^2 - 6x + 1$$

Q44. Given $u_0 = 580$, $u_1 = 556$, $u_2 = 520$ and $u_4 = 385$ find u_3.

Sol :

Given data can be tabulated as,

x	0	1	2	4
u(x)	580	556	520	385

Here

$x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$ and $y_0 = 580$, $y_1 = 556$, $y_2 = 520$, $y_3 = 385$.

By Lagrange's formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Here $x = 3$

$$f(3) = \frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)} 580 + \frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)} (556)$$

$$+ \frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)} 520 + \frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)} (385)$$

$= 145 - 556 + 780 + 96.25$

$f(3) = 465.25.$

2.5 Divided Differences Newton's General Interpolation Formula

Q45. Explain divided differences.

Ans :

Let (x_0, y_0), (x_1, y_1) ..... (x_n, y_n) be the given $(n + 1)$ points (not equally spaced x's) Then the divided differences of order 1, 2, ... n are defined by follows.

The first order divided differences of y for the arguments x_0, x_1 is written as

$$\frac{f(x_1)-f(x_0)}{x_1-x_0} \quad (\because f(x) = y)$$

$$\therefore \quad f(x_0, x_1) = [x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

Similarly

$$[x_1, x_2] = \frac{f(x_2)-f(x_1)}{x_2-x_1} \text{ and so on.}$$

The second order divided differences of y for the arguments x_0, x_1, x_2 is written as

$$\frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$$

$$\therefore \quad f(x_0, x_1, x_2) = [x_0, x_1, x_2] = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$$

$$[x_0, x_1, x_2] = \frac{[x_1,x_2]-[x_0,x_1]}{x_2-x_0}$$

In this way we define the higher order divided differenced.

Note :

$$[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f(x_0)-f(x_1)}{x_0-x_1} = [x_1, x_0]$$

$\therefore$ $[x_0, x_1] = [x_1, x_0]$ and so on.

Divided differences table.

x	y = f(x)	1st order	2nd order	3rd order	4th order
x_0	y_0				
		$[x_0, x_1]$			
x_1	y_1		$[x_0, x_1, x_2]$		
		$[x_1, x_2]$		$[x_0, x_1, x_2, x_3]$	
x_2	y_2		$[x_1, x_2, x_3]$		$[x_0, x_1, x_2, x_3, x_4]$
		$[x_2, x_3]$		$[x_1, x_2, x_3, x_4]$	
x_3	y_3		$[x_2, x_3, x_4]$		
		$[x_3, x_4]$			
x_4	y_4				

Newton's General Interpolation Formula

Q46. Derive Newton's general interpolation formula.

Sol :

Let

$y_0, y_1, y_2 ..... y_n$ be the values of y corresponding to $x_0, x_1, x_2, ... x_n$ of x (let $y = f(x)$).

By defination of divided differences, we have

$$f(x, x_0) = [x, x_0] = \frac{f(x)-f(x_0)}{x-x_0} = \frac{y-y_0}{x-x_0}$$

$$\Rightarrow \quad [x, x_0] = \frac{y - y_0}{x - x_0}$$

$$\Rightarrow \quad y = (x - x_0)[x, x_0] + y_0$$

$$\Rightarrow \quad y = y_0 + (x - x_0)[x, x_0] \qquad ..... (1)$$

Again $[x, x_0, x_1] = \dfrac{[x, x_0] - [x_0, x_1]}{x - x_1}$

$$\Rightarrow \quad [x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in (1), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \qquad ..... (2)$$

But

$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

and so

$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$$

Equation (2) Now gives,

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] +$$
$$(x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2] \qquad ..... (3)$$

Proceeding in this way, ϕ we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + ...$$
$$+ (x - x_0)(x - x_1)(x - x_2) ..... (x - x_n)[x_0, x_1, x_2, ...x_n] \quad ..... (4)$$

This formula is called Newton's general interpolation formula with divided differences, the last term being the remainder term after (n + 1) terms.

Q47. Certain corresponding values of x and $\log_{10}x$ are (300, 2.4771), (304, 2.4829), (305, 2.4843) and (307,2.4871) Find $\log_{10}$ 301. By using the Newton's divided difference formula.

Sol :

The divided difference table is.

x	y = $\log^x_{10}$	1st order	2nd order
300	2.4771		
		0.00145	
304	2.4829		–0.00001
		0.00140	
305	2.4843		0
		0.00140	
307	2.4871		

The Newton's divided difference formula B,

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)[x - x_1](x - x_2)[x_0, x_1, x_2, x_3] + ...$$

Here

$$x = 301$$

$$y = 2.4771 + (301 - 300)(0.00145) + (301 - 300)(301 - 304)(-0.00001)$$

$$y = 2.4786.$$

Q48. Using the following table find f(x) as a polynomial in x.

x	−1	0	3	6	7
f(x)	3	−6	39	822	1611

Sol :

The divided difference table is,

x	y = f(x)	1st order	2nd order	3rd order	4th order
−1	3				
		−9			
0	−6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

The Newton's divided difference formula is,

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)[x - x_1](x - x_2)[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)[x_0, x_1, x_2, x_3, x_4] + ...$$

$$y = 3 + (x + 1)(-9) + (x + 1)(x - 0)(6) + (x + 1)(x - 0)(x - 3)(5) + (x + 1)(x - 0)(x - 3)(x - 6)(1)$$

$$y = x^4 - 3x^3 + 5x^2 - 6$$

Q49. Using Newton's divided difference formula find the values of f(2); f(8) and f(15) given the following table.

x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

Sol :

We form the divided difference table since intervals are unequal.

x	y = f(x)	1st order	2nd order	3rd order	4th order
4	48				
		$\frac{100-48}{5-4} = 52$			
5	100		$\frac{97-52}{7-4} = 15$		
		$\frac{294-100}{7-5} = 97$		$\frac{21-15}{10-4} = 1$	
7	294		$\frac{202-97}{10-5} = 21$		0
		$\frac{900-294}{10-7} = 202$		$\frac{27-21}{11-5} = 1$	
10	900		$\frac{310-202}{11-7} = 27$		0
		$\frac{1210-900}{11-10} = 310$		$\frac{33-27}{13-7} = 1$	
11	1210		$\frac{409-310}{13-10} = 33$		
		$\frac{2028-1210}{13-11} = 409$			
13	2028				

By Newton's divided difference formula is,

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$

$$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + ...$$

$$y = 48 + (x - 4)(52) + (x - 4)(x - 5)(15) + (x - 4)(x - 5)(x - 7)(1)$$

$$f(2) = 48 - 104 + 90 - 30 = 4$$

$$f(8) = 48 + (4)(52) + (4)(3)(15) + (4)(3)(1)(1) = 448$$

$$f(15) = 48 + 11(52) + (11)(10)(15) + 11(10)(8) = 3150$$

Q50. Find $\log_{10}$ 323.5 given

x	321.0	322.8	324.2	325
$\log_{10} x$	2.50651	2.50893	2.51081	2.51188

Sol :

The divided difference table.

x	$y = \log^x_{10}$	1st order	2nd order	3rd order
321	2.50651			
		$\frac{2.50893 - 2.50651}{322.8 - 321} = 0.00134$		
322.8	2.50893		0	
		0.00134		0
324.2	2.51081		0	
		0.00134		
325.0	2.51188			

Here

$x = 323.5$

By Newton's divided difference formula,

$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + ...$

$y = 2.50651 + (323.5 - 321)(0.00134) + 0$

$y = 2.50651 + 0.00334$

$y = 2.50985.$

Q51. The values of y and x are given as below.

x	5	6	9	11
y	12	13	14	16

Find the value of y when x = 10

Sol :

The divided difference table is.

x	y	1st order	2nd order	3rd order
5	12			
		1		
6	13		-0.167	
		0.333		0.050
9	14		0.133	
		1		
11	16			

Here x = 10

By Newton's divided difference formula,

$y = f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$

$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + ...$

$= 12 + (10 - 5)(1) + (10 - 5)(10 - 6)(-0.167) + (10 - 5)(10 - 6)(10 - 9)(0.05)$

$= 12 + 5 - 3.340 + 1$

$y = f(10) = 14.66$

The value of y when $x = 10$ is 14.66

Q52. Find a polynomial by using Newton's divided difference formula for the data.

x	−4	−1	0	2	5
f(x)	1245	33	5	9	1335

Sol :

The divided difference table is,

x	y = f(x)	1st order	2nd order	3rd order	4th order
−4	1245				
		$\frac{33-1245}{-1+4} = -404$			
			$\frac{-28+404}{0+4} = 94$		
−1	33				
		$\frac{5-33}{0+1} = -28$		$\frac{10-94}{2+4} = -14$	
0	5		$\frac{2+28}{2+1} = 0$		$\frac{13+14}{5+4} = 3$
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5+1} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1325-9}{5-2} = 442$			
5	1335				

From the table we observe that

$[x_0, x_1] = -404$; $[x_0, x_1, x_2] = 94$; $[x_0, x_1, x_2, x_3] = -14$;

$[x_0, x_1, x_2, x_3, x_4] = 3.$

By Newton's divided difference formula,

$$y = f(x) = f(x_0) + (x - x_0)\,[x, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2]$$

$$+ (x - x_0)\,[x - x_1]\,(x - x_2)\,[x_0, x_1, x_2, x_3] + ...$$

$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) + (x + 4)(x + 1)(x - 0)(-14)$$

$$+ (x + 4)(x + 1)(x - 0)(x - 2)(3)$$

$f(x) = 3x^4 - 5x^3 + 6x^2 - 14x - 5$

Q53. A function y = f(x) is given at the sample points $x = x_0, x_1$, and x_2. Show that the newton's divided difference interpolation formula and corresponding lagrange's interpolation formula are identical.

Sol :

For the function $y = f(x)$

We have x_0, x_1, x_2 are the sample points. Newton's divided difference formula is,

$$y = f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \quad ..... (1)$$

Using the defination of divided differences,

We can rewrite the equation (1) the from

$$y = f(x) = f(x_0) + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + (x - x_0)(x - x_1)$$

$$\left[\frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}\right]$$

$$= \left[1 - \frac{(x_0 - x)}{(x_0 - x_1)} + \frac{(x - x_0)(x - x_1)}{(x_0 - x_1)(x_0 - x_2)}\right] f(x_0) + \left[\frac{(x - x_0)}{(x_1 - x_0)} + \frac{(x - x_0)(x - x_1)}{(x_1 - x_0)(x_1 - x_2)}\right] f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \quad ..... (2)$$

Simplifying equation (2), It reduces to

$$y = f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

$$..... (3)$$

Which is the Lagrange's form of interpolation polynomial

Hence equation (1) and (3) are identical.

2.6 Inverse Interpolation

Q54. Define Inverse Interpolation.

Ans :

Given a set of values of x and y, the process of finding the value of x for a certain value of y is called inverse interpolation.

2.6.1 Method of Successive Approximations

Q55. Derive the method of successive approximation.

Sol :

From the Newton's forward difference formula written as,

$$y_u = u_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \quad ..... (1)$$

From this, we obtain

$$u = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_1(u_1 - 1)}{2}\Delta^2 y_0 - \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 .....\right] \quad ..... (2)$$

Neglecting the second and higher differences, we obtain the first approximation to u and this, we write.

$$u_1 = \frac{1}{\Delta y_0}(y_u - y_0) \quad ..... (3)$$

Next we obtain the second approximation to u by including the term containing the second differences.

Thus,

$$u_2 = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_1(u_1-1)}{2}\Delta^2 y_0\right] \quad ..... (4)$$

Where we have used the value of u_1, for u in the coefficient of $\Delta^2 y_0$

Simillarly, we obtain

$$u_3 = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_2(u_2-1)}{2}\Delta^2 y_0 - \frac{u_2(u_2-1)(u_2-2)}{6}\Delta^3 y_0\right] \quad ..... (5)$$

and so on.

This process should be continued till two successive approximations to v agree with each other to the required accuracy.

Q56. Tabulate y = x³ for x = 2, 3, 4 and 5 and calculate the cube root of 10 correct to three decimal places.

Sol :

x	y = x³	Δ	Δ²	Δ³
2	8			
		19		
3	27		18	
		37		6
4	64		24	
		61		
5	125			

Here

$$y_u = 10,\ y_0 = 8,\ \Delta y_0 = 19,\ \Delta^2 y_0 = 18 \text{ and } \Delta^3 y_0 = 6.$$

The successive approximation to u are,

$$u_1 = \frac{1}{\Delta y_0}(y_u - y_0) = \frac{1}{19}(10-8) = 0.1$$

$$u_2 = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_1(u_1-1)}{2}\Delta^2 y_0\right]$$

$$u_2 = \frac{1}{19}\left[10-8-\frac{(0.1-1)0.1}{2}(18)\right] = 0.15$$

$$u_3 = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_2(u_2-1)}{2}\Delta^2 y_0 - \frac{u_2(u_2-1)(u_2-2)}{6}\Delta^3 y_0\right]$$

$$= \frac{1}{19}\left[10-8-\frac{0.15(0.15-1)}{2}(18)-\frac{0.15(0.15-1)(0.15-2)}{6}(6)\right]$$

u_3 = 0.1532.

$$u_4 = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_3(u_3-1)}{2}\Delta^2 y_0 - \frac{u_3(u_3-1)(u_3-2)}{6}\Delta^3 y_0\right]$$

$$= \frac{1}{19}\left[10-8-\frac{0.1532(0.1532-1)}{2}(18)-\frac{0.1532(0.1532-1)(0.1532-2)}{6}(6)\right]$$

u_4 = 0.1541

$$u_5 = \frac{1}{19}\left[2-\frac{0.1541(0.1541-1)}{2}(18)-\frac{0.1541(0.1541-1)(0.1541-2)}{6}(6)\right]$$

u_5 = 0.1542.

∴ Take u = 0.154 correct to three decimal places.

Hence the value of x (which is corresponds to y = 10)

i.e, the cube root of 10 is x_0 + uh = 2.154.

Q57. From the table of values of x and e^x are (1.4, 4.0552), (1.5, 4.4817), (1.6, 4.9530), (1.7, 5.4739) find x when e^x = 4.7115 using the method of successive approximations.

Sol :

x	y = e^x	Δy	$\Delta^2 y$	$\Delta^3 y$
1.4	4.0552			
		0.4265		
1.5	4.4817		0.0448	
		0.4713		0.0048
1.6	4.9530		0.0496	
		0.5209		
1.7	5.4739			

Here y_u = 4.7115, y_0 = 4.0552, Δy_0 = 0.4265,

$\Delta^2 y_0$ = 0.0448, $\Delta^3 y_0$ = 0.0048

The successive approximation to u are,

$$u_1 = \frac{1}{\Delta y_0}(y_u - y_0) = \frac{1}{0.4265}(4.7115 - 4.0552) = 1.5388$$

$$u_2 = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_1(u_1-1)}{2}\Delta^2 y_0\right]$$

$$= \frac{1}{0.4265}\left[4.7115 - 4.0552 - \frac{1.5388(1.5388-1)}{2}0.0448\right]$$

$$= \frac{1}{0.4265}[0.6563 - 0.0186] = 1.4952$$

$$u_3 = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_2(u_2-1)}{2}\Delta^2 y_0 - \frac{u_2(u_2-1)(u_2-2)}{6}\Delta^3 y_0\right]$$

$$= \frac{1}{0.4265}\left[0.6563 - \frac{1.4952(1.4952-1)}{2}0.0448 - \frac{1.4952(1.4952-1)(1.4952\text{-}2)}{6}0.0048\right]$$

$$= \frac{1}{0.4265}[0.6563 - 0.0186 - 0.0003]$$

$u_3 = 1.4945.$

$$u_4 = \frac{1}{0.4265}\left[0.6563 - \frac{1.4945(1.4945-1)}{2}0.0448 - \frac{1.4945(1.4945-1)(1.4945-2)}{6}(0.0048)\right]$$

$$= \frac{1}{0.4265}[0.6563 - 0.0166 - 0.0003]$$

$u_4 = 1.4992.$

$$u_5 = \frac{1}{0.4265}\left[0.6563 - \frac{1.4992(1.4992-1)}{2}0.0448 - \frac{1.4992(1.4992-1)(1.4992-2)}{6}(0.0048)\right]$$

$$= \frac{1}{0.4265}[0.6563 - 0.0168 - 0.0003]$$

$u_5 = 1.4987 = 1.499.$

Take $u = 1.499$

i.e, The value of e^x is $x_0 + uh = 1.4 + 1.499\ (0.1) = 1.5499$

$= 1.55.$

Choose the Correct Answers

1. The relation between Δ & E is ________ [c]

   (a) $\Delta \equiv E - 1$ (b) $E \equiv \Delta + 1$

   (c) Both   (d) None

2. Let $(x_0, y_0),(x_1, y_1)$ be the given two points the divided difference of order i is ________ [b]

   (a) $[x_0, x_1] = \dfrac{y_0 - y_1}{x_0 - x_1}$ (b) $[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0}$

   (c) $[x_0, x_1] = \dfrac{y_0 - y_1}{x_1 - x_0}$ (d) None

3. $E^{-\frac{1}{2}}$ = ______ [b]

   (a) $\mu + \dfrac{\delta}{2}$ (b) $\mu - \dfrac{\delta}{2}$

   (c) $\dfrac{\mu}{2} + \delta$ (d) $\dfrac{\mu}{2} - \delta$

4. The forward difference operation is ________. [a]

   (a) Δ (b) ∇

   (c) E   (d) δ

5. The n^{th} forward differences are defined by the formula ________. [b]

   (a) $\Delta^n y_r = \Delta^{n-1} y_r - \Delta^{n-1} y_{r+1}$ (b) $\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r$

   (c) $\Delta^n y_r = \Delta^{n-1} y_r - \Delta^{n-2} y_{r-1}$ (d) None

6. The backward difference operator is ________. [b]

   (a) Δ (b) ∇

   (c) E   (d) δ

7. $E\nabla = \nabla E$ = ______ [a]

   (a) Δ (b) ∇

   (c) E   (d) None

8. If $x_0 = 0.75825$, $x = 0.759$ and $h = 0.00005$ then n = ______ [b]

   (a) 1.5   (b) 15

   (c) 2.5   (d) 25

9. Newton's forward interpolation formula can be used ________. [a]

   (a) Only for equally spaced intervals   (b) Only for unequally spaced intervals

   (c) a & b   (d) None

10. Using Newton's forward formula, find sin (0.1604) from the following table : [b]

x	0.160	0.161	0.162
f(x)	0.15932	0.1603	0.1613

   (a) 0.1697   (b) 0.1597

   (c) 0.1587   (d) 0.1687

Fill in the Blanks

1. The central difference operator is ________.
2. In Newton's forward interpolation, p = ________.
3. Newton's Backward interpolation formula is ________ $\frac{p(p+1)}{2!}$ D2yn + ...
4. $(1 + \nabla)(1 - \nabla)$ = ________.
5. The first order divided differences $[x_0, x_1]$ = ________.
6. If $x_0 = 0.6$, n = 2.6 and h = 0.2 then x = ________.
7. If f(1.0) = 0.7651977 and f(1.3) = 0.6200860 then the $f[x_0, x_1]$ = ________.
8. The co-efficents of Newton's forward difference form of two interpolating poly are tabulated along the ________. in the table.
9. The backward difference, for the sequence $\{p_n\}_{n=0}^{\infty}$ is defined by ∇p_n = ________.
10. If $f(x) = e^{2x}$ then the value of f(0.43) = ________.

Answers

1. δ
2. $\frac{x - x_0}{h}$
3. $y_n(x) = y_n + p\nabla y_n +$
4. 1
5. $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$
6. 1.12
7. –0.4837057
8. Diagonal
9. $p_n - p_{n-1}$
10. 2.36316

UNIT III	**Curve Fitting:** Least Square Curve Fitting: Fitting a Straight Line-Nonlinear Curve Fitting. **Numerical Differentiation and Integration:** Numerical Differentiation - Numerical Integration: Trapezoidal Rule-Simpson's 1/3rd-Rule and Simpson's 3/8th-Rule - Boole's and Weddle's Rule - Newton's Cotes Integration Formulae.

3.1 Curve Fitting

3.1.1 Least Square Curve Fitting a Straight Line

Q1. Explain least square curve fitting.

Sol :

Let the set of data points be (x_i, y_i), i = 1, 2, ... m and let the curve given by y = f(x) be fitted to this data. At $x = x_i$, the given ordinate is y_i and the corresponding value on the fitting curve is $f(x_i)$.

If e_1 is the error of approximation at $x = x_i$, then

we have, $e_i = y_i - f(x_i)$ ... (1)

If we write

$$S = [y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + ... + [y_m - f(x_m)]^2$$

$$S = e_1^2 + e_2^2 + ... + e_m^2 \quad ... (2)$$

then the method of least squares consists in minimizing S. i.e, the sum of the squares of the errors.

Q2. Write the procedure for fitting a straight line by the method of least square.

Sol :

Let $y = a_0 + a_1x$ be the straight libe to the fitted to the given data.

Then, we have

$$S = [y_1 - (a_0 + a_1x_1)]^2 + [y_2 - (a_0 + a_1x_2)]^2 + ... + [y_m - (a_0 + a_1x_m)^2 \quad ... (1)$$

For S to be minimum, we have

$$\frac{\partial S}{\partial a_0} = 0 = -2[y_1 - (a_0 + a_1x_1)] - 2[y_2 - (a_0 + a_1x_2)] + ... - 2[ym - (a_0 + a_1x_m)] \quad ... (2)$$

and $$\frac{\partial S}{\partial a_1} = 0 = -2x_1[y_1 - (a_0 + a_1x_1)] - 2x_2[y_2 - (a_0 + a_1x_2)] .... - 2x_m[y - (a_0 + a_1x_m)] \quad ... (3)$$

The above equations simplify to

$$\left.\begin{aligned} ma_0 + a_1(x_1 + x_2 + ... + x_m) &= y_1 + y_2 + ... + y_m \\ \text{and } a_0(x_1 + x_2 + ... + x_m) + a_1(x_1^2 + x_2^2 + ... + x_m^2) &= x_1y_1 + x_2y_2 + ... + x_my_m \end{aligned}\right\} \quad .... (4)$$

or, more compactly to

$$\left.\begin{aligned} ma_0 + a_1 \sum_{i=1}^{m} x_i &= \sum_{i=1}^{m} y_i \\ \text{and } a_0 + \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} x_1^2 &= \sum_{i=1}^{m} x_i y_i \end{aligned}\right\} \quad ... (5)$$

Since the x_i and y_i are known quantities, equations (4) and (5), called the normal equation, can be solved for the unknown a_0 and a_1.

We can obtain easily,

$$a_1 = \frac{m\sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} x_i . \sum_{i=1}^{m} y_i}{m\sum_{i=1}^{m} x_i^2 - \left(\sum_{i=1}^{m} x_i\right)^2} \quad ... (6)$$

and then $a_0 = \bar{y} - a_1\bar{x}$ ... (7)

Since $\frac{\partial^2 S}{\partial a_0^2}$ and $\frac{\partial^2 S}{\partial a_1^2}$ are both positive at the points a_0 and a_1.

$\bar{x}$ and $\bar{y}$ are the means of x and y respectively.

From equation (7) we have

$$\bar{y} = a_0 + a_1\bar{x}$$

Which shows that the fitted straight line passes through the centroid of the data points.

The correlation coefficient (cc) is defined as,

$$cc = \sqrt{\frac{S_y - S}{S_y}} \quad ... (8)$$

where $S_y = \sum_{i=1}^{m} (y_i - \bar{y})^2$ ... (9)

and S is defined in equation (1)

If cc is close to 1, then the fit is considered to be good, although this is not always true.

Q3. The Table gives the temperature T(in ^{0}c) and lengths l(in mm) of a heated rod. If 1 = a_0 + a_1T. Find the best values for a_0 and a_1.

T	20^0	30^0	40^0	50^0	60^0	70^0
l	800.3	800.4	800.6	800.7	800.9	801.0

Sol :

We require ΣT, Σl, ΣT^2 and ΣTl

T	l	T^2	Tl
20	800.3	400	16006
30	800.4	900	24012
40	800.6	1600	32024
50	800.7	2500	40035
60	800.9	3600	48054
70	801.0	4900	56070
ΣT = 270	Σl = 4803.9	ΣT^2 = 139000	ΣTl = 16201

The normal equations are, $ma_0 + a_1\sum_{i=1}^{m} T_i = \sum_{i=1}^{m} l_i$ ... (1)

$$a_0\sum_{i=1}^{m} T_i + a_1\sum_{i=1}^{m} T_i^2 = \sum_{i=1}^{m} T_i l_i \quad ... (2)$$

Substituting the corresponding values in equations (1) & (2)

$\Rightarrow$ $6a_0 + 270a_1 = 4803.9$ ... (3)

$270a_0 + 13900a_1 = 216201$ ... (4)

Solving equations (3) and (4)

we get $a_0 = 800$ and $a_1 = 0.0146$

Q4. Certain experimental values of x and y are given below : (0, – 1), (2, 5), (5, 12), (7, 20). If the straight line Y = a_0 + a_1x is fitted to the above data, find the approximate values of a_0 and a_1.

Sol :

Let the straight line be Y = $a_0 + a_1x$

The table of values is given below :

x	y	x^2	xy
0	–1	0	0
2	5	4	10
5	12	25	60
7	20	49	140
14	36	78	210

The normal equations are $ma_0 + a_1\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i$ ... (1)

$$a_0\sum_{i=1}^{m} x_i + a_1\sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i y_i \quad \text{... (2)}$$

Substituting corresponding values in (1) & (2)

$\Rightarrow$ $4a_0 + 14a_1 = 36$ ... (3)

and $14a_0 + 78a_1 = 210$ ... (4)

Now solving equation (3) and (4)

(3) × 14 – (4) × 4

$$\Rightarrow \begin{array}{r} \cancel{56a_0} + 196a_1 = 504 \\ \cancel{56a_0} + 312a_1 = 840 \\ - \quad - \quad - \\ \hline \end{array}$$

$-116a_1 = -336$

$a_1 = 2.8966$

Sub a_1 in (3), we get

$a_0 = -1.1381$

Hence, the best straight line fit is given by

$Y = -1.1381 + x(2.8966)$

Q5. Explain the method of least squares to fit a straight line of the $Y = a_0 + a_1x$ to the data (x_i, y_i) :

x	**1**	**2**	**3**	**4**	**6**	**8**
y	**2.4**	**3.1**	**3.5**	**4.2**	**5.0**	**6.0**

Sol :

Let the equation of the straight line be $Y = a_0 + a_1x$

x	**y**	**x^2**	**xy**
1	2.4	1	2.4
2	3.1	4	6.2
3	3.5	9	10.5
4	4.2	16	16.8
6	5.0	36	30
8	6.0	64	48
$\Sigma x = 24$	$\Sigma y = 24.2$	$\Sigma x^2 = 130$	$\Sigma xy = 113.9$

Now, the normal equations are,

$$ma_0 + a_1\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i \qquad ... (1)$$

$$a_0\sum_{i=1}^{m} x_i + a_1\sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i y_i \qquad ... (2)$$

Substituting the corresponding values in equation (1) & (2),

$\Rightarrow$ $6a_0 + 24a_1 = 24.2$

$24a_0 + 130a_1 = 113.9$

Solving the above equations we get

$a_0 = 2.016, a_1 = 0.503$

Q6. Find the values of a_0 and a_1 so that $Y = a_0 + a_1x$ fits the data given in the table.

x	0	1	2	3	4
y	1.0	2.9	4.8	6.7	8.6

Sol :

x	y	x^2	xy
0	1.0	0	0
1	2.9	1	2.9
2	4.8	4	9.6
3	6.7	9	20.1
4	8.6	16	34.4
$\Sigma x = 10$	$\Sigma y = 24$	$\Sigma x^2 = 30$	$\Sigma xy = 67$

The normal equations are,

$$ma_0 + a_1\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i \qquad ... (1)$$

$$a_0\sum_{i=1}^{m} x_i + a_1\sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i y_i \qquad ... (2)$$

Substituting the corresponding values in equation (1) & (2),

$\Rightarrow$ $5a_0 + 10a_1 = 24$ ... (3)

$10a_0 + 30a_1 = 67$ ... (4)

Solving equation (3) & (4)

(3) × 2 – (4)

$\Rightarrow \quad \cancel{10a_0} + 20a_1 = 48$

$\cancel{10a_0} + 30a_1 = 67$

$- \quad - \quad -$

$-10a_1 = -19$

$a_1 = \frac{19}{10} = 1.9$

Sub a_1 in (3) we get

$a_0 = 1$

$\therefore \quad a_0 = 1$ and $a_1 = 1.9$

Q7. Fit a straight line to the following data :

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

by the method of least squares.

Sol :

Let the required straight line be $y = a_0 + a_1x$

We calculate Σx, Σy, Σx^2, Σxy from the following table.

x	y	x^2	xy
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
$\Sigma x = 10$	$\Sigma y = 16.9$	$\Sigma x^2 = 30$	$\Sigma xy = 47.1$

The normal equations are

$ma_0 + a_1\Sigma x_i = \Sigma y_i$ ... (1)

$a_0\Sigma x_i + a_1\Sigma x_i^2 = \Sigma x_i y_i$ ... (2)

Substituting the above values in equations (1) & (2) we get

$5a_0 + 10a_1 = 16.9$ ... (3)

$10a_0 + 30a_1 = 47.1$ ... (4)

Solving (3) & (4), we get

$a_0 = 0.72$ and

$a_1 = 1.33$

$\therefore$ The required straight line is $Y = 0.72 + 1.33x$

Q8. Fit a straight line to the form $Y = a_0 + a_1x$ for the following data :

x	0	5	10	15	20	25
y	12	15	17	22	24	30

Sol :

x	y	x^2	xy
0	12	0	0
5	15	25	75
10	17	100	170
15	22	225	330
20	24	400	480
25	30	625	750
$\Sigma x = 75$	$\Sigma y = 120$	$\Sigma x^2 = 1375$	$\Sigma xy = 1805$

To fit a straight line of the form $Y = a_0 + a_1x$.

We have the normal equations are

$$ma_0 + a_1\Sigma x_i = \Sigma y_i \quad \text{... (1)}$$

$$a_0\Sigma x_i + a_1 \Sigma x_i^2 = \Sigma x_i y_i \quad \text{... (2)}$$

Substituting the above values in equations (1) & (2)

We get,

$$6a_0 + 75a_1 = 120 \quad \text{... (3)}$$

$$75a_0 + 1375a_1 = 1805 \quad \text{... (4)}$$

Solving (3) & (4), we get

$a_0 = 11.2862$ and $a_1 = 0.6971$

$\therefore$ The fitted straight line is $Y = 11.2862 + 0.6971x$.

Q9. Find the values of a_0 and a_1, so that $Y = a_0 + a_1x$ fits the data given in the table.

x	0	2	5	7
y	– 1	5	12	20

Sol :

We prepare the following table.

x	y	x^2	xy
0	–1	0	0
2	5	4	10
5	12	25	60
7	20	49	140
$\Sigma x = 14$	$\Sigma y = 36$	$\Sigma x^2 = 78$	$\Sigma xy = 210$

The normal equations are

$ma_0 + a_1\Sigma x_i = \Sigma y_i$ ... (1)

$a_0\Sigma x_i + a_1\Sigma x_i^2 = \Sigma x_i y_i$ ... (2)

Substituting the above values in equations (1) & (2) we get

$4a_0 + 14a_1 = 36$ ... (3)

$14a_0 + 78a_1 = 210$ ... (4)

Now,

(1) × 14 – (2) × 4 gives

$16a_1 = 366 \Rightarrow a_1 = 2.8966$

Sub a_1 in (3) we get

$a_0 = -1.1381$

$\therefore$ $a_0 = -1.1381$ and $a_1 = 2.8966$.

Q10. By the method of least squares, find the straight line by the following data.

x	1	2	3	4	5
y	14	27	40	55	68

Sol :

Let the required line of the form $Y = a_0 + a_1x$.

x	y	x^2	xy
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340
$\Sigma x = 15$	$\Sigma y = 204$	$\Sigma x^2 = 55$	$\Sigma xy = 748$

The normal equations are

$$ma_0 + a_1\Sigma x_i = \Sigma y_i \quad ... (1)$$

$$a_0\Sigma x_i + a_1 \Sigma x_i^2 = \Sigma x_i y_i \quad ... (2)$$

Substituting the above values in equations (1) & (2) we get

$$5a_0 + 15a_1 = 204 \quad ... (3)$$

$$15a_0 + 55a_1 = 748 \quad ... (4)$$

Solving this equations we get $a_0 = 0$, $a_1 = 13.6$

Hence, the required line is Y = 13.6x.

3.1.2 Non -Linear Curve Fitting

Q11. Write the types of Non - Linear curve fitting

Sol :

A polynomial of n^{th} degree and an exponential function to fit the given data points (x_i, y_i), i = 1, 2, ... m.

1) Power Function :

Let $y = ax^c$ is the function to be fitted using the given data.

Taking logarithms on both sides, we get

$$\log y = \log a + c \log x \quad ... (1)$$

Which is of the form $Y = a_0 + a_1x$

Where $a_0 = \log a$, $a_1 = c$ and $Y = \log y$ and $X = \log x$.

Using the procedure of fitting a straight line we can find a_0 and a_1 and Hence c and a.

2) Polynomial of n^{th} Degree :

Consider that the n^{th} degree polynomial

$$Y = a_0 + a_1x + a_2x^2 + ... + a_nx^n \quad ... (2)$$

be fitted to the data points (x_i, y_i), i = 1, 2, ... m.

Then we have

$$S = [y_1 - (a_0 + a_1x_1 + ... + a_n x_1^n]^2 + [y_2 - (a_0 + a_1x_2 + ... + a_n x_2^n]^2 + ... + [y_m - (a_0 + a_1x_m + ... + a_n x_m^n]^2.$$

We equate the first partial derivaties of S to zero and after simplication, we get the following normal equations.

$$\left.\begin{array}{l} ma_0 + a_1 \sum_{i=1}^{m} x_i + ... + a_n \sum_{i=1}^{m} x_i^n = \sum_{i=1}^{m} y_i \\ a_0 \sum_{i=1}^{m} x_i + a_1 \sum_{i=1}^{m} x_i^2 + ... + a_n \sum_{i=1}^{m} x_i^{n+1} = \sum_{i=1}^{m} x_i y_i \\ ------------------------ \\ a_0 \sum_{i=1}^{m} x_i^n + a_1 \sum_{i=1}^{m} x_i^{n+1} + ... + a_n \sum_{i=1}^{m} x_i^{2n} = \sum_{i=1}^{m} x_i^n y_i \end{array}\right\} \quad ... (3)$$

These are (m + 1) equations in (m + 1) unknowns.

Hence these can be solved for a_0, a_1, ... a_n.

Substituting these values in (2), we get the required polynomial of n^{th} degree.

3) Exponential Function :

Let the curve $y = a_0 e^{a_1 x}$ ... (4)

be fitted to the given data. Then, taking log on both sides of equation (4),

$$\log y = \log a_0 e^{a_1 x}$$

$$\log y = \log a_0 + a_1 x \quad ... (5)$$

Which can be written in the form,

$$Z = A + Bx$$

where $Z = \log y$, $A = \log a_0$ and $B = a_1$

The problem therefore reduces to finding a least squares straight line through the given data.

Q12. Fit a polynomial of the second degree to the data points given in the following table.

x	**0**	**1.0**	**2.0**
y	**1.0**	**6.0**	**17.0**

Sol :

Let the required second degree polynomial is,

$$Y = a_0 + a_1 x + a_2 x^2$$

The normal equations are,

$$\left.\begin{array}{r} ma_0 + a_1 \Sigma x_i + a_2 \Sigma x_i^2 = \Sigma y_i \\ a_0 \Sigma x_i + a_1 \Sigma x_i^2 + a_2 \Sigma x_i^3 = \Sigma x_i y_i \\ a_0 \Sigma x_i^2 + a_1 \Sigma x_i^3 + a_3 \Sigma x_i^4 = \Sigma x_i^2 y_i \end{array}\right\} \quad ... (1)$$

x	**y**	**x^2**	**x^3**	**x^4**	**xy**	**x^2y**
0	1	0	0	0	0	0
1.0	6	1	1	1	6	6
2.0	17	4	8	16	34	68
$\Sigma x = 3$	$\Sigma y = 24$	$\Sigma x^2 = 5$	$\Sigma x^3 = 9$	$\Sigma x^4 = 17$	$\Sigma xy = 40$	$\Sigma x^2 y = 74$

Substitute these values in equation (1),

$3a_0 + 3a_1 + 5a_2 = 24$ ... (2)

$3a_0 + 5a_1 + 9a_2 = 40$ ... (3)

$5a_0 + 9a_1 + 17a_2 = 74$ ... (4)

From (2) & (3),

$$\begin{array}{l} \cancel{3a_0} + 3a_1 + 5a_2 = 24 \\ \cancel{3a_0} + 5a_1 + 9a_2 = 40 \\ \underset{-}{} \quad \underset{-}{} \quad \underset{-}{} \quad \underset{-}{} \\ \hline -2a_1 - 4a_2 = -16 \\ \hline \end{array}$$

$a_1 + 2a_2 = 8$ ... (5)

Now,

(3) × 5 – (4) × 3

$$\begin{array}{l} \cancel{15a_0} + 25a_1 + 45a_2 = 200 \\ \cancel{15a_0} + 27a_1 + 51a_2 = 222 \\ \underset{-}{} \quad \underset{-}{} \quad \underset{-}{} \quad \underset{-}{} \\ \hline -2a_1 - 6a_2 = -22 \\ \hline \end{array}$$

$a_1 + 3a_2 = 11$ ... (6)

From (5) and (6)

$$\begin{array}{l} \cancel{a_1} + 2a_2 = 8 \\ \cancel{a_1} + 3a_2 = 11 \\ \underset{-}{} \quad \underset{-}{} \quad \underset{-}{} \\ \hline -a_2 = -3 \\ \hline \end{array}$$

$a_2 = 3$

Sub a_2 in (6), $a_1 + 3(3) = 11$

$a_1 = 11 – 9$

$a_1 = 2$

Sub a_1, a_2 in (1), $3a_0 + 3(2) + 5(3) = 24$

$3a_0 + 6 + 15 = 24$

$3a_0 + 21 = 24$

$3a_0 = 3$

$a_0 = 1$

Hence, the required second degree polynomial is

$Y = 1 + 2x + 3x^2$.

Q13. Find the values a_0, a_1 and a_2 so that $Y = a_0 + a_1x + a_2x^2$ is the best fit to the data.

x	0	1	2	3	4
y	1	0	3	10	21

Sol :

The normal equations are,

$$\left.\begin{aligned} ma_0 + a_1\Sigma x_i + a_2\Sigma x_i^2 &= \Sigma y_i \\ a_0\Sigma x_i + a_1\Sigma x_i^2 + a_2\Sigma x_i^3 &= \Sigma x_i y_i \\ a_0\Sigma x_i^2 + a_1\Sigma x_i^3 + a_3\Sigma x_i^4 &= \Sigma x_i^2 y_i \end{aligned}\right\} \quad \text{... (1)}$$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	0	1	1	1	0	0
2	3	4	8	16	6	12
3	10	9	27	81	30	90
4	21	16	64	256	84	336
$\Sigma x = 10$	$\Sigma y = 35$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$	$\Sigma xy = 120$	$\Sigma x^2y = 438$

Substitute the above values in equation (1),

$5a_0 + 10a_1 + 30a_2 = 35$ ... (2)

$10a_0 + 30a_1 + 100a_2 = 120$ ... (3)

$30a_0 + 100a_1 + 354a_2 = 438$ ... (4)

Now solving equation (2), (3) and (4)

(2) × 2 – (3)

$$\begin{array}{l} \cancel{10a_0} + 20a_1 + 60a_2 = 70 \\ \cancel{10a_0} + 30a_1 + 100a_2 = 120 \\ \quad - \quad\;\; - \quad\;\; - \qquad\; - \\ \hline \quad -10a_1 - 40a_2 = -50 \\ \hline \end{array}$$

$\Rightarrow$ $10a_1 + 40a_2 = 50$ ... (5)

(3) × 3 – (4)

$$\begin{array}{l} \cancel{30a_0} + 90a_1 + 300a_2 = 360 \\ \cancel{30a_0} + 100a_1 + 354a_2 = 438 \\ \quad - \quad\;\; - \quad\;\; - \qquad\; - \\ \hline \quad -10a_1 - 54a_2 = -78 \\ \hline \end{array}$$

$10a_1 + 54a_2 = 78$ ... (6)

From (5), (6)

$$\begin{aligned} 10a_1 + 40a_2 &= 50 \\ 10a_1 + 54a_2 &= 78 \\ - \quad - \quad & - \\ \hline -14a_2 &= -28 \end{aligned}$$

$-14a_2 = -28$

$a_2 = \frac{28}{14}$

$a_2 = 2$

Sub a_2 in (5),

$10a_1 + 40(2) = 50$

$10a_1 + 80 = 50$

$\Rightarrow \quad 10a_1 = 50 - 80$

$a_1 = -3$

Sub a_1, a_2 in (2),

$5a_0 + 10(-3) + 30(2) = 35$

$5a_0 - 30 + 60 = 35$

$5a_0 + 30 = 35$

$5a_0 = 5$

$a_0 = 1$

$\therefore \quad a_0 = 1, a_1 = -3$ and $a_2 = 2$

Q14. Determine the constants a and b by the method of least squares such that $y = ae^{bx}$ fits the following data.

x	2	4	6	8	10
y	4.077	11.084	30.128	81.897	222.62

Sol :

The given relation is $y = ae^{bx}$

Taking log on both sides, we obtain

$\log y = \log ae^{bx}$

$\log y = \log a + bx$ ... (1)

Let $\log y = Y$, $x = X$, $\log a = a_0$ and $b = a_1$

The relation (1) takes the form

$y = a_0 + a_1 x$

which is a straight line.

X = x	Y = log y	X^2	XY
2	1.405	4	2.810
4	2.405	16	9.620
6	3.405	36	20.430
8	4.405	64	35.240
10	5.405	100	54.050
$\Sigma X = 30$	$\Sigma Y = 17.05$	$\Sigma X^2 = 220$	$\Sigma XY = 122.150$

Normal equations are,

$$\left.\begin{aligned} ma_0 + a_1\Sigma x_i &= \Sigma y_i \\ a_0\Sigma x_i + a_1\Sigma x_i^2 &= \Sigma x_i y_i \end{aligned}\right\} \quad \text{... (2)}$$

Substitute the corresponding values in (2)

$$5a_0 + 30a_1 = 17.025 \quad \text{... (3)}$$

$$30a_0 + 220a_1 = 122.150 \quad \text{... (4)}$$

Now solving (3) & (4)

(3) × 6 – (4)

$$\begin{aligned} 30a_0 + 180a_1 &= 102.150 \\ 30a_0 + 220a_1 &= 122.150 \\ - \quad - \quad & \quad - \\ \hline \Rightarrow \quad -40a_1 &= -20 \end{aligned}$$

$$a_1 = \frac{-20}{-40} = \frac{1}{2} = 0.5$$

$$a_1 = 0.5$$

Substitute a_1 in (3),

$$5a_0 + 30(0.5) = 17.025$$

$$5a_0 + 15 = 17.025$$

$$5a_0 = 17.025 - 15$$

$$5a_0 = 2.025$$

$$a_0 = 0.405$$

Since $\log a = a_0$

$a = e^{a_0} = e^{0.405} = 1.499$

$b = a_1 = 0.5$

Hence, the required curve is,

$y = 1.499e^{0.5x}$.

Q15. Find the parabola of the form y = a + bx + cx² passing through the points (–1, 2), (0, 1) and (1, 4).

Sol :

x	y	x^2	x^3	x^4	xy	x^2y
–1	2	1	–1	1	–2	2
0	1	0	0	0	0	0
1	4	1	1	1	4	4
$\Sigma x_i = 0$	$\Sigma y = 7$	$\Sigma x^2 = 2$	$\Sigma x^3 = 0$	$\Sigma x^4 = 2$	$\Sigma xy = 2$	$\Sigma x^2y = 6$

The normal equations are,

$$\left.\begin{aligned} ma + b\Sigma x_i + c\Sigma x_i^2 &= \Sigma y_i \\ a\Sigma x_i + b\Sigma x_i^2 + c\Sigma x_i^3 &= \Sigma x_i y_i \\ a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma_i^4 &= \Sigma x_i^2 y_i \end{aligned}\right\} \quad \dots (1)$$

Substitute the values, we get

$3a + 2c = 7$ ... (2)

$2b = 2$ ... (3)

$2a + 2c = 6$ ... (4)

Solving (2), (3) and (4)

From (3), b = 1

From (2) and (4), a = 1

From (4) $\Rightarrow$ c = 2

Hence the required parabola is $y = 1 + x + 2x^2$.

Q16. Find the curve of best fit of the type y = ae^bx to the following data by the method of least squares.

x	1	5	7	9	12
y	10	15	12	15	21

Sol :

The curve of the form $y = ae^{bx}$

Apply log on both sides

$\log y = \log (ae^{bx})$

$\log y = \log a + bx$ ... (1)

let $\log y = Y$, $x = X$, $\log a = a_0$ and $b = a_1$.

Then equation (1), becomes $Y = a_0 + a_1 x$.

which is a straight line.

X = x	y	Y = log y	X^2	XY
1	10	2.3026	1	2.3026
5	15	2.7080	25	13.54
7	12	2.4849	49	17.3943
9	15	2.7080	81	24.372
12	21	3.0445	144	36.534
$\Sigma X = 34$		$\Sigma Y = 13.248$	$\Sigma X^2 = 300$	$\Sigma XY = 94.1429$

Substitute these values in Normal equation, we get

$5a_0 + 34a_1 = 13.248$

$34a_0 + 300a_1 = 94.1429$

Solving the above two equations, we get

$a_0 = 2.2484$ and $a_1 = 0.059$

$\therefore \quad a = e^{a_0} = e^{2.2484} = 9.4725$ and $b = a_1 = 0.059$

Hence, the required curve is $y = 9.4725\, e^{0.059x}$.

3.2 Numerical Differentiation and Integration

3.2.1 Numerical Differentiation

Q17. Derive the derivatives using Newton's Forward Difference formula.

Sol :

Suppose that given a set values (x_i, y_i) $i = 0, 1, 2, ... n$.

We want to find the derivative of $y = f(x)$ passing through the $(n + 1)$ points, at a point nearer to the starting value $x = x_0$.

Consider Newton's forward difference formula :

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ... \quad ... (1)$$

$$\text{Where } u = \frac{x - x_0}{h} \quad ... (2)$$

Differentiating equation (1) with respect to u, we have

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + ...$$

Differentiating equation (2) w.r.to x, we have

$$\frac{du}{dx} = \frac{1}{h}$$

Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + ...\right] \quad ... (3)$$

Equation (3) gives the value of $\frac{dy}{dx}$ at any point x.

Which may be anywhere in the interval.

At $x = x_0$, we obtain $u = 0$ From equation (2).

Hence equation (3) gives

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ...\right] \quad ... (4)$$

Differentiating equation (3) once again, we obtain

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6u-6}{6}\Delta^3 y_0 + \frac{12u^2-36u+22}{24}\Delta^4 y_0 + ...\right] \quad ... (5)$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + ...\right]$$

lly $\left[\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^3 y_0 + \frac{3}{2}\Delta^4 y_0 + ...\right]$

These Formulae for computing higher derivatives may be obtained by successive differentiation.

Q18. Derive the derivatives using Newton's Backward Difference formula.

Sol :

Newton's Backward difference formula is,

$$y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + ... \quad ... (1)$$

Where $u = \frac{x - x_n}{h}$ ... (2)

Differentiating equation (1) w.r.to u, we get

$$\frac{dy}{du} = \nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n + \frac{4u^3+18u^2+22u+6}{4!}\nabla^4 y_n + ...$$

Diff equation (2) w.r.to x,

$$\frac{du}{dx} = \frac{1}{h}$$

Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{h^2}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n + \frac{4u^3+18u^2+22u+6}{4!}\nabla^4 y_n + ...\right] ... (3)$$

at $x = x_n$,

From equation (2), we get $u = 0$

Hence, equation (3) gives

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + ...\right] \quad ... (4)$$

Diff equation (3), we obtain

$$\frac{d^2y}{dx^2} = \frac{1}{h}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \frac{6u^2+18u+11}{12}\nabla^4 y_n + ...\right]$$

put $u = 0$, we get

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + ...\right]$$

lly $\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + ...\right]$

Q19. Derive the derivatives using stirling's formula.

Sol:

Stirling's formula gives,

$$y = y_0 + \frac{u}{2}[\Delta y_0 + \Delta y_{-1}] + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u^3 - u}{12}[\Delta^3 y_{-1} + \Delta^3 y_{-2}] + \frac{u^4 - u^2}{24}\Delta^4 y_{-2} + ... \quad ... (1)$$

where $u = \dfrac{x - x_0}{h}$ ... (2)

diff equation (1) w.r.to u, we get

$$\frac{dy}{du} = \frac{1}{2}(\Delta y_0 + \Delta y_{-1}) + u\,\Delta^2 y_{-1} + \frac{3u^2 - 1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{12}(2u^3 - u)\,\Delta^4 y_{-2} + \frac{5u^4 - 15u^2 + 4}{240}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + ...$$

Diff equation (2) w.r.to x, we get

$$\frac{du}{dx} = \frac{1}{h}$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{h}\left[\frac{1}{2}\left(\Delta y_0 + \Delta y_{-1}\right) + u\Delta^2 y_{-1} + \frac{3u^2 - 1}{12}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right) + \frac{1}{12}\left(2u^3 - u\right)\Delta^4 y_{-2} + \frac{5u^4 - 15u^2 + 4}{240}\left(\Delta^5 y_{-2} + \Delta^5 y^{-3}\right) + ...\right]$$

At $x = x_0 \Rightarrow u = 0$

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}\left(\Delta y_0 + \Delta y_{-1}\right) - \frac{1}{12}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right) + \frac{1}{60}\left(\Delta^5 y_{-2} + \Delta^5 y_{-3}\right) + ...\right]$$

$$\text{Illy } \left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + ...\right]$$

$$\text{and } \left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\frac{1}{2}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right) + ...\right]$$

Q20. From the following table of values of x and y, obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for x = 1.2 and $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ for x = 2.2 and also $\frac{dy}{dx}$ at x = 2.0.

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Sol :

The difference table is,

x	y	Δy	Δ^2y	Δ^3y	Δ^4y	Δ^5y	Δ^6y
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

a) Here $x_0 = 1.0$, $y_0 = 2.7183$ amd $h = 0.2$

If $x = 1.2$, $u = \dfrac{x - x_0}{h} = \dfrac{1.2 - 1.0}{0.2} = 1$

By Newton's forward difference formula is,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + ...\right]$$

sub $u = 1$, in $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}\Delta^2 y_0 - \frac{1}{6}\Delta^3 y_0 + \frac{1}{12}\Delta^4 y_0 - \frac{1}{20}\Delta^5 y_0 + ...\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.2} = \frac{1}{0.2}\left[0.6018 + \frac{1}{2}(0.1333) - \frac{1}{6}(0.0294) + \frac{1}{12}(0.0069) - \frac{1}{20}(0.0013)\right]$$

$$= \frac{1}{0.2}\left[0.6018 + 0.0667 - 0.0049 + 0.0006 - 0.0001\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.2} = 3.3205$$

$$\left[\frac{d^2y}{dx^2}\right] = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + ...\right]$$

Sub u = 1 in $\frac{d^2y}{dx^2}$

$$\left[\frac{d^2y}{dx^2}\right] = \frac{1}{h^2}\left[\Delta^2 y_0 - \frac{1}{12}\Delta^4 y_0 + \frac{1}{12}\Delta^5 y_0\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{(0.2)^2}\left[0.1333 - \frac{1}{12}(0.0067) + \frac{1}{12}(0.0013)\right]$$

$$= 3.31854$$

$$\left[\frac{d^2y}{dx^2}\right] = 3.32$$

b) Here x_n = 2.2, y_n = 9.0250, h = 0.2

By Newton's backward difference formula,

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + ...\right]$$

$$= \frac{1}{0.2}\left[1.6359 + \frac{1}{2}(0.2964) + \frac{1}{3}(0.0535) + \frac{1}{4}(0.0094) + \frac{1}{5}(0.0014)\right]$$

$$\left[\frac{dy}{dx}\right]_{x=2.2} = 9.0228.$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + ...\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=2.2} = \frac{1}{(0.2)^2}\left[0.2964 + 0.0535 + \frac{11}{12}(0.0094) + \frac{5}{6}(0.0014)\right] = 8.992$$

c) To find $\frac{dy}{dx}$ at x = 2.0

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + ...\right]$$

$$= \frac{1}{0.2}\left[1.3395 + \frac{1}{2}(0.2429) + \frac{1}{3}(0.0441) + \frac{1}{4}(0.0080)\right] = 7.3896.$$

Q21. From the following table of values of x and y, obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for x = 1.6.

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Sol :

The difference table is

see in the above problem.

$$h = 0.2,\ x_0 = 1.6$$

By stiriling formula,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_{-1}+\Delta y_0}{2} - \frac{1}{6}\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-3}+\Delta^5 y_{-2}}{2} + ...\right]$$

$$= \frac{1}{0.2}\left[\frac{0.8978+1.0966}{2} - \frac{1}{6}\,\frac{0.0361+0.0441}{2} + \frac{1}{30}\,\frac{0.0013+0.0014}{2}\right]$$

$$= 4.9530.$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + ...\right]$$

$$= \frac{1}{(0.2)^2}\left[0.1988 - \frac{1}{12}(0.0080) + \frac{1}{90}(0.0001)\right]$$

$$= 4.9525$$

Q22. The following table gives angular displacement θ (in radius) at different times t (seconds):

(0, 0.052), (0.02, 0.105), (0.04, 0.168), (0.06, 0.242), (0.08, 0.327), (0.10, 0.408), (0.12, 0.489)

Calculate the angular velocity at t = 0.06.

Sol :

We know that the angular velocity is $\frac{d\theta}{dt}$.

The differnce table is

t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$	$\Delta^6\theta$
0	0.052 y_{-3}						
		0.053 Δy_{-3}					
0.02	0.105 y_{-2}		0.01 $\Delta^2 y_{-3}$				
		0.063 Δy_{-2}		+ 0.001 $\Delta^4 y_{-3}$			
0.04	0.168 y_{-1}		0.011 $\Delta^2 y_{-2}$		−0.001 $\Delta^4 y_{-3}$		
		0.074 Δy_{-1}		0.000 $\Delta^3 y_{-2}$		−0.014 $\Delta^5 y_{-3}$	
0.06	0.242 y_0		0.011 $\Delta^2 y_{-1}$		−0.015 $\Delta^4 y_{-2}$		0.048
		0.085 Δy_0		−0.015 $\Delta^3 y_{-1}$	0.034		
0.08	0.327		−0.004 $\Delta^2 y_0$		0.019 $\Delta^4 y_{-1}$		
		0.081 Δy_1		0.004 $\Delta^3 y_0$			
0.10	0.408		0.000 $\Delta^2 y_1$				
		0.081 Δy_2					
0.12	0.489						

Here $x_0 = 0.06$, $h = 0.02$

Now we find $\frac{d\theta}{dt}$ at t = 0.06

By using striling formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_{-1} + \Delta y_0}{2} - \frac{1}{6}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2} + ...\right]$$

$$\left(\frac{d\theta}{dt}\right)_{x=0.06} = \frac{1}{0.02}\left[\frac{0.074 + 0.085}{2} - \frac{1}{6}\frac{0.000 - 0.015}{2} + \frac{1}{30}\frac{-0.014 + 0.034}{2}\right]$$

$$= \frac{1}{0.02}\ [0.0795 + 0.0013 + 0.0003] = 4.055$$

Q23. From the following values of x and y. find $\frac{dy}{dx}$ at x = 0.6

(0.4, 1.5836), (0.5, 1.7974), (0.6, 2.0442), (0.7, 2.3275), (0.8, 2.6511)

Sol :

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.4	1.5836 (y_{-2})				
		0.2138 (Δy_{-2})			
0.5	1.7974 (y_{-1})		0.033 ($\Delta^2 y_{-2}$)		
		0.2468 (Δy_{-1})		0.0035 ($\Delta^3 y_{-2}$)	
0.6	2.0442 (y_0)		0.0365 ($\Delta^2 y_{-1}$)		0.0003 ($\Delta^4 y_{-2}$)
		0.2833 (Δy_0)		0.0038 ($\Delta^3 y_{-1}$)	
0.7	2.3275		0.0403 ($\Delta^2 y_0$)		
		0.3236			
0.8	2.6511				

Here x_0 = 0.6, h = 0.1

By using striling formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_{-1}+\Delta y_0}{2} - \frac{1}{6}\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-3}+\Delta^5 y_{-2}}{2} + ...\right]$$

$$\left(\frac{dy}{dx}\right)_{x=0.6} = \frac{1}{0.1}\left[\frac{0.2468+0.2833}{2} - \frac{1}{6}\frac{0.0035+0.0038}{2}\right]$$

$$= \frac{1}{0.1}[0.26505 - 0.00061]$$

$$= 2.6444$$

Q24. The distances (x cm) traversed by a particle at different times (t seconds) are given below.

t	**0.0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**
x	**3.01**	**3.16**	**3.29**	**3.36**	**3.40**	**3.38**	**3.32**

Find the velocity of the particle at t = 0.3 seconds.

Sol :

The difference table is

t	**x**	**Δx**	**Δ^2x**	**Δ^3x**	**Δ^4x**	**Δ^5x**	**Δ^6x**
0.0	3.01						
		0.15					
0.1	3.16		– 0.02				
		0.13		– 0.04			
0.2	3.29		– 0.06		0.07		
		0.07		0.03		– 0.07	
0.3	3.36		– 0.03		0		0.12
		0.04		– 0.03		0.05	
0.4	3.40		– 0.06		0.05		
		– 0.02		0.02			
0.5	3.38		– 0.04				
		– 0.06					
0.6	3.32						

Here t = 0.3, h = 0.1

By striling formula,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_{-1}+\Delta y_0}{2} - \frac{1}{6}\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-3}+\Delta^5 y_{-2}}{2} + ...\right]$$

$$\left[\frac{dt}{dx}\right]_{x=0.3} = \frac{1}{0.1}\left[\frac{0.07+0.04}{2} - \frac{1}{6}\frac{0.03-0.03}{2} + \frac{1}{30}\frac{-0.07+0.05}{2}\right]$$

$$= \frac{1}{0.1}[0.055 - 0.0003]$$

$$= 0.5467$$

Q25. From the following values of x and y, find $\frac{dy}{dx}$ when

(a) x = 1, (b) x = 3 (c) x = 6 and (d) $\frac{d^2y}{dx^2}$ at x = 3.

x	0	1	2	3	4	5	6
y	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506	9.0309

Sol :

The difference table is,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	6.9897						
		0.4139					
1	7.4036		– 0.036				
		0.3779		0.0057			
2	7.7815		– 0.0303		– 0.0011		
		0.3476		0.0046		– 0.0001	
3	8.1291		– 0.0257		– 0.0012		0.0009
		0.3219		0.0034		0.0008	
4	8.4510		– 0.0223		– 0.0004		
		0.2996		0.003			
5	8.7506		– 0.0193				
		0.2803					
6	9.0309						

a) Here $x_0 = 0, h = 1$

If x = 1, $u = \frac{x - x_0}{h} = \frac{1-0}{1} = 1$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 + ...\right]$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}\Delta^2 y_0 - \frac{1}{6}\Delta^3 y_0 + \frac{1}{12}\Delta^4 y_0 + ...\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{1}\left[0.4139 + \frac{1}{2}(-0.036) - \frac{1}{6}(0.0057) + \frac{1}{12}(-0.0011)\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1} = 0.3948$$

b) Since x = 3, we will use striling formula,

Hence h = 1, x_0 = 3.

By stirling's formula,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_{-1} + \Delta y_0}{2} - \frac{1}{12}\left(\Delta^3 y_{-2} + \Delta^3 y_{-1}\right) + \frac{1}{60}\left(\Delta^5 y_{-2} + \Delta^5 y_{-3}\right) + ...\right]$$

$$\left[\frac{dy}{dx}\right]_{x=3} = \frac{1}{1}\left[\frac{0.3219 + 0.3476}{2} - \frac{1}{12}(0.0034 + 0.0046) + \frac{1}{60}(0.0008 - 0.0001)\right]$$

$$= 0.33475 - 0.00066 - 0.000012$$

$$= 0.334075$$

$$\left[\frac{dy}{dx}\right]_{x=3} = 0.3341$$

and $$\left[\frac{d^2y}{dx^2}\right]_{x=3} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + ...\right]$$

$$= \frac{1}{1^2}\left[-0.0257 - \frac{1}{12}(0.0012) + \frac{1}{90}(-0.0007)\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=3} = -0.0256.$$

c) Since x = 6, we will use backward difference formula.

Here h = 1, x_n = 6,

By Newton's backward difference formula,

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + ...\right]$$

$$\left[\frac{dy}{dx}\right]_{x=0} = \frac{1}{1}\left[0.2803 + \frac{1}{2}(-0.0193) + \frac{1}{3}(0.003) + \frac{1}{4}(-0.0004) + \frac{1}{5}(-0.0008) + \frac{1}{6}(-0.0007)\right]$$

$$\left[\frac{dy}{dx}\right]_{x=0} = 0.2713$$

3.2.1.1 Errors in Numerical Differentiation

Q26. Write the types of errors in Numerical Differentiation.

Sol :

The numerical computation of derivatives involves two types of errors. There are namely, truncation error and rounding error.

Truncation Error

The truncation error is caused by replacing the tabulated function by means of an interpolating polynomial.

However, the truncation error in any numerical differentiation formula can be easily estimated in the following way.

Suppose,that the tabulated function is such that its differences of a certain order are small and that the tabulated function is well approximated by the polynomial.

In stirling formula, we can write

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{\Delta y_{-1} + \Delta y_0}{2h} + T_1 \text{ where } T_1 \text{ is Truncation error.}$$

T_1 is given by,

$$T_1 = \frac{1}{6h}\left|\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right|$$

lly

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\Delta^2 y_{-1} + T_2$$

where $T_2 = \frac{1}{12h^2}|\Delta^4 y_{-2}|$

Rounding Error

The rounding error is inversely proportional to h in case of first derivative, inversely proportional to h^2 in the case of second derivatives and so on.

Thus, the rounding error increases as h decreases.

In stirling formula,

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{\Delta y_{-1} + \Delta y_0}{2h} - \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{12h}$$

$$= \frac{y_{-2} - 8y_{-1} + 8y_1 - y_2}{12h}$$

has the maximum rounding error

$$\frac{18E}{12h} = \frac{3E}{2h}$$

Finally, the formula,

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{\Delta^2 y_{-1}}{h^2} + ... = \frac{y_{-1} - 2y_0 + y_1}{h^2} + ...$$

has the maximum rounding error $\frac{4E}{h^2}$.

Q27. Assuming that the function values given in the table are correct to the accuracy given, estimate the errors in the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 1.6.

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Sol :

The difference table is

Refer question No (4),

Since the values are correct to 4D, it follows that $E < 0.00005 = 0.5 \times 10^{-4}$.

Value of $\frac{dy}{dx}$ at x = 1.6

$$\text{Truncation error} = \frac{1}{6h}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_0}{2}\right]$$

$$= \frac{1}{6(0.2)}\left[\frac{0.0361 + 0.0441}{2}\right]$$

$$= 0.03342$$

and

$$\text{Rounding error} = \frac{3E}{2h}$$

$$= \frac{3(0.5)10^{-4}}{0.4}$$

$$= 0.00038$$

Hence,

Total error = 0.03342 + 0.00038 = 0.0338

Using stirling's formula, with the first differences,

We obtain

$$\left[\frac{dy}{dx}\right]_{x=1.6} = \frac{\Delta y_{-1} + \Delta y}{2h} = \frac{0.8978 + 1.0966}{0.4} = 4.9860$$

The exact value is 4.9530

So that the error in the above solution is 0.0330 (4.9860 – 4.9530)

Value of $\frac{d^2y}{dx^2}$ at x = 1.6

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.6} = \frac{\Delta^2 y_{-1}}{h^2} = \frac{0.1988}{0.04} = 4.9700$$

So that the error = 4.9700 – 4.9530 = 0.0170

Also,

$$\text{Truncation error} = \frac{1}{12h^2}|\Delta^4 y_{-2}| = \frac{1}{12(0.04)}(0.0080) = 0.01667$$

and

$$\text{Rounding error} = \frac{4E}{h^2} = \frac{4(0.5)\times 10^{-4}}{0.04} = 0.0050$$

Hence,

$$\text{Total error in } \left[\frac{d^2y}{dx^2}\right]_{x=1.6} = 0.0167 + 0.0050 = 0.0217.$$

3.2.1.2 Cubic Spline Method

Q28. Solve y(x) = sinx in [0, π] by cubic spline method.

Sol :

We consider the function

y(x) = sin x in [0, π]

Here $M_0 = M_N = 0$

let N = 2,

i.e, $h = \frac{\pi}{2}$

Then $y_0 = y_2 = 0, y_1 = 1$ and $M_0 = M_2 = 0$

We have

$$M_0 + 4M_1 + M_2 = \frac{6}{h^2}(y_0 - 2y_1 + y_2)$$

or $$M_1 = \frac{-12}{\pi^2}$$

Thus, in $0 \leq x \leq \frac{\pi}{2}$

We obtain

$$S(x) = \frac{2}{\pi}\left[\frac{-2x^3}{\pi^2} + \frac{3x}{2}\right]$$

Which gives

$$S'(x) = \frac{2}{\pi}\left[\frac{-2}{\pi^2}(3x^2) + \frac{3}{2}\right] \quad \text{... (1)}$$

Hence, $$S'\left(\frac{\pi}{4}\right) = \frac{2}{\pi}\left[-\frac{6}{\pi^2}\left(\frac{\pi^2}{16}\right) + \frac{3}{2}\right] = \frac{9}{4\pi} = 0.71619725.$$

Exact value of $S'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{12} = 0.70710681.$

The percentage error in the completed value of $S'\left(\frac{\pi}{4}\right) = 1.28\%.$

From (i), $$S''(x) = -\frac{24}{\pi^3}x$$

and hence, $$S''\left(\frac{\pi}{4}\right) = -\frac{24}{\pi^3} \cdot \frac{\pi}{4} = -\frac{6}{\pi^2} = -0.60792710.$$

Since the exact value is $\frac{-1}{\sqrt{2}}$, the percentage error in the result is 14.03%.

3.2.1.3 Differentiation Formulae with Function Values

Q29. Write the differentiation formulae with function values.

Sol :

The numerical differentiation formulae in terms of function values. The differentiation formulae for use in numerical computations.

i) Forward Differences :

$$y'(x_i) = \frac{y_{i+1} - y_i}{h} \;;\; y'(x_i) = \frac{-y_{i+2} + 4y_{i+1} - 3y_i}{2h} + 0(h^2)$$

$$y''(x_i) = \frac{y_i - 2y_{i+1} + y_{i+2}}{h^2} \;;\; y''(x_i) = \frac{-y_{i+3} + 4y_{i+2} - 5y_{i+1} + 2y_i}{h^2}$$

ii) Backward Differences :

$$y'(x_i) = \frac{y_i - y_{i-1}}{h} \;;\; y'(x_i) = \frac{3y_i - 4y_{i-1} + y_{i-2}}{2h}$$

$$y''(x_i) = \frac{y_i - 2y_{i-1} + y_{i-2}}{h^2} \;;\; y''(x_i) = \frac{2y_i - 5y_{i-1} + 4y_{i-2} - y_{i-3}}{h^2}$$

iii) Central Differences :

$$y'(x_i) = \frac{y_{i+1} - y_{i-1}}{2h} \;;\; y'(x_i) = \frac{-y_{i+2} + 8y_{i+1} - 8y_{i-1} + y_{i-2}}{12h}$$

$$y''(x_i) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \;;\; y''(x_i) = \frac{-y_{i+2} + 16y_{i+1} - 30y_i + 16y_{i-1} - y_{i-2}}{12h^2}$$

These formulae can be derived by using Taylor series expansion of the functions.

3.2.2 Numerical Integration

3.2.2.1 Trapezodial Rule

Q30. Write the steps involved in Trapezodial Rule.

Sol :

1. Let the given integral be denoted by $I = \int_{x_0}^{x_n} y\,dx$.
2. Determine the integral width h by using the expression, $h = \frac{b-a}{n}$.
3. Then $\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n]$.

3.2.2.2 Simpson's 1/3rd - Rule

Q31. Write the steps involved in Simpson's $\frac{1}{3}$ - Rule.

Sol :

1. Let the given integral be denoted by $I = \int_{x_0}^{x_n} y\,dx$.
2. Determine the integral width h by using the expression, $h = \frac{b-a}{n}$.
3. Then $\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + y_6 + ... + y_{n-2}) + y_n]$.

3.2.2.3 Simpson's 3/8th - Rule

Q32. Write the steps involved in Simpson's $\frac{3}{8}$ - Rule.

Sol :

1. Let the given integral be denoted by $I = \int_{x_0}^{x_n} y\,dx$.

2. Determine the integral width h by using the expression, $h = \frac{b-a}{n}$.

3. Then $\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + ... + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$.

3.2.2.4 Boole's and Weddle's Rule

Q33. Write the formulae for Boole's and Weddle's rule.

Sol :

If we wish to retain differences upto those of the fourth order, we should integrate between x_0 and x_4 and obtain Boole's formula

$$\int_{x_0}^{x_4} y\,dx = \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

If, on the other hand, we integrate between x_0 and x_6 retaining differences upto those of the sixth order,

We obtain Weddles's rule,

$$\int_{x_0}^{x_6} y\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Q34. Evaluate $I = \int_0^1 \frac{1}{1+x}dx$. correct to three decimal places.

Sol :

We solve the given integral is both the trapezodial and simpson's rule with h = 0.5, 0.25 and 0.125 respectively.

i) Let h = 0.5

x	0	0.5	1
y(x)	1.0000	0.6667	0.5000

$$y_0 = y(x) = \frac{1}{1+x} = \frac{1}{1+0} = \frac{1}{1} = 1$$

$$y_1 = y(0.5) = \frac{1}{1+0.5} = \frac{1}{1.5} = 0.6667$$

$$y_2 = y(1) = \frac{1}{1+1} = \frac{1}{1+1} = 0.5000$$

a) Trapezodial rule

$$I = \frac{h}{2}[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n]$$

$$= \frac{0.5}{2}[1 + 2(0.6667) + 0.5] = 0.70835$$

b) Simpson's rule

$$I = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2}) + y_n]$$

$$= \frac{0.5}{3}[1 + 4(0.6667) + 0.5] = 0.6945$$

ii) Let h = 0.25

x	0	0.25	0.5	0.75	1
y	1	0.8000	0.6667	0.5714	0.5000

$$y_0 = y(0) = \frac{1}{1+x} = \frac{1}{1+0} = \frac{1}{1} = 1$$

$$y_1 = y(0.25) = \frac{1}{1+0.25} = \frac{1}{1.25} = 0.8000$$

$$y_2 = y(0.5) = \frac{1}{1+0.5} = \frac{1}{1.5} = 0.6667$$

$$y_3 = y(0.75) = \frac{1}{1+0.75} = \frac{1}{1.75} = 0.5714$$

$$y_n = y(1) = \frac{1}{1+1} = \frac{1}{2} = 0.5000$$

a) By Trapezodial rule,

$$I = \frac{h}{2}[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n]$$

$$= \frac{0.25}{2}[1 + 2(0.8 + 0.6667 + 0.5714) + 0.5]$$

$$= 0.6970$$

b) By Simpson's rule

$$I = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2}) + y_n]$$

$$= \frac{0.25}{3}[1 + 4(0.8 + 0.5714) + 2(0.6667) + 0.5]$$

$$I = 0.6932$$

iii) let h = 0.125

x	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1.0
y	1	0.8889	0.8000	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

$$y_0 = y(0) = \frac{1}{1+0} = \frac{1}{1} = 1$$

$$y_1 = y(0.125) = \frac{1}{1+0.125} = \frac{1}{1.125} = 0.8889$$

$$y_2 = y(0.250) = \frac{1}{1+0.250} = \frac{1}{1.250} = 0.8000$$

$$y_3 = y(0.375) = \frac{1}{1+0.375} = \frac{1}{1.375} = 0.7273$$

$$y_4 = y(0.5) = \frac{1}{1+0.5} = \frac{1}{1.5} = 0.6667$$

$$y_5 = y(0.625) = \frac{1}{1+0.625} = \frac{1}{1.625} = 0.6154$$

$$y_6 = y(0.750) = \frac{1}{1+0.750} = \frac{1}{1.750} = 0.5714$$

$$y_7 = y(0.875) = \frac{1}{1+0.875} = \frac{1}{1.875} = 0.5333$$

$$y_8 = y(1) = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

a) Trapezodial rule,

$$I = \frac{h}{2}[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n]$$

$$= \frac{0.125}{2}[1 + 2(0.8889 + 0.8 + 0.7273 + 0.6667 + 0.6154 + 0.5714 + 0.5333 + 0.5]$$

$$I = 0.6941$$

b) Simpson's rule,

$$I = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2}) + y_n]$$

$$= \frac{0.125}{3}[1 + 4(0.8889 + 0.7273 + 0.6154 + 0.5333 + 2(0.8 + 0.6667 + 0.5714) + 0.5]$$

$$I = 0.6932$$

Q35. Using Simpson's $\frac{1}{3}$ - rule with h = i, evaluate the integral I = $\int_3^7 x^2 \log x\, dx$.

Sol :

$$I = \int_3^7 x^2 \log x\, dx$$

$h = 1, x_0 = 3$ and $x_n = 7$.

$y_0 = y(x_0) = y(3) = 3^2 \log 3 = 9.8875$

$y_1 = y(x_1) = y(x_0 + h) = y(4) = 4^2 \log 4 = 22.1807$

$y_2 = y(x_2) = y(x_1 + h) = y(5) = 5^2 \log 5 = 40.2359$

$y_3 = y(x_3) = y(x_2 + h) = y(6) = 6^2 \log 6 = 64.5033$

$y_4 = y(x_4) = y(x_3 + h) = y(7) = 7^2 \log 7 = 95.3496$

By Simpson's rule,

$$I = \frac{1}{3}[9.8875 + 4(22.1807 + 64.5033) + 2(40.2359) + 95.3496]$$

$$= \frac{1}{3}[9.8875 + 346.7360 + 80.4718 + 95.3496]$$

$I = 177.4816$

Q36. Evaluate $\int_0^2 \frac{dx}{x^3 + x + 1}$ by simpson's $\frac{1}{3}$ rule with h = 0.25.

Sol :

$$I = \int_0^2 \frac{dx}{x^3 + x + 1}$$

$h = 0.25, x_0 = 0, x_n = 2$

$y_0 = y(x_0) = y(0) = \frac{1}{0 + 0 + 1} = 1$

$y_1 = y(x_1) = y(x_0 + h) = y(0.25) = \frac{1}{(0.25)^3 + 0.25 + 1} = 0.7901$

$y_2 = y(x_2) = y(x_1 + h) = y(0.5) = \frac{1}{(0.5)^3 + 0.5 + 1} = 0.6154$

$y_3 = y(x_3) = y(x_2 + h) = y(0.75) = \frac{1}{0.75^3 + 0.75 + 1} = 0.4604$

$y_4 = y(x_4) = y(x_3 + h) = y(1) = \frac{1}{1^3+1+1} = 0.3333$

$y_5 = y(x_5) = y(x_4 + h) = y(1.25) = \frac{1}{1.25^3+1.25+1} = 0.2379$

$y_6 = y(x_6) = y(x_5 + h) = y(1.5) = \frac{1}{1.5^3+1.5+1} = 0.1702$

$y_7 = y(x_7) = y(x_6 + h) = y(1.75) = \frac{1}{1.75^3+1.75+1} = 0.1233$

$y_8 = y(x_8) = y(x_7 + h) = y(2) = \frac{1}{2^3+2+1} = 0.0909$

By Simpson's rule,

$I = \frac{0.25}{3}[1+ 4(0.7901+0.4604 + 0.2379 + 0.1233)+ 2(0.6154 + 0.3333 + 0.1702) + 0.0909]$

$= 0.0833\ [1 + 6.4468 + 2.2378 + 0.0909]$

$I = 0.8143$

Q37. Estimate the value of the integral $\int_1^3 \frac{1}{x}dx$ by Simpson's rule with 4 strips and 8 strips respectively.

Sol :

$I = \int_1^3 \frac{1}{x}dx$

i) Let n = 4

$h = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

x	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
y	1	0.6667	0.5	0.4	0.3333

$y(1) = \frac{1}{1} = 1$

$y\left(\frac{3}{2}\right) = \frac{1}{\frac{3}{2}} = \frac{2}{3} = 0.6667$

$$y(2) = \frac{1}{2} = 0.5$$

$$y\left(\frac{5}{2}\right) = \frac{1}{\frac{5}{2}}$$

$$= \frac{2}{5} = 0.4$$

$$y(3) = \frac{1}{3} = 0.3333$$

By Simpson's rule,

$$I = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2}) + y_n]$$

$$= \frac{\frac{1}{2}}{3}[1 + 4(0.6667 + 0.4) + 2(0.5) + 0.3333]$$

$$= \frac{1}{6}[1 + 4.2668 + 1 + 0.3333]$$

I = 1.1000

ii) let n = 8

$$h = \frac{3-1}{8} = \frac{2}{8} = \frac{1}{4}$$

1	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$	$\frac{9}{4}$	$\frac{10}{4}$	$\frac{11}{4}$	3
1	0.8	0.6667	0.5714	0.5	0.4444	0.4	0.3636	0.3333

By Simpson's rule,

$$I = \frac{\frac{1}{4}}{3}[1 + 4(0.8 + 0.5714 + 0.4444 + 0.3636) + 2(0.6667 + 0.5 + 0.4) + 0.3333]$$

$$I = \frac{1}{12}[1 + 8.7176 + 3.1334 + 0.3333]$$

I = 1.0987

Q38. Evaluate $I = \int_0^{\pi/2} \sqrt{\sin x}\,dx$ using simpson's $\frac{1}{3}$ - rule with $h = \frac{\pi}{12}$.

Sol :

$$I = \int_0^{\pi/2} \sqrt{\sin x}\,dx$$

$h = \frac{\pi}{12}$, $x_0 = 0$ and $x_n = \frac{\pi}{2}$

$y_0 = y(x_0) = y(0) = \sqrt{\sin 0} = 0$

$y_1 = y(x_1) = y(x_0 + h) = y\left(\frac{\pi}{12}\right) = 0.5087$

$y_2 = y(x_2) = y(x_1 + h) = y\left(\frac{\pi}{6}\right) = 0.7071$

$y_3 = y(x_3) = y(x_2 + h) = y\left(\frac{\pi}{4}\right) = 0.8409$

$y_4 = y(x_4) = y(x_3 + h) = y\left(\frac{\pi}{3}\right) = 0.9306$

$y_5 = y(x_5) = y(x_4 + h) = y\left(\frac{5\pi}{12}\right) = 0.9828$

$y_6 = y(x_6) = y(x_5 + h) = y\left(\frac{\pi}{2}\right) = 1.0000$

By simpson's rule,

$$I = \frac{\frac{\pi}{12}}{3}[0 + 4(0.5087 + 0.8409 + 0.9828) + 2(0.7071 + 0.9306) + 1]$$

$$= \frac{\pi}{36}[0 + 9.3296 + 3.2754 + 1]$$

$$= 1.1873$$

Q39. Evaluate a) $\int_0^{\pi} x\sin x\,dx$ and $\int_{-2}^{2} \frac{x}{5+2x}dx$

Using the trapezodial rule with five ordinates.

Sol :

a) $I = \int_0^{\pi} x\sin x\,dx$

$h = \frac{x_n - x}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$ $(\because n = 4)$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	0.5554	1.5708	1.6661	0

$y_0 = y(x_0) = y(0) = 0.\sin 0 = 0$

$y_1 = y(x_1) = y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\sin\left(\frac{\pi}{4}\right) = 0.5554$

$y_2 = y(x_2) = y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = 1.5708$

$y_3 = y(x_3) = y\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4}\sin\left(\frac{3\pi}{4}\right) = 1.6661$

$y_4 = y(x_4) = y(\theta) = \theta\sin\theta = 0$

By Trapezodial rule,

$$I = \frac{h}{2}[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n]$$

$$= \frac{\frac{\pi}{4}}{2}[0 + 2(0.5554 + 1.5708 + 1.6661] + 0]$$

$$I = 2.9785$$

b) $I = \int_{-2}^{2} \frac{x}{5+2x}dx$

$h = \frac{x_n - x_0}{n} = \frac{2-(-2)}{4} = \frac{4}{4} = 1$

x	– 2	– 1	0	1	2
y	– 2	– 0.3333	0	0.1429	0.2222

$$y_0 = y(x_0) = y(-2) = \frac{-2}{5+2(-2)} = \frac{-2}{1} = -2$$

$$y_1 = y(x_1) = y(-1) = \frac{-1}{5+2(-1)} = \frac{-1}{3} = -0.3333$$

$$y_2 = y(x_2) = y(0) = \frac{0}{5+2(0)} = 0$$

$$y_3 = y(x_3) = y(1) = \frac{1}{5+2(1)} = \frac{1}{7} = 0.1429$$

$$y_4 = y(x_4) = y(2) = \frac{2}{5+2(2)} = \frac{2}{9} = 0.2222$$

By Trapezodial rule,

$$I = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + ... + y_{n-1}) + y_n]$$

$$= \frac{1}{2}[-2 + 2(-0.3333 + 0 + 0.1429) + 0.2222]$$

$$= \frac{1}{2}[-1.7778 - 0.3808]$$

$I = -1.0793.$

Q40. By using Weddle's rule to obtain an approximate value of π from the formula $\frac{\pi}{4}$ = $\int_0^1 \frac{1}{1+x^2}dx$ with h = $\frac{1}{6}$.

Evaluate the above integral by using simpson's $\frac{1}{3}$ - rule with nine ordinates and compare the results.

Sol :

Given formula $\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}dx$

Let $I = \int_0^1 \frac{1}{1+x^2}$ with $h = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
y	1	0.9730	0.9	0.8	0.6923	0.5902	0.5

$$y_0 = y(x_0) = y(0) = \frac{1}{1+0^2} = 1$$

$$y_1 = y(x_1) = y\left(\frac{1}{6}\right) = \frac{1}{1+\left(\frac{1}{6}\right)^2} = 0.9730$$

$$y_2 = y(x_2) = y\left(\frac{1}{3}\right) = \frac{1}{1+\left(\frac{1}{3}\right)^2} = 0.9000$$

$$y_3 = y(x_3) = y\left(\frac{1}{2}\right) = \frac{1}{1+\left(\frac{1}{2}\right)^2} = 0.8000$$

$$y_4 = y(x_4) = y\left(\frac{2}{3}\right) = \frac{1}{1+\left(\frac{2}{3}\right)^2} = 0.6923$$

$$y_5 = y(x_5) = y\left(\frac{5}{6}\right) = \frac{1}{1+\left(\frac{5}{6}\right)^2} = 0.5902$$

$$y_6 = y(x_6) = y(1) = \frac{1}{1+1^2} = 0.5000$$

By Weddle's rule,

$$I = \int_{x_0}^{x_6} y\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

$$I = \frac{\not{3}\left(\frac{1}{\not{6}_2}\right)}{10}(1 + 5(0.9730) + 0.9 + 6(0.8) + 0.6923 + 5(0.5902) + 0.5)$$

$$= \frac{1}{20}(1 + 4.8650 + 0.9 + 4.8 + 0.6923 + 2.9510 + 0.5)$$

$$I = 0.7854$$

But $\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\,dx$

$\frac{\pi}{4} = 0.7854$

$\pi = 3.1417$

By simpson's rule with n = 9

$h = \frac{1-0}{9} = \frac{1}{9}$

x	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$	1
y	1	0.9878	0.9529	0.9	0.8351	0.7642	0.6923	0.6231	0.5586	0.5

$y_0 = y(x_0) = y(0) = \frac{1}{1+0} = 1$

$y_1 = y(x_1) = y\left(\frac{1}{9}\right) = \frac{1}{1+\left(\frac{1}{9}\right)^2} = 0.9878$

$y_2 = y(x_2) = y\left(\frac{2}{9}\right) = \frac{1}{1+\left(\frac{2}{9}\right)^2} = 0.9529$

$y_3 = y(x_3) = y\left(\frac{1}{3}\right) = \frac{1}{1+\left(\frac{1}{3}\right)^2} = 0.9000$

$y_4 = y(x_4) = y\left(\frac{4}{9}\right) = \frac{1}{1+\left(\frac{4}{9}\right)^2} = 0.8351$

$y_5 = y(x_5) = y\left(\frac{5}{9}\right) = \frac{1}{1+\left(\frac{5}{9}\right)^2} = 0.7642$

$$y_6 = y(x_6) = y\left(\frac{2}{3}\right) = \frac{1}{1+\left(\frac{2}{3}\right)^2} = 0.6923$$

$$y_7 = y(x_7) = y\left(\frac{7}{9}\right) = \frac{1}{1+\left(\frac{7}{9}\right)^2} = 0.6231$$

$$y_8 = y(x_8) = y\left(\frac{8}{9}\right) = \frac{1}{1+\left(\frac{8}{9}\right)^2} = 0.5586$$

$$y_9 = y(x_9) = y(1) = \frac{1}{1+1^2} = 0.5000$$

By simpson's rule,

$$I = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2}) + y_n]$$

$$= \frac{\frac{1}{9}}{3}[1 + 4(0.9878 + 0.9 + 0.7642 + 0.6231) + 2(0.9529 + 0.8351 + 0.6923 + 0.5586) + 0.5)$$

$$= \frac{1}{27}[1 + 13.1004 + 6.0778 + 0.5]$$

$$I = 0.7659$$

Now, $\frac{\pi}{4} = 0.7659$

$\pi = 3.0634$

$\therefore$ By Weddle's rule the value of $\pi = 3.1417$

By Simpson's rule with the value of $\pi = 3.0634$.

Q41. Evaluate $\int_0^1 \frac{1}{1+x}dx$ with $h = \frac{1}{6}$ by simpson's rule $\frac{1}{3}$ - rule.

Sol :

$$I = \int_0^1 \frac{1}{1+x}dx$$

$$h = \frac{1}{6}$$

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
y	1	0.8571	0.75	0.6667	0.6	0.5455	0.5

$$y_0 = y(x_0) = y(0) = \frac{1}{1+0} = 1$$

$$y_1 = y(x_1) = y\left(\frac{1}{6}\right) = \frac{1}{1+\frac{1}{6}} = 0.8571$$

$$y_2 = y(x_2) = y\left(\frac{1}{3}\right) = \frac{1}{1+\frac{1}{3}} = 0.7500$$

$$y_3 = y(x_3) = y\left(\frac{1}{2}\right) = \frac{1}{1+\frac{1}{2}} = 0.6667$$

$$y_4 = y(x_4) = y\left(\frac{2}{3}\right) = \frac{1}{1+\frac{2}{3}} = 0.6000$$

$$y_5 = y(x_5) = y\left(\frac{5}{6}\right) = \frac{1}{1+\frac{5}{6}} = 0.5455$$

$$y_6 = y(x_6) = y(1) = \frac{1}{1+1} = 0.5000$$

By simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + 4(y_1 + y_3 + ... + y_{n-2}) + 2(y_2 + y_n + ... + y_{n-2}) + y_n]$$

$$= \frac{\frac{1}{6}}{3}[1 + 4(0.8571 + 0.6667 + 0.5455) + 2(0.75 + 0.6) + 0.5]$$

$$= \frac{1}{18}[1 + 8.2772 + 2.7 + 0.5]$$

$$I = 0.6932$$

3.2.3 Newton - Cotes Integration Formulae

Q42. Derive newton - cotes Integration Formulae.

Sol :

Let the integration points, x_i, be equally spaced.

i.e., $x_i = x_0 + ih, i = 0, 1, 2, ... n.$

Let the end points of the interval of integration be placed such that $x_0 = a, x_n = b, h = \frac{b-a}{n}$.

Then the definite integral

$$I = \int_a^b y\,dx \quad \text{... (1)}$$

is evaluated by an integration formula of the type

$$I_n = \sum_{i=0}^{n} c_i y_i \quad \text{... (2)}$$

where the co-eff c_i are determined completely by the abscissae x_i, Integration formulae of the type (2) are called Newton - cotes closed Integration Formulae.

Q43. Write the five simplest Newton - cotes open Integration Formulae.

Sol :

The Newton - cotes open Integration Formulae.

a) $\int_{x_0}^{x_2} y\,dx = 2hy_1 + \frac{h^3}{3} y''(\bar{x}), (x_0 < \bar{x} < x_2)$

b) $\int_{x_0}^{x_3} y\,dx = \frac{3h}{2}(y_1 + y_2) + \frac{3h^3}{4} y''(\bar{x}), (x_0 < \bar{x} < x_3)$

c) $\int_{x_0}^{x_4} y\,dx = \frac{4h}{3}(2y_1 - y_2 + 2y_3) + \frac{14}{45} h^5 y^{iv}(\bar{x}), (x_0 < \bar{x} < x_4)$

d) $\int_{x_0}^{x_5} y\,dx = \frac{5h}{24}(11y_1 + y_2 + y_3 + 11y_n) + \frac{95}{144} h^5 y^{iv}(\bar{x}), (x_0 < \bar{x} < x_5)$

e) $\int_{x_0}^{x_6} y\,dx = \frac{6h}{20}(11y_1 - 14y_2 + 26y_3 - 14y_4 + 11y_5) + \frac{41}{140} h^7 y^{vi}(\bar{x}), (x_0 < \bar{x} < x_6)$.

Q43. Write the errors of trapezodial rule, simpson's $\frac{1}{3}$ rule and simpson's $\frac{3}{8}$ rule.

Sol :

The error of trapezodial rule is,

$$E = \frac{-1}{12} h^3 n y''(\bar{x}) = -\frac{(b-a)}{12} h^2 y''(\bar{x})$$

Since $nh = b - a$

The error of simpson's $\frac{1}{3}$ - rule is,

$$\int_a^b y\,dx = -\frac{b-a}{180} h^4 y^{iv}(\bar{x})$$

Where $y^{iv}(\bar{x})$ is the largest value of the fourth derivatives.

The error of simpson's $\frac{3}{8}$ - rule is,

$$E = -\left(\frac{3}{80}\right) h^5 y^{iv}(\bar{x}).$$

Choose the Correct Answer

1. In the least square method we use ________ to find the value of unknowns. [b]

(a) Regression $eq^n s$ (b) Normal $eq^n s$

(c) General $eq^n s$ (d) Auxiliary $eq^n s$

2. Fit the curve $y = ae^{bx}$ if their normal equations are 13.1991 = 4a + 10b and 30.7134 = 10a + 30b [c]

(a) $y = -0.4569e^{4.4419x}$ (b) $y = -404419e^{0.4569}$

(c) $y = 4.4419e^{-0.4569x}$ (d) y = 1

3. Fit a straight line for the given pairs of (x,y) which are (0, 3), (1, 6), (2, 8),(3, 11), (4, 13), (5, 14) [b]

(a) y = 2.26x (b) y = 3.52 + 2.26x

(c) y = 3.52x (d) y = 4 + 3x

4. If the normal equations for a straight line y = ax + b are 26 = 4a + 6b and 34 = 6a + 4b then fit the above straight line [b]

(a) y = 5x – b (b) y = 5x + b

(c) y = x + 56 (d) y = x – 5b

5. Considering four subintervals, the value of $\int_0^1 \frac{1}{1+x} dx$ by trapezoideal rule is ________ [a]

(a) 0.6950 (b) 0.6870

(c) 0.6677 (d) 0.3597

6. For the data, [b]

$$\begin{array}{lccc} x: & 0 & 1 & 2 \\ f(x): & 8 & 5 & 6 \end{array}$$

the value of $\int_0^2 [f(x)]^2 \, dx$ by Trapezoidial rule will be

(a) 92 (b) 75

(c) 123 (d) 42

7. By simpson's $\frac{1}{3}$ rule, the value of $\int_1^7 \frac{1}{x} dx$ is [a]

(a) 1.958 (b) 1.458

(c) 1.658 (d) 1.358

8. The evaluation of the definite integrated $\int_{-1}^{1.4} x|x|\,dx$ by using simpson's $\frac{1^{rd}}{3}$ rule with step size h = 0.6 yields [d]

(a) 0.914 (b) 1.248

(c) 0.581 (d) 0.592

9. Using simpson's $\frac{3^{th}}{8}$ rule, the integral $\int_{x_0}^{x_3} y\,dx$ = ________ [a]

(a) $\frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$ (b) $\frac{h}{2}(y_0 + 4y_1 + 2y_2 + y_3)$

(c) $\frac{3h}{8}(y_0 + 3y_1 + 2y_2 + y_3)$ (d) $\frac{h}{2}(y_0 + 3y_1 + 3y_2 + y_3)$

10. Boole's formula is $\int_{x0}^{x4} ydx$ = ________ [a]

(a) $\frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$ (b) $\frac{2h}{45}(7y_0 - 32y_1 - 12y_2 - 32y_3 - 7y_4)$

(c) $\frac{2h}{45}(8y_0 - 42y_1 - 12y_2 - 32y_3 - 7y_4)$ (d) $\frac{2h}{45}(9y_0 - 18y_1 - 24y_2 - 32y_3 - 7y_4)$

Fill in the blanks

1. Fit a straight line for the given pairs of (x,y) which are (0, 3), (1, 6), (2, 8), (3, 11), (4, 13), (5, 14) is ________.
2. Trapezoidial rule gives exact value of the integral when the integrand is a ________.
3. Trapezoidial rule for the evaluation of $\int_a^b f(x)\,dx$ required the interval [a, b] to be divided into ________.
4. Simpson's $\frac{1}{3}$ rule is used to integrate the function f(x) = $\frac{3}{5}x^2 + \frac{9}{5}$ between x = 0 and x = 1 using the least number of equal sub - intervals. The value of the integral is ________.
5. Value of $\int_4^{5.2} l\,nx\,dx$ using Simpson's $\frac{1}{3}$ rule with interval size 0.3 is ________.
6. Simpson's $\frac{1}{3}$ rd rule and direct integration have given the same result if ________.
7. The formula for simpson's $\frac{3}{8}$ rule is ________.
8. The term Numerical Integration first appears in ________.
9. The weddle's rule is ________.
10. The error for simpson's $\frac{3}{8}$ rule is ________.

ANSWER

1. y = 3.52 + 2.26x
2. Linar function
3. Any number of subintervals of equal width
4. 2
5. 1.83
6. The entire curve is itself a parabola
7. $\int_{x_0}^{x_n} y^n\,dx = \frac{3h}{8}\,(y_o + y_n) + 3(y_1 + y_2 + y_4 + y_5 + ... + y_{n-1}) + 2\,(y_3 + y_6 + ... + y_{n-2})]$
8. 1915
9. $\int_{x_0}^{x_6} ydx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$
10. $\left(\frac{-3}{80}\right)h^5\,y^2\,(\bar{x})$

UNIT IV	**Numerical Solutions of Ordinary Differential Equations:** Taylor's Series Method - Picard's Method - Euler's Methods - Runge Kutta Methods.

4.1 Taylor's Series Method

Q1. Write the procedure of solution By Taylor's series.

Ans :

We consider the differential equation

$$y' = f(x,y) \quad \text{... (1)}$$

with the initial condition

$$y(x_0) = y_0 \quad \text{... (2)}$$

If y(x) is the exact solution of equation (1) then the Taylor's series for y(x) around $x = x_0$ is given by,

$$y(x) = y_0 + (x - x_0)y'_0 + \frac{(x-x_0)^2}{2!} y''_0 + \frac{(x-x_0)^3}{3!} y'''_0 + ... \quad \text{... (3)}$$

If the values of y'_0, y''_0 .... are known, then equation (3) gives a power series for y. Using the formula for total derivatives.

We can write

$$y'' = f' = f_x + y' f_y = f_x + ff_y,$$

Where the suffixes denote the partial derivatives with respect to the variable concerned.

Similarly, we obtain

$$y''' = f'' = f_{xx} + f_{xy}f + f(f_{yx} + f_{yy}f) + f_y(f_x + f_y f)$$

$$= f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2$$

and other higher derivatives of y. The method can easily be extended to simultaneous and higher - order differential equations.

Q2. From the Taylor series for y(x). Find y(0.1) correct to four decimal places if y(x) satisfies $y' = x - y^2$ and $y(0) = 1$.

Sol :

The Taylor series for y(x) is given by

$$y(x) = 1 + x y'_0 + \frac{x^2}{2} y''_0 + \frac{x^3}{6} y'''_0 + \frac{x^4}{24} y_0^{iv} + \frac{x^5}{120} y_0^{v} + ... \quad \text{... (1)}$$

The derivatives y_0', y_0'' .... etc are obtained thus :

$y'(x) = x - y^2 \Rightarrow y_0' = -1$

$y''(x) = 1 - 2yy' \Rightarrow y_0'' = 3$

$y'''(x) = -2yy'' - 2y'^2 \Rightarrow -2(1)(3) - 2(-1)^2$

$= -6 - 2 = -8$

$y^{iv}(x) = -2yy''' - 6y'y'' \Rightarrow -2(1)(-8) - 6(-1)(3)$

$= 16 + 18$

$= 34$

$y^{v}(x) = -2yy^{iv} - 8y'y''' - 6y''^2 \Rightarrow -2(1)(34) - 8(-1)(-8) - 6(3)^2$

$= -68 - 64 - 54$

$= -186$

Sub these values in (1),

We get

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(3) + \frac{x^3}{6}(-8) + \frac{x^4}{24}(34) + \frac{x^5}{120}(-186) + ...$$

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 - \frac{31}{20}x^5 + ...$$

To obtain the value of y(0.1) correct to four decimal places it is found that the terms upto x^4 should be considered.

$$y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 - \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4$$

$= 1 - 0.1 + 0.015 - 0.001333 + 0.00014166$

$y(0.1) = 0.9138$

Q3. Given the differential equation $y'' - xy' - y = 0$ with the conditions y(0) and $y'(0) = 0$. Use Taylor's series method to determine the value of y(0.1)

Sol :

We have y(x) = 1 and $y'(x) = 0$ when x = 0

The given differential equation is,

$y'' - xy' - y = 0$

$\Rightarrow \quad y''(x) = xy'(x) + y(x)$ ... (1)

Hence $y''(0) = y(0) = 1$

The successive differentiations of (1) gives

$$y'''(x) = xy''(x) + y'(x) + y'(x) = xy''(x) + 2y'(x) \quad .... (2)$$

$$y^{iv}(x) = xy'''(x) + y''(x) + y''(x) + y''(x) = xy'''(x) + 3y''(x) \quad .... (3)$$

$$y^{v}(x) = xy^{iv}(x) + y'''(x) + 3y'''(x) = xy^{iv}(x) + 4y'''(x) \quad .... (4)$$

$$y^{vi}(x) = xy^{v}(x) + y^{iv}(x) + 4y^{iv}(x) = xy^{v}(x) + 5y^{iv}(x) \quad .... (5)$$

and similarly for higher order derivatives.

Putting x = 0 in (2) to (5), we obtain

$$y'''(0) = 2y'(0) = 2(0) = 0$$

$$y^{iv}(0) = y'''(0) + 3y''(0) = 0 + 3(1) = 3$$

$$y^{v}(0) = (0)(3) + 4(0) = 0 + 0 = 0$$

$$y^{vi}(0) = (0)(0) + 5(3) = 0 + 15 = 15$$

Now By Taylor's series is,

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{iv} + \frac{(x-x_0)}{5!}y_0^{v} + ...$$

$$y(x) = y_0 + (x)y_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{iv} + \frac{x^5}{5!}y_0^{v} + \frac{x^6}{6!}y^{vi} + ...$$

$$y(0) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) + \frac{x^4}{24}y^{iv}(0) + \frac{x^5}{120}y^{v}(0) + \frac{x^6}{720}y^{vi}(0) + ...$$

Hence,

$$y(0.1) = 1 + (0.1)(0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(0) + \frac{(0.1)^4}{24}(3) + \frac{(0.1)^5}{120}(0) + \frac{(0.1)^6}{720}(15) + ...$$

$$= 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{8} + \frac{(0.1)^6}{8} + ...$$

= 1 + 0.005 + 0.0000125, neglecting the last term

= 1.0050125, correct to seven decimal places.

Q4. Given $\frac{dy}{dx}$ = 1 + xy, y(0) = 1 obtain the taylor series for y(x) and compute y(0.1), correct to four decimal places.

Sol :

Given differential equation $y' = 1 + xy$ and $y(0) = 1$

The Taylor series for y(x) is given by,

$$y(x) = 1 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{iv} + \frac{x^5}{5!}y^{v}{}_0 + ...$$

$$y(x) = 1 + xy_0' + \frac{x^2}{2}y_0'' + \frac{x^3}{6}y_0''' + \frac{x^4}{24}y_0^{iv} + \frac{x^5}{120}y^{v}{}_0 + ... \quad .... (1)$$

The derivatives y_0', y_0'', y_0''', ... etc are obtained thus :

$y'(x) = 1 + xy \Rightarrow y_1'(0) = 1 + (0)1 = 1$

$y''(x) = xy' + y \Rightarrow y''(0) = (0)(1) + 1 = 1$

$y'''(x) = xy'' + y' + y' = xy'' + 2y' \Rightarrow y'''(0) = (0)(1) + 2(1) = 2$

$y^{iv}(x) = xy''' + y'' + 2y'' = xy''' + 3y'' \Rightarrow y^{iv}(0) = 0(2) + 3(1) = 3$

$y^{v}(x) = xy''' + y''' + 3y''' = xy''' + 4y''' \Rightarrow y^{v}(0) = (0)(2) + 4(2) = 8$

Sub these values in (1)

We get,

$$y(x) = 1 + x(1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(3) + \frac{x^5}{120}(8) + ...$$

To obtain the value of y(0.1) correct to four decimal places it is found that the terms upto x^4 should be considered.

$$y(0.1) = 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8}$$

$$= 1 + 0.1 + 0.005 + 0.00033 + 0.0000125$$

$$y(0.1) = 1.1053$$

Q5. Show that the differential equation $\frac{d^2y}{dx^2}$ = – xy, y(0) = 1 and y'(0) = 0 has the series solution $y = 1 - \frac{x^3}{3!} + \frac{1\times 4}{6!}x^6 - \frac{1\times 4\times 7}{9!}x^9 + ...$

Sol :

Given differential equation $\frac{d^2y}{dx^2}$ = – xy, y(0) = 1 & y'(0) = 0.

We have $y'' = -xy$ .... (1)

Hence $y''(0) = y'(0) = 0$

The successive differentiations of equation (1) gives,

$$y'''(x) = -xy' + y(-1) = -xy' - y \Rightarrow y'''(0) = -(0)(0) - 1 = -1$$

$$y^{iv}(x) = -xy'' + y'(-1) - y' = -xy'' - 2y' \Rightarrow y^{iv}(0) = -(0)(0) - 2(0) = 0$$

$$y^{v}(x) = -xy''' + y''(-1) - 2y'' = -xy''' - 3y'' \Rightarrow y^{v}(0) = -(0)(-1) - 3(0) = 0$$

$$y^{vi}(x) = -xy^{iv} + y'''(-1) - 3y''' = -xy^{iv} - 4y''' \Rightarrow y^{vi}(0) = -(0)(0) - 4(-1) = 4$$

$$y^{vii}(x) = -xy^{v} + y^{iv}(-1) - 4y^{iv} = -xy^{v} - 5y^{iv} \Rightarrow y^{vii}(0) = -(0)(0) - 5(0) = 0$$

$$y^{viii}(x) = -xy^{vi} + y^{v}(-1) - 5y^{v} = -xy^{vi} - 6y^{v} \Rightarrow y^{viii}(0) = -(0)(4) - 6(0) = 0$$

$$y^{ix}(x) = -xy^{vii} + y^{vi}(-1) - 6y^{vi} = -xy^{vii} - 7y^{vi} \Rightarrow y^{ix}(0) = -(0)(0) - 7(4) = -28$$

Now by Taylor's series is,

$$y(x) = y_0 + (x - x_0)y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \frac{(x-x_0)^3}{3!}y'''_0 + \frac{(x-x_0)^4}{4!}y^{iv}_0 + \frac{(x-x_0)^5}{5!}y^{v}_0 + ...$$

$$y(x) = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + \frac{x^4}{4!}y^{iv}_0 + \frac{x^5}{5!}y^{v}_0 + \frac{x^6}{6!}y^{vi}_0 + \frac{x^7}{7!}y^{vii}_0 + \frac{x^8}{8!}y^{viii}_0 + \frac{x^9}{9!}y^{ix}_0 + ...$$

$$y(x) = 1 + x(0) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(4) + \frac{x^7}{7!}(0) + \frac{x^8}{8!}(0) + \frac{x^9}{9!}(-28) + ...$$

$$y(x) = 1 - \frac{x^3}{3!} + \frac{4}{6!}x^6 - \frac{28}{9!}x^9 + ...$$

$$y(x) = 1 - \frac{x^3}{3!} + \frac{1 \times 4}{6!}x^6 - \frac{1 \times 4 \times 7}{9!}x^9 + ...$$

Q6. If $\frac{dy}{dx} = \frac{1}{x^2 + y}$ with y(4) = 4. Compute the values of y(4.1) and y(4.2) by Taylor's series method.

Sol :

Given differential equation is,

$$\frac{dy}{dx} = \frac{1}{x^2 + y} \text{ with } y(4) = 4.$$

$y' = (x^2 + 1)^{-1}$ .... (1)

$$y'' = -(x^2+y)^{-2}(2x+y') = \frac{-(2x+y')}{(x^2+y)^2} \quad \text{... (2)}$$

$$y''' = -\left[\frac{(x^2+y)(2+y'') - (2x+y')2(x^2+y)(2x+y')}{(x^2+y)^4}\right]$$

$$y''' = -\left[\frac{(x^2+y)(2+y'') - 2(x^2+y)(2x+y')^2}{(x^2+y)^4}\right]$$

$$y''' = -\frac{1}{(x^2+y)^3}[(2+y'') - (2)(2x+y')^2] \quad \text{... (3)}$$

and so on

Sub x = 4 from (1) to (3)

$$y'_0(4) = (4^2 + 4)^{-1} = (20)^{-1} = 0.05$$

$$y''_0(4) = \frac{-(2(4)+0.05)}{(4^2+4)^2} = \frac{-(8+0.05)}{(20)^2} = \frac{-8.05}{400} = -0.020125$$

$$y'''_0(4) = \frac{-1}{(4^2+4)^3}[(2 - 0.020125) - 2(2(4) + 0.005)^2]$$

$$= \frac{-1}{(20)^3}[1.979875 - 2(8.05)^2]$$

$$= \frac{-1}{8000}[-127.625125] = 0.015953$$

Now by Taylor's series is,

$$y(x) = y_0 + (x-x_0)y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \frac{(x-x_0)^3}{3!}y'''_0 + ...$$

$$y(x) = 4 + (x-4)y'_0 + \frac{(x-4)^2}{2!}y''_0 + \frac{(x-4)^3}{3!}y'''_0 + ...$$

$$y(x) = 4 + (x-4)(0.05) + \frac{(x-4)^2}{2!}(-0.020125) + \frac{(x-4)^3}{3!}(0.015953) + ...$$

$$y(4.1) = 4 + (4.1-4)(0.05) + \frac{(4.1-4)^2}{2}(-0.020125) + \frac{(4.1-4)^3}{6}(0.015953) + ...$$

$$y(4.1) = 4 + (0.1)(0.05) + \frac{(0.1)^2}{2}(-0.020125) + \frac{(0.1)^3}{6}(0.015953) + ...$$

$$y(4.1) = 4 + 0.005 - 0.000100625 + 0.0000026588$$

$$y(4.1) = 4.0049$$

$$y(4.2) = 4 + (4.2-4)(0.05) + \frac{(4.2-4)^2}{2}(-0.020125) + \frac{(4.2-4)^3}{6}(0.015953)$$

$$= 4 + 0.01 - 0.0004025 + 0.0002127$$

$$y(4.2) = 4.0098$$

Q7. Using Taylor's series find y(0.1), y(0.2) and y(0.3) given $\frac{dy}{dx} = xy + y^2$, y(0) = 1.

Sol :

Given $\frac{dy}{dx} = xy + y2$ .... (1)

and $y(0) = 1$

The Successive differentiations of equation (1) gives,

since $y'(x) = xy + y^2$

$\Rightarrow \quad y'(0) = 0(1) + (1)^2 = 1$

$y''(x) = xy' + y(1) + 2y \Rightarrow xy' + 3y$

$\Rightarrow y''(0) = 0(1) + 3(1) = 3$

$y'''(x) = xy'' + y'(1) + 3y' \Rightarrow xy'' + 4y'$

$\Rightarrow y'''(0) = 0(3) + 4(1) = 4$

$y^{v}(x) = xy''' + y''(1) + 4y'' \Rightarrow xy''' + 5y''$

$\Rightarrow \quad y^{v}(0) = 0(4) + 5(3) = 15$

Now by Taylor's series is,

$$y(x) = y_0 + (x-x_0)y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \frac{(x-x_0)^3}{3!}y'''_0 + ...$$

$$y(x) = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + \frac{x^4}{4!}y^{iv} + ...$$

$$y(0.1) = 1 + (0.1)1 + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(4) + \frac{(0.1)^4}{4!}(15)$$

$$= 1 + 0.1 + 0.015 + 0.0007 + 0.00011$$

$$y(0.1) = 1.1158$$

$$y(0.2) = 1 + (0.2)1 + \frac{(0.2)^2}{2!}(3) + \frac{(0.2)^3}{3!}(4) + \frac{(0.2)^4}{4!}(15)$$

$$= 1 + 0.2 + 0.06 + 0.0053 + 0.0010$$

$$y(0.2) = 1.2663$$

$$y(0.3) = 1 + (0.3)1 + \frac{(0.3)^2}{2!}(3) + \frac{(0.3)^3}{3!}(4) + \frac{(0.3)^4}{4!}(15)$$

$$y(0.3) = 1 + 0.3 + 0.1350 + 0.018 + 0.0051 = 1.4581$$

4.2 Picard's Method

Q8. Write the procedure of Picard's Method.

Ans :

We consider the differential equation

$$y'(x) = f(x, y) \qquad \text{... (1)}$$

Integrating the above differential equation

we obtain,

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx \qquad \text{... (2)}$$

Equation (2), in which the unknown function y appears under the integral sign, is called an integral equation. Such an equations can be solved by the method of successive approximations in which the first approximation to y is obtained by putting y_0 for y on right side of equation (2) and we write

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

The integral on the right can now be solved and the resulting $y^{(1)}$ is substituted for y in the integral of equation (2) to obtain the second approximation $y^{(2)}$.

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx$$

Proceeding in this way, we obtain $y^{(3)}$, $y^{(4)}$, ... $y^{(n-1)}$ & $y^{(n)}$, where

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx \text{ with } y^{(0)} = y_0$$

Q9. Solve the equation y' = x + y², subject to the condition y = 1 when x = 0 by Picard's method.

Sol :

We start with $y^{(0)} = 1$ and obtain,

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$= 1 + \int_0^x (x + y_0^2)dx$$

$$= 1 + \int_0^x (x + 1)dx$$

$$= 1 + \left[\frac{x^2}{2} + x\right]_0^x$$

$$y^{(1)} = 1 + x + \frac{x^2}{2}$$

Then the second approximation is,

$$y^{(2)} = 1 + \int_0^x f(x, y^{(1)})dx$$

$$= 1 + \int_0^x (x + (y^{(1)})^2)dx$$

$$= 1 + \int_0^x \left[x + \left(1 + x + \frac{x^2}{2}\right)^2\right]dx$$

$$= 1 + \int_0^x \left(x + 1 + x^2 + \frac{x^4}{4} + 2x + \cancel{2}.\frac{x^2}{\cancel{2}} + \cancel{2}x.\frac{x^2}{\cancel{2}}\right)dx$$

$$= 1 + \int_0^x \left(1 + 3x + 2x^2 + x^3 + \frac{x^4}{4}\right)dx$$

$$= 1 + \left[x + 3.\frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{20}\right]_0^x$$

$$y^{(2)} = 1 + x + 3\frac{x^2}{2} + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5$$

It is obvious that the integrations might become more and more difficult as we proceed to higher approximations.

Q10. Given differential equation $\frac{dy}{dx} = \frac{x^2}{y^2+1}$ with the initial condition y = 0 when x = 0. Use Picard's method to obtain y for x = 0.25, 0.5 and 1.0 correct to three decimal places.

Sol :

We have $y^{(0)} = 0$

$$f(x, y) = \frac{x^2}{y^2+1}$$

$$f(x, y_0) = \frac{x^2}{y_0^2+1} = \frac{x^2}{0+1} = x^2.$$

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$y^{(1)} = 0 + \int_{x_0}^{x} x^2\,dx$$

$$y^{(1)} = \left[\frac{x^3}{3}\right]_0^x$$

$$y^{(1)} = \frac{1}{3}x^3$$

and

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

$$y^{(2)} = 0 + \int_0^x \frac{x^2}{{y^{(1)}}^2+1}dx = \int_0^x \frac{x^2}{\frac{x^6}{9}+1}dx$$

$$= \int_0^x \frac{x^2}{(1/9)x^6+1}dx = \tan^{-1}\left(\frac{1}{3}x^3\right)$$

$$= \frac{1}{3}x^3 - \frac{1}{81}x^9 + \ldots\ldots$$

So, $y^{(1)}$ and $y^{(2)}$ agree to the first term i.e, $\frac{1}{3}x^3$.

To find the range of values of x so that the series with the term $\left(\frac{1}{3}\right)x^3$ along will give the result correct to three placers.

We put $\frac{1}{81}x^9 \leq 0.0005$

$\Rightarrow \quad x \leq 0.7$

Hence,

$$y(0.25) = \frac{1}{3}(0.25)^3 = 0.005$$

$$y(0.5) = \frac{1}{3}(0.5)^3 = 0.042$$

$$y(1.0) = \frac{1}{3} - \frac{1}{81} = 0.321.$$

Q11. Use Picard's method to obtain y(0.1) and y(0.2) of the problem defined by $\frac{dy}{dx} = x + yx^4$, y(0) = 3.

Sol :

Since y(0) = 3

and $\frac{dy}{dx} = x + yx^4$

where $f(x, y) = x + yx^4$

We have

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$y^{(1)} = 3 + \int_{0}^{x}(x + y_0x^4)dx$$

$$= 3 + \int_{0}^{x}(x + 3x^4)dx$$

$$= 3 + \left[\frac{x^2}{2} + 3.\frac{x^5}{5}\right]_0^x$$

$$y^{(1)} = 3 + \frac{x^2}{2} + 3.\frac{x^5}{5}$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx$$

$$= 3 + \int_0^x (x + y^{(1)}x^4)dx$$

$$= 3 + \int_0^x \left(x + x^4 \left(3 + \frac{x^2}{2} + \frac{3x^5}{5} \right) \right) dx$$

$$= 3 + \int_0^x \left(x + 3x^4 + \frac{x^6}{2} + \frac{3x^9}{5} \right) dx$$

$$= 3 + \left[\frac{x^2}{2} + 3.\frac{x^5}{5} + \frac{1}{2}.\frac{x^7}{7} + \frac{3}{5}.\frac{x^{10}}{10} \right]_0^x$$

$$y^{(2)} = 3 + \frac{x^2}{2} + \frac{3x^5}{5} + \frac{x^7}{14} + \frac{3x^{10}}{50}$$

So that $y^{(1)} = y^{(2)}$ agree to the first three terms

i.e, $3 + \frac{x^2}{2} + \frac{3x^5}{5}$

To find the values of x.

Hence, $y(0.1) = 3 + \frac{(0.1)^2}{2} + \frac{(0.1)^2}{2}$

$y(0.1) = 3.005006$

$y(0.2) = 3 + \frac{(0.2)^2}{2} + \frac{3(0.2)^5}{5}$

$y(0.2) = 3.020192$

Q12. Use Picard's method to obtain y(0.1) of the problem defined by $\frac{dy}{dx} = 1 + xy$, y(0) = 1.

Sol :

Since $y(0) = 1$

and the differential equation $\frac{dy}{dx} = 1 + xy$

Where $f(x,y) = 1 + xy$

We have

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= y_0 + \int_0^x (1 + xy_0)\,dx$$

$$= 1 + \int_0^x (1 + x)\,dx$$

$$= 1 + \left[x + \frac{x^2}{2}\right]_0^x$$

$$y^{(1)} = 1 + x + \frac{x^2}{2}$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

$$= y_0 + \int_0^x (1 + xy^{(1)})\,dx = y_0 + \int_0^x 1 + x\left(1 + x + \frac{x^2}{2}\right)dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right)dx$$

$$= 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{1}{2}\left(\frac{x^4}{4}\right)\right]_0^x$$

$$y^{(2)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\,dx$$

$$= y_0 + \int_0^x (1 + xy^{(2)})\,dx$$

$$= 1 + \int_0^x \left(1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)\right)dx$$

$$= 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{1}{2}\left(\frac{x^4}{4}\right) + \frac{1}{3}\left(\frac{x^5}{5}\right) + \frac{1}{8}\left(\frac{x^6}{6}\right) \right]_0^x$$

$$y^{(3)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

$\therefore$ The series is,

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \ldots\ldots..$$

4.3 Euler's Method

Q13. Write the Euler's Method.

Sol :

We consider the differential equation

$$y' = f(x,y) \text{ with } f(x_0) = y_0 \qquad \ldots (1)$$

Suppose we wish to solve equation (1) for values y at

$$x = x_r = x_0 + rh \ (r = 1, 2, \ldots..)$$

Integrating equation (1), we obtain

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)dx \qquad \ldots (2)$$

Assuming that $f(x, y) = f(x_0, y_0)$ in $x_0 \le x \le x_1$

This gives Euler's formula,

$$y_1 \approx y_0 + hf(x_0, y_0) \qquad \ldots (3)$$

Similarly for the range $x_1 \le x \le x_2$,

we have $y_2 = y_1 + \int_{x_1}^{x_2} f(x, y)dx$

Substituting $f(x_1, y_1)$ for $f(x, y)$ in $x_1 \le x \le x_2$, we obtain

$$y_2 \approx y_1 + hf(x_1, y_1) \qquad \ldots (4)$$

Proceeding in this way, we obtain the general formula

$$y_{n+1} \approx y_n + hf(x_n, y_n), \quad n = 0, 1, 2 \ldots..$$

Q14. Using the Euler's method, solve the differential equation y' = – y with the condition y(0) = 1

Sol :

Given differential equation

$$y' = -y \Rightarrow \frac{dy}{dx} = -y$$

$\therefore$ f(x, y) = – y with y(0) = 1

Let h = 0.01

By Euler's method $y_{n+1} = y_n + hf(x_n, y_n)$, n = 0, 1, 2, 3, ...

$y_1 = y_0 + hf(x_0, y_0)$

$y_1 = y(x_1) = y(0.01) = 1 + 0.01(-y_0)$

$= 1 + 0.01(-1) = 0.99$

$y_2 = y(x_2) = y(0.02) = y_1 + hf(x_1, y_1)$

$= 0.99 + 0.01(-y_1)$

$= 0.99 + 0.01(-0.99)$

$= 0.99 - 0.0099 = 0.9801$

$y_3 = y(x_3) = y(0.03) = y_2 + hf(x_2, y_2)$

$= 0.9801 + 0.01(-y_2)$

$= 0.9801 + 0.01(-0.9801) = 0.9703$

$y_4 = y(x_4) = y(0.04) = y_3 + hf(x_3, y_3)$

$= 0.9703 + 0.01(-y_3)$

$= 0.9703 + 0.01(-0.9703) = 0.9606.$

The exact solution is $y = e^{-x}$ and from this the value at x = 0.04 is 0.9608.

Q15. Using Euler's method to solve the differential equation $\frac{dy}{dx} = 1 + y^2$**; y(0) = 0.**

Sol :

Given differential equation,

$$\frac{dy}{dx} = 1 + y^2 \; ; \; y(0) = 0$$

$f(x, y) = 1 + y^2$ and $x_0 = 0$

Let $h = 0.01$

By Euler's method,

$$y_{n+1} = y_n + hf(x_n, y_n) \text{ where } n = 0, 1, 2, 3, ...$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = y(x_1) = y(0.01) = 0 + 0.01(1+0^2)$$

$$= 0.01$$

$$y_2 = y(x_2) = y(0.02) = 0.01 + 0.01(1 + 0.01^2)$$

$$= 0.02$$

$$y_3 = y(x_3) = y(0.03) = 0.02 + 0.01(1 + 0.02^2)$$

$$= 0.03$$

The exact solution is $y = \tan x$ and for this the value at $x = 0.03$ is 0.03.

Q16. Using Euler's method to solve the differential equation $\frac{dy}{dx} = \frac{3}{5}x^3y$, $y(0) = 1$

Sol :

Given differential equation is

$$\frac{dy}{dx} = \frac{3}{5}x^3y \; ; \; y(0) = 1$$

$$f(x, y) = \frac{3}{5}x^3y \text{ and } x_0 = 0$$

let $h = 0.1$

By Euler's method,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y(x_1) = y(0.1) = y_1 = y_0 + h\left(\frac{3}{5}x_0^3y_0\right)$$

$$= 1 + 0.1\left(\frac{3}{5}(0)^3(1)\right)$$

$y_1 = y(0.1) = 1$

$y_2 = y_1 + hf(x_1, y_1)$

$\Rightarrow \quad y_2 = y(0.2) = y(x_2) = 1 + 0.1\left(\frac{3}{5}(0.1)^3(1)\right)$

$= 1.0001$

$y_3 = y_2 + hf(x_2, y_2)$

$\Rightarrow \quad y_3 = y(0.3) = y(x_3) = 1.0001 + 0.1\left(\frac{3}{5}(0.2)^3(1.0001)\right)$

$= 1.0006$

The exact solution of the given differential equation is $y = e^{\frac{3}{5}\left(\frac{x^4}{4}\right)}$ is 1.0012 when x = 0.3.

From this procedure the value of y is 1.0006 when x = 0.3

Error Estimate for the Euler's Method :

Let the true solution of the differential equation at $x = x_n$ be $y(x_n)$ .... (1)

Let the approximate solution be y_n .... (2)

Now, expanding $y(x_{n+1})$ by Taylor's series, we get

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + ...$$

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(\tau_n), \text{ where} \quad .... (3)$$

where $x_n \leq \tau_n \leq x_{n+1}$

Usually, there are two types of errors in the solution of differential equations : They are :

(i) Local errors,

and (ii) Rounding errors.

The local error is the result of replacing the given differential by means of the equation

$$y_{n+1} = y_n + h y'_n$$

This error is given by

$$L_{n+1} = -\frac{1}{2}h^2 y''(\tau_n) \quad .... (4)$$

The total error is then defined by

$$e_n = y_n - y(x_n) \quad .... (5)$$

Since y_0 is exact i.e, $y_0 = y(x_0)$

$\therefore \quad e_0 = 0$

Neglecting the rounding error, we write the total solution error as :

$$e_{n+1} = y_{n+1} - y(x_{n+1})$$

$$e_{n+1} = y_n + h y'_n - \left[y(x_n) + hy'(x_n) + \frac{h^2}{2} y''(\tau_n)\right]$$

$$= y_n + h y'_n - [y(x_n) + hy'(x_n) - L_{n+1}] \qquad \text{(from (4))}$$

$$= e_n + y(x_n) + h y'_n - [y(x_n) + hy'(x_n) - L_{n+1}] \quad (\because \text{from (5)})$$

$$= e_n + h y'_n - hy'(x_n) + L_{n+1}$$

$$= e_n + h[y'_n - y'(x_n)] + L_{n+1}$$

$$\Rightarrow \quad e_{n+1} = e_n + h[f(x_n, y_n) - y'(x_n)] + L_{n+1}$$

$(\because y' = f(x, y))$

$(\Rightarrow y'_n = f(x_n, y_n))$

$$\Rightarrow \quad e_{n+1} = e_n + h[f(x_n, y_n) - f(x_n, y(x_n))] + L_{n+1} \qquad \text{.... (6)}$$

By Mean Value Theorem, we write

$$\frac{\partial f}{\partial y}(x_n, \xi_n) = \frac{f(x_n, y_n) - f(x_n, y(x_n))}{y_n - y(x_n)},$$

$$y(x_n) \le \xi_n \le y_n$$

$$\Rightarrow \quad f(x_n, y_n) - f(x_n, y(x_n)) = [y_n - y(x_n)] \frac{\partial f}{\partial y}(x_n, \xi_n)$$

$$\Rightarrow \quad f(x_n, y_n) - f(x_n, y(x_n)) = (e_n) \frac{\partial f}{\partial y}(x_n, \xi_n) \qquad \text{.... (7)}$$

$(\because \text{ from (5)})$

Substitute (7) in (6)

$$\Rightarrow \quad e_{n+1} = e_n + h\left[(e_n)\frac{\partial f}{\partial y}(x_n, \xi_n) + L_{n+1}\right] = e_n + he_n \frac{\partial f}{\partial y}(x_n, \xi_n) + L_{n+1}$$

$$\Rightarrow \quad e_{n+1} = e_n \left[1 + h\frac{\partial f}{\partial y}(x_n, \xi_n)\right] + L_{n+1}$$

$$e_{n+1} = e_n [1 + hf_y(x_n, \xi_n)] + L_{n+1} \qquad \text{.... (8)}$$

Since $e_0 = 0$

$\Rightarrow \quad e_1 = L_1$

$e_2 = e_1[1 + hf_y(x_1, \xi_1)] + L_{1+1}$

$\Rightarrow \quad e_2 = L_1[1 + hf_y(x_1, \xi_1)] + L_2 (\because e_1 = L_1)$

lly $\quad e_3 = (L_1 + L_2) [1 + hf_y(x_1, \xi_2)] [1 + hf_y(x_1, \xi_1)] + L_3$

an so on.

Q17. We consider, the differential equation y' = – y with the condition y(0) = 1. By Euler's method solve the D.E and find the error.

Sol :

The solution of given D.E by Euler's method is 0.9606

(Refer : Q.no. (2)) by choosing h = 0.01

We have

$1 + hf_y(x_n, \xi_n) = 1 + 0.01(-1) = 0.99$

and $L_{n+1} = -\frac{1}{2} h^2 y''(\tau_n)$

$= -\frac{1}{2}(0.01)^2 y''(\tau_n)$

$\Rightarrow \quad L_{n+1} = -(0.00005)y''(\tau_n) = -(0.00005)y(\tau_n)$

In this problem, $y(\tau_n) \leq y(x_n)$, since y' is negative.

Hence we obtain successively

$|L_1| \leq 0.00005(1) = 5 \times 10^{-5}$

$|L_2| \leq (0.00005)(0.99) < 5 \times 10^{-5}$

$|L_3| \leq (0.00005)(0.9801) < 5 \times 10^{-5}$

and so on.

For computing the total solution error, we need an estimate of the rounding error. If we neglect the rounding error i.e, if we set $R_{n+1} = 0$

Then using the above bound, we obtain from equation (8) the estimates.

$e_0 = 0$

$|e_1| \leq 5 \times 10^{-5}$

$|e_2| \leq 0.99e_1 + 5 \times 10^{-5} < 10^{-4}$

$|e_3| \leq 0.99e_2 + 5 \times 10^{-5} < 10^{-4} + 5 \times 10^{-5}$

$|e_2| \leq 0.99e_3 + 5 \times 10^{-5} < 10^{-4} + 10^{-4}$

$= 2 \times 10^{-4} = 0.0002$

etc.

It can be verified that the estimate for e_4 agrees with the actual error in the value of y(0.04) obtained in problem (1).

Modified Euler's Method :

Let the differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0 \quad \text{.... (1)}$$

Integrating equation (1),

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)dx \quad \text{.... (2)}$$

Instead of approximating f(x, y) by $f(x_0, y_0)$ in equation (2) and apply the trapezodial rule to obtain.

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] \quad \text{.... (3)}$$

Thus, we obtain the iteration formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_1, y_0) + f(x_1, y_1^{(n)})], \ n = 0, 1, 2 \ ...... \quad \text{.... (4)}$$

where $y_1^{(n)}$ is the n^{th} approximation to y_1.

Iteration formula in (4) can be started by choosing $y_1^{(0)}$ from Euler's formula.

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

The general formula for Euler's modified method is,

$$y_i^{(n+1)} = y_{i-1} + \frac{h}{2}[f(x_{i-1}, y_{i-1}) + f(x_i, y_i^{(n-1)})], \ i = 1, 2, 3 \ .....$$

Q18. Determine the value of y when x = 0.1, given that y(0) = 1 and y' = x^2 + y.

Sol :

Given $\frac{dy}{dx} = x^2 + y, y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$

let h = 0.05. where $h = \frac{x_n - x_0}{n} = \frac{0.1-0}{2} = 0.05$

($\because$ n = no. of iterations)

By Euler's formula,

$y_1 = y_0 + hf(x_0, y_0)$

$y_1 = y_0 + h(x_0^2 + y_0)$

$y_1^{(0)} = 1 + 0.05(0 + 1)$

$y_1^{(0)} = 1 + 0.05$

$y_1^{(0)} = 1.05$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

$= y_0 + \frac{h}{2}[(x_0^2 + y_0) + (x_1^2 + y_1^{(0)})]$

$= 1 + \frac{0.05}{2}[(0 + 1) + (0.05^2 + 1.05)]$

$y_1^{(1)} = 1.0513.$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= y_0 + \frac{h}{2}[(x_0^2 + y_0) + (x_1^2 + y_1^{(1)})]$

$= 1 + \frac{0.05}{2}[(0 + 1) + (0.05^2 + 1.0513)]$

$y_1^{(2)} = 1.0513$

$\therefore \quad y_1^{(1)} = y_1^{(2)} = 1.0513$

Hence y_1 = 1.0513 correct to four decimal places.

Now, $y_1 = 1.0513$, h = 0.05, $x_1 = x_0 + h = 0 + 0.05 = 0.05$

$y_2 = y_1 + hf(x_1, y_1)$

$y_2^{(0)} = y_1 + h(x_1^2 + y_1)$

$= 1.0513 + (0.05)(0.05^2 + 1.0513)$

$y_2^{(0)} = 1.1040$

$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$

$y_2^{(1)} = y_1 + \frac{h}{2}[(x_1^2 + y_1) + (x_2^2 + y_2^{(0)})]$

$= 1.0513 + \frac{0.05}{2}[((0.05)^2 + 1.0513) + (1.0^2 + 1.1040)]$

$= 1.0513 + 0.0250\ [1.0538 + 2.1040]$

$= 1.1302$

$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= y_1 + \frac{h}{2}[(x_1^2 + y_1) + (x_2^2 + y_2^{(1)})]$

$= 1.0513 + \frac{0.05}{2}[((0.05)^2 + 1.0513) + (1^2 + 1.1302)]$

$= 1.0513 + 0.025\ [1.0538 + 2.1302]$

$= 1.1309$

$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= y_1 + \frac{h}{2}[(x_1^2 + y_1) + (x_2^2 + y_2^{(2)})]$

$= 1.0513 + \frac{0.05}{2}[((0.05)^2 + 1.0513) + (1^2 + 1.1309)]$

$= 1.0513 + 0.025\ [1.0538 + 2.1309]$

$= 1.1309$

$\therefore \quad y_2^{(2)} = y_2^{(3)} = 1.1309$

Hence, $y_2 = y(x_2) = y(0.1) = 1.1309$ correct to four decimal places.

Q19. Given differential equation $\frac{dy}{dx} = x^2 + y$ with y(0) = 1 compute y(0.02) using Euler's modified method.

Sol :

Given

$$\frac{dy}{dx} = x^2 + y, y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$$

Let $h = \frac{x_n - x_0}{n} = \frac{0.02 - 0}{2} = 0.01$

By Euler's formula,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= y_0 + h(x_0^2 + y_0) = 1 + 0.01(0^2 + 1)$$

$$= 1 + 0.01 = 1.01$$

$$\therefore \; y_1^{(0)} = 1.01$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})$$

$$= y_0 + \frac{h}{2}[(x_0^2 + y_0) + (x_1^2 + y_1^{(0)})]$$

$$(x_1 = x_0 + h = 0 + 0.01 = 0.01)$$

$$= 1 + \frac{0.01}{2}[(0^2 + 1) + (0.01^2 + 1.01)]$$

$$= 1 + 0.0050 [1 + 1.0101]$$

$$y_1^{(1)} = 1.0101$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})$$

$$= y_0 + \frac{h}{2}[(x_0^2 + y_0) + (x_1^2 + y_1^{(1)})]$$

$$= 1 + 0.005[1 + ((0.01)^2 + 1.0101)]$$

$$= 1 + 0.005 [1 + 1.0102]$$

$$y_1^{(2)} = 1.0101$$

$\therefore \quad y_1^{(1)} = y_1^{(2)} = 1.0101$

$\therefore \quad y_1 = 1.0101$ correct to 4 decimal places.

Now, $y_1 = 1.0101,\ h = 0.01,\ x_1 = x_0 + h = 0 + 0.01 = 0.01$

By Euler's method

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_2^{(0)} = y_1 + h(x_1^2 + y_1)$$

$$= 1.0101 + 0.01(0.01^2 + 1.0101)$$

$$y_2^{(0)} = 1.0202$$

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$$

$$= y_1 + \frac{h}{2}[(x_1^2 + y_1) + (x_2^2 + y_2^{(0)})]$$

$$(x_2 = x_1 + h = 0.01 + 0.01 = 0.02)$$

$$= 1.0101 + \frac{0.01}{2}[((0.01)^2 + 1.0101) + (0.02^2 + 1.0202)]$$

$$= 1.0101 + 0.005\,[1.0102 + 1.0206]$$

$$= 1.0203$$

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= y_1 + \frac{h}{2}[(x_1^2 + y_1) + (x_2^2 + y_2^{(1)})]$$

$$= 1.0101 + \frac{0.01}{2}[((0.01)^2 + 1.0101) + (0.02^2 + 1.0203)]$$

$$= 1.0101 + 0.005\,[1.0102 + 1.0207]$$

$$= 1.0203$$

$\therefore \quad y_2^{(1)} = y_2^{(2)} = 1.0203$

Hence, $y_2 = y(x_2) = y(0.02) = 1.0203$ correct to four decimal places.

Q20. Solve, by Euler's modified method, the problem $\frac{dy}{dx} = x + y, y(0) = 0$.

Choose h = 0.2 and compute y(0.2) and y(0.4)

Sol :

Given $\frac{dy}{dx} = x + y, y(0) = 0 \Rightarrow x_0 = 0, y_0 = 0$

here $f(x,y) = x + y$

Let $h = 0.2$

By Euler's formula,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= y_0 + h(x_0 + y_0)$$

$$= 0 + 0.2(0 + 0)$$

$$y_1^{(0)} = 0$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= y_0 + \frac{h}{2}[(x_0 + y_0) + (x_1 + y_1^{(0)})] \qquad (\because x_1 = x_0 + h = 0 + 0.2 = 0.2)$$

$$= 0 + \frac{0.2}{2}[(0 + 0) + (0.2 + 0)]$$

$$y_1^{(1)} = 0.1 [0 + 0.2] = 0.02$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= y_0 + \frac{h}{2}[(x_0 + y_0) + (x_1 + y_1^{(1)})]$$

$$= 0 + \frac{0.2}{2}[(0 + 0) + (0.2 + 0.02)]$$

$$= 0.1 [0 + 0.22]$$

$$= 0.022$$

$$y_1^{(2)} = 0.002$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= y_0 + \frac{h}{2}[(x_0 + y_0) + (x_1 + y_1^{(2)})]$$

$$= 0 + \frac{0.2}{2}[(0 + 0) + (0.2 + 0.022)]$$

$$= 0.1\,[0 + 0.222]$$

$$y_1^{(3)} = 0.0222$$

$$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$= y_0 + \frac{h}{2}[(x_0 + y_0) + (x_1 + y_1^{(3)})]$$

$$= 0 + \frac{0.2}{2}[(0 + 0) + (0.2 + 0.0222)]$$

$$= 0.1\,[0 + 0.2222]$$

$$y_1^{(4)} = 0.0222$$

$$\therefore \quad y_1^{(3)} = y_1^{(4)} = 0.0222$$

Hence $y_1 = y(x_1) = y(0.2) = 0.0222$ correct to four decimal places.

Now $y_1 = 0.0222$, $h = 0.2$, and $x_1 = 0.2$

By Euler's method

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_2^{(0)} = y_1 + h(x_1 + y_1)$$

$$= 0.0222 + 0.2(0.2 + 0.0222)$$

$$y_2^{(0)} = 0.0666$$

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$$

$$= y_1 + \frac{h}{2}[(x_1 + y_1) + (x_2 + y_2^{(0)})]$$

$$(x_2 = x_1 + h = 0.2 + 0.2 = 0.4)$$

$$= 0.0222 + \frac{0.2}{2}[(0.2 + 0.0222) + (0.4 + 0.0666)]$$

$$= 0.0222 + 0.1\,[0.2222 + 0.4666]$$

$y_2^{(1)} = 0.0911$

$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= y_1 + \frac{h}{2}[(x_1 + y_1) + (x_2 + y_2^{(1)})]$

$= 0.0222 + \frac{0.2}{2}[(0.2 + 0.0222) + (0.4 + 0.0911)]$

$= 0.0222 + 0.1\ [0.2222 + 0.4911]$

$y_2^{(2)} = 0.0935$

$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= y_1 + \frac{h}{2}[(x_1 + y_1) + (x_2 + y_2^{(2)})]$

$= 0.0222 + \frac{0.2}{2}[(0.2 + 0.0222) + (0.4 + 0.0935)]$

$= 0.0222 + 0.1\ [0.2222 + 0.4935]$

$y_2^{(3)} = 0.0938$

$y_2^{(4)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(3)})]$

$= y_1 + \frac{h}{2}[(x_1 + y_1) + (x_2 + y_2^{(3)})]$

$= 0.0222 + \frac{0.2}{2}[(0.2 + 0.0222) + (0.4 + 0.0938)]$

$= 0.0222 + 0.1\ [0.2222 + 0.4938]$

$y_2^{(4)} = 0.0938$

$y_2^{(3)} = y_2^{(4)} = 0.0938$ correct to four decimal places.

$\therefore\ y_2 = 0.0938$

$\therefore\ y_2 = y(x_2) = y(0.4) = 0.0938$

4.4 Runge - Kutta Method

Q21. Define Runge - Kutta Formulas.

Sol :

The Runge - Kutta methods are designed to give greater accuracy and they possess the advantages of requiring only the function values at some selected points on the subinterval.

We consider the differential equation

$y' = f(x_1, y)$ with the initial condition $y(x_0) = y_0$ .... (1)

Integrating equation (1), we obtain

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\,dx \quad .... (2)$$

Instead of approximating f(x, y) by $f(x_0, y_0)$ in equation (2)

we now approximate the integral given in equation (2) by means of trapezodial rule to obtain.

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] \quad .... (3)$$

If we substitute $y_1 = y_0 + hf(x_0, y_0)$ on right side of equation (3)

we obtain,

$$y_1 = y_0 + \frac{h}{2}[f_0 + f(x_0 + h, y_0 + hf_0)]$$

then the above equation becomes

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf_0$ and $k_2 = hf(x_0 + h, y_0 + k_1)$

The fourth - order runge kutta formula is,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Q22. Given $\frac{dy}{dx}$ = y – x where y(0) = 2.

find y(0.1) and y(0.2) correct to four decimal places by Runge - kutta method.

Sol

i) Runge Kutta second order formula :

Given $\frac{dy}{dx} = y - x, y(0) = 2 \Rightarrow x_0 = 0, y_0 = 2$

let h = 0.1,

here $f(x, y) = y - x$ .... (1)

By second order formula,

$$y_1 + y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf_0$ and $k_2 = hf(x_0 + h, y_0 + k_1)$

$f_0 = f(x_0, y_0) = y_0 - x_0$ (By (1))

$k_1 = hf(x_0, y_0) = h(y_0 - x_0) = 0.1\ (2 - 0)$

$= 0.1(2)$

$k_1 = 0.2$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$= h((y_0 + k_1) - (x_0 + h))$

$= 0.1((2 + 0.2) - (0 + 0.1))$

$= 0.1(2.2 - 0.1)$

$= 0.1(2.1)$

$k_2 = 0.21$

$\therefore \quad y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

$y_1 = 2 + \frac{1}{2}(0.2 + 0.21) = 2.2050$

$\therefore \quad y_1 = y(x_1) = y(0.1) = 2.2050$

$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

$k_1 = hf(x_1, y_1)$ $(\because x_1 = x_0 + h = 0 + 0.1 = 0.1)$

$= h[y_1 - x_1] = 0.1[2.2050 - 0.1]$

$= 0.2105$

$k_2 = hf(x_1 + h, y_1 + k_1)$

$= h[(y_1 + k_1) - (x_1 + h)]$

$= 0.1[(2.2050 + 0.2105) - (0.1 + 0.1)]$

$k_2 = 0.1[2.4155 - 0.2]$

$k_2 = 0.22155$

Now $y_2 = 2.2050 + \frac{1}{2}(0.2105 + 0.22155)$

$\Rightarrow \quad y_2 = 2.2050 + 0.2160$

$y_2 = 2.4210$

$\therefore \quad y_2 = y(x_2) = y(0.2) = 2.4210$

ii) Runge - kutta fourth order formula :

To determine y(0.1),

we have, $x_0 = 0, y_0 = 2, h = 0.1$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where $k_1 = hf(x_0, y_0)$

$= h[y_0 - x_0] = 0.1[2 - 0] = 0.2$

$k_2 = hf\left[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right]$

$= h\left[\left(y_0 + \frac{1}{2}k_1\right) - \left(x_0 + \frac{1}{2}h\right)\right]$

$= 0.1\left[\left(2 + \frac{1}{2}(0.2)\right) - \left(0 + \frac{1}{2}(0.1)\right)\right]$

$= 0.1[(2 + 0.1) - (0 + 0.05)]$

$= 2.205$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= h\left[\left(y_0 + \frac{1}{2}k_2\right) - \left(x_0 + \frac{1}{2}h\right)\right]$$

$$= 0.1\left[\left(2 + \frac{1}{2}(2.205)\right) - \left(0 + \frac{1}{2}(0.1)\right)\right]$$

$$= 0.1[(2 + 0.1025) - (0.05)]$$

$$k_3 = 0.2053$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= h[(y_0 + k_3) - (x_0 + h)]$$

$$= 0.1[(2 + 0.2053) - (0 + 0.1)]$$

$$= 0.1[2.2053 - 0.1]$$

$$k_4 = 0.2105$$

Now, $y_1 = y(x_1) = y(0.1) = 2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 2 + \frac{1}{6}(0.2 + 2(0.205) + 2(0.2053) + 0.2105)$$

$$= 2 + \frac{1}{6}(0.2 + 0.41 + 0.4106 + 0.2105)$$

$$= 2.2052$$

$\therefore \quad y_1 = y(x_1) = y(0.1) = 2.2052$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_1, y_1)$$

$$= h[y_1 - x_1] = 0.1(2.2052 - 0.1)$$

$$= 0.2105$$

$$k_2 = hf\left[\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)\right]$$

$$= hf\left[\left(y_1 + \frac{1}{2}k_1\right) - \left(x_1 + \frac{1}{2}h\right)\right]$$

$$= 0.1\left[\left(2.2052 + \frac{1}{2}(0.2105)\right) - \left(0.1 + \frac{1}{2}(0.1)\right)\right]$$

$$= 0.1[(2.2052 + 0.1053) - (0.1 + 0.05)]$$

$$= 0.1[2.3105 - 0.15]$$

$$k_2 = 0.2161$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right)$$

$$= h\left[\left(y_1 + \frac{1}{2}k_2\right) - \left(x_1 + \frac{1}{2}h\right)\right]$$

$$= 0.1\left[\left(2.2052 + \frac{1}{2}(0.2161)\right) - \left(0.1 + \frac{1}{2}0.1\right)\right]$$

$$= 0.1[(2.2052 + 0.1080) - (0.15)]$$

$$k_3 = 0.2163$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= h[(y_1 + k_3) - (x_1 + h)]$$

$$= 0.1[(2.2052 + 0.2163) - (0.1 + 0.1)]$$

$$= 0.1[2.4215 - 0.2]$$

$$k_4 = 0.2222$$

$$y_2 = y(x_2) = y(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2.2052 + \frac{1}{6}(0.2105 + 2(0.2161) + 2(0.2163) + 0.2222)$$

$$= 2.2052 + \frac{1}{6}(0.2105 + 0.4322 + 0.4326 + 0.2222)$$

$$y(0.2) = 2.4215$$

Q23. Given $\frac{dy}{dx} = 1 + y^2$ where y = 0 when x = 0

find y(0.2), y(0.4) and y(0.6) by using Runge - kutta fourth order formula.

Sol :

Given $\frac{dy}{dx} = 1 + y^2$, $x_0 = 0$ and $y_0 = 0$

let h = 0.2

Now $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0)$

$= h[1 + y_0^2] = 0.2\,[1 + 0^2] = 0.2$

$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

$= h\left[1 + \left(y_0 + \frac{1}{2}k_1\right)^2\right]$

$= 0.2\left[1 + \left(0 + \frac{1}{2}(0.2)\right)^2\right]$

$k_2 = 0.2020$

$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$

$= h\left[1 + \left(y_0 + \frac{1}{2}k_2\right)^2\right]$

$= 0.2\left[1 + \left(0 + \frac{1}{2}(0.202)\right)^2\right]$

$= 0.20204$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= h[1 + (y_0 + k_3)^2] = 0.2[1 + (0 + 0.20204)^2]$

$k_4 = 0.20816$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0 + \frac{1}{6}[0.2 + 2(0.2020) + 2(0.20204) + 0.20816]$$

$$= \frac{1}{6}[1.21624]$$

$$y_1 = 0.20271$$

$\therefore$ $y_1 = y(x_1) = y(0.2) = 0.2027$ correct to foue decimal places.

To compute $y_2 = y(x_2) = y(0.4)$

let $x_1 = x_0 + h = 0.2$, $y_1 = 0.2027$ and $h = 0.2$

$$k_1 = hf(x_1, y_1)$$

$$= h[1 + y^2] = 0.2[1 + 0.2027^2] = 0.2082$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) \Rightarrow k_2 = h\left[1 + \left(y_1 + \frac{1}{2}k_1\right)^2\right] = 0.2\left[1 + \left(0.2027 + \frac{1}{2}(0.2082)\right)^2\right]$$

$$= 0.2\left[1 + (0.3068)^2\right] = 0.2188$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right)$$

$$= h\left[1 + \left(y_1 + \frac{1}{2}k_2\right)^2\right] = 0.2\left[1 + \left(0.2027 + \frac{1}{2}(0.2188)\right)^2\right]$$

$$= 0.2[1 + (0.3121)^2] = 0.2195$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= h[1 + (y_1 + k_3)^2] = 0.2[1 + (0.2027 + 0.2195)^2]$$

$$= 0.2[1 + (0.4222)^2] = 0.2356$$

Now $y(0.4) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 0.2027 + \frac{1}{6}(0.2082 + 2(0.2188) + 2(0.2195) + 0.2356)$$

$y(0.4) = 0.4228$ correct to four decimal places.

To compute $y_3 = y(x_3) = y(0.6)$

Let $x_2 = x_1 + h = 0.2 + 0.2 = 0.4,\ y_2 = 0.4228$

$k_1 = hf(x_2, y_2)$

$= h[1+ y_2^2] = 0.2[1 + (0.4228)^2] = 0.2358$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right)$$

$$= h\left(1+\left(y_2 + \frac{1}{2}k_1\right)^2\right) = 0.2\left(1+\left(0.4228 + \frac{1}{2}(0.2358)\right)^2\right)$$

$= 0.2[1 + 0.5407^2] = 0.2585$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right)$$

$$= h\left(1+\left(y_2 + \frac{1}{2}k_2\right)^2\right) = 0.2\left[1+\left(0.4228 + \frac{1}{2}(0.2585)\right)^2\right]$$

$= 0.26095$

$k_4 = hf(x_2 + h, y_2 + k_3)$

$= h[1 + (y_2 + k_3)^2] = 0.2[1 + (0.4228 + 0.26095)^2]$

$= 0.2935$

Now, $y_3 = y(0.6) = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.4228 + \frac{1}{6}[0.2358 + 2(0.2585) + 2(0.26095) + 0.2935]$

$= 0.68417$

$\therefore\ y_3 = y(0.6) = 0.6842$ correct to four decimal places.

Q24. Solve the initial value problem defined by $\frac{dy}{dx} = \frac{3x+y}{x+2y}$, y(1) = 1 and find y(1.2) by Runge - kutta fourth order formula.

Sol :

$$\frac{dy}{dx} = \frac{3x+y}{x+2y},\ y(1) = 1$$

$x_0 = 1$ and $y_0 = 1$

Let take h = 0.1

By Runge - kutta fourth order formula,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad .... (1)$$

where $k_1 = hf(x_0, y_0)$

$$= h\left[\frac{3x_0 + y_0}{x_0 + 2y_0}\right] = 0.1\left[\frac{3(1)+1}{1+2(1)}\right] = 0.1\left[\frac{3+1}{3}\right] = 0.1\left[\frac{4}{3}\right] = 0.1333$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$= h\left[\frac{3\left(x_0 + \frac{1}{2}h\right) + \left(y_0 + \frac{1}{2}k_1\right)}{\left(x_0 + \frac{1}{2}h\right) + 2\left(y_0 + \frac{1}{2}k_1\right)}\right] = 0.1\left[\frac{3\left(1 + \frac{1}{2}(0.1)\right) + \left(1 + \frac{1}{2}(0.1333)\right)}{\left(1 + \frac{1}{2}(0.1)\right) + 2\left(1 + \frac{1}{2}(0.1333)\right)}\right]$$

$$= 0.1\left[\frac{3(1+0.05)+(1+0.0667)}{(1+0.05)+2(1+0.0667)}\right] = 0.1\left[\frac{4.2167}{3.1834}\right] = 0.1325$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= h\left[\frac{3\left(x_0 + \frac{1}{2}h\right) + \left(y_0 + \frac{1}{2}k_2\right)}{\left(x_0 + \frac{1}{2}h\right) + 2\left(y_0 + \frac{1}{2}k_2\right)}\right] = 0.1\left[\frac{3\left(1 + \frac{1}{2}(0.1)\right) + \left(1 + \frac{1}{2}(0.1325)\right)}{\left(1 + \frac{1}{2}(0.1)\right) + 2\left(1 + \frac{1}{2}(0.1325)\right)}\right]$$

$$= 0.1\left[\frac{3.15 + 1.0663}{1.05 + 2.1326}\right] = 0.1\left[\frac{4.2163}{3.1826}\right]$$

$$k_3 = 0.1325$$

$$k_4 = hf[x_0 + h, y_0 + k_3]$$

$$= h\left[\frac{3(x_0 + h) + (y_0 + k_3)}{(x_0 + h) + 2(y_0 + k_3)}\right] = 0.1\left[\frac{3(1+0.1)+(1+0.1325)}{(1+0.1)+2(1+0.1325)}\right]$$

$$= 0.1\left[\frac{3.3 + 1.1325}{1.1 + 2.265}\right] = 0.1\left[\frac{4.4325}{3.365}\right]$$

$$k_4 = 0.1317$$

Sub k_1, k_2, k_3, k_4 in (1)

$$y_1 = 1 + \frac{1}{6}[0.1333 + 2(0.1325) + 2(0.1325) + 0.1317]$$

$$= 1 + \frac{1}{6}[0.7950]$$

$y_1 = 1.1325$

$\therefore\ y_1 = y(x_1) = y(1.1) = 1.1325$

To determine y(1.2)

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad .... (2)$$

let $y_1 = 1.1325$, $x_1 = 1.1$

$k_1 = hf(x_1, y_1)$

$$= h\left[\frac{3x_1 + y_1}{x_1 + 2y_1}\right] = 0.1\left[\frac{3(1.1) + 1.1325}{1.1 + 2(1.1325)}\right]$$

$$= 0.1\left[\frac{4.4325}{3.3650}\right] = 0.1317$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)$$

$$= h\left[\frac{3\left(x_1 + \frac{1}{2}h\right) + \left(y_1 + \frac{1}{2}k_1\right)}{\left(x_1 + \frac{1}{2}h\right) + 2\left(y_1 + \frac{1}{2}k_2\right)}\right] = 0.1\left[\frac{3\left(1.1 + \frac{1}{2}(0.1)\right) + \left(1.1325 + \frac{1}{2}(0.1317)\right)}{\left(1.1 + \frac{1}{2}(0.1)\right) + 2\left(1.1325 + \frac{1}{2}(0.1317)\right)}\right]$$

$$= 0.1\left[\frac{3.45 + 1.1984}{1.15 + 2.3968}\right] = 0.1\left[\frac{4.6484}{3.5468}\right]$$

$k_2 = 0.1311$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right)$$

$$= h\left[\frac{3\left(x_1 + \frac{1}{2}h\right) + \left(y_1 + \frac{1}{2}k_2\right)}{\left(x_1 + \frac{1}{2}h\right) + 2\left(y_1 + \frac{1}{2}k_1\right)}\right] = 0.1\left[\frac{3\left(1.1 + \frac{1}{2}(0.1)\right) + \left(1.1325 + \frac{1}{2}(0.1311)\right)}{\left(1.1 + \frac{1}{2}(0.1)\right) + 2\left(1.1325 + \frac{1}{2}(0.1311)\right)}\right]$$

$$= 0.1\left[\frac{3.45+1.981}{1.15+2.3961}\right] = 0.1\left[\frac{4.6481}{3.5461}\right]$$

$k_3 = 0.1311$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$$= h\left[\frac{3(x_1+h)+(y_1+k_3)}{(x_1+h)+2(y_1+k_3)}\right] = 0.1\left[\frac{3(1.1+0.1)+(1.1325+0.1311)}{(1.1+0.1)+2(1.1325+0.1311)}\right]$$

$$= 0.1\left[\frac{3.6+1.2636}{1.2+2.5272}\right] = 0.1\left[\frac{4.8636}{3.7272}\right]$$

$k_4 = 0.1305$

sub k_1, k_2, k_3, k_4 in (2)

$$y_2 = 1.1325 + \frac{1}{6}[0.1317 + 2(0.1311) + 2(0.1311) + 0.1305]$$

$y_2 = 1.2636$

Choose the Correct Answer

1. Use Euler's method to calculate the approximation of y(0.2) where y(x) is the solution of the initial - value problem that is as follows. $y^{11} + xy^1 + y = 0$, y (0) = 2, $y^1(0) = 3$ [a]
   (a) $y(0.2) \approx 2.58$ (b) $y(0.2) \approx 2.458$
   (c) $y(0.2) \approx 2.5$ (d) $y(0.2) \approx 2.542$

2. Use the implicit Euler method to approximate y(3) for $y^1 = 5y$, given that y(0) = 4, using a time step of h = 1 [c]
   (a) –16 (b) $\frac{1}{32}$
   (c) $\frac{-1}{16}$ (d) 16

3. Use two steps of Euler's method with h = 0.1 on $y^1 = x\sqrt{y}$, y(1) = 4 [c]
   (a) 4.420 (b) 4.408
   (c) 4.425 (d) 4.413

4. Modified Euler's method has a transation error of the order of [b]
   (a) h (b) h^2
   (c) h^3 (d) h^4

5. Solve by using Euler's method the following differential equation for x = 1 by taking h = 0.2, $\frac{dy}{dx} = xy$, y=1 when x = 0 [b]
   (a) 1.5896 (b) 1.4593
   (c) 1.3495 (d) 0.4593

6. Which of the following methods agrees with Taylor's series solution up to term in h^4? [b]
   (a) Modified Euler's Method (b) Fourth order Runge - Kutta method
   (c) Picard's Method (d) None

7. If $\frac{dy}{dx} = x + y$, y(0) = 1 using Runge Kutta method the value of y at x = 0.2 when h = 0.2 is [a]
   (a) 1.2 (b) 1.4
   (c) 1 (d) 1.48

8. Non - linear differential equations solved by [b]
   (a) Trapezoidial rule (b) Range - kutta method
   (c) Forward Newton's Interpolation (d) None

9. Runge - kutta fourth order formula is $y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$ where K_1 = [a]
   (a) $hf(x_0, y_0)$ (b) $hf(x_0 + \frac{1}{2}, y_0)$
   (c) $kf(x_0, y_0 + \frac{1}{2})$ (d) None

10. Which method do you expect to give the more accuate sol^n [c]
   (a) Euler's method (b) Modified Euler's method
   (c) Both (d) None

Fill in the blanks

1. The Runge - kutta second order formula is y_1 = ________.
2. The taylor series expansion of $\frac{\sin x}{x}$ near origin is ________.
3. Consider the initial value problem $\frac{dy}{dx} = y^2 - t = 0$; y (0) = 1 using Euler's method with h = 1, the approximate value for y(2) is ________.
4. The taylor series expansion of 3sinx + 2cos x is ________.
5. ________ method agrees with Taylor's series solution up to term in h^4
6. Range - kutta method is used to sovle a ________.
7. The general formula for Euler's method is ________.
8. The Range - kutta fourth order formula is ________.
9. Picard's iteration formula is y_n = ________.
10. In Range - kutta fourth order formula, K_2 = ________.

ANSWERS

1. $y_0 + \frac{1}{2}(K_1 + K_2)$
2. $1 - \frac{x^2}{6} + ...$
3. 7
4. $2 + 3x - x^2 - \frac{x^3}{2} + ...$
5. Fourth - order
6. Simultaneous non - linear equation
7. $y_{n+1} = y_n + hf(x_n, y_n)$
8. $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
9. $y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$
10. $k_2 = h f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$

FACULTY OF SCIENCE

B.Sc. III Year VI Semester (CBCS) Examination

MODEL PAPER - I

NUMERICAL ANALYSIS

(MATHEMATICS)

Time : 3 Hours] [Max. Marks : 80

PART - A (8 × 4 = 32 Marks)

Note : Answer any **EIGHT** of the following questions.

ANSWERS

1. Define absolute, relative and percentage error. **(Unit-I, Q.No. 1)**
2. Derive the formula of part's false position method. **(Unit-I, Q.No. 86)**
3. Fine the relative error if $\frac{2}{3}$ is approximate the 0.667. **(Unit-I, Q.No. 3)**
4. Define interpolation. **(Unit-II, Q.No. 1)**
5. Using the method of separation of symbols, show that **(Unit-II, Q.No. 9)**

   $\Delta^n u_{x-n} = u_x - nu_{x-1} + \frac{n(n-1)}{2} u_{x-2} + ... + (-1)^n u_{x-n}$
6. Derine Gauss forward formula. **(Unit-II, Q.No. 22)**
7. Write some types of non-liner curve fitting. **(Unit-III, Q.No. 11)**
8. Write the formula for Boole's and weddle's rule. **(Unit-III, Q.No. 33)**
9. Evaluate $\int_0^1 \frac{1}{1+x} dx$ with $h = \frac{1}{6}$ by simpson's rule $\frac{1}{3}$ - rule. **(Unit-III, Q.No. 41)**
10. Write the Euler's method. **(Unit-IV, Q.No. 13)**
11. Using Taylor's series find. **(Unit-IV, Q.No. 7)**
12. If $\frac{dy}{dx} = \frac{1}{x^2 + y}$ with y(4) = 4. Compute the values of y(4.1) and **(Unit-IV, Q.No. 6)**

    y(4.2) by Taylor's series method.

PART - B (4 × 12 = 48 Marks)

Note : Answer **all** the questions.

13. (a) Find the real root of eqution $f(x) = x^3 - x - 1 = 0$ using bisection method. **(Unit-I, Q.No. 13)**

    OR

    (b) Find the root of the equation $y(x) = x^3 - 2x - 5 = 0$ which lies between 2 and 3 by muller's method. **(Unit-I, Q.No. 45)**

14. (a) Using stirling's formula and cos (0.17) given that cos 0 = 1; cos (0.05) = 0.9988, cos (0.10) = 0.9950, cos (0.15) = 0.9888, cos(0.20) = 0.9801, cos(0.25) = 0.9689 and cos(0.30) = 0.9553. **(Unit-II, Q.No. 29)**

OR

(b) Find the missing term in the following data. **(Unit-II, Q.No. 11)**

x	0	1	2	3	4
y	1	3	9	-	81

15. (a) Fit a straight line to the form $y = a_0 + a_1 x$ for the following data: **(Unit-III, Q.No. 8)**

x	0	5	10	15	20	25
y	12	15	17	22	24	30

OR

(b) Evaluate $I = \int_0^1 \frac{1}{1+x} dx$ correct to three decimal place by trapezoidals rule rule with h = 0.125. **(Unit-III, Q.No. 34)**

16. (a) Given differential equatio $\frac{dy}{dx} = x^2 + y$ with y(0) = 1 compute y(0. 02) using Euler's modified method. **(Unit-IV, Q.No. 19)**

OR

(b) Given $\frac{dy}{dx} = y - x$ where y(0) = 2 find y(0.1) and y(0.2) by Runge - kutta fourth order formula. **(Unit-IV, Q.No. 22)**

FACULTY OF SCIENCE

B.Sc. III Year VI Semester (CBCS) Examination

MODEL PAPER - II

NUMERICAL ANALYSIS

(MATHEMATICS)

Time : 3 Hours] [Max. Marks : 80

PART - A (8 × 4 = 32 Marks)

Note : Answer any **EIGHT** of the following questions.

ANSWERS

1. If $u = 3v^7 - 6v$ find the percentage error in u at $v = 1$ of the error in v is 0.05. **(Unit-I, Q.No. 9)**
2. Find a real root of $f(x) = x^3 + x^2 + x + 7 = 0$ correct to three decimal places. **(Unit-I, Q.No. 15)**
3. Find the relative and percentage error in $u = 6v^5 - 3v^4$ at $v = 1.5 \pm 0.0025$. **(Unit-I, Q.No. 10)**
4. Prove that $\nabla \equiv 1 - E^{-1}$ **(Unit-II, Q.No. 4)**
5. Derive Bessel's Formula. **(Unit-II, Q.No. 33)**
6. Show that $e^x(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + ...) = u_0 + u_1 x + u^2\frac{x^2}{2!} + ...$ **(Unit-II, Q.No. 10)**
7. Explain least square curve fitting. **(Unit-III, Q.No. 1)**
8. From the following values of x and y, find $\frac{dy}{dx}$ when **(Unit-III, Q.No. 25)**

   (a) $x = 1$, (b) $x = 3$ (c) $x = 6$ and (d) $\frac{d^2y}{dx^2}$ at $x = 3$.

x	0	1	2	3	4	5	6
y	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506	9.0309

9. Evaluate $\int_0^2 \frac{dx}{x^3 + x + 1}$ by simpson's $\frac{1}{3}$ rule with $h = 0.25$. **(Unit-III, Q.No. 36)**
10. Write the procedure of Picard's Method. **(Unit-IV, Q.No. 8)**
11. Define Runge - Kutta Formulas. **(Unit-IV, Q.No. 21)**
12. Use Picard's method to obtain y(0.1) and y(0.2) of the problem **(Unit-IV, Q.No. 11)**

    defined by $\frac{dy}{dx} = x + yx^4$, $y(0) = 3$.

PART - B (4 × 12 = 48 Marks)

Note : Answer **all** the questions.

13. (a) Find a real root of $x = \frac{1}{(x+1)^2}$ by iteration method **(Unit-I, Q.No. 23)**

OR

(b) Solve the system of equations (Unit-I, Q.No. 49)

$y^2 - 5y + 4 = 0$ and $3yx^2 - 10x + 7 = 0$ by Newton - Raphson Method.

14. (a) Derive Newton's forward difference interpolation formula. (Unit-II, Q.No. 13)

OR

(b) Find y(2) from the following data using Lagrange's formula. (Unit-II, Q.No. 39)

x	0	1	3	4	5
y	0	1	81	256	625

15. (a) Estimate the value of the integral $\int_1^3 \frac{1}{x}dx$ by Simpson's rule with 4 strips and 8 strips respectively. (Unit-III, Q.No. 37)

OR

(b) By using Weddle's rule to obtain an approximate value of π from the formula $\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}dx$ with $h = \frac{1}{6}$. Evaluate the above integral by using simpson's $\frac{1}{3}$ - rule with nine ordinates and compare the results. (Unit-III, Q.No. 40)

16. (a) Given the differential equation $y'' - xy' - y = 0$ with the conditions y(0) and $y'(0) = 0$. Use Taylor's series method to determine the value of y(0.1) (Unit-IV, Q.No. 3)

OR

(b) Solve, by Euler's modified method, the problem $\frac{dy}{dx} = x + y$, $y(0) = 0$. Choose $h = 0.2$ and compute y(0.2) and y(0.4). (Unit-IV, Q.No. 20)

FACULTY OF SCIENCE

B.Sc. III Year VI Semester (CBCS) Examination

MODEL PAPER - III

NUMERICAL ANALYSIS

(MATHEMATICS)

Time : 3 Hours] [Max. Marks : 80

PART - A (8 × 4 = 32 Marks)

Note : Answer any **EIGHT** of the following questions.

ANSWERS

1. Find the relative error in $u = \frac{5xy^2}{z^3}$ with $\partial x = \partial y = \partial z = 0.001$ and $x = y = z = 1$. **(Unit-I, Q.No. 11)**
2. Find the percentage error if 625.483 is approximated to three significant figures. **(Unit-I, Q.No. 4)**
3. An approximate value of π is given by $x_1 = 3.1428571$ and its true value is $x = 3.1415926$. Find the absolute and relative errors. **(Unit-I, Q.No. 6)**
4. Prove that $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$ **(Unit-II, Q.No. 7)**
5. Prove that $\nabla \equiv 1 - E^{-1}$ **(Unit-II, Q.No. 4)**
6. Tabulate $y = x^3$ for $x = 2, 3, 4$ and 5 and calculate the cube root of 10 correct to three decimal places. **(Unit-II, Q.No. 56)**
7. By the method of least squares, find the straight line by the following data. **(Unit-III, Q.No. 10)**

x	1	2	3	4	5
y	14	27	40	55	68

8. Write the types of errors in Numerical Differentiation. **(Unit-III, Q.No. 26)**
9. Solve $y(x) = \sin x$ in $[0, \pi]$ by cubic spline method. **(Unit-III, Q.No. 28)**
10. Using Euler's method to solve the differential equation $\frac{dy}{dx} = 1 + y^2$; $y(0) = 0$. **(Unit-IV, Q.No. 15)**
11. Solve the equation $y' = x + y^2$, subject to the condition $y = 1$ when $x = 0$ by Picard's method. **(Unit-IV, Q.No. 9)**
12. Define Runge - Kutta Formulas. **(Unit-IV, Q.No. 21)**

PART - B (4 × 12 = 48 Marks)

Note : Answer **all** the questions.

13. (a) Using Newton - Raphson method find a real root correct to 3 decimal places of the equation $\sin x = \frac{x}{2}$ given that the root lies between $\frac{\pi}{2}$ and π. **(Unit-I, Q.No. 38)**

OR

(b) Use the method of iteration correct to 4 decimal places, find a root of the equation $e^x = 3x$. **(Unit-I, Q.No. 24)**

14. (a) Using Gauss' Backward formula, find the value of f(32) given that

$f(25) = 0.2707$; $f(30) = 0.3027$; $f(35) = 0.3386$ and $f(40) = 0.3794$. **(Unit-II, Q.No. 26)**

OR

(b) From the table of values of x and e^x are (1.4, 4.0552), (1.5, 4.4817), (1.7, 5.4739) find (1.6, 4.9530), x when $e^x = 4.7115$ using the method of successive approximations. **(Unit-II, Q.No. 57)**

15. (a) Find the parabola of the form $y = a + bx + cx^2$ passing through the points (–1, 2), (0, 1) and (1, 4). **(Unit-III, Q.No. 15)**

OR

(b) Evaluate $\int_0^1 \frac{1}{1+x} dx$ with $h = \frac{1}{6}$ by simpson's rule $\frac{1}{3}$ - rule. **(Unit-III, Q.No. 41)**

16. (a) Using Taylor's series find y(0.1), y(0.2) and y(0.3) given $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$. **(Unit-IV, Q.No. 7)**

OR

(b) Given $\frac{dy}{dx} = 1 + y^2$ where y = 0 when x = 0, find y(0.2), y(0.4) and y(0.6) by using Runge - kutta fourth order formula. **(Unit-IV, Q.No. 23)**